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Intersections of an interval by a difference of a compound Poisson process and a compound renewal process

V. Kadankov, [*] T. Kadankova, [†]N. Veraverbeke [‡]

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Abstract

In this article we determine the Laplace transforms of the one-boundary characteristics and of the distribution of the number of intersections of a fixed interval by a difference of a compound Poisson process and a compound renewal process. The results obtained are applied for a particular case of this process, namely, for the difference of the compound Poisson process and the renewal process whose jumps are geometrically distributed. The advantage is that these results are in a closed form, in terms of resolvent sequences of the process. In this case, under certain assumptions, we find the limit distributions of the one-boundary and two-boundary characteristics of the process. In addition, we prove the weak convergence of these distributions to the corresponding distributions of a symmetric Wiener process.

Introduction

This article deals with the difference of the compound Poisson process and the compound renewal process. Such processes have proven to be appropriate models in many applied fields of the probability theory, such as telecommunication networks, cash management, computer networks, and in particular, in queueing theory. For instance, for a queueing system with limited waiting room, evolution of the number of the customers in this system is described by a governing process with two reflecting boundaries. In general case this process is a difference of two renewal processes. Thus, studying main characteristics of the system results to the investigating the two-boundary functionals of the governing process. For a

[*]Institute of Mathematics of the Ukrainian National Academy of Sciences 3, Tereshchenkivska st. 01601, Kyiv-4, Ukraine; phone: + 38(0) 44 452-00-55. E-mail: kadankov@voliacable.com.
[†]Hasselt University, Center for Statistics, Building D, 3590 Diepenbeek, Belgium, tel.: +32(0)11 26 82 97, e-mail: tetyana.kadankova@uhasselt.be
[‡]Hasselt University, Center for Statistics, Building D, 3590 Diepenbeek, Belgium, tel.: +32(0)11 26 82 37, e-mail: noel.veraverbeke@uhasselt.be

detailed illustration we refer to [19]. For this class of stochastic processes we determine several two-boundary characteristics, such as the first exit time from the interval and the number of intersections of the interval. This work is a continuation and a generalization of the paper [18]. The approach used for determining the Laplace transforms of the one-boundary characteristics of the process under consideration is based on the factorization methods. For determining the two-boundary characteristics of the process, we will follow the approach suggested in [13], [14] for Lévy processes. For convenience, we will use the same notation as in [18]. We first give a short overview of the existing literature. Distributions of the one-boundary functionals for the difference of renewal processes have been studied by Lindley [20], Prabhu [31] and Cohen [3]. The two-sided exit problem for such processes is closely related to the $G|G|1$ type queueing models. A summary of known results for the $G|G|1$ type model can be found in [3]. The joint distributions of the one-boundary functionals for the difference of compound renewal processes have been considered by Nasirova [25] in terms of the solutions of linear integral equations. Special cases of the difference of renewal processes have been studied by many authors. Ezhov and Kadankov [4]–[7], for instance, have employed probability-factorization methods, while Pirdzhanov and Bratiychuk [30], [2] have used factorization methods. Several two-boundary problems were solved for a semi-Markov walk with a linear drift in [17]. The one-boundary functionals for the difference of two compound Poisson processes with drifts with various jump distributions have been studied by Perry, Stadje, Zacks et. al [27], [28]. Two-boundary problems for the difference of two Poisson processes with exponential negative jumps were solved in [16]. The rest of the article is structured as follows. In Section 1 we introduce the process, notation and present preliminary results. Section 2 is concerned with one-boundary characteristics of the process. In Section 3 we study a two-sided exit problem for the process and its particular case. Section 4 deals with another two-boundary functional of the process, namely with the distribution of the number of intersections of the interval. Finally, in Section 5 we will derive some asymptotic results under certain conditions. We find the limit distributions of the two-boundary characteristics of the process and establish the weak convergence of these distributions to the corresponding distributions of a symmetric Wiener process with the dispersion σ^2.

1 Definitions, notation and preliminary results

Let $(\Omega,\ \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$ be a filtered probability space, where the filtration $\mathfrak{F} = \{\mathfrak{F}_t\}$ satisfies the usual conditions of right-continuity and completion. We assume that all random variables and stochastic processes are defined on this probability space. Let $\varkappa, \delta \in \mathbb{N} = \{1, 2, \dots\}$ be independent integer random variables, and $\eta \in (0, \infty)$ be a positive random variable independent of $\varkappa, \delta$ with the distribution function $F(x) = \mathbf{P}[\eta \le x]$, $x \ge 0$. We will assume that $\mathbf{E}\varkappa$, $\mathbf{E}\delta$, $\mathbf{E}\eta < \infty$. Introduce the sequences $\{\eta, \eta'_n\}$, $\{\varkappa, \varkappa'_n\}$, $\{\delta, \delta'_n\}$, $n \in \mathbb{N}$ of independent identically distributed (inside each sequence) variables and define the sequences

$$\begin{aligned}
&\eta_0(x) = 0, \quad \eta_1(x) = \eta_x, \quad \eta_{n+1}(x) = \eta_x + \eta'_1 + \cdots + \eta'_n, \qquad n \in \mathbb{N}, \qquad (1)\\
&\varkappa_0 = 0, \quad \varkappa_n = \varkappa'_1 + \cdots + \varkappa'_n; \qquad \delta_0 = 0, \quad \delta_n = \delta'_1 + \cdots + \delta'_n; \qquad n \in \mathbb{N},
\end{aligned}$$

where $\eta_x \in (0, \infty)$ is a random variable with the following distribution function

$$F_x(u) = \mathbf{P}\left[\eta_x \le u\right] = [F(x+u) - F(x)](1 - F(x))^{-1} \qquad u \ge 0.$$

Denote by $\{\pi(t)\}_{t\ge 0} \in \mathbb{Z}^+ = \{0, 1, \dots\}$ a compound Poisson process with the generating function of the form

$$\mathbf{E}\, \theta^{\pi(t)} = e^{tk(\theta)}, \qquad k(\theta) = \mu\left(\mathbf{E}\, \theta^{\varkappa} - 1\right), \quad |\theta| \le 1,$$

where $\mu > 0$ is the intensity of the jumps and $\varkappa$ is a jump size. For all $t \ge 0$ define a delayed renewal process generated by the random sequence $\{\eta_n(x)\}_{n\in\mathbb{Z}^+}$:

$$N_x(t) = \max\{n \in \mathbb{Z}^+ : \eta_n(x) \le t\} \in \mathbb{Z}^+, \qquad x \ge 0. \tag{2}$$

Introduce a right-continuous step process for all $x \ge 0$

$$D_x(t) = \pi(t) - \delta_{N_x(t)} \in \mathbb{Z}, \qquad t \ge 0; \quad D_x(0) = 0. \tag{3}$$

We will call the process $\{D_x(t)\}_{t\ge 0}$ a difference of the compound Poisson process and a compound renewal process. Observe, that this process is not a Markov process in general. To be able to solve two-boundary problems for this process, we will proceed as follows. We will add a linear component to this process, so that a new two-component process is a Markov process. More rigorously, for all $t \ge 0$ introduce a right-continuous process as follows:

$$\eta_x^+(t) = \begin{cases} t + x, & 0 \le t < \eta_x, \\ t - \eta_{N_x(t)}(x), & t \ge \eta_x \end{cases} \in \mathbb{R}_+ = [0, \infty), \qquad x \in \mathbb{R}_+. \tag{4}$$

The process $\{\eta_x^+(t)\}_{t\ge 0}$ increases linearly on the intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{Z}^+$, it is killed to zero at the points $\eta_n(x)$, $n \in \mathbb{N}$, and the value of the process at the instant $t_0 \ge \eta_x$ is equal to the time elapsed from the moment of the last renewal of the process (2) till t_0. We will call the process (4) a linear component. Note that this linear component is sometimes referred to as the age process (age since the last renewal). By adding this linear component to the process $\{D_x(t)\}_{t\ge 0}$, we obtain a right-continuous Markov process

$$\{X_t\}_{t\ge 0} = \left\{D_x(t),\, \eta_x^+(t)\right\}_{t\ge 0} \in \mathbb{Z} \times \mathbb{R}_+, \qquad X_0 = \{0, x\}, \quad x \in \mathbb{R}_+, \tag{5}$$

which governs the process $\{D_x(t)\}_{t\ge 0}$. The process defined in (5) is a Markov process. Note, that it is homogeneous with respect to the first component [8]. This means that if $X_{t_0} = \{k, u\}$, $k \in \mathbb{Z}$, $u \ge 0$, then the evolution of the process $\{X_t\}_{t\ge t_0}$ in the sequel does not depend on the value k of the first component. This fact will be used constantly when setting up the equations. The first jump of the process $\{\pi(t)\}_{t\ge t_0}$ (which is distributed as $\varkappa$) will occur after an exponential period of time with parameter μ. The first renewal moment of the process $\{N_x(t)\}_{t\geq t_0}$ (with a jump that is distributed as δ) will take place after elapsing of time η_u. Introduce a random sequence as follows $(n \in \mathbb{N})$

$$\xi_0(x) = 0, \quad \xi_1(x) = \pi(\eta_x) - \delta, \quad \xi_{n+1}(x) = \xi_1(x) + \sum_{i=1}^{n} \xi_i', \quad \xi_n = \xi_n(0),$$

where $\xi = \pi(\eta) - \delta \in \mathbb{Z}$, $\{\xi, \xi_n'\}$, $n \in \mathbb{N}$ is a sequence of i.i.d. random variables. We now define a right-continuous step process in the following way

$$\{S_x(t)\}_{t\geq 0} = \left\{\xi_{N_x(t)}(x)\right\}_{t\geq 0} \in \mathbb{Z}, \qquad S_x(0) = 0, \quad x \in \mathbb{R}_+. \tag{6}$$

The sample paths of the process are constant on the time intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{Z}^+$ and there occur jumps at the instants $\eta_n(x)$, $n \in \mathbb{N}$. These jumps have the same distribution as $\xi \doteq \pi(\eta) - \delta$, where $n \in \{2, 3, \dots\}$, and $\xi_1(x) \doteq \pi(\eta_x) - \delta$ for $n = 1$. Here and in the sequel we will call the process $\{S_x(t)\}_{t\geq 0}$ a semi-Markov random walk generated by the sequences $\{\eta_n(x)\}$, $\{\xi_n(x)\}$, $n \in \mathbb{Z}^+$. This name is originated from [25]. For all $x \in \mathbb{R}_+$, $|\theta| \leq 1$ denote

$$\tilde{f}_x(s) = \mathbf{E}e^{-s\eta_x}, \quad \tilde{f}(s) = \tilde{f}_0(s), \quad \tilde{f}_x(s,\theta) = \tilde{f}_x(s - k(\theta)) = \mathbf{E}\left[e^{-s\eta_x}\theta^{\pi(\eta_x)}\right].$$

Lemma 1. *Let $N_x(t)$, $D_x(t)$, $\eta_x^+(t)$, $S_x(t)$, $t \geq 0$ be random processes defined by the formulae (2)-(6), and $\nu_s \sim \exp(s)$ be an exponential random variable independent of these processes. Then for all $x \in \mathbb{R}_+$, $s > 0$, $|a| \leq 1$, $|\theta|, |b| = 1$, $p \geq 0$ the following equality holds*

$$E_x^s(a,b,\theta,p) = \mathbf{E}a^{N_x(\nu_s)}b^{D_x(\nu_s)}\theta^{S_x(\nu_s)}e^{-p\eta_x^+(\nu_s)} = \frac{se^{-px}}{s+p-k(b)}(1 - \tilde{f}_x(s+p,b))$$
$$+ \frac{sa}{s+p-k(b)}\,\tilde{f}_x(s,\theta b)\mathbf{E}(\theta b)^{-\delta}\frac{1-\tilde{f}(s+p,b)}{1-a\tilde{f}(s,\theta)\mathbf{E}(\theta b)^{-\delta}}. \tag{7}$$

In particular, for all $x \in \mathbb{R}_+$, $s > 0$ the following formulae are valid

$$\mathbf{E}\theta^{S_x(\nu_s)} = 1 - \tilde{f}_x(s) + \tilde{f}_x(s,\theta)\mathbf{E}\theta^{-\delta}\frac{1-\tilde{f}(s)}{1-\tilde{f}(s,\theta)\mathbf{E}\theta^{-\delta}}, \qquad |\theta| = 1. \tag{8}$$

Proof. It is not difficult to establish that the mathematical expectation $E_x^s(a,b,\theta,p)$ obeys the following equation for $x \in \mathbb{R}_+$, $s > 0$, $|a| \leq 1$, $|\theta|, |b| = 1$, $p \geq 0$

$$E_x^s(a,b,\theta,p) = s\frac{1-\tilde{f}_x(s+p,b)}{s+p-k(b)}e^{-px} + a\tilde{f}_x(s,\theta b)\mathbf{E}(\theta b)^{-\delta}E_0^s(a,b,\theta,p).$$

In order to write this equation, we used the total probability law, independence of the random variables δ and $\eta_1(x)$, homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component and the fact that the random time $\eta_1(x)$ is a Markov time. Setting $x = 0$ in this equation, we get

$$E_0^s(a,b,\theta,p) = \frac{s}{s+p-k(b)}\,\frac{1-\tilde{f}(s+p,\theta b)}{1-a\tilde{f}(s,\theta b)\mathbf{E}(\theta b)^{-\delta}}.$$

Substituting the expression for $E_0^s(a,b,\theta,p)$ into the previous equality, we get (7). The formula (8) follows from (7), for $p = 0$, $a = b = 1$. $\square$

In the sequel we will require one-boundary characteristics of the semi-Markov walk $\{S_x(t)\}_{t\geq 0}$ (6). We now formally define them. For all $x\in\mathbb{R}_+,\ k\in\mathbb{Z}^+$ denote by

$$\tilde{\tau}^k(x)=\inf\{t:S_x(t)>k\},\qquad \tilde{T}^k(x)=S_x(\tilde{\tau}^k(x))-k\in\mathbb{N},$$

$\tilde{\tau}^k{=}\tilde{\tau}^k(0),\ \tilde{T}^k{=}\tilde{T}^k(0)$ the instant of the first overshoot of the upper level k by the process $\{S_x(t)\}_{t\geq 0}$ and the value of the overshoot through this level; and by

$$\tilde{\tau}_k(x)=\inf\{t:S_x(t)<-k\},\qquad \tilde{T}_k(x)=-S_x(\tilde{\tau}_k(x))-k\in\mathbb{N},$$

$\tilde{\tau}_k{=}\tilde{\tau}_k(0),\ \tilde{T}_k{=}\tilde{T}_k(0)$ the instant of the first overshoot of the lower level $-k$ by the process $\{S_x(t)\}_{t\geq 0}$, and the value of the overshoot at this instant. Observe that the random variables $\tilde{\tau}^k(x),\ \tilde{\tau}_k(x)$ take their values from a countable set $\{\eta_n(x),\,n\in\mathbb{N}\}$, and they are Markov times of the process $\{S_x(t)\}_{t\geq 0}$. We now formulate and prove some results for the one-boundary characteristics of the semi-Markov walk $\{S_x(t)\}_{t\geq 0}$ which appear to be analogous to the results for usual random walks and Lévy processes (due to Spitzer [33], Rogozin and Pecherskii [32],[26]).

Lemma 2. *Let $\{S_0(t)\}_{t\geq 0}\in\mathbb{Z}$ be a semi-Markov walk (6), and*

$$S_t^+=\sup_{u\leq t}S_0(u),\qquad S_t^-=\inf_{u\leq t}S_0(u),\qquad u,t\geq 0,$$

be the supremum and the infimum of the process $\{S_0(t)\}_{t\geq 0},\ s>0$. Then

(i) *the following identity (Spitzer-Rogozin [33],[32]) is valid for the semi-Markov walk $\{S_0(t)\}_{t\geq 0}$:*

$$\mathbf{E}\theta^{S_0(\nu_s)}=\frac{1-\tilde{f}(s)}{1-\tilde{f}(s,\theta)\,\mathbf{E}\theta^{-\delta}}=\mathbf{E}\theta^{S^+_{\nu_s}}\,\mathbf{E}\theta^{S^-_{\nu_s}},\qquad |\theta|=1,\tag{9}$$

where the random variables $-S^-_{\nu_s},S^+_{\nu_s}\in\mathbb{Z}^+$ are infinitely divisible and their generating function are given as follows:

$$\mathbf{E}\theta^{S^{\pm}_{\nu_s}}=\exp\left\{\sum_{n\in\mathbb{N}}\frac{1}{n}\mathbf{E}\left[e^{-s\eta_n}\left(\theta^{\xi_n}-1\right);\pm\xi_n>0\right]\right\},\qquad |\theta|^{\pm 1}\leq 1;\tag{10}$$

(ii) *the Laplace transforms of the joint distributions of $\{\tilde{\tau}^k,\tilde{T}^k\},\ \{\tilde{\tau}_k,\tilde{T}_k\},\ k\in\mathbb{Z}^+$ of the semi-Markov walk $\{S_0(t)\}_{t\geq 0}$ obey the following equalities [26]*

$$\begin{aligned}\mathbf{E}\left[e^{-s\tilde{\tau}^k}\theta^{\tilde{T}^k};\tilde{\tau}^k<\infty\right]&=\left(\mathbf{E}\,\theta^{S^+_{\nu_s}}\right)^{-1}\mathbf{E}\left[\theta^{S^+_{\nu_s}-k};S^+_{\nu_s}>k\right],\\ \mathbf{E}\left[e^{-s\tilde{\tau}_k}\theta^{\tilde{T}_k};\,\tilde{\tau}_k<\infty\right]&=\left(\mathbf{E}\,\theta^{-S^-_{\nu_s}}\right)^{-1}\mathbf{E}\left[\theta^{-S^-_{\nu_s}-k};\,S^-_{\nu_s}<-k\right];\end{aligned}\tag{11}$$

(iii) *the integral transforms of the joint distributions of $\{S_0(\nu_s),\,S^{\pm}_{\nu_s}\}$ are such that for all $k\in\mathbb{Z}^+$*

$$\begin{aligned}\mathbf{E}\left[\theta^{S_0(\nu_s)};\,S^+_{\nu_s}\leq k\right]&=\mathbf{E}\theta^{S^-_{\nu_s}}\mathbf{E}\left[\theta^{S^+_{\nu_s}};S^+_{\nu_s}\leq k\right],\qquad |\theta|\geq 1,\\ \mathbf{E}\left[\theta^{S_0(\nu_s)};S^-_{\nu_s}\geq -k\right]&=\mathbf{E}\theta^{S^+_{\nu_s}}\mathbf{E}\left[\theta^{S^-_{\nu_s}};S^-_{\nu_s}\geq -k\right],\qquad |\theta|\leq 1.\end{aligned}\tag{12}$$

Proof. We now sketch a proof of the formulae (9)-(12). The following chain of the equalities ($s > 0$) follows from (8) for $x = 0$ and from the expansion $\ln(1-x)^{-1} = x/1 + x^2/2 + \cdots$, $|x| < 1$

$$\mathbf{E}\theta^{S_0(\nu_s)} = \frac{1-\tilde{f}(s)}{1-\tilde{f}(s,\theta)\,\mathbf{E}\theta^{-\delta}} = \exp\left\{\sum_{n\in\mathbb{N}} \frac{1}{n}\mathbf{E}\left[e^{-s\eta_n}\theta^{\xi_n} - e^{-s\eta_n}\right]\right\} =$$

$$= \exp\left\{\sum_{n\in\mathbb{N}} \frac{1}{n}\,\mathbf{E}\left[e^{-s\eta_n}\left(\theta^{\xi_n}-1\right);\, \xi_n > 0\right]\right\} \tag{13}$$

$$\times \exp\left\{\sum_{n\in\mathbb{N}} \frac{1}{n}\,\mathbf{E}\left[e^{-s\eta_n}\left(\theta^{\xi_n}-1\right);\, \xi_n < 0\right]\right\}, \quad |\theta| = 1.$$

Observe, that the first exponent in the right-hand side is an analytic function in $|\theta| \le 1$. Therefore, it can be viewed as a generating function of an infinitely divisible variable, say $\xi_s^+ \in \mathbb{Z}^+$, $\mathbf{P}[\xi_s^+ = 0] > 0$:

$$\exp\left\{\sum_{n\in\mathbb{N}} \frac{1}{n}\,\mathbf{E}\left[e^{-s\eta_n}(\theta^{\xi_n}-1);\, \xi_n > 0\right]\right\} = \mathbf{E}\theta^{\xi_s^+}, \qquad |\theta| \le 1.$$

Analogously the second exponent in the right-hand side of (13) is an analytic function in $|\theta| \ge 1$ and it is a generating function of a certain infinitely divisible variable, say $-\xi_s^- \in \mathbb{Z}^+$, $\mathbf{P}[\xi_s^- = 0] > 0$:

$$\exp\left\{\sum_{n\in\mathbb{N}} \frac{1}{n}\,\mathbf{E}\left[e^{-s\eta_n}(\theta^{\xi_n}-1);\, \xi_n < 0\right]\right\} = \mathbf{E}\theta^{\xi_s^-}, \qquad |\theta| \ge 1.$$

Let us establish the probabilistic interpretation of the variables ξ_s^+, ξ_s^- in terms of the boundary functionals of the semi-Markov walk $\{S_0(t)\}$. To do this, we need the following equality ($k \in \mathbb{Z}^+$)

$$\mathbf{E}\,\theta^{S_0(\nu_s)} = \mathbf{E}\theta^{\xi_s^+}\,\mathbf{E}\theta^{\xi_s^-} = \mathbf{E}\left[\theta^{S_0(\nu_s)};\, S_{\nu_s}^+ \le k\right] +$$

$$+\,\mathbf{E}\left[e^{-s\tilde{\tau}^k}\theta^{S_0(\tilde{\tau}^k)};\, \tilde{\tau}^k < \infty\right]\mathbf{E}\,\theta^{S_0(\nu_s)}, \quad |\theta| = 1. \tag{14}$$

To write this equation, we used the total probability law, Markov property of $\tilde{\tau}^k$, homogeneity of the process $\{S_0(t), \eta_0^+(t)\}$ with respect to the first component and properties of the exponential variable ν_s. Let us re-write the equation (14) as follows:

$$\mathbf{E}\left[\theta^{\xi_s^+-k};\xi_s^+ > k\right] - \mathbf{E}\left[e^{-s\tilde{\tau}^k}\theta^{\tilde{T}^k};\tilde{\tau}^k < \infty\right]\mathbf{E}\theta^{\xi_s^+} = \tag{15}$$

$$\mathbf{E}\left[\theta^{S_0(\nu_s)-k};S_{\nu_s}^+ \le k\right]\left(\mathbf{E}\theta^{\xi_s^-}\right)^{-1} - \mathbf{E}\left[\theta^{\xi_s^+-k};\xi_s^+ \le k\right], \quad |\theta| = 1.$$

Since $\mathbf{P}[\xi_s^- = 0] > 0$, then the function $\left(\mathbf{E}\theta^{\xi_s^-}\right)^{-1}$ is analytic and bounded in $|\theta| > 1$, also continuous on the boundary. Now we can apply the standard factorization reasoning due to Borovkov [1].

The function which enters the left-hand side of (15) is analytic and bounded in $|\theta| < 1$, and also continuous on $|\theta| = 1$. By means of this equality it can be analytically extended to the function which is analytic in $|\theta| > 1$, remaining bounded and continuous. Then by Liouville's theorem this function is a certain constant $C(s)$ (with respect to θ) in the entire complex plane. Letting $\theta \to 0$ in the left-hand side of this equality, we find that $C(s) = 0$. As a result of this reasoning we derived from (15) two formulae

$$\mathbf{E}\left[e^{-s\tilde{\tau}^k}\theta^{\tilde{T}^k}; \tilde{\tau}^k < \infty\right] = \left(\mathbf{E}\theta^{\xi_s^+}\right)^{-1}\mathbf{E}\left[\theta^{\xi_s^+ - k}; \xi_s^+ > k\right], \quad |\theta| \le 1,$$

$$\mathbf{E}\left[\theta^{S_0(\nu_s)}; S_{\nu_s}^+ \le k\right] = \mathbf{E}\theta^{\xi_s^-}\,\mathbf{E}\left[\theta^{\xi_s^+}; \xi_s^+ \le k\right], \qquad |\theta| \ge 1. \tag{16}$$

Observe, that the Laplace transforms of the joint distributions of $\{\tilde{\tau}^k, \tilde{T}^k\}$, $\{S_0(\nu_s), S_{\nu_s}^+\}$ are given in terms of the generating functions of the random variables ξ_s^+, ξ_s^-. Letting $\theta = 1$ in (16), we find that

$$\mathbf{P}\left[S_{\nu_s}^+ \le k\right] = \mathbf{P}\left[\xi_s^+ \le k\right], \qquad k \in \mathbb{Z}^+.$$

The latter equality means that the random variable ξ_s^+ is identically distributed with $S_{\nu_s}^+ = \sup_{t \le \nu_s} S_0(t)$. Applying this property to the dual random walk $-\{S_0(t)\}_{t\ge 0}$, we verify that ξ_s^- is identically distributed with $S_{\nu_s}^- = \inf_{t \le \nu_s} S_0(t)$. Hence, by means of (16), we derive the first two formulae of (11), (12). Applying (16) to $-\{S_0(t)\}_{t\ge 0}$, we obtain the second formulae of (11), (12). Note, that the method of proof can also be found in [12]. □

We will now assume that the random variable $\delta \in \mathbb{N}$ is geometrically distributed with parameter $\lambda \in [0,1)$:

$$\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}, \quad n \in \mathbb{N}, \qquad \mathbf{E}\theta^{\delta} = \theta\frac{1-\lambda}{1-\theta\lambda}, \quad |\theta| \le 1. \tag{17}$$

This assumption means that the process $\{D_x(t)\}_{t\ge 0}$ has geometrically distributed negative jumps at instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$. Here and in the sequel we will denote the distribution (17) as follows $\delta \sim ge(\lambda)$.

Lemma 3 ([18]). *Let $\tilde{f}(s) = \mathbf{E}e^{-s\eta}$. Then for $s > 0$ the equation*

$$\theta - \lambda = (1-\lambda)\tilde{f}(s - k(\theta)), \qquad |\theta| < 1 \tag{18}$$

has a unique solution $c(s)$ inside the circle $|\theta| < 1$. This solution is positive and $c(s) \in (\lambda, 1)$. If $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$, $\rho = \mu(1-\lambda)\mathbf{E}[\varkappa]\,\mathbf{E}[\eta]$, then for $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (\lambda, 1)$; and for $\rho \le 1$, $\lim_{s\to 0} c(s) = 1$.

Corollary 1. *Let $\delta \sim ge(\lambda)$, $s > 0$, $k \in \mathbb{Z}^+$. Then*

(i) *the generating functions of $S_{\nu_s}^-$, $S_{\nu_s}^+$ are such that*

$$\mathbf{E}\theta^{S_{\nu_s}^-} = \frac{1-c(s)}{1-\lambda}\,\frac{1-\lambda/\theta}{1-c(s)/\theta}, \qquad |\theta| \ge 1,$$

$$\mathbf{E}\theta^{S_{\nu_s}^+} = \frac{1-\lambda}{1-c(s)}\,\frac{(1-\tilde{f}(s))(\theta - c(s))}{\theta - \lambda - (1-\lambda)\tilde{f}(s,\theta)}, \qquad |\theta| \le 1, \tag{19}$$

(i) *the generating functions of the joint distributions of* $\{\tilde{\tau}_k, \tilde{T}_k\}$, $\{\tilde{\tau}_k, \tilde{T}_k\}$ *are given as follows*

$$\mathbf{E}\left[e^{-s\tilde{\tau}_k}; \tilde{T}_k = m\right] = (c(s)-\lambda)c(s)^k\lambda^{m-1} = \mathbf{E}e^{-s\tilde{\tau}_k}(1-\lambda)\lambda^{m-1}, \qquad (20)$$

$$\sum_{k\in\mathbb{Z}^+}\theta^k\mathbf{E}\left[e^{-s\tilde{\tau}^k}z^{\tilde{T}_k}\right] = \frac{1}{1-\theta/z}\left[1-\frac{\theta-c(s)}{z-c(s)}\,\frac{(1-\lambda)\tilde{f}(s,z)+\lambda-z}{(1-\lambda)\tilde{f}(s,\theta)+\lambda-\theta}\right].$$

Proof. The equalities (19) can be derived from (18) after taking into account the fact that the factorization expansion is unique. The formulae (20) follow from (11) of Lemma 2 and from (19).

The first equality of (20) implies that the random variable $\tilde{T}_k$ is independent of $\tilde{\tau}_k$ for all $k \in \mathbb{Z}^+$, and it is geometrically distributed with parameter λ. □

2 One-boundary characteristics of the process $\{D_x(t)\}_{t\geq 0}$

In this section we will determine the one-boundary functionals generated by the first overshoot time of a fixed level by the process $\{D_x(t)\}_{t\geq 0}$. Let $X_0 = \{0, x\}$, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. Define

$$\tau_k(x) = \inf\{t : D_x(t) < -k\}, \qquad T_k(x) = -D_x(\tau_k(x)) - k, \qquad \inf\{\emptyset\} = \infty,$$

i.e. the first overshoot time of the negative level $-k$ by the process $\{D_x(t)\}_{t\geq 0}$. We will use the convention that on the event $\{\tau_k(x) = \infty\}$ $T_k(x) = \infty$. Denote $\mathfrak{B}_k(x) = \{\tau_k(x) < \infty\}$,

$$f_k(x, m, s) = \mathbf{E}\left[e^{-s\tau_k(x)}; T_k(x) = m, \mathfrak{B}_k(x)\right], \quad m \in \mathbb{N}.$$

Lemma 4. *The generating function* $\tilde{f}_k(x,\theta,s) = \sum_{m\in\mathbb{N}}\theta^m f_k(x,m,s)$ *satisfies the following relations*

$$\begin{aligned}
\tilde{f}_k(0,\theta,s) &= \left(\mathbf{E}\,\theta^{-S^-_{\nu_s}}\right)^{-1}\mathbf{E}\left[\theta^{-S^-_{\nu_s}-k};\ S^-_{\nu_s} < -k\right],\\
\tilde{f}_k(x,\theta,s) &= \mathbf{E}\left[e^{-s\eta_x}\theta^{-\xi_1(x)-k}; \xi_1(x) < -k\right] +\\
&\quad + \sum_{i\in\mathbb{Z}^+}\mathbf{E}\left[e^{-s\eta_x}; \xi_1(x) = i-k\right]\tilde{f}_i(0,\theta,s), \qquad (21)
\end{aligned}$$

where $\xi_1(x) \doteq \pi(\eta_x) - \delta$ *and* $S^-_{\nu_s} = \inf_{t\leq\nu_s} S_0(t)$ *is the running infimum of the semi-Markov walk* $\{S_0(t)\}_{t\geq 0}$.

Proof. Observe, that the processes $D_x(t)$, $S_x(t)$, $t \geq 0$ do not decrease on the intervals $[\eta_n(x), \eta_{n+1}(x))$. It follows from the definitions of these processes (3), (6) that the negative jumps of $D_x(t)$, $S_x(t)$, $t \geq 0$ can only occur at time points $\{\eta_n(x), n \in \mathbb{N}\}$. It also follows from (3), (6) that

$$D_x(\eta_n(x)) = S_x(\eta_n(x)) = \pi(\eta_n(x)) - \delta_n, \qquad n \in \mathbb{Z}^+.$$

Thus, the stoping time $\tilde{\tau}_k(x)$ of the negative level $-k$ and the value of the overshoot $\tilde{T}_k(x)$ through this level by the semi-Markov walk $\{S_x(t)\}_{t\ge 0}$ coincide in distribution with the stopping time $\tau_k(x)$ and the value of the overshoot $T_k(x)$ by the process $\{D_x(t)\}_{t\ge 0}$. The first equality of (21) follows straightforwardly from the second formula of (11). □

Corollary 2. *Let* $\delta \sim ge(\lambda)$, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$, $m \in \mathbb{N}$, $s > 0$. *Then*

(i) *the Laplace transform of the joint distribution of* $\{\tau_k(x), T_k(x)\}$ *satisfies the following equality*

$$f_k(x, m, s) = \tilde{f}_x(s - k(c(s)))\, c(s)^k\, (1-\lambda)\lambda^{m-1}, \tag{22}$$

where $c(s) \in (\lambda, 1)$ *is the unique solution of the equation (18) inside the circle* $|\theta| < 1$, $\tilde{f}_x(s) = \mathbf{E}\, e^{-s\eta_x}$, $\tilde{f}(s) = \mathbf{E}\, e^{-s\eta} = \tilde{f}_0(s)$;

(ii) *if* $\rho > 1$, *then* $\mathbf{P}[\tau_k(x) < \infty] = \tilde{f}_x(-k(c))\, c^k < 1$, *and* $\tau_k(x)$ *for all* $k \in \mathbb{Z}^+$, $x \ge 0$ *is a defective random variable; if* $\rho \le 1$, *then* $\mathbf{P}[\tau_k(x) < \infty] = 1$, *and* $\tau_k(x)$ *is a proper variable for all* $k \in \mathbb{Z}^+$, $x \ge 0$.

Proof. It is not difficult to derive the following equality

$$\mathbf{P}[\pi(t) - \delta = i] = (1-\lambda)\mathbf{E}\left[\lambda^{\pi(t)-(i+1)}; \pi(t) > i\right], \qquad i \in \mathbb{Z}.$$

Substituting the expression for $\tilde{f}_i(0, m, s)$ from (20) into the second formula of (21) after some calculations, we obtain (22). □

A more interesting problem is determining the generating function of the joint distribution of the upper-boundary functionals of the process $\{D_x(t)\}_{t\ge 0}$. Let $X_0 = \{0, x\}$, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. Denote by

$$\tau^k(x) = \inf\{t : D_x(t) > k\}, \qquad \inf\{\emptyset\} = \infty$$

the instant of the first overshoot of the upper level k by the process $\{D_x(t)\}_{t\ge 0}$. Denote $\mathfrak{B}^k(x) = \left\{\tau^k(x) < \infty\right\}$. On the event $\mathfrak{B}^k(x)$ define

$$l_x^k = \eta_x^+(\tau^k(x)), \quad T^k(x) = D_x(\tau^k(x)) - k$$

the value of the age process and the value of the overshoot at the stopping time $\tau^k(x)$. On the event $\{\tau^k(x) = \infty\}$ we set per definition that $l_x^k = T^k(x) = \infty$. Introduce the following notation for the mathematical expectations $s > 0$, $k \in \mathbb{Z}^+$, $m \in \mathbb{N}$, $x \in \mathbb{R}_+$

$$f^k(x, dl, m, s) = \mathbf{E}\left[e^{-s\tau^k(x)}; l_x^k \in dl, T^k(x) = m, \mathfrak{B}^k(x)\right],$$

and for the generating functions ($|\theta| < 1$)

$$\Phi_\theta^s(x, dl, m) = \sum_{k\in\mathbb{Z}^+} \theta^k f^k(x, dl, m, s), \quad \Phi_\theta^s(x) = \int_0^\infty \sum_{m\in\mathbb{N}} \Phi_\theta^s(x, dl, m).$$

Let $k \in \mathbb{Z}^+$ and

$$\hat{\tau}^k = \inf\{t : \pi(t) > k\}, \quad \hat{T}^k = \pi(\hat{\tau}^k) - k$$

be the first crossing time through the upper level k by the compound Poisson process $\{\pi(t)\}_{t\geq 0}$ and the value of the overshoot at this instant. Denote by

$$\rho_k(t) = \mathbf{P}[\pi(t) = k], \qquad \sum_{k\in\mathbb{Z}^+} \theta^k \rho_k(t) = \mathbf{E}\,\theta^{\pi(t)} = e^{tk(\theta)}, \quad |\theta| \leq 1,$$

$$p_k^m(dt) = \mathbf{P}[\hat{\tau}^k \in dt, \hat{T}^k = m] = \mu \sum_{i=0}^{k} \rho_i(t)\mathbf{P}[\varkappa = k - i + m]\, dt, \quad m \in \mathbb{N}.$$

For the Laurent series $L(\theta) = \sum\limits_{k=-\infty}^{\infty} a_k\theta^k$, $|\theta| = 1$ such that $\sum\limits_{k=-\infty}^{\infty} |a_k| < \infty$ we now introduce the projectors [11] in the following way:

$$\mathfrak{P}_\theta^+[L(\theta)] = \sum_{k\in\mathbb{Z}^+} a_k\theta^k, \quad |\theta| \leq 1, \qquad \mathfrak{P}_\theta^-[L(\theta)] = \sum_{k\leq -1} a_k\theta^k, \quad |\theta| \geq 1.$$

Theorem 1. *Let $\{D_x(t)\}_{t\geq 0}$ be the difference of the compound Poisson process and compound renewal process (3), $S_{\nu_s}^+ = \sup\limits_{t\leq\nu_s} S_0(t)$, $S_{\nu_s}^- = \inf\limits_{t\leq\nu_s} S_0(t)$ be the supremum and infimum (10) of the semi-Markov walk $\{S_0(t)\}_{t\geq 0}$ on the time interval $[0, \nu_s]$. Then*

the generating functions of the Laplace transform of the joint distribution of $\{\tau^k(x), l_x^k, T^k(x)\}$ satisfy the following formula on the event $\mathfrak{B}^k(x)$

$$\Phi_\theta^s(x, dl, m) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)}\, \mathbf{I}_{\{l>x\}} \Pi_\theta^m(d(l - x)) +$$
$$+ e^{-sl}[1 - F(l)]\, \frac{\tilde{f}_x(s, \theta)}{1 - \tilde{f}(s)}\, \mathbf{E}\theta^{S_{\nu_s}^+} \mathfrak{P}_\theta^+ \left[\mathbf{E}\theta^{-\delta + S_{\nu_s}^-} \Pi_\theta^m(dl)\right], \tag{23}$$

where $\Pi_\theta^m(dl) = \sum\limits_{k\in\mathbb{Z}^+} \theta^k p_k^m(dl) = \mu e^{lk(\theta)} \mathbf{E}\left[\theta^{\varkappa - m}; \varkappa \geq m\right] dl$; and in particular,

$$\Phi_\theta^s(x) = \frac{1 - \tilde{\Pi}_\theta^s}{1 - \theta} + \frac{\tilde{f}_x(s, \theta)}{1 - \tilde{f}(s)}\, \mathbf{E}\theta^{S_{\nu_s}^+} \mathfrak{P}_\theta^+ \left[\left(\mathbf{E}\theta^{-\delta} - 1\right) \mathbf{E}\theta^{S_{\nu_s}^-} \frac{1 - \tilde{\Pi}_\theta^s}{1 - \theta}\right], \tag{24}$$

where $\tilde{\Pi}_\theta^s = \mathbf{E}\theta^{\pi(\nu_s)} = s/(s - k(\theta))$,

Proof. Due to the total probability law, the Markov property of the random times $\{\eta_n(x),\ n \in \mathbb{N}\}$ and the homogeneity of the process $\{X_x(t)\}_{t\geq 0}$ with respect to the first component, the function $f^k(x, dl, m, s)$, $k \in \mathbb{Z}^+$, $x \in \mathbb{R}_+$, $s > 0$ satisfies the following equation:

$$f^k(x, dl, m, s) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m(d(l - x))$$
$$+ \sum_{i=0}^{k} \mathbf{E}[e^{-s\eta_x}; \pi(\eta_x) = i] \sum_{r=1}^{\infty} \mathbf{P}\left[\delta = r\right] f^{k-i+r}(0, dl, m, s), \quad m \in \mathbb{N}. \tag{25}$$

Multiplying the equation (25) by θ^k, $|\theta| < 1$ and summing over $k \in \mathbb{Z}^+$, we derive the equation

$$\Phi^s_\theta(x, dl, m) = e^{-s(l-x)}\,\frac{1 - F(l)}{1 - F(x)}\mathbf{I}\{l > x\}\Pi^m_\theta(d(l - x)) + \\ + \tilde{f}_x(s, \theta)\mathfrak{P}^+_\theta\left[\mathbf{E}\theta^{-\delta}\Phi^s_\theta(0, dl, m)\right]. \tag{26}$$

Letting $x = 0$ in (26), we derive

$$\Phi^s_\theta(0, dl, m) = e^{-sl}\,[1 - F(l)]\Pi^m_\theta(dl) + \tilde{f}(s, \theta)\mathfrak{P}^+_\theta\left[\mathbf{E}\theta^{-\delta}\Phi^s_\theta(0, dl, m)\right], \tag{27}$$

i.e. an equation with respect to $\Phi^s_\theta(0, dl, m)$. To solve this equation, we will use the Wiener-Hopf factorization. The first step is to determine the auxiliary function $\mathbf{I}^+_\theta(dl, m, s) = \mathfrak{P}^+_\theta\left[\mathbf{E}\theta^{-\delta}\Phi^s_\theta(0, dl, m)\right]$. Once the function $\mathbf{I}^+_\theta(dl, m, s)$ is found, we will employ the equalities (26), (27) to determine the function $\Phi^s_\theta(x, dl, m)$. Multiplying the equation (26) by $\mathbf{E}\theta^{-\delta}$, and then setting $|\theta| = 1$, we find that

$$\mathbf{E}\theta^{-\delta}\Phi^s_\theta(0, dl, m) = e^{-sl}\,[1 - F(l)]\Pi^m_\theta(dl)\mathbf{E}\theta^{-\delta} + \tilde{f}(s, \theta)\mathbf{E}\theta^{-\delta}\mathbf{I}^+_\theta(dl, m, s).$$

Now employing the identity (9) and the equality $\mathbf{I}^+_\theta(dl, m, s) + \mathbf{I}^-_\theta(dl, m, s) = \mathbf{E}\theta^{-\delta}\Phi^s_\theta(0, dl, m)$, we rewrite this equality as follows:

$$\mathbf{I}^+_\theta(dl, m, s)\left(\mathbf{E}\theta^{S^+_{\nu_s}}\right)^{-1} - e^{-sl}\frac{1 - F(l)}{1 - \tilde{f}(s)}\mathfrak{P}^+_\theta\left[\mathbf{E}\theta^{-\delta + S^-_{\nu_s}}\Pi^m_\theta(dl)\right] = \\ = e^{-sl}\frac{1 - F(l)}{1 - \tilde{f}(s)}\mathfrak{P}^-_\theta\left[\mathbf{E}\theta^{-\delta + S^-_{\nu_s}}\Pi^m_\theta(dl)\right] - \frac{1}{1 - \tilde{f}(s)}\mathbf{I}^-_\theta(dl, m, s)\mathbf{E}\theta^{S^-_{\nu_s}}, \quad |\theta| = 1.$$

Using the standard factorization reasoning approach for this equation, we determine the auxiliary function $\mathbf{I}^+_\theta(dl, m, s)$:

$$\mathbf{I}^+_\theta(dl, m, s) = e^{-sl}\frac{1 - F(l)}{1 - \tilde{f}(s)}\mathbf{E}e^{-pS^+_{\nu_s}}\mathfrak{P}^+_\theta\left[\mathbf{E}\theta^{-\delta + S^-_{\nu_s}}\Pi^m_\theta(dl)\right], \quad |\theta| \le 1.$$

Inserting the latter expression into the right-hand side of (26), we derive the formulae (23). Integrating (23) with respect to $l \in \mathbb{R}_+$ and summing over $m \in \mathbb{N}$, we get the formula (24). □

The joint distribution of $\{\tau^k(0), T^k(0)\}$ was studied by Gusak [9]. The integral transforms of this distribution [9] differ from our results (the formula (24) for $x = 0$). However, to solve the two-boundary problem, we require a more general one-boundary functional of the process $\{D_x(t)\}_{t\ge 0}$, i.e. the joint distribution of $\{\tau^k(x), l^k_x, T^k(x)\}$, $k, x \in \mathbb{R}_+$. Hence, in the sequel we will use the results of Theorem 1.

Corollary 3 ([18]). *Let $\delta \sim ge(\lambda)$, $\{D_x(t)\}_{t\ge 0}$ be the difference of the compound Poisson process and the compound renewal process (3) whose jumps are geometrically distributed, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. Then*

(i) *the Laplace transforms of the joint distribution of* $\{\tau^k(x), l_x^k, T^k(x)\}$ *satisfy the following equality*

$$f^k(x, dl, m, s) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m(d(l-x))$$

$$+ \Phi_\lambda^s(0, dl, m)\, Q_k^s(x) - e^{-sl}\,[1 - F(l)] \sum_{i=0}^{k} Q_i^s(x)\, p_{k-i}^m(dl), \qquad (28)$$

where $\Phi_\lambda^s(0, dl, m) = e^{-sl}[1 - F(l)] \sum_{k \in \mathbb{Z}^+} c(s)^k p_k^m(dl)$, $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$ *is the resolvent sequence of the process* $\{D_x(t)\}_{t\geq 0}$ *[18]*

$$Q_k^s(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{d\theta}{\theta^{k+1}} \frac{(1-\lambda)\tilde{f}_x(s - k(\theta))}{(1-\lambda)\tilde{f}(s - k(\theta)) + \lambda - \theta}, \quad \alpha \in (0, c(s)); \quad (29)$$

(ii) *for the Laplace transform of the first crossing time through the upper level* k *by the process* $\{D_x(t)\}_{t\geq 0}$ *for all* $k \in \mathbb{Z}^+$, $s, x \in \mathbb{R}_+$ *the following formula holds*

$$\mathbf{E}e^{-s\tau^k(x)} = 1 - \frac{s}{s - k(c(s))} \frac{Q_k^s(x)}{1-\lambda} + \sum_{i=0}^{k} \tilde{\rho}_i(s) \left[\frac{Q_{k-i}^s(x)}{1-\lambda} - 1\right], \qquad (30)$$

where $\tilde{\rho}_k(s) = s\int_0^\infty e^{-st}\rho_k(t)\,dt$;

(iii) *for* $\mathbf{E}[\varkappa]$, $\mathbf{E}[\eta] < \infty$ *and* $\rho < 1$, $\tau^k(x)$ *is a defective random variable and*

$$\mathbf{P}[\tau^k(x) < \infty] = 1 - (1-\rho)(1-\lambda)^{-1} Q_k(x) < 1,$$

where $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$ *is the resolvent sequence of the process* $\{D_x(t)\}_{t\geq 0}$, *given by (29) for* $s = 0$:

$$Q_k(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{d\theta}{\theta^{k+1}} \frac{(1-\lambda)\tilde{f}_x(-k(\theta))}{(1-\lambda)\tilde{f}(-k(\theta)) + \lambda - \theta}, \quad \alpha \in (0, c(0)); \quad (31)$$

if $\rho \geq 1$, *then for all* $k \in \mathbb{Z}^+$, $x \in \mathbb{R}_+$ $\tau^k(x)$ *is a proper random variable.*

Proof. In case when $\delta \sim ge(\lambda)$ the formulae (17), (19) imply

$$\mathfrak{P}_\theta^+ \left[\mathbf{E}\theta^{-\delta + S_{\nu_s}^-} \Pi_\theta^m(dl)\right] = \mathfrak{P}_\theta^+ \left[\frac{1 - c(s)}{\theta - c(s)} \Pi_\theta^m(dl)\right] = \frac{1 - c(s)}{\theta - c(s)} \left(\Pi_\theta^m(dl) - \Pi_{c(s)}^m(dl)\right).$$

Substituting this projector and $\mathbf{E}\theta^{S_{\nu_s}^+}$ (19) into (23) yields

$$\Phi_\theta^s(x, dl, m) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}_{\{l \geq x\}} \Pi_\theta^m(d(l-x)) -$$

$$- e^{-sl}[1 - F(l)] \frac{(1-\lambda)\tilde{f}_x(s,\theta)}{(1-\lambda)\tilde{f}(s,\theta) + \lambda - \theta} \left(\Pi_\theta^m(dl) - \Pi_{c(s)}^m(dl)\right).$$

Employing the definition of the resolvent (29) and comparing the coefficients of $\theta^k,\ \ k \in \mathbb{Z}^+$ in both sides of this equality, we get the formula (28) of Corollary 3. Analogously we calculate

$$\mathfrak{P}_\theta^+ \left[\left(\mathbf{E}\theta^{-\delta} - 1 \right) \mathbf{E}\theta^{S_{\nu_s}^-} \frac{1 - \mathbf{E}\theta^{\pi(\nu_s)}}{1-\theta} \right] = \frac{1-c(s)}{1-\lambda} \frac{\mathbf{E}\left[c(s)^{\pi(\nu_s)} - \theta^{\pi(\nu_s)} \right]}{\theta - c(s)},$$

$$\Phi_\theta^s(x) = \frac{1-\mathbf{E}\theta^{\pi(\nu_s)}}{1-\theta} - \frac{\tilde{f}_x(s,\theta)}{(1-\lambda)\tilde{f}(s,\theta) + \lambda - \theta} \mathbf{E}\left[c(s)^{\pi(\nu_s)} - \theta^{\pi(\nu_s)} \right]. \qquad (32)$$

Taking into account the definition of the resolvent (29) we compare the coefficients of $\theta^k,\ \ k \in \mathbb{Z}^+$ in both sides of this equality, which yields the formula (30) of Corollary 3. Note that the formulae of the corollary were obtained by other methods in [18]. $\square$

3 Two-sided exit problem for the process $\{D_x(t)\}_{t\geq 0}$

In this section we will determine the Laplace transforms of the joint distribution of the first exit time from the interval, the value of the overshoot and the value of the age process at this instant. These Laplace transforms will be used to determine the main characteristic of interest, i.e. the distribution of the number of the intersection of the interval. Let $B \in \mathbb{Z}^+$ be fixed, $k \in \{0, \ldots, B\}$, $r = B - k$, $X_0 = \{0, x\}$, $x \geq 0$, and introduce the random variable

$$\chi = \inf\{t : D_x(t) \notin [-r, k]\}$$

the first exit time from the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\geq 0}$. This random variable is a Markov time of the process $\{X_t\}_{t\geq 0}$. Exit from the interval can occur either through the upper boundary k, or through the lower boundary $-r$. In view of this remark we introduce the events:
$\mathfrak{A}^k = \{D_x(\chi) > k\}$ i.e. the process $\{D_x(t)\}_{t\geq 0}$ exits the interval $[-r, k]$ through the upper boundary k;
$\mathfrak{A}_r = \{D_x(\chi) < -r\}$ i.e. the process $\{D_x(t)\}_{t\geq 0}$ exits the interval $[-r, k]$ through the lower boundary $-r$. Denote by

$$T = (D_x(\chi) - k)\mathbf{I}_{\mathfrak{A}^k} + (-D_x(\chi) - r)\mathbf{I}_{\mathfrak{A}_r}, \quad L = \eta_x^+(\chi)\mathbf{I}_{\mathfrak{A}^k} + 0 \cdot \mathbf{I}_{\mathfrak{A}_r}, \quad \mathbf{P}[\mathfrak{A}^k + \mathfrak{A}_r] = 1$$

the value of the overshoot through the boundaries of the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\geq 0}$ and the value of the age process at the instant of the first exit. For all $k \in \{0, \ldots, B\}$, $r = B - k$, $i \in \mathbb{N}$, $x \in \mathbb{R}_+$ denote

$$F^k(x, dl, i, s) = f^k(x, dl, m, s) - \sum_{j\in\mathbb{N}} f_r(x, j, s) f^{j+B}(0, dl, i, s),$$

$$F_r(x, i, s) = f_r(x, i, s) - \sum_{j\in\mathbb{N}} \int_0^\infty f^k(x, dl, j, s) f_{j+B}(l, i, s).$$

Observe, that the functions $f_r(x, i, s)$, $f^k(x, dl, i, s)$ are given by (21), (23).

Theorem 2 ([18])**.** *Let* $\{D_x(t)\}_{t\ge 0}$ *be the difference of the compound Poisson process and the compound renewal process (3) and* $B\in\mathbb{Z}^+$, $k\in\{0,\dots,B\}$, $r=B-k$, $X_0=\{0,x\}$, $x\ge 0$. *Then the Laplace transforms*

$$V^k(x,dl,i,s)=\mathbf{E}\left[e^{-s\chi};L\in dl,T=i,\mathfrak{A}^k\right],\quad V_r(x,i,s)=\mathbf{E}\left[e^{-s\chi};T=i,\mathfrak{A}_r\right]$$

of the joint distribution of $\{\chi,L,T\}$ *satisfy the following formulae for* $s>0$

$$\begin{aligned}
V^k(x,dl,i,s)&=F^k(x,dl,i,s)+\sum_{j\in\mathbb{N}}\int_0^\infty F^k(x,d\nu,j,s)\mathfrak{K}^+_{\nu,j}(dl,i,s),\\
V_r(x,i,s)&=F_r(x,i,s)+\sum_{j\in\mathbb{N}}F_r(x,j,s)\,\mathfrak{K}^-_j(i,s),\quad i\in\mathbb{N},
\end{aligned}\tag{33}$$

where

$$\mathfrak{K}^+_{\nu,j}(dl,i,s)=\sum_{n\in\mathbb{N}}K^+_{\nu,j}(dl,i,s)^{(n)},\quad \mathfrak{K}^-_j(i,s)=\sum_{n\in\mathbb{N}}K^-_j(i,s)^{(n)}\tag{34}$$

are uniformly convergent series of the iterations, and

$$\begin{aligned}
&K^+_{\nu,j}(dl,i,s)^{(1)}\stackrel{\text{def}}{=}K^+_{\nu,j}(dl,i,s),\qquad K^-_j(i,s)^{(1)}\stackrel{\text{def}}{=}K^-_j(i,s),\\
&K^+_{\nu,j}(dl,i,s)^{(n+1)}=\sum_{m\in\mathbb{N}}\int_0^\infty K^+_{\nu,j}(du,m,s)K^+_{u,m}(dl,i,s)^{(n)},\quad n\in\mathbb{N}\\
&K^-_j(i,s)^{(n+1)}=\sum_{m\in\mathbb{N}}K^-_j(m,s)\,K^-_m(i,s)^{(n)},\quad n\in\mathbb{N}
\end{aligned}\tag{35}$$

are the successive iterations of the kernels $K^+_{\nu,j}(dl,i,s)$, $K^-_j(i,s)$, *which are given by the following defining formulae*

$$\begin{aligned}
K^+_{\nu,j}(dl,i,s)&=\sum_{m\in\mathbb{N}}f_{i+B}(\nu,m,s)\,f^{m+B}(0,dl,i,s),\\
K^-_j(i,s)&=\sum_{m\in\mathbb{N}}\int_0^\infty f^{j+B}(0,dl,m,s)\,f_{m+B}(l,i,s).
\end{aligned}\tag{36}$$

Corollary 4 ([18])**.** *Let* $\delta\sim ge(\lambda)$, *and* $\{D_x(t)\}_{t\ge 0}$ *be the difference of the compound Poisson process and the compound renewal process (3) whose jumps are geometrically distributed,* $\{Q^s_k(x)\}_{k\in\mathbb{Z}^+}$ *be the resolvent sequence of the process given by (29),* $Q^s_k\stackrel{\text{def}}{=}Q^s_k(0)$. *Then*

(i) *the Laplace transforms of the joint distribution of* $\{\chi,L,T\}$ *satisfy the following equalities for all* $x,s\in\mathbb{R}_+$, $m\in\mathbb{N}$

$$\begin{aligned}
V_r(x,m,s)&=\frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{B+\delta}}(1-\lambda)\lambda^{m-1},\\
V^k(x,dl,m,s)&=f^k(x,dl,m,s)-\frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{B+\delta}}\,\mathbf{E}f^{B+\delta}(dl,m,s),
\end{aligned}\tag{37}$$

where the function $f^k(x, dl, m, s)$ *is given by (28),*

$$\mathbf{E}\, Q^s_{B+\delta} = \sum_{k\in\mathbb{N}} (1-\lambda)\lambda^{k-1} Q^s_{k+B},$$
$$\mathbf{E} f^{B+\delta}(dl, m, s) = \sum_{k\in\mathbb{N}} (1-\lambda)\lambda^{k-1} f^{k+B}(0, dl, m, s);$$

(ii) *for the Laplace transforms of the first exit time* χ *from the interval by the process* $\{D_x(t)\}_{t\geq 0}$ *the following formulae hold*

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_r\right] = \frac{Q^s_k(x)}{\mathbf{E}\, Q^s_{B+\delta}}, \tag{38}$$
$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^k\right] = 1 + A^s_k(x) - \frac{Q^s_k(x)}{\mathbf{E}\, Q^s_{B+\delta}}\left(1 + \mathbf{E}\, A^s_{B+\delta}(0)\right),$$

where $\mathbf{E}\, A^s_{B+\delta} = \sum_{k\in\mathbb{N}} (1-\lambda)\lambda^{k-1} A^s_{k+B}(0)$,

$$A^s_k(x) = \sum_{i=0}^{k} \tilde{\rho}_i(s)\left[\frac{Q^s_{k-i}(x)}{1-\lambda} - 1\right], \quad \tilde{\rho}_i(s) = s\int_0^\infty e^{-st}\mathbf{P}[\pi(t) = i]\, dt.$$

4 Intersections of the interval by the process $\{D_x(t)\}_{t\geq 0}$

In this section we determine the distribution of the number of intersections of a fixed interval.

Let $B \in \mathbb{Z}^+$ be fixed, $k \in \{0, \ldots, B\}$, $r = B - k$, $X_0 = \{0, x\}$, $x \in \mathbb{R}_+$. Introduce the random sequences $(n \in \mathbb{N})$

$$i^{\pm}_0 = 0, \quad i^-_n = \inf\{t > i^+_{n-1} : D_x(t) < -r\}, \quad i^+_n = \inf\{t > i^-_n : D_x(t) > k\},$$

where $\inf\{\emptyset\} = \infty$. Denote by

$$\alpha^+_t = \max\{n \in \mathbb{Z}^+ : i^+_n \leq t\} \in \mathbb{Z}^+$$

the number of upward intersections of the interval $[-r, k]$ by the process $D_x(t)$ up to time t. Introduce random sequences

$$j^{\pm}_0 = 0, \quad j^+_n = \inf\{t > j^-_{n-1} : D_x(t) > k\}, \quad j^-_n = \inf\{t > j^+_n : D_x(t) < -r\}.$$

Denote by

$$\alpha^-_t = \max\{n \in \mathbb{Z}^+ : j^-_n \leq t\} \in \mathbb{Z}^+,$$

the number of the downward intersections of the interval $[-r, k]$ by the process $D_x(t)$ on the time interval $[0, t]$. For all $i, j, n \in \mathbb{Z}^+$, $|i - j| \leq 1$ denote by

$$p^j_i(t) = \mathbf{P}\left[\alpha^+_t = i, \alpha^-_t = j\right], \quad p^{\pm}_n(t) = \mathbf{P}\left[\alpha^{\pm}_t = n\right]. \tag{39}$$

the distributions of the number of intersections. It is worth mentioning that in [21], [22] the distributions $p^{\pm}_n(\nu_s)$, $n \in \mathbb{Z}^+$, χ were obtained in terms of the projective operators applied to the factorization components. These distributions were determined for random walks defined on the Markov chain by Lotov and Orlova [23], [24]. The authors employed the matrix factorization components of the random walk. In addition, the asymptotic expansions for the distributions of χ, $p_n^{\pm}(\nu_s)$ were derived for Lévy processes and random walks which satisfy Cramér's condition. The distribution $p_i^j(\nu_s)$ was also found in [14] in terms of the joint distributions of $\{\chi, T\}$ and successive iterations. In the same vain as in [14] we will derive the distributions (39) in terms of the joint distribution of $\{\chi, L, T\}$ (33) and the successive iterations $K_{\nu,i}^{+}(dl, m, s)^{(n)}$, $K_i^{-}(m,s)^{(n)}$, $n \in \mathbb{N}$ of the process (35). For all $t \geq 0$, $v \in \mathbb{N}$ introduce the processes

$$D_x^v(t) = v + D_x(t), \quad X_t^v = \left\{D_x^v(t), \, \eta_x^+(t)\right\}, \qquad X_0^v = \{v, x\}. \tag{40}$$

For the process $\{D_x^v(t)\}_{t\geq 0}$, $v > k$ introduce the random sequences $(i_0^{\pm}(v) = 0)$

$$i_n^-(v) = \inf\{t > i_{n-1}^+(v) : D_x^v(t) < -r\}, \quad i_n^+(v) = \inf\{t > i_n^-(v) : D_x^v(t) > k\},$$

and define the random variables

$$\alpha_t^+(v) = \max\{n \in \mathbb{Z}^+ : i_n^+(v) \leq t\}, \quad \alpha_t^-(v) = \max\{n \in \mathbb{Z}^+ : i_n^-(v) \leq t\},$$

i.e. the number of the upward and downward intersections of the interval $[-r, k]$ by the process $D_x^v(t)$ on the time interval $[0, t]$. Similarly, for the process $\{D_x^v(t)\}_{t\geq 0}$, $v < -r$ introduce the random sequences $(j_0^{\pm}(v) = 0)$

$$j_n^+(v) = \inf\{t > j_{n-1}^-(v) : D_x^v(t) > k\}, \quad j_n^-(v) = \inf\{t > j_n^+(v) : D_x^v(t) < -r\}$$

and define the random variables

$$\alpha_t^+(v) = \max\{n \in \mathbb{Z}^+ : j_n^+(v) \leq t\}, \quad \alpha_t^-(v) = \max\{n \in \mathbb{Z}^+ : j_n^-(v) \leq t\},$$

i.e. the number of the upward and downward intersections of the interval $[-r, k]$ by the process $D_x^v(t)$ on the time interval $[0, t]$.

Let $i \in \mathbb{N}$, $a, b \in [0, 1]$, and introduce the generating functions as follows

$$\begin{aligned}
h^i(x, a, b, s) &= \mathbf{E}\left[a^{\alpha_{\nu_s}^+(k+i)} b^{\alpha_{\nu_s}^-(k+i)}\right] = h^i(x, s), \\
h_i(x, a, b, s) &= \mathbf{E}\left[a^{\alpha_{\nu_s}^+(-r-i)} b^{\alpha_{\nu_s}^-(-r-i)}\right] = h_i(x, s), \\
h(x, a, b, s) &= \mathbf{E}\left[a^{\alpha_{\nu_s}^+} b^{\alpha_{\nu_s}^-}\right] = h(x, s).
\end{aligned}$$

Theorem 3. *Let $\{D_x(t)\}_{t\geq 0}$ be the difference of the compound Poisson process and the compound renewal process (3). Then*

(i) *the generating functions of the joint distribution of $\{\alpha_{\nu_s}^+, \alpha_{\nu_s}^-\}$ are such that*

$$\begin{aligned}
h^i(x, a, b, s) = &\sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} \mathfrak{K}_{x,i}^+(dl, j, a, b, s) \left(1 - f_{j+B}(l, s)\right) + \\
&+ b \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} \mathfrak{K}_{x,i}^+(dl, j, a, b, s) \sum_{m\in\mathbb{N}} f_{j+B}(l, m, s) \left(1 - f^{m+B}(0, s)\right),
\end{aligned} \tag{41}$$

$$h_i(0,a,b,s) = \sum_{j\in\mathbb{N}} \mathfrak{K}_i^-(j,a,b,s)\,\left(1 - f^{j+B}(0,s)\right) +$$

$$+\, a \sum_{j\in\mathbb{N}} \mathfrak{K}_i^-(j,a,b,s) \sum_{m\in\mathbb{N}} \int_{\mathbb{R}_+} f^{j+B}(0,dl,m,s)\,(1 - f_{m+B}(l,s)), \qquad (42)$$

$$h(x,a,b,s) = 1 - \mathbf{E}e^{-s\chi} +$$

$$+ \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x,dl,i,s) h^i(l,a,b,s) + \sum_{i\in\mathbb{N}} V_r(x,i,s) h_i(0,a,b,s), \qquad (43)$$

where $\delta_{i,j}$ *is the Kronecker symbol and* $\delta(x)$ *is the delta function,* $f^k(x,s) = \mathbf{E}e^{-s\tau^k(x)}$, $f_r(x,s) = \mathbf{E}e^{-s\tau_r(x)}$,

$$\mathfrak{K}_{x,i}^+(dl,j,a,b,s) = \delta_{i,j}\,\delta(x-l)\,dl + \sum_{n\in\mathbb{N}} (ab)^n K_{x,i}^+(dl,j,s)^{(n)},$$

$$\mathfrak{K}_i^-(j,a,b,s) = \delta_{i,j} + \sum_{n\in\mathbb{N}} (ab)^n K_i^-(j,s)^{(n)};$$

(ii) *the joint distribution* $\tilde{p}_i^j(s) = p_i^j(\nu_s)$, $i,j \in \mathbb{Z}^+$, $|i-j| \le 1$ *of* $\{\alpha_{\nu_s}^+, \alpha_{\nu_s}^-\}$ *obeys the following formulae for* $n \in \mathbb{Z}^+$

$$\tilde{p}_n^{n+1}(s) = \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x,dl,i,s)$$

$$\times \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} K_{l,i}^+(d\nu,j,s)^{(n)} \sum_{m\in\mathbb{N}} f_{j+B}(\nu,m,s)(1 - f^{m+B}(0,s)), \qquad (44)$$

$$\tilde{p}_{n+1}^n(s) = \sum_{i\in\mathbb{N}} V_r(x,i,s) \sum_{j\in\mathbb{N}} K_i^-(j,s)^{(n)} \sum_{v\in\mathbb{N}} \int_{\mathbb{R}_+} f^{j+B}(0,dl,v,s)\,(1 - f_{v+B}(l,s)),$$

$$\tilde{p}_0^0(s) = 1 - \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x,dl,i,s) f_{i+B}(l,s) - \sum_{i\in\mathbb{N}} V_r(x,i,s) f^{i+B}(0,s),$$

$$\tilde{p}_n^n(s) = \sum_{i\in\mathbb{N}} V_r(x,i,s) \sum_{j\in\mathbb{N}} K_i^-(j,s)^{(n)} \left(1 - f^{j+B}(0,s)\right) +$$

$$+ \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x,d\nu,i,s) \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} K_{i,\nu}^+(dl,j,s)^{(n)}\,(1 - f_{j+B}(l,s));$$

(iii) *the distributions* $\tilde{p}_n^{\pm}(s) = \mathbf{P}\left[\alpha_{\nu_s}^{\pm} = n\right]$ *satisfy the following relations*

$$\tilde{p}_0^+(s) = 1 - \sum_{i\in\mathbb{N}} V_r(x,i,s) f^{i+B}(0,s) - \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x,dl,i,s) K_{l,i}^+(s)^{(1)},$$

$$\tilde{p}_n^+(s) = \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x,dl,i,s) \left[K_{l,i}^+(s)^{(n)} - K_{l,i}^+(s)^{(n+1)}\right] \qquad (45)$$

$$+ \sum_{i\in\mathbb{N}} V_r(x,i,s) \sum_{j\in\mathbb{N}} \left[K_i^-(j,s)^{(n-1)} - K_i^-(j,s)^{(n)}\right] f^{j+B}(0,s), \quad n \in \mathbb{N},$$

$$\tilde{p}_0^-(s) = 1 - \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x, dl, i, s) f_{i+B}(l, s) - \sum_{i\in\mathbb{N}} V_r(x, i, s) K_i^-(s)^{(1)},$$

$$\tilde{p}_n^-(s) = \sum_{i\in\mathbb{N}} V_r(x, i, s) \left[K_i^-(s)^{(n)} - K_i^-(s)^{(n+1)}\right] \tag{46}$$

$$+ \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x, dl, i, s) \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} \left[K_{l,i}^+(dv, j, s)^{(n-1)} - K_{l,i}^+(dv, j, s)^{(n)}\right] f_{j+B}(v, s),$$

where $K_i^-(j, s)^{(0)} = \delta_{i,j}, \;\; K_{x,i}^+(dl, j, s)^{(0)} = \delta_{i,j}\, \delta(x - l)\, dl,$

$$K_i^-(s)^{(n)} = \sum_{j\in\mathbb{N}} K_i^-(j, s)^{(n)} \quad K_{x,i}^+(s)^{(n)} = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} K_{x,i}^+(dl, j, s)^{(n)}.$$

Proof. Taking into account the total probability law, the homogeneity property of the process X_t with respect to the first component, the Markov property of $\tau^k(x), \;\tau_k(x), \;\chi,$ we can write the following system for the functions $h^i(x, a, b, s)$, $h_i(0, a, b, s), \;\; h(x, a, b, s)$

$$h^i(x, a, b, s) = 1 - f_{i+B}(x, s) + b \sum_{j\in\mathbb{N}} f_{i+B}(x, j, s) h_j(0, a, b, s),$$

$$h_j(0, a, b, s) = 1 - f^{j+B}(0, s) + a \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} f^{j+B}(0, dl, i, s) h^i(l, a, b, s), \tag{47}$$

$$h(x, a, b, s) = 1 - \mathbf{E}e^{-s\chi} + \sum_{i\in\mathbb{N}} \int_{\mathbb{R}_+} V^k(x, dl, i, s) h^i(l, s) + \sum_{i\in\mathbb{N}} V_r(x, i, s) h_i(0, s).$$

Substituting the expression for the function $h_j(0, a, b, s)$ from the second equation into the first one yields

$$h^i(x, a, b, s) = 1 - f_{i+B}(x, s) + b \sum_{j\in\mathbb{N}} f_{i+B}(x, j, s) \left(1 - f^{j+B}(0, s)\right)$$

$$+ ab \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} K_{x,i}^+(dl, j, s)^{(n)} h^j(l, a, b, s), \qquad i \in \mathbb{N},$$

a linear integral equation with respect to the function $h^i(x, a, b, s)$. Solving it by the method of successive iterations [29] results in the following:

$$h^i(x, a, b, s) = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} \mathfrak{K}_{x,i}^+(dl, j, a, b, s) \left(1 - f_{j+B}(l, s)\right) +$$

$$+ b \sum_{j\in\mathbb{N}} \int_{\mathbb{R}_+} \mathfrak{K}_{x,i}^+(dl, j, a, b, s) \sum_{m\in\mathbb{N}} f_{j+B}(l, m, s) \left(1 - f^{m+B}(0, s)\right).$$

Hence, we obtained the equality (41) of Theorem 3. The function $h_i(0, a, b, s)$ can be found analogously, and the function $h(x, a, b, s)$ is then derived from the third equation of (47). Comparing the coefficients of $a^m b^n, \;\; m, n \in \mathbb{Z}^+, \;\; |m - n| \leq 1$ in both sides of the equalities in the first part of Theorem 3, we get the formulae of the second part of Theorem 3.

We now verify the formulae (45),(46). Letting $b=1$ in (41), we derive the following relation for the generating function $h^i(x,s,a,1)=\mathbf{E}\left[a^{\alpha^+_{\nu_s}(k+i)}\right]$, $a\in[0,1]$,

$$h^i(x,s,a,1)=1-K^+_{x,i}(s)^{(1)}+\sum_{n\in\mathbb{N}}a^n\left[K^+_{x,i}(s)^{(n)}-K^+_{x,i}(s)^{(n+1)}\right]. \tag{48}$$

Analogously, taking into account (42) for $b=1$, we can write for the generating function $h_i(0,s,a,1)$

$$\begin{aligned}h_i(0,s,a,1)&=1-f^{i+B}(0,s)\\&+\sum_{n\in\mathbb{N}}a^n\sum_{j\in\mathbb{N}}\left[K^-_i(j,s)^{(n-1)}-K^-_i(j,s)^{(n)}\right]f^{j+B}(0,s).\end{aligned} \tag{49}$$

Substituting the right-hand sides of (48), (49) into the third equality of (47), we derive the relation for the generating function $h(x,s,a,1)$ of the distribution $\mathbf{P}\left[\alpha^+_{\nu_s}=n\right]$

$$\begin{aligned}h(x,s,a,1)=1&-\sum_{i\in\mathbb{N}}V_r(x,i,s)f^{i+B}(0,s)-\sum_{i\in\mathbb{N}}\int_{\mathbb{R}_+}V^k(x,i,dl,s)K^+_{l,i}(s)^{(1)}\\&+\sum_{n\in\mathbb{N}}a^n\sum_{i\in\mathbb{N}}\int_{\mathbb{R}_+}V^k(x,i,dl,s)\left[K^+_{l,i}(s)^{(n)}-K^+_{l,i}(s)^{(n+1)}\right]\\&+\sum_{n\in\mathbb{N}}a^n\sum_{i\in\mathbb{N}}V_r(x,i,s)\sum_{j\in\mathbb{N}}\left[K^-_i(j,s)^{(n-1)}-K^-_i(j,s)^{(n)}\right]f^{j+B}(0,s).\end{aligned}$$

Comparing the coefficients of a^n, $n\in\mathbb{Z}^+$, in both sides of this equality, we obtain (45). Letting $a=1$ in (41)-(43) yields

$$\begin{aligned}&h(x,s,1,b)=1-\sum_{i\in\mathbb{N}}V_r(x,i,s)K^-_i(s)^{(1)}-\sum_{i\in\mathbb{N}}\int_{\mathbb{R}_+}V^k(x,i,dl,s)f_{i+B}(l,s)+\\&\sum_{n\in\mathbb{N}}b^n\sum_{i\in\mathbb{N}}\int_{\mathbb{R}_+}V^k(x,i,dl,s)\sum_{j\in\mathbb{N}}\int_{\mathbb{R}_+}\left[K^+_{l,i}(dv,j,s)^{(n-1)}-K^+_{l,i}(dv,j,s)^{(n)}\right]f_{j+B}(v,s)\\&+\sum_{n\in\mathbb{N}}b^n\sum_{i\in\mathbb{N}}V_r(x,i,s)\left[K^-_i(s)^{(n)}-K^-_i(s)^{(n+1)}\right].\end{aligned}$$

Comparing the coefficients of b^n, $n\in\mathbb{Z}^+$ in both sides, we get (46). □

Corollary 5. *Let $\delta\sim ge(\lambda)$. Then*

$$\begin{aligned}\tilde{p}^{n+1}_n(s)&=[1-F(s)]\left(\varphi^k_x(s)-\frac{Q^s_k(x)}{\mathbf{E}Q^s_{B+\delta}}T(s)\right)T(s)^n,\\\tilde{p}^n_{n+1}(s)&=[F(s)-T(s)]\frac{Q^s_k(x)}{\mathbf{E}Q^s_{B+\delta}}T(s)^n,\\\tilde{p}^n_n(s)&=\mathbf{I}_{\{n=0\}}-\left(\varphi^k_x(s)+\frac{Q^s_k(x)}{\mathbf{E}Q^s_{B+\delta}}[F(s)-T(s)]\right)T(s)^n\\&\quad+\mathbf{I}_{\{n\in\mathbb{N}\}}\left(\varphi^k_x(s)F(s)+\frac{Q^s_k(x)}{\mathbf{E}Q^s_{B+\delta}}[1-F(s)]T(s)\right)T(s)^{n-1},\end{aligned} \tag{50}$$

where $c_x(s) = \tilde{f}_x(s - k(c(s)))$, $r(s) = 1 + (1-\lambda)k'(c(s))\tilde{f}'(s - k(c(s)))$;

$$\begin{aligned} \varphi_x^k(s) &= c_x(s)c(s)^{B-k} - c(s)^{B+1}r(s)\frac{Q_k^s(x)}{1-\lambda}, \\ T(s) &= \mathbf{E}\varphi_0^{B+\delta}(s) = 1 - c(s)^{B+1}r(s)\frac{\mathbf{E}Q_{B+\delta}^s}{1-\lambda}, \\ F(s) &= \mathbf{E}f^{B+\delta}(0,s) = (1-\lambda)\sum_{k\in\mathbb{N}}\lambda^{k-1}f^{k+B}(0,s), \end{aligned} \tag{51}$$

and the Laplace transform $f^k(0,s)$ *is given by (30);*

the distributions $\hat{p}_n^{\pm}(s) = \mathbf{P}\left[\alpha_{\nu_s}^{\pm} \geq n\right]$, $n \in \mathbb{N}$ *of the random variables* $\alpha_{\nu_s}^{\pm}$ *are such that*

$$\mathbf{P}\left[\alpha_{\nu_s}^{+} = 0\right] = 1 - F(s)f_r(x,s), \quad \hat{p}_n^{+}(s) = F(s)f_r(x,s)T(s)^{n-1}, \tag{52}$$

$$\mathbf{P}\left[\alpha_{\nu_s}^{-} = 0\right] = 1 - \varphi_x^k(s), \quad \hat{p}_n^{-}(s) = \varphi_x^k(s)T(s)^{n-1}. \tag{53}$$

where $f_r(x,s) = \mathbf{E}e^{-s\tau_r(x)} = c_x(s)c(s)^r$.

Proof. The formulae of the corollary follow straightforwardly from the equalities (44)–(46) of Theorem 3 when $\delta \sim ge(\lambda)$. To illustrate this, we verify the formulae (53) from (46). The successive iterations (34) were calculated in [18] when $\delta \sim ge(\lambda)$:

$$\begin{aligned} K_i^-(m,s)^{(n)} &= \varphi_0^{i+B}(s)(1-\lambda)\lambda^{m-1}\left(\mathbf{E}\varphi_0^{B+\delta}(s)\right)^{n-1}, \\ K_{\nu,i}^+(dl,m,s)^{(n)} &= c_\nu(s)c(s)^{i+B}\mathbf{E}f^{B+\delta}(0,dl,m,s)\left(\mathbf{E}\varphi_0^{B+\delta}(s)\right)^{n-1}, \end{aligned} \tag{54}$$

where $\varphi_x^k(s) = \sum\limits_{i\in\mathbb{N}}\int\limits_{\mathbb{R}+} f^k(x,dl,i,s)c_l(s)c(s)^{i+B}$. Substituting the expression for the successive iterations (54) and the expression for $V_r(x,m,s)$, $V^k(x,dl,m,s)$ (37) into (46) and performing some calculations, we find

$$\tilde{p_0}^-(s) = 1 - \varphi_x^k(s), \quad \tilde{p_n}^-(s) = \varphi_x^k(s)(1-T(s))T(s)^{n-1}, \qquad n \in \mathbb{N}.$$

The formulae (50), (52) can be verified analogously. □

5 Asymptotic results

In this section we study the asymptotic behavior of the two-boundary characteristics of the process. More specifically, we prove weak convergence of the distributions of these characteristics for the difference of the compound Poisson process and the compound renewal process to the corresponding distributions for the Wiener process under certain conditions. Here and in the sequel we will assume that $\delta \sim ge(\lambda)$ and that the following conditions are satisfied:

(A) $\rho = (1-\lambda)\mu\mathbf{E}\eta\mathbf{E}\varkappa = 1, \quad \sigma^2 = \mu\left[\mathbf{E}\varkappa(\varkappa-1) + \frac{\mathbf{E}\varkappa\mathbf{E}\eta^2}{(1-\lambda)(\mathbf{E}\eta)^2}\right] < \infty.$

Before stating the main results of this section, we present auxiliary results. In the sequel we will require the following expansions:

$$\tilde{f}_x(s) = 1 - s\mathbf{E}\eta_x + \frac{1}{2}s^2\mathbf{E}\eta_x^2 + o(s^2), \quad x \in \mathbb{R}_+,$$
$$\mathbf{E}e^{-p\varkappa} = 1 - p\mathbf{E}\varkappa + \frac{1}{2}p^2\mathbf{E}\varkappa^2 + o(p^2), \tag{55}$$

which are valid for small s, p.

Lemma 5. *Let* $x, k \in \mathbb{R}_+$, $s > 0$. *The following limit equalities hold*

$$\lim_{B\to\infty} \mathbf{E}e^{-s\tau_{[kB]}(x)/B^2} = \lim_{B\to\infty} \mathbf{E}e^{-s\tau^{[kB]}(x)/B^2} = e^{-k\sqrt{2s}/\sigma}, \tag{56}$$

where $[a]$ *stands for an integer part of the number* a;

$$\lim_{B\to\infty} \frac{1}{B} Q^{s/B^2}_{[kB]}(x) = \lim_{B\to\infty} \frac{1}{B}\mathbf{E}Q^{s/B^2}_{[kB]+\delta}(x) = \frac{1}{\mathbf{E}\eta}\frac{2}{\sigma\sqrt{2s}}\sinh(k\sqrt{2s}/\sigma). \tag{57}$$

Proof. It follows from (55) and (18) $(1-\lambda)\tilde{f}(s - k(c(s))) = c(s) - \lambda$ that the following representation is valid for $c(s/B^2)$ as $B \to \infty$

$$c(s/B^2) = 1 - \frac{1}{B}\sqrt{2s}/\sigma + o\left(\frac{1}{B}\right). \tag{58}$$

The formula (22) and this asymptotic equality imply the first part of (56). We now turn to the asymptotic properties of the resolvent sequence $\{Q^s_k(x)\}_{k\in\mathbb{Z}^+}$. It follows from the definition (29) for $\theta = e^{-p}$, $p > -ln\, c(s)$ that

$$\mathbb{Q}^s_\theta(x) = \sum_{k\in\mathbb{Z}^+} \theta^k Q^s_k(x)\Big|_{\theta=e^{-p}} = \int_0^\infty e^{-p[k]} Q^s_{[k]}(x)\, dk$$
$$= \int_0^\infty e^{\{k\}p} e^{-pk} Q^s_{[k]}(x)\, dk = \mathbb{Q}^s_{e^{-p}}(x), \quad p > -ln\, c(s),$$

where $\{a\}$ is a fractional part of the number a. It is clear that

$$\mathfrak{Q}^s_p(x) \le \mathbb{Q}^s_{e^{-p}}(x) \le e^p \mathfrak{Q}^s_p(x), \tag{59}$$

where $\mathfrak{Q}^s_p(x) = \int_0^\infty e^{-pk} Q^s_{[k]}(x)\, dk,\quad p > -ln\, c(s)$ is the Laplace transform of the function $Q^s_{[k]}(x)$, $k \in \mathbb{R}_+$. The definition (29) and (55) imply for $p > \sqrt{2s}/\sigma$ that

$$\lim_{B\to\infty} \frac{1}{B^2}\mathbb{Q}^{s/B^2}_{e^{-p/B}}(x) = \lim_{B\to\infty} \frac{1}{B^2} \frac{(1-\lambda)\tilde{f}_x(s/B^2 - k(e^{-p/B}))}{(1-\lambda)\tilde{f}(s/B^2 - k(e^{-p/B})) + \lambda - e^{-p/B}}$$
$$= \frac{1}{\mathbf{E}\eta}\frac{1}{\frac{1}{2}p^2\sigma^2 - s}, \qquad p > \sqrt{2s}/\sigma.$$

It follows from the chain (59) that

$$\lim_{B\to\infty} \frac{1}{B^2}\mathfrak{Q}^{s/B^2}_{p/B}(x) = \lim_{B\to\infty} \frac{1}{B^2}\mathbb{Q}^{s/B^2}_{e^{-p/B}}(x) = \frac{1}{\mathbf{E}\eta}\frac{1}{\frac{1}{2}p^2\sigma^2 - s}, \quad p > \sqrt{2s}/\sigma.$$

Inverting the Laplace transforms in both sides, we obtain

$$\lim_{B\to\infty}\frac{1}{B}Q^{s/B^2}_{[kB]}(x)=\frac{1}{\mathbf{E}\eta}\frac{2}{\sigma\sqrt{2s}}\sinh(k\sqrt{2s}/\sigma),$$

i.e. the first part of (57). It is not difficult to derive the following representation

$$\tilde{Q}^s_\theta=\sum_{k\in\mathbb{Z}^+}\theta^k\mathbf{E}Q^s_{k-\delta}=\frac{1-\lambda}{(1-\lambda)\tilde{f}(s-k(c(s)))+\lambda-\theta},\quad \theta\in(0,c(s)).$$

The latter equality and (55) imply that for $p>\sqrt{2s}/\sigma$

$$\lim_{B\to\infty}\frac{1}{B^2}\tilde{Q}^{s/B^2}_{e^{-p/B}}=\lim_{B\to\infty}\frac{1}{B^2}\int_0^\infty e^{-p[k]/B}\mathbf{E}Q^{s/B^2}_{[k]-\delta}\,dk=\frac{1}{\mathbf{E}\eta}\frac{1}{\frac{1}{2}p^2\sigma^2-s}.$$

This asymptotic equality implies the second part of (57). We will now verify the second part of (56). Taking into account the formulae (32),(55), we derive

$$\lim_{B\to\infty}\frac{1}{B}\Phi^{s/B^2}_{e^{-p/B}}(x)=\lim_{B\to\infty}\frac{1}{B}\int_0^\infty e^{-p[k]/B}\mathbf{E}e^{-s\tau^{[k]}(x)/B^2}\,dk=\frac{1}{p+\sqrt{2s}/\sigma}$$

It is obvious that $\tilde{\Phi}^s_p(x)\le\Phi^s_{e^{-p}}(x)\le e^p\tilde{\Phi}^s_p(x)$, where

$$\tilde{\Phi}^s_p(x)=\int_0^\infty e^{-pk}\mathbf{E}e^{-s\tau^{[k]}(x)}\,dk\quad p>-ln\,c(s)$$

is the Laplace transform of the function $\mathbf{E}e^{-s\tau^{[k]}(x)}$, $k\in\mathbb{R}_+$. Hence, $\lim\limits_{B\to\infty}\frac{1}{B}\tilde{\Phi}^{s/B^2}_{p/B}(x)=\lim\limits_{B\to\infty}\frac{1}{B}\Phi^{s/B^2}_{e^{-p/B}}(x)$ and

$$\lim_{B\to\infty}\frac{1}{B}\int\limits_0^\infty e^{\frac{-pk}{B}}\mathbf{E}e^{-s\tau^{[k]}(x)/B^2}\,dk=\lim_{B\to\infty}\int\limits_0^\infty e^{-pk}\mathbf{E}e^{-s\tau^{[kB]}(x)/B^2}\,dk=\frac{1}{p+\sqrt{2s}/\sigma}.$$

The latter equality implies the second part of (56). □

Denote by $\{w_t;\ t\ge 0\}$ a standard Wiener process, $\mathbf{E}w_1=0$, $\mathbf{V}ar\ w_1=\sigma^2>0$, and let

$$\chi^*=\inf\{t:\ w_t\notin(-r,k)\},\qquad k\in(0,1),\quad r=1-k,$$

be the first exit time from the interval $(-r,k)$ by the process w_t. It is well-known (see for instance [10]) that the Laplace transforms of χ^* are such that

$$\mathbf{E}\left[e^{-s\chi^*};A^k\right]=\frac{\sinh\left(r\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)},\qquad \mathbf{E}\left[e^{-s\chi^*};A_r\right]=\frac{\sinh\left(k\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)},$$

where $A^k=\{w_{\chi^*}=k\}$, $A_r=\{w_{\chi^*}=-r\}$ are the events denoting the exit from the interval $(-r,k)$ through the upper boundary k and through the lower boundary $-r$.

Corollary 6. *Assume that the conditions (A) are satisfied,* $\delta \sim ge(\lambda)$, *and let* $\{D_x(t)\}_{t\geq 0}$ *be the difference of the compound Poisson process and the compound renewal process (3),*

$$\chi(B) = \inf\{t : D_x(t) \notin [-rB, kB]\}, \qquad k \in (0,1), \quad r = 1-k, \quad B \in \mathbb{R}_+,$$

$\mathfrak{A}^k(B) = \{D_x(\chi(B)) > kB\}$, $\mathfrak{A}_r(B) = \{D_x(\chi(B)) < -rB\}$. *Then the following limiting equalities hold for* $B \to \infty$

$$\mathbf{P}\left[\frac{\chi(B)}{B^2} \in dt; \mathfrak{A}^k(B)\right] \to \mathbf{P}\left[\chi^* \in dt; A^k\right] = \pi\sigma^2 \sum_{n\in\mathbb{N}} n e^{-\frac{t}{2}(\sigma\pi n)^2} \sin(k\pi n)\, dt,$$
$$\mathbf{P}\left[\frac{\chi(B)}{B^2} \in dt; \mathfrak{A}_r(B)\right] \to \mathbf{P}\left[\chi^* \in dt; A_r\right] = \pi\sigma^2 \sum_{n\in\mathbb{N}} n e^{-\frac{t}{2}(\sigma\pi\nu)^2} \sin(r\pi n)\, dt. \quad (60)$$

The limiting exit probabilities admit the following representations for $B \to \infty$

$$\mathbf{P}\left[\mathfrak{A}^k(B)\right] \to \frac{2}{\pi}\sum_{n\in\mathbb{N}} \frac{\sin(k\pi n)}{n} = r, \quad \mathbf{P}\left[\mathfrak{A}_r(B)\right] \to \frac{2}{\pi}\sum_{n\in\mathbb{N}} \frac{\sin(r\pi n)}{n} = k.$$

Proof. The first formula of (38) and (57) imply that

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}_r(B)\right] = \lim_{B\to\infty} \frac{Q^{s/B^2}_{[kB]}(x)}{\mathbf{E}Q^{s/B^2}_{[B]-\delta}} = \frac{\sinh\left(k\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)} = \mathbf{E}\left[e^{-s\chi^*}; A_r\right].$$

Inverting the Laplace transform in the right-hand side of this equality, we derive the first equality of (60). Taking into account the definition of the function $A^s_k(x)$ (38), we have

$$\mathbb{A}^s_\theta(x) = \sum_{k\in\mathbb{Z}^+} \theta^k A^s_k(x) = \frac{s}{s-k(\theta)}\left(\frac{\tilde{f}_x(s-k(\theta))}{(1-\lambda)\tilde{f}(s-k(\theta))+\lambda-\theta} - \frac{1}{1-\theta}\right),$$

where $\theta \in (0, c(s))$. The latter equality and (55) imply for $p > \sqrt{2s}/\sigma$ that

$$\lim_{B\to\infty} \frac{1}{B}\mathbb{A}^{s/B^2}_{e^{-p/B}}(x) = \lim_{B\to\infty}\frac{1}{B}\int_0^\infty e^{\{k\}p/B} e^{-pk/B} A^{s/B^2}_{[k]}(x)\, dk = \frac{1}{p}\,\frac{s}{\frac{1}{2}p^2\sigma^2 - s}.$$

It is clear that $\mathfrak{A}^s_p(x) \le \mathbb{A}^s_{e^{-p}}(x) \le e^p\mathfrak{A}^s_p(x)$, where $\mathfrak{A}^s_p(x) = \int_0^\infty e^{-kp}A^s_{[k]}(x)\, dk$ is the Laplace transform of the function $A^s_{[k]}(x)$, $k \in \mathbb{R}_+$. Hence, $\lim\limits_{B\to\infty} \frac{1}{B}\mathfrak{A}^{s/B^2}_{p/B}(x) = \lim\limits_{B\to\infty} \frac{1}{B}\mathbb{A}^{s/B^2}_{e^{-p/B}}(x)$ and, thus,

$$\lim_{B\to\infty}\frac{1}{B}\mathfrak{A}^{s/B^2}_{p/B}(x) = \lim_{B\to\infty}\int_0^\infty e^{-kp}A^{s/B^2}_{[kB]}(x)\, dk = \frac{1}{p}\,\frac{s}{\frac{1}{2}p^2\sigma^2 - s}.$$

Inverting the Laplace transforms in the both sides, we get

$$\lim_{B\to\infty} A^{s/B^2}_{[kB]}(x) = \cosh\left(k\sqrt{2s}/\sigma\right) - 1, \quad p > \sqrt{2s}/\sigma. \quad (61)$$

Analogously we derive that for all $k \in \mathbb{R}_+$

$$\lim_{B\to\infty} \mathbf{E}A^{s/B^2}_{[kB]-\delta}(0) = \cosh\left(k\sqrt{2s}/\sigma\right) - 1, \quad p > \sqrt{2s}/\sigma. \tag{62}$$

where $\mathbf{E}\,A^s_{[u]-\delta} = \sum\limits_{i\in\mathbb{N}}(1-\lambda)\lambda^{i-1}A^s_{[u]+i}(0)$. It follows from the second formula of (38) and from the asymptotic equations (61), (62) that

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}^k(B)\right] = \cosh\left(k\sqrt{2s}/\sigma\right) - \frac{\sinh\left(k\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)}\cosh\left(\sqrt{2s}/\sigma\right)$$
$$= \frac{\sinh\left(r\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)} = \mathbf{E}\left[e^{-s\chi^*}; A^k\right].$$

Inverting the Laplace transforms in the both sides, we obtain the second equality of (60). It is worth noticing that by means of (60) we established the weak convergence of $\chi(B)/B^2$ to χ^* as $\to\infty$. □

Corollary 7. *Suppose that the conditions (A) are fulfilled, $\delta \sim ge(\lambda)$, and let $\{D_x(t)\}_{t\ge 0}$ be the difference of the compound Poisson process and the compound renewal process (3) whose jumps are geometrically distributed, $k \in (0,1)$, $r = 1-k$, $B > 0$. Denote by*

$\tilde{\alpha}^+_{tB^2}$ the number of the upward intersections of the interval $[-rB, kB]$ by the process $D_x(\cdot)$ on the time interval $[0, tB^2]$;

$\tilde{\alpha}^-_{tB^2}$ the number of the downward intersections of the interval $[-rB, kB]$ by the process $D_x(\cdot)$ on the time interval $[0, tB^2]$. Then

(i) *the distributions $p^j_i(t,B) = \mathbf{P}\left[\tilde{\alpha}^+_{tB^2} = i, \tilde{\alpha}^-_{tB^2} = j\right]$, $|i-j| \le 1$ obey the following relations for $n \in \mathbb{Z}^+$ as $B \to \infty$*

$$p^{n+1}_n(t,B) \to 2\sum_{i\ge 2(n+1)} (-1)^i \mu_t(-r+i, r+i), \tag{63}$$
$$p^n_{n+1}(t,B) \to 2\sum_{i\ge 2(n+1)} (-1)^i \mu_t(-k+i, k+i),$$

$$p^0_0(t,B) \to 1 - 2\sum_{i\in\mathbb{N}}(-1)^{i-1}\left[\mu_t(k+i) + \mu_t(r+i)\right],$$
$$p^n_n(t,B) \to 2\sum_{i\ge 2n+1}(-1)^{i-1}\left[\mu_t(-k+i, k+i) + \mu_t(-r+i, r+i)\right],$$

where $\mu_t(a) = \mu_t(a,\infty) = \mathbf{P}\left[w_t > a\right]$,

$$\mu_t(a,b) = \mathbf{P}\left[w_t \in (a,b)\right] = \frac{1}{\sigma\sqrt{2\pi t}}\int_a^b e^{-x^2/2t\sigma^2}\,dx;$$

(ii) *for the distributions $\hat{p}^\pm_n(t,B) = \mathbf{P}\left[\tilde{\alpha}^\pm_{tB^2} \ge n\right]$, $n \in \mathbb{N}$ the following asymptotic equalities hold as $B \to \infty$*

$$\mathbf{P}\left[\tilde{\alpha}^+_{tB^2} = 0\right] \to 1 - 2\mu_t(2-k), \quad \hat{p}^+_n(t,B) \to 2\mu_t(2n-k),$$
$$\mathbf{P}\left[\tilde{\alpha}^-_{tB^2} = 0\right] \to 1 - 2\mu_t(2-r), \quad \hat{p}^-_n(t,B) \to 2\mu_t(2n-r).$$

Proof. We first establish the asymptotic properties of the functions (51) which will be used in the sequel. It follows from (30) that

$$F(s) = 1 - \frac{s}{s - k(c(s))} \frac{\mathbf{Q}^s_{B+\delta}}{1-\lambda} + \mathbf{A}^s_{B+\delta}(0).$$

Employing the asymptotic equality $k(c(s/B^2) = \frac{1}{B}\mu\mathbf{E}\varkappa + o(\frac{1}{B})$ and the formulae (57), (62), we find that

$$\lim_{B\to\infty} F(s/B^2) = 1 - \sinh\left(\sqrt{2s}/\sigma\right) + \cosh\left(\sqrt{2s}/\sigma\right) - 1 = e^{-\sqrt{2s}/\sigma}.$$

The identity $(1-\lambda)\tilde{f}(s - k(c(s))) = c(s) - \lambda$ and the relation (58) imply that

$$\lim_{B\to\infty} Br(s/B^2) = (1-\lambda)\mathbf{E}\eta\sigma\sqrt{2s}.$$

Taking into account the latter equality and the formulae (56), (57), we calculate

$$\lim_{B\to\infty} \varphi_x^{[kB]}(s/B^2) = e^{-(k+1)\sqrt{2s}/\sigma}, \qquad \lim_{B\to\infty} T(s/B^2) = e^{-2\sqrt{2s}/\sigma}.$$

Now we can obtain the asymptotic equalities of Corollary 7 by employing the formulae of Corollary 5. Let us verify (63). It follows from the first formula of (50) that

$$\int_0^\infty e^{-st} p_n^{n+1}(t,B)\,dt =$$
$$= \frac{1}{s}\left[1 - F\left(\frac{s}{B^2}\right)\right]\left(\varphi_x^{[kB]}\left(\frac{s}{B^2}\right) - \frac{Q^{s/B^2}_{[kB]}(x)}{\mathbf{E}Q^{s/B^2}_{B+\delta}} T\left(\frac{s}{B^2}\right)\right) T\left(\frac{s}{B^2}\right)^n.$$

Calculating the limits in the right-hand side of this equality, we derive

$$\lim_{B\to\infty}\int_0^\infty e^{-st} p_n^{n+1}(t,B)\,dt = \frac{1}{s}\frac{e^{r\sqrt{2s}/\sigma} - e^{-r\sqrt{2s}/\sigma}}{1 + e^{-\sqrt{2s}/\sigma}} e^{-2(n+1)\sqrt{2s}/\sigma}. \tag{64}$$

Let $\tau^a = \inf\{t > 0 : w_t > a\}$ denote the first crossing time of the level $a \in \mathbb{R}_+$ by the Wiener process whose Laplace exponent is given by $k(p) = \frac{1}{2}p^2\sigma^2$. Then the well-known relation [10] $\mathbf{P}[\tau^a < t] = 2\mathbf{P}[w_t > a]$ implies the following equality for the Laplace transforms

$$\frac{1}{s}e^{-a\sqrt{2s}/\sigma} = 2\int_0^\infty e^{-st}\mathbf{P}[w_t > a]\,dt.$$

Taking into account the latter equality and the expansion $(1 + e^{-\sqrt{2s}/\sigma})^{-1} = \sum_{i\in\mathbb{Z}^+}(-1)^i e^{-i\sqrt{2s}/\sigma}$ and inverting the Laplace transforms in both sides of (64), we obtain

$$\lim_{B\to\infty} p_n^{n+1}(t,B) = 2\sum_{i\geq 2(n+1)}(-1)^i\mu_t(-r+i, r+i), \quad n \in \mathbb{Z}^+$$

i.e. the first equality of (63). Other equalities can be verified analogously. Note, that the probabilities which enter the right-hand sides of the formulae of the corollary are the distributions of the number of the intersections of the interval $(-r,k)$, $k \in (0,1)$, $r = 1-k$ by the Wiener process w_t on the time interval $[0,t]$ (see [13]). $\square$

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