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Group-cograded multiplier Hopf (*-)algebras

A.T. Abd El-hafez (1), *L. Delvaux* (2) and *A. Van Daele* (3)

Abstract

Let G be a group and assume that $(A_p)_{p\in G}$ is a family of algebras with identity. We have a *Hopf G-coalgebra* (in the sense of Turaev) if, for each pair $p, q \in G$, there is given a unital homomorphism $\Delta_{p,q} : A_{pq} \to A_p \otimes A_q$ satisfying certain properties.

Consider now the direct sum A of these algebras. It is an algebra, without identity, except when G is a finite group, but the product is non-degenerate. The maps $\Delta_{p,q}$ can be used to define a coproduct Δ on A and the conditions imposed on these maps give that (A, Δ) is a multiplier Hopf algebra. It is G-cograded as explained in this paper.

We study these so-called *group-cograded multiplier Hopf algebras*. They are, as explained above, more general than the Hopf group-coalgebras as introduced by Turaev. Moreover, our point of view makes it possible to use results and techniques from the theory of multiplier Hopf algebras in the study of Hopf group-coalgebras (and generalizations).

In a separate paper, we treat the quantum double in this context and we recover, in a simple and natural way (and generalize) results obtained by Zunino. In this paper, we study integrals, in general and in the case where the components are finite-dimensional. Using these ideas, we obtain most of the results of Virelizier on this subject and consider them in the framework of multiplier Hopf algebras.

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(1) Department of Mathematics, Mansoura University, Mansoura 35516 (Egypt). E-mail: a_t_amer@yahoo.com

(2) Department of Mathematics, L.U.C., Universiteitslaan, B-3590 Diepenbeek (Belgium). E-mail: Lydia.Delvaux@luc.ac.be

(3) Department of Mathematics, K.U. Leuven, Celestijnenlaan 200B, B-3001 Heverlee (Belgium). E-mail: Alfons.VanDaele@wis.kuleuven.ac.be

0. Introduction

Let G be a (discrete) group. Assume that we have a family of algebras $(A_p)_{p\in G}$ indexed over G. In this paper, we will only consider algebras over $\mathbb{C}$, with or without identity, but with a non-degenerate product. Consider the *direct sum* $\oplus_{p\in G}A_p$ of these algebras and denote it by A. Elements in A are functions a on G so that $a(p)\in A_p$ for all $p\in G$ and $a(p)=0$ except for finitely many p. Pointwise operations make A into an algebra. It will not have an identity, except when all the components have an identity and when the group G is finite. However, the product will still be non-degenerate. Therefore, we can consider the multiplier algebra $M(A)$ of A (see [VD1, Definition A.1] for a precise definition). In this case, $M(A)$ will consist of *all* functions a on G with $a(p)\in M(A_p)$ for all $p\in G$. Recall that $M(A_p)=A_p$ if A_p has an identity. So, if all components have an identity, elements in $M(A)$ are functions a on G such that $a(p)\in A_p$ for all p, without further restrictions.

We will also use $A\otimes A$ and its multiplier algebra $M(A\otimes A)$. Elements in $A\otimes A$ are functions a on $G\times G$ such that $a(p,q)\in A_p\otimes A_q$ for all $p,q\in G$ and such that a has finite support. On the other hand, elements in $M(A\otimes A)$ are functions a on $G\times G$ such that $a(p,q)\in M(A_p\otimes A_q)$ for all p,q, without further restrictions.

In this paper, we study *G-cograded multiplier Hopf (*-)algebras*. This notion will be introduced in Section 1 (cf. Definition 1.1). A multiplier Hopf algebra (A,Δ) is called G-cograded if $A=\oplus_{p\in G}A_p$ as above and

$$\Delta(A_{pq})(1\otimes A_q)=A_p\otimes A_q$$
$$(A_p\otimes 1)\Delta(A_{pq})=A_p\otimes A_q$$

for all $p,q\in G$. Observe that we consider the algebras A_p as sitting inside A in the obvious way. When we have a multiplier Hopf *-algebra, we require the components also to be *-subalgebras.

A trivial example is obtained when all algebras A_p are just $\mathbb{C}$. In this case A is $K(G)$, the algebra of complex functions with finite support on G (with pointwise sum and product). The product in G gives rise to a coproduct Δ_G on $K(G)$ defined by $(\Delta_G(f))(p,q)=f(pq)$ where $f\in K(G)$ and $p,q\in G$. Notice that Δ_G maps $K(G)$ into the multiplier algebra $M(K(G)\otimes K(G))$ where first $K(G)\otimes K(G)$ is identified with $K(G\times G)$ and then $M(K(G)\otimes K(G))$ is identified with the algebra of all complex functions on $G\times G$. With this coproduct, $K(G)$ is a multiplier Hopf algebra. It is trivially G-cograded. In fact, it is a G-cograded multiplier Hopf *-algebra if we take the complex conjugate of a function as the involution.

Non-trival examples will be given (see e.g. Example 1.6), but the above example is important because of the following result (Theorem 1.2). It is shown that a multiplier Hopf algebra (A,Δ) is G-cograded if and only if there is a non-degenerate injective homomorphism $\gamma: K(G)\to M(A)$ such that γ actually maps into the center of $M(A)$ and moreover $\Delta\circ\gamma=(\gamma\otimes\gamma)\circ\Delta_G$. Here, Δ_G is the coproduct on $K(G)$ as defined in the preceding paragraph. We also need the unique extension of $\gamma\otimes\gamma$ from $K(G)\otimes K(G)$ to a homomorphism on $M(K(G)\otimes K(G))$ (which exists because γ is assumed to be non-degenerate - see e.g. [VD1, Proposition A.5], and later in this introduction). The proof of this result is not very hard. Given a G-cograded multiplier Hopf algebra (A, Δ), the map γ is given by $(\gamma(f))(p) = f(p)1_p$ where 1_p is the identity in the algebra $M(A_p)$. Conversely, given such a map γ, the algebra A_p is defined by $\gamma(f_p)A$ where f_p is the function with value 1 on p and 0 everywhere else. In the case of a *-algebra, the imbedding γ must be a *-homomorphism.

As a special case, we get the Hopf group-coalgebras as introduced by Turaev in [T, Section 11] and studied further by Virelizier in [V] and others ([Z], [W1] and [W2]). The relation is as follows (see Theorem 1.5). As in [T], let G be a group and let $(A_p)_{p\in G}$ be a family of algebras. Now it is assumed that all these algebras have an identity. Given are also unital homomorphisms $\Delta_{p,q} : A_{pq} \to A_p \otimes A_q$ for all $p, q \in G$. It is assumed that the family is coassociative in the sense that

$$(\Delta_{p,q} \otimes \iota)\Delta_{pq,r} = (\iota \otimes \Delta_{q,r})\Delta_{p,qr}$$

on A_{pqr} for all triples $p, q, r \in G$. We use ι to denote the identity map on all A_p. If there is also a *counit* and an *antipode* (see Theorem 1.5 for a precise definiton), this system is called a Hopf G-coalgebra (see [T, Section 11]).

Now assume that we have such a Hopf G-coalgebra. Let A be the direct sum of the algebras $(A_p)_{p\in G}$ as before. Define $\Delta : A \to M(A \otimes A)$ by

$$(\Delta(a))(p, q) = \Delta_{p,q}(a(pq))$$

when $a \in A$ for all $p, q \in G$. Recall that $a(pq) \in A_{pq}$ and that $\Delta_{p,q} : A_{pq} \to A_p \otimes A_q$ so that the right hand side of the above equation indeed belongs to $A_p \otimes A_q$ and therefore, $\Delta(a)$ is really defined in $M(A \otimes A)$ by the above formula. It is easy to see that Δ is a non-degenerate homomorphism from A in $M(A\otimes A)$. It is coassociative because the family $(\Delta_{p,q})_{p,q}$ is assumed to be coassociative.

It will be shown in Theorem 1.5 that (A, Δ), as defined above, is actually a multiplier Hopf algebra (in the sense of [VD1]). This is so because of the existence of the counit and the antipode above and it is proven in a more or less standard way. It is also shown that the original direct sum decomposition makes it into a G-cograded multiplier Hopf algebra. In general however, not every G-cograded multiplier Hopf algebra A comes from a Hopf G-coalgebra. For this to happen, it is necessary and sufficient that A is regular and that all the components are algebras with identity. All this is proven in Theorem 1.5.

We see that, roughly speaking, the Hopf G-coalgebras, introduced by Turaev and studied further by others, are in fact nothing else but multiplier Hopf algebras (A, Δ) with a G-cograding, or equvalently, with a distinguished embedding of $K(G)$ into the center of $M(A)$.

This point of view is important. It allows the results and techniques from the theory of multiplier Hopf algebras to be used in the study of Hopf group-coalgebras (and its generalizations).

In a separate paper ([D-VD]), we follow these ideas to study the quantum double for Hopf group-coalgebras. We generalize the setting, considered in the paper by Zunino [Z], and construct a family of associated multiplier Hopf algebras. One of them is the usual Drinfel'd double, the other one is the double constructed in [Z].

In this paper however, we mainly study integrals on group-cograded multiplier Hopf algebras and we obtain our results by applying known result about multiplier Hopf algebras in general. We obtain most of the results by Virelizier on this subject ([V]), in a simpler, natural way (and in greater generality).

Let us briefly summarize the *content* of the paper.

In *Section 1* we introduce the notion of a group-cograded multiplier Hopf (*-)algebra (as explained earlier in this introduction). We focus on the relation with the Hopf group-coalgebras, introduced by Turaev. In *Section 2*, we apply the known results about integrals on multiplier Hopf algebras to our group-cograded multiplier Hopf algebras. This is a small section because the results are quite obvious and easy to obtain. Finally, in the last section, *Section 3*, we fully make use of known techniques in the theory of multiplier Hopf algebras, more precisely about multiplier Hopf algebras of discrete type, to construct integrals on a group-cograded multiplier Hopf algebra when the components are all finite-dimensional. We recover results obtained by Virelizier [V]. We also give an explicit formula for the integrals.

The main material, needed for reading this paper, is found in the basic references on (regular) multiplier Hopf (*-)algebras ([VD1], [VD-Z2]) and multiplier Hopf algebras with integrals [VD4]. We will freely use notions and results of these papers. However, if convenient, we will recall the main concepts so as to make the paper self-contained to a certain extend. We will also use the Sweedler notation in the case of multiplier Hopf algebras. It is known that this is justified and often, it makes formulas and arguments more easy to understand. For the general results on Hopf algebras, we refer to the standard works of Abe [A] and Sweedler [S].

Finally, let us recall some of the *conventions* and *notations* that will be used. We use 1 for the identity in various algebras while we use e for the identity element in a group. The symbol ι will be reserved for the identity map, usually here on algebras and subalgebras. We will not give these symbols an index to indicate e.g. in what algebra we have the identity, except when it is really necessary to avoid confusion. Usually however, things should be clear from the context. Similarly, we will always use Δ to denote a comultiplication.

Because this plays an important role, we will say something more about non-degenerate homomorphisms and their extensions. Take two algebras A and B, both with or without identity, but wit a non-degenerate product. Consider their multiplier algebras $M(A)$ and $M(B)$ respectively. Let $\alpha : A \to M(B)$ be a homomorphism (i.e. an algebra map). It is called non-degenerate if $\alpha(A)B = B\alpha(A) = B$. Then, it is possible to extend α to a unital homomorphism from $M(A)$ to $M(B)$. This extension is uniquely defined and therefore, it makes sense to denote also the extension with the same symbol α. This can e.g. be applied to the comultiplication Δ on a multiplier Hopf algebra A. The extension is a unital homomorphism from $M(A)$ to $M(A \otimes A)$. It can also be applied to the counit. Then the extension becomes a homomorphism from $M(A)$ to $\mathbb{C}$. Finally, also the antipode S of a (possibly non-regular) multiplier Hopf algebra can be extended to a map from $M(A)$ to itself. In this case one has to apply the general result with $B = A$ as a vector space, but endowed with the opposite multiplication so as to make $S : A \to M(B)$ a homomorphism. That the antipode S is non-degenerate follows easily from the formulas, characterizing the antipode, given by

$$m((S \otimes \iota)(\Delta(a)(1 \otimes b))) = \varepsilon(a)b$$
$$m((\iota \otimes S)((b \otimes 1)\Delta(a))) = \varepsilon(a)b$$

for $a, b \in A$. In these formulas, we have used m to denote the multiplication in A, as a map from $A \otimes A$ to A, and extended to a map on $M(A) \otimes A$ and on $A \otimes M(A)$. For properties of non-degenerate homomorphisms and extensions, we refer to the appendix of [VD1].

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1. Group-cograded multiplier Hopf (*-)algebras

Let G be any (discrete) group. We now introduce the notion of a G-cograded multiplier Hopf (*-)algebra. It is the main object in this paper.

1.1 Definition Let (A, Δ) be a multiplier Hopf algebra. Assume that there is a family of (non-trivial) subalgebras $(A_p)_{p\in G}$ of A so that
i) $A = \sum_{p\in G} \oplus_{p\in G} A_p$ with $A_p A_q = 0$ whenever $p, q \in G$ and $p \neq q$,
ii) $\Delta(A_{pq})(1 \otimes A_q) = A_p \otimes A_q$ and $(A_p \otimes 1)\Delta(A_{pq}) = A_p \otimes A_q$ for all $p, q \in G$.

Then we call (A, Δ) a G-*cograded* multiplier Hopf algebra.

If (A, Δ) is a multiplier Hopf *-algebra, we will assume that all these subalgebras are *-subalgebras and call it a G-cograded multiplier Hopf *-algebra. Examples of G-cograded multiplier Hopf algebras will be given later in this section (see Example 1.6).

In the following theorem we characterize a G-cograded multiplier Hopf algebra by using the multiplier Hopf algebra $(K(G), \Delta)$ as defined in the introduction.

1.2 Theorem A multiplier Hopf algebra (A, Δ) is G-cograded if and only if there is a non-degenerate algebra embedding $\gamma : K(G) \to M(A)$ satisfying
i) $\gamma(K(G))$ belongs to the centre $Z(M(A))$ of $M(A)$,
ii) γ is also a coalgebra map, that is, $\Delta(\gamma(f)) = (\gamma \otimes \gamma)(\Delta(f))$ for all $f \in K(G)$.

In the case of a G-cograded multiplier Hopf *-algebra, we have that γ is a *-map.

Proof: First assume that A is a G-cograded multiplier Hopf algebra. We write $A = \oplus_{p\in G} A_p$ as in Definition 1.1. It is not hard to see that $M(A) = \prod_{p\in G} M(A_p)$.

Define a map $\gamma : K(G) \to M(A)$ by $(\gamma(f))(p) = f(p)1_p$ for any $p \in G$ where 1_p is the identity in $M(A_p)$. It is easy to see that γ is a homomorphism and that it maps into the centre of $M(A)$.

To prove that it is non-degenerate, take any $a \in A$. Because A is the direct sum of the algebras A_p, we have a finite subset F of G such that $a(p) = 0$ for $p \notin F$. If $f \in K(G)$ is such that $f(p) = 1$ when $p \in F$, we have that $\gamma(f)a = a$. Therefore, γ is a non-degenerate homomorphism from $K(G)$ to $M(A)$.

For any $f \in K(G)$ and $p, q \in G$ we get

$$((\gamma \otimes \gamma)(\Delta(f)))(p, q) = (\Delta(f))(p, q)(1_p \otimes 1_q) = f(pq)\Delta(1_{pq})$$

and we see that $\Delta(\gamma(f)) = (\gamma \otimes \gamma)(\Delta(f))$. This completes the proof of one direction.

Conversely, suppose that we have a multiplier Hopf algebra (A, Δ) with a non-degenerate homomorphism $\gamma : K(G) \to M(A)$ satisfying the conditions of the theorem. Define the subalgebra A_p by $\gamma(f_p)A$ where $p \in G$ and f_p is the function on G satisfying $f_p(p) = 1$ and $f_p(q) = 0$ when $q \neq p$. Because γ maps into the centre, A_p will be a subalgebra of A. It is non-trivial for each p because γ is assumed to be injective. The product in each subalgebra is still non-degenerate. And because γ is assumed to be non-degenerate, we have $A = \gamma(K(G))A$ and it follows that $A = \oplus_{p \in G} A_p$. Clearly also $A_p A_q = 0$ if $p \neq q$. The identity 1_p in $M(A_p)$ is $\gamma(f_p)$.

From the fact that γ respects the comultiplications in the sense that $\Delta(\gamma(f)) = (\gamma \otimes \gamma)(\Delta(f))$ for all $f \in K(G)$, we get

$$\begin{aligned} \Delta(A_{pq})(1 \otimes A_q) &= \Delta(\gamma(f_{pq})A)(1 \otimes \gamma(f_q)A)) \\ &= (\gamma \otimes \gamma)(\Delta(f_{pq})(1 \otimes f_q))(\Delta(A)(1 \otimes A)) \\ &= ((\gamma \otimes \gamma)(f_p \otimes f_q))(A \otimes A) = A_p \otimes A_q \end{aligned}$$

for all $p, q \in G$. Similarly, the other equality is proven. This completes the proof of the other implication in the theorem.

We will use this charaterization to look at the counit and the antipode of a group-cograded multiplier Hopf algebra. First we prove the following lemma. We use, as explained already in the introduction, extensions to the multiplier algebra of non-degenerate homomorphisms and use the same symbol to denote these extension.

1.3 Lemma Let A and B be multiplier Hopf algebras. Assume that $\alpha : A \to M(B)$ is a non-degenerate homomorphism that respects the comultiplication in the sense that $\Delta(\alpha(a)) = (\alpha \otimes \alpha)\Delta(a)$ for all $a \in A$. Then α preserves the unit, the counit and the antipode, that is,

$$\alpha(1) = 1 \qquad \varepsilon(\alpha(a)) = \varepsilon(a) \qquad S(\alpha(a)) = \alpha(S(a))$$

for all $a \in A$.

Proof: As α is a non-degenerate homomorphism from $A \to M(B)$, the extension is a unital homomorphism from $M(A)$ to $M(B)$ as we have said already in the introduction.

To show that α respects the counits, take any $a, a' \in A$ and $b \in B$. We can apply the counit in B on the first leg of the equation

$$\Delta(\alpha(a))(1 \otimes \alpha(a')b) = ((\alpha \otimes \alpha)(\Delta(a)(1 \otimes a')))(1 \otimes b)$$

to get $\alpha(aa')b = \sum_{(a)} \varepsilon(\alpha(a_{(1)}))\alpha(a_{(2)}a')b$. On the other hand, we also have $aa' = \sum_{(a)} \varepsilon(a_{(1)})a_{(2)}a'$. Then we use the fact that $\Delta(A)(1 \otimes A) = A \otimes A$ and we can conclude that $\varepsilon(c)\alpha(c')b = \varepsilon(\alpha(c))\alpha(c')b$ for all $c, c' \in A$ and $b \in B$. Now because $\alpha(A)B = B$ we get that $\varepsilon(c) = \varepsilon(\alpha(c))$ for all $c \in A$.

Next we show that α converts one antipode to the other. Again take any $a, a' \in A$ and $b \in B$ and consider the equation

$$\Delta(\alpha(a))(1 \otimes \alpha(a')b) = ((\alpha \otimes \alpha)(\Delta(a)(1 \otimes a')))(1 \otimes b).$$

Now, we apply the map $m \circ (S \otimes \iota)$ on both sides of this equation. Notice that we have elements in $M(B) \otimes B$ and that it is also possible to extend the antipode to $M(B)$ (as we mentioned in the introduction). We get

$$\varepsilon(\alpha(a))\alpha(a')b = \sum_{(a)} S(\alpha(a_{(1)}))\alpha(a_{(2)}a')b.$$

Because $\varepsilon(\alpha(a) = \varepsilon(a)$, we obtain

$$\sum_{(a)} \alpha(S(a_{(1)}))\alpha(a_{(2)}a')b = \sum_{(a)} S(\alpha(a_{(1)}))\alpha(a_{(2)}a')b.$$

Again, because $\Delta(A)(1 \otimes A) = A \otimes A$ we conclude that $\alpha(S(c))\alpha(c')b = S(\alpha(c))\alpha(c')b$ for all $c, c' \in A$ and $b \in B$. Using once more that α is non-degenerate, we get $\alpha(S(c)) = S(\alpha(c))$ for all $c \in A$. This completes the proof.

The reader may notice that the above arguments are very similar to the onces used to show that the counit and the antipode in multiplier Hopf algebras are unique. This is not surprising. It would be a consequence of the result in the lemma that the counit and the antipode is unique. Simply take $A = B$ with the identity map for α.

Remark that we do not need the image of α to belong to the center for the above lemma.

It is now easy to obtain the following result.

1.4 Proposition Let A be a G-cograded multiplier Hopf algebra with decomposition $A = \oplus_{p \in G} A_p$ as in Definition 1.1. Then $\varepsilon(A_p) = 0$ when $p \neq e$ and $S(A_p) \subseteq M(A_{p^{-1}})$ for all p. If the multiplier Hopf algebra is regular, we have $S(A_p) = A_{p^{-1}}$.

Proof: By Theorem 1.2 there is a non-degenerate central imbedding $\gamma : K(G) \to M(A)$ which respects the comultiplications. So Lemma 1.3 applies and the result follows from the fact that the counit and the antipode on $K(G)$ are given by the formulas $\varepsilon(f) = f(e)$ and $(S(f))(p) = f(p^{-1})$ for $f \in K(G)$ and $p \in G$.

In the following theorem, we compare this notion with the Hopf group-coalgebras as introduced by Turaev in [T, Section 11]. It shows that, roughly speaking, Hopf group-coalgebras are multiplier Hopf algebras with some extra structure. This observation throws a new light on the notion introduced by Turaev. We will see some of the consequences in the rest of the paper (see Section 2 and 3). We also refer to our work on the quantum double in this context [D-VD].

1.5 Theorem Let G be a group. Assume that for all $p \in G$, we have given an algebra A_p, with or without identity, but with a non-degenerate product. Moreover, we require the following:

i) For all $p, q \in G$ we have a homomorphism $\Delta_{p,q} : A_{pq} \to M(A_p \otimes A_q)$ satisfying $\Delta_{p,q}(A_{pq})(1 \otimes A_q) \subseteq A_p \otimes A_q$ and $(A_p \otimes 1)\Delta_{p,q}(A_{pq}) \subseteq A_p \otimes A_q$, as well as $(1 \otimes A_q)\Delta_{p,q}(A_{pq}) \subseteq A_p \otimes A_q$ and $\Delta_{p,q}(A_{pq})(A_p \otimes 1) \subseteq A_p \otimes A_q$. Furthermore, it is assumed that these maps form a 'coassociative' family in the sense that for all $p, q, r \in G$ we have

$$(c \otimes 1 \otimes 1)((\Delta_{p,q} \otimes \iota)(\Delta_{pq,r}(a)(1 \otimes b))) = ((\iota \otimes \Delta_{q,r})((c \otimes 1)\Delta_{p,qr}(a)))(1 \otimes 1 \otimes b)$$

whenever $a \in A_{pqr}$, $b \in A_r$ and $c \in A_p$.

ii) There is a homomorphism $\varepsilon_e : A_e \to \mathbb{C}$ so that for all $p \in G$ we get

$$(\iota \otimes \varepsilon_e)((a \otimes 1)\Delta_{p,e}(b)) = ab$$
$$(\varepsilon_e \otimes \iota)(\Delta_{e,p}(a)(1 \otimes b)) = ab$$

whenever $a, b \in A_p$.

iii) For all $p \in G$, there is a anti-isomorphism $S : A_p \to A_{p^{-1}}$ so that

$$m(S_{p^{-1}} \otimes \iota)(\Delta_{p^{-1},p}(a)(1 \otimes b)) = \varepsilon_e(a)b$$
$$m(\iota \otimes S_{p^{-1}})((b \otimes 1)\Delta_{p,p^{-1}}(a)) = \varepsilon_e(a)b$$

for all $p \in A_e$ and $b \in A_p$

Under these assumptions, the algebra A, defined as $\oplus_{p \in G} A_p$, can be given a comultiplication in a natural way, making it into a regular multiplier Hopf algebra.

Conversely, assume that we have a regular G-cograded multiplier Hopf algebra A with decomposition $A = \oplus_{p \in G} A_p$ as in Definition 1.1. Then, the family of subalgebras $(A_p)_{p \in G}$ can be endowed (in a natural way) with the above objects and they will satisfy the above conditions i), ii) and iii).

Proof: First, assume that we have such a family of algebras, provided with these data satisfying the condition i), ii) and iii) above. Define the algebra $A = \oplus_{p \in G} A_p$

as before. Recall that this is an algebra with non-degenerate product, that hence we can consider the multiplier algebra $M(A)$ and that this is given by the product $\prod_{p\in G} M(A_p)$. Also $A\otimes A = \oplus_{p,q}(A_p \otimes A_q)$ while $M(A\otimes A) = \prod_{p,q} M(A_p\otimes A_q)$. We can define a map $\Delta : A \to M(A\otimes A)$ by

$$(\Delta(a))(p,q) = \Delta_{p,q}(a(pq))$$

whenever $p,q \in G$. Because $a(pq) \in A_{pq}$ and $\Delta_{p,q}$ maps $A_{p,q}$ into $M(A_p \otimes A_q)$, this map is well-defined.

Because of all the conditions on these maps $\Delta_{p,q}$, it follows easily that $\Delta(A)(1\otimes A)$ and $(1\otimes A)\Delta(A)$, as well as $\Delta(A)(A\otimes 1)$ and $(1\otimes A)\Delta(A)$, all actually are subsets of $A\otimes A$. Because the maps $\Delta_{p,q}$ are assumed to be homomorphisms, the same is true for Δ. Finally, coassociativity of Δ follows by the 'coassociativity' of the family $(\Delta_{p,q})_{p,q}$.

We will now use Proposition 2.9 from [VD4] to show that the pair (A,Δ) is indeed a (regular) multiplier Hopf algebra. To use this proposition, we have to define the counit and the antipode on A. We first define ε on A by $\varepsilon(a) = \varepsilon_e(a(e))$. It is straigthforward to verify that it is a homomorphism and satisfies the requirements of a counit. The antipode S is defined on A by $(S(a))(p^{-1}) = S_p(a(p))$ for all $p \in G$. Again it is straightforward to show that this is a anti-homomorphism from A to itself and that it satisfies the requirements of an antipode. Hence, by Proposition 2.9 of [VD4], we do have a regular multiplier Hopf algebra. It is clearly G-cograded in the sense of Definition 1.1.

Conversely, now assume that we have a G-cograded multiplier Hopf algebra A with $A = \oplus_{p\in G} A_p$ as in the definition. We now define $\Delta_{p,q} : A_{pq} \to M(A_p \otimes A_q)$ by the restricting Δ to A_{pq}. These maps will satisfy the requirements. Moreover, because of Theorem 1.2 and Lemma 1.3, we must have that the counit ε and antipode S satifsy $\varepsilon(a) = 0$ if $a \in A_p$ and $p \neq e$, as well as $S(a) \in A_{p^{-1}}$ when $a \in A_p$. Then we can define ε_e on A_e simply by restricting ε to A_e and we can define S_p on A_p also by taking the restriction of S. Again it is all straightforward to verify that the assumptions in i) and ii) are satisfied.

This proves the result.

As a special case of the above theorem, consider a Hopf group-coalgebra as defined by Turaev in [T]. This is exactly the situation of Theorem 1.5 with all the components unital algebras.

Before we pass to examples, we would like to make some comment on the notion of regularity and why it is needed for the above result. Recall first that a multiplier Hopf algebra (A,Δ) is called regular if also (A,Δ^{op}) is a multiplier Hopf algebra (see Definition 2.3 in [VD1]). This is exactly the case when the antipode S maps A into itself (and not only in $M(A)$ as it does in general) and when it is bijective. This is used in the previous proof because we refer to Proposition 2.9 in [VD4] to show that we actually get a multiplier Hopf algebra. Probably, it is possible to formulate some similar result that holds also when we do not have regularity.

On the other hand, for the Hopf group-coalgebras of Turaev, we know that the antipodes are assumed to be bijections. And also, in the case of a Hopf *-algebra, regularity is automatic (see again [VD1]).

We finish this section by looking at some examples.

We have already considered the basic but trivial example where the multiplier Hopf algebra is $K(G)$, the algebra of functions with finite support on a group G, and all the components just the trivial algebra $\mathbb{C}$. It is not hard to use this to construct other, still rather trivial examples. Indeed, take any multplier Hopf algebra A_0 and any group G. Now let $A = K(G) \otimes A_0$ with the tensor product algebra structure and the tensor product coalgebra structure. So, $(f \otimes a)(g \otimes b) = fg \otimes ab$ when $f, g \in K(G)$ and $a, b \in A_0$. Also $\Delta(f \otimes a) = (\iota \otimes \sigma \otimes \iota)(\Delta(f) \otimes \Delta(a))$ where $f \in K(G)$ and $a \in A_0$ and where σ is the flip from $K(G) \otimes A_0$ to $A_0 \otimes K(G)$. In this case, the imbedding $\gamma : K(G) \to M(A)$ is of course given by $\gamma(f) = f \otimes 1$ where 1 is the identity in $M(A_0)$. All the components are A_0 and all the maps $\Delta_{p,q}$ are simply the comultiplication on A_0.

A non-trivial example is obtained from the above by deforming the comultiplication with the help of an action of the group on A. This gives the following non-trivial example. It is known as a multiplier Hopf algebra (see e.g. Example 3.3 in [D]), but we present it here as an example of a group-cograded multiplier Hopf algebra.

1.6 Example Let (A_0, Δ_0) be a multiplier Hopf algebra. Let G be a group and assume that G acts on the multiplier Hopf algebra A_0 in the following sense. For each $p \in G$, we have an automorphism α_p of the algebra A_0 such that also $\Delta_0(\alpha_p(a)) = (\alpha_p \otimes \alpha_p)\Delta_0(a)$ for all $a \in A_0$ and all $p \in G$. We also assume that $\alpha_e = \iota$ and $\alpha_p(\alpha_q(a)) = \alpha_{pq}(a)$ for all $a \in A_0$ and $p, q \in G$.

Now, we consider again the algebra A, defined as $K(G) \otimes A_0$ with the tensor product algebra structure. We will consider elements in A as functions on G with finite support and values in A_0. This algebra is made into a multiplier Hopf algebra by defining the coproduct Δ by

$$(\Delta(a))(p, q) = (\alpha_q \otimes \iota)(\Delta_0(a(pq)))$$

where $a \in A$ and $p, q \in G$. It is straightforward to show that this is a multiplier Hopf algebra. The counit ε on A is given by $\varepsilon(a) = \varepsilon_0(a(e))$ where ε_0 is the counit on A_0. The antipode S on A is given by $S(a)(p) = S_0(\alpha_p(a(p^{-1})))$. If the action is trivial, we get the tensor product coalgebra structure as discussed above.

Now, we have the imbedding $\gamma : K(G) \to M(A)$ given by $\gamma(f) = f \otimes 1$ where again 1 stands for the identity in $M(A_0)$. Of course, this is a central imbedding and it is trivial to verify that it is compatible with the comultiplications. So, we have a G-cograded multiplier Hopf algebra. Again, all the components are the same, namely A_0. However, not all the maps $\Delta_{p,q}$ are equal. We have $\Delta_{p,q} = (\alpha_q \otimes \iota)\Delta_0$ for all $p, q \in G$.

It is not hard to find more concrete examples using the above construction. Take e.g. another group H and assume that G acts on H by means of automorphisms ρ_p, with

$p \in G$. Take $A_0 = K(H)$ with its comultiplication coming from the product in the group H. Then an action α of G on A_0 is given by the formula $(\alpha_p(f))(h) = f(\rho_{p^{-1}}(h))$ for $f \in K(H)$ and $h \in H$ and $p \in G$. It is not hard to see that we get the algebra of complex functions with finite support on the semi-direct product of G with H.

One can still make it more concrete if we let $H = G$ and take the adjoint action ρ of G on itself given by $\rho_p(q) = pqp^{-1}$ when $p, q \in G$. Then A is $K(G \times G)$ while the coproduct is given by

$$(\Delta(f))(p, h, q, k) = f(pq, q^{-1}hqk)$$

whenever $p, q, k, h \in G$. The embedding γ is now given by $(\gamma(f))(p, h) = f(p)$.

It is also possible to give examples of related situations. Consider e.g. the discrete quantum group $su_q(2)$, the Pontryagin dual of the compact quantum group $SU_q(2)$. As an algebra, it is the direct sum A of the matrix algebras $M_n(\mathbb{C})$ where $n = 1, 2, 3, \ldots$. The comultiplication is rather complicated on this level. Inside the multiplier algebra $M(A)$, we have the elements k, e, f satisfying the commutation rules $ke = \lambda ek$, $kf = \lambda^{-1} fk$ and $ef - fe = k^2 - k^{-2}$. Here λ is the deformation parameter and $0 < \lambda < 1$. It is possible to construct an imbedding γ of $K(\mathbb{Z})$ into the multiplier algebra $M(A)$ of this discrete quantum group. It sends the delta function f_p, defined as 1 in p and 0 anywhere else, to the spectral projection of k associated with the eigenvalue $\lambda^{\frac{1}{2}p}$. This imbedding respects the comultiplication. However, it is not central as k is not a central element in A.

This example shows that more general objects than the group-cograded multiplier Hopf algebras are also of interest. The requirement that $K(G)$ is imbedded in the center of $M(A)$ may be somewhat restrictive. In [L-VD], related objects are studied without this restriction so that the above example (with the discrete quantum group $su_q(2)$) fits into the theory.

2. Integrals on group-cograded multiplier Hopf (*-)algebras

In this section, we consider a regular multiplier Hopf algebra (A, Δ). Regular multiplier Hopf algebras with integrals are studied in [VD4]. We recall some of the results.

First observe that for any linear functional φ on A and any element $a \in A$, we can define a multiplier $(\iota \otimes \varphi)\Delta(a)$ in $M(A)$ by the formulas

$$((\iota \otimes \varphi)\Delta(a))b = (\iota \otimes \varphi)(\Delta(a)(b \otimes 1))$$
$$b((\iota \otimes \varphi)\Delta(a)) = (\iota \otimes \varphi)((b \otimes 1)\Delta(a))$$

where $b \in A$. Because we assume our multiplier Hopf algebra to be regular, we have not only $(b \otimes 1)\Delta(a) \subseteq A \otimes A$ but also $\Delta(a)(b \otimes 1) \subseteq A \otimes A$ and so the above formulas make sense.

2.1 Definition A linear functional φ on A is called *left invariant* if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ in $M(A)$ for all $a \in A$. A non-zero left invariant functional is called a *left integral* on

A. Similarly, a non-zero linear function ψ satisfying $(\psi\otimes\iota)\Delta(a) = \psi(a)1$ for all $a \in A$ is called a *right integral* on A.

There are various data (with many relations among them) associated with left and right integrals. We recall the following definition and results from [VD4].

2.2 Theorem Let (A, Δ) be a regular multiplier Hopf algebra with a left integral φ. Any other left integral is a scalar multiple of φ. There exists also a right integral, unique up to a scalar, given by $\varphi \circ S$. There is a scalar ν (the *scaling constant*) in $\mathbb{C}$ defined by $\varphi \circ S^2 = \nu\varphi$. There is a grouplike multiplier δ in $M(A)$ (the *modular element*) such that $(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta$ for all $a \in A$. The integral is faithful in the sense that for any $a \in A$ we have $a = 0$ if either $\varphi(ab) = 0$ for all $b \in A$ or $\varphi(ba) = 0$ for all b. Also the right integral is faithful. Finally, there are automorphisms σ and σ' (the *modular automorphims*) satisfying $\varphi(ab) = \varphi(b\sigma(a))$ and $\psi(ab) = \psi(b\sigma'(a))$ for all $a, b \in A$.

2.3 Definition As before, let (A, Δ) be a regular multiplier Hopf algebra with a left integral φ. Denote by $\hat{A}$ the space of linear functionals on A of the form $x \mapsto \varphi(xa)$ for some $a \in A$. Then $\hat{A}$ is made into a regular multiplier Hopf algebra with the product and coproduct in $\hat{A}$ dual to the coproduct and product resp. in A. This is called the *dual* of A.

It is again a multiplier Hopf algebra with integrals. Moreover, the dual of $\hat{A}$ is canonically isomorphic with the original multiplier Hopf algebra. This is some form of the Pontryagin duality for multiplier Hopf algebras with integrals. It is a generalization of the duality theorem for finite-dimensional Hopf algebras.

We have two important special cases. First, we have the multiplier Hopf algebras of *compact type*. These are nothing else but Hopf algebras with integrals. The other ones are the multiplier Hopf algebras of *discrete type*. A multiplier Hopf algebra is called of discrete type if it has a co-integral. A *left co-integral* is a non-zero element $h \in A$ satisfying $ah = \varepsilon(a)h$ for all $a \in A$.

In [VD-Z1, Theorem 2.10], it is proven that there always exists an integral on a multiplier Hopf algebra of discrete type. Moreover, the dual $\hat{A}$ of a multiplier Hopf algebra of discrete type is of compact type, that is a Hopf algebra (with integrals); see again Proposition 2.1 in [VD-Z1]. We will use some of these results in the next section.

We now will apply the above results about integrals to regular group-cograded multiplier Hopf algebras as they were introduced in the previous section.

Let G be a group and (A, Δ) a regular G-cograded multiplier Hopf algebra with decomposition $A = \oplus_{p\in G} A_p$. The (full) linear dual A' of A is given by $\prod_{p\in G}(A_p)'$. Here elements are given by functions f on G such that $f(p) \in A'_p$ for all $p \in G$ (with no restrictions on the support). We will write f_p in stead of $f(p)$. The multiplier algebra $M(A)$ is given by $\prod_{p\in G} M(A_p)$. Again, when $m \in M(A)$, we will consider the associated elements m_p in $M(A_p)$. Let $\Delta_{p,q}$ be the associated map from A_{pq} to $M(A_p \otimes A_q)$, as given in the proof of Proposition 1.5.

The following results are easy to obtain.

2.4 Proposition Let A be a regular G-cograded multiplier Hopf algebra as before. Let φ be a left integral on A. Then all the components φ_p are faithful. We have $(\iota \otimes \varphi_q)(\Delta_{p,q}(a)) = \varphi_{pq}(a)1_p$ for all $a \in A_{pq}$ and all $p, q \in G$.

2.5 Proposition Let A and φ be as in the previous proposition. For the components (δ_p) of the modular element δ, we have $(\varphi_p \otimes \iota)\Delta_{p,q}(a) = \varphi_{pq}(a)\delta_q$ when $a \in A_{pq}$ and $p, q \in G$. Also $\Delta_{p,q}(\delta_{pq})(b \otimes c) = \delta_p b \otimes \delta_q c$ for all $a \in A_{pq}$, $b \in A_p$, $c \in A_q$ and $p, q \in G$.

There is not much one can say about the scaling constant, except that it is the same for all components. Indeed, as the square of the antipode leaves each component globally invariant, we have trivially $\varphi_p(S^2(a)) = \nu\varphi_p(a)$ whenever $a \in A_p$ and $p \in G$.

Also, when we look at the modular automorphisms, we see that they again must leave the components globally invariant. If we denote by σ_p the restriction of σ to A_p we have $\varphi_p(ab) = \varphi_p(b\sigma_p(a))$ for all $a, b \in A_p$ and all $p \in G$.

In fact, all the relations among the different data associated with a regular multiplier Hopf algebra A with integrals (a so-called algebraic quantum group), like δ, σ, σ' and their dual objects related with the dual $\hat{A}$, will give rise to similar relations among the components.

Let us finish by looking briefly at the example in the previous section (Example 1.6). Suppose that the multiplier Hopf algebra A_0 has a left integral φ_0. By the uniqueness of integrals, we must have a homomorphism $\mu : G \to \mathbb{C}$, given by $\varphi(\alpha_p(a)) = \mu(p)\varphi(a)$ for all $a \in A$ and $p \in G$. The new multiplier Hopf algebra A will also have a left integral φ and it will be given by $\varphi(a) = \sum_{p \in G} \varphi_0(a(p))$ where $a \in A$ and where we consider a as a function on G with finite support and with values in A_0 as in Example 1.6. For the modular element δ in $M(A)$ we get components $\delta_p = \mu(p)\delta_0$ for all p.

3. Group-cograded multiplier Hopf (*-) algebras with finite-dimensional components

In this section, we consider regular group-cograded multiplier Hopf algebras with finite-dimensional components. So, as before, let G be a group and let (A, Δ) be a G-cograded multiplier Hopf algebra. Let $A = \oplus_{p \in G} A_p$ as in the definition and assume now that every subalgebra A_p has a unit and is finite-dimensional. As we have seen in Section 1, we then have a Hopf G-coalgebra. It is called 'finite-dimensional' by Turaev in [T, Section 11].

We will now use results from the general theory of multiplier Hopf algebras to obtain properties in this special case. We will show that in this case, the multiplier Hopf algebra A is of discrete type in the sense of [VD-Z1]. It then follows that there are integrals on A. We recover the results of Virelizier obtained in [V]. Moreover, we give an explicit formula for the left integral φ in terms of a basis and a dual basis in A_p and its dual A'_p for each

$p \in G$. Here, we are also inspired by the results about the integrals on finite-dimensional Hopf algebras as found in [VD3].

We first need to notice that the definition of a Hopf group-coalgebra as given by Virelizier in [V, Definition 1.3] is somewhat more general than the original definition given by Turaev in [T, Definition 11.2]. However, in the finite-dimensional case that we consider here, Virelizier showed that the two notions coincide (see [V, Corollary 3.7]). His proof of this result makes use of the integral on the Hopf group-coalgebra. We will give a more direct proof of this result in Lemma 3.2 below.

For convenience of the reader, we recall the following definition, given by Virelizier (Definition 1.3 in [V]).

3.1 Definition Let G be a group. Consider a family $(A_p)_{p\in G}$ of finite-dimensional unital algebras. Suppose we have the following.
i) For all $p, q \in G$, we have a unital homomorphism $\Delta_{p,q} : A_{pq} \to A_p \otimes A_q$ satisfying 'coassociativity' in the sense that

$$(\Delta_{p,q} \otimes \iota)\Delta_{pq,r} = (\iota \otimes \Delta_{q,r})\Delta_{p,qr}$$

on A_{pqr} for all p, q, r in G.
ii) There is a unital homomorphism $\varepsilon_e : A_e \to \mathbb{C}$ such that for all p

$$(\iota \otimes \varepsilon)\Delta_{p,e}(a) = (\varepsilon \otimes \iota)\Delta_{e,p}(a) = a$$

when $a \in A_p$.
iii) There exists a *linear map* $S_p : A_p \to A_{p^{-1}}$ for all p such that for all $a \in A_e$ we get

$$m(S_{p^{-1}} \otimes \iota)(\Delta_{p^{-1},p}(a)) = \varepsilon_e(a)1$$
$$m(\iota \otimes S_{p^{-1}})(\Delta_{p,p^{-1}}(a)) = \varepsilon_e(a)1$$

for all p (where m denotes multiplication in A_p).

With these conditions, the family $(A_p)_{p\in G}$ is called a 'finite-dimensional' Hopf G-coalgebra (in the sense of Virelizier).

Remark the similarity with the notions in Theorem 1.5 in Section 1. Also observe that this notion is close to the definition given by Turaev, the difference being that here it is not assumed that the maps S_p are anti-isomorphisms. It is just required that they are linear maps. It is well-known in Hopf algebra theory that it is sufficient to assume that the antipode is linear. It follows that it is an anti-homomorphism. This is also the case here. And because the components are finite-dimensional, we even get also bijectivity of the maps S_p. Again, this is also expected from Hopf algebra theory (see [A] and [S]). As mentioned before, we obtain these result directly, in the following lemma.

3.2 Lemma With the notations and assumptions of Definition 3.1, we get that $S_p : A_p \to A_{p^{-1}}$ is a anti-isomorphism for all $p \in G$.

Proof: Denote by A'_p the dual space of A_p for all p. Consider the direct sum $A^* = \oplus_{p\in G} A'_p$ of these dual spaces (and consider the components as subspaces). It is a subspace of the full dual A' of the direct sum $A = \oplus_{p\in G} A_p$. We will now make A^* into a Hopf algebra and show that it has integrals. Then we will use results from usual Hopf algebra theory to prove the lemma.

First we make A^* into a unital algebra. We define the product fg of elements $f \in A'_p$ and $g \in A'_q$ by $(fg)(x) = (f \otimes g)(\Delta_{p,q}(x))$ for $x \in A_{pq}$ and we put fg equal to 0 on other components. So in fact, $fg \in A'_{pq}$. The product is associative because of condition i) in Definition 3.1. The element ε_e in A'_e is an identity for A^* because of condition ii) in Definition 3.1.

Now, we define a coproduct Δ on this algebra A^* by dualizing the product. We let $\Delta : A'_p \to A'_p \otimes A'_p$ be given by $(\Delta(f))(x \otimes y) = f(xy)$ when $x, y \in A_p$. This map is a unital homomorphism and it will be coassociative. A counit ε on A^* is given by $\varepsilon(f) = f(1)$ where 1 is the identity in A_p when $f \in A'_p$. An antipode S on A^* is given by the formula $(S(f))(x) = f(S_{p^{-1}}(x))$ where $x \in A_{p^{-1}}$ and $f \in A'_p$. So, $S : A'_p \to A'_{p^{-1}}$ for all p. That it is an antipode follows from condition iii) in Definition 3.1.

Hence, we have made A^* into a Hopf algebra. Recall that we only need the antipode to be linear. It automatically follows that it is a anti-homomorphism.

It also follows that $\Delta \circ S = (S \otimes S) \circ \Delta^{\mathrm{op}}$ where Δ^{op} is the opposite comultiplication, obtained by applying the flip automorphism (see e.g. [S, Proposition 4.0.1]). From this it easily follows that the map S_p is a anti-homomorphism from A_p to $A_{p^{-1}}$ for all p.

It remains to show that the maps S_p are bijections.

To prove this, first observe that $(A_e, \Delta_{e,e})$ is a finite-dimensional Hopf algebra. This is an easy consequence of the conditions in Definition 3.1. We know that there must be a left cointegral h in A_e. This is an element $h \in A_e$ satisfying $ah = \varepsilon_e(a)h$ for all $a \in A_e$. We will now use this element to get an integral on A^*. We define $\varphi(f) = f(h)$ for $f \in A_e$ and $\varphi(f) = 0$ on other components. This defines a linear functional on A^*. It is not hard to show that it is a left integral.

It is known that the antipode in a Hopf algebra with integrals must be injective (see e.g. [S, Corollary 5.1.7]). In particular, the maps $S_p : A'_p \to A'_{p^{-1}}$ will be injective. And because this also is the case for $S_{p^{-1}} : A'_{p^{-1}} \to A'_p$, and the spaces are finite-dimensional, we must have that A'_p and $A'_{p^{-1}}$ have the same dimensions and that these maps are bijections. Then also the adjoint maps $S_p : A_p \to A_{p^{-1}}$ will be bijections.

This completes the proof of the lemma.

It is an immediate consequence of this lemma that the direct sum A of these algebras is a regular G-cograded multiplier Hopf algebra because the assumptions of Theorem 1.5 are now fulfilled.

So, in the rest of this section, we consider a regular G-cograded multiplier Hopf algebra A with finite-dimensional unital components $(A_p)_{p\in G}$.

We get the following result.

3.3 Proposition Assume that (A, Δ) is a regular G-cograded multiplier Hopf algebra with finite-dimensional unital components. Then A is a multiplier Hopf algebra of discrete type.

Proof: Recall from Theorem 1.5 that the family of components is provided with these homomorphims $\Delta_{p,q} : A_{pq} \to A_p \otimes A_q$, just like in the Definition 3.1 above. Again $(A_e, \Delta_{e,e})$ will be a finite-dimensional Hopf algebra. As in the proof of the previous proposition, let h be a left cointegral in $(A_e, \Delta_{e,e})$. Because of the structure of the algebra A, being a direct sum of the algebras $(A_p)_{p\in G}$, we consider h as sitting in A and it will still satisfy $ah = \varepsilon(a)h$, for all $a \in A$. So, h is also a left cointegral for the multiplier Hopf algebra (A, Δ). Then, by definition, this multiplier Hopf algebra is of discrete type (see Definition 5.2 in [VD4]).

The following is then an immediate consequence (see [VD-Z1]).

3.4 Corollary With the notations and assumptions of Proposisition 3.3 we have the following.
i) The multiplier Hopf algebra A has integrals (Theorem 2.10 in [VD-Z]).
ii) The dual $\hat{A}$ of A is a (usual) Hopf algebra. In this case, $\hat{A} = A^*$ where $A^* = \oplus_{p\in G} A'_p$. The Hopf algebra structure on A^* is as in the proof of Lemma 3.2.

Combining the previous results, we recover the result (on the existence of integrals) of Virelizier (Section 3 in [V]). Recall that the uniqueness of the integrals is a general property, already stated in the previous section.

The rest of this section is now devoted to obtain an explicit formula for the left integral on this G-cograded multiplier Hopf algebra with finite-dimensional unital components. The formula we obtain is of the same nature as the formula for the left integral on a finite-dimensional Hopf algebra as given in [VD3]; see also Section 5 in [VD4].

Remark that the dual space A' of A (the space of all linear functionals on A) is given by $\prod_{p\in G} A'_p$ (as we have seen before, in Section 2). In general, for a multiplier Hopf algebra A, the dual space A' can not be made into an algebra by dualizing the product. In this case, we have $\hat{A} = A^*$ with $A^* = \oplus_{p\in G} A'_p$ as we saw in Corollary 3.4 and we have an algebra structure on $\hat{A}$. We can extend this product. Indeed, it is possible to define $fg \in A'$ for $f, g \in A'$ as soon as one of the factors actually sits in the reduced dual A^* (in fact, this can be done for any regular multiplier Hopf algebra, see [VD1]). The product is still given by $(fg)(a) = (f \otimes g)\Delta(a)$, an expression that makes sense if e.g. $f \in A^*$ because then the first leg of $\Delta(a)$ will be covered.

Using this definition, it is easily seen that a linear function φ on A will be left invariant if and only if $f\varphi = f(1_p)\varphi$ for all $f \in (A_p)'$ and for all $p \in G$. We will use this to prove left invariance below.

Recall that all the components A_p are assumed to be finite-dimensional. Therefore we can introduce the following.

3.5 Notations For each $p \in G$, let $(e_{p,i})_i$ be a basis in A_p and $(f_{p,i})_i$ a dual basis in the dual space A'_p. For $f \in A'$ and $a \in A_p$, denote by $f(\,\cdot\, a)$ the linear functional on A_p given by $x \mapsto f(xa)$.

We need the following lemmas.

3.6 Lemma With the notations and assumptions from before, we have the following two formulas.
i) For all p, q we have

$$\sum_i f_{(pq),i} \otimes \Delta_{p,q}(e_{(pq),i}) = \sum_{j,k} f_{p,j} f_{q,k} \otimes e_{p,j} \otimes e_{q,k}$$

in $A^* \otimes A_p \otimes A_q$.
ii) For all p we have

$$\sum_i \Delta(f_{p,i}) \otimes e_{p,i} = \sum_{j,k} f_{p,j} \otimes f_{p,k} \otimes e_{p,j} e_{p,k}$$

in $A^* \otimes A^* \otimes A_p$.

The proof is straightforward. It uses the facts that the product in A^* is dual to the coproduct in A (to prove i)) and that the coproduct in A^* is dual to the product in A (to prove ii)). Of course, also the basic property of dual bases has to be used several times. It says that for all p and all $a \in A_p$ we have $\sum_i f_{p,i}(a) e_{p,i} = a$.

3.7 Lemma Again with the notations and assumptions from before, we get the following.
i) For all $f \in A^*$, $g \in A'$ and $a \in A$ we have

$$(fg)(\,\cdot\, S(a)) = \sum_{(a)} f(\,\cdot\, S(a_{(2)})) g(\,\cdot\, S(a_{(1)})).$$

ii) For all $f \in A'$, $a \in A$ and $p \in G$ we have

$$\sum_i f_{p,i}(f(\,\cdot\, S(a))) \otimes e_{p,i} = \sum_{i,(a)} (f_{p,i} f)(\,\cdot\, S(a_{(1)})) \otimes e_{p,i} a_{(2)}.$$

Proof: First observe that f is a reduced functional on A and therefore, the element $S(a_{(2)})$ will be covered in the first formula. Moreover, the product fg is well defined as a product of an element in A^* with an element in A'. The result is in A', but as we include $S(a)$ in the argument, the functional in the left hand side of the first formula is an element in A^*. In the right hand side of this formula, we have a sum of products of elements in A^* and so the result is again in A^*. So, the first formula in the lemma makes sense within A^*. The proof of i) is an immediate consequence of the definition of these products.

Next we consider ii). Now, $a_{(2)}$ is covered by the elements $e_{p,i}$. Similarly as in i), we have products of functionals with at least one factor in A^* and anyway, in both sides of the equation, we have a result with functionals in A^*. Using i) the right hand side of the formula in ii) can be written as

$$\sum_{i,(a)} f_{p,i}(\,\cdot\, S(a_{(2)}))(f(\,\cdot\, S(a_{(1)})) \otimes e_{p,i}a_{(3)}.$$

Now we use ii) of Lemma 3.6 to see that this expression gives

$$\begin{aligned}\sum_{i,j,(a)} f_{p,j}(S(a_{(2)}))(f_{p,i}(f(\,\cdot\, S(a_{(1)})))) &\otimes e_{p,i}e_{p,j}a_{(3)} \\ &= \sum_{i,(a)} f_{p,i}(f(\,\cdot\, S(a_{(1)}))) \otimes e_{p,i}S(a_{(2)})a_{(3)} \\ &= \sum_{i} f_{p,i}(f(\,\cdot\, S(a))) \otimes e_{p,i}.\end{aligned}$$

This proves ii).

Now that we have obtained the main technical results, we will define a candidate for the left integral. The previous lemmas will make it rather easy to prove that we do have a non-zero left invariant functional, defined as follows.

3.8 Definition Let f be any linear functional in A'_e. Define a linear functional φ_f on A by $\varphi_f(a) = \sum_{p,i}(f_{p,i}f)(aS^2(e_{p,i}))$.

Observe that, because $f \in A'_e$, the products $f_{p,i}f$ are again functionals on A_p and that S^2 leaves A_p globally invariant so that $aS^2(e_{p,i}) \in A_p$. Because $a \in \oplus_{p\in G}A_p$, the element a will only have finitely many components and therefore, we have a sum over only finitely many p above. So, this definition makes sense.

Compare this formula with the one in Proposition 1.1 in [VD3]. It is a dual version of this formula, generalized to the case we study here.

Therefore, the following should not come as a surprise.

3.9 Theorem Let (A, Δ) be a regular G-cograded multiplier Hopf algebra with finite-dimensional unital components. For any $f \in A'_e$, the linear functional φ_f on A, as defined in Definition 3.8, is left invariant on (A, Δ).

Proof: As explained before, to prove left invariance of φ_f, we need to show that $g\varphi_f = g(1_p)\varphi_f$ in A' for all $p \in G$ and $g \in A'_p$. This is equivalent with requiring $\sum_i f_{p,i}\varphi_f \otimes e_{p,i} = \varphi_f \otimes 1_p$ in $A' \otimes A_p$ for all p. We use 1_p to denote the identity in A_p.

Now, let $p, q \in G$. We have, using the definition of φ_f,

$$\sum_i (f_{p,i}\varphi_f)(\,\cdot\,1_{pq}) \otimes e_{p,i} = \sum_i f_{p,i}(\varphi_f(\,\cdot\,1_q)) \otimes e_{p,i}$$
$$= \sum_{i,j} f_{p,i}((f_{q,j}f)(\,\cdot\,S^2(e_{q,j}))) \otimes e_{p,i}.$$

By Lemma 3.7.ii), this last expression equals

$$\sum_{i,j,(e_{q,j})} (f_{p,i}f_{q,j}f)(\,\cdot\,S^2(e_{q,j\,(2)})) \otimes e_{p,i}S(e_{q,j\,(1)}).$$

Then, using Lemma 3.6.i) we get

$$\sum_{i,j,k}(f_{p,i}f_{p^{-1},j}f_{pq,k}f)(\,\cdot\,S^2(e_{pq,k})) \otimes e_{p,i}S(e_{p^{-1},j})$$
$$= \sum_k (f_{pq,k}f)(\,\cdot\,S^2(e_{pq,k})) \otimes 1_p$$
$$= \varphi_f(\,\cdot\,1_{pq}) \otimes 1_p.$$

This is true for all q and hence we obtain the required formula $\sum_i f_{p,i}\varphi_f \otimes e_{p,i} = \varphi_f \otimes 1_p$ in $A' \otimes A_p$ for all p. This completes the proof.

In order to get a left integral on A with this formula, we need to show that for at least one f, the functional $\varphi_f \neq 0$. By the uniqueness of left integrals, then we know that φ_f will be a scalar multiple of the left integral. This problem is similar as with the formula that we used in [VD3]. We solve it in the following proposition.

3.10 Proposition Take $p \in G$ and any $g \in A'_p$. Then $g(a) = \sum_i \varphi_{f_{p^{-1},i}g}(aS(e_{p^{-1},i}))$ for all $a \in A_p$. Consequently, $\varphi_h \neq 0$ for some $h \in A'_e$.

Proof: From equation i) in Lemma 3.6 we get

$$\varepsilon_e \otimes 1_p = \sum_{i,j}(f_{p,j}f_{p^{-1},i} \otimes S(e_{p^{-1},i})S^2(e_{p,j}))$$

for all p. Therefore, any $g \in A'_p$ has the form

$$g = \sum_{i,j}(f_{p,j}f_{p^{-1},i}g)(\,\cdot\,S(e_{p^{-1},i})S^2(e_{p,j}))$$

or equivalently

$$g = \sum_i \varphi_{f_{p^{-1},i}g}(\,\cdot\,S(e_{p^{-1},i})).$$

Now, it follows from this formula that, if $g \neq 0$, we must have that for some i also $\varphi_h \neq 0$ where $h = f_{p^{-1},i} g$.

So, we have shown that φ_f will be non-zero for at least some $f \in A'_e$. We proved already that any G-cograded multiplier Hopf algebra with finite-dimensional unital components has integrals (Proposition 3.3 and Corollary 3.4). Now, we also have a constructive proof of this result.

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