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CLEFT EXTENSIONS OF KOSZUL TWISTED CALABI-YAU ALGEBRAS

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Abstract. Let H be a twisted Calabi-Yau (CY) algebra and σ a 2-cocycle on H. Let A be an N-Koszul twisted CY algebra such that A is a graded H^σ-module algebra. We show that the cleft extension $A\#_\sigma H$ is also a twisted CY algebra. This result has two consequences. Firstly, the smash product of an N-Koszul twisted CY algebra with a twisted CY Hopf algebra is still a twisted CY algebra. Secondly, the cleft objects of a twisted CY Hopf algebra are all twisted CY algebras. As an application of this property, we determine which cleft objects of $U(\mathcal{D}, \lambda)$, a class of pointed Hopf algebras introduced by Andruskiewitsch and Schneider, are Calabi-Yau algebras.

Introduction

We work over a fix a field $\Bbbk$. Without otherwise stated, all vector spaces, algebras are over $\Bbbk$. Given a 2-cocycle σ on a Hopf algebra H (Definition 1.3), we can construct the algebras H^σ and ${}_\sigma H$. Their products are deformed from the product of H by

$$x * y = \sigma(x_1, y_1)x_2y_2\sigma^{-1}(x_3, y_3)$$

$$x \bullet_\sigma y = \sigma(x_1, y_1)x_2y_2,$$

for any $x, y \in H$ respectively. The algebra H^σ together with its original coalgebra structure form a Hopf algebra, called a cocycle deformation of H. On the one hand, the algebra ${}_\sigma H$ together with the original regular coaction ${}_\sigma H \to {}_\sigma H \otimes H$ form a right H-cleft extension over the field $\Bbbk$. It is called a right cleft object. On the other hand, ${}_\sigma H$ is a left H^σ-cleft object with respect to the original coalgebra ${}_\sigma H \to H^\sigma \otimes {}_\sigma H$. Therefore, ${}_\sigma H$ is an (H^σ, H)-bicleft object. The Hopf algebra H^σ is characterized as the Hopf algebra L such that ${}_\sigma H$ is an (L, H)-biGalois object ([33]).

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In [28], Masuoka studied cocycle deformations and cleft objects of a class of pointed Hopf algebras. This class of algebras includes the pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of finite Cartan type introduced by Andruskiewitsch and Schneider ([5]). The Hopf algebras $U(\mathcal{D}, \lambda)$ consists of pointed Hopf algebras with finite Gelfand-Kirillov dimension, which are domains with finitely generated abelian groups of group-like elements, and generic infinitesimal braiding ([1]). By results in [28], we know that a pointed Hopf algebra $U(D, \lambda)$ and its associated graded Hopf algebra $U(D, 0)$ are cocycle deformations of each other.

The Calabi-Yau (CY for short) property of the algebras $U(\mathcal{D}, \lambda)$ are discussed in [39]. CY algebras were introduced by Ginzburg [19] in 2006. They were studied in recent years because of their applications in algebraic geometry and mathematical physics. More general than CY algebras are so-called twisted CY algebras, which form a large class of algebras possessing the similar homological properties as the CY algebras and include CY algebras as a subclass. Associated to a twisted CY algebra, there exists a so-called Nakayama automorphism. This automorphism is unique up to an inner automorphism. A twisted CY algebra is CY if and only if its Nakayama automorphism is an inner automorphism.

For the Hopf algebra $U(\mathcal{D}, \lambda)$, both $U(\mathcal{D}, \lambda)$ itself and its associated graded Hopf algebra $U(\mathcal{D}, 0)$ are twisted CY algebras ([39, Theorem 3.9]). A more interesting phenomenon is that the CY property of $U(D, \lambda)$ is dependent only on the CY property of $U(D, 0)$. In other words, if $U(D, 0)$ is CY, then any lifting $U(D, \lambda)$ is CY. Note that $U(D, \lambda)$ is a cocycle deformation of $U(D, 0)$. This raises a natural question whether a cocycle deformation of a graded pointed (twisted) CY Hopf algebra is still a (twisted) CY algebra. For a Hopf algebra H and its cocycle deformation H^σ, the algebra ${}_\sigma H$ can be viewed as the "connection" between H and H^σ as it defines a Morita tensor equivalence between the comodule categories over the two Hopf algebras. To understand the relation between the twisted CY property of H and that of H^σ, we shall first answer the question whether ${}_\sigma H$ is a twisted CY algebra when H is.

The algebra ${}_\sigma H$ can be viewed as the crossed product $\Bbbk \#_\sigma H$ (the definition of a crossed product will be reviewed in Section 1.2). More generally, one could ask whether the crossed product $A\#_\sigma H$ will be a twisted CY algebra when both A and H are twisted CY algebras. In this paper, we are able to answer the question when A is a graded N-Koszul algebra. We note here that to form an algebra $A\#_\sigma H$, it is only required that σ is an invertible map in $\mathrm{Hom}(H \otimes H, A)$ satisfying the cocycle condition and A is a twisted H-module. When A is a graded N-Koszul algebra, the assumption that σ has its image in $\Bbbk$ is necessary to make sure that the obtained crossed product $A\#_\sigma H$ is still a graded algebra. In this case σ is just a 2-cocycle on H and A is a left graded H^σ-module algebra. Here A is a left graded H^σ-module algebra means that A is a left H^σ-module algebra such that each graded piece A_i is a left H^σ-module. The following theorem is our main result (see Theorem 2.18):

Theorem 0.1. *Let H be a twisted CY Hopf algebra with homological integral $\int_H^l = \Bbbk_\xi$, where $\xi : H \to \Bbbk$ is an algebra homomorphism and σ a 2-cocycle on H. Let A be a N-Koszul graded twisted CY algebra with Nakayama automorphism μ such that A is a left graded H^σ-module algebra. Then $A\#_\sigma H$ is a twisted CY algebra with Nakayama automorphism ρ defined by $\rho(a\#h) = \mu(a)\#\operatorname{hdet}_{H^\sigma}(h_1)(S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(h_2)))\xi(h_3)$ for all $a\#h \in A\#_\sigma H$.*

Here, $\operatorname{hdet}_{H^\sigma}$ denotes the homological determinant of the H^σ-action. The homological integral of a twisted CY Hopf algebra will be given in Section 2. The notion $S_{\sigma,\tau}$ will be recalled in Section 1.1. Examples of Theorem 0.1 will be provided in Section 4.

Theorem 0.1 has two consequences. Firstly, in Theorem 0.1, if we let the cocycle σ be trivial, then the crossed product $A\#_\sigma H$ is just the smash product $A\#H$. Therefore, we obtain the following result on smash products.

Theorem 0.2. *Let H be a twisted CY Hopf algebra with homological integral $\int_H^l = \Bbbk_\xi$, where $\xi : H \to \Bbbk$ is an algebra homomorphism and A an N-Koszul graded twisted CY algebra with Nakayama automorphism μ such that A is a left graded H-module algebra. Then $A\#H$ is a twisted CY algebra with Nakayama automorphism ρ defined by $\rho(a\#h) = \mu(a)\#\operatorname{hdet}_H(h_1)(S^{-2}(h_2))\xi(h_3)$, for any $a\#h \in A\#H$.*

This generalizes the results in [23] and [32]. The smash products of CY algebras has been studied quite broadly. For instance, see [16], [20], [23], [38], [32]. The results in [23] and [32] are probably two of the most general results in this direction. [23] states that when H is an involutory Hopf CY algebra and A is an N-Koszul CY algebra, the smash product $A\#H$ is CY if and only if the homological determinant of the H-action on A is trivial. One of the main results in [32] states that the smash product $A\#H$ is a twisted CY algebra when A is a graded twisted CY algebra and H a finite dimensional Hopf algebra acting on A. The Nakayama automorphism of $A\#H$ is determined by the ones of A and H, along with the homological determinant of the H-action.

Secondly, in Theorem 0.1, if we let the algebra A be $\Bbbk$, we obtain the following description of the twisted CY property of cleft objects.

Theorem 0.3. *Let H be a twisted CY Hopf algebra with $\int_H^l = {}_\xi\Bbbk$, and σ a 2-cocycle on H. Then the right cleft object ${}_\sigma H$ is a twisted CY algebra with Nakayama automorphism μ defined by*

$$\mu(x) = S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(x_1))\xi S(x_2)$$

for any $x \in {}_\sigma H$.

As an application of Theorem 0.3, we study the CY property of the cleft objects of the Hopf algebras $U(\mathcal{D}, \lambda)$ in Section 3. It turns out that all cleft objects of the algebra $U(\mathcal{D}, \lambda)$ are twisted CY algebras. Their Nakayama automorphisms are given explicitly in Proposition 3.7. Hence we are able to characterize when a clefts object is CY. It is interesting that a cleft object of $U(\mathcal{D}, \lambda)$ could be a CY algebra even when $U(\mathcal{D}, \lambda)$ itself is not. We give such an example at the end of Section 3.

Our motivating examples are the algebras of the form $A\#_\sigma \Bbbk G$, where A is a polynomial algebra, G is a finite group acting on A, and $\sigma : G \times G \to \mathbb{C}^\times$ is a 2-cocycle on G. Such crossed products are of interest in geometry due to their relationship with corresponding orbifolds (for e.g., see [2], [12], [36]). In Section 4, we show that these crossed products are all twisted CY algebras. PBW deformations of the crossed product $A\#_\sigma \Bbbk G$ are the twisted Drinfeld Hecke algebras defined in [37]. If the cocycle is trivial, then $A\#\Bbbk G$, the skew group algebra, is just the Drinfeld Hecke algebras defined by V. Drinfeld [14]. They have been studied by many authors, for example [15], [6], [25]. Quantum Drinfeld Hecke algebras are anther generalizations of Drinfeld Hecke algebras by replacing polynomial algebras by quantum polynomial algebras [27], [31]. More generally, Naidu defined twisted quantum Drinfeld Hecke algebras in [30]. A twisted quantum Drinfeld Hecke algebra is an algebra of the form $A\#_\sigma \Bbbk G$, where A is a quantum polynomial algebra, G is a finite group acting on A, and σ is a 2-cocycle on G. Twisted quantum Drinfeld Hecke algebras are generalizations of both twisted Drinfeld Hecke algebras and quantum Drinfeld Hecke algebras. A quantum polynomial algebra is a Koszul algebra. If PBW deformations of the algebra $A\#_\sigma H$ in Theorem 0.1 are still twisted CY algebras, then twisted quantum Drinfeld Hecke algebras will all be twisted CY algebras. We will discuss this problem in our upcoming paper.

1. Preliminaries

Throughout this paper, the unadorned tensor $\otimes$ means $\otimes_\Bbbk$ and Hom means $\mathrm{Hom}_\Bbbk$.

Given an algebra A, we write A^{op} for the opposite algebra of A and A^e for the enveloping algebra $A \otimes A^{op}$. An A-bimodule can be identified with a left A^e-module or a right A^e-module.

For an A-bimodule M and two algebra automorphisms μ and ν, we let ${}^{\mu}M^{\nu}$ denote the A-bimodule such that ${}^{\mu}M^{\nu} \cong M$ as vector spaces, and the bimodule structure is given by

$$a \cdot m \cdot b = \mu(a)m\nu(b),$$

for all $a, b \in A$ and $m \in M$. If one of the automorphisms is the identity, we will omit it. It is well-known that $A^{\mu} \cong {}^{\mu^{-1}}A$ as A-A-bimodules. $A^{\mu} \cong A$ as A-A-bimodules if and only if μ is an inner automorphism.

We assume that the Hopf algebras considered in this paper have bijective antipodes. For a Hopf algebra H, we use Sweedler's (sumless) notation for the comultiplication and coaction of H.

1.1. Cogroupoid.

Definition 1.1. a *cocategory* $\mathcal{C}$ consists of:

- A set of objects $\mathrm{ob}(\mathcal{C})$.
- For any $X, Y \in \mathrm{ob}(\mathcal{C})$, an algebra $\mathcal{C}(X, Y)$.
- For any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, algebra homomorphisms

  $$\Delta^Z_{XY} : \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y) \text{ and } \varepsilon_X : \mathcal{C}(X, X) \to \Bbbk$$

  such that for any $X, Y, Z, T \in \mathrm{ob}(\mathcal{C})$, the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{C}(X,Y) & \xrightarrow{\Delta^Z_{X,Y}} & \mathcal{C}(X,Z) \otimes \mathcal{C}(Z,Y) \\
{\scriptstyle \Delta^T_{X,Y}}\downarrow & & \downarrow{\scriptstyle \Delta^T_{X,Z} \otimes 1} \\
\mathcal{C}(X,T) \otimes \mathcal{C}(T,Y) & \xrightarrow{1 \otimes \Delta^Z_{T,Y}} & \mathcal{C}(X,T) \otimes \mathcal{C}(T,Z) \otimes \mathcal{C}(Z,Y)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{C}(X,Y) & & \\
{\scriptstyle \Delta^Y_{X,Y}}\downarrow & \searrow{=} & \\
\mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Y) & \xrightarrow{1 \otimes \varepsilon_Y} & \mathcal{C}(X,Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}(X,Y) & & \\
{\scriptstyle \Delta^X_{X,Y}}\downarrow & \searrow{=} & \\
\mathcal{C}(X,X) \otimes \mathcal{C}(X,Y) & \xrightarrow{\varepsilon_X \otimes 1} & \mathcal{C}(X,Y).
\end{array}$$

Thus a cocategory with one object is just a bialgebra.

A cocategory $\mathcal{C}$ is said to be *connected* if $\mathcal{C}(X, Y)$ is a non zero algebra for any $X, Y \in \mathrm{ob}(\mathcal{C})$.

Definition 1.2. A *cogroupoid* $\mathcal{C}$ consists of a cocategory $\mathcal{C}$ together with, for any $X, Y \in \mathrm{ob}(\mathcal{C})$, linear maps

$$S_{X,Y} : \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(Y,X)$$

such that for any $X, Y \in \mathcal{C}$, the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{C}(X,X) & \xrightarrow{\varepsilon_X} & \Bbbk & \xrightarrow{u} & \mathcal{C}(X,Y) \\ \downarrow{\scriptstyle \Delta^Y_{X,X}} & & & & \uparrow{\scriptstyle m} \\ \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,X) & & \xrightarrow{1\otimes S_{Y,X}} & & \mathcal{C}(X,Y)\otimes\mathcal{C}(X,Y) \end{array}$$

$$\begin{array}{ccccc} \mathcal{C}(X,X) & \xrightarrow{\varepsilon_X} & \Bbbk & \xrightarrow{u} & \mathcal{C}(Y,X) \\ \downarrow{\scriptstyle \Delta^Y_{X,X}} & & & & \uparrow{\scriptstyle m} \\ \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,X) & & \xrightarrow{S_{X,Y}\otimes 1} & & \mathcal{C}(Y,X)\otimes\mathcal{C}(Y,X) \end{array}$$

We refer to [8] for basic properties of cogroupoids.

In this paper, we are mainly concerned with the 2-cocycle cogroupoid of a Hopf algebra.

Definition 1.3. Let H be a Hopf algebra. A *(right) 2-cocycle* on H is a convolution invertible linear map $\sigma : H \otimes H \to \Bbbk$ satisfying

$$\sigma(h_1, k_1)\sigma(h_2k_2, l) = \sigma(k_1, l_1)\sigma(h, k_2l_2) \tag{1}$$

$$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h) \tag{2}$$

for all $h, k, l \in H$. The set of 2-cocycles on H is denoted $Z^2(H)$.

The convolution inverse of σ, denote σ^{-1}, satisfies

$$\sigma^{-1}(h_1k_1, l)\sigma^{-1}(h_2, k_2) = \sigma^{-1}(h, k_1l_1)\sigma^{-1}(k_2, l_2) \tag{3}$$

$$\sigma^{-1}(h, 1) = \sigma^{-1}(1, h) = \varepsilon(h) \tag{4}$$

for all $h, k, l \in H$. Such a convolution invertible map is called a *left 2-cocycle* on H. Conversely, the convolution inverse of a left 2-cocycle is just a right 2-cocycle.

The set of 2-cocycles defines the 2-cocycle cogroupoid of H.

Let $\sigma, \tau \in Z^2(H)$. The algebra $H(\sigma, \tau)$ is defined to be the vector space H together with the multiplication given by

$$h \centerdot k = \sigma(h_1, k_1)h_2k_2\tau^{-1}(h_3, k_3), \tag{5}$$

for any $h, k \in H$.

Now we recall the necessary structural maps for the 2-cocycle cogroupoid on H. For any $\sigma, \tau, \omega \in Z^2(H)$, define the following maps:

$$\begin{aligned} \Delta^{\omega}_{\sigma,\tau} = \Delta : H(\sigma,\tau) &\longrightarrow H(\sigma,\omega) \otimes H(\omega,\tau) \\ h &\longmapsto h_1 \otimes h_2. \end{aligned} \tag{6}$$

$$\varepsilon_\sigma = \varepsilon : H(\sigma,\sigma) \longrightarrow \Bbbk. \tag{7}$$

$$\begin{aligned} S_{\sigma,\tau} : H(\sigma,\tau) &\longrightarrow H(\tau,\sigma) \\ h &\longmapsto \sigma(h_1, S(h_2))S(h_3)\tau^{-1}(S(h_4), h_5). \end{aligned} \tag{8}$$

It is routine to check that the inverse of $S_{\sigma,\tau}$ is given as follows:

$$\begin{aligned} S^{-1}_{\sigma,\tau} : H(\tau,\sigma) &\longrightarrow H(\sigma,\tau) \\ h &\longmapsto \sigma^{-1}(h_5, S^{-1}(h_4))S^{-1}(h_3)\tau(S^{-1}(h_2), h_1). \end{aligned} \tag{9}$$

The 2-*cocycle cogroupoid* of H, denoted by $\underline{H}$, is the cogroupoid defined as follows:

(i) $\mathrm{ob}(\underline{H}) = Z^2(H)$.
(ii) For $\sigma, \tau \in Z^2(H)$, the algebra $\underline{H}(\sigma,\tau)$ is the algebra $H(\sigma,\tau)$ defined in (5).
(iii) The structural maps $\Delta^{\bullet}_{\bullet,\bullet}$, $\varepsilon_\bullet$ and $S_{\bullet,\bullet}$ are defined in (6), (7) and (8) respectively.

[8, Lemma 3.13] shows that the maps $\Delta^{\bullet}_{\bullet,\bullet}$, $\varepsilon_\bullet$ and $S_{\bullet,\bullet}$ indeed satisfy the conditions required for a cogroupoid. It is clear that a 2-cocycle cogroupoid is connected. The following lemma follows from basis properties of cogroupoids.

Lemma 1.4. [8, Proposition 2.13]*Let $\underline{H}$ be the 2-cocycle cogroupoid, and let $\sigma, \tau \in ob(\underline{H})$.*

(i) *$S_{\sigma,\tau} : H(\sigma,\tau) \to H(\tau,\sigma)^{op}$ is an algebra homomorphism.*
(ii) *For any $\omega \in ob(\underline{H})$ and $h \in H$, we have*

$$\Delta^{\omega}_{\tau,\sigma}(S_{\sigma,\tau}(h)) = S_{\omega,\tau}(h_1) \otimes S_{\sigma,\omega}(h_2).$$

The Hopf algebra $H(1,1)$ (where 1 stands for $\varepsilon \otimes \varepsilon$) is just the Hopf algebra H itself. Let σ be a 2-cocycle. We write ${}_\sigma H$ for the algebra $H(\sigma, 1)$. Similarly, we write $H_{\sigma^{-1}}$ for the algebra $H(1,\sigma)$. To make the presentation clear, we let $\cdot_\sigma$ and $\cdot_{\sigma^{-1}}$ denote the multiplications in ${}_\sigma H$ and $H_{\sigma^{-1}}$ respectively.

The Hopf algebra $H(\sigma,\sigma)$ is just the *cocycle deformation* H^σ of H defined by Doi in [13]. The comultiplication of H^σ is the same as the comultiplication of H. However, the multiplication and the antipode are deformed:

$$h * k = \sigma(h_1, k_1)h_2k_2\sigma^{-1}(h_3, k_3),$$

$$S_{\sigma,\sigma}(h) = \sigma(h_1, S(h_2))S(h_3)\sigma^{-1}(S(h_4), h_5)$$

for any $h, k \in H^\sigma$. In the following, $S_{\sigma,\sigma}$ is denoted by S^σ for simplicity.

1.2. **Cleft extensions.** A Hopf algebra H is said to *measure* an algebra A if there is a $\Bbbk$-linear map $H \otimes A \to A$, given by $h \otimes a \mapsto h \cdot a$, such that $h \cdot 1 = \varepsilon(h)$ and $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$ for all $h \in H$, $a, b \in A$.

Definition 1.5. Let H be a Hopf algebra and A an algebra. Assume that H measures A and that σ is an invertible map in $\mathrm{Hom}(H \otimes H, A)$. The *crossed product* $A\#_\sigma H$ of A with H is defined on the vector space $A \otimes H$ with multiplication given by

$$(a\#h)(b\#k) = a(h_1 \cdot b)\sigma(h_2, k_1)\#h_3k_2$$

for all $h, k \in H$, $a, b \in A$. Here we write $a\#h$ for the tensor product $a \otimes h$.

The following lemma is well-known (cf. [29, Lemma 7.1.2]).

Lemma 1.6. *$A\#_\sigma H$ is an associative algebra with identity element $1\#1$ if and only if the following two conditions are satisfied:*

(i) *A is a twisted H-module. That is, $1 \cdot a = a$ for all $a \in A$, and*

$$h \cdot (k \cdot a) = \sigma(h_1, k_1)(h_2k_2 \cdot a)\sigma^{-1}(h_3, k_3),$$

for all $h, k \in H$, $a \in A$.

(ii) *σ is a cocycle. That is, $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1$ for all $h \in H$ and*

$$[h_1 \cdot \sigma(k_1, m_1)]\sigma(h_2, k_2m_2) = \sigma(h_1, k_1)\sigma(h_2k_2, m)$$

for all $h, k, m \in H$.

Note that if σ is trivial, that is, $\sigma(h, k) = \varepsilon(h)\varepsilon(k)1$, for all $h, k \in H$. Then the crossed product $A\#_\sigma H$ is just the smash product $A\#H$.

Remark 1.7. Let $A\#_\sigma H$ be a crossed product and σ an invertible map in $\mathrm{Hom}(H \otimes H, \Bbbk)$. Then $A\#_\sigma H$ is an associative algebra if and only if σ is a 2-cocycle and A is an H^σ-module algebra.

Definition 1.8. Let $A \subseteq B$ be an extension of algebras, and H a Hopf algebra.

(i) $A \subseteq B$ is called a *(right) H-extension* if B is a right H-comodule algebra such that $B^{coH} = A$.

(ii) The H-extension $A \subseteq B$ is said to be *H-cleft* if there exists a right H-comodule morphism $\gamma : H \to B$ which is (convolution) invertible. Note that this γ can be chosen such that $\gamma(1) = 1$.

If $\Bbbk \subseteq B$ is H-cleft, then B is called a *(right) cleft object*. Left cleft extensions and left cleft objects can be defined similarly.

Lemma 1.9. [29, Theorem 7.2.2, Proposition 7.2.3, Proposition 7.2.7] *Let H be a Hopf algebra. An H-extension $A \subseteq B$ is H-cleft with right convolution invertible H-comodule morphism $\gamma : H \to B$ if and only if $B \cong A\#_\sigma H$ as algebras with a convolution invertible map $\sigma : H \otimes H \to A$. The twisted H-module action on A is given by*

$$h \cdot a = \gamma(h_1)a\gamma^{-1}(h_2),$$

for all $a \in A$, $h \in H$. Moreover, γ and σ are constructed each other by

$$\sigma(h,k) = \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2)$$

and

$$\gamma(h) = 1\#h, \quad \gamma^{-1}(h) = \sigma^{-1}(Sh_2, h_3)\#Sh_1$$

for all $h,k \in H$, $a \in A$.

From this lemma, we see that right cleft objects of a Hopf algebra H are just the algebras ${}_\sigma H$, where σ is a 2-cocycle on H.

1.3. **AS-Gorenstein algebras.** In this paper, unless otherwise stated, a graded algebra will always mean an $\mathbb{N}$-graded algebra. An $\mathbb{N}$-graded algebra $A = \oplus_{i\geqslant 0}A_i$ is called connected if $A_0 = \Bbbk$.

Definition 1.10. A connected graded algebra A is called *AS-Gorenstein* if the following conditions hold:

(i) A has finite injective dimension d on both sides,

(ii) $\operatorname{Ext}^i_A({}_A\Bbbk, {}_AA) \cong \begin{cases} 0, & i \neq d; \\ \Bbbk(l), & i = d, \end{cases}$ where l is an integer,

(iii) The right version of (ii) holds.

If, in addtion,

(iv) A is of finite global dimension d, then A is called *AS-regular*.

Noe that an AS-Gorenstein (regular) algebra can be defined on an augmented algebra in general, see [10]. For an algebra A, if the injective dimension of ${}_AA$ and A_A are both finite, then these two integers are equal by [40, Lemma A]. We call this common value the *injective dimension* of A. The left global dimension and the right global dimension of a Noetherian algebra are equal. When the global dimension is finite, then it is equal to the injective dimension.

Definition 1.11. (cf. [10, defn. 1.2]). Let A be a Noetherian algebra with a fixed augmentation map $\varepsilon : A \to \Bbbk$.

(i) The algebra A is said to be *AS-Gorenstein*, if
   (a) $\operatorname{injdim}{}_A A = d < \infty$,
   (b) $\dim \operatorname{Ext}^i_A({}_A\Bbbk, {}_AA) = \begin{cases} 0, & i \neq d; \\ 1, & i = d, \end{cases}$
   (c) the right versions of (a) and (b) hold.

(ii) If, in addition, the global dimension of A is finite, then A is called *AS-regular*.

The concept of a homological integral for an AS-Gorenstein Hopf algebra was introduced by Lu, Wu and Zhang in [24] to study infinite dimensional Noetherian Hopf algebras. It is a generalization of the concept of an integral of a finite dimensional Hopf algebra. It turns out that homological integrals are useful in describing homological properties of Hopf algebras (see e.g. [18, Theorem 2.3]).

Definition 1.12. Let A be an AS-Gorenstein algebra with injective dimension d. Then $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$ is a 1-dimensional right A-module. Any nonzero element in $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$ is called a *left homological integral* of A. We write $\int^l_A$ for $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$. Similarly, $\operatorname{Ext}^d_A(\Bbbk_A, A_A)$ is a 1-dimensional left A-module. Any nonzero element in $\operatorname{Ext}^d_A(\Bbbk_A, A_A)$ is called a *right homological integral* of A. Write $\int^r_A$ for $\operatorname{Ext}^d_A(\Bbbk_A, A_A)$.

$\int^l_A$ and $\int^r_A$ are called *left and right homological integral modules* of A respectively.

The left integral module $\int^l_A$ is a 1-dimensional right A-module. Thus $\int^l_A \cong \Bbbk_\xi$ for some algebra homomorphism $\xi : A \to \Bbbk$. Similarly, $\int^r_A \cong {}_\eta\Bbbk$ for some algebra homomorphism η.

1.4. N**-Koszul algebras.** Let V be a finite dimensional vector space, and $T(V) = \Bbbk \otimes V \otimes V^{\otimes 2} \otimes \cdots$ be the tensor algebra with the usual grading. A graded algebra $T(V)/\langle R\rangle$ is called *N-homogenous* if R is a subspace of $V^{\otimes N}$. Let V^* be the dual space $\operatorname{Hom}(V, \Bbbk)$. The algebra $A^! = T(V^*)/\langle R^\perp\rangle$ is called the *homogeneous dual* of A, where $R^\perp$ is the orthogonal subspace of R in $(V^*)^{\otimes N}$.

Remark 1.13. Let ϕ be the map defined as follows:

$$\begin{array}{rccl} \phi: & (V^*)^{\otimes n} & \to & (V^{\otimes n})^* \\ & f_n \otimes f_{n-1} \otimes \cdots \otimes f_1 & \mapsto & \phi(f_n \otimes f_{n-1} \otimes \cdots \otimes f_1), \end{array}$$

where $\phi(f_n \otimes f_{n-1} \otimes \cdots \otimes f_1)(x_1 \otimes \cdots x_{n-1} \otimes x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$, for any $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in V^{\otimes n}$. This map ϕ is a bijection. Throughout, we identify $(V^*)^{\otimes n}$ with $(V^{\otimes n})^*$ via this bijection.

Let $\mathbf{n} : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$\mathbf{n}(i) = \begin{cases} Nk, & i = 2k \\ Nk+1, & i = 2k+1. \end{cases}$$

An N-homogenous algebra A is called *N-Koszul* if the trivial module ${}_A\Bbbk$ admits a graded projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to {}_A\Bbbk \to 0$$

such that P_i is generated in degree $\mathbf{n}(i)$ for all $i \geqslant 0$. A Koszul algebra is a just 2-Koszul algebra.

The Koszul bimodule complex of a Koszul algebra is constructed by Van den Bergh in [35]. This complex was generalized to N-Koszul case in [7]. Now let $A = T(V)/\langle R \rangle$ be an N-Koszul algebra. Let $\{e_i\}_{i=1,2,\cdots,n}$ be a basis of V and $\{e_i^*\}_{i=1,2,\cdots,n}$ the dual basis. Define two N-differentials

$$d_l, d_r : A \otimes (A^!_p)^* \otimes A \to A \otimes (A^!_{p-1})^* \otimes A$$

as follows:

$$d_l(x \otimes \omega \otimes y) = \textstyle\sum_{i=1}^n xe_i \otimes e_i^* \cdot \omega \otimes y$$
$$d_r(x \otimes \omega \otimes y) = \textstyle\sum_{i=1}^n x \otimes \omega \cdot e_i^* \otimes e_i y,$$

for $x \otimes \omega \otimes y \in A \otimes (A^!_p)^* \otimes A$. The left action $e_i^* \cdot \omega$ is defined by $[e_i^* \cdot \omega](\alpha) = \omega(\alpha e_i^*)$ for any $\alpha \in (A^!_{p-1})^*$. The right action $\omega \cdot e_i^*$ is defined similarly. One can check that d_l and d_r commute. Fix a primitive N-th root of unity q. Define $d : A \otimes (A^!_p)^* \otimes A \to A \otimes (A^!_{p-1})^* \otimes A$ by $d = d_l - q^{p-1}d_r$. We obtain the following N-complex:

$$\mathbf{K_{l-r}(A)} : \cdots \xrightarrow{d_l - d_r} A \otimes (A^!_N)^* \otimes A \xrightarrow{d_l - q^{N-1}d_r} \cdots \xrightarrow{d_l - qd_r} A \otimes V \otimes A \xrightarrow{d_l - d_r} A \otimes A \to 0.$$

The bimodule Koszul complex $\mathbf{K_b(A)}$ is a contraction of $\mathbf{K_{l-r}(A)}$. It is obtained by keeping the arrow $A \otimes V \otimes A \xrightarrow{d_l - d_r} A \otimes A$ at the far right, then putting together the $N-1$ consecutive ones, and continuing alternately:

$$\mathbf{K_b(A)} : \cdots \xrightarrow{d^{N-1}} A \otimes (A^!_{N+1})^* \otimes A \xrightarrow{d} A \otimes (A^!_N)^* \otimes A \xrightarrow{d^{N-1}} A \otimes V \otimes A \xrightarrow{d} A \otimes A \to 0.$$

Here $d = d_l - d_r$ and $d^{N-1} = d_l^{N-1} + d_l^{N-2}d_r + \cdots + d_l d_r^{N-2} + d_r^{N-1}$.

An N-homogenous algebra is N-Koszul if and only if the complex $\mathbf{K_b(A)} \to A \to 0$ is exact via the multiplication $A \otimes A \to A$ [7, Theorem 4.4]. Moreover, in such a case, $\mathbf{K_b(A)} \to A \to 0$ is a minimal bimodule free resolution of A.

1.5. **Calabi-Yau algebras.**

Definition 1.14. An algebra A is called a *twisted Calabi-Yau algebra of dimension d* if

(i) A is *homologically smooth*, that is, A has a bounded resolution of finitely generated projective A^e-modules;
(ii) There is an automorphism μ of A such that

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d \\ A^{\mu}, & i = d \end{cases} \tag{10}$$

as A^e-modules.

If such an automorphism μ exists, it is unique up to an inner automorphism and is called the *Nakayama automorphism* of A. A *Calabi-Yau algebra* is a twisted Calabi-Yau algebra whose Nakayama automorphism is an inner automorphism.

A *Graded twisted CY algebra* can be defined in a similar way. That is, we should consider the category of graded modules and condition (10) should be replaced by

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A_{\mu}(l), & i = d, \end{cases}$$

where l is an integer and $A_{\mu}(l)$ is the shift of A_{μ} by degree l.

We end this section with the following lemma, which shows that AS-regular Hopf algebras are just twisted CY Hopf algebras.

Lemma 1.15. *Let A be a Noetherian AS-regular Hopf algebra with $\int_A^l = \Bbbk_{\xi}$, where $\xi : A \to \Bbbk$ is an algebra homomorphism. The followings hold:*

(i) [32, Lemma 1.3] *The algebra A is twisted CY with Nakayama automorphism μ defined by $\mu(x) = S^{-2}(x_1)\xi(x_2)$ for any $x \in A$. (Alternatively, the algebra automorphism ν defined by $\nu(x) = \xi(x_1)S^2(x_2)$ is also a Nakayama automorphism of A).*
(ii) [18, Theorem 2.3] *The algebra A is CY if and only if $\xi = \varepsilon$, and S^2 is an inner automorphism.*

2. The CY property of Cleft extension

Let H be a Hopf algebra, σ a 2-cocycle on H and A an N-Koszul H^{σ}-module algebra. Then the crossed product $A\#_{\sigma}H$ is an associative algebra. In this section we show that $A\#_{\sigma}H$ is a twisted CY algebra if both A and H are twisted CY algebras. This generalizes [23, Theorem 2.12] and [32, Theorem 0.2],

The following definition is inspired by "H_{S^i}-equivariant A-bimodule" introduced in [32, Definition 2.2], where H is a Hopf algebra and i is an even integer.

Definition 2.1. Let H be a Hopf algebra and A a left H-module algebra. For a given even integer i, we define an algebra $A^e \rtimes_{S^i} H$. As vector spaces, $A^e \rtimes_{S^i} H = A \otimes A \otimes H$. The multiplication is given by

$$(a \otimes b \otimes g)(a' \otimes b' \otimes h) = a(S^i g_1 \cdot a') \otimes (g_3 \cdot b')b \otimes g_2 h,$$

for any $a \otimes b \otimes g, a' \otimes b' \otimes h \in A \otimes A \otimes H$.

Remark 2.2. (i) When $i = 0$, $A^e \rtimes_{S^i} H$ is just the algebra $A^e \rtimes H$ introduced by Kaygun [21].

(ii) An $A^e \rtimes_{S^i} H$-module M is a vector space such that it is both an A^e-module and an H-module satisfying

$$h \cdot (amb) = ((S^i h_1) \cdot a)(h_2 \cdot m)(h_3 \cdot b), \tag{11}$$

for any $h \in H$, $a, b \in A$ and $m \in M$.

Lemma 2.3. *Let M be an $A^e \rtimes_{S^i} H$-module and N an $(A\#H)^e$-module.*

(i) *The space $\mathrm{Hom}_{A^e}(M, N)$ is a left H-module with the H-action defined by*

$$(h \rightharpoonup f)(m) = (S^i h_3) f[(S^{-1} h_2) \cdot m](S^{-1} h_1) \tag{12}$$

for any $h \in H$, $f \in \mathrm{Hom}_{A^e}(M, N)$ and $m \in M$.

(ii) *The space $M \otimes_{A^e} N$ is a left H-module with the H-action given by*

$$h \cdot (m \otimes n) = h_2 \cdot m \otimes h_3 n (S^{i+1} h_1) \tag{13}$$

for any $h \in H$ and $m \otimes n \in M \otimes N$.

Proof. The proof is routine and quite similar to the proofs of Lemma 1.8 and Lemma 1.9 in [23].

Remark 2.4. Keep the notations as in Lemma 2.3, $\mathrm{Hom}_{A^e}(M, N)$ can be made into a right H-module by defining $f \leftharpoonup h = Sh \rightharpoonup f$ for any $h \in H$ and $f \in \mathrm{Hom}_{A^e}(M, N)$. That is,

$$(f \leftharpoonup h)(m) = S^{i+1} h_1 f(h_2 \cdot m) h_3. \tag{14}$$

Since A is a left H-module algebra, the algebra A^e is an $(A\#H)^e$-module with the following module structure:

$$(a\#h)\cdot(x\otimes y)=a(h\cdot x)\otimes y,\ \ (x\otimes y)\cdot(b\#g)=x\otimes(S^{-1}g)\cdot(yb) \tag{15}$$

for any $x\otimes y\in A^e$ and $a\#h,\ b\#g\in A\#H$.

By Lemma 2.3, $\mathrm{Hom}_{A^e}(M,A^e)$ is a left H-module for any $A^e\rtimes H$-module M. Furthermore, the A^e-bimodule structure of A^e induces a left A^e-module structure on $\mathrm{Hom}_{A^e}(M,A^e)$. That is,

$$[(a\otimes b)\cdot f](x)=f(x)(b\otimes a), \tag{16}$$

for any $a\otimes b\in A^e$, $f\in\mathrm{Hom}_{A^e}(M,A^e)$ and $x\in M$.

In [23] the authors showed that if H is involutory, then $\mathrm{Hom}_{A^e}(M,A^e)$ is again an $A^e\rtimes H$-module for any $A^e\rtimes H$-module M. In general, we have the following.

Lemma 2.5. *Let M be an $A^e\rtimes H$-module. Then $\mathrm{Hom}_{A^e}(M,A^e)$ is an $A^e\rtimes_{S^{-2}} H$-module.*

In [23, Theorem 2.4] the Van den Bergh duality was generalized to algebras with a Hopf action from an involutory Hopf algebra. In fact, we can drop the condition "involutory".

Proposition 2.6. *Let H be a Hopf algebra and A a left H-module algebra. Assume that A admits a finitely generated A^e-projective resolution of finite length such that it is a complex of $A^e\rtimes H$-modules. Suppose there exists an integer d such that*

$$\mathrm{Ext}^i_{A^e}(A,A^e)=\begin{cases}0, & i\neq d;\\ U, & i=d,\end{cases}$$

where U is an invertible A^e-module. Then for any $(A\#H)^e$-module N, we have

$$\mathrm{HH}^i(A,N)\cong{}_{S^{-2}}\mathrm{HH}_{d-i}(A,U\otimes_A N)$$

as left H-modules.

Proof. Suppose that P is an $A^e\rtimes H$-module such that it is finitely generated and projective as an A^e-module, and N is an $(A\#H)^e$-module. By Lemma 2.5, $\mathrm{Hom}_{A^e}(P,A^e)$ is an $A^e\rtimes_{S^{-2}} H$-module. So $\mathrm{Hom}_{A^e}(P,A^e)\otimes_{A^e}N$ is an H-module with the module structure given by (13). Moreover, the equation (12) defines an H-module structure on $\mathrm{Hom}_{A^e}(P,N)$. With these H-actions, one can check that the canonical isomorphism

$$\Psi:\mathrm{Hom}_{A^e}(P,A^e)\otimes_{A^e}N\to\mathrm{Hom}_{A^e}(P,N)$$

is also an H-isomorphism. Therefore, the proof of [23, Theorem 2.4] works for non-involutory Hopf algebras. But for a non-involutory Hopf algebra H, the module U is an $A^e \rtimes_{S^{-2}} H$-module by Lemma 2.5. Thus, $U \otimes_A N$ is an $(A\#H)^e$-module with module structure defined by

(17) $(a\#h)\cdot(u\otimes n) = a((S^2h_1)\cdot u)\otimes(S^2h_2)\cdot n,\;\; (u\otimes n)\cdot(b\#g) = u\otimes n\cdot(b\#g),$

for any $a\#h$, $b\#g \in A\#H$ and $u\#n \in U \otimes N$. Consequently, we have the following H-isomorphisms:

$$\begin{aligned}
\mathrm{HH}^i(A,N) &\cong \mathrm{Ext}^i_{A^e}(A,N)\\
&\cong \mathrm{H}^i(\mathrm{RHom}_{A^e}(A,N))\\
&\cong \mathrm{H}^i(\mathrm{RHom}_{A^e}(A,A^e)^L \otimes_{A^e} N)\\
&\cong \mathrm{H}^i(U[-d]^L \otimes_{A^e} N)\\
&\cong \mathrm{H}^{i-d}(U^L \otimes_{A^e} N)\\
&\cong \mathrm{H}^{i-d}({}_{S^{-2}}[A \otimes_{A^e} (U^L \otimes_A N)])\\
&\cong {}_{S^{-2}}\,\mathrm{HH}_{d-i}(A, U\otimes_A N).
\end{aligned}$$

□

In the rest of this section, we work with the category of graded modules. Let A be a graded algebra, and let A-GrMod denote the category of graded left A-modules and graded homomorphisms of degree zero. For any $M, N \in A$-GrMod, $\mathrm{Hom}_A(M,N)$ is the graded vector space consisting of graded A-module homomorphisms. That is,

$$\mathrm{Hom}_A(M,N) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{A\text{-GrMod}}(M, N(i)).$$

Let H be a Hopf algebra. We say that a graded algebra A is a left graded H-module algebra if it is a left H-module algebra such that each A_i is an H-module. Let σ is a 2-cocycle on H. The cocycle deformation H^σ is a Hopf algebra. If A is a left graded H^σ-module algebra, then we have the algebra $A\#H^\sigma$. Moreover, we can construct the algebra $A\#_\sigma H$ by Remark 1.7. It is easy to see that both $A\#H^\sigma$ and $A\#_\sigma H$ have natural graded algebra structures.

Now, we fix a Hopf algebra H and a 2-cocycle σ on H. Let V be a left H^σ-module and $A = T(V)/\langle R\rangle$ an N-Koszul graded H^σ-module algebra. The dual V^* is a right H^σ-module with the module structure given by

(18) $$(\alpha \triangleleft h)(x) = \alpha(h\cdot x).$$

for $\alpha \in V^*$, $h \in H$ and $x \in V$.

Remark 2.7. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of V. Suppose that $h\cdot e_i = \sum_{j=1}^n c^h_{ji} e_j$ with $c^h_{ji} \in \Bbbk$. Then we have $e^*_i \triangleleft h = \sum_{j=1}^n c^h_{ij} e^*_j$.

We extend the action "$\triangleleft$" on V^* to $(V^*)^{\otimes n}$:

$$(\alpha_n \otimes \alpha_{n-1} \otimes \cdots \otimes \alpha_1) \triangleleft h = (\alpha_n \triangleleft h_n) \otimes (\alpha_{n-1} \triangleleft h_{n-1}) \otimes \cdots \otimes (\alpha_1 \triangleleft h_1).$$

It is easy to check that $R^\perp \triangleleft h \subseteq R^\perp$. Consequently, $A^!$ is a right H^σ-module algebra with the action "$\triangleleft$". In fact, one can make $A^!$ into a left H^σ-module algebra as follows:

$$h \cdot \beta = \beta \triangleleft (S^{\sigma^{-1}} h), \tag{19}$$

for any $\beta \in A^!$ and $h \in H$.

Thanks to Lemma 2.5, we obtain the following proposition generalizing [23, Proposition 2.2].

Proposition 2.8. *Let H be a Hopf algebra, σ a 2-cocycle on H, and A a left graded H^σ-module algebra. If A is an N-Koszul graded twisted CY algebra of dimension d with Nakayama automorhism μ, then as $A^e \rtimes_{S^{\sigma^{-2}}} H^\sigma$-modules*

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A_\mu \otimes A^!_{\mathbf{n}(d)}, & i = d, \end{cases}$$

where the $A^e \rtimes_{S^{\sigma^{-2}}} H^\sigma$-module structure on $A_\mu \otimes A^!_{\mathbf{n}(d)}$ is given by

$$(a \otimes b \otimes h)(x \otimes \alpha) = a((S^{\sigma^{-2}} h_1) \cdot x)\mu(b) \otimes h_2 \cdot \alpha, \tag{20}$$

for any $a \otimes b \otimes h \in A^e \rtimes_{S^{-2}} H$ and $x \otimes \alpha \in A \otimes A^!_{\mathbf{n}(d)}$.

Proof. The algebra H^σ is a Hopf algebra and the algebra A is a left H^σ-module algebra. Proposition 2.1 in [23] shows that the A^e-projective resolution $\mathbf{K_b(A)} \to A \to 0$ of A is an $A^e \rtimes H^\sigma$-module complex. The $A^e \rtimes H^\sigma$-module structure is defined as follows. Each term in $\mathbf{K_b(A)}$ is of the form $A \otimes (A^!_p)^* \otimes A$. Since $A^!_p$ is a right H^σ-module with the action "$\triangleleft$" defined in (18), $(A^!_p)^*$ is a natural left H^σ-module. That is,

$$(h \cdot \omega)(x) = \omega(x \triangleleft h),$$

for any $h \in H^\sigma$, $\omega \in (A^!_p)^*$ and $x \in A^!_p$. Each $A \otimes (A^!_p)^* \otimes A$ is an $A^e \rtimes H^\sigma$-module with the module structure defined by

$$(a \otimes b \otimes h) \cdot (x \otimes \omega \otimes y) = a(h_1 \cdot x) \otimes h_2 \cdot \omega \otimes (h_3 \cdot y)b, \tag{21}$$

where $a \otimes b \otimes h \in A^e \rtimes H$ and $x \otimes \omega \otimes y \in A \otimes (A^!_p)^* \otimes A$.

Now we recall another bimodule complex constructed in [7]. First, we define two N-differentials:

$$\delta_l, \delta_r : A \otimes A^!_p \otimes A \to A \otimes A^!_{p+1} \otimes A$$

as follows:

$$\delta_l(x \otimes \alpha \otimes y) = \sum_{i=1}^{n} xe_i \otimes e_i^* \alpha \otimes y, \text{and } \delta_r(x \otimes \alpha \otimes y) = \sum_{i=1}^{n} x \otimes \alpha e_i^* \otimes e_i y,$$

for $x \otimes \alpha \otimes y \in A \otimes A^!_p \otimes A$. It is easy to check that δ_l and δ_r commute. Fix a primitive N-th root of unity q. The complex

$$\mathbf{L_{l-r}(A)} : A \otimes A \xrightarrow{\delta_r - \delta_l} A \otimes V^* \otimes A \xrightarrow{\delta_r - q\delta_l} \cdots \xrightarrow{\delta_r - q^{N-1}\delta_l} A \otimes A^!_N \otimes A \xrightarrow{\delta_r - \delta_l} \cdots$$

is an N-complex. The complex $\mathbf{L_b(A)}$ is the contraction of $\mathbf{L_{l-r}(A)}$. It is obtained by keeping the arrow $A \otimes A \xrightarrow{\delta_r - \delta_l} A \otimes V^* \otimes A$ at the far left, then putting together the $N-1$ following ones, and continuing alternately:

$$\mathbf{L_b(A)} : A \otimes A \xrightarrow{\delta} A \otimes V^* \otimes A \xrightarrow{\delta^{N-1}} A \otimes A^!_N \otimes A \xrightarrow{\delta} A \otimes A^!_{N+1} \otimes A \xrightarrow{\delta^{N-1}} \cdots,$$

where $\delta = \delta_r - \delta_l$ and $\delta^{N-1} = \delta_r^{N-1} + \delta_r^{N-2}\delta_l + \cdots + \delta_r \delta_l^{N-2} + \delta_l^{N-1}$. When the Hopf algebra H^σ is involutory, Proposition 2.2 in [23] shows that the complex $\mathrm{Hom}_{A^e}(\mathbf{K_b(A)}, A^e)$ and the complex $\mathbf{L_b(A)}$ are isomorphic as $A^e \rtimes H^\sigma$-complexes.

When H^σ is not involutory, $\mathrm{Hom}_{A^e}(\mathbf{K_b(A)}, A^e)$ is a complex of $A^e \rtimes_{S^{\sigma^{-2}}} H^\sigma$-modules by Lemma 2.5. In this case, $\mathrm{Hom}_{A^e}(\mathbf{K_b(A)}, A^e)$ and $\mathbf{L_b(A)}$ are isomorphic as $A^e \rtimes_{S^{\sigma^{-2}}} H^\sigma$-module complexes. The $A^e \rtimes_{S^{\sigma^{-2}}} H^\sigma$-module structure of each term $A \otimes A^!_p \otimes A$ in $\mathbf{L_b(A)}$ is given by

$$(a \otimes b \otimes h) \cdot (x \otimes \alpha \otimes y) = a((S^{\sigma^{-2}} h_1) \cdot x) \otimes h_2 \cdot \alpha \otimes (h_3 \cdot y)b,$$

for any $a \otimes b \otimes h \in A^e \rtimes_{S^{\sigma^{-2}}} H^\sigma$ and $x \otimes \alpha \otimes y \in A \otimes A^!_p \otimes A$.

Now we can use the complex $\mathbf{L_b(A)}$ to compute $\mathrm{Ext}^*_{A^e}(A, A^e)$. The method is the same as the one in the proof of Proposition 2.2 in [23].

Since the algebra A is an N-Koszul graded twisted CY algebra, A is AS-regular (see [32, Lemma 1.2]). The Ext algebra $E(A)$ of A is graded Frobenius by Corollary 5.12 in [7]. Thus, there exists an automorphism ϕ of $E(A)$, such that

$$E(A)_\phi \cong E(A)^*(-d)$$

as $E(A)$-bimodules.

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of $A_1 = V$, and $\{e_1^*, e_2^*, \cdots, e_n^*\}$ the corresponding dual basis. Suppose that ϕ is given by

$$\phi(e_1^*, e_2^*, \cdots, e_n^*) = (e_1^*, e_2^*, \cdots, e_n^*)Q,$$

for some invertible matrix Q. Define an automorphism of A via

$$\varphi(e_1, e_2, \cdots, e_n) = (e_1, e_2, \cdots, e_n)Q^T,$$

where Q^T is the transpose of Q. It is obvious that the restriction of ϕ to V^* and the restriction of φ to V are dual to each other.

Let ϵ be the automorphism of A defined by $\epsilon(a) = (-1)^i a$ for any homogeneous element $a \in A_i$. By assumption, we have $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ for $i \neq d$. Now we compute $\mathrm{Ext}^d_{A^e}(A, A^e)$. Suppose $N \geqslant 3$. Then the dimension d must be odd. We consider the following sequence

$$A \otimes A^!_{\mathbf{n}(d)-1} \otimes A \xrightarrow{\delta} A \otimes A^!_{\mathbf{n}(d)} \otimes A \xrightarrow{u} A_\mu \otimes A^!_{\mathbf{n}(d)} \to 0, \tag{22}$$

where $\mu = \epsilon^{d+1}\varphi$ and the morphism u is given by $u(x \otimes \alpha \otimes y) = x\mu(y) \otimes \alpha$, for any $x \otimes \alpha \otimes y \in A \otimes A^!_{\mathbf{n}(d)} \otimes A$. Since $E(A)$ is Frobenius with Nakayama automorphism ϕ, by [7, Proposition 3.1], we have $e_i^*\alpha = \alpha\phi(e_i^*)$, for any $\alpha \in A^!_{\mathbf{n}(d)-1}$. Now for any $x \otimes \alpha \otimes y \in A \otimes A^!_{\mathbf{n}(d)-1} \otimes A$, we have:

$$\begin{aligned}
u\delta(x \otimes \alpha \otimes y) &= u(\textstyle\sum_{i=1}^n x \otimes \alpha e_i^* \otimes e_i y - \sum_{i=1}^n x e_i \otimes e_i^* \alpha \otimes y) \\
&= \textstyle\sum_{i=1}^n x\mu(e_i y) \otimes \alpha e_i^* - \sum_{i=1}^n x e_i \mu(y) \otimes e_i^* \alpha \\
&= \textstyle\sum_{i=1}^n x\mu(e_i)\mu(y) \otimes \alpha e_i^* - \sum_{i=1}^n x e_i \mu(y) \otimes \alpha\phi(e_i^*) \\
&= \textstyle\sum_{i=1}^n x\mu(e_i)\mu(y) \otimes \alpha e_i^* - \sum_{i=1}^n x e_i \mu(y) \otimes \alpha(\sum_{j=1}^n q_{ji} e_j^*) + \\
&= \textstyle\sum_{i=1}^n x\mu(e_i)\mu(y) \otimes \alpha e_i^* - \sum_{i=1}^n \sum_{j=1}^n q_{ji} x e_i \mu(y) \otimes \alpha e_j^* \\
&= \textstyle\sum_{i=1}^n x\mu(e_i)\mu(y) \otimes \alpha e_i^* - \sum_{i=1}^n x\varphi(e_i)\mu(y) \otimes \alpha e_i^* \\
&= \textstyle\sum_{i=1}^n (-1)^{d+1} x\varphi(e_i)\mu(y) \otimes \alpha e_i^* - \sum_{i=1}^n x\varphi(e_i)\mu(y) \otimes \alpha e_i^* \\
&= 0.
\end{aligned}$$

Therefore, the sequence (22) is a complex. Hence, it is exact by [7, Proposition 4.1].

Similar to the proof of [23, Prop 2.2], we can show that (20) defines an $A^e \rtimes_{S_\sigma^{-2}} H^\sigma$-module structure on $A \otimes A^!_d$ and u is an $A^e \rtimes_{S_\sigma^{-2}} H^\sigma$-homomorphism. Therefore, $\mathrm{Ext}^d_{A^e}(A, A^e) \cong A_\mu \otimes A^!_{\mathbf{n}(d)}$ as $A^e \rtimes_{S_\sigma^{-2}} H^\sigma$-modules.

For the case $N = 2$, the proof is similar. □

Let H be a Hopf algebra, σ a 2-cocycle on H, and A a graded H^σ-module algebra. Let P be an $A^e \rtimes H^\sigma$-module. $\mathrm{Hom}_{A^e}(P, A^e)$ is a right H^σ-module as defined in (14). Then we can define a right H-module structure on $\mathrm{Hom}_{A^e}(P, A^e) \otimes {}_\sigma H \otimes {}_\sigma H$:

$$(f \otimes k \otimes l) \leftarrow h = f \leftharpoonup h_2 \otimes (S_{1,\sigma} h_1) \cdot_\sigma k \otimes l \cdot_\sigma h_3 \tag{23}$$

for all $f \otimes k \otimes l \in \mathrm{Hom}_{A^e}(P, A^e) \otimes {}_\sigma H \otimes {}_\sigma H$ and $h \in H$. Recall that H can be viewed as the algebra $H(1,1)$. Here $h_1 \otimes h_2 \otimes h_3 = (\Delta^\sigma_{1,\sigma} \otimes \mathrm{id})\Delta^\sigma_{1,1}(h)$. Both $\Delta^\sigma_{1,\sigma}$ and $\Delta^\sigma_{1,1}$ are algebra homomorphisms. So this H-module is well-defined. We denote this H-module by $\mathrm{Hom}_{A^e}(P, A^e)_* \otimes {}_{*\sigma} H \otimes {}_\sigma H_*$.

The right H-module structure of H induces a natural H-module structure on $\mathrm{Hom}_{A^e}(P,A^e)\otimes{}_\sigma H\otimes H$. That is,

$$(f\otimes k\otimes l)\leftarrow h=f\otimes k\otimes lh \tag{24}$$

for all $f\otimes k\otimes l\in\mathrm{Hom}_{A^e}(P,A^e)\otimes_\sigma H\otimes H$ and $h\in H$. We denote this H-module by $\mathrm{Hom}_{A^e}(P,A^e)\otimes{}_\sigma H\otimes H_*$.

We can define an $(A\#_\sigma H)^e$-module structure on $\mathrm{Hom}_{A^e}(P,A^e)\otimes{}_\sigma H\otimes H_*$ as follows:

$$\begin{aligned}(a\#h)\cdot(f\otimes k\otimes l)&=a((S^{\sigma^2}h_1)\rightharpoonup f)\otimes S_{1,\sigma}(S_{\sigma,1}(h_2))\cdot_\sigma k\otimes h_3l,\\ (f\otimes k\otimes l)\cdot(b\#g)&=f(k_1\cdot b)\otimes k_2\cdot_\sigma g\otimes l,\end{aligned} \tag{25}$$

for any $a\#h$, $b\#g\in A\#_\sigma H$ and $f\otimes k\otimes l\in\mathrm{Hom}_{A^e}(P,A^e)\otimes{}_\sigma H\otimes H_*$. Recall that the left H^σ-module structure of $\mathrm{Hom}_{A^e}(P,A^e)$ is defined in (12). Here $h_1\otimes h_2\otimes h_3=(\Delta^\sigma_{\sigma,1}\otimes\mathrm{id})\Delta^\sigma_{\sigma,1}(h)$ and $k_1\otimes k_2=\Delta^\sigma_{\sigma,1}(k)$. We first check that the left $A\#_\sigma H$-module structure is well-defined. We have the following equations:

$$\begin{aligned}&(b\#g)\cdot[(a\#h)\cdot(f\otimes k\otimes l)]\\ =\;&(b\#g)\cdot[a((S^{\sigma^2}h_1)\rightharpoonup f)\otimes S_{1,\sigma}(S_{\sigma,1}(h_2))\cdot_\sigma k\otimes h_3l]\\ =\;&b[(S^{\sigma^2}g_1)\rightharpoonup(a((S^{\sigma^2}h_1)\rightharpoonup f))]\otimes S_{1,\sigma}(S_{\sigma,1}(g_2))\cdot_\sigma S_{1,\sigma}(S_{\sigma,1}(h_2))\cdot_\sigma k\otimes g_3h_3l\\ \overset{(11)}{=}\;&b[(g_1\cdot a)((S^{\sigma^2}g_2)*(S^{\sigma^2}h_1))\rightharpoonup f]\otimes S_{1,\sigma}(S_{\sigma,1}(g_3))\cdot_\sigma S_{1,\sigma}(S_{\sigma,1}(h_2))\cdot_\sigma k\otimes g_4h_3l\\ =\;&b(g_1\cdot a)((S^{\sigma^2}g_2)*(S^{\sigma^2}h_1))\rightharpoonup f\otimes S_{1,\sigma}(S_{\sigma,1}(g_3))\cdot_\sigma S_{1,\sigma}(S_{\sigma,1}(h_2))\cdot_\sigma k\otimes g_4h_3l\\ =\;&b(g_1\cdot a)(S^{\sigma^2}(g_2*h_1))\rightharpoonup f\otimes S_{1,\sigma}(S_{\sigma,1}(g_3\cdot_\sigma h_2))\cdot_\sigma k\otimes g_4h_3l\\ =\;&[b(g_1\cdot a)\#g_2\cdot_\sigma h]\cdot(f\otimes k\otimes l)\\ =\;&[(b\#g)(a\#h)]\cdot(f\otimes k\otimes l).\end{aligned}$$

By Lemma 1.4 we know that $S_{1,\sigma}\circ S_{\sigma,1}$ is an algebra homomorphism of ${}_\sigma H$. Therefore, the fifth equation holds. The sixth equation follows from the fact that $\Delta^\sigma_{\sigma,1}$ is an algebra homomorphism. It follows that $\mathrm{Hom}_{A^e}(P,A^e)\otimes_\sigma H\otimes H_*$ is a left $A\#_\sigma H$-module. Similarly, we can see that $\mathrm{Hom}_{A^e}(P,A^e)\otimes{}_\sigma H\otimes H_*$ is a right $A\#_\sigma H$-module and for any $a\#h,b\#g\in A\#_\sigma H$, and $f\otimes k\otimes l\in\mathrm{Hom}_{A^e}(P,A^e)\otimes{}_\sigma H\otimes H_*$,

$$[(a\#h)(f\otimes k\otimes l)](b\#g)=(a\#h)[(f\otimes k\otimes l)(b\#g)].$$

In conclusion, $\mathrm{Hom}_{A^e}(P,A^e)\otimes{}_\sigma H\otimes H_*$ is indeed an $(A\#_\sigma H)^e$-module as defined in (25).

The module $\mathrm{Hom}_{A^e}(P,A^e)_*\otimes{}_{*\sigma}H\otimes{}_\sigma H_*$ is also an $(A\#_\sigma H)^e$-module with the module structure defined by

$$\begin{aligned}(a\#h)\cdot(f\otimes k\otimes l)&=(S^{\sigma^{-1}}(h_1l_1)\cdot a)f\otimes k\otimes h_2\cdot_\sigma l_2,\\ (f\otimes k\otimes l)\cdot(b\#g)&=f(k_1\cdot b)\otimes k_2\cdot_\sigma g\otimes l,\end{aligned} \tag{26}$$

where $h_1\otimes h_2=\Delta^\sigma_{\sigma,1}(h)$, $l_1\otimes l_2=\Delta^\sigma_{\sigma,1}(l)$ and $k_1\otimes k_2=\Delta^\sigma_{\sigma,1}(k)$.

Now both $\mathrm{Hom}_{A^e}(P, A^e)_* \otimes {}_{*\sigma}H \otimes H_*$ and $\mathrm{Hom}_{A^e}(P, A^e) \otimes {}_{\sigma}H \otimes H_*$ are right $H \otimes (A\#_\sigma H)^e$-modules.

Lemma 2.9. *Let H be a Hopf algebra, σ a 2-cocycle on H, and A a graded left H^σ-module algebra. If P is an $A^e \rtimes H^\sigma$-module, then the following Ψ and Φ are $H \otimes (A\#_\sigma H)^e$-module isomorphisms*

$$\mathrm{Hom}_{A^e}(P, A^e)_* \otimes {}_{*\sigma}H \otimes {}_{\sigma}H_* \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathrm{Hom}_{A^e}(P, A^e) \otimes H \otimes H_*,$$

where the module structures are given by (23), (24), (25) and (26), Ψ and Φ are defined as follows:

$$\begin{aligned} \Psi(f \otimes k \otimes l) &= f \leftharpoonup S^\sigma(l_1) \otimes S_{1,\sigma}(S_{\sigma,1}(l_2)) \cdot_\sigma k \otimes l_3, \\ \Phi(f \otimes k \otimes l) &= f \leftharpoonup l_2 \otimes S_{1,\sigma}(l_1) \cdot_\sigma k \otimes l_3. \end{aligned}$$

Moreover, Ψ and Φ are inverse to each other.

Lemma 2.10. *Let H be a Hopf algebra, σ a 2-cocycle on H, and A a graded left H^σ-module algebra. Let P be an $A^e \rtimes H^\sigma$-module, and M an $(A\#_\sigma H)^e$-bimodule. Then $\mathrm{Hom}_{A^e}(P, M)$ is a right H-module defined by*

$$(f \leftarrow h)(x) = S_{1,\sigma}(h_1) f(h_2 x) h_3$$

for any $h \in H$, $f \in \mathrm{Hom}_{A^e}(P, M)$ and $x \in P$. Here $h_1 \otimes h_2 \otimes h_3 = (\Delta^\sigma_{1,\sigma} \otimes \mathrm{id})\Delta^\sigma_{1,1}(h)$.

Proof. For any $h, k \in H$ and $f \in \mathrm{Hom}_{A^e}(P, M)$, the following equations hold:

$$\begin{aligned} [(f \leftarrow h) \leftarrow k](x) &= S_{1,\sigma}(k_1)(f \leftarrow h)(k_2 x) k_3 \\ &= S_{1,\sigma}(k_1)[S_{1,\sigma}(h_1) f(h_2(k_2(x))) h_3] k_3 \\ &= [S_{1,\sigma}(k_1) \cdot_\sigma S_{1,\sigma}(h_1)] f((h_2 * k_2)(x))(h_3 \cdot_\sigma k_3) \\ &= [S_{1,\sigma}(h_1 \cdot_{\sigma^{-1}} k_1)] f((h_2 * k_2)(x))(h_3 \cdot_\sigma k_3) \\ &= [f \leftarrow (hk)](x). \end{aligned}$$

The third equation holds since M is an $A\#_\sigma M$-bimodule. The fourth equation follows from Lemma 1.4(i). The last equation follows from the fact that both $\Delta^\sigma_{1,\sigma}$ and $\Delta^\sigma_{1,1}$ are algebra homomorphisms. □

Remark 2.11. Since A is a graded left H^σ-module algebra, A is naturally an $A^e \rtimes H^\sigma$-module. Hence, $\mathrm{Hom}_{A^e}(A, M)$ is a right H-module for any $(A\#_\sigma H)^e$-bimodule M. H is just the algebra $H(1,1)$. From the fact that $S_{1,\sigma}(h_1)h_2 = \varepsilon(h)$ for any $h \in H$, it is easy to check that

$$\mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_{A^e}(A, M)) \cong \mathrm{Hom}_{(A\#_\sigma H)^e}(A\#_\sigma H, M),$$

for any $(A\#_\sigma H)^e$-bimodule M.

From Lemma 2.10 we see that $\mathrm{Hom}_{A^e}(P,(A\#_\sigma H)^e)$ is a right H-module. Moreover, the inner structure of $(A\#_\sigma H)^e$ induces a right $(A\#_\sigma H)^e$-module structure on $\mathrm{Hom}_{A^e}(P,(A\#_\sigma H)^e)$. That is,

$$[f\cdot(a\#h)\otimes(b\#g)](x)=f(x)_1(a\#h)\otimes(b\#g)f(x)_2$$

for any $f\in\mathrm{Hom}_{A^e}(P,(A\#_\sigma H)^e)$ and $a\#h, b\#g\in A\#_\sigma H$.

Lemma 2.12. *Let P be an $A^e\rtimes H^\sigma$-module.*

(i) *There is a right $H\otimes(A\#_\sigma H)^e$-module homomorphism*

$$\begin{array}{rcl}\Theta:\mathrm{Hom}_{A^e}(P,A^e)_*\otimes{}_{*\sigma}H\otimes{}_\sigma H_* & \to & \mathrm{Hom}_{A^e}(P,(A\#_\sigma H)^e)\\ f\otimes k\otimes l & \mapsto & \Theta(f\otimes k\otimes l)\end{array}$$

where $\Theta(f\otimes k\otimes l)(x)=f(x)_1\#k\otimes l_1f(x)_2\#l_2$ for any $x\in P$. Here $l_1\otimes l_2=\Delta^\sigma_{\sigma,1}(l)$.

(ii) *If P is finitely generated projective when viewed as an A^e-module, then Θ is an isomorphism.*

In [34], Stefan showed the relation between the Hochschild cohomologies of A and B, where B/A is a Hopf-Galois extension. When $B=A\#_\sigma H$ is a cleft extension, we have the following lemma:

Lemma 2.13. [34, Theorem 3.3] *Let H be a Hopf algebra, σ a 2-cocycle on H. Let A be a graded H^σ-module algebra and N an $(A\#_\sigma H)^e$-bimodule. Then there is a spectral sequence*

$$E_2^{p,q}=\mathrm{Ext}^p_{H^e}(H,\mathrm{Ext}^q_{A^e}(A,N))\Longrightarrow\mathrm{Ext}^{p+q}_{(A\#_\sigma H)^e}(A\#_\sigma H,N)$$

which is natural in N. The right H-module $\mathrm{Ext}^q_{A^e}(A,N)$ is viewed as H^e-module via the trivial action on the left side.

Lemma 2.14. *Let H be a Hopf algebra, σ a 2-cocycle on H and A a left H^σ-module algebra. If both A and H are homologically smooth, then so is $A\#_\sigma H$.*

Proof. Let I be an injective $A\#_\sigma H$-module. $\mathrm{Hom}_{A^e}(A,I)$ is a right H-module by Remark 2.11. From the proof of [34, Proposition 3.2], we see that $\mathrm{Hom}_{A^e}(A,I)$ is an injective H-module. Moreover, we see in Remark 2.11 that

$$\mathrm{Hom}_H(\Bbbk,\mathrm{Hom}_{A^e}(A,M))\cong\mathrm{Hom}_{(A\#_\sigma H)^e}(A\#_\sigma H,M)$$

for any $A\#_\sigma H$-bimodule M. Now the proof of Proposition 2.11 in [23] is valid for the cleft extension $A\#_\sigma H$. We obtain that $A\#_\sigma H$ is homologically smooth. □

The following lemma is probably well-known, for the convenience of the reader, we provide a proof here.

Lemma 2.15. *Let H be an augmented algebra such that H is a twisted CY algebra of dimension d with Nakayama automorphism ν. Then H is of global dimension d. Moreover, there is an isomorphism of right H-modules*

$$\mathrm{Ext}^i_H({}_H\Bbbk, {}_HH) \cong \begin{cases} 0, & i \neq d; \\ k_\xi, & i = d, \end{cases}$$

where $\xi : H \to \Bbbk$ is the homomorphism defined by $\xi(h) = \varepsilon(\nu(h))$ for any $h \in H$.

Proof. If H is an augmented algebra, then ${}_H\Bbbk$ is a finite dimensional module. By [9, Remark 2.8], H has global dimension d.

It follows from [9, Proposition 2.2] that H admits a projective bimodule resolution

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to H \to 0,$$

where each P_i is finitely generated as an H-H-bimodule. Tensoring with functor $\otimes_H \Bbbk$, we obtain a projective resolution of ${}_H\Bbbk$:

$$0 \to P_d \otimes_H \Bbbk \to \cdots \to P_1 \otimes_H \Bbbk \to P_0 \otimes_H \Bbbk \to {}_H\Bbbk \to 0.$$

Since each P_i is finitely generated, the following isomorphisms of right H-modules holds:

$$\Bbbk \otimes_H \mathrm{Hom}_{H^e}(P_i, H^e) \cong \mathrm{Hom}_H(P_i \otimes_H \Bbbk, H).$$

Therefore, the complex $\mathrm{Hom}_H(P_\bullet \otimes_H \Bbbk, H)$ is isomorphic to the complex $\Bbbk \otimes_H \mathrm{Hom}_{H^e}(P_\bullet, H^e)$. The algebra H is twisted CY with Nakayama automorphism ν. So the following H-H-bimodule complex is exact,

$$0 \to \mathrm{Hom}_{H^e}(P_0, H^e) \to \cdots \to \mathrm{Hom}_{H^e}(P_{d-1}, H^e) \to \mathrm{Hom}_{H^e}(P_d, H^e) \to H^\nu \to 0.$$

Thus the complex $\Bbbk \otimes_H \mathrm{Hom}_{H^e}(P_\bullet, H^e)$ is exact except at $\Bbbk \otimes_H \mathrm{Hom}_{H^e}(P_d, H^e)$, whose homology is $\Bbbk \otimes_H H^\nu$. It is easy to see that $\Bbbk \otimes_H H^\nu \cong \Bbbk_\xi$, where $\xi : H \to \Bbbk$ is the algebra homomorphism defined by $\xi(h) = \varepsilon(\nu(h))$ for any $h \in H$. In conclusion, we obtain the following isomorphisms right H-modules

$$\mathrm{Ext}^i_H({}_H\Bbbk, {}_HH) \cong \begin{cases} 0, & i \neq d; \\ \Bbbk_\xi, & i = d. \end{cases}$$

□

Remark 2.16. In a similar way, we can also obtain the following isomorphisms of left H-modules:

$$\mathrm{Ext}^i_H(\Bbbk_H, H_H) \cong \begin{cases} 0, & i \neq d; \\ {}_\eta k, & i = d, \end{cases}$$

where $\eta : H \to \Bbbk$ is the homomorphism defined by $\eta = \varepsilon \circ \nu^{-1}$. Therefore, if H is a twisted CY augmented algebra, then H has finite global dimension and satisfy the AS-Gorenstein condition. However, H is not necessarily Noetherian. It is not AS-regular in the sense of Definition 1.11. We still call $\mathrm{Ext}^i_H({}_H\Bbbk, {}_HH)$ and $\mathrm{Ext}^d_H(\Bbbk_H, H_H)$ left and right homological integral of H and denoted them by $\int^l_H$ and $\int^r_H$ respectively.

Lemma 2.17. *Let H be a twisted CY Hopf algebra with homological integral $\int^l_H = \Bbbk_\xi$, where $\xi : H \to \Bbbk$ is an algebra homomorphism. Then the Nakayama automorphism ν of H is given by $\nu(h) = \xi(h_1)S^2(h_2)$ for any $h \in H$. If the right homological integral of H is $\int^l_H = {}_\eta\Bbbk$, then $\eta = \xi \circ S$.*

Proof. Proposition 4.5(a) in [10] holds true when the Hopf algebra is not necessarily Noetherian. So we obtain that the Nakayama automorphism ν satisfies $\nu(h) = \xi(h_1)S^2(h_2)$ for any $h \in H$. From Remark 2.16, we see that $\eta = \varepsilon \circ \nu^{-1}$. Note that for every $h \in H$, $\nu^{-1}(h) = \xi(Sh_1)S^{-2}(h_2)$ and $\xi \circ S^2(h) = \xi(h)$. Therefore, we obtain that $\eta = \xi \circ S$. □

Theorem 2.18. *Let H be a twisted CY Hopf algebra with homological integral $\int^l_H = \Bbbk_\xi$, where $\xi : H \to \Bbbk$ is an algebra homomorphism and let σ be a 2-cocycle on H. Let A be an N-Koszul graded twisted CY algebra with Nakayama automorphism μ such that A is a left graded H^σ-module algebra. Then $A\#_\sigma H$ is a graded twisted CY algebra with Nakayama automorphism ρ defined by*

$$\rho(a\#h) = \mu(a)\#\,\mathrm{hdet}_{H^\sigma}(h_1)(S^{-1}_{\sigma,1}(S^{-1}_{1,\sigma}(h_2)))\xi(h_3)$$

for all $a\#h \in A\#_\sigma H$.

Proof. Assume that the CY dimensions of H and A are d_1 and d_2 respectively. Take the Koszul complex $\mathbf{K_b(A)} \to A \to 0$. In the proof of Proposition 2.8, we see that $\mathbf{K_b(A)} \to A \to 0$ is a complex of $A^e \rtimes H^\sigma$-modules. It follows from Lemma 2.9 and Lemma 2.12 that the following isomorphisms of $H \otimes (A\#H)^e$-module complexes hold:

$$\begin{aligned}\mathrm{Hom}_{A^e}(\mathbf{K_b}(A), (A\#_\sigma H)^e) &\cong \mathrm{Hom}_{A^e}(\mathbf{K_b}(A), (A^e))_* \otimes {}_{*\sigma}H \otimes {}_\sigma H_* \\ &\cong \mathrm{Hom}_{A^e}(\mathbf{K_b}(A), (A^e)) \otimes {}_\sigma H \otimes H_*.\end{aligned}$$

After taking cohomologies, we obtain that

$$\mathrm{Ext}^q_{A^e}(A, (A\#_\sigma H)^e) \cong \mathrm{Ext}^q_{A^e}(A, A^e) \otimes {}_\sigma H \otimes H_*$$

as $H\otimes(A\#_\sigma H)^e$-modules, for any $q\geqslant 0$.

If we view the right H-module $\mathrm{Ext}^q_{A^e}(A,(A\#_\sigma H)^e)$ as H^e-module via the trivial action on the left side, then

$$\begin{aligned}\mathrm{Ext}^p_{H^e}(H,\mathrm{Ext}^q_{A^e}(A,(A\#H)^e)) &\cong \mathrm{Ext}^p_H(\Bbbk,\mathrm{Ext}^q_{A^e}(A,(A\#_\sigma H)^e))\\ &\cong \mathrm{Ext}^p_H(\Bbbk,\mathrm{Ext}^q_{A^e}(A,A^e)\otimes{}_\sigma H\otimes H_*)\\ &\cong \mathrm{Ext}^q_{A^e}(A,A^e)\otimes{}_\sigma H\otimes \mathrm{Ext}^p_H(\Bbbk_H,H_H).\end{aligned}$$

By Lemma 2.13, $\mathrm{Ext}^i_{(A\#H)^e}(A\#H,(A\#H)^e))=0$, for $i\neq d_1+d_2$ and

$$\mathrm{Ext}^{d_1+d_2}_{(A\#H)^e}(A\#H,(A\#H)^e))\cong \mathrm{Ext}^{d_2}_{A^e}(A,A^e)\otimes{}_\sigma H\otimes \mathrm{Ext}^{d_1}_H(\Bbbk_H,H_H).$$

It is an isomorphism of $(A\#_\sigma H)^e$-bimodules if the $(A\#_\sigma H)^e$-bimodule on $\mathrm{Ext}^{d_2}_{A^e}(A,A^e)\otimes{}_\sigma H\otimes\mathrm{Ext}^{d_1}_H(\Bbbk,H)$ is given by

$$\begin{aligned}&(a\#h)\cdot(x\otimes k\otimes l)=a((S^{\sigma^2}h_1)\rightharpoonup x)\otimes S_{1,\sigma}(S_{\sigma,1}(h_2))\cdot_\sigma k\otimes\xi(Sh_3)l,\\ &(x\otimes k\otimes l)\cdot(b\#g)=x(k_1\cdot b)\otimes k_2\cdot_\sigma g\otimes l,\end{aligned}$$

for any $a\#h,b\#g\in A\#_\sigma H$ and $x\otimes k\otimes l\in\mathrm{Ext}^{d_2}_{A^e}(A,A^e)\otimes_\sigma H\otimes\mathrm{Ext}^{d_1}_H(\Bbbk_H,H_H)$. Note that $\mathrm{Ext}^{d_1}_H(\Bbbk_H,H_H)\cong{}_\eta\Bbbk$, where $\eta=\xi\circ S$ (Lemma 2.17).

By Proposition 2.8, we obtain the following isomorphism:

$$\mathrm{Ext}^{d_1+d_2}_{(A\#H)^e}(A\#H,(A\#H)^e))\cong A_\mu\otimes A^!_{d_2}\otimes H\otimes{}_{\xi\circ S}\Bbbk.$$

Since the algebra A is N-Koszul graded twisted CY of dimension d_2, it is AS-regular of global dimension d_2. By [22, Lemma 5.10], we obtain that $A^!_{d_2}\cong\mathrm{Ext}^{d_2}_A(\Bbbk,\Bbbk)$ is one dimensional. Let t be a nonzero element in $A^!_{d_2}$. The left H^σ-action on $A^!_{d_2}$ is given by

$$h\cdot t=\mathrm{hdet}(S^{\sigma^{-1}}h)t,$$

for any $h\in H$. Therefore, the $(A\#H)^e$-module structure on $A_\mu\otimes A^!_{d_2}\otimes H\otimes_{\xi\circ S}\Bbbk$ is given by

$$\begin{aligned}&(a\#h)\cdot(x\otimes t\otimes k\otimes y)\\ &=a(h_1\cdot x)\otimes\mathrm{hdet}_{H^\sigma}(S^\sigma h_2)t\otimes(S_{1,\sigma}(S_{\sigma,1}h_3))\cdot_\sigma k\otimes\xi(Sh_4)y\\ &(x\otimes t\otimes k\otimes y)\cdot(b\#g)\\ &=x\mu(k_1\cdot b)\otimes t\otimes k_2\cdot_\sigma g\otimes y,\end{aligned}\tag{27}$$

for $(x\otimes t\otimes k\otimes y)\in A_\mu\otimes A^!_{d_2}\otimes H\otimes{}_{\xi\circ S}\Bbbk$ and $a\#h,b\#g\in A\#H$.

Now we prove that $A_\mu\otimes A^!_{d_2}\otimes{}_\sigma H\otimes{}_{\xi\circ S}\Bbbk\cong(A\#_\sigma H)^\rho$ as $(A\#_\sigma H)^e$-modules for some automorphism ρ of $A\#_\sigma H$.

It is straightforward to check that for any $x\in A$, $k\in H$, we have:

$$\begin{aligned}x\otimes t\otimes k\otimes 1 &= [x\#\,\mathrm{hdet}_{H^\sigma}(k_1)S^{-1}_{\sigma,1}(S^{-1}_{1,\sigma}(k_2))\xi(k_3)]\cdot(1\otimes t\otimes 1\otimes 1)\\ &=(1\otimes t\otimes 1\otimes 1)\cdot(\mu^{-1}(x)\#k).\end{aligned}$$

This implies that $(1 \otimes t \otimes 1 \otimes 1)$ is a left and right $A\#_\sigma H$-module generator of $A_\mu \otimes A^!_{d_2} \otimes {}_\sigma H \otimes {}_{\xi\circ S}\mathbb{k}$. The same formula implies that no nonzero element of $A\#H$ annihilates $(1 \otimes t \otimes 1 \otimes 1)$. Therefore, $A_\mu \otimes A^!_{d_2} \otimes {}_\sigma H \otimes {}_{\xi\circ S}\mathbb{k}$ is a free $A\#_\sigma H$-module of rank 1 on each side. So $A_\mu \otimes A^!_{d_2} \otimes {}_\sigma H \otimes {}_{\xi\circ S}\mathbb{k} \cong (A\#H)^\rho$ as $(A\#_\sigma H)^e$-modules for some automorphism ρ of $A\#_\sigma H$. Next we compute ρ. For any $h \in H$,

$$\begin{aligned}(1 \otimes t \otimes 1 \otimes 1) \cdot (1\#h) &= 1 \otimes t \otimes h \otimes 1\\ &= (1\#\, \mathrm{hdet}_{H^\sigma}(h_1) S^{-1}_{\sigma,1}(S^{-1}_{1,\sigma}(h_2))\xi(h_3)) \cdot (1 \otimes t \otimes 1 \otimes 1).\end{aligned}$$

This shows that $\rho(h) = \mathrm{hdet}(h_1)(S^{-1}_{\sigma,1}(S^{-1}_{1,\sigma}(h_2)))\xi(h_3)$.

On the other hand, for any $a \in A$, we have:

$$\begin{aligned}(1 \otimes t \otimes 1 \otimes 1) \cdot (a\#1) &= \mu(a) \otimes t \otimes 1 \otimes 1\\ &= (\mu(a)\#1) \cdot (1 \otimes t \otimes 1 \otimes 1).\end{aligned}$$

So $\rho(a) = \mu(a)$. It follows that the automorphism ρ of $A\#_\sigma H$ is give by

$$\rho(a\#h) = \mu(a)\#\, \mathrm{hdet}_{H^\sigma}(h_1)(S^{-1}_{\sigma,1}(S^{-1}_{1,\sigma}(h_2)))\xi(h_3)$$

for any $a\#h \in A\#H$ and $A_\mu \otimes A^!_{d_2} \otimes {}_\sigma H \otimes \mathbb{k}_\xi \cong (A\#_\sigma H)^\rho$. To summarize, we obtain the following isomorphisms of $(A\#H)^e$-modules:

$$\mathrm{Ext}^i_{(A\#_\sigma H)^e}(A\#_\sigma H, (A\#_\sigma H)^e) \cong \begin{cases} 0, & i \neq d_1 + d_2;\\ (A\#_\sigma H)^\rho, & i = d_1 + d_2.\end{cases}$$

By Lemma 2.14, $A\#_\sigma H$ is homologically smooth. The proof is completed. □

Let H be a Hopf algebra. For an algebra homomorphism $\xi : H \to \mathbb{k}$, We write $[\xi]^l$ for the *left winding homomorphism* of ξ defined by

$$[\xi]^l(h) = \xi(h_1)h_2,$$

for any $h \in H$. The *right winding automorphism* $[\xi]^r$ of ξ can be defined similarly. It is well-known that both $[\xi]^l$ and $[\xi]^r$ are algebra automorphisms of H. In Theorem 2.18, if we take the 2-cocycle to be trivial, we obtain the following result about smash products.

Theorem 2.19. *Let H be a twisted CY Hopf algebra with homological integral $\int^l_H = \mathbb{k}_\xi$, where $\xi : H \to \mathbb{k}$ is an algebra homomorphism and A an N-Koszul graded twisted CY algebra with Nakayama automorphism μ such that A is a left graded H-module algebra. Then $A\#H$ is a twisted CY algebra with Nakayama automorphism $\rho = \mu\#(S^{-2} \circ [\mathrm{hdet}_H]^l \circ [\xi]^r)$.*

Proof. From Theorem 2.18, we see that $A\#H$ is a graded twisted CY algebra with Nakayama automorphism ρ defined by

$$\rho(a\#h) = \mu(a)\#\,\mathrm{hdet}_H(h_1)(S^{-2}(h_2))\xi(h_3)$$

for all $a\#h \in A\#_\sigma H$. That is, $\rho = \mu\#(S^{-2} \circ [\mathrm{hdet}_H]^l \circ [\xi]^r)$. □

Corollary 2.20. *With the same assumption as in Theorem 2.19, the algebra $A\#H$ is a CY algebra if and only if $\mathrm{hdet}_H = \xi \circ S$ and $\mu\#S^{-2}$ is an inner automorphism of $A\#H$.*

Proof. Since $\mu\#(S^{-2} \circ [\mathrm{hdet}_H]^l \circ [\xi]^r) = (\mu\#S^{-2}) \circ (\mathrm{id}\,\#([\mathrm{hdet}_H]^l \circ [\xi]^r))$, the sufficiency part is clear.

In the proof of Theorem 2.18, if we let the cocycle σ be trivial, then the proof is just a modification of the proof of the sufficiency part of [23, Theorem 2.12]. If we modify the proof of the necessary part, we obtain that $\xi \star \mathrm{hdet}_H = \varepsilon$, where $\star$ stands for the convolution product. It is easy to see that $\xi \circ S$ and ξ are inverse to each other with respect to the convolution product. Therefore, we obtain that $\mathrm{hdet}_H = \xi \circ S$. Now $\mu\#(S^{-2} \circ [\mathrm{hdet}_H]^l \circ [\xi]^r) = \mu\#S^{-2}$. It follows from Theorem 2.19 that $\mu\#S^{-2}$ is an inner automorphism. □

In case A is an N-Koszul graded CY algebra and H is a CY Hopf algebra, we have the following consequence.

Corollary 2.21. *Let H be a CY Hopf algebra, and let A be an N-Koszul graded CY algebra and a left graded H-module algebra. Then $A\#H$ is a graded CY algebra if and only if the homological determinant of the H-action on A is trivial and $\mathrm{id}\,\#S^2$ is an inner automorphism of $A\#H$.*

Proof. Since H is a CY Hopf algebra, by Lemma 1.15 (ii), the algebra H satisfies $\int_H^l = \Bbbk$. Now the corollary follows immediately from Corollary 2.20. □

Remark 2.22. From Lemma 2.15 and Lemma 2.17, it is not hard to see that if H is CY Hopf algebra, then S^2 is an inner automorphism of H. However, $\mathrm{id}\,\#S^2$ is not necessarily an inner automorphism of $A\#H$ even if $A\#H$ is CY. Example 4.2 in Section 4 is a counterexample. It also shows that the smash product $A\#H$ could be a CY algebra when A itself is not.

In Theorem 2.18, if we let the algebra A be $\Bbbk$, then we obtain the following result about the twisted CY property of cleft objects.

Theorem 2.23. *Let H be a twisted CY Hopf algebra with $\int_H^l = {}_\xi\Bbbk$. Suppose ${}_\sigma H$ is a right cleft object of H. Then ${}_\sigma H$ is a twisted CY algebra with Nakayama automorphism μ defined by*

$$\mu(x) = S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(x_1))\xi S(x_2)$$

for any $x \in {}_\sigma H$.

3. Cleft objects of $U(\mathcal{D}, \lambda)$

The pointed Hopf algebras $U(\mathcal{D}, \lambda)$ introduced in [5] are generalizations of the quantized enveloping algebras $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra. Chelma showed that the algebras $U_q(\mathfrak{g})$ are CY algebras [11, Theorem 3.3.2]. The CY property of the algebras $U(\mathcal{D}, \lambda)$ were discussed in [39]. In this section we will show that the cleft objects of the algebras $U(\mathcal{D}, \lambda)$ are all twisted CY algebras.

3.1. **The Hopf algebra** $U(\mathcal{D}, \lambda)$**.** We refer to [3] for a detailed discussion about braided Hopf algebras and Yetter-Drinfeld modules. For a group Γ, we denote by ${}_\Gamma^\Gamma\mathcal{YD}$ the category of Yetter-Drinfeld modules over the group algebra $\Bbbk\Gamma$. If Γ is an abelian group, then it is well-known that a Yetter-Drinfeld module over the algebra $\Bbbk\Gamma$ is just a Γ-graded Γ-module.

We fix the following terminology.

- a free abelian group Γ of finite rank s;
- a Cartan matrix $\mathbb{A} = (a_{ij}) \in \mathbb{Z}^{\theta\times\theta}$ of finite type, where $\theta \in \mathbb{N}$. Let $(d_1, \cdots, d_\theta)$ be a diagonal matrix of positive integers such that $d_i a_{ij} = d_j a_{ji}$, which is minimal with this property;
- a set $\mathcal{X}$ of connected components of the Dynkin diagram corresponding to the Cartan matrix $\mathbb{A}$. If $1 \leqslant i, j \leqslant \theta$, then $i \sim j$ means that they belong to the same connected component;
- a family $(q_I)_{I\in\mathcal{X}}$ of elements in $\Bbbk$ which are *not* roots of unity;
- elements $g_1, \cdots, g_\theta \in \Gamma$ and characters $\chi_1, \cdots, \chi_\theta \in \hat{\Gamma}$ such that

$$\chi_j(g_i)\chi_i(g_j) = q_I^{d_i a_{ij}},\ \chi_i(g_i) = q_I^{d_i},\ \text{for all } 1 \leqslant i, j \leqslant \theta,\ I \in \mathcal{X}. \tag{28}$$

For simplicity, we write $q_{ji} = \chi_i(g_j)$. Then Equation (28) reads as follows:

$$q_{ii} = q_I^{d_i} \text{ and } q_{ij}q_{ji} = q_I^{d_i a_{ij}} \text{ for all } 1 \leqslant i, j \leqslant \theta, I \in \mathcal{X}. \tag{29}$$

Let $\mathcal{D}$ be the collection $\mathcal{D}(\Gamma, (a_{ij})_{1\leqslant i,j\leqslant\theta}, (q_I)_{I\in\mathcal{X}}, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta})$. A *linking datum* $\lambda = (\lambda_{ij})$ for $\mathcal{D}$ is a collection of elements $(\lambda_{ij})_{1\leqslant i<j\leqslant\theta, i\nsim j} \in \Bbbk$ such that $\lambda_{ij} = 0$ if $g_ig_j = 1$ or $\chi_i\chi_j \neq \varepsilon$. We write the datum $\lambda = 0$, if $\lambda_{ij} = 0$ for

all $1 \leqslant i < j \leqslant \theta$. The datum $(\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), q_{_I}, (g_i), (\chi_i), (\lambda_{ij}))$ is called a *generic datum of finite Cartan type* for group Γ.

A generic datum of finite Cartan type for a group Γ defines a Yetter-Drinfeld module over the group algebra $\Bbbk\Gamma$. Let V be a vector space with basis $\{x_1, x_2, \cdots, x_\theta\}$. We set

$$|x_i| = g_i, \;\; g(x_i) = \chi_i(g)x_i, \;\; 1 \leqslant i \leqslant \theta, g \in \Gamma,$$

where $|x_i|$ denote the degree of x_i. This makes V a Yetter-Drinfeld module over the group algebra $\Bbbk\Gamma$. We write $V = \{x_i, g_i, \chi_i\}_{1\leqslant i\leqslant\theta} \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$. The braiding is given by

$$c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, \;\; 1 \leqslant i, j \leqslant \theta.$$

The tensor algebra $T(V)$ on V is a natural graded braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. The smash product $T(V)\#\Bbbk\Gamma$ is a usual Hopf algebra. It is also called a bosonization of $T(V)$ by $\Bbbk\Gamma$.

Definition 3.1. Given a generic datum of finite Cartan type $(\mathcal{D}, \lambda)$ for a group Γ. Define $U(\mathcal{D}, \lambda)$ as the quotient Hopf algebra of the smash product $T(V)\#\Bbbk\Gamma$ modulo the ideal generated by

$$(\mathrm{ad}_c x_i)^{1-a_{ij}}(x_j) = 0, \;\; 1 \leqslant i \neq j \leqslant \theta, \;\; i \sim j,$$

$$x_ix_j - \chi_j(g_i)x_jx_i = \lambda_{ij}(g_ig_j - 1), \;\; 1 \leqslant i < j \leqslant \theta, \;\; i \nsim j,$$

where ad_c is the braided adjoint representation defined in [5, Sec. 1].

The algebra $U(\mathcal{D}, \lambda)$ is a pointed Hopf algebra with

$$\Delta(g) = g \otimes g, \; \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \;\; g \in \Gamma, 1 \leqslant i \leqslant \theta.$$

To present the CY property of the algebras $U(\mathcal{D}, \lambda)$, we recall the concept of root vectors. Let Φ be the root system corresponding to the Cartan matrix $\mathbb{A}$ with $\{\alpha_1, \cdots, \alpha_\theta\}$ a set of fix simple roots, and $\mathcal{W}$ the Weyl group. We fix a reduced decomposition of the longest element $w_0 = s_{i_1} \cdots s_{i_p}$ of $\mathcal{W}$ in terms of the simple reflections. Then the positive roots are precisely the followings,

$$\beta_1 = \alpha_{i_1}, \;\; \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}).$$

For $\beta_i = \sum_{i=1}^{\theta} m_i\alpha_i$, we write

$$g_{\beta_i} = g_1^{m_1} \cdots g_\theta^{m_\theta} \text{ and } \chi_{\beta_i} = \chi_1^{m_1} \cdots \chi_\theta^{m_\theta}.$$

Lusztig defined the root vectors for a quantum group $U_q(\mathfrak{g})$ in [26]. Up to a non-zero scalar, each root vector can be expressed as an iterated braided commutator. In [4, Sec. 4.1], the root vectors were generalized on a pointed

Hopf algebras $U(\mathcal{D},\lambda)$. For each positive root β_i, $1 \leqslant i \leqslant p$, the root vector x_{β_i} is defined by the same iterated braided commutator of the elements $x_1, \cdots, x_\theta$, but with respect to the general braiding.

Remark 3.2. If $\beta_j = \alpha_l$, then we have $x_{\beta_j} = x_l$. That is, $x_1, \cdots, x_\theta$ are the simple root vectors.

Lemma 3.3. *Let $(\mathcal{D},\lambda)$ be a generic datum of finite Cartan type for a group Γ, and H the Hopf algebra $U(\mathcal{D},\lambda)$. Let s be the rank of Γ and p the number of the positive roots of the Cartan matrix.*

(i) *The algebra H is Noetherian AS-regular of global dimension $p+s$. The left homological integral module $\int_H^l$ of H is isomorphic to $\Bbbk_\zeta$, where $\zeta : H \to \Bbbk$ is an algebra homomorphism defined by $\zeta(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$ and $\zeta(x_k) = 0$ for all $1 \leqslant k \leqslant \theta$.*
(ii) *The algebra H is twisted CY with Nakayama automorphism μ defined by $\mu(x_k) = q_{kk}x_k$, for all $1 \leqslant k \leqslant \theta$, and $\mu(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$.*
(iii) *The algebra H is CY if and only if $\prod_{i=1}^p \chi_{\beta_i} = \varepsilon$ and S^2 is an inner automorphism.*

Proof. (i) This is Theorem 2.2 in [39].

(ii) By Lemma 1.15(i), we conclude that the algebra H is twisted CY with Nakayama automorphism μ defined by $\mu(x_k) = S^{-2}(x_k) = q_{kk}x_k$ for $1 \leqslant k \leqslant \theta$ and $\mu(g) = \xi(g)g = (\prod_{i=1}^p \chi_{\beta_i})(g)g$ for $g \in \Gamma$.

(iii) This follows directly from (i) and Lemma 1.15 (ii). □

Remark 3.4. Theorem 2.3 in [39] showed that the Nakayama automorphism of the algebra $U(\mathcal{D},\lambda)$ is the algebra automorphism ν defined by $\nu(x_k) = \prod_{i=1, i\neq j_k}^p \chi_{\beta_i}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$, and $\nu(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$, where each j_k, $1 \leqslant k \leqslant \theta$, is the integer such that $\beta_{j_k} = \alpha_k$. Now we show that the algebra automorphisms μ and ν only differ by an inner automorphism.

By a similar discussion to the one in the proof of Lemma 4.1 in [39], we see that

$$\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)) = \prod_{i=1, i\neq j_k}^{p} \chi_k^{-1}(g_{\beta_i})$$

for each $1 \leqslant k \leqslant \theta$. Therefore,

$$\begin{aligned}[\prod_{i=1}^{p} g_{\beta_i}]^{-1}(\mu(x_k))[\prod_{i=1}^{p} g_{\beta_i}] &= \prod_{i=1}^{p} \chi_k^{-1}(g_{\beta_i})q_{kk}x_k \\ &= \prod_{i=1, i\neq j_k}^{p} \chi_k^{-1}(g_{\beta_i})x_k \\ &= \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) \\ &= \nu(x_k)\end{aligned}$$

for $1 \leqslant k \leqslant \theta$. Moreover, Γ is abelian, so $[\prod_{i=1}^{p} g_{\beta_i}]^{-1}(\mu(g))[\prod_{i=1}^{p} g_{\beta_i}] = \mu(g) = \nu(g)$ for all $g \in \Gamma$. This shows that μ and ν indeed differ by an inner automorphism.

In [28], the author classified the cleft objects of a class of pointed Hopf algebras. This class of algebras contains the algebras $U(\mathcal{D}, \lambda)$.

Now we fix a generic datum of finite Cartan type

$$(\mathcal{D}, \lambda) = (\Gamma, (a_{ij})_{1\leqslant i,j\leqslant\theta}, (q_I)_{I\in\mathcal{X}}, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (\lambda_{ij})_{1\leqslant i<j\leqslant\theta, i\nsim j}),$$

where Γ is a free abelian group of rank s.

Let $\sigma \in Z^2(\Bbbk\Gamma)$ be a 2-cocycle for the group algebra $\Bbbk\Gamma$. Define $\chi_i^\sigma(g) = \frac{\sigma(g,g_i)}{\sigma(g_i,g)}\chi_i(g)$. From [28, Proposition 1.11], we obtain that

$${}_\sigma V = \{x_i, g_i, \chi_i^\sigma\}_{1\leqslant i\leqslant\theta} \in {}^{\Gamma}_{\Gamma}\mathcal{YD}.$$

The associated braiding is given by

$$c^\sigma(x_i \otimes x_j) = q_{ij}^\sigma x_j \otimes x_i,$$

where $q_{ij}^\sigma = \frac{\sigma(g_i,g_j)}{\sigma(g_j,g_i)}q_{ij}$.

Define

$$\Xi(\sigma) = \{(i,j) \mid i<j, i \nsim j, \chi_i^\sigma\chi_j^\sigma = 1\}.$$

Given the braided vector space ${}_\sigma V$, we have the tensor algebra $T({}_\sigma V)$ and the smash product $T({}_\sigma V)\#\Bbbk\Gamma$. The 2-cocycle σ for the group algebra $\Bbbk\Gamma$ can be regarded as a 2-cocycle for $T({}_\sigma V)\#\Bbbk\Gamma$ through the projection $T({}_\sigma V)\#\Bbbk\Gamma \to \Bbbk\Gamma$. Then we have the crossed product $T({}_\sigma V)\#_\sigma\Bbbk\Gamma$. The difference between the crossed product and the smash product $T({}_\sigma V)\#\Bbbk\Gamma$ is given by

$$\overline{g}\overline{g'} = \sigma(g,g')\overline{gg'}, \ \ g,g' \in \Gamma, \forall g \in G.$$

Here $g \in T({}_\sigma V)\#\Bbbk\Gamma$ is denoted by $\overline{g} \in T({}_\sigma V)\#_\sigma\Bbbk\Gamma$ to avoid confusion.

Definition 3.5. Given $\pi = (\pi_{ij}) \in \Bbbk^{\Xi(\sigma)}$. Define $B^\lambda(\sigma,\pi)$ to be the quotient algebra of $T({}_\sigma V)\#_\sigma\Bbbk\Gamma$ modulo the ideal generated by

$$(\mathrm{ad}_{c^\sigma}\, x_i)^{1-a_{ij}}(x_j) = 0, \ \ 1 \leqslant i \neq j \leqslant \theta, \ \ i \sim j,$$

$$(\mathrm{ad}_{c^\sigma}\, x_i)(x_j) - \lambda_{ij}\overline{g}_i\overline{g}_j + \pi_{ij} = 0, \ \ 1 \leqslant i < j \leqslant \theta, i \nsim j,$$

where we set $\pi_{ij} = 0$ if $(i,j) \notin \Xi(\sigma)$.

Let $\mathcal{Z} = \mathcal{Z}(\Gamma, \Xi, \Bbbk)$ denote the set of all pairs (σ, π), where $\sigma \in Z^2(\Bbbk\Gamma)$ and $\pi = (\pi_{ij}) \in \Bbbk^{\Xi(\sigma)}$. For two pairs (σ, π) and (σ', π'), define $(\sigma, \pi) \sim (\sigma', \pi')$, if there is an invertible map $f : \Bbbk\Gamma \to \Bbbk$ such that

$$\sigma'(g,h) = f^{-1}(g)f^{-1}(h)\sigma(g,h)f(gh), \quad g,h \in \Gamma;$$

$$\pi'_{ij} = f^{-1}(g_i)f^{-1}(g_j)\pi_{ij}, \quad (i,j) \in \Xi(\sigma).$$

This defines an equivalence relation on $\mathcal{Z}$. We write $\mathcal{H}(\Gamma, \Xi, \Bbbk) = \mathcal{Z}/\sim$.

The following Lemma is the right version of Theorem 6.3 in [28]. It describes the isomorphism classes of right cleft objects of the algebras $U(\mathcal{D}, \lambda)$.

Lemma 3.6. *The map defined by*

$$\begin{array}{rcl} \mathcal{H}(\Gamma, \Xi, \Bbbk) & \longrightarrow & \mathrm{Cleft}(U(\mathcal{D}, \lambda)) \\ (\sigma, \pi) & \longmapsto & B^\lambda(\sigma, \pi) \end{array}$$

is a bijection, where $\mathrm{Cleft}(U(\mathcal{D}, \lambda))$ *denotes the set of the isomorphism classes the right cleft objects of* $U(\mathcal{D}, \lambda)$.

Proposition 3.7. *Given a pair* $(\sigma, \pi) \in \mathcal{Z}(\Gamma, \Xi, \Bbbk)$. *The algebra* $B^\lambda(\sigma, \pi)$ *is twisted CY with Nakayama automorphism defined by* $\mu(x_k) = q_{kk}x_k$ *for all* $1 \leqslant k \leqslant \theta$ *and* $\mu(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ *for all* $g \in \Gamma$.

In particular, the algebra $B^\lambda(\sigma, \pi)$ *is CY if and only if there is en element* $h \in \Bbbk\Gamma$ *such that* $\frac{\sigma(h,g)}{\sigma(g,h)} = (\prod_{i=1}^p \chi_{\beta_i})(g)$, *for all* $g \in \Gamma$ *and* $(\prod_{i=1, i \neq j_k}^p \chi_{\beta_i})(g)\chi_k(h) = 1$ *for each* $1 \leqslant k \leqslant \theta$, *where each* j_k, $1 \leqslant k \leqslant \theta$, *is the integer such that* $\beta_{j_k} = \alpha_k$.

Proof. Let $H = U(\mathcal{D}, \lambda)$. Without loss of generality, we may assume that σ satisfies that

$$\sigma(g, g^{-1}) = \sigma(g^{-1}, g) = 1$$

for all $g \in \Gamma$. This follows from Lemma 3.6 and the fact that for each pair (σ, π), there is a pair (σ', π') such that $(\sigma, \pi) \sim (\sigma', \pi')$ and σ' satisfies $\sigma'(g, g^{-1}) = \sigma'(g^{-1}, g) = 1$ for all $g \in \Gamma$. The algebra $B_q^\lambda(\sigma, \pi)$ is a cleft object of H. Then $B_q^\lambda(\sigma, \pi) \cong {}_\tau H$, for some 2-cocycle τ. The 2-cocycle τ can be calculated using Lemma 1.9. We conclude that τ satisfies the following:

$$\begin{array}{rcl} \tau(g, g') & = & \sigma(g, g'), \\ \tau(g, x_i) & = & \tau(x_i, g) = 0, \quad 1 \leqslant i \leqslant \theta, g, g' \in \Gamma. \\ \tau(x_i, x_j) & = & \begin{cases} \lambda_{ij}\sigma(g_i, g_j) - \pi_{ij}, & i < j, i \nsim j \\ 0, & \textit{otherwise}. \end{cases} \end{array}$$

Lemma 3.3 shows that the algebra $H = U(\mathcal{D}, \lambda)$ is Noetherian AS-regular. The left homological integral module $\int_H^l$ of H is isomorphic to $\Bbbk_\zeta$, where $\zeta : H \to \Bbbk$ is an algebra homomorphism defined by $\zeta(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$ and $\zeta(x_k) = 0$ for all $1 \leqslant k \leqslant \theta$.

Since H is AS-regular, by Theorem 2.23, $B_q(\sigma, \pi) \cong {}_\tau H$ is a twisted CY algebra. Its Nakayama automorphism can be calculated as follows. For $g \in \Gamma$,

$$\begin{aligned} \mu(g) &= S_{\tau,1}^{-1}(S_{1,\tau}^{-1}(g))\zeta(g) = S_{\tau,1}^{-1}(g^{-1}\sigma(g^{-1}, g))\zeta(g) \\ &= S_{\tau,1}^{-1}(g^{-1})\zeta(g) = \sigma(g, g^{-1})g\zeta(g) = \zeta(g)g \\ &= (\textstyle\prod_{i=1}^p \chi_{\beta_i})(g)g. \end{aligned}$$

For each $1 \leqslant k \leqslant \theta$,

$$\begin{aligned} \mu(x_k) &= S_{\tau,1}^{-1}(S_{1,\tau}^{-1}(x_k)) = S_{\tau,1}^{-1}(-g_k^{-1}x_k\sigma(g_k^{-1}, g_k)) \\ &= S_{\tau,1}^{-1}(-g_k^{-1}x_k) = \sigma(g_k^{-1}, g_k)q_{kk}x_k \\ &= q_{kk}x_k. \end{aligned}$$

The algebra $B^\lambda(\sigma, \pi)$ is CY if and only if the algebra automorphism μ is inner. Since the algebra $U(\mathcal{D}, \lambda)$ is a domain [5, Theorem 4.3], the invertible elements of $B^\lambda(\sigma, \pi)$ fall in $\Bbbk\Gamma$. In $B^\lambda(\sigma, \pi)$, for $l, g \in \Gamma$ and $1 \leqslant k \leqslant \theta$, we have

$$\bar{l}\bar{g} = \frac{\sigma(l, g)}{\sigma(g, l)}\bar{g}\bar{l}, \quad \bar{l}x_k = \chi_k^\sigma(l)x_k\bar{l} = \frac{\sigma(l, g_k)}{\sigma(g_k, l)}\chi_k(l)x_k\bar{l}.$$

With these facts, we see that the automorphism μ is an inner automorphism if and only if there exists an element $h \in \Bbbk\Gamma$ such that

$$\frac{\sigma(h, g)}{\sigma(g, h)} = (\prod_{i=1}^p \chi_{\beta_i})(g), \quad \frac{\sigma(h, g_k)}{\sigma(g_k, h)}\chi_k(h) = q_{kk}, \tag{30}$$

for all $g \in \Gamma$ and $1 \leqslant k \leqslant \theta$. Note that if $\frac{\sigma(h,g)}{\sigma(g,h)} = (\prod_{i=1}^p \chi_{\beta_i})(g)$ holds for any $g \in \Gamma$, then $\frac{\sigma(h,g_k)}{\sigma(g_k,h)} = (\prod_{i=1}^p \chi_{\beta_i})(g)$. So the condition (30) is equivalent to

$$\frac{\sigma(h, g)}{\sigma(g, h)} = (\prod_{i=1}^p \chi_{\beta_i})(g), (\prod_{i=1, i\neq j_k}^p \chi_{\beta_i})(g)\chi_k(h) = 1,$$

for all $g \in \Gamma$ and $1 \leqslant k \leqslant \theta$, where each j_k, $1 \leqslant k \leqslant \theta$, is the integer such that $\beta_{j_k} = \alpha_k$. □

We end this section by giving some examples. We first need the following lemma.

Lemma 3.8. *Let Γ be an abelian group, σ a 2-cocycle for the group algebra $\Bbbk\Gamma$. For any $g, k, h \in \Gamma$, we have*

$$\frac{\sigma(gk, h)}{\sigma(h, gk)} = \frac{\sigma(g, h)}{\sigma(h, g)}\frac{\sigma(k, h)}{\sigma(h, k)}.$$

Proof. Since σ is a 2-cocycle, the following equations hold for any $g, h, k \in \Gamma$.

$$\sigma(g,k)\sigma(gk,h) = \sigma(k,h)\sigma(g,kh) \tag{31}$$

$$\sigma(g,k)\sigma(h,gk) = \sigma(h,g)\sigma(hg,k) \tag{32}$$

$$\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk) \tag{33}$$

$$\sigma(h,k)\sigma(g,hk) = \sigma(g,h)\sigma(gh,k) \tag{34}$$

By (31) and (32), we obtain

$$\begin{aligned}\frac{\sigma(gk,h)}{\sigma(h,gk)} &= \frac{\sigma(k,h)\sigma(g,kh)}{\sigma(h,g)\sigma(hg,k)} \\ &\overset{(33,34)}{=} \frac{\sigma(k,h)\frac{\sigma(g,h)\sigma(gh,k)}{\sigma(h,k)}}{\sigma(h,g)\frac{\sigma(h,k)\sigma(g,hk)}{\sigma(g,h)}} \\ &= \frac{\sigma(g,h)}{\sigma(h,g)}\frac{\sigma(k,h)}{\sigma(h,k)}\frac{\sigma(g,h)\sigma(gh,k)}{\sigma(h,k)\sigma(g,hk)} \\ &\overset{(33)}{=} \frac{\sigma(g,h)}{\sigma(h,g)}\frac{\sigma(k,h)}{\sigma(h,k)}.\end{aligned}$$

□

Now we give an example in which the algebra $U(\mathcal{D},\lambda)$ is CY, but the algebra $B^\lambda(\sigma,\pi)$ is not necessarily CY.

Example 3.9. Let $(\mathcal{D},\lambda)$ be the datum given by

- $\Gamma = \langle y_1, y_2\rangle$, a free abelian group of rank 2;
- The Cartan matrix is of type $A_2 \times A_2$;
- $g_1 = g_3 = y_1, g_2 = g_4 = y_2$;
- $\chi_1(y_1) = q^2, \chi_1(y_2) = q^{-1}, \chi_2(y_1) = q^{-1}, \chi_2(y_2) = q^{-2}$, and $\chi_3 = \chi_1^{-1}, \chi_4 = \chi_2^{-1}$, where q is not a root of unity;
- $\lambda = (\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}) = (0,1,1,0)$.

Then the algebra $U(\mathcal{D},\lambda)$ is just the quantized enveloping algebra $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra corresponding to the Cartan matrix of type A_2. Therefore, $U(\mathcal{D},\lambda)$ is CY ([11, Theorem 3.3.2]). In fact, we have that

$$\beta_1 = \alpha_1,\ \ \beta_2 = \alpha_1+\alpha_2,\ \ \beta_3 = \alpha_2,\ \ \beta_4 = \alpha_3,\ \ \beta_5 = \alpha_3+\alpha_4,\ \ \beta_6 = \alpha_4$$

are the positive roots, where α_i $(1 \leqslant i \leqslant 4)$ are the simple roots. Hence $\prod_{i=1}^6 \chi_{\beta_i} = \chi_1^2\chi_2^2\chi_3^2\chi_4^2 = \varepsilon$. Moreover, $(y_1^{-2}y_2^{-2})x_i(y_1^2y_2^2) = q_{ii}^{-1}x_i = S^2(x_i)$ for $1 \leqslant i \leqslant 4$.

Let σ be a 2-cocycle such that $u_{12} = \frac{\sigma(y_2,y_1)}{\sigma(y_1,y_2)}$ is not a root of unity. Let $u_{21} = u_{12}^{-1}$. We claim that the algebra $B^\lambda(\sigma,\pi)$ can not be a CY algebra. Otherwise, by Proposition 3.7, there is an element $y_1^iy_2^j \in \Gamma$ such that for any $y_1^ky_2^l \in \Gamma$, $\frac{\sigma(y_1^iy_2^j, y_1^ky_2^l)}{\sigma(y_1^ky_2^l, y_1^iy_2^j)} = u_{21}^{il}u_{12}^{jk} = 1$, where the first equation follows from

Lemma 3.8 and the second equation holds because $\prod_{i=1}^{6}\chi_{\beta_i} = \varepsilon$. Now let $k = l = 1$. We obtain that $u_{21}^{i}u_{12}^{j} = u_{12}^{i-j} = 1$, Since u_{12} is not a root of unity, we have that $i = j$. Then $u_{21}^{il}u_{12}^{jk} = u_{12}^{k-l}$ can not equal to 1 when $k \neq l$. This is a contradiction.

The next example shows that the algebra $U(\mathcal{D}, \lambda)$ is not CY, but some cleft objects are CY.

Example 3.10. Let $(\mathcal{D}, \lambda)$ be the datum given by

- $\Gamma = \langle y_1, y_2\rangle$, a free abelian group of rank 2;
- The Cartan matrix $\mathbb{A}$ is of type $A_1 \times A_1$;
- $g_1 = y_1, g_2 = y_2$;
- $\chi_1(g_1) = q^2, \chi_1(g_2) = q^{-4}, \chi_2(g_1) = q^4, \chi_2(g_2) = q^{-2}$, where q is not a root of unity;
- $\lambda = 0$.

The positive roots of $\mathbb{A}$ are just the simple roots. Since $\chi_1\chi_2 \neq \varepsilon$, the algebra $H = U(\mathcal{D}, \lambda)$ is not CY (Lemma 3.3 (c)).

Let $B^0(\sigma, \pi)$ be a cleft object of H such that the 2-cocycle σ satisfies $u_{12} = \frac{\sigma(g_2,g_1)}{\sigma(g_1,g_2)} = q^3$. We also put $u_{21} = u_{12}^{-1}$. Choose an element $h = g_1^2g_2^2 \in \Gamma$. Then

$$\frac{\sigma(h, g_1)}{\sigma(g_1, h)} = \frac{\sigma(g_1^2g_2^2, g_1)}{\sigma(g_1, g_1^2g_2^2)} = u_{12}^2 = q^6 = \chi_1\chi_2(g_1),$$

where the second equation also follows from Lemma 3.8. Similarly,

$$\frac{\sigma(h, g_2)}{\sigma(g_2, h)} = \frac{\sigma(g_1^2g_2^2, g_2)}{\sigma(g_2, g_1^2g_2^2)} = u_{21}^2 = q^{-6} = \chi_1\chi_2(g_2).$$

Moreover,

$$\chi_2(g_1)\chi_1(h) = \chi_2(g_1)\chi_1(g_1^2g_2^2) = 1,$$

$$\chi_1(g_2)\chi_2(h) = \chi_1(g_2)\chi_2(g_1^2g_2^2) = 1.$$

By Proposition 3.7, the algebra $B^0(\sigma, \pi)$ is a CY algebra.

4. More Examples

In this section, we give some examples of Theorem 2.18.

The following example shows that it is possible that the crossed product of CY algebras might be a CY algebra, while their smash product is not CY.

Example 4.1. Let $A = \mathbb{k}\langle x_1, x_2\rangle/(x_1x_2 - x_2x_1)$ be the polynomial algebra with two variables. Then A is a CY algebra. Let Γ be the free abelian group of rank 2 with generators g_1 and g_2. There is a Γ-action on A as follows:

$$g_1 \cdot x_1 = qx_1, \quad g_2 \cdot x_1 = q^{-1}x_1,$$
$$g_1 \cdot x_2 = qx_2, \quad g_2 \cdot x_2 = q^{-1}x_2,$$

where q is not a root of unity. The homological determinant of this Γ-action is not trivial, namely, $\text{hdet}(g_1) = q^2$, $\text{hdet}(g_2) = q^{-2}$. The algebra $A\#\mathbb{k}\Gamma$ is not a CY algebra by Theorem 2.12 in [23].

Let σ be a 2-cocycle on Γ such that $\frac{\sigma(g_2,g_1)}{\sigma(g_1,g_2)} = q$. Without loss of generality, we may assume that $\sigma(g, g^{-1}) = \sigma(g^{-1}, g) = 1$ for $g \in \Gamma$. Then the algebra $A\#_\sigma\mathbb{k}\Gamma$ is a twisted CY algebra with Nakayama automorphism ρ defined by $\rho(a\#g) = \text{hdet}(h)a\#g$ for any $a\#g \in A\#_\sigma\mathbb{k}\Gamma$. Choose an element $h = g_1^2g_2^2 \in \Gamma$. By Lemma 3.8,

$$\frac{\sigma(h,g_1)}{\sigma(g_1,h)} = \frac{\sigma(g_1^2g_2^2,g_1)}{\sigma(g_1,g_1^2g_2^2)} = (\frac{\sigma(g_2,g_1)}{\sigma(g_1,g_2)})^2 = q^2 = \text{hdet}(g_1),$$

$$\frac{\sigma(h,g_2)}{\sigma(g_2,h)} = \frac{\sigma(g_1^2g_2^2,g_2)}{\sigma(g_2,g_1^2g_2^2)} = (\frac{\sigma(g_1,g_2)}{\sigma(g_2,g_1)})^2 = q^{-2} = \text{hdet}(g_2).$$

Moreover, $h \cdot x_i = x_i$, $1 \leqslant i \leqslant 2$. Therefore, $\rho(a\#g) = h(a\#g)h^{-1}$, for any $a\#g \in A\#_\sigma\mathbb{k}\Gamma$. The automorphism ρ is an inner automorphism. So the algebra $A\#_\sigma\mathbb{k}\Gamma$ is a CY algebra.

In the followings, we provide some examples involving the algebras $U(\mathcal{D}, \lambda)$. The definitions of algebras $U(\mathcal{D}, \lambda)$ are recalled in Section 3.1.

The following example shows that the smash product $A\#H$ is a CY algebra while A itself is not.

Example 4.2. Let H be $U(\mathcal{D}, \lambda)$ with the datum $(\mathcal{D}, \lambda)$ given by

- $\Gamma = \langle g\rangle$, a free abelian group of rank 1;
- The Cartan matrix is of type $A_1 \times A_1$;
- $g_1 = g_2 = g$;
- $\chi_1(g) = q^2$, $\chi_2(g) = q^{-2}$, where q is not a root of unity;
- $\lambda_{12} = \frac{1}{q-q^{-1}}$.

The algebra H is isomorphic to the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$.

Let $A = \mathbb{k}\langle u, v\rangle/(uv - qvu)$ be the quantum plane. There is an H-action on A as follows:

$$x_1 \cdot u = 0, \quad x_2 \cdot u = qv, \quad g \cdot u = qu,$$
$$x_1 \cdot v = u, \quad x_2 \cdot v = 0, \quad g \cdot v = q^{-1}v.$$

The algebra $A\#H$ is isomorphic to the quantized symplectic oscillator algebra of rank 1 [17].

It is well known that the algebra A is a twisted CY algebra with Nakayama automorphism μ given by

$$\mu(u) = qu, \quad \mu(v) = q^{-1}v,$$

and the algebra H is a CY Hopf algebra ([11, Theorem 3.3.2]). One can easily check that the homological determinant of the H-action is trivial and for any $x \in A\#H$, $[\mu\#S^{-2}](x) = gxg^{-1}$. That is, the automorphism $\mu\#S^{-2}$ is an inner automorphism. Therefore, $A\#H$ is a CY algebra.

The invertible elements of $A\#H$ are $\{g^m\}_{m\in\mathbb{Z}}$. Therefore, one can see that the automorphism $\mathrm{id}\,\#S^2$ of $A\#H$ can not be an inner automorphism, although, S^2 is an inner automorphism of H.

More generally, we have the following example.

Example 4.3. Let H be $U(\mathcal{D}, \lambda)$ with the datum $(\mathcal{D}, \lambda)$ given by

- $\Gamma = \langle y_1, y_2, \cdots, y_n\rangle$, a free abelian group of rank n;
- The Cartan matrix $\mathbb{A}$ is of type $A_n \times A_n$;
- $g_i = g_{n+i} = y_i$, $1 \leqslant i \leqslant n$;
- $\chi_i(g_j) = q^{a_{ij}}$, $\chi_{n+i}(g_j) = q^{-a_{ij}}$, $1 \leqslant i \leqslant n$, where q is not a root of unity;
- $\lambda_{ij} = \delta_{n+i,j}\frac{1}{q-q^{-1}}$, $1 \leqslant i < j \leqslant 2n$.

Then H is isomorphic to the algebra $U_q(\mathfrak{sl}_n)$. It is also a CY Hopf algebra.

Let A be the quantum polynomial algebra

$$\Bbbk\langle u_1, u_2, \cdots, u_{n+1} \mid u_j u_i - q u_i u_j, 1 \leqslant i < j \leqslant n+1\rangle.$$

There is an H-action on A as follows:

$$x_i \cdot u_j = \delta_{ij} u_{i+1}, 1 \leqslant i \leqslant n; \qquad x_i \cdot u_j = \delta_{i+1,j} q u_i, n+1 \leqslant i \leqslant 2n$$

$$y_i \cdot u_j = \begin{cases} q^{-1}u_j, & j = i; \\ qx_j, & j = i+1; \\ x_j, & \textit{otherwise}. \end{cases}$$

It is well known that the algebra A is a twisted CY algebra with Nakayama automorphism μ given by $\mu(u_i) = q^{n+2-2i}u_i$, $1 \leqslant i \leqslant n+1$.

One can also check that the homological determinant of the H-action is trivial. The automorphism $\mu\#S^{-2}$ is an inner automorphism. For any $x \in A\#H$,

$[\mu\#S^{-2}](x) = gxg^{-1}$, where $g = y_1^n y_2^{2n-2} \cdots y_i^{in-i(i-1)} \cdots y_n^{n^2-n(n-1)}$. Therefore, $A\#H$ is a CY algebra.

Let H^0 be the algebra $U(\mathcal{D},0)$. The algebra H is a cocycle deformation of $U(\mathcal{D},0)$. Actually, $H \cong (H^0)^\sigma$, where σ is a 2-cocycle on H^0 such that $\sigma(h_1,h_2)=1$, $\sigma(x_i,h_1)=\sigma(h_2,x_i)=0$, for all $h_1,\ h_2 \in \Gamma$ and $1 \leqslant i \leqslant n+1$, and

$$\sigma(x_i,x_j) = \begin{cases} \lambda_{ij}, & j=n+i; \\ 0, & otherwise. \end{cases}$$

Then we have the crossed product $A\#_\sigma H^0$. By Theorem 2.18, $A\#_\sigma H^0$ is a twisted CY algebra with Nakayama automorphism η defined by $\eta(a\#h) = \mu(a)\#h$, for all $a\#h \in A\#H$. In fact, η is an inner automorphism. For any $x \in A\#_\sigma H^0$, $\eta(x) = gxg^{-1}$. So $A\#_\sigma H^0$ is also a CY algebra.

Example 4.4. Let $H = U(\mathcal{D},\lambda)$, where $(\mathcal{D},\lambda)$ is the datum given by

- $\Gamma = \langle y_1, y_2\rangle$, a free abelian group of rank 2;
- The Cartan matrix $\mathbb{A}$ is of type $A_1 \times A_1$;
- $g_1 = y_1, g_2 = y_2$;
- $\chi_1(g_1) = q^2, \chi_1(g_2) = q^{-4}, \chi_2(g_1) = q^4, \chi_2(g_2) = q^{-2}$, where q is not a root of unity;
- $\lambda = 0$.

The algebra H is a twisted CY algebra with homological integral ${}_{\xi_1}\Bbbk$, where ξ_1 is the algebra homomorphism given by

$$\xi_1(g_1) = q^6 g_1, \xi_1(g_2) = q^{-6} g_2, \text{ and } \xi_1(x_i) = 0, i = 1,2.$$

Let σ be a 2-cocycle on H such that $\frac{\sigma(g_1,g_2)}{\sigma(g_2,g_1)} = q^3$, $\sigma(x_i,g_j) = \sigma(g_j,x_i) = 0$, $1 \leqslant i,j \leqslant 2$, and $\sigma(x_1,x_2) = \frac{1}{q-1}, \sigma(x_2,x_1) = 0$. Then the cocycle deformation H^σ is just the algebra $U(\mathcal{D}',\lambda')$, where $(\mathcal{D}',\lambda')$ is the datum given by

- $\Gamma = \langle y_1, y_2\rangle$, a free abelian group of rank 2;
- The Cartan matrix is of type $A_1 \times A_1$;
- $g_1 = y_1, g_2 = y_2$;
- $\chi_1(g_1) = q^{-2}, \chi_1(g_2) = q, \chi_2(g_1) = q^{-1}, \chi_2(g_2) = q^2$, where q is not a root of unity;
- $\lambda_{12} = \frac{1}{q-1}$.

The algebra H^σ is a twisted CY algebra with homological integral ${}_{\xi_2}\Bbbk$, where ξ_2 is the algebra homomorphism given by

$$\xi_2(g_1) = q^{-3} g_1, \xi_2(g_2) = q^3 g_2, \text{ and } \xi_2(x_i) = 0, i = 1,2.$$

Let $A = \Bbbk\langle u, v\rangle/(uv - q^2vu)$ be the quantum plane. There is an H^σ-action on A as follows:

$$\begin{array}{llll} x_1 \cdot u = 0, & x_2 \cdot u = v, & g_1 \cdot u = q^{-1}u, & g_2 \cdot u = q^2 u \\ x_1 \cdot v = u, & x_2 \cdot v = 0, & g_1 \cdot v = qv, & g_2 \cdot v = q^{-2}v. \end{array}$$

We have mentioned in Example 4.2 that A is a twisted CY algebra with Nakayama automorphism μ given by

$$\mu(u) = q^2u, \ \ \mu(v) = q^{-2}v.$$

One can check that the homological determinant of the H action is trivial. Now we can form the algebras $A\#H^\sigma$ and $A\#_\sigma H$. By Theorem 2.19, the algebra $A\#H^\sigma$ is a twisted CY algebra with Nakayama automorphism $\mu\#(S^{-2}\circ[\xi]^r)$. This automorphism cannot be an inner automorphism. That is, $A\#H^\sigma$ is not a CY algebra. Theorem 2.18 shows that the algebra the algebra $A\#_\sigma H$ is a twisted CY algebra with Nakayama automorphism ρ defined by $\rho(a) = \mu(a)$, $a \in A$, $\rho(x_1) = q^{-2}x_1$, $\rho(x_2) = q^2x_2$, and $\rho(g_i) = \xi(g_i)g_i$, $i = 1, 2$. The automorphism ρ is an inner automorphism. For any $x \in A\#_\sigma H$, $\rho(x) = (g_1^2g_2^2)^{-1}x(g_1^2g_2^2)$. Therefore, the algebra $A\#_\sigma H$ is a CY algebra.

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