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Weak and Strong Convergence of Amarts in Fréchet Spaces

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Several new characterizations of nuclearity in Fréchet spaces are proved. The most important one states tat a Fréchet space is nuclear if and only if every mean bounded amart is strongly a.s. convergent. This extends the result in [A. Bellow, *Proc. Nat. Acad. Sci. USA* **73**, No. 6 (1976), 1798–1799] in a more positive way, and gives a different proof of it. The results of Brunel and Sucheston [*C. R. Acad. Sci. Paris Ser. A* (1976), 1011–1014], are extended to yield the same characterization of reflexivity of a Fréchet space in terms of weak convergence a.s. of weak amarts.

1. INTRODUCTION

The extension of the Banach space-valued Bochner-integral to Fréchet spaces is well known (see, e.g. [12]). The space of all Fréchet-valued integrable functions on a probability space $(\Omega, \mathscr{F}, P)$ is denoted by L_E^1, where E is the Fréchet space. Let $(\mathscr{F}_n)_{n=1}^{\infty}$ be an increasing sequence of sub-σ-algebras of $\mathscr{F}$, and $(X_n, \mathscr{F}_n)_{n=1}^{\infty}$ an adapted sequence (i.e., for every $n \in \mathbb{N}$: $X_n \in L_E^1(\Omega, \mathscr{F}_n, P/\mathscr{F}_n)$). Denote by T the set of all bounded stopping times with respect to $(\mathscr{F}_n)$. $(X_n, \mathscr{F}_n)$ is called an amart (resp. weak amart, abbreviated (W) amart) if $(\int_\Omega X_\tau)_{\tau \in T}$ converges strongly (resp. weakly). $(X_n, \mathscr{F}_n)_{n=1}^{\infty}$ is called a weak sequential amart ((WS) amart) if for every increasing sequence $(\tau_n)_{n=1}^{\infty}$ in T, $(\int_\Omega X_{\tau_n})_{n=1}^{\infty}$ converges weakly. As in the Banach-space case it can be seen that every amart is a (WS) amart and that every (WS) amart is a (W) amart. (W) and (WS) amarts were introduced by Brunel and Sucheston [3]. Amongst other result, they proved in [3, 4]:

THEOREM A. *Suppose l^1 is not contained in E, then every (W) amart is a (WS) amart iff E is reflexive.*

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THEOREM B. *Suppose E has (RNP) and that E' is separable. Then every (WS) amart of class (B) converges weakly a.s. $(X_n, \mathscr{F}_n)$ is of class (B) means*: $\sup_{\tau \in T} \int \|X_\tau\| \, dP < \infty$.

THEOREM C. *E is reflexive iff every (W) amart of class (B) (or uniformly bounded) is convergent, weakly a.s.*

As a remark I can include the following improvement of Theorem A:

THEOREM A′. *Let E be a Banach space. The following assertions are equivalent*:

(i) *E is weakly sequentially complete.*

(ii) *Every (W) amart is a (WS) amart.*

(iii) *For every uniformly bounded (W) amart $(X_n, \mathscr{F}_n)$, and every A in $\bigcup_n \mathscr{F}_n$, the sequence $(\int_A X_n)_{n=1}^\infty$ is weakly convergent.*

Proof. (i) → (ii). Suppose E is a weakly sequentially complete Banach space, and let $(X_n, \mathscr{F}_n)$ be a (W) amart. So, for every x' in E', $(x'(X_n), \mathscr{F}_n)$ is an amart. Hence for every increasing sequence (τ_n) in T, the sequence

$$\left(\int_\Omega x'(X_{\tau_n}) \, dP\right)_{n=1}^\infty$$

is convergent as is well known for amarts. So $(\int_\Omega X_{\tau_n} dP)_{n=1}^\infty$ is weakly Cauchy and thus weakly convergent, proving that $(X_n, \mathscr{F}_n)$ is a (WS) amart.

(ii) → (iii). Let $A \in \bigcup_n \mathscr{F}_n$. So, there is a $n_0 \in \mathbb{N}$ such that $A \in \mathscr{F}_{n_0}$. Define τ_n for $n \geqslant n_0$ by

$$\begin{aligned} \tau_n &= n \quad \text{on } A \\ &= n_0 \quad \text{on } \Omega \backslash A. \end{aligned}$$

So $\tau_n \in T$, $\forall\, n \geqslant n_0$. Furthermore $(\tau_n)_{n \geqslant n_0}$ is increasing. Thus $(\int_\Omega X_{\tau_n})_{n=n_0}^\infty$ is weakly convergent. But

$$\begin{aligned} \int_\Omega X_{\tau_n} &= \int_A X_{\tau_n} + \int_{\Omega \backslash A} X_{\tau_n} \\ &= \int_A X_n + \int_{\Omega \backslash A} X_{n_0} \end{aligned}$$

for every n in $\mathbb{N}$. Since $\int_{\Omega \backslash A} X_{n_0}$ is constant, the sequence $(\int_A X_n)_{n=1}^\infty$ converges weakly.

(iii) → (i) (inspired by [4]). Suppose $(x_n)_{n=1}$ is a weak Cauchy sequence, which is not weakly convergent. Choose: $X: \Omega \to \{-1, +1\}((\Omega, \mathscr{F}, P)$ arbitrary probability space), such that $P\{X=1\} = P\{X=-1\} = \frac{1}{2}$. Put $X_n = X \cdot x_n$. For each n, we take the same partitions $\pi_n = \{\{X=-1\}, \{X=1\}\}$. Now $(X_n, \sigma(\pi_n))$ is a (W) amart. But for $A = \{X=1\}$, we have $X_n|_A = X_n \cdot 1_A$. So $\int_A X_n = x_n/2$. So $(\int_A X_n)_{n=1}^{\infty}$ is not weakly convergent. ∎

The proof of Theorem A′ does not use Rosenthal's result [15]. To obtain Theorem A form Theorem A′ we do use his result. So this method to prove Theorem A sheds more light on this result.

Let us suppose now that E is a Fréchet space. The Pettis-topology on L_E^1 is generated by the seminorms $P_U: L_E^1 \to \mathbb{R}^+$,

$$P_U(X) = \sup_{x' \in U^0} \int_\Omega |x'(X)| \, dP,$$

where U is an arbitrary zero-neighborhood, and U^0 denotes the polar of U with respect to the duality $\langle E, E' \rangle$ (see [16]). A Pettis-bounded sequence in L_E^1 is a sequence (X_n) such that

$$\sup_{n \in \mathbb{N}} \sup_{x' \in U^0} \int_\Omega |x'(X_n)| \, dP < \infty$$

for every 0-neighborhood U. A Pettis-uniformly integrable sequence in L_E^1 is a sequence (X_n) such that

$$\lim_{P(A) \to 0} \sup_{x' \in U^0} \int_A |x'(X_n)| \, dP = 0$$

uniformly in $n \in \mathbb{N}$, for every U, 0-neighborhood. Amongst other results, we proved in [12]:

THEOREM 1. *Let E be a Fréchet space. The following assertions are equivalent*:

(i) *E is nuclear.*

(ii) *Every Pettis-bounded and Pettis-uniformly integrable amart is mean convergent.*

(iii) *On L_E^1, the Pettis-topology is the same as the topology of mean-convergence.*

Whenever we say "mean-convergence" we mean of course convergence with respect to the family of seminorms

$$q(X) = \int_\Omega p(X)\, dP,$$

where p is an arbitrary continuous seminorm on E. We stated as an "added in proof" the following result:

THEOREM 2. *Let E be a Fréchet space. The following assertions are equivalent*:

(i) *E is nuclear.*

(ii) *Every mean bounded amart $(X_n, \mathscr{F}_n)$ is of class (B), i.e.,*

$$\operatorname{Sup}_{\tau \in T} \int_\Omega p(X_\tau)\, dP < \infty$$

for every continuous seminorm p on E.

(iii) *For every mean bounded and mean-uniformly integrable amart $(X_n, \mathscr{F}_n)$, and for every continuous seminorm p on E: $(p(X_n), \mathscr{F}_n)$ is an amart.*

Since the proof of Theorem 2 is non-trivial and not included in [12], and since we use Theorem 2 crucially in our main new result, we prove it here in Section 2. In Section 3 we give the main result:

THEOREM 3. *Let E be a Fréchet space. The following assertions are equivalent*:

(i) *E is nuclear.*

(ii) *Every mean bounded amart is strongly a.s. convergent.*

To conclude the paper we note that Theorems B and C are valid also for Fréchet spaces (where we use the strong topology on E'), yielding a characterization of reflexivity in Fréchet spaces.

2. PROOF OF THEOREM 2

(i) → (ii). Suppose E is a nuclear Fréchet space and let $(X_n, \mathscr{F}_n)$ be a mean bounded amart. By [14, 4.1.5], there exists, for every continuous seminorm p on E, a zero-neighborhood V and a Radon measure ν on the polar V^0, such that for every $\omega \in \Omega$ and $\tau \in T$,

$$p(X_\tau(\omega)) \leqslant \int_{V^0} |\langle X_\tau(\omega), x' \rangle| \, d\nu(x').$$

Applying Fubini's theorem,

$$\int p(X_\tau(\omega))\, dP(\omega) \leqslant \sup_{x' \in V^0} \left[\int_\Omega |\langle X_\tau(\omega), x' \rangle| \, dP(\omega) \right] \nu(V^0)$$

$$\leqslant 4 \sup_{x' \in V^0} \sup_{A \in \mathscr{F}_\tau} \left| x' \left(\int_A X_\tau(\omega)\, dP(\omega) \right) \right| \nu(V^0).$$

The last inequality is true since X_τ is $\mathscr{F}_\tau$-measurable, where

$$\mathscr{F}_\tau = \{ A \in \mathscr{F} \,\|\, A \cap \{\tau = n\} \in \mathscr{F}_n, \ \forall n \in \mathbb{N} \}.$$

Denoting $p_V(\cdot) = \sup_{x' \in V^0} |x'(\cdot)|$, we have

$$\operatorname*{Sup}_{\tau \in T} \int_\Omega p(X_\tau)\, dP \leqslant 4 \sup_{\tau \in T} \sup_{A \in \mathscr{F}_\tau} p_V \left(\int_A X_\tau \, dP \right) \nu(V^0).$$

Since (X_n) is an L^1_E-bounded amart, it is well known that (see Theorem 1 in [1], applied for every seminorm) the right-hand side of the last inequality is finite, proving this part of the theorem.

(ii) → (i). Suppose E is not nuclear. By [14, 4.2.5], there is a summable sequence (x_n) (i.e., $\sum x_n$ is commutatively convergent) which is not absolutely convergent. Choose any strictly decreasing sequence (γ_n) in $[0, \frac{1}{2}]$, converging to 0. Put $\delta_n = \gamma_n - \gamma_{n-1}$. Define, for every n in $\mathbb{N}$,

$$A_0^n = [0, \gamma_{n+1}), \qquad B^n = [\gamma_{n+1}, \gamma_n), \qquad A_1^n = [\gamma_n, 1),$$

and

$$\pi_1 = \{A_0^1, B^1, A_1^1\},$$
$$\pi_2 = \pi_1 \vee \{A_0^2, B^2, A_1^2\},$$
$$\cdot\ \pi_n = \pi_{n-1} \vee \{A_0^n, B^n, A_1^n\}.$$

Put

$$X_n = \frac{x_n}{\delta_n} 1_{B^n}$$

and

$$\mathscr{F}_n = \sigma(\pi_n).$$

We claim that the sequence $(X_n, \mathscr{F}_n)_{n=1}^{\infty}$ has the following properties:

(a) It is an amart.

(b) It is mean convergent to 0.

(c) It is not of class (B).

Indeed, for every τ in T,

$$\int_0^1 X_\tau = \sum_{l=n_\tau}^{n'_\tau} \frac{x_k}{\delta_k} P(B^l \cap \{\tau = l\}),$$

where

$$n_\tau = \min\{\tau(\omega) \,\|\, \omega \in [0, 1)\},$$
$$n'_\tau = \max\{\tau(\omega) \,\|\, \omega \in [0, 1)\}.$$

Since $P(B^l \cap \{\tau = l\}) \leqslant \delta_l$, and by the unconditional convergence of $\sum x_n$, we have

$$\lim_{\tau \in T} \int_0^1 X_\tau \, dP = 0$$

(see [14, 1.3.6, p. 26–27]). This proves (a). Now $\int_0^1 p(X_n) = p(x_n)$, for every continuous seminorm p on E. Hence (b) follows, by the convergence of $\sum x_n$. Now suppose that p_0 is a continuous seminorm on E for which $\sum_{n=1}^{\infty} p_0(x_n) = \infty$. Then

$$\int_0^1 p_0(X_\tau) = \sum_{l=n_\tau}^{n'_\tau} p_0(x_l) \cdot \frac{P(B^l \cap \{\tau = l\})}{\delta_l},$$

Choose

$$\begin{aligned} \tau_n &= 1 && \text{on } B^1 \\ &\cdots && \cdots \\ &= n && \text{on } B^n \\ &= n+1 && \text{on } [0,1) \Big\backslash \left(\bigcup_{i=1}^n B^i \right). \end{aligned}$$

Since

$$\int_0^1 p_0(X_{\tau_n}) = \sum_{l=1}^n p_0(x_l) + \int_{[0,1)\backslash \bigcup_{l=1}^n B^l} p_0(X_{n+1})$$
$$\geqslant \sum_{l=1}^n p_0(x_l)$$

it follows that, since $\tau_n \in T$, for every n in $\mathbb{N}$,

$$\sup_{\tau \in T} \int_0^1 p_0(X_\tau) \geqslant \sup_n \int_0^1 p_0(X_{\tau_n}) \geqslant \sum_{l=1}^{\infty} p_0(x_l) = \infty,$$

proving (c); hence proving (ii) → (i).

(i) → (iii). Let $(X_n, \mathscr{F}_n)$ be a mean bounded and uniformly integrable amart. From [13, 6] it follows that E has (RNP). Hence it is immediately seen that the limit measure $F: \mathscr{F} \to E$: $\forall A \in \mathscr{F}$,

$$F(A) = \lim_{n \to \infty} \int_A X_n \, dP$$

has a RN-derivative in L_E^1, where $\mathscr{F} = \sigma(\bigcup_n \mathscr{F}_n)$. We denote this derivative by X. As in the Banach-space case it is immediately seen that the conditional expectation of X with respect to every $\mathscr{F}_\tau$ exists. Denote this by $E^{\mathscr{F}_\tau}X$. Applying [14, 4.1.5] to $X_\tau - E^{\mathscr{F}_\tau}X$, we have with the same notations as in the proof of (i) → (ii), $\forall \tau \in T$, $\forall \omega \in \Omega$, for every continuous seminorm p on E,

$$p(X_\tau(\omega) - E^{\mathscr{F}_\tau}X(\omega)) \leqslant \int_{V^0} |\langle X_\tau(\omega) - E^{\mathscr{F}_\tau}X(\omega), x' \rangle| \, d\nu(x').$$

Again, using Fubini,

$$\int_\Omega p(X_\tau(\omega) - E^{\mathscr{F}_\tau}X(\omega)) \, dP(\omega)$$
$$\leqslant \sup_{x' \in V^0} \left[\int |\langle X_\tau(\omega) - E^{\mathscr{F}_\tau}X(\omega), x' \rangle| \, dP(\omega) \right] \cdot \nu(V^0).$$

Now we have

$$\int_\Omega p(X_\tau(\omega) - E^{\mathscr{F}_\tau}X(\omega)) \, dP(\omega)$$
$$\leqslant 4\nu(V^0) \sup_{x' \in V^0} \sup_{A \in \mathscr{F}_\tau} \left| x' \left(\int_A X_\tau \, dP - \int_A E^{\mathscr{F}_\tau}X \, dP \right) \right|.$$

Denoting $P_V(\cdot) = \sup_{x' \in V^0} |x'(\cdot)|$, we see

$$(*) \quad \int p(X_\tau(\omega) - E^{\mathscr{F}_\tau}X(\omega)) \, dP(\omega)$$
$$\leqslant 4\nu(V^0) \sup_{A \in \mathscr{F}_\tau} p_V \left(\int_A X_\tau \, dP - F(A) \right).$$

It is well known (see Theorem 1 in [1], applied seminormwise) that the right-hand side of this last inequality is small for τ large enough. Furthermore, since $(E^{\mathscr{F}_\tau}X, \mathscr{F}_\tau)_{\tau \in T}$ is a martingale, generated by one integrable function, it is obvious that $(E^{\mathscr{F}_\tau}X)_{\tau \in T}$ converges in the mean to X. So for every continuous seminorm p on E:

$$\lim_{\tau \in T} \int_\Omega p(E^{\mathscr{F}_\tau}X - X)\, dP = 0.$$

So by (*) we finally have

$$\lim_{\tau \in T} \int_\Omega p(X_\tau)\, dP = \int_\Omega p(X)\, dP$$

and so $(p(X_n), \mathscr{F}_n)$ is an amart.

Remark. In (i) → (iii) we in fact proved (i) → (iii)′ with (iii)′: For every mean-bounded and uniformly integrable amart $(X_n, \mathscr{F}_n)$, $(X_\tau)_{\tau \in T}$ is mean convergent. This result will be extended in Theorem 4, generalizing (i) → (iii). However, (i) → (iii) is used in the proof of Theorem 3 so that it is not superfluous.

(iii) → (i) (see also [12]). Suppose that E is not nuclear. By [14, 4.2.5], there exists a summable sequence (x_n) which is not absolutely summable. Take $\Omega = [0, 1)$. Since there exists a continuous seminorm p on E such that $\sum_{n=1}^{\infty} p(x_n) = \infty$, there exists K_1 in $\mathbb{N}$ such that $\alpha_1 = \sum_{n=1}^{K_1} p(x_n) > 1$. So $\sum_{n=1}^{K_1} p(y_n) = 1$ with $y_n = x_n/\alpha_1$ $(n = 1,..., K_1)$. Let K_2 in $\mathbb{N}$ be the smallest natural number, larger than K_1 such that

$$\alpha_2 = \sum_{n=K_1+1}^{K_2} p(x_n) > p(y_1).$$

So again $\sum_{n=K_1+1}^{K_2} p(y_n) = p(y_1)$ with $y_n = (x_n/\alpha_2)\, p(y_1)$ $(n = K_1 + 1,..., K_2)$. We do the same argument with $p(y_2),..., p(y_{K_1})$, and start all over again, in the same way with $p(y_{K_1+1})$, and so on. By this construction, we run an infinite number of times through [0, 1), so we have that $\Sigma\, p(y_n) = \infty$. But since every y_n is x_n times a scalar in (0, 1], we have that (y_n) is still summable. Every time we run through [0, 1) we obtain a refinement of the foregoing division. So we obtain an increasing sequence of finite partitions denoted by

$$\pi_0 = \{[0, 1]\},$$

$$\pi_1 = \{A_1,..., A_{K_1}\},$$

$$\pi_2 = \{A_1,..., A_{K_{K_1+1}}\},$$

etc. We made it so that $P(A_i) = p(y_i)$, for every i in $\mathbb{N}$. We now put

$$X_n = \frac{y_n}{p(y_n)} 1_{A_n}.$$

The sequence $(X_n, \sigma(\pi_n))_{n=1}^{\infty}$ has the following properties:

(a) It is an amart.

(b) It is mean convergent to 0.

(c) $(p(X_n), \sigma(\pi_n))_{n=1}^{\infty}$ is not an amart.

Indeed, for every $\tau \in T$,

$$\int_0^1 X_\tau = \sum_{l=n_\tau}^{n'_\tau} \int_{\{\tau = l\}} \frac{y_l}{p(y_l)} 1_{A_l}\, dP.$$

So

$$\int_0^1 X_\tau = \sum_{l=n_\tau}^{n'_\tau} \alpha_l y_l,$$

where $|\alpha_l| \leqslant 1$. Since $\sum y_n$ is unconditionally convergent we have that $\lim_{\tau \in T} \int_0^1 X_\tau = 0$, proving (a). Property (b) is immediate since for every continuous seminorm p' on X:

$$\lim_n \int_0^1 p'(X_n) = \lim_n p'(x_n) = 0.$$

From (b) it follows that $(X_n, \sigma(\pi_n))$ is mean bounded and uniformly integrable. We now prove (c). Define a stopping time τ_k on A_j (for $A_j \in \pi_k$) as j. Then we see that

$$\int_0^1 p(X_{\tau_k}) = 1$$

for every k in $\mathbb{N}$. Now define τ'_k on A_j (for $A_j \in \pi_k$) as $\rho_k(j)$, where $\rho_k(j)$ is a fixed permutation of the indices j appearing for the sets A_j in π_k, such that ρ_k has no fixed point. Now

$$\int_0^1 p(X_{\tau'_k}) = \sum_{j=n_{\tau'_k}}^{n'_{\tau'_k}} \int_{\{\tau'_k = j\}} p(X_j).$$

Now $\{\tau'_k = j\} = A_{\rho_k^{-1}(j)}$. Since $\rho_k^{-1}(j) \neq j$, the intersection $A_{\rho_k^{-1}(j)} \cap A_j$ is void. Hence

$$\int_0^1 p(X_{\tau'_k}) = 0$$

for every k in $\mathbb{N}$. Now both sequences (τ_k) and (τ'_k) are cofinal in T. So $\int_0^1 p(X_\tau)$ cannot converge, and thus $(p(X_n), \sigma(\pi_n))$ is not an amart.

Remark. In the proof of (iii) → (i), we could also use the simpler sequence constructed in the proof of (ii) → (i). However, the sequence constructed in the proof of (iii) → (i) given above has the advantage to be uniformly bounded in case E is a Banach space (cf. also [12]). Also this less trivial sequence is needed in Theorems 3 and 5 further on. Incidentally, the construction in (iii) → (i) of Theorem 2 gives rise to a result in pramart-theory. We recall the definition of a pramart (stated here in Fréchet spaces).

Definition. Let E be a Fréchet space. An E-valued adapted sequence $(X_n, \mathscr{F}_n)$ is called a pramart if for every continuous seminorm p on E, $(p(X_\sigma - E^{\mathscr{F}_\sigma} X_\tau))_{\sigma \leqslant \tau, \sigma, \tau \in T}$ converges to zero in probability.

Theorem 3. *Let E be a Fréchet space. The following assertions are equivalent*:

(i) *E is nuclear.*

(ii) *Every amart is a pramart.*

Proof. (i) → (ii). Let $(X_n, \mathscr{F}_n)$ be an amart. By the well-known difference-property (trivially extended to our case), $(E^{\mathscr{F}_\sigma} X_\tau - X_\sigma)_{\sigma \leqslant \tau, \sigma, \tau \in T}$ converges to zero in Pettis-sense; hence in L^1_E-sense, by Theorem 1 (so $(X_n, \mathscr{F}_n)$ is uniform amart). Since L^1_E-convergence implies convergence in probability, we are done.

(ii) → (i). Consider again the example developed in the proof of (iii) → (i) of Theorem 2, supposing E not nuclear. With the same meaning of $(\tau_k)_{k=1}^\infty$ and $(\tau'_k)_{k=1}^\infty$ as in that proof, we have that $\tau'_k > \tau_{k-1}$ for every k in $\mathbb{N}$, and both (τ_k) and (τ'_k) are cofinal in T. Furthermore $p(E^{\mathscr{F}_{\tau_{k-1}}} X_{\tau'_k} - X_{\tau_{k-1}}) = p(X_{\tau_{k-1}}) = 1$ for every k in $\mathbb{N}$. So $(X_n, \mathscr{F}_n)$ is not a pramart, but it was an amart. ∎

Remark. The Banach-space case of the above theorem was proved differently in [9]. We also remark that the example in the foregoing theorem is not a mil and not a game fairer with time.

Remark. The example constructed in the proof of (ii) → (i) in Theorem 2 shows that if every mean bounded pramart is of class (B), the Fréchet space

E must be nuclear. However, the converse is not true, not even for $E = \mathbb{R}$, as the following example shows: $\Omega = [0, 1]$, $X_n = 2^n 1_{(1/2^n, 1/2^{n-1}]}$, $n = 1, 2, \ldots$, $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$. Then $(X_n, \mathscr{F}_n)$ is a pramart and $\int_0^1 X_n = 1$ for every n in $\mathbb{N}$. Consider

$$
\begin{aligned}
\tau_n &= 1 && \text{on } \left(\frac{1}{2}, 1\right] \\
&= 2 && \text{on } \left(\frac{1}{2^2}, \frac{1}{2}\right] \\
&\cdots \\
&= n && \text{on } \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \\
&= n+1 && \text{on } \left[0, \frac{1}{2^{n+1}}\right].
\end{aligned}
$$

Then $\tau_n \in T$, for every n in $\mathbb{N}$. Now $\int X_{\tau_n} > n$, for every n in $\mathbb{N}$. Hence $\sup_{\tau \in T} \int |X_\tau| = \sup_{\tau \in T} \int X_\tau = \infty$.

3. Strong Convergence a.s. of Amarts

We are now able to attack the problem of strong convergence a.s. of amarts in Fréchet spaces. In Banach spaces, Bellow [2] proved:

THEOREM D. *Let E be a Banach space. The following assertions are equivalent*:

(i) $\dim E < \infty$.

(ii) *Every mean bounded amart is strongly a.s. convergent.*

(iii) *Every uniformly bounded amart is strongly a.s. convergent.*

Theorem 1 is an extension of (iii) $\rightarrow$ (i) in Theorem D to Fréchet spaces. Theorem D is very interesting, however, it is a negative result: except for the finite dimensional case, there is no strong convergence of amarts! In Fréchet spaces, however, the class of nuclear spaces is much larger than the class of finite dimensional spaces, and consists of spaces, important both for functional analysts and for probabilists (see [13], for examples). So the next result is more positive, in the sense that it allows strong convergence of amarts in nuclear Fréchet spaces:

THEOREM 4. *Let E be a Fréchet space. The following assertions are equivalent*:

(i) *E is nuclear.*

(ii) *Every Pettis-bounded amart is strongly a.s. convergent.*

(iii) *Every mean-convergent (and uniformly bounded in case E is a Banach space) amart is strongly a.s. convergent.*

Remark. (iii) → (i) is new even in Banach spaces.

Proof of Theorem 4. (ii) → (iii). Trivial.

(iii) → (i). Suppose that E is not nuclear. The example constructed in the proof of (iii) → (i) in Theorem 2 yields an amart which is mean convergent to 0. However, $(X_n, \mathscr{F}_n)$ is not strongly convergent since for every $\omega \in [0, 1]: X_n(\omega) = 0$ for a cofinal set of indices n as well as $X_n(\omega) = x_n/p(x_n)$ for a cofinal set of indices n.

(i) → (ii). We will prove a more general result in order to prove (i) → (ii):

THEOREM 5. *Let E be a nuclear Fréchet space. Then every mean bounded amart $(X_n, \mathscr{F}_n)$ is the sum of a martingale $(Y_n, \mathscr{F}_n)$ which is mean bounded and strongly convergent and an amart $(Z_n, \mathscr{F}_n)$ such that $(Z_\tau)_{\tau \in T}$ converges in the mean sense and in the strong sense a.s. to* 0.

Proof. The Riesz-decomposition theorem in [8] is easily extended to our case, since a nuclear Fréchet space does have (RNP). We thus have

$$X_n = Y_n + Z_n,$$

where $(Y_n, \mathscr{F}_n)$ is a martingale, mean bounded, and $(Z_\tau)_{\tau \in T}$ goes to 0 in Pettis-sense. By Theorem 1, $(Z_\tau)_{\tau \in T}$ goes to 0 in the mean sense. Noting the fact that a nuclear Fréchet space is the projective limit of a sequence of Hilbert spaces, we easily see that the martingale $(Y_n, \mathscr{F}_n)$ is strongly convergent a.s. in E. So the only thing to show is that $Z_n \to 0$ a.s. (strongly). By the mean convergence of (Z_n) we have the mean boundedness, and so, by Theorem 2, (Z_n) is of class (B). Suppose $E = \varprojlim g_{mn} H_n$, with projection maps g_n. Then for every fixed m in $\mathbb{N}$, $(g_m(Z_n), \mathscr{F}_n)_{n=1}^\infty$ is an amart of class (B) in H_m. By [5], $(g_m(Z_n))_{n=1}^\infty$ is weakly a.s. convergent, say to $Z^{(m)}$ on $\Omega \backslash N_1^m$, where $P(N_1^m) = 0$. Since Z_n is mean convergent to 0 it follows from Theorem 2 that for every fixed m, $(\| g_m(Z_n) \|_m, \mathscr{F}_n)_{n=1}^\infty$ is a real amart (where $\| \cdot \|_m$ denotes the norm in H_m). This is true since the system $\{\| g_m(\cdot) \|_m \,\|\, m \in \mathbb{N}\}$ is a generating family continuous seminorms for E. By the real amart convergence theorem [7, p. 203], $(\| g_m(Z_n) \|_m)_{n=1}^\infty$ converges a.s., say on $\Omega \backslash N_2^m$, with $\Omega(N_2^m) = 0$. Hence on $\Omega \backslash (N_1^m \cup N_2^m)$ we have $(g_m(Z_n))_{n=1}^\infty$ converges weakly and $(\| g_m(Z_n) \|_m)_{n=1}^\infty$ converges. Since $Z_n \to 0$ in L_E^1-sense, $(g_m(Z_n))_{n=1}^\infty \to 0$ in $L_{H_m}^1$-sense for every fixed m in $\mathbb{N}$. So there is a pointwise a.e. convergent subsequence $(g_m(Z_{n_k}))_{k=1}^\infty$ converging strongly to

0 in H_m, say on $\Omega \backslash N^m$, with $P(N^m) = 0$. This and the above give: On $\Omega \backslash (N^m \cup N_1^m \cup N_2^m)$,

$$w\text{-}\lim_{n \to \infty} g_m(Z_n) = 0,$$

$$\lim_{n \to \infty} \| g_m(Z_n) \|_m = 0.$$

Since H_m is a Hilbert space this implies

$$\lim_{n \to \infty} g_m(Z_n) = 0$$

strongly.

So on $\Omega \backslash \bigcup_{m=1}^{\infty} [N^m \cup N_2^m \cup N_2^m]$

$$\lim_{n \to \infty} g_m(Z_n) = 0$$

for every m in $\mathbb{N}$. This means $Z_n \to 0$ strongly a.s. in E.

4. Weak Convergence a.s. of Weak and Weak Sequential Amarts

In this section we are going to prove in Fréchet spaces the results proved by Brunel and Sucheston in [3]. We shall indicate only the differences in proof.

THEOREM 5. *Let E be a Fréchet space with (RNP) and such that the strong dual E'_β is separable. Let $(X_n, \mathscr{F}_n)$ be a (WS) amart of class (B). Then (X_n) converges weakly a.s.*

Proof. This is a non-trivial modification of the proof on [3, p. 1012–1013]. Consider E as the projective limit of the sequence of Banach spaces $(E_m)_{m=1}^{\infty}: E = \varprojlim g_{nm} E_m$, with projection maps g_m. So it is immediate that $(g_m(X_n), \mathscr{F}_n)$, for every fixed m in $\mathbb{N}$, is a (WS) amart in E_m. Fix $\varepsilon > 0$. For every fixed m in $\mathbb{N}$, there exists, following [5, p. 57], a set $A_m \in \mathscr{F}$, $P(\Omega \backslash A_m) < \varepsilon / 2^m$ and a (WS) amart $(Y_n^m, \mathscr{F}_n)_{n=1}^{\infty}$ with $\sup_n \| Y_n^m \|_m \in L^1$ ($\| \cdot \|_m$ denotes the norm in E_m) such that $Y_n^m = g_m(X_n)$ on A_m. Using the fact that $\sup_n \| Y_n^m \|_m \in L^1$ and Lemma 1 on [3, p. 1012], we get that the weak limit measure μ_m of $(Y_n^m)_{n=1}^{\infty}$ exists on Ω. Hence $(*)$

$$\mu^m |_{A_m} = w\text{-}\lim_{n \to \infty} \int_{\cdot} g_m(X_n),$$

where the integral is restricted to $\mathscr{F}$-measurable sets in A_m. Put $A = \bigcap_{m=1}^{\infty} A_m$. So $A \in \mathscr{F}$ and $P(\Omega \backslash A) < \varepsilon$. Put $\mu = (\mu_m)_{m=1}^{\infty}$. We have

$$\mu_A = (\mu_m |_A)_{m=1}^{\infty} = \left(w\text{-}\lim_{n \to \infty} \int_{\cdot} g_m(X_n) \right)_{m=1}^{\infty},$$

where the integral is restricted to subsets of A, in $\mathscr{F}$. Hence $\mu_A(\cdot) \in E$, and so μ_A is the weak limit measure of (X_n) on A. Furthermore every μ_m is of bounded variation and P-continuous. Hence so is μ_A on $(A, \mathscr{F}|_A, P|_A)$. So, E having (RNP), the RN-derivative $X_{\infty,A}$ exists in $L_E^1(A, \mathscr{F}|_A, P|_A)$. Now for every $f \in E'$, $(f(X_n), \mathscr{F}_n)$ is an amart. By the real amart theory, we see that $(f(X_n))$ converges a.e. and on A necessarily to $f(X_{\infty,A})$ a.s. Furthermore since $(X_n, \mathscr{F}_n)$ is of class (B), we have by Lemma 1 in [9] that for every continuous seminorm p on E: $\limsup_n p(X_n) \in L^1$. So $\sup_n p(X_n) < \infty$ a.s. Now we have that $X_{\infty,A} = w\text{-}\lim_{n \to \infty} X_n|_A$ a.e. Since ε was arbitrary this shows that (X_n) converges w.a.e.

THEOREM 6. *Let E be a Fréchet Space. The following assertions are equivalent*:

(i) *E is reflexive.*

(ii) *Every (W) amart of class (B) is weakly a,s, convergent.*

(iii) *Every uniformly bounded (W) amart is weakly a.s. convergent.*

Proof. (i) → (ii). Let $(X_n, \mathscr{F}_n)$ be a (W) amart of class (B). By Theorem A it is a (WS) amart. Since

$$F = \overline{\text{span}}\{X_n(\omega) \,\|\, n \in \mathbb{N}, \omega \in \Omega\}$$

is a separable closed subset of E, F is reflexive. So $F = (F'_\beta)'_\beta$ is separable. Hence every bounded subset in F'_β is separable (see [11, p. 607, Ex. 2]). But F is a Fréchet space; hence F'_β contains a countable fundamental system of bounded sets. So F'_β is separable. Therefore we have only to apply Theorem 5 (and [16, p. 135]).

(ii) → (iii). Trivial.

(iii) → (i). The proof given in [3, p.1013–1014] extends in a fairly trivial way, noting only the fact that Rosenthal's theorem [15] has been extended to Fréchet spaces by Lohman, as remarked to me by W. B. Johnson.

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