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Abstract. Let A be a Koszul Artin–Schelter regular algebra and σ an algebra homomorphism from A to $M_{2\times 2}(A)$. We compute the Nakayama automorphisms of a trimmed double Ore extension $A_P[y_1, y_2; \sigma]$ [introduced in Zhang and Zhang (J Pure Appl Algebra 212:2668–2690, 2008)]. Using a similar method, we also obtain the Nakayama automorphism of a skew polynomial extension $A[t; \theta]$, where θ is a graded algebra automorphism of A. These lead to a characterization of the Calabi–Yau property of $A_P[y_1, y_2; \sigma]$, the skew Laurent extension $A[t^{\pm 1}; \theta]$ and $A[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$ with σ a diagonal type.

Introduction

Nakayama automorphisms play an important role in noncommutative algebraic geometry especially in noncommutative invariant theory [4,14,19]. Let A be a Koszul Artin–Schelter regular algebra with Nakayama automorphism ν in the sense of [2]. The Nakayama automorphism and Calabi–Yau property of Ore extensions and of skew polynomial extensions were studied in [1,8–10,12,19]. In this paper, we compute the Nakayama automorphisms of certain double Ore extension $A_P[y_1, y_2; \sigma]$ of A; the general notion of a double Ore extension was introduced by Zhang and Zhang [24]. Then we study the Calabi–Yau property of $A_P[y_1, y_2; \sigma]$, a skew Laurent extension $A[t^{\pm 1}; \theta]$, where $\theta \in \mathrm{GrAut}(A)$, and $A[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$ with σ a diagonal type.

It is well-known that a graded Ore extension of a Koszul algebra is also Koszul (see [17, Corollary 1.3] for example). For a Koszul Artin–Schelter regular algebra, Van den Bergh proposed an effective method to compute the Nakayama automorphism through the Yoneda Ext algebra (see Proposition 1.4 or [21, Theorem 9.2]). Inspired by these two facts, we first show the following:

Theorem 1. (Theorem 2.1) *Let A be a Koszul algebra and $B = A_P[y_1, y_2; \sigma]$ be a trimmed double Ore extension of A. Then, B is a Koszul algebra.*

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By describing the Yoneda Ext algebra, we are able to compute the Nakayama automorphism of a trimmed double Ore extension of a Koszul Artin–Schelter regular algebra.

Theorem 2. (Proposition 3.11 and Theorem 3.12) *Let A be a Koszul Artin–Schelter regular algebra with Nakayama automorphism ν, and $B = A_P[y_1, y_2; \sigma]$ a trimmed double Ore extension of A. Then,*

(1) *The restriction of the Nakayama automorphism ν_B of B to A equals* $(\det_r \sigma)^{-1}\nu$, *and*

$$\nu_B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (\operatorname{hdet}\sigma)\mathbb{P}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $\det_r \sigma$ *is an algebra automorphism induced by* σ, $\operatorname{hdet}\sigma \in M_2(\Bbbk)$ *is determined by* σ, *and* $\mathbb{P} \in M_2(\Bbbk)$ *is determined by the data* P *(see Eqs. 1.6, 1.8 and Definition 2.5 for their definitions);*

(2) *B is Calabi–Yau if and only if* $\det_r \sigma = \nu$ *and* $\operatorname{hdet}\sigma = \mathbb{P}$.

In a similar way, one can obtain the analogous results on the Nakayama automorphism and the Calabi–Yau property of the skew polynomial extension of a Koszul Artin–Schelter regular algebra (see Proposition 3.15 and Theorem 3.16).

Farinati [5, Theorem 6] showed that the Calabi–Yau property is preserved by noncommutative localizations. Here, we characterize the Calabi–Yau property of the localization of both the skew polynomial extension with respect to the Ore set $\{t^i, i \in \mathbb{N}\}$ (called the skew Laurent extension) and the iterated skew polynomial extension. The third main result reads as follows:

Theorem 3. (Theorems 4.2 and 4.5) *Let A be a Koszul Artin–Schelter regular algebra with Nakayama automorphism ν.*

(1) *The skew Laurent extension $A[t^{\pm 1}; \theta]$ of A is Calabi–Yau if and only if there exists an integer n such that $\theta^n = \nu$ and the homological determinant* $\operatorname{hdet}(\theta)$ *of θ equals* 1.

(2) *Given two automorphisms τ and ξ of A, let, $Q = A[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$, where $\sigma = diag(\tau, \xi)$ is a map from A to $M_{2\times 2}(A)$. Then, Q is Calabi–Yau if and only if there exists two integers m, n such that $\tau^m \xi^n = \nu$ and* $\operatorname{hdet}(\tau) = \operatorname{hdet}(\xi) = 1$.

In fact, Part (2) of Theorem 3 is a special case of what is proved in Theorem 4.5. The aforementioned results and their proofs indicate that there exists a strong relation between the Nakayama automorphisms of those extensions and the homological determinants of the automorphisms which determine those extensions (see Theorems 2 and 3). The Nakayama automorphisms of the right coideal subalgebras of the quantized enveloping algebras were explicitly computed [16]. In fact, those coideal subalgebras are special iterated Ore extensions. The general case of iterated Ore extensions and their relation with double Ore extensions were discussed in [3]. So it would be interesting to study the Nakayama automorphism and the Calabi–Yau property of double Ore extensions and those of the localizations of iterated skew polynomial extensions in general.

The paper is organised as follows. In Sect. 1, we recall the definitions and the properties, including the relation between the Nakayama automorphism of a Koszul Artin–Schelter regular algebra and its Yoneda Ext algebra. Section 2 prepares necessary means for computing the Nakayama automorphisms of trimmed double Ore extensions of Koszul algebras.

In Sect. 3, we mainly compute the Nakayama automorphism and study the Calabi–Yau property of trimmed double Ore extensions of Koszul Artin–Schelter regular algebras. In Sect. 4, apart from what we mentioned in Theorem 3, the Calabi–Yau property of the skew Laurent extensions and the Calabi–Yau property of a localization of iterated Ore extensions are studied. Necessary and sufficient conditions for those algebras to be Calabi–Yau are determined, see Theorem 4.6.

Throughout, $\mathbb{k}$ is a field and all algebras are $\mathbb{k}$-algebras; unadorned $\otimes$ means $\otimes_{\mathbb{k}}$ and * always denotes the dual over $\mathbb{k}$.

1. Preliminaries

An $\mathbb{N}$-graded algebra $A = \bigoplus_{i \geqslant 0} A_i$ is called connected if $A_0 = \mathbb{k}$. By a graded algebra we mean a locally finite graded algebra generated in degree 1. Let $A^e = A \otimes A^{op}$ denotes the enveloping algebra of A. A module means a left (graded) module. The shifting of a graded module is denoted ( ). For a module M over A, ${}^{\varphi}M$ stands for a twisted module by an algebra automorphism φ, where the action is defined by $a \cdot m := \varphi(a)m$. Similarly, M^{φ} and ${}^{1}M^{\varphi}$ denote the twisted right module and the twisted bimodule respectively.

Let V be a finite-dimensional vector space, and $T_{\mathbb{k}}(V)$ be the tensor algebra with the usual grading. A connected graded algebra $A = T_{\mathbb{k}}(V)/\langle R \rangle$ is called a quadratic algebra if R is a subspace of $V^{\otimes 2}$. The homogeneous dual of A is then defined as $A^! = T_{\mathbb{k}}(V^*)/\langle R^{\perp} \rangle$, where

$$R^{\perp} = \{\lambda \in V^* \otimes V^* \mid \lambda(r) = 0, \quad \text{for all} \quad r \in R\}.$$

Here, we identify $(V \otimes V)^*$ with $V^* \otimes V^*$ by

$$(\alpha \otimes \beta)(x \otimes y) = \alpha(x)\beta(y) \tag{1.1}$$

for $\alpha, \beta \in V^*$ and $x, y \in V$. For more detail, see [20].

Definition 1.1. A quadratic algebra A is called Koszul if the trivial A-module ${}_A\mathbb{k}$ admits a projective resolution

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_A\mathbb{k} \longrightarrow 0$$

such that P_n is generated in degree n for all $n \geq 0$.

For more detail about Koszul algebras and the Koszul duality, we refer the reader to [18, Ch.2]. Now, we recall the definitions of an Artin–Schelter regular algebra, a Nakayama automorphism and a Calabi–Yau algebra.

Definition 1.2. A connected graded algebra A is called Artin–Schelter (AS, for short) Gorenstein of dimension d with parameter l for some integers d and l, if

(i) $\operatorname{inj.dim}({}_A A) = \operatorname{inj.dim}(A_A) = d$; and

(ii) $\operatorname{Ext}^i_A(\Bbbk, A) \cong \operatorname{Ext}^i_{A^{op}}(\Bbbk, A) \cong \begin{cases} 0, & i \neq d, \\ \Bbbk(l), & i = d. \end{cases}$

If, in addition, A has a finite global dimension, then A is called AS-regular.

Definition 1.3. [2,6] A graded algebra A is called twisted Calabi–Yau of dimension d if

(i) A is homologically smooth, i.e., A, as an A^e-module, has a finitely generated projective resolution of finite length.

(ii) $\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d \\ A^\nu(l), & i = d \end{cases}$ as A^e-modules for some automorphism ν of A and some integers d, l.

The automorphism ν is called *the Nakayama automorphism* of A. If, in addition, A^ν is isomorphic to A as A^e-modules, or equivalently, ν is inner, then A is called *Calabi–Yau of dimension* d. Ungraded Calabi–Yau algebras are defined similarly but without degree shift.

Let E be a Frobenius algebra. By definition, there is an isomorphism $\varphi : E \longrightarrow E^*$ of right E-modules. This is equivalent to the existence of a nondegenerate bilinear form, often called Frobenius pair, $\langle -, - \rangle : E \times E \to \Bbbk$ such that $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in E$ (where the bilinear form is defined by $\langle a, b\rangle := \varphi(b)(a)$). By the nondegeneracy of the bilinear form, there exists an automorphism μ, unique up to an inner automorphism, such that

$$\langle a, b\rangle = \langle \mu(b), a\rangle \tag{1.2}$$

for all $a, b \in E$. Thus, φ becomes an isomorphism of E-bimodules ${}^\mu E \cong E^*$. The automorphism μ is usually called the Nakayama automorphism of E. For more detail, see [20].

Now, there are two notions of Nakayama automorphisms: one for twisted Calabi–Yau algebras and one for Frobenius algebras. We use ν for the former and μ for the latter if there is no confusion. In fact, the notion of a Nakayama automorphism in [2] can be defined for algebras with finite injective dimension, and it coincides with the classical Nakayama automorphism of a Frobenius algebra. But in this paper, we focus ourselves on twisted Calabi–Yau algebras (or equivalently, AS-regular algebras in the connected graded case [19, Lemma 1.2]). It is well known that a connected graded algebra A is AS-regular if and only if its Yoneda Ext algebra is Frobenius [13, Corollary D]. In this case, the two notions of Nakayama automorphisms will coincide in the sense of the Koszul duality, see Proposition 1.4. To get there, we need the following preparation.

Let $A = T_\Bbbk(V)/\langle R \rangle$ be a Koszul algebra. Then its Yoneda Ext algebra $E(A) := \bigoplus_{i\in\mathbb{N}} \operatorname{Ext}^i_A(\Bbbk, \Bbbk)$ is isomorphic to $A^! = T_\Bbbk(V^*)/\langle R^\perp \rangle$, see [20]. For a graded automorphism θ of A, we define a map $\theta^* : V^* \to V^*$ by $\theta^*(f)(x) = f(\theta(x))$ for each $f \in V^*$ and $x \in V$. It is easy to see that θ^* induces a graded automorphism of $A^!$ because θ is assumed to preserve the relation space R. We still use the notation θ^* for this algebra automorphism. Suppose that $\{e_1, e_2, \dots, e_n\}$ is a $\Bbbk$-linear basis of

V and $\{e_1^*, e_2^*, \ldots, e_n^*\}$ is the corresponding dual basis of V^*. If $\theta(e_i) = \sum_j c_{ij} e_j$ for $c_{ij} \in \Bbbk\,(1 \leq i, j \leq n)$, then we have:

$$\theta^*(e_i^*) = \sum_j c_{ji} e_j^*. \tag{1.3}$$

Moreover, for each $i, j = 1, 2, \ldots, n$, we have:

$$\theta^*(e_i^*)(e_j) = e_i^*(\theta(e_j)). \tag{1.4}$$

Proposition 1.4. [21, Theorem 9.2] *Let A be a Koszul AS-regular algebra of dimension d. Then, the Nakayama automorphism ν of A is equal to $\epsilon^{d+1}\mu^*$, where μ is the Nakayama automorphism of the Frobenius algebra $A^!$ and ϵ is the automorphism of A defined by $a \mapsto (-1)^{\deg a} a$, for each homogeneous element $a \in A$.* □

Next, we recall the definition and some basic properties of a double Ore extension.

Definition 1.5. [24,25] Let A be a subalgebra of a $\Bbbk$-algebra B.

(1) B is called a right double Ore extension of A if
 (i) B is generated by A together with two variables y_1 and y_2;
 (ii) y_1 and y_2 satisfy the relation:

$$y_2 y_1 = p y_1 y_2 + q y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0$$

 for some $p, q \in \Bbbk$ and $\tau_1, \tau_2, \tau_0 \in A$;
 (iii) B is a free left A-module with basis $\{y_1^i y_2^j; i, j \geq 0\}$;
 (iv) $y_1 A + y_2 A \subseteq A y_1 + A y_2 + A$.

(2) B is called a left double Ore extension of A if:
 (i) B is generated by A and two new variables y_1 and y_2;
 (ii) y_1 and y_2 satisfy the relation

$$y_1 y_2 = p' y_2 y_1 + q' y_1^2 + y_1 \tau_1' + y_2 \tau_2' + \tau_0'$$

 for some $p', q' \in \Bbbk$ and $\tau_1', \tau_2', \tau_0' \in A$;
 (iii) B is a free right A-module with basis $\{y_2^i y_1^j; i, j \geq 0\}$;
 (iv) $A y_1 + A y_2 \subseteq y_1 A + y_2 A + A$.

(3) B is called a double Ore extension of A if it is a left and a right double Ore extension of A with the same generating set $\{y_1, y_2\}$.

Note that Condition (1).(iv) in the Definition 1.5 is equivalent to the existence of two maps:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : A \to M_{2\times 2}(A) \quad \text{and} \quad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} : A \to M_{2\times 1}(A)$$

subject to

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} a = \sigma(a) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \delta(a) \tag{1.5}$$

for all $a \in A$, where $\sigma_{ij}, \delta_i \in \mathrm{End}_{\Bbbk}(A)$. *In case B is a right double Ore extension of A, we will write $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = (p, q) \in \Bbbk^2$, $\tau = (\tau_0, \tau_1, \tau_2) \in A^3$, and σ, δ as above*. Along with the datum P, we define a matrix $\mathbb{P}$ in $M_2(\Bbbk)$ as follows:

$$\mathbb{P} = \begin{pmatrix} p & 0 \\ -(1 + \frac{1}{p})q & \frac{1}{p} \end{pmatrix} \tag{1.6}$$

Like in an Ore extension, here σ is a homomorphism of algebras and δ is a σ-derivation, that is, δ is $\Bbbk$-linear and satisfies $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in A$. The double Ore extensions that we shall consider mainly in this work are the so-called *trimmed* double Ore extensions.

Definition 1.6. A double Ore extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ is called a *trimmed double Ore extension*, if δ is the zero map and $\tau = \{0, 0, 0\}$. In this case, we use the short notation $A_P[y_1, y_2; \sigma]$ for a trimmed double Ore extension.

Condition (2).(iv) in Definition 1.6 is equivalent to the existence of two maps

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \to M_{2\times 2}(A) \quad \text{and} \quad \delta' = \begin{pmatrix} \delta_1' & \delta_2' \end{pmatrix} : A \to M_{1\times 2}(A)$$

satisfying

$$a \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \phi(a) + \delta'(a) \tag{1.7}$$

for all $a \in A$. For a double Ore extension, the connection between σ and ϕ in Eqs. (1.5) and (1.7) can be seen in the following definition and lemma.

Definition 1.7. [24] Let $\sigma : A \to M_{2\times 2}(A)$ be an algebra homomorphism. We say that σ is invertible if there is an algebra homomorphism $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \to M_{2\times 2}(A)$ satisfies the following conditions:

$$\sum_{k=1}^{2} \phi_{jk}(\sigma_{ik}(r)) = \begin{cases} r, & i = j \\ 0, & i \neq j \end{cases} \quad \text{and} \quad \sum_{k=1}^{2} \sigma_{kj}(\phi_{ki}(r)) = \begin{cases} r, & i = j \\ 0, & i \neq j \end{cases}$$

for all $r \in A$. The map ϕ is called the inverse of σ.

The following lemma gives the relation of the condition that σ is invertible and the condition a right double Ore extension being a double Ore extension.

Lemma 1.8. [24, Lemma 1.9 and its proof, Proposition 2.1] *Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double Ore extension of A.*

(1) *If B is a double Ore extension, then σ is invertible with the inverse ϕ such that the Eq.* (1.7) *holds for some δ'.*
(2) *Suppose that both A and B are connected graded algebras. If $p \neq 0$ and σ is invertible, then B is a double Ore extension.* □

Next, we list the identities induced by commuting the equation $y_2y_1 = py_1y_2 + qy_1^2$ with element $r \in A$. Explicitly, since $y_2y_1r = (py_1y_2 + qy_1^2)r$ for each $r \in A$, so we get the relations R3.1-R3.3 in [24, p. 2674] (as we only consider the trimmed double Ore extension here). Dually, we have the following

(R′3.1) $\phi_{11}(\phi_{12}(r)) + q\phi_{11}(\phi_{22}(r)) = p\phi_{12}(\phi_{11}(r)) + q\phi_{11}(\phi_{11}(r)) + pq\phi_{12}(\phi_{21}(r)) + q^2\phi_{11}(\phi_{21}(r))$
(R′3.2) $\phi_{21}(\phi_{12}(r)) + p\phi_{11}(\phi_{22}(r)) = p\phi_{22}(\phi_{11}(r)) + q\phi_{21}(\phi_{11}(r)) + p^2\phi_{12}(\phi_{21}(r)) + pq\phi_{11}(\phi_{21}(r))$
(R′3.3) $\phi_{21}(\phi_{22}(r)) = p\phi_{22}(\phi_{21}(r)) + q\phi_{21}(\phi_{21}(r))$

In order to study the regularity of double Ore extensions, Zhang and Zhang introduced an invariant of σ, called the (right) determinant of σ, which is similar to the quantum determinant of the 2×2-matrix. As we will see, this invariant will play an important role in the description of the Nakayama automorphism of the trimmed double Ore extension.

Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double Ore extension of A. The right determinant of σ is defined to be the map:

$$\det_r \sigma : \; A \longrightarrow A, \; a \mapsto -q\sigma_{12}(\sigma_{11}(a)) + \sigma_{22}(\sigma_{11}(a)) - p\sigma_{12}(\sigma_{21}(a)) \tag{1.8}$$

for $a \in A$. If σ is invertible with the inverse ϕ, then the left determinant of ϕ is defined by:

$$\det_l \phi := -q\phi_{11} \circ \phi_{21} + \phi_{11} \circ \phi_{22} - p\phi_{12} \circ \phi_{21}.$$

We remark that when $q = 0$ the above expression of $\det_l \phi$ coincides with the one in [24] after E2.1.6. The following properties of the determinant of σ were given in [24].

Proposition 1.9. [24, proof of Proposition 2.1] *Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a double Ore extension of A such that σ is invertible with inverse ϕ. Then,*

(1) $\det_r \sigma$ *is an algebra endomorphism of A;*
(2) *if $p \neq 0$, then*

$$\det_r \sigma = \frac{q}{p}\sigma_{11} \circ \sigma_{12} + \sigma_{11} \circ \sigma_{22} - \frac{1}{p}\sigma_{21} \circ \sigma_{12},$$
$$\det_l \phi = \frac{q}{p}\phi_{21} \circ \phi_{11} + \phi_{22} \circ \phi_{11} - \frac{1}{p}\phi_{21} \circ \phi_{12};$$

(3) $\det_r \sigma$ *is invertible with inverse* $\det_l \phi$.

Remark that the equation $\det_l \phi = \frac{q}{p}\phi_{21} \circ \phi_{11} + \phi_{22} \circ \phi_{11} - \frac{1}{p}\phi_{21} \circ \phi_{12}$ follows from the relation R′3.2. Here, we use the notions of $\det_r$ and $\det_l$ in order to differ the morphisms determined by σ and ϕ. Note that there is a print typos in the formula of [24, line-11, p. 2677], where the minus sign of the first term should be dropped. In fact, that can be verified by using [24, R3.2, p. 2674].

Double Ore extensions are used to construct higher dimensional AS-regular algebras from lower dimensional ones because of the following result which will be used later.

Lemma 1.10. [24, Theorem 0.2] *Let A be an AS-regular algebra. If B is a connected graded and a double Ore extension of A, then B is AS-regular and* $\operatorname{gldim} B = \operatorname{gldim} A + 2$. □

2. Koszul algebra and homological determinant

In this section, we make necessary preparation for computing the Nakayama automorphism of a trimmed double Ore extension. To this aim, we first prove that the Koszul property is preserved by making a trimmed double Ore extension. We then introduce the homological determinant of an algebra homomorphism $\sigma : A \to M_{2\times 2}(A)$ for a Koszul algebra A and study its properties.

Theorem 2.1. *Let A be a Koszul algebra and $B = A_P[y_1, y_2; \sigma]$ be a trimmed double Ore extension of A. Then, B is a Koszul algebra.*

Proof. Suppose that M is a B-A-bimodule and φ is an automorphism of A. Recall that ${}^1M^{\varphi}$ is the twisted bimodule on the $\Bbbk$-space M with

$$b \cdot m \cdot a = bm\varphi(a)$$

for all $m \in M, b \in B$ and $a \in A$. On the space $M \oplus M$, there is another right A-module structure defined by using σ as follows:

$$(m, n) \circ a = (m, n) \begin{pmatrix} \sigma_{11}(a) & \sigma_{12}(a) \\ \sigma_{21}(a) & \sigma_{22}(a) \end{pmatrix} = (m\sigma_{11}(a) + n\sigma_{21}(a), m\sigma_{12}(a) + n\sigma_{22}(a)) \tag{2.1}$$

for all $m, n \in M$ and $a \in A$. Since σ is an algebra homomorphism, $M \oplus M$ is a B-A-bimodule. Denote by $(M \oplus M)^{\sigma}$ this B-A-bimodule. By [24, Theorem 2.2], there is an exact sequence of B-A-bimodules

$$0 \to B^{\det_r \sigma} \xrightarrow{g} (B \oplus B)^{\sigma} \xrightarrow{f} B \xrightarrow{\varepsilon} A \to 0, \tag{2.2}$$

where, f maps (s, t) to $sy_1 + ty_2$, g sends r to $(r(qy_1 - y_2), rpy_1)$ and the last term A is identified with $B/(y_1, y_2)$. Moreover, (2.2) is a linear resolution of ${}_BA$ in case both y_1 and y_2 are of degree 1.

Now, by assumption, ${}_A\Bbbk$ admits a projective resolution:

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_A\Bbbk \longrightarrow 0 \tag{2.3}$$

with P_n generated in degree n for each $n \geq 0$. We consider the third quadrant bicomplex:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
\cdots \arrow[r] & B \otimes_A P_2 \arrow[r] \arrow[u] & B \otimes_A P_1 \arrow[r] \arrow[u] & B \otimes_A P_0 \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & (B \oplus B)^{\sigma} \otimes_A P_2 \arrow[r] \arrow[u] & (B \oplus B)^{\sigma} \otimes_A P_1 \arrow[r] \arrow[u] & (B \oplus B)^{\sigma} \otimes_A P_0 \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & B^{\det_r \sigma} \otimes_A P_2 \arrow[r] \arrow[u] & B^{\det_r \sigma} \otimes_A P_1 \arrow[r] \arrow[u] & B^{\det_r \sigma} \otimes_A P_0 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

It follows that $\det_r \sigma$ is an automorphism of A and that B is a right free A-module, $B^{\det_r \sigma}$ is projective as a right A-module. Now, for the right A-module $(B \oplus B)^{\sigma}$, we are going to show it is also projective as a right A-module. Since we have the following general result: if M, P, Q are projective in the exact sequence

$$0 \to M \to N \to P \to Q \to 0,$$

then so is N. For this end, we take K to be the kernel of $P \to Q$ and we get two short exact sequence

$$0 \to M \to N \to K \to 0, \qquad 0 \to K \to P \to Q \to 0.$$

Since P and Q are projective, the second sequence is split and K is projective. Therefore, the first sequence is split and N is projective. Hence, each term in the sequence (2.2) is projective as a right A-module. Further, all the rows of the bicomplex are exact except at the (-1)-st column. Thus, the homology along the rows yields a single nonzero column, that is,

$$\cdots \to 0 \to B^{\det_r \sigma} \otimes_A \Bbbk \to (B \oplus B)^{\sigma} \otimes_A \Bbbk \to B \otimes_A \Bbbk \to 0. \tag{2.4}$$

Moreover, the sequence (2.2) is a split exact sequence. Therefore, the homology of (2.4) is ${}_B A \otimes_A \Bbbk = {}_B \Bbbk$. Namely, the total complex of the bicomplex is a projective resolution of the B-module ${}_B\Bbbk$. Finally, both sequence (2.2) and (2.3) are linear resolutions, so is the total complex of the bicomplex. The proof is completed. □

Remark 2.2. (1) Theorem 2.1 generalizes the well-known result that a graded Ore extension of a Koszul algebra is again Koszul (see [17, Corollary 1.3]).

(2) It was proved in [25, Theorem 0.1(b)] that a graded double Ore extension of an AS-regular algebra of dimension 2 is Koszul. Since an AS-regular algebra of dimension 2 is always Koszul, Theorem 2.1 generalizes [25, Theorem 0.1(b)] in the trimmed case.

For an AS-Gorenstein algebra A, Jørgensen and Zhang proposed the notion of the homological determinant of a graded automorphism in [11] in order to study the noncommutative invariant theory. Roughly speaking, for an AS-Gorenstein algebra A, the homological determinant, denoted $hdet$, is a homomorphism from the graded automorphism group $\mathrm{GrAut}(A)$ of A to the multiplicative group $\Bbbk\backslash\{0\}$ generalizing the usual determinant of a matrix. For the precise definition and its application, we refer to [11,19]. Here, we just need the following characterization of the homological determinant of an automorphism of a Koszul algebra.

Proposition 2.3. [22, Proposition 1.11] *Let A be a Koszul AS-regular algebra of global dimension d. Suppose that θ is a graded automorphism of A and θ^* is its corresponding dual graded automorphism of the dual algebra $A^!$. Then, we have $\theta^*(u) = (\mathrm{hdet}\,\theta)u$ for any $u \in \mathrm{Ext}_A^d(\Bbbk, \Bbbk)$.* □

Suppose that $A = T_{\Bbbk}(V)/\langle R\rangle$ is a Koszul algebra. Let $\sigma : A \to M_{2\times 2}(A)$ be an algebra homomorphism. Then,

$$\begin{pmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{21}^* & \sigma_{22}^* \end{pmatrix} : V^* \to M_{2\times 2}(V^*)$$

defines a $\Bbbk$-linear map, denoted by σ^*, where σ_{ij}^* is the dual of σ_{ij} on the space V^* (see the paragraph before Proposition 1.4) for each pair (i, j) with $i, j \in \{1, 2\}$. Extend σ^* to an algebra homomorphism $\sigma^* : T_{\Bbbk}(V^*) \to M_{2\times 2}(T_{\Bbbk}(V^*))$ by letting:

$$\sigma^*(xy) := \sigma^*(x)\sigma^*(y)$$

for each $x, y \in V^*$. In particular, for $e_i^*, e_j^* \in V^*$

$$\begin{aligned}\sigma^*(e_i^*e_j^*) &= \begin{pmatrix} \sigma_{11}^*(e_i^*) & \sigma_{12}^*(e_i^*) \\ \sigma_{21}^*(e_i^*) & \sigma_{22}^*(e_i^*) \end{pmatrix} \begin{pmatrix} \sigma_{11}^*(e_j^*) & \sigma_{12}^*(e_j^*) \\ \sigma_{21}^*(e_j^*) & \sigma_{22}^*(e_j^*) \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{11}^*(e_i^*)\sigma_{11}^*(e_j^*) + \sigma_{12}^*(e_i^*)\sigma_{21}^*(e_j^*) & \sigma_{11}^*(e_i^*)\sigma_{12}^*(e_j^*) + \sigma_{12}^*(e_i^*)\sigma_{22}^*(e_j^*) \\ \sigma_{21}^*(e_i^*)\sigma_{11}^*(e_j^*) + \sigma_{22}^*(e_i^*)\sigma_{21}^*(e_j^*) & \sigma_{21}^*(e_i^*)\sigma_{12}^*(e_j^*) + \sigma_{22}^*(e_i^*)\sigma_{22}^*(e_j^*) \end{pmatrix}.\end{aligned}$$

For any $e_k, e_l \in V$,

$$\begin{aligned}&(\sigma_{11}^*(e_i^*)\sigma_{11}^*(e_j^*) + \sigma_{12}^*(e_i^*)\sigma_{21}^*(e_j^*))(e_ke_l) \\ &\overset{\text{by}(1.1)}{=} \sigma_{11}^*(e_i^*)(e_k)\sigma_{11}^*(e_j^*)(e_l) + \sigma_{12}^*(e_i^*)(e_k)\sigma_{21}^*(e_j^*)(e_l) \\ &\overset{\text{by}(1.4)}{=} e_i^*(\sigma_{11}(e_k))e_j^*(\sigma_{11}(e_l)) + e_i^*(\sigma_{12}(e_k))e_j^*(\sigma_{21}(e_l)) \\ &\overset{\text{by}(1.1)}{=} e_i^*e_j^*(\sigma_{11}(e_k)\sigma_{11}(e_l) + \sigma_{12}(e_k)\sigma_{21}(e_l)) \\ &= e_i^*e_j^*\big((\sigma_{11}\sigma_{11} + \sigma_{12}\sigma_{21})(e_ke_l)\big).\end{aligned}$$

Then, $\sigma_{11}^*(r') \in R^{\perp}$ for any $r' \in R^{\perp}$. For this end, assuming that $r' = \sum_{i,j} c_{ij}e_i^*e_j^* \in R^{\perp}$, then for any $r = \sum_{k,l} d_{kl}e_ke_l \in R$ it follows from the above computation that

$$\begin{aligned}\sigma_{11}^*(r')(r) &= \sum_{i,j} c_{ij}(\sigma_{11}^*(e_i^*)\sigma_{11}^*(e_j^*) + \sigma_{12}^*(e_i^*)\sigma_{21}^*(e_j^*))\left(\sum_{k,l} d_{kl}e_ke_l\right)\\ &= \sum_{i,j} c_{ij}e_i^*e_j^*\left((\sigma_{11}\sigma_{11} + \sigma_{12}\sigma_{21})\left(\sum_{k,l} d_{kl}e_ke_l\right)\right)\\ &= r'(\sigma_{11}(r)).\end{aligned}$$

Since σ_{11} is an algebra endomorphism of $A = T_{\Bbbk}(V)/\langle R\rangle$, we obtain that $\sigma_{11}(r) \in R$. Hence, $\sigma_{11}^*(r')(r) = r'(\sigma_{11}(r)) = 0$. It is shown that $\sigma_{11}^*(r') \in R^{\perp}$ for any $r' \in R^{\perp}$. That is, σ_{11}^* induces an algebra endomorphism of $A^! = T_{\Bbbk}(V^*)/\langle R^{\perp}\rangle$. Similarly, the same claims for σ_{12}^*, σ_{21}^* and σ_{22}^* hold by computation. Furthermore, σ^* induces an algebra homomorphism from $A^!$ to $M_{2\times 2}(A^!)$. We still use the same notation σ^* for this algebra homomorphism if no confusion occurs. The following property is easy to check.

Lemma 2.4. *Let A be a Koszul algebra and $\sigma : A \to M_{2\times 2}(A)$ an algebra homomorphism. Then σ is invertible (in the sense of Definition 1.7) with inverse ϕ if and only if σ^* is invertible with inverse ϕ^*. Here both σ^* and ϕ^* are algebra homomorphisms from $A^!$ to $M_{2\times 2}(A^!)$.*

Let x_0 be a base element of the highest nonzero component $A^!_d$, which is 1-dimensional $\Bbbk$-space, of $A^!$. We assume that:

$$\sigma^*(x_0) = \begin{pmatrix} Wx_0 & Xx_0 \\ Yx_0 & Zx_0 \end{pmatrix}, \quad \phi^*(x_0) = \begin{pmatrix} W'x_0 & X'x_0 \\ Y'x_0 & Z'x_0 \end{pmatrix} \tag{2.5}$$

for some $W, X, Y, Z, W', X', Y', Z' \in \Bbbk$.

Inspired by Proposition 2.3, we may introduce the following:

Definition 2.5. Let A be a Koszul AS-regular algebra. Suppose that σ is an algebra homomorphism from A to $M_{2\times 2}(A)$ and σ^* is its dual algebra homomorphism from $A^!$ to $M_{2\times 2}(A^!)$. The homological determinant of σ, denoted $\operatorname{hdet}\sigma$, is defined by

$$\operatorname{hdet}\sigma := \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

where W, X, Y and Z are determined by (2.5).

The following property follows directly from Lemma 2.4.

Lemma 2.6. *Let A be a Koszul AS-regular algebra. Suppose that σ and ϕ are two algebra homomorphism from A to $M_{2\times 2}(A)$ such that they are inverse of each other in the sense of Definition 1.7. Then*

$$\operatorname{hdet}\sigma\,(\operatorname{hdet}\phi)^t = I_2,$$

or equivalently,

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\begin{pmatrix} W' & Y' \\ X' & Z' \end{pmatrix} = I_2,$$

where M^t is the transpose of a matrix M and I_2 is the 2×2 identity matrix. □

Example 2.7. Let A be a Koszul AS-regular algebra and $B = A_P[y_1, y_2; \sigma]$ be a trimmed double Ore extension of A with $\sigma = \begin{pmatrix} \tau & 0 \\ 0 & \xi \end{pmatrix}$. Then, both τ and ξ are automorphisms of A and $\tau\xi = \xi\tau$ (see Proposition 4.4 for its proof). Moreover, B is an iterated Ore extension of A by [3, Theorem 2.2]. It is easy to see that

$$\operatorname{hdet}\sigma = \begin{pmatrix} \operatorname{hdet}\tau & 0 \\ 0 & \operatorname{hdet}\xi \end{pmatrix}.$$

3. Nakayama automorphisms

In this section, we study the Yoneda Ext algebra of a trimmed double Ore extension of a Koszul AS-regular algebra, and compute the Nakayama automorphism of the trimmed double Ore extension. This leads to the characterization of the Calabi–Yau property of a trimmed double Ore extension. As consequences, we recover several known results on the Calabi–Yau property of a skew polynomial extension.

Throughout this section, $A = T_{\Bbbk}(V)/\langle R\rangle$ is a Koszul AS-regular algebra of global dimension d with Nakayama automorphism ν, and $B = A_P[y_1, y_2; \sigma]$ is a trimmed double Ore extension of A, where $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ is an algebra morphism subject to (1.5). Let $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ be the inverse of σ in the sense of (1.7), $\operatorname{hdet}\sigma = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$, and $\operatorname{hdet}\phi = \begin{pmatrix} W' & X' \\ Y' & Z' \end{pmatrix}$ throughout this section. We choose a basis $\{e_1, \dots, e_n\}$ of V, and let $\{e_1^*, \dots, e_n^*\}$ be the corresponding dual basis of V^*. For the the Frobenius algebra $A^!$, we fix a base element x_0 of the 1-dimensional $\Bbbk$-space $A^!_d$. By [20, Lemma 3.2], $A^!$ possesses a nondegenerate bilinear form given by

$$\langle a, b\rangle = c_{ab} \tag{3.1}$$

where c_{ab} is the coefficient of x_0 in the product of ab. We can pick a $\Bbbk$-linear basis $\{\eta_1, \eta_2, \dots, \eta_n\}$ of $A^!_{d-1}$ such that $e_i^*\eta_j = \delta_{ij}x_0$. Then $\eta_i e_j^* = \lambda_{ij}x_0$ for some $\lambda_{ij} \in \Bbbk$. Or equivalently,

$$\langle e_i^*, \eta_j\rangle = \delta_{ij}, \quad \langle \eta_i, e_j^*\rangle = \lambda_{ij}$$

for $i, j = 1, 2, \ldots, n$. Then, it follows from (1.2) that the Nakayama automorphism $\mu_{A^!}$ of $A^!$ is given by:

$$\mu_{A^!}(e_i^*) = \sum_j \lambda_{ji} e_j^*. \tag{3.2}$$

Now we assume that the algebra homomorphism $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \to M_{2\times 2}(A)$ is given by

$$\phi_{ij}(e_l) = \sum_k \phi_{ij}^{lk} e_k \tag{3.3}$$

for each l, where $\phi_{ij}^{lk} \in \Bbbk$. Then, we have

$$\phi_{ij}^*(e_l^*) = \sum_k \phi_{ij}^{kl} e_k^*. \tag{3.4}$$

Now B is a Koszul algebra and it can be presented by generators and relations as $B = T_{\Bbbk}(V \oplus \Bbbk y_1 \oplus \Bbbk y_2)/\langle R_B \rangle$, where R_B consists of three types of relations:

(R1) the relations defining A;
(R2) $y_2 y_1 - p y_1 y_2 - q y_1^2$;
(R3) $\{y_j e_i - \sigma_{j1}(e_i) y_1 - \sigma_{j2}(e_i) y_2;\ j = 1, 2, i = 1, \ldots, n\}$.

Note that from Definition 1.5 and Definition 1.7 it follows that the relation (R3) is equivalent to

(R3') $\{e_i y_j - y_1 \phi_{1j}(e_i) - y_2 \phi_{2j}(e_i);\ j = 1, 2, i = 1, \ldots, n\}$.

Let $C := \Bbbk\langle y_1, y_2\rangle / \langle y_2 y_1 - p y_1 y_2 - q y_1^2 \rangle (p \neq 0)$. We need the following well-known property of the algebra C.

Proposition 3.1. *The algebra C is Koszul AS-regular of dimension 2. Its Yoneda Ext algebra $C^!$ is $\Bbbk\langle y_1^*, y_2^*\rangle / \langle (y_1^*)^2 + q y_2^* y_1^*, y_1^* y_2^* + p y_2^* y_1^*, (y_2^*)^2 \rangle$.*

Proof. The algebra is known as the Jordan plane ($q \neq 0$) or quantum plane ($q = 0$) which are both Koszul AS-regular of dimension 2. Its Yoneda Ext algebra $E(C) := \bigoplus_{i \in \mathbb{N}} \operatorname{Ext}_C^i(\Bbbk, \Bbbk)$ is isomorphic to $C^! = T_{\Bbbk}(V^*)/\langle R^{\perp} \rangle$, see [20, Theorem 5.9]. □

Next, we can describe the algebra $B^!$ in terms of generators and relations. It is obvious that $\{e_1^*, e_2^*, \ldots, e_n^*, y_1^*, y_2^*\}$ forms a $\Bbbk$-linear basis of $B_1^!$.

Lemma 3.2. *The algebra $B^!$ is generated by elements $\{e_1^*, e_2^*, \ldots, e_n^*, y_1^*, y_2^*\}$ with the relations:*

($\perp 1$) *the relations for $A^!$;*
($\perp 2$) *the relations for $C^!$;*
($\perp 3$) $\{y_j^* e_i^* + \phi_{j1}^*(e_i^*) y_1^* + \phi_{j2}^*(e_i^*) y_2^*;\ j = 1, 2, i = 1, \ldots, n\}$, *where ϕ is the inverse of σ.*

Proof. Since B is Koszul, we have $B^! = T_{\Bbbk}(V^* \oplus \Bbbk y_1^* \oplus \Bbbk y_2^*)/\langle (R_B)^{\perp}\rangle$. According to the defining relations of B, it is easy to see that relations ($\perp$1) and ($\perp$2) belong to $(R_B)^{\perp}$. Now we show that ($\perp$3) also belongs to $(R_B)^{\perp}$. It suffices to verify that for every i, j, we have:

$$\big(y_j^* e_i^* + \phi_{j1}^*(e_i^*) y_1^* + \phi_{j2}^*(e_i^*) y_2^*\big)(r) = 0$$

for each $r \in R_B$ by the definition of $(R_B)^{\perp}$ for each i, j. But this is trivial since the generating relations of B are given by (R1), (R2) and (R3).

On the other hand, each element in $(V^*)^{\otimes 2}$ has the form $f + g + h$, where $f = \sum_i k_i e_i^* y_1^* + l_i e_i^* y_2^* + m_i y_1^* e_i^* + n_i y_2^* e_i^*$, $g = \sum c_{ij} e_i^* e_j^*$ and $h = a(y_1^*)^2 + b y_1^* y_2^* + c y_2^* y_1^* + d(y_2^*)^2$. Assume that $f + g + h \in (R_B)^{\perp}$. Then, it is easy to see g is in the span of ($\perp$1) and h is in the span of ($\perp$2). For the rest, we need to show that every element $f = \sum_i k_i e_i^* y_1^* + l_i e_i^* y_2^* + m_i y_1^* e_i^* + n_i y_2^* e_i^* \in (R_B)^{\perp}$ can be written as

$$f = \sum a_i(y_1^* e_i^* + \phi_{11}^*(e_i^*) y_1^* + \phi_{12}^*(e_i^*) y_2^*) + b_i(y_2^* e_i^* + \phi_{21}^*(e_i^*) y_1^* + \phi_{22}^*(e_i^*) y_2^*),$$

for $a_i, b_i \in \Bbbk$. Firstly, we have

$$k_i = \sum_j m_j e_j^*(\phi_{11}(e_i)) + n_j e_j^*(\phi_{21}(e_i))$$

and

$$l_i = \sum_j m_j e_j^*(\phi_{12}(e_i)) + n_j e_j^*(\phi_{22}(e_i))$$

for any i. Further,

$$\sum_i e_j^*(\phi_{11}(e_i)) e_i^* = \phi_{11}^*(e_j^*)$$

by the definition of ϕ_{11}^*. Hence, we have

$$\begin{aligned} f &= \sum_j m_j \phi_{11}^*(e_j^*) y_1^* + n_j \phi_{21}^*(e_j^*) y_1^* \\ &\quad + \sum_j m_j \phi_{12}^*(e_j^*) y_2^* + n_j \phi_{22}^*(e_j^*) y_2^* \\ &\quad + \sum_i m_i y_1^* e_i^* + n_i y_2^* e_i^* \\ &= \sum_i m_i (y_1^* e_i^* + \phi_{11}^*(e_i^*) y_1^* + \phi_{12}^*(e_i^*) y_2^*) \\ &\quad + n_i (y_2^* e_i^* + \phi_{21}^*(e_i^*) y_1^* + \phi_{22}^*(e_i^*) y_2^*), \end{aligned}$$

which completes the proof. □

Remark 3.3. The third type of relation ($\perp$3) of $B^!$ can be replaced by

$(\perp 3')$ $\{e_i^* y_j^* + y_1^* \sigma_{1j}^*(e_i^*) + y_2^* \sigma_{2j}^*(e_i^*);\ j=1,2, i=1,\ldots,n\}$
since the relation R3 can be replaced by R3$'$.

Proposition 3.4. *Suppose that A is a Koszul algebra and $B = A_P[y_1, y_2; \sigma]$ is a trimmed double Ore extension of A. Then,*
(1) *$A^!$ is a subalgebra of $B^!$;*
(2) *$B^!$ is a free right (and left) $A^!$-module with a basis $\{1, y_1^*, y_2^*, y_1^* y_2^*\}$.*

Proof. The statement (1) is a consequence of Lemma 3.2. Moreover, there is a surjective algebra homomorphism $\pi : A^! \coprod C^! \to B^!$ from the coproduct of $A^!$ and $C^!$ to $B^!$. Hence, as a left $A^!$-module, $B^!$ is generated by $1, y_1^*, y_2^*$ and $y_1^* y_2^*$. By Lemma 3.2 and Remark 3.3, the kernel of π is the ideal generated by

$$\{e_i^* y_j^* + y_1^* \sigma_{1j}^*(e_i^*) + y_2^* \sigma_{2j}^*(e_i^*);\ j=1,2, i=1,\ldots,n\}.$$

Therefore, the elements $1, y_1^*, y_2^*$ and $y_1^* y_2^*$ are also the generators of $B^!$ as a right $A^!$-module.

Next, since B is a free left A-module with basis $\{y_1^i y_2^j;\ i, j \geq 0\}$ by definition, the Hilbert series of B is equal to the Hilbert series of $A \otimes \Bbbk[y_1, y_2]$, i.e.,

$$\mathrm{H}_B(t) = \frac{\mathrm{H}_A(t)}{(1-t)^2}.$$

It is well known that there is a functional equation on Hilbert series

$$\mathrm{H}_S(t)\,\mathrm{H}_{S^!}(-t) = 1$$

for any Koszul algebra S. Since both A and B are Koszul algebras by Theorem 2.1, so we have

$$\mathrm{H}_{B^!}(t) = (1+t)^2\,\mathrm{H}_{A^!}(t). \tag{3.5}$$

Therefore, $B^!$ is a free left(also right) $A^!$-module, with a basis $\{1, y_1^*, y_2^*, y_1^* y_2^*\}$. $\square$

In order to compute the Nakayama automorphism of $B^!$, we need the following:

Lemma 3.5. *With notations and assumptions as in the second paragraph of this section, we have*
(1) *$\varepsilon := x_0 y_1^* y_2^*$ is a basis element of the 1-dimensional space $B^!_{d+2}$.*
(2) *For any $1 \leq i, j \leq n$ and $m = 1, 2$, the following equations hold:*

$$\begin{cases} e_i^* \eta_j y_1^* y_2^* \overset{(a)}{=} \delta_{ij}\varepsilon, & \eta_i y_1^* y_2^* e_j^* \overset{(b)}{=} \sum\limits_{k,l} (\frac{q}{p}\phi_{21}^{kj}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{kj}\phi_{12}^{lk} + \phi_{22}^{kj}\phi_{11}^{lk})\lambda_{il}\varepsilon, \\ e_i^* x_0 y_m^* \overset{(c)}{=} 0, & x_0 y_m^* e_i^* \overset{(d)}{=} 0, \\ y_1^* x_0 y_2^* \overset{(e_1)}{=} (-1)^d W'\varepsilon, & y_1^* x_0 y_1^* \overset{(e_2)}{=} (-1)^d (\frac{q}{p} W' - \frac{1}{p} X')\varepsilon, \\ y_2^* x_0 y_2^* \overset{(e_3)}{=} (-1)^d Y'\varepsilon, & y_2^* x_0 y_1^* \overset{(e_4)}{=} (-1)^d (\frac{q}{p} Y' - \frac{1}{p} Z')\varepsilon, \\ x_0 y_1^* y_1^* \overset{(f_1)}{=} \frac{q}{p}\varepsilon, & x_0 y_1^* y_2^* \overset{(f_2)}{=} \varepsilon, \\ x_0 y_2^* y_1^* \overset{(f_3)}{=} -\frac{1}{p}\varepsilon, & x_0 y_2^* y_2^* \overset{(f_4)}{=} 0, \\ y_m^* \eta_j y_1^* y_2^* \overset{(g)}{=} 0, & \eta_j y_1^* y_2^* y_m^* \overset{(h)}{=} 0, \end{cases}$$

where W', X', Y' and Z' are given in (2.5).

Proof. Part (1) is obvious by Proposition 3.4 (2). Since $e_i^*\eta_j = \delta_{ij}x_0$, we have

$$e_i^*\eta_j y_1^* y_2^* = \delta_{ij}x_0 y_1^* y_2^* = \delta_{ij}\varepsilon.$$

So Equation (a) holds. Since $A^! \to B^!$ is injective, Equation (c) holds naturally. Equation (d) holds due to the relation ($\perp$3) of Lemma 3.2. Equations (g) and (h) follow from the relations ($\perp$2) and ($\perp$3) of Lemma 3.2. As for Equation (b), by relation ($\perp$3) of Lemma 3.2 and Proposition 3.1, we have:

$$\begin{aligned}
y_1^* y_2^* e_j^* &= -y_1^*(\phi_{21}^*(e_j^*)y_1^* + \phi_{22}^*(e_j^*)y_2^*)\\
&= -\sum_k (\phi_{21}^{kj} y_1^* e_k^* y_1^* + \phi_{22}^{kj} y_1^* e_k^* y_2^*)\\
&= \sum_k (\phi_{21}^{kj}\phi_{11}^*(e_k^*)y_1^* y_1^* + \phi_{21}^{kj}\phi_{12}^*(e_k^*)y_2^* y_1^* + \phi_{22}^{kj}\phi_{11}^*(e_k^*)y_1^* y_2^*)\\
&= \sum_{k,l} (\phi_{21}^{kj}\phi_{11}^{lk} e_l^* y_1^* y_1^* + \phi_{21}^{kj}\phi_{12}^{lk} e_l^* y_2^* y_1^* + \phi_{22}^{kj}\phi_{11}^{lk} e_l^* y_1^* y_2^*)\\
&= \sum_{k,l} \left(\frac{q}{p}\phi_{21}^{kj}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{kj}\phi_{12}^{lk} + \phi_{22}^{kj}\phi_{11}^{lk}\right) e_l^* y_1^* y_2^*.
\end{aligned}$$

Thus, for each i

$$\begin{aligned}
\eta_i y_1^* y_2^* e_j^* &= \sum_{k,l} \left(\frac{q}{p}\phi_{21}^{kj}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{kj}\phi_{12}^{lk} + \phi_{22}^{kj}\phi_{11}^{lk}\right) \eta_i e_l^* y_1^* y_2^*\\
&= \sum_{k,l} \left(\frac{q}{p}\phi_{21}^{kj}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{kj}\phi_{12}^{lk} + \phi_{22}^{kj}\phi_{11}^{lk}\right) \lambda_{il}\varepsilon,
\end{aligned}$$

where the second equation follows from $\eta_i e_l^* = \lambda_{il}x_0$ by the assumption. Next, we show the rest equations. For a fixed j, suppose that $\eta_j = \sum_m \lambda_m e_{m_1}^* e_{m_2}^* \dots e_{m_{d-1}}^*$, where $\lambda_m \in \Bbbk$. Then,

$$\begin{aligned}
\begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} x_0 &= \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_j^*\eta_j\\
&= \sum_m \lambda_m \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_j^* e_{m_1}^* e_{m_2}^* \dots e_{m_{d-1}}^*\\
&= -\sum_m \lambda_m \phi^*(e_j^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_{m_1}^* e_{m_2}^* \dots e_{m_{d-1}}^*\\
&= (-1)^2 \sum_m \lambda_m \phi^*(e_j^*)\phi^*(e_{m_1}^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_{m_2}^* \dots e_{m_{d-1}}^*\\
&= \cdots\\
&= (-1)^d \sum_m \lambda_m \phi^*(e_j^*)\phi^*(e_{m_1}^*)\phi^*(e_{m_2}^*) \dots \phi^*(e_{m_{d-1}}^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix}\\
&= (-1)^d \sum_m \lambda_m \phi^*(e_j^* e_{m_1}^* e_{m_2}^* \dots e_{m_{d-1}}^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix}
\end{aligned}$$

$$= (-1)^d \phi^*(e_j^* \sum_m \lambda_m e_{m_1}^* e_{m_2}^* \dots e_{m_{d-1}}^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix}$$
$$= (-1)^d \phi^*(e_j^* \eta_j) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = (-1)^d \phi^*(x_0) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix}.$$

It follows from the definition of ϕ^* that we obtain:

$$\begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} x_0 = (-1)^d \begin{pmatrix} \phi_{11}^*(x_0) & \phi_{12}^*(x_0) \\ \phi_{21}^*(x_0) & \phi_{22}^*(x_0) \end{pmatrix} \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = (-1)^d \begin{pmatrix} W' x_0 y_1^* + X' x_0 y_2^* \\ Y' x_0 y_1^* + Z' x_0 y_2^* \end{pmatrix}.$$

Thus, we have proved Equations (e_i), $i = 1, 2, 3, 4$. Finally, the equations (f_i), $i = 1, \dots, 4$ and (g), (h) follow from Proposition 3.1. □

Since $B^!$ is Frobenius, we may apply the Frobenius pair (3.1) on the equations in Lemma 3.5(2).

Corollary 3.6. *The following equations hold:*

$$\begin{cases} \langle e_i^*, \eta_j y_1^* y_2^* \rangle \overset{(a')}{=} \delta_{ij}, & \langle \eta_i y_1^* y_2^*, e_j^* \rangle \overset{(b')}{=} \sum_{k,l} \left(\frac{q}{p} \phi_{21}^{kj} \phi_{11}^{lk} - \frac{1}{p} \phi_{21}^{kj} \phi_{12}^{lk} + \phi_{22}^{kj} \phi_{11}^{lk} \right) \lambda_{il}, \\ \langle e_i^*, x_0 y_m^* \rangle \overset{(c')}{=} 0, & \langle x_0 y_m^*, e_i^* \rangle \overset{(d')}{=} 0, \\ \langle y_1^*, x_0 y_2^* \rangle \overset{(e_1')}{=} (-1)^d W', & \langle y_1^*, x_0 y_1^* \rangle \overset{(e_2')}{=} (-1)^d \left(\frac{q}{p} W' - \frac{1}{p} X' \right), \\ \langle y_2^*, x_0 y_2^* \rangle \overset{(e_3')}{=} (-1)^d Y', & \langle y_2^*, x_0 y_1^* \rangle \overset{(e_4')}{=} (-1)^d \left(\frac{q}{p} Y' - \frac{1}{p} Z' \right), \\ \langle x_0 y_1^*, y_1^* \rangle \overset{(f_1')}{=} \frac{q}{p}, & \langle x_0 y_1^*, y_2^* \rangle \overset{(f_2')}{=} 1, \\ \langle x_0 y_2^*, y_1^* \rangle \overset{(f_3')}{=} -\frac{1}{p}, & \langle x_0 y_2^*, y_2^* \rangle \overset{(f_4')}{=} 0, \\ \langle y_m^*, \eta_j y_1^* y_2^* \rangle \overset{(g')}{=} 0, & \langle \eta_j y_1^* y_2^*, y_m^* \rangle \overset{(h')}{=} 0. \end{cases}$$

Corollary 3.7. *The vector set $\{\eta_1 y_1^* y_2^*, \eta_2 y_1^* y_2^*, \dots, \eta_n y_1^* y_2^*, x_0 y_1^*, x_0 y_2^*\}$ forms a $\Bbbk$-linear basis of $B_{d+1}^!$.*

Proof. Suppose that:

$$a_1 \eta_1 y_1^* y_2^* + \cdots + a_n \eta_n y_1^* y_2^* + b_1 x_0 y_1^* + b_2 x_0 y_2^* = 0$$

for some coefficients $a_1, \dots, a_n, b_1, b_2 \in \Bbbk$. For each $i = 1, 2, \dots, n$, we have

$$\begin{aligned} 0 &= \langle e_i^*, a_1 \eta_1 y_1^* y_2^* + \cdots + a_n \eta_n y_1^* y_2^* + b_1 x_0 y_1^* + b_2 x_0 y_2^* \rangle \\ &= \sum_{j=1}^n a_j \langle e_i^*, \eta_j y_1^* y_2^* \rangle + b_1 \langle e_i^*, x_0 y_1^* \rangle + b_2 \langle e_i^*, x_0 y_2^* \rangle \\ &= a_i. \qquad \text{(by Equations } (a') \text{ and } (c')) \end{aligned}$$

Similarly, we have:

$$\begin{aligned}0 &= \langle y_1^*, a_1\eta_1 y_1^* y_2^* + \cdots + a_n\eta_n y_1^* y_2^* + b_1 x_0 y_1^* + b_2 x_0 y_2^*\rangle \\ &= \sum_{j=1}^n a_j \langle y_1^*, \eta_j y_1^* y_2^*\rangle + b_1\langle y_1^*, x_0 y_1^*\rangle + b_2\langle y_1^*, x_0 y_2^*\rangle \\ &= b_1(-1)^d\left(\frac{q}{p}W' - \frac{1}{p}X'\right) + b_2(-1)^d W', \quad \text{(by Equations } (g'), (e_1') \text{ and } (e_2')).\end{aligned}$$

and

$$b_1(-1)^d\left(\frac{q}{p}Y' - \frac{1}{p}Z'\right) + b_2(-1)^d Y' = 0$$

obtained in a similar way. So we obtain a system of linear equations:

$$\begin{cases}\left(\frac{q}{p}W' - \frac{1}{p}X'\right)b_1 + W'b_2 = 0,\\ \left(\frac{q}{p}Y' - \frac{1}{p}Z'\right)b_1 + Y'b_2 \quad = 0.\end{cases}$$

The determinant of the matrix $\begin{pmatrix}\frac{q}{p}W' - \frac{1}{p}X' & W'\\ \frac{q}{p}Y' - \frac{1}{p}Z' & Y'\end{pmatrix}$ is nonzero by Lemma 2.6. Hence, $b_1 = b_2 = 0$. Thus, the vectors $\eta_1 y_1^* y_2^*, \eta_2 y_1^* y_2^*, \ldots, \eta_n y_1^* y_2^*, x_0 y_1^*$ and $x_0 y_2^*$ are linear independent. On the other hand, by Eq. (3.5), we have $\dim B^!_{d+1} = 2\dim A^!_d + \dim A^!_{d-1} = n+2$. That is, these vectors form a $\Bbbk$-linear basis of $B^!_{d+1}$. □

Now, we are ready to compute the Nakayama automorphism μ of the Frobenius algebra $B^!$. This automorphism is determined by the equation

$$\langle a, b\rangle = \langle \mu(b), a\rangle$$

for any $a, b \in B^!$ (see (1.2)). Note that $B^!$ is generated by the degree 1 elements: $e_1^*, e_2^*, \ldots, e_n^*, y_1^*, y_2^*$. Hence, we just need to describe the images of those elements under the Nakayama automorphism. By Corollary 3.7, we see that $\{\eta_1 y_1^* y_2^*, \eta_2 y_1^* y_2^*, \ldots, \eta_n y_1^* y_2^*, x_0 y_1^*, x_0 y_2^*\}$ forms a basis of $B^!_{d+1}$. Due to the fact that the Nakayama automorphism is graded, we can use the equations in Corollary 3.6 to determine the Nakayama automorphism.

Proposition 3.8. *The restriction of the Nakayama automorphism $\mu_{B^!}$ of $B^!$ to $A^!$ equals $\mu_{A^!}(\det_l \phi)^*$.*

Proof. Suppose that

$$\mu_{B^!}(e_i^*) = k_{i,1}e_1^* + \cdots + k_{i,n}e_n^* + k_{i,n+1}y_1^* + k_{i,n+2}y_2^*.$$

Since $\langle -, -\rangle$ is a Frobenius pair,

$$\langle x_0 y_m^*, e_i^*\rangle = \langle \mu_{B^!}(e_i^*), x_0 y_m^*\rangle$$

for $m = 1, 2$. From Equations (d') and (c') in Corollary 3.6, we obtain:

$$\begin{aligned}
0 &= \langle \mu_{B^!}(e_i^*), x_0 y_m^* \rangle \\
&= \langle k_{i,1}e_1^* + \cdots + k_{i,n}e_n^* + k_{i,n+1}y_1^* + k_{i,n+2}y_2^*, x_0 y_m^* \rangle \\
&= \langle k_{i,n+1}y_1^* + k_{i,n+2}y_2^*, x_0 y_m^* \rangle \\
&= k_{i,n+1}\langle y_1^*, x_0 y_m^* \rangle + k_{i,n+2}\langle y_2^*, x_0 y_m^* \rangle
\end{aligned}$$

From Equations (e_1')-(e_4') in Corollary 3.6 , we obtain the following system of linear equations:

$$\begin{cases} \left(\frac{q}{p}W' - \frac{1}{p}X'\right) k_{i,n+1} + \left(\frac{q}{p}Y' - \frac{1}{p}Z'\right) k_{i,n+2} = 0, \\ W'k_{i,n+1} + Y'k_{i,n+2} = 0, \end{cases}$$

Since the determinant of the matrix $\begin{pmatrix} \frac{q}{p}W' - \frac{1}{p}X' & \frac{q}{p}Y' - \frac{1}{p}Z' \\ W' & Y' \end{pmatrix}$ is nonzero by Lemma 2.6, we have:

$$k_{i,n+1} = 0 = k_{i,n+2}$$

for each i. Following the definition of the Nakayama automorphism (see (1.2)) and Equations (a') and (b'), we arrive at:

$$\mu_{B^!}(e_i^*) = \sum_j \left(\sum_{k,l} \left(\frac{q}{p}\phi_{21}^{ki}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{ki}\phi_{12}^{lk} + \phi_{22}^{ki}\phi_{11}^{lk} \right) \lambda_{il} \right) e_j^*.$$

On the other hand, we claim that

$$(\det_l \phi)^*(e_i^*) = \sum_{k,l} \left(\frac{q}{p}\phi_{21}^{ki}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{ki}\phi_{12}^{lk} + \phi_{22}^{ki}\phi_{11}^{lk} \right) e_l^*.$$

Since for any e_m,

$$\begin{aligned}
(\det_l \phi)^*(e_i^*)(e_m) &= e_i^*(\det_l \phi(e_m)) \\
&= e_i^* \left(\frac{q}{p}\phi_{21} \circ \phi_{11}(e_m) + \phi_{22} \circ \phi_{11}(e_m) - \frac{1}{p}\phi_{21} \circ \phi_{12}(e_m) \right) \\
&= e_i^* \left(\frac{q}{p}\phi_{21} \left(\sum_k \phi_{11}^{mk} e_k \right) + \phi_{22} \left(\sum_k \phi_{11}^{mk} e_k \right) - \frac{1}{p}\phi_{21} \left(\sum_k \phi_{12}^{mk} e_k \right) \right) \\
&= e_i^* \left(\frac{q}{p} \sum_{k,l} \phi_{11}^{mk}\phi_{21}^{kl} e_l + \sum_{k,l} \phi_{11}^{mk}\phi_{22}^{kl} e_l - \frac{1}{p} \sum_{k,l} \phi_{12}^{mk}\phi_{21}^{kl} e_l \right) \\
&= \frac{q}{p} \sum_k \phi_{11}^{mk}\phi_{21}^{ki} + \sum_k \phi_{11}^{mk}\phi_{22}^{ki} - \frac{1}{p} \sum_k \phi_{12}^{mk}\phi_{21}^{ki},
\end{aligned}$$

which coincides the value of $\sum_{k,l} \left(\frac{q}{p}\phi_{21}^{ki}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{ki}\phi_{12}^{lk} + \phi_{22}^{ki}\phi_{11}^{lk} \right) e_l^*(e_m)$.

It follows that

$$\begin{aligned}\mu_{A^!}(\det_l \phi)^*(e_i^*) &= \mu_{A^!}\left(\sum_{k,l}\left(\frac{q}{p}\phi_{21}^{ki}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{ki}\phi_{12}^{lk} + \phi_{22}^{ki}\phi_{11}^{lk}\right)e_l^*\right)\\ &= \sum_{k,l}\left(\frac{q}{p}\phi_{21}^{ki}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{ki}\phi_{12}^{lk} + \phi_{22}^{ki}\phi_{11}^{lk}\right)\mu_{A^!}(e_l^*)\\ &= \sum_{k,l}\left(\frac{q}{p}\phi_{21}^{ki}\phi_{11}^{lk} - \frac{1}{p}\phi_{21}^{ki}\phi_{12}^{lk} + \phi_{22}^{ki}\phi_{11}^{lk}\right)\sum_j \lambda_{jl} e_j^*.\end{aligned}$$

That is, $\mu_{B^!}(e_i^*) = \mu_{A^!}(\det_l \phi)^*(e_i^*)$, for all i. □

We need the following technical result although the proof is obvious.

Lemma 3.9. *Let $E = \Bbbk \oplus E_1 \oplus \cdots \oplus E_m$ be a graded Frobenius algebra which is generated in degree 1. Suppose that $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ are $\Bbbk$-linear bases of E_1 and E_{m-1} respectively. Let*

$$\begin{cases} \langle \alpha_1, \beta_1\rangle = a, & \langle \beta_1, \alpha_1\rangle = e,\\ \langle \alpha_1, \beta_2\rangle = b, & \langle \beta_2, \alpha_1\rangle = f,\\ \langle \alpha_2, \beta_1\rangle = c, & \langle \beta_1, \alpha_2\rangle = g,\\ \langle \alpha_2, \beta_2\rangle = d, & \langle \beta_2, \alpha_2\rangle = h.\end{cases}$$

Then, the Nakayama automorphism of E is given by:

$$\begin{aligned}\mu(\alpha_1) &= \frac{de-cf}{ad-bc}\alpha_1 + \frac{af-be}{ad-bc}\alpha_2,\\ \mu(\alpha_2) &= \frac{dg-ch}{ad-bc}\alpha_1 + \frac{ah-bg}{ad-bc}\alpha_2.\end{aligned}$$

Proof. Note that the Frobenius pair $\langle -, -\rangle$ is a nondegenerate bilinear form. It follows that $ad - bc \neq 0$. Since the Nakayama automorphism is graded and E is generated in degree 1, the Nakayama automorphism is determined by the assumed equations. we are only to determine the image of elements of degree 1. The conclusion follows from a direct computation. □

Proposition 3.10. *The image of y_1^* and y_2^* under Nakayama automorphism $\mu_{B^!}$ are given as follows:*

$$\begin{aligned}\mu_{B^!}(y_1^*) &= (-1)^{d+1}\big((qX + \frac{q}{p}X + \frac{1}{p}W)y_1^* + (qZ + \frac{q}{p}Z + \frac{1}{p}Y)y_2^*\big),\\ \mu_{B^!}(y_2^*) &= (-1)^{d+1}(pXy_1^* + pZy_2^*).\end{aligned}$$

where W, X, Y and Z form the homological determinant of σ.

Proof. The proof is similar to the one of Proposition 3.8. Suppose that

$$\mu_{B^!}(y_1^*) = k_1 e_1^* + \cdots + k_n e_n^* + k_{n+1} y_1^* + k_{n+2} y_2^*.$$

Since the equation $\langle \eta_j y_1^* y_2^*, y_1^* \rangle = \langle \mu_{B^!}(y_1^*), \eta_j y_1^* y_2^* \rangle$, where $j = 1, 2, \ldots, n$, we have:

$$
\begin{aligned}
0 &= \langle \eta_j y_1^* y_2^*, y_1^* \rangle \\
&= \langle \mu_{B^!}(y_1^*), \eta_j y_1^* y_2^* \rangle \\
&= \sum_{i=1}^{n} k_i \langle e_i, \eta_j y_1^* y_2^* \rangle + k_{n+1} \langle y_1^*, \eta_j y_1^* y_2^* \rangle + k_{n+2} \langle y_2^*, \eta_j y_1^* y_2^* \rangle \\
&= \sum_{i=1}^{n} k_i \delta_{ij} = k_j.
\end{aligned}
$$

It follows that $\mu_{B^!}(y_1^*) = k_{n+1} y_1^* + k_{n+2} y_2^*$. Similarly, $\mu_{B^!}(y_2^*) = l_{n+1} y_1^* + l_{n+2} y_2^*$ for some $l_{n+1}, l_{n+2} \in \Bbbk$. Hence, both $\mu_{B^!}(y_1^*)$ and $\mu_{B^!}(y_2^*)$ are completely determined by the values in Equations (e_1')-(e_4') and (f_1')-(f_4') in Corollary 3.6. Thus, we arrive at the case of Lemma 3.9. It follows that

$$
\mu_{B^!}(y_1^*) = (-1)^d \left(\frac{qY' + \frac{q}{p} Y' - \frac{1}{p} Z'}{W'Z' - X'Y'} y_1^* + \frac{-qW' - \frac{q}{p} W' + \frac{1}{p} X'}{W'Z' - X'Y'} y_2^* \right),
$$

$$
\mu_{B^!}(y_2^*) = (-1)^d \left(\frac{pY'}{W'Z' - X'Y'} y_1^* + \frac{-pW'}{W'Z' - X'Y'} y_2^* \right).
$$

Finally, the statement follows from the equation:

$$
\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \frac{1}{W'Z' - X'Y'} \begin{pmatrix} Z' & -Y' \\ -X' & W' \end{pmatrix},
$$

a consequence of Lemma 2.6. □

Proposition 3.11. *The restriction of the Nakayama automorphism ν_B of B to A equals* $(\det_r \sigma)^{-1} \nu$, *and*

$$
\nu_B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (\operatorname{hdet} \sigma) \mathbb{P}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
$$

where $\mathbb{P}$ is given by (1.6).

Proof. By Proposition 3.8 and Proposition 3.10, the restriction of Nakayama automorphism $\mu_{B^!}$ to $B_1^!$ has the form $\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where $Q_1 \in M_d(\Bbbk)$ and $Q_2 \in M_2(\Bbbk)$. By Proposition 1.4 and Eq. 1.3, the Nakayama automorphism of B is also of this type. Combining Proposition 1.4, Proposition 3.8 and Proposition 1.9(3) we obtain the first statement. By Proposition 1.4 and Proposition 3.10, we have:

$$
\begin{aligned}
\nu_B(y_1) &= \left(qX + \frac{q}{p} X + \frac{1}{p} W \right) y_1 + pX y_2, \\
\nu_B(y_2) &= \left(qZ + \frac{q}{p} Z + \frac{1}{p} Y \right) y_1 + pZ y_2.
\end{aligned}
$$

Thus, the second conclusion follows from the definition of the homological determinant of σ in Definition 2.5 and Eq. 1.6. □

Now we are ready to characterize the Calabi–Yau property of a trimmed double Ore extension of a Koszul AS-regular algebra.

Theorem 3.12. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν. Let $B = A_P[y_1, y_2; \sigma]$ be a trimmed double Ore extension of A. Then B is Calabi–Yau if and only if* $\det_r \sigma = \nu$ *and* $\operatorname{hdet}\sigma = \mathbb{P}$.

Proof. Since B is Koszul and is of finite global dimension, the Koszul B^e-bimodule complex provides a finitely generated projective resolution of B of finite length. That is, B is homologically smooth. Because B is connected graded, its only inner automorphism is the identity. So for B to be Calabi–Yau, its Nakayama automorphism must be the identity. Therefore, the statement is a consequence of Proposition 3.11. □

Remark 3.13. For a Koszul AS-regular algebra A with Nakayama automorphism ν, there exists a unique skew polynomial extension D such that D is Calabi–Yau, see [8–10, 12, 19]. Here, we consider the existence and the uniqueness of a Calabi–Yau trimmed double Ore extension of a Koszul AS-regular algebra.

(1). For any Koszul AS-regular algebra A with Nakayama automorphism ν, consider the trimmed double Ore extension $B = A_P[y_1, y_2; \sigma]$ with $P = (1, 0)$ and $\sigma = \begin{pmatrix} \nu & 0 \\ 0 & id \end{pmatrix}$. Then B is Calabi–Yau. But it is easy to see that B is an iterated Ore extension of A (see [25, Proposition 3.6] or its proof). Hence, we ask if there exists a nontrivial double Ore extension B (not an iterated one) such that B is Calabi–Yau? The answer is negative from the following example.

Let $A = \Bbbk\langle x_1, x_2\rangle/(x_2x_1 - x_1x_2 - x_1^2)$ be the Jordan plane. Its Nakayama automorphism ν is given by $\nu(x_1) = x_1$ and $\nu(x_2) = 2x_1 + x_2$. Then, there is only one nontrivial double Ore extension by the classification in [25], namely, the type $\mathbb{H} := A_P[y_1, y_2; \sigma]$ with $P = (-1, 0)$ and σ given by the matrix $\begin{pmatrix} 0 & h & 0 & 0 \\ h & 0 & 0 & 0 \\ 0 & hf & 0 & h \\ hf & 0 & h & 0 \end{pmatrix}$ with $0 \neq h \in \Bbbk$ and $f \in \Bbbk$. Now, $\det_r(\sigma)$ is an automorphism given by $\det_r(\sigma)(x_1) = h^2x_1$ and $\det_r(\sigma)(x_2) = 2h^2 f x_1 + h^2x_2$. Let x_0 to be a base element of the 1-dimensional space $A_2^!$. Then

$$\sigma^*(x_0) = \begin{pmatrix} h^2x_0 & 0 \\ 0 & h^2x_0 \end{pmatrix}.$$

That is, $W = h^2$, $X = 0$, $Y = 0$ and $Z = h^2$. By Proposition 3.11, the Nakayama automorphism of $\mathbb{H}$ is

$$\begin{aligned} \nu : & x_1 \to h^{-2}x_1 \\ & x_2 \to h^{-2}((2 - 2f)x_1 + x_2) \\ & y_1 \to -h^2y_1 \\ & y_2 \to -h^2y_2. \end{aligned}$$

Therefore, there is no Calabi–Yau algebra in the class of the type $\mathbb{H}$.

(2). For the uniqueness, let $A = \Bbbk\langle x_1, x_2\rangle/(x_2x_1 + x_1x_2)$ be the quantum plane whose Nakayama automorphism is given by $\nu(x_1) = -x_1$ and $\nu(x_2) = -x_2$. Suppose that $B := A_P[y_1, y_2; \sigma]$ with $P = (-1, 0)$, where σ is given by the matrix $\begin{pmatrix} 0 & 0 & -g & f \\ 0 & 0 & f & -g \\ g & f & 0 & 0 \\ f & g & 0 & 0 \end{pmatrix}$ with $f, g \in \Bbbk$ and $f^2 \neq g^2$. So B is of type $\mathbb{N}$ in the classification of [25]. Now, $\det_r(\sigma)$ is an automorphism given by $\det_r(\sigma)(x_1) = (f^2 - g^2)x_1$ and $\det_r(\sigma)(x_2) = (f^2 - g^2)x_2$. Let x_0 be a base element of the 1-dimensional space $A^!_2$. Then we have:

$$\sigma^*(x_0) = \begin{pmatrix} (f^2 - g^2)x_0 & 0 \\ 0 & (f^2 - g^2)x_0 \end{pmatrix}.$$

In this case, $W = f^2 - g^2$, $X = 0$, $Y = 0$ and $Z = f^2 - g^2$. Thus, the Nakayama automorphism of B is equal to $(g^2 - f^2)\,\mathrm{id}$ by Proposition 3.11. Hence, B is Calabi–Yau if and only if $g^2 - f^2 = 1$. Therefore, a trimmed double Ore extension, which is Calabi–Yau, of a Koszul AS-regular algebra may not be unique if it exists.

Remark 3.14. In the first example in Remark 3.13, we know that $\det_r \sigma = \nu_A$ if and only if $h^2 = f = 1$. Moreover, $\mathrm{hdet}\,\sigma = \begin{pmatrix} h^2 & 0 \\ 0 & h^2 \end{pmatrix}$. But, $\mathbb{P} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore, the condition $\det_r \sigma = \nu_A$ and the condition $\mathrm{hdet}\,\sigma = \mathbb{P}$ in Theorem 3.12 are independent. More examples can be constructed from iterated Ore extensions, see Example 2.7.

To end this section, we return to discuss the Nakayama automorphism and the Calabi–Yau property of the skew polynomial extension. For a twisted Calabi–Yau algebra A with Nakayama automorphism ν, it was proved in [12, Theorem 3.3] that the Nakayama automorphism of an Ore extension $D = A[t; \theta, \delta]$ is given by

$$\nu_D(x) = \begin{cases} \theta^{-1} \circ \nu(x), & x \in A; \\ ax + b, & x = t, \end{cases}$$

for some $a, b \in A$ with a invertible. It was also remarked there that if $\delta = 0$, then $\nu_D(t) = at$. Now if we restrict to Koszul algebras, we can describe the Nakayama automorphism more explicitly as follows.

Proposition 3.15. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν, θ is a graded algebra automorphism of A and $D = A[t; \theta]$. The Nakayama automorphism ν_D of D is given by:*

$$\nu_D(x) = \begin{cases} \theta^{-1} \circ \nu(x), & x \in A \\ (\mathrm{hdet}\,\theta)x, & x = t. \end{cases}$$

Proof. We only give a sketch of the proof since it is similar to the one of Proposition 3.11. Suppose that $D = T(V \oplus \Bbbk t)/\langle R_D\rangle$. The generating relations in D are of two types: $te_i - \theta(e_i)t$ $(1 \leq i \leq n)$ and the relations from A. Obviously, $\{e_1^*, e_2^*, \ldots, e_n^*, t^*\}$ forms a $\Bbbk$-linear basis for $D^!_1$. By [15, Proposition 2.4], the defining relations for $D^!$ consist of the following three types:

(1) the relations from $A^!$;
(2) $\{t^*e_i^* + (\theta^{-1})^*(e_i^*)t^* \mid 1 \le i \le n\}$;
(3) $\{(t^*)^2\}$.

By [15, Proposition 2.5], $D^!$ is a free $A^!$-module with basis $\{1, t^*\}$. Hence, x_0t^* is a base element of the 1-dimensional space $D^!_{d+1}$, denoted ε, where x_0 is a base element of the 1-dimensional $\Bbbk$-space $A^!_d$. Now let $(b_{ij})_{n\times n}$ be the matrix of the restriction of θ^{-1} to V, i. e.,

$$\theta^{-1}(e_i) = \sum_j b_{ij}e_j \tag{3.6}$$

for each i. Then, we have

(1) $\{\eta_1t^*, \eta_2t^*, \dots, \eta_nt^*, x_0\}$ is a $\Bbbk$-linear basis of $D^!_d$;
(2) the following equations hold:

$$\begin{cases} e_i^*\eta_jt^* = \delta_{ij}\varepsilon, & \eta_it^*e_j^* = -\sum_k b_{kj}\lambda_{ik}\varepsilon, \\ e_i^*x_0 = 0, & x_0e_i^* = 0, \\ t^*x_0 = (-1)^d(\operatorname{hdet}(\theta))^{-1}\varepsilon, & x_0t^* = \varepsilon, \\ t^*\eta_jt^* = 0, & \eta_jt^*t^* = 0. \end{cases}$$

Using the same argument in the proof of Proposition 3.8 and Proposition 3.10, one obtains that the Nakayama automorphism $\mu_{D^!}$ of $D^!$ is given by:

$$\mu_{D^!}(\alpha) = \begin{cases} -\mu_{A^!}\circ(\theta^{-1})^*(\alpha), & \alpha \in A^! \\ (-1)^d(\operatorname{hdet}\theta)\alpha, & \alpha = t^*. \end{cases}$$

The last step is to transfer $\mu_{D^!}$ to the Nakayama automorphism ν_D of D by Proposition 1.4. □

Note that the homological determinant of the Nakayama automorphism of a Koszul AS-regular algebra is equal to 1 [19, Theorem 0.4]. Thus, we arrive at the following result which was proved in [8–10, 12, 19]:

Theorem 3.16. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν, θ is a graded algebra automorphism of A and $D = A[t;\theta]$. Then, D is Calabi–Yau if and only if $\theta = \nu$.* □

4. Skew Laurent extensions

In this section, we consider the Calabi–Yau property of the Ore localizations of both $A[t;\theta]$ and $A_P[y_1, y_2;\sigma]$ with some conditions. For a skew polynomial extension $A[t;\theta]$ of an algebra A, the multiplicatively closed set $\{t^i; i \in \mathbb{N}\}$ is an Ore set. The localization of $A[t;\theta]$ with respect to this Ore set is just the skew Laurent polynomial extension $A[t^{\pm 1};\theta]$. Farinati proposed a general notion of a noncommutative localization in [5]. It was proved there that the Van den Bergh duality is preserved by such a localization and the corresponding dualizing module is also explicitly described. The Ore localization is an example of a noncommutative localization [5, Example 8].

Proposition 4.1. *Suppose that A is a Koszul AS-regular algebra of dimension d and $D = A[t; \theta]$ is a skew polynomial extension of A. Then, the Nakayama automorphism $\widetilde{\nu}$ of $A[t^{\pm 1}; \theta]$ is given by*

$$\widetilde{\nu}(x) = \begin{cases} \nu_D(x), & x \in D \\ \frac{1}{\operatorname{hdet}\theta} x, & x = t^{-1}. \end{cases}$$

Proof. By assumption and [5, Theorem 6], we have:

$$\operatorname{Ext}^i_{E^e}(E, E^e) \cong \begin{cases} 0, & i \neq d+1 \\ E \otimes_D D^{\nu} \otimes_D E(d+1), & i = d+1, \end{cases}$$

where E stands for the algebra $A[t^{\pm 1}; \theta]$. Thus, the claim follows from the description of the Nakayama automorphism of D in Proposition 3.15. □

Theorem 4.2. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν and θ is a graded algebra automorphism of A. Then, $A[t^{\pm 1}; \theta]$ is graded Calabi–Yau if and only if there exists an integer n such that $\theta^n = \nu$ and the homological determinant of θ equals* 1.

Proof. It follows from the proof of [5, Theorem 6] that $A[t^{\pm 1}; \theta]$ is homologically smooth. Thus, the proof focuses on the description of the Nakayama automorphisms of algebras $A[t; \theta]$ and $A[t^{\pm 1}; \theta]$ as showed in Proposition 3.15 and Proposition 4.1 respectively. Note that the only invertible elements in $\Bbbk[t^{\pm 1}]$ are monomials. Suppose that $A[t^{\pm 1}; \theta]$ is Calabi–Yau. Then, its Nakayama automorphism $\widetilde{\nu}$ is inner, i.e., there exists an integer $n \in \mathbb{Z}$ such that $\widetilde{\nu}(x) = t^n x t^{-n}$ for each $x \in A[t^{\pm 1}; \theta]$. In particular, $\widetilde{\nu}(t) = t$. Therefore, $\operatorname{hdet}(\theta) = 1$ by Proposition 3.15. If n is nonnegative, then for each $x \in A$ we have

$$\begin{aligned} \widetilde{\nu}(x) = \theta^{-1}\nu(x) &= t^n x t^{-n} \\ &= t^n (t^{-1}\theta(x) t) t^{-n} \\ &= t^{n-1}\theta(x) t^{1-n} \\ &= \cdots = \theta^n(x). \end{aligned}$$

Hence, $\nu(x) = \theta^{n+1}(x)$. Similarly, the claim also holds for the case when n is a negative integer.

Conversely, if $\theta^n = \nu$ for some integer n and the homological determinant of σ equals 1, then $\widetilde{\nu}(t) = t$. Next, for each $x \in A$, we have

$$\widetilde{\nu}(x) = \theta^{-1}\nu(x) = \theta^{n-1}(x).$$

But in $A[t^{\pm 1}; \theta]$, $\theta(x) = t x t^{-1}$. That is, both θ and its inverse are inner. Therefore, $\widetilde{\nu}$ is an inner automorphism. The proof is completed. □

Example 4.3. Let $A = \Bbbk\langle x, y\rangle/(yx - xy - x^2)$ be the Jordan plane. It is a twisted Calabi–Yau algebra of dimension 2 whose Nakayama automorphism ν is given by $\nu(x) = x$ and $\nu(y) = 2x + y$. Then, $A[t; \theta]$ is Calabi–Yau if and only if $\theta = \nu$ by Theorem 3.16. It is not hard to see that each graded automorphism θ of A has the form $\theta(x) = ax$ and $\theta(y) = bx + ay$ for some $a, b \in \Bbbk$. By Proposition 2.3, the homological determinant of θ is equal to a^2. Thus, $A[t^{\pm 1}; \theta]$ is Calabi–Yau if and only if θ is either given by

$$\begin{cases} \theta(x) = x \\ \theta(y) = \frac{2}{n}x + y \end{cases}$$

for some nonzero integer n, or given by

$$\begin{cases} \theta(x) = -x \\ \theta(y) = \frac{2}{n}x - y \end{cases}$$

for some even integer n.

Finally, we consider the localization or the quotient ring of the double Ore extension B with respect to the Ore set generated by new generators. However, we can only do this in some special case as follows.

Proposition 4.4. *Let $B = A_P[y_1, y_2; \sigma]$ be a trimmed double Ore extension with $P = (p, 0)$ and $\sigma = \begin{pmatrix} \tau & 0 \\ 0 & \xi \end{pmatrix}$. Then,*

(1) *Both τ and ξ are automorphisms of A. Moreover, they commutate with each other.*
(2) *The multiplicatively closed set $S := \{ay_1^{n_1} y_2^{n_2}; a \in k, n_1, n_2 \in \mathbb{Z}_{\geq 0}\}$ is an Ore set.*
(3) *The quotient ring B_S of B with respect to S exists.*

Proof. Since B is a trimmed double Ore extension of A, σ is invertible according to Lemma 1.8. Hence, both τ and ξ are automorphisms of A. By the definition of the right determinant of σ (see (1.8)) and its equivalent description in Proposition 1.9, we have $\tau\xi = \xi\tau$. The rest of the proof is straightforward.

In fact, the algebra $B = A_P[y_1, y_2; \sigma]$ considered above is an iterated skew polynomial extension $A[y_1; \tau][y_2; \xi']$ where ξ' is the automorphism of $A[y_1; \tau]$ defined as follows

$$\xi'(x) = \begin{cases} \xi(x), & x \in A; \\ px, & x = y_1. \end{cases}$$

If $p = 1$, then the quotient ring B_S is isomorphic to the iterated skew Laurent ring $A[y_1^{\pm 1}, y_2^{\pm 1}; \tau, \xi]$ (see [7, pp. 23–24]). In the case of $p \neq 1$, we can also construct the iterated skew Laurent ring, denoted $A_P[y_1^{\pm 1}, y_2^{\pm 1}; \tau, \xi]$ or just $A_P[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$. Similarly, the quotient ring B_S in the above Proposition is isomorphic to the iterated skew Laurent ring $A_P[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$

Theorem 4.5. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν, $B = A_P[y_1, y_2; \sigma]$ is a trimmed double Ore extension with $P = (p, 0)$ and $\sigma = \begin{pmatrix} \tau & 0 \\ 0 & \xi \end{pmatrix}$ and $B_S = A_P[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$. Then, B_S is Calabi–Yau if and only if there exist two integers m, n such that the following conditions are satisfied:*

(1) $\tau^n \xi^m = \nu$;
(2) $\operatorname{hdet}(\tau) = p^m$ *and* $\operatorname{hdet}(\xi) = 1/p^n$.

Proof. The homologically smoothness of B_S also follows from the proof of [5, Theorem 6]. Observe that for the given homomorphism $\sigma : A \to M_{2\times 2}(A)$, the induced algebra homomorphism σ^* form $A^!$ to $M_{2\times 2}(A^!)$ has the form $\begin{pmatrix} \tau^* & 0 \\ 0 & \xi^* \end{pmatrix}$, where τ^* and ξ^* are automorphisms of $A^!$ induced by τ and ξ respectively. By Example 2.7 and Proposition 3.11 we obtain that the Nakayama automorphism of B is given as follows:

$$\nu_B(x) = \begin{cases} (\tau\xi)^{-1} \circ \nu(x), & x \in A; \\ \frac{1}{p}(\operatorname{hdet} \tau)x, & x = y_1; \\ p(\operatorname{hdet} \xi)x, & x = y_2. \end{cases}$$

Thus, it follows from [5, Theorem 6] that the Nakayama automorphism $\widetilde{\nu}$ of B_S is given by

$$\widetilde{\nu}(x) = \begin{cases} \nu_B(x), & x \in B \\ \frac{p}{\operatorname{hdet} \tau} x, & x = y_1^{-1} \\ \frac{1}{p \operatorname{hdet} \xi} x, & x = y_2^{-1} \end{cases}$$

Note that the only invertible elements in B_S are monomials $a_{nm} y_1^n y_2^m$ for some $a_{nm} \in \Bbbk$ and $n, m \in \mathbb{Z}$. Suppose that B_S is Calabi–Yau. Then, its' Nakayama automorphism $\widetilde{\nu}$ is inner, i.e., there exists integer $m, n \in \mathbb{Z}$ such that $\widetilde{\nu}(x) = y_1^n y_2^m x y_2^{-m} y_1^{-n}$ for each $x \in B_S$. In particular, $\widetilde{\nu}(y_1) = y_1^n y_2^m y_1 y_2^{-m} y_1^{-n} = \frac{1}{p}(\operatorname{hdet} \tau) y_1$. It follows that $\operatorname{hdet}(\tau) = p^{m+1}$ since y_1 and y_2 satisfy $y_2 y_1 = p y_1 y_2$. Similarly, we have $\operatorname{hdet}(\xi) = 1/p^{n+1}$. Now, without loss of generality, we may assume that both n and m are nonnegative. For any element $x \in A$, we have

$$\begin{aligned} (\tau\xi)^{-1} \circ \nu(x) &= \widetilde{\nu}(x) \\ &= y_1^n y_2^m x y_2^{-m} y_1^{-n} \\ &= y_1^n y_2^{m-1} \xi(x) y_2^{1-m} y_1^{-n} \\ &= \cdots \\ &= y_1^n \xi^m(x) y_1^{-n} \\ &= \cdots \\ &= \tau^n \xi^m(x). \end{aligned}$$

Hence, $\nu = \tau^{n+1} \xi^{m+1}$.

The proof of the converse is similar. □

In general, if $\theta_1, \ldots, \theta_m$ are commuting graded automorphisms of A, one can construct an iterated skew polynomial extension $A[y_1, \ldots, y_m; \theta_1, \ldots, \theta_m]$ as follows. Let $R_1 = A[y_1; \theta_1]$. Then, extend θ_2 to an algebra automorphism θ_2' of R_1 such that $\theta_2'|_A = \theta_2$ and $\theta_2'(y_1) = y_1$. Now let $R_2 = A[y_1; \theta_1][y_2; \theta_2']$. In this way, one can construct R_i for $i = 1, 2, \ldots m$, such that, for $i < m$, $R_{i+1} = R_i[y_{i+1}, \theta_{i+1}']$, where θ'_{i+1} is the automorphism of R_i satisfying $\theta'_{i+1}|_A = \theta_{i+1}$ and $\theta'_{i+1}(y_j) = y_j$ for $j = 1, \ldots, i$. Finally, let

$$R_m = A[y_1; \theta_1][y_2; \theta'_2] \cdots [y_m; \theta'_m].$$

In order to describe the basic data that determine R_m, one writes R_m in a different way as follows:

$$R_m = A[y_1, \ldots, y_m; \theta_1, \ldots, \theta_m].$$

Note that $y_i y_j = y_j y_i$, $y_i a = \theta_i(a) y_i$ for all i, j and any $a \in A$.

Now, let $R = R_m$ for some positive m. The quotient ring R_S of R with respect to the multiplicatively closed set $S := \{y_1^{n_1} \cdots y_m^{n_m}; n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}\}$ exists and is isomorphic to the iterated skew Laurent ring $A[y_1^{\pm 1}, \ldots, y_m^{\pm 1}; \theta_1, \ldots, \theta_m]$. For more details, we refer to [7, p. 23-24]. In the following, we will give a criterion for such an iterated skew polynomial extension of a Koszul AS-regular algebra to be Calabi–Yau.

Theorem 4.6. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν, $R = A[y_1, \ldots, y_m; \theta_1, \ldots, \theta_m]$ and $S := \{y_1^{n_1} \cdots y_m^{n_m}; n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}\}$. Then,*

(1) *the Nakayama automorphism ν_R of R is given by*

$$\nu_R(x) = \begin{cases} (\theta_1 \circ \cdots \circ \theta_m)^{-1} \circ \nu(x), & x \in A \\ (\mathrm{hdet}\, \theta_i) x, & x = y_i, 1 \leq i \leq m; \end{cases}$$

(2) *R is Calabi–Yau if and only if $\theta_1 \circ \cdots \circ \theta_m = \nu$ and* $\mathrm{hdet}\, \theta_i = 1$ *for all i;*

(3) *R_S is Calabi–Yau if and only if*
   (i) $\mathrm{hdet}(\theta_i) = 1$ *for all i, and*
   (ii) *there exist integers $k_1, \ldots, k_m$ such that $\theta_1^{k_1} \cdots \theta_m^{k_m} = \nu$.*

Proof. It is well-known that a skew polynomial extension of a Koszul algebra is again Koszul, c.f. [17, Corollary 1.3]). So both R and R_S are homologically smooth. By Proposition 3.15, the Nakayama automorphism ν_{R_2} of R_2 is given by

$$\nu_{R_2}(x) = \begin{cases} (\theta'_2)^{-1} \circ \nu_{R_1}(x), & x \in R_1 \\ (\mathrm{hdet}\, \theta'_2) x, & x = y_2. \end{cases}$$

It follows from the construction of θ'_2 and the description of the Nakayama automorphism ν_{R_1} of R_1 that

$$\nu_{R_2}(x) = \begin{cases} (\theta_2 \theta_1)^{-1} \circ \nu(x), & x \in A; \\ (\mathrm{hdet}\, \theta_1) x, & x = y_1; \\ (\mathrm{hdet}\, \theta'_2) x, & x = y_2. \end{cases}$$

On the other hand, according to the proof of Theorem 4.5, $\nu_{R_2}(y_2) = (\mathrm{hdet}\, \theta_2) y_2$. Hence, $\mathrm{hdet}\, \theta'_2 = \mathrm{hdet}\, \theta_2$. Repeating this process, we obtain Part (1). Part (2) follows from Part (1). The proof of Part (3) is similar to the proof of Theorem 4.5. $\square$

Note that a typical example of R_S is the smash product of a Koszul AS-regular algebra with a free abelian group algebra. For example, those Hopf algebras in the classification of Calabi–Yau pointed Hopf algebras of finite Cartan type in [23].

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