



Cyclicity of common slow–fast cycles

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Abstract

We study the limit cycles of planar slow–fast vector fields, appearing near a given slow–fast cycle, formed by an arbitrary sequence of slow parts and fast parts, and where the slow parts can meet the fast parts in a nilpotent contact point of arbitrary order. Using the notion slow divergence integral, we delimit a large subclass of these slow–fast cycles out of which at most one limit cycle can perturb, and a smaller subclass out of which exactly one limit cycle will perturb. Though the focus lies on common slow–fast cycles, i.e. cycles with only attracting or only repelling slow parts, we present results that are valid for more general slow–fast cycles. We also provide examples of attracting common slow–fast cycles out of which more than one limit cycle can perturb, one of which is repelling.

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1. Introduction

This paper is devoted to the study of limit cycles appearing in singularly perturbed families of planar vector fields (or more generally vector fields X_ϵ on an orientable 2-manifold). We consider an ϵ -family of vector fields on a 2-manifold that, for $\epsilon = 0$, has a curve of singular points. Such a curve will be called a slow curve and will be denoted S . In general, S consists of hyperbolically attracting points, hyperbolically repelling points and contact points, depending on whether the linear part of the vector field at that point of S has a negative nonzero eigenvalue, a

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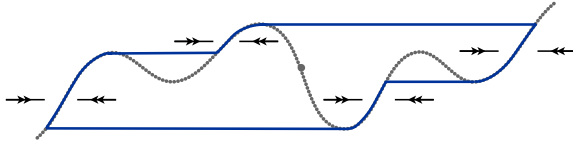


Fig. 1. An example of a common slow–fast cycle.

positive nonzero eigenvalue, or 2 zero eigenvalues. In this paper, we will only consider contact points of nilpotent type, subject to some mild extra conditions (see later). The main question that we deal with is to describe the dynamics near a so-called *slow–fast cycle*, focusing first on *common slow–fast cycles*. In particular we study the limit cycles that can perturb from such a slow–fast cycle. Let us explain the notion “slow–fast cycle” and “common slow–fast cycle”. It is well-known that orbits of X_ϵ , for $\epsilon > 0$ small, are Hausdorff close to a succession of fast orbits and parts of the slow curve. Fast orbits are regular orbits of X_0 . The presence of a curve of singular points of X_0 typically reduces the complexity of the phase portrait: points near the curve are either attracted or repelled away from the slow curve. In the neighborhood of the slow curve, the behavior of X_ϵ , for $\epsilon > 0$ small, is determined by a slow drift along the slow curve. This is called the *slow dynamics*.

A *slow–fast cycle* is a succession of slow parts and fast orbits, such that the end points coincide. For $\epsilon = 0$ we of course have no limit cycles; limit cycles may exist for $\epsilon > 0$ small, bifurcating from a slow–fast cycle. A *common slow–fast cycle* is a slow–fast cycle where the slow parts are either all attracting or all repelling, see for example in Fig. 1. This notion is seen in contrast with a canard cycle, which is a slow–fast cycle where both attracting and repelling slow parts are present. In this paper, we obtain results on both common and canard-type slow–fast cycles in a quite general setting, but to present the ideas, we will focus in the introduction on attracting common slow–fast cycles (the case of repelling common cycles is completely similar by inversion of time). Furthermore, we will first suppose that the slow dynamics is regular as well outside as at the contact points. If this is the case, we speak of a regular common slow–fast cycle. Later in the paper we will obtain a result for a more general class of slow–fast cycles.

There are two main results in this paper. On one hand we prove that one can have at most one limit cycle perturbing from an attracting common slow–fast cycle Γ ; on the other hand we show the occurrence of a (unique) limit cycle of X_ϵ when perturbing from a *strongly* common slow–fast cycle. For the first result (that in fact will be proven to hold for a slightly more general type of slow–fast cycles than the attracting common one), we essentially study the first return map along Γ , and prove that it has a derivative strictly between 0 by 1 (for $\epsilon > 0$). To that end, we relate this derivative of the first return map to the exponential of the integral of the divergence of the vector field. We prove that this divergence integral is dominated by the contributions it has from the slow parts (i.e. the fast parts are negligible, and so are the parts in the vicinity of contact points). Since the slow parts of a common cycle are all attracting or all repelling, the divergence integral surely has a fixed sign, from which the required results on the derivative of the first return map are derived. Such a bound on the divergence integral will be obtained for arbitrary regular common slow–fast cycles, showing that at most one limit cycle may perturb from it. We then generalize the result to a slightly larger class of slow–fast cycles.

On the other hand, we will introduce the notion “strongly common slow–fast cycle”, and for this subset of common cycles, we will show that necessarily a (unique) limit cycle perturbs from it. The existence part in this statement will be done by covering the slow–fast cycle with a chain of limits of flow box neighborhoods. More details will be given in Sections 5 and 6.

The result on the upper bound for the number of limit cycles is the hardest to obtain: the main technique consists in showing a fixed sign for the divergence integral. To prove that the part of the divergence integral near contact points is negligible, we will now impose some (rather mild) conditions on the contact point. Besides assuming that the contact point has a nilpotent linear part, we assume that it is a regular contact point of finite order or a singular contact point of index +1 (precise definitions are formulated in Section 4). Locally near the contact point, the family of vector fields X_ϵ can be written (in an appropriate local chart of the 2-manifold, and after a suitable time rescaling) as

$$\begin{cases} \dot{x} = y - f(x) \\ \dot{y} = \epsilon(g(x, \epsilon) + (y - f(x))h(x, y, \epsilon)), \end{cases}$$

where f, g and h are assumed to be C^∞ near $x = y = \epsilon = 0$. (The reduction of vector fields near nilpotent contact points to the above normal form will be done in Section 4.) For the moment we require that f only depends on x . Later on we will permit that f also depends on extra parameters. A contact point of finite order is a contact point where $f(x)$ is non-flat:

$$f(x) = cx^n + O(x^{n+1}), \quad c \neq 0 \text{ and } n \geq 2.$$

We call n the *order* of the contact point. The contact point is called *regular* when

$$g(0, 0) \neq 0.$$

In Section 4, we show that the two notions “order of a contact point”, and “regular contact point” can be defined in an intrinsic way. In the literature, a regular contact point is often called a *jump point*. A typical and well-known example of a family of vector fields having a common slow–fast cycle is the Van der Pol oscillator (without control parameter)

$$\begin{cases} \dot{x} = y + x - \frac{1}{3}x^3 \\ \dot{y} = -\epsilon x. \end{cases}$$

In the Van der Pol example, two contact points are present; these are both regular and of second order. The limit cycle, which exists for small $\epsilon > 0$, is a combination of a fast, a slow, a fast and another slow part; it is therefore called a relaxation oscillation. In this paper, we discuss, among others, common slow–fast cycles with an arbitrary number of successions of slow and fast parts, through regular contact points of any (odd or even) finite order. To illustrate the generality of the common slow–fast cycles, we observe that it can in general be composed of ten different types of local passages, only four of which are present in the study of the Van der Pol oscillator.

The introductory section essentially dealt with families of vector fields X_ϵ , but the remainder of the paper will deal with more general families of vector fields, depending on extra parameters besides ϵ . The extra parameters actually show the strength of the results in this paper: when the family of vector fields, including f , depends on additional parameters λ , the shape of the common slow–fast cycle may change in a non-differentiable way under influence of λ . A contact point of 4th order might for example bifurcate in two contact points of 2nd order. The examples in Section 8 show that the study of attracting common slow–fast cycles is highly nontrivial when extra parameters are present.

2. Definitions and statement of the results

In order to state the main results, we will introduce notions like slow–fast systems, contact points, the slow dynamics, the slow divergence integral, common slow–fast cycles and canard

slow–fast cycles. Most of these concepts are well-known but are not often written in the language of manifolds.

We consider a two-dimensional smooth orientable manifold (without boundary) M . Let $X_{\epsilon,\lambda}$ be a smooth family of vector fields on M , defined for $\epsilon \in [0, \epsilon_1]$ (for a given $\epsilon_1 > 0$) and for $\lambda \in \Lambda$. Here and in the rest of the paper, Λ is a compact subset of an euclidean space. We assume that $X_{\epsilon,\lambda}$ is of slow–fast type, with singular parameter ϵ , meaning that there exists a smooth λ -family of 1-dimensional embedded manifolds S_λ consisting entirely of singularities of $X_{0,\lambda}$. We remark that S_λ could in general correspond to only a subset of the set of singularities of $X_{0,\lambda}$, and that we do not require S_λ to be connected; in fact one could have several connected components (curves).

Definition 1. A point $p \in S_\lambda$ is called a *normally hyperbolic* (resp. normally attracting or normally repelling) point of $X_{0,\lambda}$ if the linear part of $X_{0,\lambda}$ at p has a nonzero (resp. negative or positive) eigenvalue. It is called a *contact point* when the linear part has two zero eigenvalues. In that case, we distinguish between a nilpotent and a degenerate contact point, depending on whether the differential $D(x_0, \lambda)$ at p is nilpotent or is zero.

We remark that we will only treat passages near nilpotent contact points. We do this for several reasons. First, it is proven that families of vector fields with nilpotent singularities in the plane can be completely desingularized, see [10]. Second, near nilpotent singularities, the set of singularities of $X_{0,\lambda}$ corresponds exactly to the slow curve, i.e. one cannot have transcritical intersections. Indeed, let p be a nilpotent contact point of X_{0,λ_0} . Then we show in Section 4 that there exists a local $((\epsilon, \lambda)$ -dependent) chart where $X_{\epsilon,\lambda}$ is given, up to C^∞ equivalence, by

$$\begin{cases} \dot{x} = y - f(x, \lambda) \\ \dot{y} = \epsilon(g(x, \epsilon, \lambda) + (y - f(x, \lambda))h(x, y, \epsilon, \lambda)) \end{cases} \tag{1}$$

for (ϵ, λ) sufficiently close to $(0, \lambda_0)$. Here, the functions f, g and h are smooth near $(x, y, \epsilon, \lambda) = (0, 0, 0, \lambda_0)$ and $f(0, \lambda_0) = \frac{\partial f}{\partial x}(0, \lambda_0) = 0$. The family of slow curves S_λ is locally given by the smooth family of graphs $y = f(x, \lambda)$.

Definition 2. A nilpotent contact point of X_{0,λ_0} is called regular if

$$g(0, 0, \lambda_0) \neq 0.$$

It is called of finite order when there exists an $n \geq 2$ (called the order of the nilpotent singularity) for which

$$f(x, \lambda_0) = cx^n + O(x^{n+1}), \quad x \rightarrow 0, \quad c \neq 0.$$

A nilpotent contact point is called singular when $g(0, 0, \lambda_0) = 0$.

In case $g(0, 0, \lambda_0) = 0$ and $\frac{\partial g}{\partial x}(0, 0, \lambda_0) \neq 0$, then a single singularity persists from the contact point for $\epsilon > 0$. This singularity has index ± 1 when

$$\text{sign} \frac{\partial g}{\partial x}(0, 0, \lambda_0) = \mp 1.$$

In that case, we say the contact point is of singularity index ± 1 ; an intrinsic definition is given later in Definition 8.

Let us now define the *slow vector field*. At normally hyperbolic points p , we write the family of vector fields $X_{\epsilon,\lambda}$ as a family of vector fields lifted to the manifold $M \times [0, \epsilon_1]$:

$$\bar{X}_\lambda = X_{\epsilon,\lambda} + 0 \frac{\partial}{\partial \epsilon}.$$

At a normally hyperbolic point $(p, 0)$ of \bar{X}_λ , the family has a λ -family of two-dimensional invariant manifolds (center manifolds), that in local coordinates can be written as a graph

$$y = \psi(x, \epsilon, \lambda), \quad \psi(0, 0) = 0.$$

(The center manifold theorem yields C^k -graphs, for any k .) The slow curve is of course given by $y = \psi(x, 0, \lambda)$, which means that $X_{\epsilon,\lambda}(x, \psi(x, \epsilon, \lambda))$ is divisible by ϵ . Consider now the vector field

$$\frac{1}{\epsilon} X_{\epsilon,\lambda}(x, \psi(x, \epsilon, \lambda)).$$

Since $y = \psi(x, \epsilon, \lambda)$ is locally invariant, the result is a family of vector fields tangent to the center manifold, and therefore

$$X_\lambda^{\text{slow}} := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} X_{\epsilon,\lambda}(x, \psi(x, \epsilon, \lambda))$$

is a vector field that, in points of the slow curve, is tangent to the slow curve. Of course, this definition does not depend on the choice of the chart or on the choice of the chosen center manifold. As an example, one can easily find that the slow vector field of (1) is given by

$$x' = \frac{g(x, f(x, \lambda), 0, \lambda)}{\frac{\partial f}{\partial x}(x, \lambda)}, \quad x \neq 0. \tag{2}$$

Here, x' stands for $\frac{dx}{ds}$ where s is the “slow time” $s = \epsilon t$. In principle, the slow dynamics is undefined at contact points.

We focus in essence on families of vector fields where the slow vector field is nonzero everywhere. (Nevertheless, **Theorem 1** will be formulated in a more general framework, treating some classes of slow–fast cycles with singularities on the slow arcs, we come back to this point later.)

Definition 3. Let the family of slow curves be given by the family of embeddings $S_\lambda: I \subset \mathbf{R} \rightarrow M$. For any $[p, q] \subset I$ (with $p \neq q$), the curves $S_\lambda|_{[p,q]}$ (identified with their images) are called slow arcs when $S_\lambda|_{]p,q[}$ consists of normally hyperbolic points of $X_{0,\lambda}$ only. Also the “closed” and “half-open” curves $S_\lambda|_{[p,q]}$, $S_\lambda|_{]p,q]}$, $S_\lambda|_{]p,q[}$ are called slow arcs when they consist of normally hyperbolic points. The points p and q (identified with their image under S_λ) are called the end points of the slow arc (but these could be either points of a slow arc, or could be contact points if they do not belong to the slow arc).

Let F_λ be a family of orbits of $X_{0,\lambda}$, parameterized by fast time t . For any $t_1, t_2 \in \mathbf{R}$ (with $t_1 < t_2$), the curves $F_\lambda|_{[t_1,t_2]}$ are called *fast orbits* if they represent regular orbits. In that case also $F_\lambda|_{[t_1,+\infty[}$, $F_\lambda|_{]-\infty,+\infty[}$ and $F_\lambda|_{]-\infty,t_2]}$ are called fast orbits when the α -limit and/or the ω -limit of the orbit is a point of a slow curve S_λ .

Associated to slow arcs, we intend to define the notion of *slow divergence integral*. Though the notion divergence of integral does not appear in the formulation of our main theorem, the concept is central in the whole paper, and we therefore prefer to introduce it here.

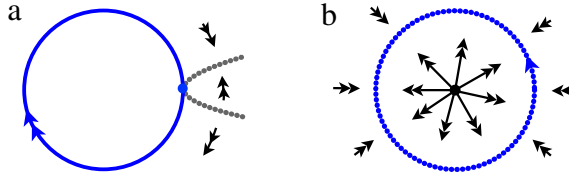


Fig. 2. A cycle without slow arcs on the left, is by definition not a slow–fast cycle. A common slow–fast cycle without fast parts on the right.

Fixing a volume form on the manifold M , we can define the notion of “divergence of a vector field”. Let $\text{div}X_{\epsilon,\lambda}$ be the divergence of $X_{\epsilon,\lambda}$ (for the chosen volume form). We define the *slow divergence integral* along the slow arc defined by the end points p and q as

$$I(p, q, \lambda) = \int_p^q \text{div}X_{0,\lambda} ds,$$

where we integrate along the slow arc of the slow curve from p to q w.r.t. the so-called slow time s , i.e. the time induced by the slow vector field X_λ^{slow} . To clarify, in (1), let x_p and x_q be the x -coordinates of the end points p and q , then $\frac{dx}{ds}$ is given in (2) and

$$I(p, q, \lambda) = - \int_{x_p}^{x_q} \frac{\left(\frac{\partial f}{\partial x}(x, \lambda)\right)^2}{g(x, f(x, \lambda), 0, \lambda)} dx. \tag{3}$$

The importance of the slow divergence integral lies in the fact that it is shown to be a first-order approximation of the divergence integral along orbits of the slow–fast system $X_{\epsilon,\lambda}$. The divergence integral along orbits of the slow–fast system is of interest because of the well-known relation between the exponential of this integral and the derivative of the transition map between two sections transversely cutting the slow curve at p and q .

Remark. The divergence of a vector field, calculated at its singular points, does not depend on the volume form. Also the slow divergence integral is not depending on the volume form, nor on the local chart. It is therefore a well-defined coordinate-free notion.

Let us now define the notions slow–fast cycle, canard slow–fast cycle and common slow–fast cycle. All these concepts are piecewise-smooth curves that form an invariant set of $X_{0,\lambda}$.

Definition 4. Given $\lambda_0 \in \Lambda$. A subset Γ , diffeomorphic to a piecewise smooth circle consisting of a finite number of fast orbits of X_{0,λ_0} and a finite number of slow arcs of X_{0,λ_0} , is called a *slow–fast cycle* of X_{0,λ_0} if it contains at least one slow arc. Moreover, it must be possible to orientate the circle in a way that the orientation is compatible to the natural orientation on the fast orbits and such that on all slow arcs it agrees with the orientation of the slow dynamics. The slow dynamics should be regular except for at most a finite number of isolated points.

A *regular* slow–fast cycle is a slow–fast cycle without singularities on the slow arcs and such that each of its contact points is regular (see Definition 2).

By imposing that the cycle should have at least one slow arc, we avoid complicated situations such as displayed in Fig. 2(a).

Definition 5. A *common slow–fast cycle* is a slow–fast cycle for which the slow arcs are either all attracting or all repelling. A *canard cycle* is a slow–fast cycle that is not common.

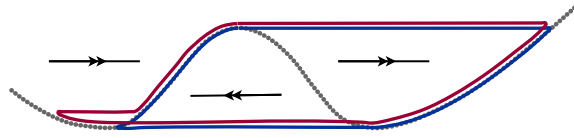


Fig. 3. Common slow–fast cycle and a nearby canard cycle (the nearby canard is shifted a bit for the sake of clarity).

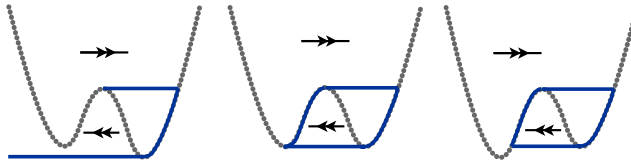


Fig. 4. A common slow–fast cycle, in the middle picture, breaking under influence of a parameter.

It is interesting to observe that for any given common slow–fast cycle Γ of X_{ϵ, λ_0} , there exists a neighborhood of Γ where Γ is the only common slow–fast cycle. (Although it is not difficult to prove, we prefer not to work this out.) However, slow–fast cycles of canard type can exist in an arbitrarily small neighborhood of Γ ; in fact there are common cycles that are the Hausdorff limit of canard cycles, see Fig. 3. When perturbing with a perturbation parameter λ , there can be either no or exactly one common slow–fast cycle Γ of $X_{\epsilon, \lambda}$ near Γ , see Fig. 4 for an example where the cycle could be broken after perturbation.

For a regular common slow–fast cycle Γ , the slow divergence integral along a slow arc is finite; associated to Γ , one can define the slow divergence integral along Γ as the sum of all slow divergence integrals along each slow arc of Γ . Of course, as can be seen from (3), a slow divergence integrals may be undefined when e.g. an end point of the slow arc is a singular contact point. It can be shown however that when the singularity order of the contact point is 1 (see Definition 2; for a general definition of singularity order see Definitions 7 and 8), the integral still remains defined.

Lemma 1. *Given a regular slow–fast cycle Γ . Then the total slow divergence integral $I(\Gamma)$, defined as the sum of all slow divergence integrals along the slow arcs, is a well-defined (finite) number. Furthermore, if Γ is common then $I(\Gamma) < 0$ when it is attracting and $I(\Gamma) > 0$ when it is repelling. The statement is also true if one or more of the contact points are singular with $\frac{\partial g}{\partial x}(0, 0, \lambda_0) \neq 0$ in the normal form (see Definition 2) at the contact point.*

The proof of this lemma is straightforward using (3).

If we have a continuous family of common slow–fast cycles Γ_λ , then it should be clear that the divergence integral $I(\Gamma_\lambda)$ is only piecewise continuous (where it is defined).

The main problem is the following: given any slow–fast cycle for a value of the parameter λ_0 , can one determine the number of limit cycles that can bifurcate from it, keeping (ϵ, λ) close to $(0, \lambda_0)$? At present, we do not deal with this question in full generality, but we will show that at most one limit cycle can bifurcate from the slow–fast cycle under the conditions that the slow–fast cycle is of regular common type, and that all contact points are nilpotent, regular and of finite order. We will also treat regular canard cycles, subject to a non-degeneracy condition, and some common slow–fast cycles with singularities in the slow dynamics or with singular contact points. We in fact treat all classes where one can in general expect the occurrence of at most one limit cycle.

The proof focuses on the treatment of attracting regular common slow–fast cycles; the extension to other types of cycles is discussed afterward. The technique for dealing with limit cycles near an attracting regular common slow–fast cycle is as follows: choose a section Σ transverse to the common slow–fast cycle, and consider a neighborhood V of the cycle. For the set of all points of Σ that have a first-return map toward Σ , while staying inside V , it will be shown that the derivative of the first-return map is strictly bounded by an expression roughly related to $\exp(I(T)/\epsilon)$, where $I(T)$ is the slow divergence integral along the common slow–fast cycle T . Since $I(T) < 0$, there can be at most one periodic orbit in V . The first return map will be seen as a composition of a finite number of so-called elementary transition maps: we choose a finite number of sections $\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_N = \Sigma$, transverse to the flow for $\epsilon \neq 0$ and compose the transition maps $\Sigma_i \rightarrow \Sigma_{i+1}$. Any regular *attracting* common slow–fast cycle can, for a fixed value λ_0 , be decomposed in so-called elementary *attracting* slow–fast segments of the following types:

- “regular fast”: segment of a regular orbit of the fast vector field X_{0,λ_0} .
- “regular slow”: a slow arc.
- “fast–fast”: two (regular) fast segments with a regular contact point of even order in the middle. The orientation in the first fast segment is toward the contact point, and in the second away from the contact point. We have two subtypes that are essentially different in the analysis. We call them “funnel fast–fast” and “canard fast–fast”. Observe that the “canard fast–fast” segment is part of an attracting slow–fast cycle but contains repelling slow–fast segments in any neighborhood. In other words, a common cycle with a “canard fast–fast” segment has canard cycles nearby.
- “odd slow–slow”: two slow arcs, with a regular contact point of odd order in between. The orientation in the first slow arc is toward the contact point, and in the second arc away from the contact point.
- “slow–fast” (or “jump”): a slow arc and a fast orbit, glued together by means of a regular contact point of even order. The orientation in the slow arc is toward the contact point, and in the fast orbit away from the contact point.
- “fast–slow”: a fast orbit, oriented toward a regular contact point, glued together with a slow arc with an orientation away from the contact point. We distinguish several subcases:
 - “hyperbolic fast–slow”: The fast orbit ends at a normally hyperbolic point
 - “odd fast–slow, cuspidal type” and “odd fast–slow, regular type”: the fast fiber ends at a contact point of odd order. Depending on the shape of the slow curve, the slow–fast segment looks like a cusp or like a regular curve
 - “canard fast–slow”: the fast fiber ends at a contact point of even order. The slow–fast segment makes a cusp-like connection with an attracting branch of the slow curve. In analogy with the canard fast–fast type, this segment is of canard type because in any Hausdorff neighborhood of this segment, one can find slow–fast segments of canard type.

The two slow–fast segments “canard fast–fast” and “canard fast–slow” are called *canard-type slow–fast segments*, despite the fact that they may be part of a common cycle. Slow–fast segments not of canard type will be called *strongly common slow–fast segments*.

Slow–fast cycles of canard type (i.e. cycles that are not common) may contain additional types of segments, as will be pointed out in later sections. Also slow–fast cycles that are not regular, i.e. contain singularities in the slow dynamics on the slow arcs, or slow–fast cycles that contain singular contact points may be composed out of additional types of segments. Instead of describing these types in full detail, we prefer to postpone this to later sections and to restrict to the essentials.

Theorem 1. *Let Γ be a slow–fast cycle of X_{ϵ, λ_0} for a given $\lambda_0 \in \Lambda$. We suppose that*

1. *All the contact points are nilpotent and of finite order. They are regular contact points, or of singularity index $+1$ (see Definition 2).*
2. *If Γ does not contain singularities of the slow dynamics (outside contact points), then*

$$\int_{\Gamma} \operatorname{div} X_{0, \lambda_0} ds \neq 0.$$

3. *If Γ contains singularities of the slow dynamics (outside contact points), they are all located on hyperbolic arcs of the same type (all attracting or all repelling).*

Let q_1, \dots, q_ℓ denote the singular contact points and let W_1, \dots, W_ℓ be mutually disjoint open sets with $q_j \in W_j$ for $j = 1, \dots, \ell$.

Under assumptions (1)–(3), there exists a neighborhood \mathcal{T} of Γ , an $\epsilon_0 > 0$ and a neighborhood Λ_0 of λ_0 such that for any $(\epsilon, \lambda) \in]0, \epsilon_0] \times \Lambda_0$, there is at most one closed orbit of $X_{\epsilon, \lambda}$ inside \mathcal{T} that does not lie entirely in one of the W_j ; if it appears, this closed orbit is a hyperbolic limit cycle: attracting if Γ is attracting and repelling if Γ is repelling.

Remark. 1. After introducing a Riemannian metric on the surface it is possible to reformulate Theorem 1 requiring that \mathcal{T} be a δ -neighborhood of Γ , for some $\delta > 0$. From the proof of Theorem 1 it will also be clear that a closed orbit which is not contained in one of the W_j is necessarily δ -Hausdorff close to Γ .

2. There may be other slow–fast cycles that are partly inside \mathcal{T} , from which might also perturb limit cycles. These limit cycles however do not completely lie inside \mathcal{T} .
3. In case one starts with an analytic family of vector fields, all nilpotent contact points are automatically of finite order.
4. If, in Theorem 1, one permits the slow dynamics to have extra singularities other than the ones admitted in the current statement, then one quickly runs into problems. If one e.g. permits that singularities occur both on attracting and repelling slow arcs, then the cyclicity of the slow–fast cycle is in general at least two. For examples, see [1].
5. Even for a regular common slow–fast cycle complications occur when the contact points are no longer as required in the theorem. When the singularity at the contact point is not simple of index $+1$ (i.e. $g'(0) \geq 0$), then we believe that the cyclicity of the common slow–fast cycle will be strictly larger than one, at least if one considers a sufficiently general perturbation.
6. We will provide in Section 8 two simple examples of attracting regular common slow–fast cycles that can be approached by a generic saddle–node bifurcation of limit cycles, hence also by repelling limit cycles. This result might seem quite unexpected but is due to the occurrence of hyperbolic saddles in the unfolding of the contact point.
7. In Section 9 we will formulate and prove Theorem 5, which is a substantial extension of Theorem 1. It deals no longer with cycles (that are homeomorphic to circles) but to a very large class of closed slow–fast paths. These will be arbitrary successions of fast and slow curves, with similar properties as in Theorem 1, except for the fact that the subset in M that is the image of the path is not necessarily a topological circle. A precise definition and a precise formulation of a result, generalizing the statement of Theorem 1, requires a lot more definitions, which will be formulated throughout the text.

Theorem 1 gives upper bound 1 for the number of limit cycles near any regular common slow–fast cycle Γ . We will complement this result with an existence result of limit cycles, for a

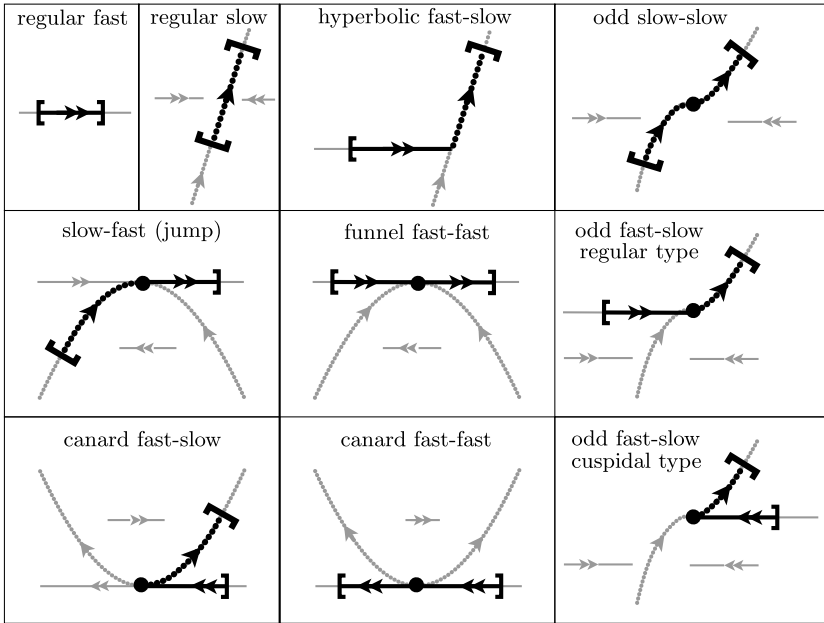


Fig. 5. Elementary slow–fast segments. In the pictures, the double arrows indicate the fast flow, the single arrows indicate the slow flow. The large dots show the contact points. In all pictures, the slow flow is upwards.

more restrictive class of slow–fast cycles:

Definition 6. A strongly common slow–fast cycle (or in short a strongly common cycle) is a regular common slow–fast cycle that only contains strongly common slow–fast segments (and hence is without slow–fast segments of canard type), see Fig. 5.

We recall that in this paper, the contact points appearing in slow–fast cycles are all assumed to be nilpotent and of finite order. Since a strongly common slow–fast cycle is a regular slow–fast cycle, we additionally have that the slow dynamics on Γ has no singularities outside the contact points and that each contact point is a regular contact point.

Theorem 2. Let Γ be a strongly common slow–fast cycle of X_{ϵ, λ_0} for a given $\lambda_0 \in \Lambda$.

Then there exists an $\epsilon_0 \in]0, \epsilon_1]$, a neighborhood $\Lambda_0 \subset \Lambda$ of λ_0 and a neighborhood \mathcal{T} of Γ such that for $\epsilon \in]0, \epsilon_0]$ and $\lambda \in \Lambda_0$, the vector field $X_{\epsilon, \lambda}$ has a limit cycle in \mathcal{T} . This limit cycle is unique and hyperbolic, and tends in Hausdorff sense toward Γ as $(\epsilon, \lambda) \rightarrow (0, \lambda_0)$.

The statement is not true for arbitrary regular common slow–fast cycles: the presence of a canard fast–fast or canard fast–slow segment in the slow–fast cycle may cause the first return map to be undefined (see eg. Fig. 4).

Remark. The canard fast–fast and canard fast–slow segments are also different in the following sense: suppose one considers a common slow–fast cycle Γ for a specific value of the parameter λ_0 . If Γ does not contain segments of the two given canard types, then one has the property that given any neighborhood V of Γ , the family $X_{\epsilon, \lambda}$ has another common slow–fast cycle in V , provided taking λ close enough to λ_0 . In other words, the presence of a common slow–fast cycle

near Γ , is guaranteed. This guarantee is not given for common slow–fast cycles containing the canard fast–fast or canard fast–slow type segments.

3. Layout of the paper

In Section 4 we establish a strong normal form for equivalence near contact points of nilpotent type. This normal form not only forms the basis for the treatment of most types of slow–fast segments, but also makes it clear that notions such as the order of the contact point, regular contact point, contact point with a singularity of index ± 1 , are intrinsic notions.

In Section 5, we consider any point of a slow–fast cycle, distinguishing between contact points, regular or singular points on the slow arcs, and introduce *adapted neighborhoods* near that point. Different kinds of adapted neighborhoods will allow us to cover the entire slow–fast cycle by a finite number of these neighborhoods. The construction of these neighborhoods is basic and is an essential part in the proof of both main results.

In Section 6 we prove that [Theorem 2](#) is a consequence of [Theorem 1](#). Using the results from Section 5, it will follow that for $\epsilon > 0$ small enough, the vector fields behave like a succession of flow boxes near Γ . This way, we are able to show that the first return map in the neighborhood of any strongly common slow–fast cycle is well-defined and it is a mapping from a closed interval into itself. At the end of Section 6, we also prove the existence of limits of flow box neighborhoods for the two slow–fast segments of canard type, but with different properties than for the other slow–fast segments.

In Section 7 we prove [Theorem 1](#). We stress that the proof of [Theorem 1](#) does not rely on [Theorem 2](#). In the proof of [Theorem 1](#), one deals with all types of slow–fast segments that may occur in a slow–fast cycle of the type that is described in the formulation of [Theorem 1](#). Of course, most attention goes to the treatment of segments with a contact point. A distinction is made between regular contact points and contact points with a singularity of index $+1$. The regular contact points can be studied without using the technique of blow-up, and heavily rely on the use of the normal forms obtained in Section 4. The technique used to deal with regular contact points can however not be extended to singular contact points. The study of such singular contact points is more involved, and a recursive application of the blow-up technique is necessary.

In Section 8 we give two examples making clear that by allowing more singularities than the ones admitted in the formulation of [Theorem 1](#), one could have more than one limit cycle in the neighborhood of a common slow–fast cycle Γ , which are all Hausdorff-close to Γ .

In Section 9 we state and prove [Theorem 5](#), which is a generalization of [Theorem 1](#) as referred to in the 7th remark after [Theorem 1](#).

4. Normal form near a contact point

In Section 2 we have stated that any nilpotent contact point may be brought into the form specified in (1). The proofs of the main results all rely on this normal form, whose existence we prove in this section. Furthermore, it will be clear from the results in this section that notions such as the order of the contact point, and regular contact point, are well-defined in an intrinsic way. Throughout this section, we assume that $X_{\epsilon,\lambda}$ is a family of slow–fast vector fields, as defined in the beginning of Section 2.

Proposition 1. *Consider a smooth slow–fast system $X_{\epsilon,\lambda}$ on a smooth surface M . Let p be a nilpotent contact point for a parameter value $\lambda = \lambda_0$. There exist smooth local coordinates*

(x, y) such that $p = (0, 0)$, and in which, up to multiplication by a smooth strictly positive function, the system $X_{\epsilon, \lambda}$ is written in the following normal form:

$$\begin{cases} \dot{x} = y - f(x, \lambda) \\ \dot{y} = \epsilon(g(x, \epsilon, \lambda) + (y - f(x, \lambda))h(x, y, \epsilon, \lambda)), \end{cases} \tag{4}$$

for smooth functions f, g, h and $f(0, \lambda_0) = \frac{\partial f}{\partial x}(0, \lambda_0) = 0$. This means that $X_{\epsilon, \lambda}$ is C^∞ -equivalent to the normal form (4).

Proof. We will obtain the normal form (4) by a succession of several steps. We will assume that each successive coordinate system is chosen such that $p = (0, 0)$.

(1) For $\{\epsilon = 0\}$ the system can be written $X_{0, \lambda} = F_\lambda(x, y)Z_\lambda$ in a neighborhood of (p, λ_0) , with $F_\lambda(0, 0) = 0, dF_\lambda(0, 0) \neq 0$ and Z_λ a family of vector fields such that $Z_{\lambda_0}(0, 0) \neq 0$. It follows from the flow box theorem that, in a neighborhood of (p, λ_0) , one can choose the coordinates such that $Z_\lambda \equiv \frac{\partial}{\partial x}$.

(2) The point p is a nilpotent contact point, so one can suppose that the 1-jet of the vector field at $(\epsilon, \lambda) = (0, \lambda_0)$ is given by $j^1 X_{0, \lambda_0}(0, 0) = y \frac{\partial}{\partial x}$. This means that one has: $F_{\lambda_0}(0, 0) = 0$ and $\frac{\partial F_{\lambda_0}}{\partial y}(0, 0) \neq 0$. Then the slow curve is locally a graph $\{y = f_\lambda(x)\}$ for a smooth family of functions f_λ and one can write $F_\lambda(x) = U(x, y, \lambda)(y - f_\lambda(x))$ for a smooth function U such $U(0, 0, 0) > 0$. We will choose a neighborhood in which $U(x, y, \lambda) > 0$ for every (x, y, λ) . After division by the function U , the system is now locally written:

$$\begin{cases} \dot{x} = y - f(x, \lambda) + \epsilon G_1(x, y, \epsilon, \lambda) \\ \dot{y} = \epsilon G_2(x, y, \epsilon, \lambda), \end{cases} \tag{5}$$

for smooth functions G_1, G_2 .

(3) The mapping $(x, y) \rightarrow (x, y + \epsilon G_1(x, y, \epsilon, \lambda))$ is a smooth family of diffeomorphisms for $(x, y, \epsilon, \lambda) \sim (0, 0, 0, \lambda_0)$. It defines a conjugacy which brings (5) into:

$$\begin{cases} \dot{x} = y - f(x, \lambda) \\ \dot{y} = \epsilon \tilde{G}_2(x, y, \epsilon, \lambda), \end{cases}$$

for a new smooth function \tilde{G}_2 .

(4) As x and $y - f(x, \lambda)$ are independent functions at $(x, y) = (0, 0)$, one can write $\tilde{G}_2(x, y, \epsilon, \lambda) = g(x, \epsilon, \lambda) + (y - f(x, \lambda))h(x, y, \epsilon, \lambda)$ for smooth functions g, h . \square

We will write $f(x, \lambda) = f_\lambda(x)$ and $g(x, \epsilon, \lambda) = g_{\epsilon, \lambda}(x)$. It is easy to verify that the orders of functions f_{λ_0} and g_{0, λ_0} at $x = 0$ are independent of the choice of the normal form (4) up to smooth equivalence:

Proposition 2. *Let $X_{\epsilon, \lambda}$ be a germ of slow-fast systems $X_{\lambda, \epsilon}$ at a nilpotent contact point p and for the value λ_0 of the parameter. Consider two normal forms (4): $N_{\epsilon, \lambda}, \tilde{N}_{\epsilon, \lambda}$ of $X_{\epsilon, \lambda}$, associated to the functions $f_\lambda, g_{\epsilon, \lambda}$ and $\tilde{f}_\lambda, \tilde{g}_{\epsilon, \lambda}$ respectively. Then:*

$$\text{Ord}|_0(f_{\lambda_0}) = \text{Ord}|_0(\tilde{f}_{\lambda_0}) \quad \text{and} \quad \text{Ord}|_0(g_{0, \lambda_0}) = \text{Ord}|_0(\tilde{g}_{0, \lambda_0}).$$

Proof. $N_{\epsilon, \lambda}$ is sent to $\tilde{N}_{\epsilon, \lambda}$ by a diffeomorphism family $H_{\epsilon, \lambda}(x, y)$ such that $H_{0, \lambda_0}(0, 0) = (0, 0)$ and multiplication by a positive function $U(x, y, \epsilon, \lambda)$. For $\epsilon = 0$, the diffeomorphism $H_{0, \lambda}$ preserves globally the trajectories of the vector field $\frac{\partial}{\partial x}$. This means that:

$$H_{0, \lambda} = (h(x, y, \lambda), \varphi(y, \lambda)),$$

for smooth functions h, φ such that $h(0, 0, \lambda_0) = 0, \varphi(0, \lambda_0) = 0$ and $\frac{\partial \varphi}{\partial y}(0, \lambda_0), \frac{\partial h}{\partial x}(0, 0, \lambda_0) \neq 0$.

(1) $\text{Ord}|_0(f_{\lambda_0}) = \text{Ord}|_0(\tilde{f}_{\lambda_0})$. The diffeomorphism H_{0,λ_0} sends the line of zeros of $N_{0,\lambda_0} : \{y = f(x, \lambda_0)\}$ to the line of zeros of $\tilde{N}_{0,\lambda_0} : \{y = \tilde{f}(x, \lambda_0)\}$ and preserves the axis Ox . The result follows from the fact that a diffeomorphism preserves contact orders between regular curves.

(2) $\text{Ord}|_0(g_{0,\lambda_0}) = \text{Ord}|_0(\tilde{g}_{0,\lambda_0})$. The diffeomorphism $H_{\epsilon,\lambda}(x, y)$ may be written $H_{\epsilon,\lambda} : X = h(x, y, \lambda) + O(\epsilon), Y = \varphi(y, \lambda) + O(\epsilon)$. The second line of the system $\tilde{N}_{\epsilon,\lambda}$ is then proportional to:

$$\dot{Y} = \left(\frac{\partial \varphi}{\partial y} + O(\epsilon) \right) \dot{y} + O(\epsilon) \dot{x} = \left(\frac{\partial \varphi}{\partial y} + O(\epsilon) \right) \epsilon (g(x, \epsilon, \lambda) + O(y - f(x, \lambda))).$$

On the other side we have that $\dot{Y} = \epsilon(\tilde{g}(X, \epsilon, \lambda) + O(Y - \tilde{f}(X, \lambda)))$ and that $Y - \tilde{f}(X, \lambda) = O(y - f(x, \lambda))$. From this, it follows that

$$\tilde{g}(X, 0, \lambda_0) = g(x, 0, \lambda_0) + O(y - f(x, \lambda_0)). \tag{6}$$

On the other side we have that:

$$\begin{aligned} \tilde{g}(X, 0, \lambda_0) &= \tilde{g}(h(x, y, 0, \lambda_0), 0, \lambda_0) \\ &= \tilde{g}(h(x, f(x, \lambda_0), 0, \lambda_0), 0, \lambda) + O(y - f(x, \lambda_0)). \end{aligned}$$

Combining this with (6) we obtain, by unicity of the division by $y - f(x, \lambda_0)$, that:

$$g(x, 0, \lambda_0) \equiv \tilde{g}(h(x, f(x, \lambda_0), 0, \lambda_0), 0, \lambda_0).$$

But, as $f(x, \lambda_0) = O(x^2)$, the function $\tilde{h}(x) = h(x, f(x, \lambda_0), 0, \lambda_0)$ is a diffeomorphism near $x = 0$ in the sense that $\tilde{h}(0) = 0$ and $\frac{d\tilde{h}}{dx}(0) \neq 0$. This implies that $\text{Ord}|_0(g_{0,\lambda_0}) = \text{Ord}|_0(\tilde{g}_{0,\lambda_0})$. \square

Consider a system written in the normal form (4). Outside the set of contact points $C_\lambda = \{(x, y) \mid \frac{\partial f_\lambda}{\partial x}(x) = 0, y = f_\lambda(x)\}$ the slow dynamic is defined on the slow curve S_λ by the equation

$$\frac{\partial f_\lambda}{\partial x}(x) \dot{x} = g(x, \lambda).$$

Then the zeros of g_λ on $S_\lambda \setminus C_\lambda$ are the zeros of the slow dynamics with as order the order of the zero of g_λ . This order has then an intrinsic meaning, independent of the choice of the normal form. Proposition 2 extends this property at contact points, allowing the following definition of invariants for a contact point:

Definition 7. Consider any normal form (4) near a contact point p of a slow-fast system $X_{\epsilon,\lambda}$ for the value λ_0 . The order $\text{Ord}|_0(f_{\lambda_0})$, which is ≥ 2 , is called *order of the contact point*. The order $\text{Ord}|_0(g_{0,\lambda_0})$, which is ≥ 0 , is called the *singularity order* at the contact point. We will say that the contact point p is *regular* if the singularity order at this point is zero (i.e. if $g_{0,\lambda_0}(0) \neq 0$) and *singular* if not.

Remark. 1. The singularity order represents the algebraic multiplicity of the (isolated) singularities that we will encounter near $(0, 0)$ for $\epsilon > 0$.

2. If $\frac{\partial f_\lambda}{\partial x}(0) \neq 0$ in (4), we could say that the point $p = (0, 0)$ has an order equal to 1. This point is then a regular point of the slow curve S_{λ_0} , and not a contact point.

Definition 8. Consider any normal form (4) near a singular contact point p of singularity order 1. The point is said to be a singular contact point of index ± 1 when

$$\text{sign} \frac{\partial g_{0,\lambda_0}}{\partial x}(0) = \mp 1.$$

For such a singular contact point, the family of vector fields $X_{\epsilon,\lambda}$ has, for (ϵ, λ) close to $(0, \lambda_0)$, a singular point $(x, y) = (x_0(\epsilon, \lambda), y_0(\epsilon, \lambda))$ tending to $(0, 0)$ as $(\epsilon, \lambda) \rightarrow (0, \lambda_0)$. This singularity is non-degenerate for $\epsilon > 0$ sufficiently small. It is of saddle type when it is a singularity of index -1 , and of center/focus type when it is a singularity of index $+1$. From these observations, it is clear that the singularity index, as defined in Definition 2, also has an intrinsic meaning.

5. Flow-box neighborhoods and other adapted neighborhoods

In the proofs of Theorems 1 and 2, we will cover the slow–fast cycle Γ by so-called adapted neighborhoods. In the case of a regular slow–fast cycle, one can cover the cycle by a set of neighborhoods that are flow boxes for $\epsilon \neq 0$. In case singularities appear on the slow arcs, or in case the cycle contains singular contact points, this is not possible, but in that case we cover the singular points by a type of neighborhoods that have some of the properties of a flow box.

Let us first focus on regular slow–fast cycles. The main idea is the observation that for $\epsilon > 0$ small enough, and sufficiently close to the regular slow–fast cycle Γ , the vector field has no singular points. This shows that near the slow–fast cycle, the flow of the vector field is the flow in a succession of flow boxes. As $\epsilon \rightarrow 0$, one loses this property. We will construct flow box neighborhoods, for $\epsilon > 0$, that stay of a fixed size as $\epsilon \rightarrow 0$. To that end, we define a flow box neighborhood:

Definition 9. Given a smooth vector field X on a smooth manifold M . A flow box neighborhood V of X of a point (or a set) p is a neighborhood of p that is the image of a smooth diffeomorphism $\phi: [0, 1] \times [0, 1] \rightarrow M$ having the following properties: both $\phi(\{0\} \times [0, 1])$ and $\phi(\{1\} \times [0, 1])$ are parts of orbits of X , the vector field X is pointing into V along $\phi([0, 1] \times \{0\})$ (called the “inset”), and out of V along $\phi([0, 1] \times \{1\})$ (called the “outset”). Furthermore, X has no singular points in $\phi([0, 1] \times]0, 1[)$.

A slow–fast family of flow box neighborhoods of p for a slow–fast family $X_{\epsilon,\lambda}$ is a family of sets $V_{\epsilon,\lambda} = \phi_{\epsilon,\lambda}([0, 1] \times [0, 1])$ with $\phi_{\epsilon,\lambda}: [0, 1] \times [0, 1] \rightarrow M$, defined for $(\epsilon, \lambda) \in [0, \epsilon_0] \times W$ with $\epsilon_0 > 0$ and W a neighborhood of λ_0 , ϕ a smooth mapping with the property that each $\phi_{\epsilon,\lambda}$ is a diffeomorphism, $V_{\epsilon,\lambda}$ is a flow box neighborhood for $X_{\epsilon,\lambda}$ when $(\epsilon, \lambda) \in]0, \epsilon_0[\times W$, and $\bigcap_{(\epsilon,\lambda)} V_{\epsilon,\lambda}$ is a neighborhood of p .

Let Γ be a regular slow–fast cycle of X_{0,λ_0} , and let also $V_{\epsilon,\lambda}$ be a family of flow box neighborhoods of $p \in \Gamma$ as in Definition 9. As noticed above, $V_{\epsilon,\lambda}$ contains no singular point of $X_{\epsilon,\lambda}$ for $\epsilon \neq 0$. Then it follows from the Poincaré–Bendixson theory that, for $\epsilon \neq 0$, the trajectory starting at any point of the inset must reach the outset. It is the reason why we call $V_{\epsilon,\lambda}$ a flow box neighborhood.

It is clear that the best way to describe flow box neighborhoods for slow–fast segments, is using normal forms (for equivalence). Should one need more than one normal form neighborhood to cover a single slow–fast segment, then it is always possible to break down the slow–fast segment in several smaller pieces, each of them again being a slow–fast segment. For the regular fast segment, we assume to work in a flow box (of course), for the regular slow and hyperbolic

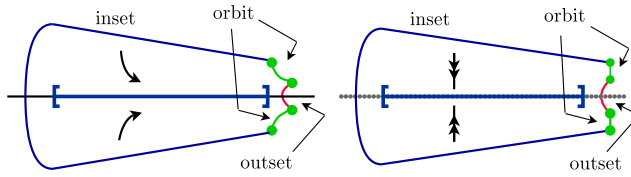


Fig. 6. Flow box neighborhood near a regular slow segment. To the left for $\epsilon > 0$; the limit as $\epsilon \rightarrow 0$ to the right.

fast–slow segments, we use Takens normal form coordinates to reduce the family of vector fields to $\{\dot{x} = \epsilon, \dot{y} = -y\}$. For all other slow–fast segments, we use the normal forms near contact points that we discussed in previous sections, i.e. we use the normal form (1). It should be clear that one can take the slow–fast segments containing a contact point as close as required (in Hausdorff sense) to the contact point.

Proposition 3. *Given a strongly common slow–fast segment S (for a fixed parameter value λ_0), assume that the entire segment is seen in a single appropriate normal form, as explained above.*

Given any neighborhood \mathcal{V} of S , there exists, for (ϵ, λ) sufficiently close to $(0, \lambda_0)$, a slow–fast family of flow box neighborhoods $V_{\epsilon, \lambda}$ of S , lying completely inside \mathcal{V} .

This proposition is already quite useful but we need a slight refinement. We need a way of making the outset as close as required but without affecting the inset (possibly by restricting ϵ to a sufficiently small neighborhood of 0). Due to the exponential attraction toward the slow curve, this is surely possible in a slow–fast context. There are however two exceptions: the regular fast segment and the funnel fast–fast segment. In these two situations, it is possible that reducing the outset in size can only be achieved by also reducing the distance of the inset to S . For all other segments, we have the following refinement of Proposition 3.

Proposition 4. *Given a “strongly common” slow–fast segment S (for a fixed parameter value λ_0), that is not a regular fast segment or a funnel fast–fast segment (see Fig. 5).*

There exists a fixed neighborhood A of the starting point of the slow–fast segment, so that for any neighborhood B of the end point of the slow–fast segment, the flow box neighborhoods obtained in Proposition 3 can be chosen so that additionally $A \subset V_{\epsilon, \lambda}$ and that the outset of $V_{\epsilon, \lambda}$ lies completely inside B , by choosing $\epsilon > 0$ small enough.

Proof of Propositions 3 and 4

For the regular fast segment, we use flow box coordinates as normal form, so this is straightforward. For the regular slow segment, the Takens normal form is given by $\{\dot{x} = \epsilon, \dot{y} = -y\}$. A regular slow segment is then the compact interval on the y -axis from $x = 0$ to $x = 1$. In this normal form, the shape of the flow box neighborhood is given as in Fig. 6. The outset is simply a small vertical segment a little bit to the right of $\{x = 1\}$, and bent a little near S (the slight bending will be explained in a minute). Connected to this outset are short parts of orbits, that tend to vertical fibers as $\epsilon \rightarrow 0$. These orbits together with the outset can be chosen as close as required to $(x, y) = (1, 0)$. For the inset, there is a lot of freedom in the choice. Along the entire length of the segment, we choose a line of any slope, and to the left of $\{x = 0\}$, we connect both ends in a convex way. Since the slopes of the inset above $x \in [0, 1]$ can be anything, it is clear that we can ensure that the flow box neighborhood fits in any given neighborhood of the segment. Furthermore, it is clear that we can shrink the outset without affecting the relevant part of the inset, as announced in Proposition 4. Of course, it would be possible to give analytic

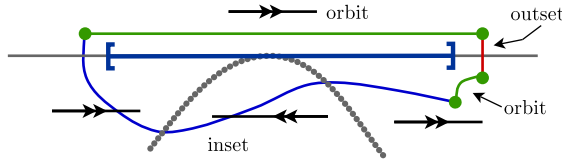


Fig. 7. Flow box neighborhood near a funnel fast–fast segment.

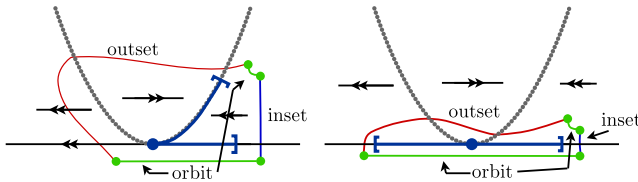


Fig. 8. Flow box neighborhood near a canard fast–slow segment (left) and a canard fast–fast segment (right).

descriptions of the flow box neighborhoods, but this would give no extra insight. Let us now explain why we introduce a slight bending in the outset, and not just a vertical segment. This is to ensure that even in the limit, the outset makes a true angle with the two orbits connected to it; indeed both orbits themselves tend to vertical segments, so if the outset is bent, we are sure that the inside of the region described by the outset, the inset and the two orbits, are diffeomorphic to a square.

An important observation is that one can make an inset that crosses the line of singular points: away from the slow curve, it is the fast dynamics $\dot{y} = -y$ that ensures the transversality; near the slow curve, it is the slow dynamics $\dot{x} = \epsilon$ that helps.

Also the funnel fast–fast is quite instructive to deal with: the flow box neighborhood is shown in Fig. 7.

As before the outset can be chosen as small as required. One of the two orbits connecting to it is similar to the orbit used in the regular slow segment, but the second one is just a fast orbit over a larger length. It is the inset that is somewhat more difficult: observe that the fast fibers are horizontal, so away from the slow curve, the inset must cross the fibers transversely. As we have seen in the regular fast segment, it is possible to cross the slow curve by making the inset tangent to the fibers at the crossing.

The other cases are treated similarly, using the same guidelines. This finishes the proof of Propositions 3 and 4. \square

For the sake of completeness, and in order to show that the two canard-type segments are essentially different from the strongly common ones, we show in Fig. 8 that one can find neighborhoods for the two canard-type segments, but from the picture it should be clear that the outset cannot be taken small: the outset will always cut two branches of the curve of singular points. In other words, we have the following result.

Proposition 5. *Given a “canard-type” slow–fast segment S (for a fixed parameter value λ_0). Assume that the entire segment lies in a neighborhood where one can take a single normal form (1). Then there is a slow–fast family of flow box neighborhoods of S .*

When we consider slow–fast cycles with singularities on the slow arcs, or with singular nilpotent contact points, one cannot expect to have slow–fast families of flow box neighborhoods.

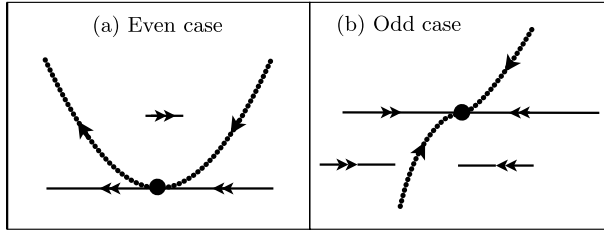


Fig. 9. Singular contact points of index +1.

Instead, we will work with another type of *well-adapted neighborhoods*. We make a distinction between adapted neighborhoods around singular points on slow arcs and adapted neighborhoods around singular contact points. We first treat singular contact points, and afterward singularities on the slow arcs.

Adapted neighborhoods near singular contact points

Recall the normal form for nilpotent contact points

$$\begin{cases} \dot{x} = y - f(x, \lambda) \\ \dot{y} = \epsilon(g(x, \epsilon, \lambda) + (y - f(x, \lambda))h(x, y, \epsilon, \lambda)), \end{cases}$$

with $f(0, \lambda_0) = \frac{\partial f}{\partial x}(0, \lambda_0) = 0$. The contact point $(0, 0)$ is singular when $g(0, 0, \lambda_0) = 0$. It is a simple singularity of index ± 1 when $\frac{\partial g}{\partial x}(0, 0, \lambda_0) \neq 0$. In that case, the implicit function theorem gives a singularity $(x_0(\epsilon, \lambda), y_0(\epsilon, \lambda)) = (0, 0) + O(\epsilon, \lambda - \lambda_0)$ for (ϵ, λ) sufficiently close to $(0, \lambda_0)$. The linear part of the vector field at this point has determinant $-\epsilon(\frac{\partial g}{\partial x}(0, 0, \lambda_0) + o(1))$, showing that contact points with index +1 have the property

$$\frac{\partial g}{\partial x}(0, 0, \lambda_0) < 0.$$

Under these conditions, we can even improve the normal form: one can suppose that $\frac{\partial f}{\partial x} > 0$ for $x > 0$ sufficiently small, by changing $(x, y) \mapsto (-x, -y)$ if necessary. We can then distinguish two cases: the case of a contact point of even order (called the even case) and the case of a contact point of odd order (called the odd case), see Fig. 9. It is clear that the odd case cannot appear inside a slow–fast cycle, due to the compatibility condition on the orientations of the slow arcs. We have shown

Lemma 2. *A nilpotent contact point with a simple singularity of index +1, appearing inside a slow–fast cycle is always a contact point of even type, as in Fig. 9(a).*

We can distinguish three types of slow–fast segments passing through an even contact point, see Fig. 10.

Recalling that the contact point will persist as an isolated singularity for $\epsilon \neq 0$, it is clear that one cannot expect to have flow boxes as neighborhoods. We introduce another kind of adapted neighborhood: instead of using diffeomorphic images of a unit square, as is the case in flow box neighborhoods, we will use diffeomorphic images of a *standard whirlpool neighborhood*:

Definition 10. A *standard whirlpool neighborhood SW* is a plane area bounded by the top half of the circle C_1 of radius 1 around the origin, the bottom half of the circle C_2 of radius 2 around the origin and the two horizontal segments $I_- = [-2, -1] \times \{0\}$ and $I_+ = [1, 2] \times \{0\}$ connecting both arcs, see Fig. 11.

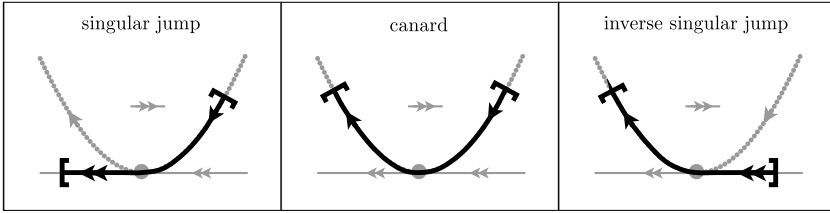


Fig. 10. Slow-fast segments through a contact point of even type (common or not common).

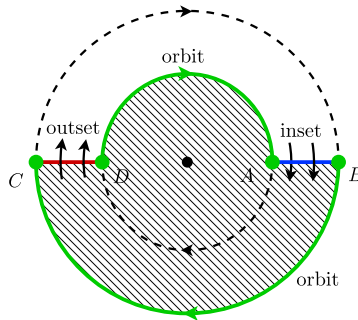


Fig. 11. Standard whirlpool neighborhood and the flow of the harmonic oscillator.

The standard whirlpool neighborhood \mathbf{SW} is for the harmonic oscillator ($\dot{x} = y, \dot{y} = -x$) much like the flow box neighborhood is for a regular flow. Indeed, the boundary of \mathbf{SW} is composed of an inset, an outset, and two orbits. Nevertheless, such a set is essentially different from a square (in the sense that there is no diffeomorphism sending the unit square to \mathbf{SW}). The standard whirlpool neighborhood forms the basis for the definition of an adapted neighborhood that is essentially used to cover contact points of singularity index $+1$.

Definition 11. A slow-fast family of whirlpool neighborhoods of p for a slow-fast family $X_{\epsilon,\lambda}$ is a family of sets $V_{\epsilon,\lambda} = \phi_{\epsilon,\lambda}(\mathbf{SW})$ with $\phi_{\epsilon,\lambda}: \mathbf{SW} \rightarrow M$, defined for $(\epsilon, \lambda) \in [0, \epsilon_0] \times W$ with $\epsilon_0 > 0$ and W a neighborhood of λ_0 , ϕ a smooth mapping with the property that each $\phi_{\epsilon,\lambda}$ is a diffeomorphism, $\bigcap_{(\epsilon,\lambda)} V_{\epsilon,\lambda}$ is a neighborhood of p and such that for each (ϵ, λ) , with $\epsilon \neq 0$, the image of \mathbf{C}_1 and \mathbf{C}_2 are orbits, the image of \mathbf{I}_+ is an inset and the image of \mathbf{I}_- is an outset.

Proposition 6. Let p be a singular contact point of singularity index $+1$ for the slow-fast family of vector fields X_{ϵ,λ_0} . Inside any given neighborhood of p , there exist a slow-fast family of whirlpool neighborhoods. Furthermore, for each fixed (ϵ, λ) sufficiently close to $(0, \lambda_0)$, the points of the inset that reach the outset in finite time by following $X_{\epsilon,\lambda}$ -orbits form a connected part of the inset, and the saturation of it by the $X_{\epsilon,\lambda}$ -orbits (up to the outset) form a flow box inside the whirlpool neighborhood.

Such a flow box inside a whirlpool neighborhood could be called a “total passage flow box”. In Fig. 12 we represent the three possibilities depending on the flow inside the whirlpool neighborhood.

A whirlpool neighborhood is clearly not a flow box, for two reasons. First, the interior contains an isolated singularity of the vector field. Second, in a flow box, the two (fast) orbits on the square

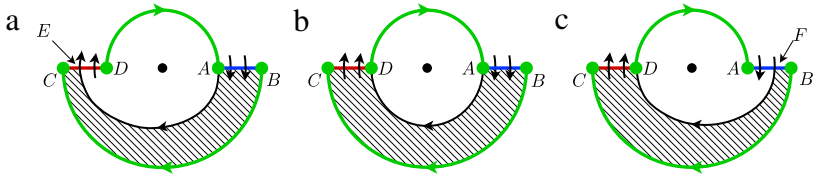


Fig. 12. Three possibilities for the “total passage flow box” inside a whirlpool neighborhood.

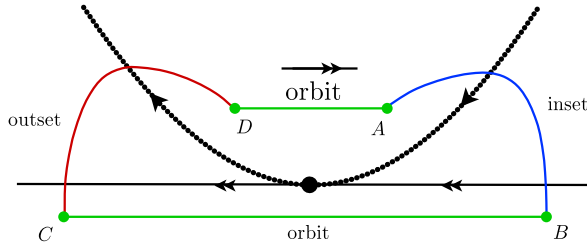


Fig. 13. Whirlpool neighborhood near a contact point of even type.

both point from the inset to the outset. Here, one of the two orbits points from the outset to the inset.

We remark that a vector field with a whirlpool neighborhood around p can in general be quite complicated: most of the conditions in the definition only have implications on the boundary. Nevertheless, the necessary presence of singularities inside the whirlpool can be deduced using index theory. We see that a closed orbit of $X_{\epsilon,\lambda}$ can be confined to such a whirlpool neighborhood, but if it is not, then it cuts both the inset and the outset. The last sentence in Proposition 6 is an extra property, and it allows us to treat the neighborhood essentially like a flow box, in studying limit cycles that are not confined to the whirlpool neighborhood but cut both the inset and the outset of it.

Proof of Proposition 6. We introduce a whirlpool neighborhood as drawn in Fig. 13. It is clear that the drawn neighborhood is diffeomorphic to a SW, has two orbits, an inset and an outset (for $\epsilon > 0$). Observe that the way the inset/outset intersects the curve of singular points is well-chosen, using the same guidelines as for the construction of the flow boxes above.

Let us now show that the part of the inset that reaches the outset, for a fixed (ϵ, λ) , is a connected set. To that end, consider the 4 vertices of SW and its image in the neighborhood, denoted A, B, C, D , ordered clockwise starting from the top-right. We consider the orbit through A in positive time. Since it enters the neighborhood, it must leave somewhere on the outset in a point E , either equal to C or between C and D , or it must accumulate to a singular point or limit cycle O in the interior. In the case E lies between C and D (or in case $E = C$), the region delimited by the orbit AE , the inset AB , the orbit BC and the outset CE is a flow box. In the second case, we consider the orbit through D in negative time. Since it cannot accumulate to O as well, it must reach the inset in finite negative time in a point F between A and B . We can again delimit a flow box region delimited by the orbit FD , the inset FB , the orbit BC and the outset CD . (Note that in the first case, the orbit in negative time from D will accumulate to the singular point or limit cycle O in the interior, except when $E = D$.)

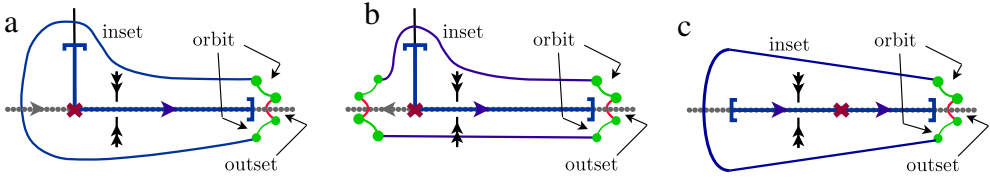


Fig. 14. Singular segments and associated “adapted neighborhoods”. The \times symbol indicates a singularity of the slow dynamics.

Of course, when the parameters (ϵ, λ) change, the flow box region inside the adapted neighborhood may change drastically, but at least for each (ϵ, λ) fixed, the part of the inset that reaches the outset is a connected set. \square

Adapted neighborhoods near singular points on slow arcs

Let p be a singular point for the slow dynamics of X_{0,λ_0} on a slow arc in a slow–fast family of vector fields $X_{\epsilon,\lambda}$. Due to the potential presence of singularities for $\epsilon \neq 0$, we cannot use flow box neighborhoods to cover p . We will again introduce an adapted neighborhood, but this time we will not introduce a formal definition and a formal proposition, since the ideas are more or less identical to the ones already given. An adapted neighborhood (or a slow–fast family of adapted neighborhoods), will in this case also have an inset, an outset and two orbits as boundary, it is a diffeomorphic image of a square, but we do not impose conditions like the absence of singularities on the interior of the neighborhood.

Using Takens normal forms, we have, near a singular point on a slow arc, the following normal form for equivalence for the family of vector fields

$$\begin{cases} \dot{x} = \epsilon h(x, \epsilon, \lambda) \\ \dot{y} = -y \end{cases} \tag{7}$$

where $h(0, 0, \lambda_0) = 0$, and $h(x, 0, \lambda_0) \neq 0$ for $x \neq 0$. We distinguish two types of slow–fast segments that can belong to a slow–fast cycle near which periodic movements are possible:

1. *Singular fast–slow*: a fast orbit $\{0\} \times [0, 1]$ together with a slow part $[0, 1] \times \{0\}$. One assumes $h(x, 0, \lambda_0) > 0$ for $x \in]0, 1[$, see Fig. 14(a) or (b). In (a), the singularity in the slow dynamics is a saddle–node, whereas in (b), the singularity is a saddle singularity.
2. *Singular slow*: an arc $[-1, 1] \times \{0\}$, under the assumption that $h(x, 0, \lambda_0) > 0$ for $x \in [-1, 1] \setminus \{0\}$, see Fig. 14(c).

The way to obtain slow–fast families of adapted neighborhoods for these segments, with the above mentioned properties, is completely analogous to the construction of slow–fast families of flow boxes for the hyperbolic fast–slow and regular slow segments.

Remark. The part of the inset that reaches the outset may not be connected. This gives a slight complication in the proof of Theorem 1, but we refer to Section 7.4 on how this technicality is dealt with.

6. Existence of relaxation oscillations

In this section, we prove that Theorem 2 is a consequence of Theorem 1. We first prove a proposition, formulated below, that together with Theorem 1 implies Theorem 2. The reader will

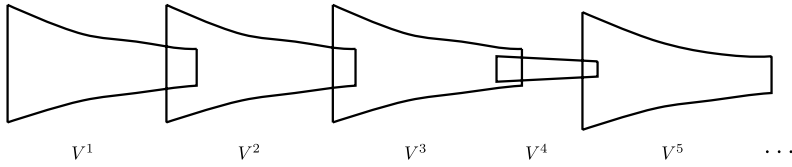


Fig. 15. Chain of flow box neighborhoods near some segments.

observe that the proposition constitutes the first part of [Theorem 2](#):

Proposition 7. *Let Γ be a strongly common slow–fast cycle of X_{0,λ_0} in a smooth slow–fast family of vector fields $X_{\epsilon,\lambda}$, for a given $\lambda_0 \in \Lambda$, and let \mathcal{T}_0 be a given neighborhood of Γ . Then there exists an $\epsilon_0 \in]0, \epsilon_1]$, a neighborhood $\Lambda_0 \subset \Lambda$ of λ_0 and a neighborhood $\mathcal{T} \subset \mathcal{T}_0$ of Γ such that for $\epsilon \in]0, \epsilon_0]$ and $\lambda \in \Lambda_0$, the vector field $X_{\epsilon,\lambda}$ has a closed orbit in \mathcal{T} .*

To see how this implies [Theorem 2](#) we refer to the end of this section.

Proof of Proposition 7. First, we divide the strongly common slow–fast cycle Γ in a finite number of slow–fast segments, each slow–fast segment being of the “strongly common” type. Taking ϵ small enough, we use [Proposition 3](#) to cover Γ by a finite number of slow–fast flow box neighborhoods $V_{\epsilon,\lambda}^k$, $k = 1, \dots, N$. Because we can make the outlets as close as required to the end points of the segments, we can ensure that the outlet of V^{N-1} lies inside V^N . Similarly, by adjusting V^{N-2} if necessary, we can ensure that its outlet lies inside V^{N-1} . We can continue this way to form a chain, as represented in [Fig. 15](#). To obtain that the outlet of V^N lies inside V^1 , we however need to adjust its outlet in a way without affecting the inset. This is precisely the subject of [Proposition 4](#). One just needs to take an N -th part that is not a regular fast or funnel fast–fast segment (because then we have a flow box like V^4 in [Fig. 15](#), where on respectively two sides or on one side of the slow–fast segment the size of the outlet stays $O(1)$ if the inset is chosen to be of size $O(1)$). Since a slow–fast cycle contains at least one slow arc, this is not a problem.

This way, the first return map from the inset of V^1 to itself is well-defined and is a smooth mapping from a closed interval into itself. Therefore, it has a fixed point, which corresponds to a closed orbit of $X_{\epsilon,\lambda}$. This finishes the proof of [Proposition 7](#). \square

Let us now comment on how [Theorem 2](#) follows from the combination of [Theorem 1](#) and [Proposition 7](#):

Proof of Theorem 2. We notice that we can apply [Theorem 1](#) to any strongly common slow–fast cycle. This gives us a neighborhood \mathcal{T}_0 and a neighborhood Λ_0 of $(\epsilon, \lambda) = (0, \lambda_0)$ with the properties mentioned in the formulation of [Theorem 1](#) and observing that a strongly common slow–fast cycle does not have singular contact points, the vector field $X_{\epsilon,\lambda}$ has, for those values of $(\epsilon, \lambda) \in \Lambda_0$, at most one closed orbit inside \mathcal{T}_0 and if it exists, it is hyperbolic. We can now apply [Proposition 7](#) to prove that the limit cycle actually exists, after possibly reducing Λ_0 to a smaller neighborhood of $(0, \lambda_0)$ and reducing \mathcal{T}_0 to a smaller neighborhood of the slow–fast cycle. \square

Remark. We recall that the proof of [Theorem 1](#) does not rely on [Theorem 2](#); we have opted to give the proof of [Theorem 2](#) before that of [Theorem 1](#) merely for didactic purposes.

7. Proof of Theorem 1

7.1. Regular slow–fast cycles

In this section, we prove [Theorem 1](#), under the condition that the slow–fast cycle Γ is a regular slow–fast cycle. We recall that the contact points of the slow–fast cycles considered in this paper are all nilpotent and of finite order. Regular slow–fast cycles have the additional properties that the contact points are regular and that the slow dynamics on the slow arcs have no singularities. Of course, [Theorem 1](#) is more general, since it also allows situations where singularities appear, either in the slow dynamics or at the contact points. These situations are however not treated in this subsection, but are postponed to [Sections 7.2–7.4](#).

We recall from [Lemma 1](#) that, if a slow–fast cycle is regular, then one can define a slow divergence integral along any segment of Γ and in particular along the whole Γ . We will write it $\int_{\Gamma} \operatorname{div}(X_{0,\lambda_0})ds$ (s is the slow time).

We assume that

$$\int_{\Gamma} \operatorname{div}(X_{0,\lambda_0})ds \neq 0.$$

Remark. The assumption on the slow divergence integral is always fulfilled for regular *common* cycles. A lot of studies have been devoted to the case of a slow–fast cycle where slow divergence integral vanishes. In this case, in general more than one limit cycle may bifurcate from the slow–fast cycle (see [[5–8,1](#)], ...).

We first show that [Theorem 1](#) (or at least the part of the theorem treated in this section) is a direct consequence of

Theorem 3. *Let Γ be a regular slow–fast cycle, with regular nilpotent contact points of finite contact order. Then, given any $\eta > 0$, there exist $\mathcal{T}, \epsilon_0, \Lambda_0$ as in [Theorem 1](#) such that if $C_{\epsilon,\lambda}$ is any closed orbit of $X_{\epsilon,\lambda}$ contained in \mathcal{T} , for $(\epsilon, \lambda) \in]0, \epsilon_0] \times \Lambda_0$, then we have*

$$\left| \epsilon \int_{C_{\epsilon,\lambda}} \operatorname{div}(X_{\epsilon,\lambda})dt - \int_{\Gamma} \operatorname{div}(X_{0,\lambda_0})ds \right| \leq \eta.$$

The proof of [Theorem 3](#) is done in [Section 7.1.3](#). This is done using some preparatory results on divergence integrals near several types of slow–fast segments in [Section 7.1.2](#). But first, we show in [Section 7.1.1](#) that the part of [Theorem 1](#) concerned with regular slow–fast cycles is indeed a consequence of [Theorem 3](#).

7.1.1. Deduction of [Theorem 1](#) from [Theorem 3](#), in case of regular slow–fast cycles

We recall that [Section 7.1](#) is concerned with the proof of [Theorem 1](#) for *regular* slow–fast cycles. By reversing time, an attracting slow–fast cycle as in [Theorem 1](#) is transformed into a repelling one. Then, it is sufficient to consider an attracting slow–fast cycle. We will hence assume this property for the given slow–fast cycle Γ , i.e. we assume

$$I := \int_{\Gamma} \operatorname{div}(X_{0,\lambda_0})ds < 0.$$

We can apply [Theorem 3](#) with an $\eta < |I|$. Then, for any closed orbit $C_{\epsilon,\lambda}$ of $X_{\epsilon,\lambda}$, as in [Theorem 3](#), we have that

$$\int_{C_{\epsilon,\lambda}} \operatorname{div}(X_{\epsilon,\lambda})dt < -\frac{K}{\epsilon}$$

with $K = |I| - \eta > 0$.

We choose the neighborhood \mathcal{T} small enough such that, for $(\epsilon, \lambda) \in]0, \epsilon_0] \times \Lambda$ we have that

1. $X_{\epsilon,\lambda}$ has no singular point in \mathcal{T} .
2. If $C_{\epsilon,\lambda}^1$ and $C_{\epsilon,\lambda}^2$ are two closed orbits of $X_{\epsilon,\lambda}$, these two closed orbits bound an annulus in \mathcal{T} .

Let $C_{\epsilon,\lambda}$ be any closed orbit in \mathcal{T} . Let Σ be a transverse section of $C_{\epsilon,\lambda}$, parameterized by $h \sim 0$, with $\{h = 0\} = \Sigma \cap \Gamma_{\epsilon,\lambda}$. Let $P_{\epsilon,\lambda}(h)$ be the Poincaré mapping of $X_{\epsilon,\lambda}$ defined on Σ , for $h \sim 0$. We have that $P_{\epsilon,\lambda}(0) = 0$ and the derivative $P'_{\epsilon,\lambda}(0)$ is given by the Poincaré–Leontovitch–Sotomayor formula:

$$P'_{\epsilon,\lambda}(0) = \exp \left[\int_{C_{\epsilon,\lambda}} \operatorname{div}(X_{\epsilon,\lambda})dt \right].$$

Then $P'_{\epsilon,\lambda}(0) \leq \exp[-\frac{K}{\epsilon}] < 1$. This proves that any closed orbit of $X_{\epsilon,\lambda}$ is a hyperbolic and attracting limit cycle.

To finish, let us suppose that for $X_{\epsilon,\lambda}$ we have more than one limit cycle in \mathcal{T} . Then, as each limit cycle is isolated, we can find two limit cycles $C_{\epsilon,\lambda}^1$ and $C_{\epsilon,\lambda}^2$ bounding an annulus A which does not contain any more limit cycle in its interior. But, as A does not contain any singular point of $X_{\epsilon,\lambda}$ and as $C_{\epsilon,\lambda}^1$ and $C_{\epsilon,\lambda}^2$ are attracting, it follows from the Poincaré–Bendixson theory that A must contain another limit cycle in its interior, which leads to a contradiction. This proves that [Theorem 1](#), under the extra conditions imposed in [Section 7.1](#), is a consequence of [Theorem 3](#). \square

7.1.2. Divergence integrals near slow–fast segments

In this section, we estimate divergence integrals passing from the inset to the outset of adapted neighborhoods of the different types of slow–fast segments.

For a regular fast segment, the part of an orbit from the inset to the outset has just an $O(1)$ contribution (as $\epsilon \rightarrow 0$) in the divergence integral. It remains to deal with the regular slow segment, the hyperbolic fast–slow segment, and segments near contact points.

To estimate the integral $\int \operatorname{div}(X_{\epsilon,\lambda})dt$ near regular slow and hyperbolic fast–slow segments, we need the following result whose proof is direct, using the ideas developed in eg. [2].

Lemma 3. *We consider the following data associated to X_{0,λ_0} :*

1. A segment $[m, n]$ contained in an hyperbolic attracting arc ℓ of the slow curve C_{λ_0} , such that the slow dynamics of X_{0,λ_0} goes from m to n .
2. A point M outside the slow curve and such that its ω -limit for the field X_{0,λ_0} is the point m . We call γ the closure of the positive orbit starting at M (it is the fast segment with end points m, M).
3. A transverse section Σ to the slow curve at the point n , disjoint from γ .

We choose a compact W , neighborhood of each γ and disjoint from the Σ . We also assume that any point $q \in W$ has a unique ω -limit $\pi(q) \in \ell$ (the mapping π is smooth and $\pi(M) = m$).

Then, for $\epsilon > 0$ small enough, the positive orbit of $X_{\epsilon,\lambda}$ through a point $q \in W$, cuts transversely Σ at a unique point $n(q, \epsilon, \lambda)$. We write $\gamma(q, \epsilon, \lambda)$ the segment of this orbit between q and $n(q, \epsilon, \lambda)$. We have that

$$\epsilon \int_{\gamma(q,\epsilon,\lambda)} \operatorname{div}(X_{\epsilon,\lambda})dt \rightarrow \int_{[\pi(q),n]} \operatorname{div}(X_{0,\lambda})ds$$

when $(\epsilon, \lambda) \rightarrow (0, \lambda_0)$ ($[\pi(q), n]$ is a segment on ℓ). The convergence is uniform in $q \in W$.

Now, let us focus on estimating divergence integrals of orbits passing near segments containing contact points. Of course, the segment can be chosen arbitrarily small (i.e. arbitrarily close to the contact point in Hausdorff sense).

Let p be a regular contact point for a parameter value $\lambda = \lambda_0$. We can find smooth local coordinates (x, y) such that $p = (0, 0)$, in which the system $X_{\epsilon,\lambda}$ is written in the following normal form:

$$\begin{cases} \dot{x} = U(x, y, \epsilon, \lambda)(y - f(x, \lambda)) \\ \dot{y} = \epsilon U(x, y, \epsilon, \lambda)G(x, y, \epsilon, \lambda) \end{cases}$$

for smooth functions U, f, G such that

$$f(0, \lambda_0) = \frac{\partial f}{\partial x}(0, \lambda_0) = 0, \quad U(0, 0, 0, \lambda_0) \neq 0 \quad \text{and} \quad G(0, 0, 0, \lambda_0) \neq 0.$$

We choose a compact neighborhood W of $(0, 0)$ in the coordinate domain, an $\epsilon_0 > 0$ and a neighborhood A of λ_0 such that $U(x, y, \epsilon, \lambda)G(x, y, \epsilon, \lambda) \neq 0$ for all $(x, y, \epsilon, \lambda) \in W \times [0, \epsilon_0] \times A$. Then, there exists a $K > 0$ such that:

$$|U(x, y, \epsilon, \lambda)G(x, y, \epsilon, \lambda)| \geq K \tag{8}$$

on $W \times [0, \epsilon_0] \times A$.

A volume form is chosen on M and for a smooth vector field X we write $\operatorname{div}(X)$ the corresponding divergence function. The divergence function $\operatorname{div}(X_{\epsilon,\lambda})$ is a smooth function in $(x, y, \epsilon, \lambda)$. Then, there exists a finite bound C such that

$$|\operatorname{div}(X_{\epsilon,\lambda})| \leq C \tag{9}$$

on $W \times [0, \epsilon_0] \times A$.

Lemma 4. *Let W, ϵ_0, A chosen as above. For some $(\epsilon, \lambda) \in]0, \epsilon_0] \times A$ we consider an arc of trajectory γ of $X_{\epsilon,\lambda}$, contained in W , with end points (x_0, y_0) and (x_1, y_1) . Then:*

$$\left| \int_{\gamma} \operatorname{div}(X_{\epsilon,\lambda})dt \right| \leq \frac{C}{\epsilon K} |y_1 - y_0|.$$

Proof. It follows from (8) that we can see γ as a graph of a smooth function $x(y)$, defined for $y \in [y_0, y_1]$. Then, we can write

$$\int_{\gamma} \operatorname{div}(X_{\epsilon,\lambda})dt = \int_{y_0}^{y_1} \frac{\operatorname{div}(X_{\epsilon,\lambda})}{\epsilon UG}(x(y), y)dy.$$

As $\gamma \subset W$, we can use the inequalities (8) and (9) along γ to obtain that

$$\left| \int_{\gamma} \operatorname{div}(X_{\epsilon,\lambda})dt \right| \leq \left| \int_{y_0}^{y_1} \frac{C}{\epsilon K} dy \right| = \frac{C}{\epsilon K} |y_1 - y_0|. \quad \square$$

Consider now a regular slow–fast cycle Γ defined for the value λ_0 . Let p a regular contact point belonging to Γ . To the data (Γ, p) corresponds one of the possible types of local transition which are represented in Fig. 5. To this type of transition are associated specific pairs of local sections Σ_0, Σ_1 to Γ , near p (respectively entrance and exit section).

We have the following easy corollary of the previous lemma:

Proposition 8. *Choose any $\delta > 0$. Then there exist a pair of local sections $\Sigma_0, \Sigma_1, \epsilon_0 > 0$ and a neighborhood Λ of λ_0 with the following properties. Let us consider any orbit $\Gamma_{\epsilon,\lambda}$ of $X_{\epsilon,\lambda}$, cutting Σ_0 , with $(\epsilon, \lambda) \in]0, \epsilon_0] \times \Lambda$. Then:*

1. *The sections Σ_0, Σ_1 cut $\Gamma_{\epsilon,\lambda}$ transversely, each of them at a unique point.*
2. *Let $\gamma_{\epsilon,\lambda}$ be the oriented arc of $\Gamma_{\epsilon,\lambda}$, beginning on Σ_0 and ending on Σ_1 . We have that*

$$\left| \int_{\gamma_{\epsilon,\lambda}} \operatorname{div}(X_{\epsilon,\lambda}) dt \right| \leq \frac{\delta}{\epsilon}. \tag{10}$$

Proof. First we choose neighborhoods W, ϵ_0, Λ as in Lemma 4 and moreover we impose the following supplementary properties:

1. W is chosen small enough in a normal form domain of coordinates (x, y) such that if $(x_0, y_0), (x_1, y_1) \in W$, then $\frac{C}{K} |y_1 - y_0| < \delta$.
2. Choose now in W any pair of local sections (Σ_0, Σ_1) , adapted to the transition along Γ . We write $p_0 = \Gamma \cap \Sigma_0, p_1 = \Gamma \cap \Sigma_1$. Let γ be the slow–fast arc in Γ with end points p_0, p_1 (of course $\gamma \subset W$). Consider an orbit $\Gamma_{\epsilon,\lambda}$ of $X_{\epsilon,\lambda}$ starting at a point $q_0 \in \Sigma_0$. We know that this orbit is transverse to Σ_0 at q and, if ϵ and $|\lambda - \lambda_0|$ are small enough, it cuts also transversely Σ_1 at a unique point $q_1(\epsilon, \lambda)$ and the arc $\gamma_{\epsilon,\lambda}$ of $\Gamma_{\epsilon,\lambda}$ between q_0 and $q_1(\epsilon, \lambda)$, must be contained in W (the orbit $\Gamma_{\epsilon,\lambda}$ converges in a C^0 -way toward any closed interval I of $\gamma \setminus \{p_0\}$ and in a C^∞ -way if $p \notin I$). Let us choose ϵ_0, Λ to have these properties.

We apply Lemma 4 in W to the arc $\gamma_{\epsilon,\lambda}$. This gives (10). \square

7.1.3. Proof of Theorem 3

We consider a slow–fast cycle Γ as in the statement of Theorem 3 and fix a constant $\eta > 0$. Let p_1, \dots, p_k be the contact points contained in Γ . For each $i = 1, \dots, k$, we choose a neighborhood W_i and $\epsilon_0^i, \Lambda_0^i$ as in Proposition 8 for a constant $0 < \delta < \frac{\eta}{3k}$. We take $\epsilon_0 = \inf_i \epsilon_0^i$ and $\Lambda = \cap_i \Lambda^i$. In each W^i we choose an inset Σ_0^i and an outset Σ_1^i as in Proposition 8. We choose S_i to be the segment in W^i with end points $q_0^i = \Gamma \cap \Sigma_0^i$ and $q_1^i = \Gamma \cap \Sigma_1^i$.

The segment R_i of Γ between the points q_1^i and q_0^{i+1} can be seen as an elementary slow–fast segment of type regular slow or hyperbolic fast–slow (see Fig. 5); we do not need to use regular fast segments (of course, we assume that we have $k + 1 \equiv 1$ in a cyclic way). We can now enter in the proof of Theorem 3. First, we notice that in Proposition 8 it is possible to choose arbitrarily short sections Σ_0^i, Σ_1^i . So, we can choose the sections Σ_1^i with the following property: if the successive segment R_i starting at q_1^i is a hyperbolic fast–slow segment, then all the ω -limits $\pi(q)$ of points $q \in \Sigma_1^i$ are near the starting m_i and moreover:

$$\left| \int_{[\pi(q), m]} \operatorname{div}(X_{0,\lambda}) ds \right| < \frac{\eta}{3k} \tag{11}$$

for all $q \in \Sigma_1^i$.

We now choose the neighborhood \mathcal{T} . Let us suppose that a metric is chosen on M . We choose \mathcal{T} as a μ -neighborhood of Γ for this metric, with μ sufficiently small that:

1. \mathcal{T} is an annulus. For ϵ and $|\lambda - \lambda_0|$ are small enough, each limit cycle in \mathcal{T} is isotopic to Γ in \mathcal{T} .
2. The sections Σ_0^i and Σ_1^i are transverse to the boundary of \mathcal{T} and then their end points are outside \mathcal{T} .

Consider now a limit cycle $\Gamma_{\epsilon,\lambda}$ of $X_{\epsilon,\lambda}$ contained into \mathcal{T} . If ϵ and $|\lambda - \lambda_0|$ are small enough, it follows from the choice of \mathcal{T} that $\Gamma_{\epsilon,\lambda}$ must cut each section Σ_0^i and Σ_1^i at a unique point. The limit cycle is then cut in $2k$ segments: $s_{\epsilon,\lambda}^i$ near S_i , between Σ_0^i and Σ_1^i and $r_{\epsilon,\lambda}^i$ near R_i , between Σ_1^i and Σ_0^{i+1} , for $i = 1, \dots, k$. As $s_{\epsilon,\lambda}^i$ cuts Σ_0^i , it must be contained in W_i and we know from the above choice of W_i that

$$\epsilon \left| \int_{s_{\epsilon,\lambda}^i} \operatorname{div}(X_{\epsilon,\lambda}) ds \right| < \frac{\eta}{3k} \tag{12}$$

for $i = 1, \dots, k$, if ϵ and $|\lambda - \lambda_0|$ are small enough. It follows from the above lemma that if ϵ and $|\lambda - \lambda_0|$ are small enough, we have

$$\left| \epsilon \int_{r_{\epsilon,\lambda}^i} \operatorname{div}(X_{\epsilon,\lambda}) dt - \int_{[\pi(q_1^i), q_0^{i+1}]} \operatorname{div}(X_{0,\lambda}) ds \right| < \frac{\eta}{3k} \tag{13}$$

if R_i is a hyperbolic fast–slow segment, for $i = 1, \dots, k$. Moreover, in this case and by the choice of the sections Σ_1^i , we have that

$$\left| \int_{R_i} \operatorname{div}(X_{0,\lambda}) ds - \int_{[\pi(q_1^i), q_0^{i+1}]} \operatorname{div}(X_{0,\lambda}) ds \right| < \frac{\eta}{3k}. \tag{14}$$

If R_i is a regular slow segment, we can write the inequality (13) with the interval $[\pi(q_1^i), q_0^{i+1}]$ replaced by the segment R_i itself.

Finally, putting together the inequalities (12)–(14) we have that the integral $\epsilon \int_{\Gamma_{\epsilon,\lambda}} \operatorname{div}(X_{\epsilon,\lambda}) dt$ written as the sum of the integrals $\epsilon \int_{r_{\epsilon,\lambda}^i} \operatorname{div}(X_{\epsilon,\lambda}) dt$ and $\epsilon \int_{s_{\epsilon,\lambda}^i} \operatorname{div}(X_{\epsilon,\lambda}) dt$ has a difference with the slow–fast integral $\int_{\Gamma} \operatorname{div}(X_{0,\lambda}) dt$ which is less than η , if ϵ and $|\lambda - \lambda_0|$ are small enough. This finishes the proof of [Theorem 3](#). \square

7.2. Singular contact points of index +1

To finish the proof of [Theorem 1](#), we still have to deal with slow–fast cycles with singular contact points, and with singularities on the slow arcs. In this section, and in [Section 7.3](#), we give a first step in that direction: we consider slow–fast cycles without singularities on the slow arcs, but with one or more singular nilpotent contact points. The presence of singularities on the slow arcs will be dealt with in [Section 7.4](#).

In case the slow–fast cycle contains a singular nilpotent contact point, it is assumed in the formulation of [Theorem 1](#) that it is a simple singularity, with singularity index +1. We can repeat the technique from the previous two sections, covering the slow–fast cycle this time using whirlpool neighborhoods near the singular contact points and flow box neighborhoods elsewhere.

A result like in [Proposition 8](#) can be shown in the case of singular contact points:

Proposition 9. *Given a singular nilpotent contact point of singularity index +1. For each $\eta > 0$, there exists a slow–fast family of whirlpool neighborhoods of the contact point (for (ϵ, λ) sufficiently close to $(0, \lambda_0)$) so that the divergence integral along any orbit that goes from the inset to the outset has a divergence integral bounded in absolute value by η/ϵ .*

A direct consequence is that we can prove the following analog of [Theorem 3](#) for slow–fast cycles with one or more singular contact points of index +1:

Theorem 4. *Let Γ be a slow–fast cycle, having regular contact points and/or contact points of singularity index +1, and let the slow dynamics on the slow arcs of Γ be without singularities. Let q_1, \dots, q_ℓ denote the singular contact points and let W_1, \dots, W_ℓ be mutually disjoint open sets with $q_j \in W_j$ for $j = 1, \dots, \ell$.*

Then, given any $\eta > 0$, there exist $\mathcal{T}, \epsilon_0, \Lambda_0$ as in [Theorem 1](#) such that if $(\epsilon, \lambda) \in]0, \epsilon_0] \times \Lambda_0$, and if $C_{\epsilon, \lambda}$ is any closed orbit of $X_{\epsilon, \lambda}$ contained in \mathcal{T} but not contained entirely in one of the W_j , then we have

$$\left| \epsilon \int_{C_{\epsilon, \lambda}} \operatorname{div}(X_{\epsilon, \lambda}) dt - \int_{\Gamma} \operatorname{div}(X_{0, \lambda_0}) ds \right| \leq \eta.$$

As the part of [Theorem 1](#) concerned with regular slow–fast cycles was shown to be a direct consequence of [Theorem 3](#), the part of [Theorem 1](#) concerned with slow–fast cycles with singular contact points is shown to be a direct consequence of [Theorem 4](#). Hence, we will finish the proof of [Theorem 1](#), in case of absence of singularities on the slow arcs, by proving [Proposition 9](#).

Remark. After introducing a Riemannian metric on the surface it is possible to reformulate [Theorem 4](#) requiring that \mathcal{T} be a δ -neighborhood of Γ , for some $\delta > 0$. It should be clear that the covering of Γ by flow box neighborhoods and sufficiently small whirlpool neighborhoods shows that a closed orbit in \mathcal{T} which is not contained in one of the W_j is necessarily δ -Hausdorff close to Γ . This remark justifies the first remark made after the formulation of [Theorem 1](#), at least for slow–fast cycles without singularities on the slow arcs.

7.3. Divergence integral near a singular contact point of index +1

In this subsection we prove [Proposition 9](#). Using the assumed conditions on the contact point, we have shown (see [Lemma 2](#) and the text below the lemma), that one has a normal form

$$\begin{cases} \dot{x} = y - f(x, \lambda) \\ \dot{y} = \epsilon(g(x, \epsilon, \lambda) + (y - f(x, \lambda))h(x, y, \epsilon, \lambda)), \end{cases} \tag{15}$$

with $f(0, \lambda_0) = \frac{\partial f}{\partial x}(0, \lambda_0) = g(0, 0, \lambda_0) = 0, \frac{\partial g}{\partial x}(0, 0, \lambda_0) < 0$ and $\frac{\partial f}{\partial x}(x, \lambda_0) > 0$ for $x > 0$. Using a rescaling, we can make $\frac{\partial g}{\partial x}(0, 0, \lambda_0) = -1$.

We will study such contact points by means of blow-up. Instead of studying this family of vector fields, we will study a more general, versal family of vector fields

$$\begin{cases} \dot{x} = y - f(x, a, \mu) \\ \dot{y} = \epsilon(g(x, b, \mu) + (y - f(x, a, \mu))h(x, y, \mu)), \end{cases} \tag{16}$$

where

$$f(x, a, \mu) = \sum_{k=1}^{n-1} a_k x^k + x^n + x^{n+1} R(x, \mu), \quad g(x, a, b, \mu) = b - x + x^2 S(x, \mu).$$

Here, $(a, b) \sim (0, 0)$, $b \in \mathbf{R}$, $a = (a_1, \dots, a_{n-1}) \in \mathbf{R}^{n-1}$, μ lies in a compact euclidean parameter space and R, S and h are smooth functions. The “contact order” n is even.

It is clear that any family (15) appears as a subfamily of a well-chosen versal family (16). Of course, as $(\epsilon, \lambda) \rightarrow (0, \lambda_0)$ in (15), we have $(\epsilon, a, b, \mu_0) \rightarrow (0, 0, 0, \mu_0)$. In the remainder of the section, we will consider the limit $(\epsilon, a, b) \rightarrow (0, 0, 0)$ and keep μ in a compact.

Using a nonlinear coordinate change in the x -direction, i.e. writing $x = \alpha(\tilde{x})$ for some diffeomorphism α , and after a suitable time change, we can even ensure that $S = 0$. We will hence assume $S = 0$ in the remainder of this section.

Observe also that the perturbation parameters a_1 and b serve the same purpose: by linear translation, one can make $b = 0$, so we will eliminate b . We now make a rescaling in parameter space

$$(\epsilon, a_1, a_2, \dots, a_{n-1}) = (v^{2n-2} E, v^{n-1} A_1, v^{n-2} A_2, \dots, v A_{n-1}),$$

where $v \geq 0$ and $(E, A_1, \dots, A_{n-1}) \in S^{n-1}$ (keeping $E \geq 0$): we then have

$$\begin{cases} \dot{x} = y - \left(\sum_{k=1}^{n-1} v^{n-k} A_k x^k + x^n + x^{n+1} R(x, \mu) \right) \\ \dot{y} = v^{2n-2} E \left(-x + \left(y - \left(\sum_{k=1}^{n-1} v^{n-k} A_k x^k + x^n + x^{n+1} R(x, \mu) \right) \right) h(x, y, \mu) \right), \end{cases} \tag{17}$$

and we study the limit $v \rightarrow 0$.

Instead of working with $(E, A) \in S^{n-1}$, we will work with an equivalent compact set $(E, A) \in \mathcal{S}$ where \mathcal{S} is given by

$$\mathcal{S} = \{E = 1, (A_1, \dots, A_{n-1}) \in \mathcal{A}\} \cup \{E \in [0, E_0], (A_1, \dots, A_{n-1}) \in S^{n-2}\},$$

where \mathcal{A} is a compact set, whose size depends on the choice of E_0 (\mathcal{A} grows as E_0 decreases). We will identify the first component of \mathcal{S} as the regular component, and the second as the singular component of the parameter space:

$$\begin{aligned} \mathcal{S}_{\text{reg}} &= \{E = 1, (A_1, \dots, A_{n-1}) \in \mathcal{A}\} \\ \mathcal{S}_{\text{sing}} &= \{E \in [0, E_0], (A_1, \dots, A_{n-1}) \in S^{n-2}\}. \end{aligned}$$

We prove Proposition 9 by recursion on the order of the contact point.

We make a blow-up of the origin in (x, y, v) -space:

$$(x, y, v) = (u\bar{x}, u^n \bar{y}, u\bar{v})$$

where $u \geq 0$ and $(\bar{x}, \bar{y}, \bar{v}) \in S^2$, keeping $\bar{v} \geq 0$.

We already know that there exists, arbitrarily close to the origin, a family of whirlpool neighborhoods, we will denote it by $ABCD$, where A, B, C, D serve as vertices both for the orbits and for the inset respectively outset. Similarly, we also have the existence of an arbitrarily large family of whirlpool neighborhoods in the family chart, denoted $A'B'C'D'$. A picture of a pair of such whirlpool neighborhoods is shown in Fig. 16. The existence of such whirlpool

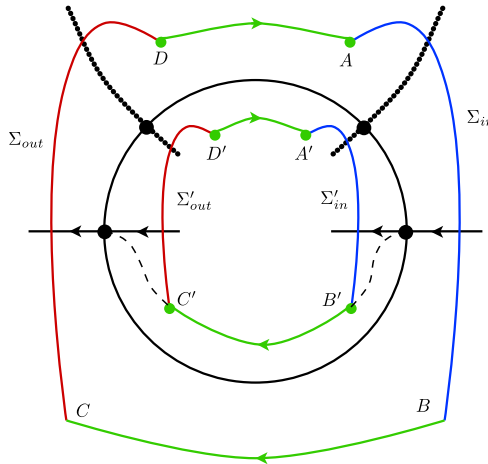


Fig. 16. Whirlpool neighborhoods inside and outside of the blow-up locus.

neighborhoods follows directly from the qualitative study of the circle at infinity on the blow-up locus. The proof of Proposition 9 will now be divided in three parts:

- (1) The study of the divergence integrals along orbits as they pass from Σ_{in} to Σ'_{in} (and by symmetry the orbits from Σ'_{out} to Σ_{out}).
- (2) The study of the divergence integrals along orbits inside $A'B'C'D'$, as they pass from Σ'_{in} to Σ'_{out} .
- (3) The study of the divergence integrals along orbits from Σ_{in} to Σ_{out} that never pass $A'B'C'D'$ (the bottom part).

The study of part (2) will be done in the family rescaling chart, where we will use an induction argument. Parts (1) and (3) are done in the phase-directional rescaling charts.

7.3.1. The family rescaling chart

First, we deal with (2), by examining the family rescaling chart $\{\bar{v} = 1\}$ of the blow-up. The intention is to identify the singular set as a compact subset of $A'B'C'D'$, and to cover it completely by adapted neighborhoods. The passage of each orbit from the inset to the outset will be treated in each adapted neighborhood individually and it will be shown that any such passage has a contribution of at most $o(\frac{1}{\epsilon})$ in the divergence integral (as $v \rightarrow 0$). Outside the singular set, we trivially use flow box neighborhoods. Of course, any orbit that goes from Σ'_{in} to Σ'_{out} can enter any of the adapted neighborhoods only once: orbits that pass two times a single adapted neighborhood are easily seen to get trapped and hence never reach Σ'_{out} . Since the number of neighborhoods is finite, the number of passages through such neighborhoods is also finite. A passage from Σ'_{in} to Σ'_{out} has therefore at most a finite number of $o(\frac{1}{\epsilon})$ contributions in the divergence integral, so the result remains $o(\frac{1}{\epsilon})$ as $v \rightarrow 0$.

In the family chart, we obtain, after division by the positive factor u^{n-1} ,

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \left(\sum_{k=1}^{n-1} A_k \bar{x}^k + \bar{x}^n \right) + O(u) \\ \dot{\bar{y}} = -E(\bar{x} + O(u)). \end{cases}$$

We intend to relate the divergence integral of the blown-up vector field to a divergence integral of the original vector field.

Lemma 5. *Fixing $u > 0$ and $v > 0$ and a set of parameters, Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$ be two points and let $\bar{p} = (\bar{x}_p, \bar{y}_p)$ and $\bar{q} = (\bar{x}_q, \bar{y}_q)$ be the corresponding blown-up coordinates. Let γ be an orbit from p to q , and let $\bar{\gamma}$ be the orbit in blown-up coordinates. Let I be the divergence integral of the vector field calculated along γ , and let \bar{I} be the divergence integral of the blown-up vector field calculated along $\bar{\gamma}$. Then*

$$I = \bar{I}.$$

Proof. Define \tilde{X} to be the blown-up vector field *before* division by u^{n-1} . Then \tilde{X} is obtained by the original vector field after a linear change of coordinates. In that case it is immediately clear that $I = \tilde{I}$. Next, the fact that $\tilde{I} = \bar{I}$ is a direct consequence of the relation of the transition map between two transverse sections with the exponential of the divergence integral, a property that is independent of time when the time change is a factor that is a constant of motion. \square

We keep (\bar{x}, \bar{y}) in a compact set, and consider (E, A) on \mathcal{S} . We keep u on a sufficiently small compact interval $[0, u_0]$. In $\mathcal{S}_{\text{sing}}$, we find a singular set consisting of

$$\left\{ E = 0, \bar{y} = \left(\sum_{k=1}^{n-1} A_k x^k + x^n \right) + O(u) \right\}.$$

In the parameter space \mathcal{S}_{reg} , where $E = 1$, we find the singular set consisting of

$$\{(\bar{x}, \bar{y}) = (\bar{x}_0(u), \bar{y}_0(u))\},$$

where both $\bar{x}_0(u)$ and $\bar{y}_0(u)$ are $O(u)$ (the singular point also depends on (A, μ) , but we prefer not to write this dependence explicitly). We will first cover the singular set in $\mathcal{S}_{\text{sing}}$ by means of adapted neighborhoods (flow box, whirlpool and other kinds). Of course, in this part of the parameter space, the blown-up vector field is a slow-fast vector field with singular parameter E , and the singular set is the slow curve. Near compact pieces of normally hyperbolic parts of the slow curve, we can find adapted neighborhoods, and it is clear that the contribution of the divergence integral between the inset and the outset of such adapted neighborhoods is $O(\frac{c}{E})$, for some $c > 0$. Keeping in mind that $\epsilon = v^{2n-2}E$, any $O(\frac{c}{E})$ expression is $o(\frac{1}{\epsilon})$ as $v \rightarrow 0$. Similarly, near any *regular* contact points, we have already seen the existence of adapted neighborhoods and we already know that in such neighborhoods, the contribution to the divergence integral of an orbit from the inset to the outset is of order $O(\frac{1}{E})$ and hence of order $o(\frac{1}{\epsilon})$.

Observe however, that there is a possibility of again having a contact point with a simple singularity inside the blow-up locus when $n \geq 4$. If this is the case, the contact point is found in an $O(u)$ -neighborhood of the origin, and from the fact that in $\mathcal{S}_{\text{sing}}$, we keep $(A_1, \dots, A_{n-1}) \in S^{n-2}$, it is clear that the contact point has at most order $n - 1$ in that case.

When the contact order is odd, the contact point has a neighborhood that cannot be passed through, as has already been observed in the text before [Lemma 2](#). When the contact order is even, the order is at most $n - 2$. This implies that we can use [Proposition 9](#) by recursion to find an adapted neighborhood and to find that the contribution of the divergence integral between the inset and the outset of this neighborhood is $O(\frac{1}{E})$, and hence $o(\frac{1}{\epsilon})$ as $v \rightarrow 0$.

Let us now focus on the singular set in \mathcal{S}_{reg} and let $(\bar{x}_0(u), \bar{y}_0(u))$ be the unique singularity. The equation for $u \sim 0$ is given by

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \left(\sum_{k=1}^{n-1} A_k \bar{x}^k + \bar{x}^n \right) + u O(\bar{x}^{n+1}) \\ \dot{\bar{y}} = -\bar{x} + u O(\bar{x}^2) + u^2 O(\bar{x}, \bar{y}). \end{cases}$$

We introduce polar coordinates $\{\bar{x} = \bar{x}_0(u) + r \cos \theta, \bar{y} = \bar{y}_0(u) + r \sin \theta\}$. We find

$$\begin{cases} \dot{r} = O(r) \\ \dot{\theta} = -1 + O(u, r, A_1). \end{cases}$$

So for (u, r, A_1) small enough, we have an orbit making a complete rotation around the origin. This could be a cycle, a part of an outwards spiral, or a part of an inwards spiral. In all cases, an orbit making one turn can easily be used to delimit a neighborhood where orbits from the inset Σ_{in} to the outset Σ_{out} cannot pass. When A_1 is not small, the singularity at the origin is hyperbolically attracting or repelling.

This means that we have a uniformly-sized neighborhood of the singular set in \mathcal{S}_{reg} where orbits from the inset Σ_{in} to the outset Σ_{out} cannot pass (uniformly-sized means independent of the parameters $(A_1, \dots, A_{n-1}) \in S^{n-1}$ and μ). We hence need not cover this part of the phase space.

To conclude, we have covered the entire of the whirlpool set $A'B'C'D'$, or at least the part of this neighborhood that consists of orbits from Σ'_{in} to Σ'_{out} , by flow box neighborhoods or other adapted neighborhoods, and in each of these neighborhoods the contribution of a passage through such neighborhood in the computation of the divergence integral is $o(\frac{1}{\epsilon})$. As mentioned before, the number of adapted neighborhoods is finite, and any orbit from Σ'_{in} to Σ'_{out} can only pass each neighborhood at most once. This implies:

Proposition 10. *For each arbitrarily large whirlpool neighborhood $A'B'C'D'$ as in Fig. 16, a divergence integral along an orbit of (16) passing from Σ'_{in} to Σ'_{out} , is $o(\frac{1}{\epsilon})$, as $(\epsilon, a, b) \rightarrow (0, 0, 0)$, the o -property not depending on the choice of the orbit and not depending on the parameters (ϵ, a, b, μ) , for (ϵ, a, b) close enough to $(0, 0, 0)$ and μ in a compact subset.*

7.3.2. The phase-directional rescaling charts

We study the different charts $\{\bar{y} = \pm 1\}$ and $\{\bar{x} = \pm 1\}$. The study of the chart $\{\bar{x} = -1\}$ is completely analogous to the study of the chart $\{\bar{x} = +1\}$, so we will only deal with the latter. The study of the chart $\{\bar{y} = -1\}$ is trivial, since no singular points are encountered there.

We would like to use the same idea as in the previous section: covering the inside of $ABCD$ by adapted neighborhoods. However, near the circle at infinity, we have a three-dimensional vector field and the notion of adapted neighborhood should be modified. Instead of introducing new terminology, we prefer to speak just of adapted neighborhoods, keeping in mind that we will only use the following property of adapted neighborhoods: any orbit from Σ_{in} to Σ_{out} can only pass the neighborhood once.

Near a nonsingular point on the circle at infinity, it is easy to find such neighborhood that stays of a uniform size (in blown-up coordinates), we just need to take care of the singular points on the circle at infinity.

7.3.2.1. The chart $\{\bar{x} = +1\}$. We blow up using

$$(x, y, v) = (u, u^n \bar{y}, u \bar{v})$$

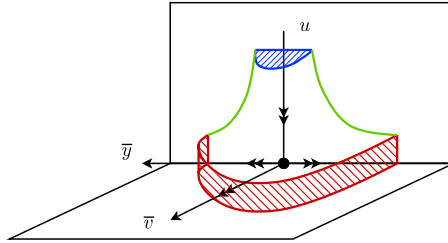


Fig. 17. Flow box near the hyperbolic saddles at the blow-up locus.

and study the equations for (u, \bar{y}, \bar{v}) near $(0, 0, 0)$. We find the family of vector fields, after division by u^{n-1} ,

$$\begin{cases} \dot{u} = -u(1 - \bar{y} + O(u, \bar{v})) \\ \dot{\bar{v}} = \bar{v}(1 - \bar{y} + O(u, \bar{v})) \\ \dot{\bar{y}} = E\bar{v}^{2n-2}(-1 + O(u)) + n\bar{y}(1 - \bar{y} + O(u, \bar{v})) \end{cases}$$

The only singularity that we want to treat now is the hyperbolic singularity at the origin; there is a second singularity near $\bar{y} = 1$, but we prefer to treat it in the chart $\bar{y} = 1$. The singularity at the origin, and nearby integrals along orbits could be studied using normal form theory, and this is the usual way of dealing with such points. In this case, the singularity is very specific and we only need very rough bounds, so we prefer to work with an explicit, ad hoc, method.

We consider the inset

$$\Sigma_{in}^{local} = \{u = u_0, \bar{y}^2 + \bar{v}^2 \leq \delta\}$$

and the outset

$$\Sigma_{out}^{local} = \{\bar{y}^2 + \bar{v}^2 = r_0\},$$

for sufficiently small and positive u_0, r_0, δ , keeping $\delta < r_0$. It is easy to see that the union of orbits through Σ_{in}^{local} , taken until they intersect Σ_{out}^{local} , form a neighborhood of the origin (in the octant $u \geq 0, \bar{v} \geq 0$), see Fig. 17. To that end, it helps to see that the function

$$V = \frac{1}{2}\bar{v}^2 + \frac{1}{2}\bar{y}^2$$

is strictly increasing along orbits (which implies that in some neighborhood of the origin, all orbits intersect Σ_{out}^{local} in positive time and Σ_{in}^{local} in negative time). Indeed, using O as a general shortcut for a function that is $O(u, \bar{v}, \bar{y})$, we have

$$\begin{aligned} \dot{V} &= \bar{v}^2(1 + O) + n\bar{y}^2(1 + O) - \bar{y}E\bar{v}^{2n-2}(1 + O) \\ &= \bar{v}^2(1 + O) + \left(n\bar{y} - \frac{1}{2n}E\bar{v}^{2n-2}(1 + O)\right)^2(1 + O) \\ &\geq CV \end{aligned}$$

for some $C > 0$. The last inequality can be easily seen by treating the cases $|\bar{y}| \leq |\bar{v}|$ and the cases $|\bar{y}| \geq |\bar{v}|$ separately.

Let us now bound the divergence integral of the original vector field, very roughly. Consider the divergence as an $O(1)$ term, we bound the transition time. Recalling that the blown-up vector

field is obtained after division by u^{n-1} , we have to bound

$$\int \frac{dV}{u^{n-1}\dot{V}}.$$

On each orbit, we have $u\bar{v} = v$, so $u \geq \frac{v}{\bar{v}} \geq \frac{v}{\sqrt{2V}}$. The integral is hence bounded by

$$\frac{\tilde{C}}{v^{n-1}} \int V^{(n-1)/2-1} dV \geq \frac{\tilde{C}}{v^{n-1}}$$

keeping in mind that $n \geq 2$. In any case, this is $O(\frac{1}{\sqrt{\epsilon}})$, so $o(\frac{1}{\epsilon})$ as $v \rightarrow 0$.

7.3.2.2. *The chart $\{\bar{y} = +1\}$.* We blow up using

$$(x, y, v) = (u\bar{x}, u^n, u\bar{v})$$

and study the equations for (u, \bar{v}) near $(0, 0)$ and keeping \bar{y} in a large compact. We find the family of vector fields, after division by u^{n-1} ,

$$\begin{cases} \dot{u} = \frac{1}{n} u \bar{v}^{2n-2} E(-\bar{x} + O(u)) \\ \dot{\bar{v}} = -\frac{1}{n} \bar{v}^{2n-1} E(-\bar{x} + O(u)) \\ \dot{\bar{x}} = 1 - \sum_{k=1}^{n-1} \bar{v}^{n-k} A_k \bar{x}^k - \bar{x}^n + u O(\bar{x}^{n+1}) - \frac{1}{n} \bar{x} \bar{v}^{2n-2} E(-\bar{x} + O(u)). \end{cases}$$

In terms of the original integration time there is a $C > 0$ for which

$$\dot{u} \leq -C\epsilon u^{2-n} < 0$$

for $\bar{x} \in [\frac{1}{2}, \frac{3}{2}]$, and for u and \bar{v} small enough. On the other hand, the divergence of the original vector field is easily seen to be bounded by Du^{n-1} , for some $D > 0$, so the divergence integral is bounded by

$$\int_0^{u_0} \frac{Du^{n-1}}{C\epsilon u^{2-n}} du = \frac{1}{\epsilon} \int_0^{u_0} \frac{D}{C} u^{2n-3} du.$$

This is clearly $O(\frac{1}{\epsilon}) \cdot O(u_0)$. This means that any orbit that passes through the region $\{\bar{x} \in [\frac{1}{2}, \frac{3}{2}]\}$ with (u, \bar{v}) sufficiently small will either stay forever inside or will leave after some time. If it leaves then the part of the orbit between the entry and the exit has a contribution of the given order in the divergence integral.

This finishes the proof of [Proposition 9](#). \square

7.4. Singularities on slow arcs

Here, we prove [Theorem 1](#) in case one or more isolated singularities appear on the slow arcs. We recall that in that case, we assume in [Theorem 1](#) that the singularities on the slow arcs are either all on the attracting arcs, or all on the repelling ones. Without loss of generality, we assume that all singularities appear on *attracting* slow arcs. The idea is to show that the divergence integral has a fixed sign, after which the same logic as in the previous section can be applied to show that the number of limit cycles that can perturb from the slow-fast cycle is at most one. Indeed, looking back at the proof of [Theorem 1](#) for regular slow-fast cycles, one can identify

the following two crucial elements: a fixed sign for the divergence integral, and the fact that two limit cycles sufficiently close to the slow–fast cycle (and for parameter values sufficiently close to $(\epsilon, \lambda) = (0, \lambda_0)$), will delimit an annulus without singular points inside, thereby forming a contradiction with the information from the divergence integral.

Although it is not necessary, in this section, we choose a Riemannian metric on the surface M , in order to define a Hausdorff metric. This Hausdorff metric will be used to cover the singular slow–fast cycle by a neighborhood within Hausdorff distance δ of the slow–fast cycle. This simplifies the presentation of the proof.

Let Γ be the slow–fast cycle. We can cover Γ by flow box neighborhoods and whirlpool neighborhoods, except at the singularities on the slow arcs, where we use the adapted “neighborhoods” as introduced at the end of Section 5.

Note that the covering of Γ by these neighborhoods does not have a chain property in the sense that the outset of the i -th neighborhood lies inside the $(i + 1)$ -th neighborhood, but this is not necessary in view of proving [Theorem 1](#).

We first prove that the divergence integral along any limit cycle inside the neighborhood has a divergence integral of a fixed, in this case negative, sign.

Just like in the proof of [Theorems 3](#) and [4](#), we can prove that the contribution outside the finite number of adapted neighborhoods near the singularities on the slow arcs is $O(\frac{K}{\epsilon})$, for some $K > 0$. We have that near each of the singularities on the slow arc, the contribution of the divergence integral near these points has the property

$$\epsilon \cdot I \rightarrow -\infty, \quad (\delta, \epsilon, \lambda) \rightarrow (0, 0, \lambda_0), \tag{18}$$

where δ is the Hausdorff distance between the neighborhood and Γ . Property (18) can be easily shown using the normal form (7) near a singular point on a slow arc, or we could refer to Proposition 4.6 in [1] where this property is shown. The intuition behind the proof of (18) is that as $\delta \rightarrow 0$, any periodic orbit near Γ is forced to spend some time in an $O(\delta)$ -neighborhood of the singularity on the slow arc. Knowing that the divergence of $X_{\epsilon,\lambda}$ in such a neighborhood is uniformly bounded away from 0, it is not hard, using normal forms, to give a rough bound like (18) on the divergence integral. This shows that by choosing the neighborhood sufficiently small, the total divergence integral of any periodic near Γ has a fixed sign.

Let us now comment on the second ingredient: we prove that any two periodic orbits sufficiently close to Γ delimit an annulus without singular points inside, at least for (ϵ, λ) sufficiently close to $(0, \lambda_0)$. We suppose Γ is covered entirely by adapted neighborhoods (flow box, whirlpool or other adapted neighborhoods), and we suppose the two periodic orbits $C_{\epsilon,\lambda}^1$ and $C_{\epsilon,\lambda}^2$ lie completely inside this covering. The absence of singularities can be shown in each adapted neighborhood individually. In flow box neighborhoods, it is clear. For whirlpool neighborhoods, it suffices to recall that they are only used to cover contact points of index +1, using [Proposition 6](#); the same proposition tells us that passing through a whirlpool happens in a flow box, namely the total passage flow box.

So it remains to deal with neighborhoods of singularities on the slow arcs. We recall the normal form for smooth equivalence that we have near such points:

$$\begin{cases} \dot{x} = \epsilon h(x, \epsilon, \lambda) \\ \dot{y} = -y \end{cases} \tag{19}$$

where $h(0, 0, \lambda_0) = 0$, and $h(x, 0, \lambda_0) \neq 0$ for $x \neq 0$. The periodic orbits can follow two types of slow–fast segment near the singular point: “singular slow” ([Fig. 18\(a\)](#)), or “singular

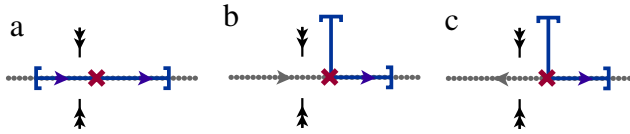


Fig. 18. Singularities on the slow arcs.

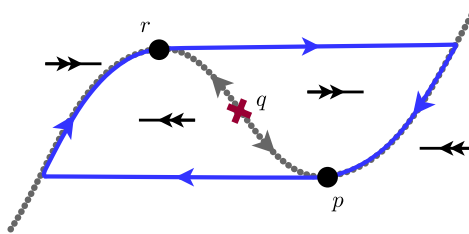


Fig. 19. (Example 1) Slow-fast cycle with BT contact point. The \times symbol means a singularity on the slow arcs.

fast-slow” (see Fig. 18(b) or (c)). For each individual value of the parameters (ϵ, λ) , the equation in normal form has a singular point, and when it has, we denote with s the x -coordinate of the rightmost singularity. In the case $C_{\epsilon, \lambda}^i$ follow a singular slow segment, the presence of a singularity at $x = s$ is a contradiction with the fact the orbits $C_{\epsilon, \lambda}^i$ are able to reach the outset; the absence of a singularity would imply that the part between the two periodic orbits forms a flow box inside the adapted neighborhood. In the case $C_{\epsilon, \lambda}^i$ follow a singular fast-slow segment, the presence of a singularity at $x = s$ would imply that both orbits $C_{\epsilon, \lambda}^i$ are found in the half plane $\{x > s\}$, thus also in this case the part between $C_{\epsilon, \lambda}^1$ and $C_{\epsilon, \lambda}^2$ forms a flow box inside the adapted neighborhood. Also the absence of a singularity would imply the required property.

We have shown that in all cases, two periodic orbits $C_{\epsilon, \lambda}^1$ and $C_{\epsilon, \lambda}^2$ near Γ delimit an annulus without singular points, and therefore all necessary ingredients are obtained to conclude Theorem 1 in the case of slow-fast cycles with singularities on the slow arcs. \square

8. Examples of two limit cycles near common slow-fast cycles

This section is devoted to providing two examples of attracting common slow-fast cycles, perturbing to more than one limit cycle. Of course, these common slow-fast cycles do not satisfy the assumptions of Theorem 1. It shows that a generalization of Theorem 1 to slow-fast cycles with more or other kinds of singular points is not possible.

We refer to [1] for results on slow-fast cycles (not common) with singularities on both attracting and repelling slow arcs. In this section, we prove that even for common slow-fast cycles, one can find counterexamples when the contact points are of a different kind than allowed in Theorem 1. In the examples below, one of the contact points of the common slow-fast cycle has a singularity of index -1 or a singularity of higher multiplicity.

8.1. Example 1

In a first example, the common slow-fast cycle looks like in Fig. 19, where at the point p we have a slow-fast Bogdanov-Takens point, as defined and studied in [3]. An explicit example of

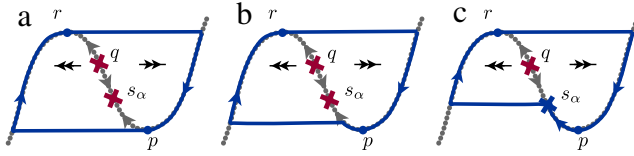


Fig. 20. Canard cycles for (21).

such a slow–fast cycle and related unfolding is given by

$$X_{\epsilon,b}: \begin{cases} \dot{x} = y - x^2 - x^3 \\ \dot{y} = \epsilon(b_0 + b_1x - x^2 - 3x^3) \end{cases} \tag{20}$$

with $(b_0, b_1) \sim (0, 0)$, $\epsilon > 0$, $\epsilon \sim 0$. It hence occurs for Liénard systems with cubic forcing and quadratic damping, of which (20) is an expression in the Liénard plane. The slow–fast cycle Γ is a cycle for the parameter values $(b_0, b_1) = (0, 0)$. In Eq. (20) there is, for $\epsilon > 0$, a hyperbolic repelling node q , located at $(-\frac{1}{3}, \frac{2}{27})$ for $(b_0, b_1) = (0, 0)$. The common slow–fast cycle under consideration looks like the traditional common Van der Pol cycle, except that p is no longer a regular contact point (i.e. a jump point), but a BT-point. In (20), the point p is positioned at the origin for $(b_0, b_1) = (0, 0)$. We see that $g(x) = -x^2 - 3x^3$, hence $g'(0) = 0$. The assumptions on g mean that the contact point p is a singular nilpotent contact point of multiplicity 2, violating the assumptions of Theorem 1. At $r = (-\frac{2}{3}, \frac{4}{27})$ the slow–fast cycle has a regular jump point. We will show below that such a common slow–fast cycle can be approached (in the Hausdorff metric) by generic saddle–node bifurcations of limit cycles.

Proposition 11. *Let Γ be the slow–fast cycle of $X_{\epsilon,0}$, as shown in Fig. 19. The slow–fast cycle Γ can, for (ϵ, b_0, b_1) close to $(0, 0, 0)$, be approached (for the Hausdorff metric) by generic saddle–node bifurcations of limit cycles.*

Proof. We first consider (20) with $b_0 = 0$ and $b_1 = -\alpha$, keeping $\alpha > 0$, $\alpha \sim 0$:

$$\begin{cases} \dot{x} = y - x^2 - x^3 \\ \dot{y} = \epsilon x(-\alpha - x - 3x^2). \end{cases} \tag{21}$$

For $\epsilon > 0$, we see that (21) has singularities on $\{y = x^2 + x^3\}$ situated at $x = 0$, $x \sim -\frac{1}{3}$ and at $x = \beta$, with

$$\beta = -\alpha + O(\alpha^2).$$

The singularity at $s_\alpha := (\beta, \beta^2 + \beta^3)$ is a hyperbolic saddle. In Fig. 20 we draw the layer equation for a fixed $\alpha > 0$ indicating the slow dynamics with a single arrow and the layer dynamics with a double arrow. The layer equation of (21) contains three types of canard (slow–fast) cycles: one as represented in (a), one as represented in (c) and an infinity as represented in (b). All three types have been studied before and we know that the essential information to study their cyclicity is contained in the slow divergence integral, that is obtained by adding the slow divergence integrals along the different slow arcs. One of these arcs is the arc (p, s_α) along which the layer equation is normally repelling; we parameterize it by $X \in]\beta, 0[$, recalling that $\beta < 0$. Let us denote by Γ_X the canard cycle containing the fast orbit related to this parameter value X . The slow divergence

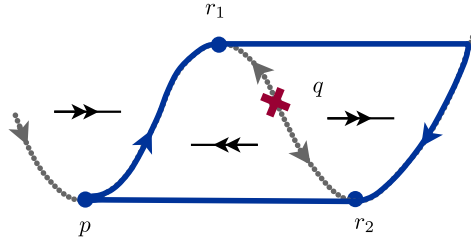


Fig. 21. Slow–fast cycle with saddle-type contact point (p). The \times symbol means a singularity on the slow arc.

integral between p and the point at the parameter value X is given by

$$I(X) = - \int_0^X \frac{(2x + 3x^2)^2}{\alpha + x + 3x^2} dx = \int_X^0 \frac{(2x + 3x^2)^2}{3(x - \beta)(x - \gamma(\beta))} dx,$$

where $\gamma(\beta)$ represents the zero of the denominator close to $-\frac{1}{3}$.

The integral is clearly strictly positive for $X \in]\beta, 0[$; it tends to zero for $X \rightarrow 0$ and it tends to $+\infty$ for $X \rightarrow \beta$. Moreover,

$$\frac{dI}{dX} = - \frac{(2X + 3X^2)^2}{3(X - \beta)(X - \gamma(\beta))} < 0,$$

or in other words, the slow divergence integral along this repelling arc (p, s_α) strictly increases from 0 to $+\infty$ when we let X move from 0 to β .

On the other hand, the slow divergence integral along the totality of slow arcs of Γ_X clearly also increases (becomes less negative) when X moves from 0 to β . As such the total slow divergence integral of Γ_X increases strictly monotonically when X moves from 0 to β , staying at a strictly negative value (for $X = 0$) and tending to $+\infty$ for $X \rightarrow \beta$.

There is hence a unique value $X_0 \in]\beta, 0[$ where the total slow divergence integral of Γ_{X_0} is zero and this is a simple zero of the total slow divergence integral.

Given the presence of b_0 in (20) and the fact that $b_1 = -\alpha < 0$ for $\alpha > 0$, we have a Hopf breaking mechanism at the origin (see [4]). It now follows from former papers (see [4] for an account exactly fit for this situation) that there exist systems (20) with $b_0 \sim 0$, exhibiting, Hausdorff-close to Γ_{X_0} , a saddle–node bifurcation of limit cycles.

This happens for all $\alpha > 0, \alpha \sim 0$, proving our claim with respect to systems (20). \square

If instead of merely treating Example (20), we would like to treat a general slow–fast cycle like in Fig. 19, subject to a generic unfolding of the BT-point, i.e. a slow–fast BT-bifurcation (see [3]), then the result is still the same. The proof now relies on an appropriate normal form for the slow–fast BT-bifurcation and is essentially similar to the one that we provided above.

8.2. Example 2

In the second example, the common slow–fast cycle looks like in Fig. 21, where this time at p we have a saddle-type contact point of singularity index -1 . An explicit example of such a slow–fast cycle and related unfolding is given by

$$\begin{cases} \dot{x} = y - (2(2 + a)x^2 - 4x^3 + x^4) \\ \dot{y} = \epsilon(x + b) \left(\frac{3}{2} - x \right), \end{cases} \tag{22}$$

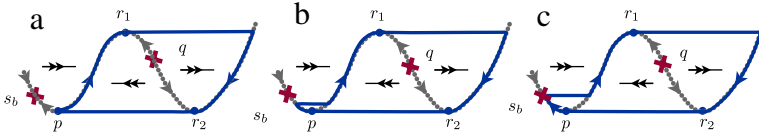


Fig. 22. Canard cycles for (23).

with $(a, b) \sim (0, 0)$, $\epsilon > 0$, $\epsilon \sim 0$. It hence occurs for Liénard systems with quadratic forcing and cubic damping, of which (22) is an expression in the Liénard plane.

In Eq. (22) there is, for $\epsilon > 0$ and $a = 0$, a hyperbolic repelling node at $q = (\frac{3}{2}, \frac{9}{17})$. In (22) the point p is positioned at the origin and, for $b = 0$, we see that $g(x) = x(\frac{3}{2} - x)$, hence $g'(0) > 0$. Both at $r_1 = (1, 1)$ and $r_2 = (2, 0)$, the slow–fast cycle (at $(a, b) = (0, 0)$) has a regular jump point. We will now prove the following proposition:

Proposition 12. *Let Γ be the slow–fast cycle of $X_{\epsilon,0}$, as shown in Fig. 21. The slow–fast cycle Γ can, for (ϵ, a, b) close to $(0, 0, 0)$, be approached (for the Hausdorff metric) by generic saddle–node bifurcations of limit cycles.*

Proof. We first consider (22) with $a = 0$ and $b > 0$, $b \sim 0$:

$$\begin{cases} \dot{x} = y - (4x^2 - 4x^3 + x^4) \\ \dot{y} = \epsilon(x + b) \left(\frac{3}{2} - x \right). \end{cases} \tag{23}$$

For $\epsilon > 0$ we see that (23) has singularities on $\{y = 4x^2 - 4x^3 + x^4\}$ situated at $x = \frac{3}{2}$ and at $x = -b$. The singularity at $s_b := (-b, 4b^2 + 4b^3 + b^4)$ is a hyperbolic saddle. In Fig. 22 we draw the layer equation (for a fixed $b > 0$, $b \sim 0$) indicating the slow dynamics with a single arrow and the layer dynamics with a double arrow. The layer equation of (23) contains three types of canard (slow–fast) cycles: one as represented in (a), one as represented in (c), and an infinity as represented in (b). The three types have been studied before and we know that the essential information to study their cyclicity is contained in the total slow divergence integral.

Using that s_b is an hyperbolic saddle, we now proceed exactly like in the treatment of Example 1 in Section 8.1 in order to show the existence of a canard cycle Γ_{X_0} of type (b) at which the total slow divergence integral has a simple zero.

This canard cycle contains a jump breaking mechanism between r_2 and p and it is easy to see that the connection between r_2 and p breaks in a regular way when varying the parameter a in Eq. (22). For that we write

$$f_a = f(x, a) = 2(2 + a)x^2 - 4x^3 + x^4.$$

The left minimum of f_a is situated at $x = 0$, while the right one is situated at $r_2(a) = -\frac{1}{2}(3 + \sqrt{1 - 4a})$. The breaking parameter of the jump mechanism (see [9]) hence can be expressed as

$$B(a) = f(r_2(a), a)$$

and clearly

$$\frac{dB}{da}(0) = 8 > 0.$$

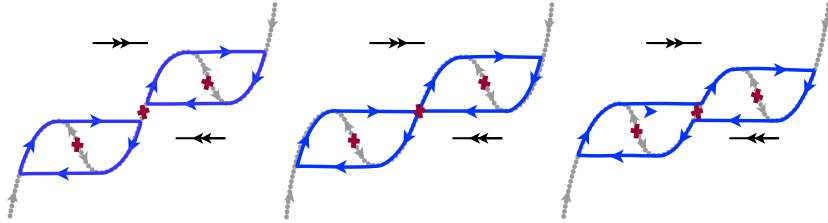


Fig. 23. Figure-eight limit periodic set (for $a = 0$) in the middle, together with its unfolding ($a > 0$ to the left and $a < 0$ to the right).

It now follows from [9] (see also [4]) that there exist systems (22) with $a \sim 0$, exhibiting Hausdorff near Γ_{X_0} , a saddle–node bifurcation of limit cycles. This happens for all $b > 0$, $b \sim 0$, proving our claim with respect to systems (22). \square

If instead of merely treating (22) we would like to treat a general slow–fast cycle like in Fig. 21, subject to a generic breaking of the connection between r_2 and p , then the result is still the same. The proof relies on using an appropriate normal form near p and is essentially similar to the one that we provided for treating (22).

9. Slow–fast paths that are not cycles

Besides the slow–fast cycles defined in this paper, a slow–fast family of vector field may have other kinds of slow–fast limit periodic sets out of which periodic orbits can bifurcate. As a motivating example, consider the slow–fast family of vector fields

$$\begin{cases} \dot{x} = y - \left((5 + a)x - 2x^3 + \frac{1}{5}x^5 \right) \\ \dot{y} = \epsilon \left(x(4 - x^2) \right). \end{cases} \tag{24}$$

For $a \sim 0$, there are 4 contact points $(p, F(p))$, $(q, F(q))$ and their reflections $(-p, -F(p))$, $(-q, -F(q))$, where $F(x) = (5 + a)x - 2x^3 + \frac{1}{5}x^5$ and

$$p = \sqrt{5} - \frac{\sqrt{5}}{40}a + O(a^2), \quad q = 1 + \frac{1}{8}a + O(a^2).$$

It is also useful to know that $F(p) = \sqrt{5}a + O(a^2)$ and $F(q) = \frac{16}{5} + O(a)$. For $a = 0$, we have a figure-eight “(common) limit periodic set” as shown in Fig. 23, with a saddle singularity in the slow dynamics at the origin. For $a > 0$, the figure-eight breaks down in two separated cycles; for $a < 0$, the figure-eight opens up to a single cycle. So for a given sufficiently small neighborhood of the figure-eight, there exists an $a_0 > 0$, $\epsilon_0 > 0$ so that for $(a, \epsilon) \in]0, a_0[\times]0, \epsilon_0[$, the vector field has exactly 2 periodic orbits, both hyperbolically attracting, and for $(a, \epsilon) \in]-a_0, 0[\times]0, \epsilon_0[$, the vector field has exactly one periodic orbit, and it is hyperbolically attracting. In both cases, the periodic orbits lie in the given neighborhood of the figure-eight.

This means that Theorem 1 may not be valid for limit periodic sets other than cycles. In this section, we state, and prove, a generalization of Theorem 1. Recall that a slow–fast cycle is a succession of slow arcs and regular (fast) orbits that together form a piecewise-smooth curve homeomorphic to a circle. We generalize slow–fast cycles to *slow–fast paths* in the following definition.

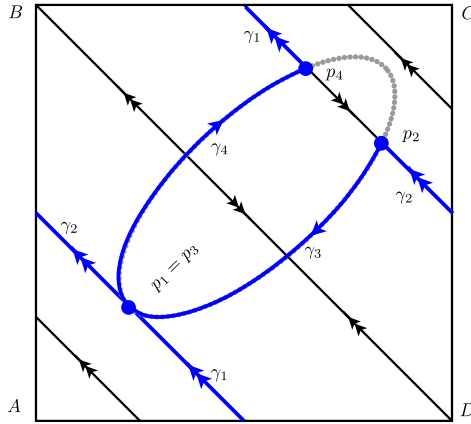


Fig. 24. Example of a closed slow-fast path (of canard type) on a torus.

Definition 12. Consider a slow-fast vector field $X_{\epsilon,\lambda}$, and given $\lambda_0 \in \Lambda$. A slow-fast path is a succession of regular slow arcs and regular (fast) orbits $P = (\gamma_i)_{i=1,\dots,n}$ of X_{0,λ_0} (with at least one slow arc) so that for each $i = 1, \dots, n - 1$, we have that $p_i = \overline{\gamma_i} \cap \overline{\gamma_{i+1}}$ is a contact point, the corner in a hyperbolic fast-slow segment, or a singularity of the slow dynamics on a regular arc, and so that each $\gamma_i \cup \{p_i\} \cup \gamma_{i+1}$ is a slow-fast segment of the following type:

1. the slow-fast segments with a contact point p_i in Fig. 5, and their time-reversed counterparts;
2. the slow-fast segments with a contact point p_i in Fig. 10;
3. the slow-fast segments with a singularity in the slow arc p_i in Fig. 18, and their time-reversed counterparts,

and so that the orientation of the slow dynamics on the slow arcs and the natural orientation on the fast orbits is compatible.

A closed slow-fast path is a slow-fast path where the above conditions is also satisfied for $i = n$, with the notional convention $\gamma_1 = \gamma_{n+1}$.

Remark. 1. In this definition, the slow arcs are assumed to be regular. Should one want to study a slow-fast path with some arc γ_i that has an isolated singularity of the slow dynamics, then one can write $\gamma_i = \gamma_{i,1} \cup \{p\} \cup \gamma_{i,2}$ where p is the singularity. Then replace γ_i by the two regular slow arcs $\gamma_{i,1}$ and $\gamma_{i,2}$ in order to define a slow-fast path satisfying the requirements of Definition 12.

2. The slow arcs and fast orbits are open in the sense of embedded manifolds without boundary.
3. We use the notation P to stress the difference with a slow-fast cycle Γ , which is a subset of the manifold M . We can associate a subset of M to any slow-fast path P by defining $\Gamma(P) = \cup_i \overline{\gamma_i}$. This subset $\Gamma(P)$ does not need to be homeomorphic to a circle; it can be a figure-eight, or a more complicated closed set. We also stress that the Definition 12 does not require the γ_i to be pairwise disjoint.
4. There are a lot of possibilities to form closed slow-fast paths, for example on a torus, see Fig. 24. Here, the side AB is identified to the side DC; the side BC is identified to the side AD.

As for slow-fast cycles, one can define the notion “slow divergence integral” under the condition that all contact points appearing in the slow-fast segments are regular nilpotent contact points, or nilpotent contact points of singularity index $+1$ and supposing that there are no

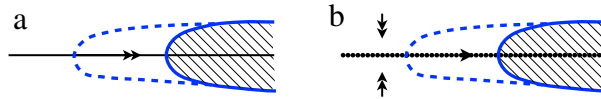


Fig. 25. Extending adapted neighborhoods along fast orbits or slow arcs.

singularities on the slow dynamics (outside contact points), by simply adding the slow divergence integrals along each slow arc appearing in P . We denote the slow divergence integral by $I(P)$. A closed slow–fast path is called attracting (resp. repelling) when $I(P) < 0$ (resp. > 0). A closed slow–fast path is also called attracting (resp. repelling) when it has singularities on the slow dynamics, all located on attracting (resp. repelling) branches.

Near each p_i (p_i being either a contact point, a corner in a hyperbolic fast–slow segment, or a singularity on the slow arcs), one can define families of adapted neighborhoods, the inset cutting γ_i transversely and the outset cutting γ_{i+1} transversely. The following lemma states that with well-chosen such neighborhoods, it is possible to cover the entire slow–fast path.

Lemma 6. *Let $P = (\gamma_1, \dots, \gamma_n)$ be a closed slow–fast path, with $\overline{\gamma_i} \cap \overline{\gamma_{i+1}} = \{p_i\}$ (using the convention $\gamma_{n+1} = \gamma_1$). Let $(V_i)_{i=1, \dots, n}$ be families of adapted neighborhoods near the points p_i . Then, taking (ϵ, λ) close enough to $(0, \lambda_0)$, the adapted neighborhoods can be extended to adapted neighborhoods $(W_i)_{i=1, \dots, n}$ of p_i (with $W_i \supset V_i$) covering the entire set $\Gamma(P)$. More particularly, $\overline{\gamma_{i+1}} \subset W_i \cup W_{i+1}$, with the notational convention $\gamma_{n+1} = \gamma_n$ and $W_{n+1} = W_1$. Moreover, the neighborhoods W_i can be fit inside any a priori neighborhood \mathcal{T} of $\Gamma(P) = \cup_i \overline{\gamma_i}$ that contains the V_i .*

Proof. Given $i \in \{1, \dots, n\}$. Then γ_{i+1} is a slow arc or a fast orbit between the points p_i and p_{i+1} , and of course the inset of V_{i+1} and the outset of V_i intersect γ_i transversely. We aim to cover $\overline{\gamma_{i+1}}$ by $W_i \cup W_{i+1}$, by extending either V_i or V_{i+1} .

When γ_{i+1} is a fast orbit, it is possible to extend V_{i+1} along γ_{i+1} , in the sense that V_{i+1} can be replaced by $W_{i+1} \supset V_{i+1}$, the inset of W_{i+1} cutting γ_{i+1} at a point farther away from p_{i+1} , as shown in Fig. 25(a). We do not give a detailed proof, but it is for sure possible, using local flow box coordinates, to replace the inset by another inset that cuts the fast orbits transversely everywhere, moving the point of intersection with γ_{i+1} .

When γ_{i+1} is an attracting slow arc, it is possible to use the same strategy, replacing the inset of V_{i+1} by a deformed inset, that is transverse to the fast flow everywhere except at the intersection with the slow arc. A well-chosen inset, like the one shown in Fig. 25(b), will also be transverse to the flow of the slow–fast vector field when $\epsilon > 0$, due to the slow dynamics on the slow arc.

When γ_{i+1} is a repelling slow arcs, it is in general not possible to extend the inset of V_{i+1} , but it is possible to extend the outset of V_i (this can be proved simply using a time reversal). \square

Theorem 5. *Consider a slow–fast vector field $X_{\epsilon, \lambda}$ and, for a given $\lambda_0 \in \Lambda$, a closed slow–fast path $P = (\gamma_1, \dots, \gamma_n)$. Let $\Gamma(P) = \cup_i \overline{\gamma_i}$, and let $p_i = \overline{\gamma_i} \cap \overline{\gamma_{i+1}}$, with the convention $\gamma_{n+1} = \gamma_1$. We suppose that*

1. *All the contact points appearing in $\Gamma(P)$ are nilpotent and of finite order. They are regular contact points, or of singularity index $+1$.*
2. *If $\Gamma(P)$ does not contain singularities of the slow dynamics, then $I(P) \neq 0$.*
3. *If $\Gamma(P)$ contains singularities of the slow dynamics, they are all located on hyperbolic arcs of the same type (all attracting or all repelling).*

Under assumptions (1)–(3), there exists an $\epsilon_0 > 0$ and a neighborhood Λ_0 of λ_0 , and families of adapted neighborhoods $(V_i)_{i=1,\dots,n}$ of p_i so that

$$\overline{\gamma_{i+1}} \subset V_i \cup V_{i+1}, \quad \forall i = 1, \dots, n.$$

Moreover, for any fixed $(\epsilon, \lambda) \in]0, \epsilon_0] \times \Lambda_0$, there is at most one closed orbit of $X_{\epsilon, \lambda}$ inside $\cup_i V_i$ that meets the inset of each V_i in increasing order of i as time increases (within one period). If such a closed orbit appears, it is a hyperbolic limit cycle: attracting if P is attracting and repelling if P is repelling.

Remark. If $p_i = p_j$ for some $i < j$, then possibly $V_i = V_j$, but it might also be that $V_i \neq V_j$. In any case, the closed orbits that we consider need to pass through V_i when coming close to p_i and (once more) through V_j when coming close to p_j .

Proof of Theorem 5. Suppose there exist two closed orbits. Given $i \in \{1, \dots, n\}$, and define $V_{n+1} = V_1$. The two closed orbits intersect the inset of V_i and the inset of V_{i+1} ; the parts of the two orbits between the insets, and the part of the insets between the two orbits delimit a flow box. This is done for each i , to show that the two closed orbits delimit an annulus on the manifold M without singular points inside. The remainder of the proof is similar to the proof of [Theorem 1](#).

Remark. In the example depicted in [Fig. 24](#), [Theorem 5](#) can be applied to $P = (\gamma_1, \gamma_4, \gamma_1)$, but also to the slow–fast paths $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_1)$, to $(\gamma_1, \gamma_2, \gamma_3, \gamma_2, \gamma_3, \gamma_4, \gamma_1)$, and to many other slow–fast paths contained in [Fig. 24](#), as long as the relevant divergence integral is nonzero.

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