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ON NONDIAGONAL FINITE QUASI-QUANTUM GROUPS OVER FINITE ABELIAN GROUPS†

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Dedicated to Professor Shaoxue Liu on the occasion of his 90th birthday

Abstract. In this paper, we initiate the study of nondiagonal finite quasi-quantum groups over finite abelian groups. We mainly study the Nichols algebras in the twisted Yetter-Drinfeld module category ${}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^{\Phi}$ with Φ a nonabelian 3-cocycle on a finite abelian group G. A complete clarification is obtained for the Nichols algebra $B(V)$ in case V is a simple twisted Yetter-Drinfeld module of nondiagonal type. This is also applied to provide a complete classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras over abelian groups of odd order and confirm partially the generation conjecture of pointed finite tensor categories due to Etingof, Gelaki, Nikshych and Ostrik.

1. Introduction

This is a further contribution to the classification problem of finite quasi-quantum groups and pointed finite tensor categories over finite abelian groups beyond some previous works [10, 11] by the first two authors jointly with Liu and Ye. Throughout, we work over an algebraically closed field $\Bbbk$ of characteristic zero. Unless stated otherwise, in this paper all spaces, maps, (co)algebras, (co)modules, and categories, etc., are over $\Bbbk$.

The theory of finite tensor categories started at [6]. In their pioneering work [4], Etingof and Gelaki proposed to classify pointed finite tensor categories which are nonsemisimple. Via the Tannakian formalism [5], pointed finite tensor categories are naturally related to finite quasi-quantum groups, i.e. quasi-Hopf and coquasi-Hopf algebras. This theory of pointed finite tensor categories is a natural generalization of the deep and beautiful theory of finite-dimensional pointed Hopf algebras. It has attracted much interest in recent years. Systems of new examples and pointed finite tensor categories and the related finite quasi-quantum groups have been obtained.

In [10, 11], finite quasi-quantum groups of diagonal type are classified. A key observation is that the study of such algebras can be transformed to that of finite-dimensional pointed Hopf algebras over abelian groups. The latter has been successfully developed and featured with many powerful tools such as Nichols algebras, Weyl groupoids, and

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arithmetic root systems, see e.g. [2, 3, 7, 8]. In this paper, we initiate the investigation of finite quasi-quantum groups of nondiagonal type. In the following we use some concrete notations to provide more explicit explanations.

Once and for all, let G be a finite abelian group and Φ be a 3-cocycle on G. A complete understanding of the Nichols algebras in the twisted Yetter-Drinfeld module category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is the crux for the classification of finite-dimensional pointed coquasi-Hopf algebras. A twisted Yetter-Drinfeld module $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is said to be of diagonal type, if it is a direct sum of 1-dimensional twisted Yetter-Drinfeld modules. The associated Nichols algebra $B(V)$ and the coquasi-Hopf algebra $B(V)\#\Bbbk G$ are called diagonal if V is so. It is shown in [11] that, all $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ are diagonal if and only if the 3-cocycle Φ is abelian, and if and only if there exists a bigger finite abelian group $\mathbb{G}$ with canonical projection $\pi : \mathbb{G} \to G$ such that $\pi^*(\Phi)$ is a 3-coboundary on $\mathbb{G}$. If this is the case, then the diagonal Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ can essentially be reduced to those in ${}^{\Bbbk \mathbb{G}}_{\Bbbk \mathbb{G}}\mathcal{YD}^{\pi^*(\Phi)}$, and thus in ${}^{\Bbbk \mathbb{G}}_{\Bbbk \mathbb{G}}\mathcal{YD}$ as $\pi^*(\Phi)$ is a 3-coboundary. So to go further beyond [11], we shall consider the case with Φ nonabelian and $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ nondiagonal.

To this end, in principle we need to develop a theory for the Nichols algebras of semisimple twisted Yetter-Drinfeld modules. The Hopf version of such a theory was developed in [1, 9]. However, at present it seems not easy to extend this theory to the quasi-Hopf case directly. As a trial step, firstly we study the Nichols algebras of semisimple twisted Yetter-Drinfeld modules with few summands. It turns out that if the number of summands is less than or equal to 2, then we are able to make a connection from this to the diagonal case. The main idea is to consider the support groups of such easy Yetter-Drinfeld modules and carry out the base group change as in our previous works [10, 11]. More precisely, if $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is nondiagonal and has at most 2 simple summands, then its support group G_V is either a cyclic group or the direct product of two cyclic groups. Moreover, the Nichols algebra $B(V) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is essentially nothing other than $B(V) \in {}^{\Bbbk G_V}_{\Bbbk G_V}\mathcal{YD}^{\Phi|_{G_V}}$. In this situation, all 3-cocycles on G_V are abelian and then [10, 11] can be applied.

Our first main result is a complete clarification of the Nichols algebra $B(V)$ when V is a simple twisted Yetter-Drinfeld module of nondiagonal type. In particular, we provide an explicit necessary and sufficient condition on V for $B(V)$ to be finite-dimensional. The same idea and process can be applied to $B(V)$ when V is a direct sum of 2 simple twisted Yetter-Drinfeld modules. As this will not provide more insights for our ultimate aim, we do not include a detailed discussion of this case. Instead, we present several simple examples to offer the reader some flavor. Surprisingly, the result on $B(V)$ with V simple is already enough for us to achieve half of our final aim. Our second main result is a complete classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras over abelian groups of odd order. The key observation is that $B(V) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is infinite-dimensional for any simple nondiagonal twisted Yetter-Drinfeld module V if the order of G is odd. As an application, we also prove that any pointed finite tensor category over an abelian group of odd order is tensor generated by objects of length 2, which partially confirms the generation conjecture [5, Conjecture 5.11.10.] of pointed finite tensor categories due to Etingof, Gelaki, Nikshych and Ostrik.

The paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3, we consider mainly the Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ with Φ nonabelian. A full description of the Nichols algebra of a simple nondiagonal twisted Yetter-Drinfeld module is obtained. This is applied in Section 4 to the generation problem and a complete classification of finite-dimensional pointed coquasi-Hopf algebras over finite abelian groups of odd order. Finally, in Section 5 we provide some further examples and problems of finite-dimensional pointed coquasi-Hopf algebras over finite abelian groups of even order.

2. Preliminaries

In this section, we recall some necessary notions and basic facts on pointed finite tensor categories, pointed coquasi-Hopf algebras, twisted Yetter-Drinfeld modules, Nichols algebras, and arithmetic root systems. The reader is referred to [5, 10, 11] for any unexplained concepts and notations.

2.1. **Pointed finite tensor categories and coquasi-Hopf algebras.**

A finite tensor category is called pointed if every simple object is invertible. According to [6], every pointed finite tensor category is tensor equivalent to the category of comodules of a finite-dimensional pointed coquasi-Hopf algebras.

Recall that a coquasi-Hopf algebra is a coalgebra $(\mathrm{M}, \Delta, \varepsilon)$ equipped with a compatible quasi-algebra structure and a quasi-antipode. Namely, there exist two coalgebra homomorphisms

$$m : \mathrm{M} \otimes \mathrm{M} \longrightarrow \mathrm{M},\ a \otimes b \mapsto ab \quad \text{and} \quad \mu : \Bbbk \longrightarrow \mathrm{M},\ \lambda \mapsto \lambda 1_{\mathrm{M}},$$

a convolution-invertible map $\Phi : \mathrm{M}^{\otimes 3} \longrightarrow \Bbbk$ called associator, a coalgebra antimorphism $S : \mathrm{M} \longrightarrow \mathrm{M}$ and two functions $\alpha, \beta : \mathrm{M} \longrightarrow \Bbbk$ such that for all $a, b, c, d \in \mathrm{M}$ the following equalities hold:

$$\begin{gathered} a_1(b_1c_1)\Phi(a_2, b_2, c_2) = \Phi(a_1, b_1, c_1)(a_2b_2)c_2, \\ 1_{\mathrm{M}}a = a = a1_{\mathrm{M}}, \\ \Phi(a_1, b_1, c_1d_1)\Phi(a_2b_2, c_2, d_2) = \Phi(b_1, c_1, d_1)\Phi(a_1, b_2c_2, d_2)\Phi(a_2, b_3, c_3), \\ \Phi(a, 1_{\mathrm{M}}, b) = \varepsilon(a)\varepsilon(b). \\ S(a_1)\alpha(a_2)a_3 = \alpha(a)1_{\mathrm{M}}, \quad a_1\beta(a_2)S(a_3) = \beta(a)1_{\mathrm{M}}, \\ \Phi(a_1, S(a_3), a_5)\beta(a_2)\alpha(a_4) = \Phi^{-1}(S(a_1), a_3, S(a_5))\alpha(a_2)\beta(a_4) = \varepsilon(a). \end{gathered}$$

The triple (S, α, β) is called a quasi-antipode. M is called a **pointed coquasi-Hopf algebra** if $(\mathrm{M}, \Delta, \varepsilon)$ is a pointed coalgebra, i.e., every simple comodule of M is 1-dimensional.

Let C be a coalgebra, the coradical C_0 of C is the sum of all simple subcoalgebras of C. Fix a coalgebra C with coradical C_0, define C_n inductively as follows: for each

$n \geq 1$, define

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C).$$

Then we get a filtration $C_0 \subset C_1 \subset \cdots \mathbb{C}_n \subset \cdots$, which is called the coradical filtration of C. A coquasi-Hopf algebra also has a coradical filtration since every coquasi-Hopf algebra is a coalgebra.

Given a coquasi-Hopf algebra $(\mathrm{M}, \Delta, \varepsilon, m, \mu, \Phi, S, \alpha, \beta)$, let $\{\mathrm{M}_n\}_{n\geq 0}$ be its coradical filtration, and let

$$\mathrm{gr}\,\mathrm{M} = \mathrm{M}_0 \oplus \mathrm{M}_1/\mathrm{M}_0 \oplus \mathrm{M}_2/\mathrm{M}_1 \oplus \cdots,$$

the corresponding coradically graded coalgebra. Then naturally $\mathrm{gr}\,\mathrm{M}$ inherits from M a graded coquasi-Hopf algebra structure. The corresponding graded associator $\mathrm{gr}\,\Phi$ satisfies $\mathrm{gr}\,\Phi(\bar{a}, \bar{b}, \bar{c}) = 0$ for all homogeneous $\bar{a}, \bar{b}, \bar{c} \in \mathrm{gr}\,\mathrm{M}$ unless they all lie in M_0. Similar conditions hold for $\mathrm{gr}\,\alpha$ and $\mathrm{gr}\,\beta$. A coquasi-Hopf algebra M is called **coradically graded** if $M \cong \mathrm{gr}(M)$ as coquasi-Hopf algebras.

Here is an example with some useful terms and notations for our later investigations.

Example 2.1. *Let G be a group. Clearly the group algebra $\Bbbk G$ is a Hopf algebra with $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$ and $\varepsilon(g) = 1$ for any $g \in G$. Let ω be a normalized 3-cocycle on G, i.e.*

$$\omega(ef, g, h)\omega(e, f, gh) = \omega(e, f, g)\omega(e, fg, h)\omega(f, g, h), \tag{2.1}$$

$$\omega(f, 1, g) = 1 \tag{2.2}$$

for all $e, f, g, h \in G$. By linearly extending, $\omega\colon (\Bbbk G)^{\otimes 3} \to \Bbbk$ becomes a convolution-invertible map. Define two linear functions $\alpha, \beta\colon \Bbbk G \to \Bbbk$ by

$$\alpha(g) := \varepsilon(g) \quad \text{and} \quad \beta(g) := \frac{1}{\omega(g, g^{-1}, g)}$$

for any $g \in G$. Then kG together with these ω, α and β makes a coquasi-Hopf algebra, which will be written as $(\Bbbk G, \omega)$ in the following. By definition, the Gr-category Vec_G^{ω} is just the category of comodules of $(\Bbbk G, \omega)$.

It is well known that a pointed fusion category over $\Bbbk$ is equivalent to a Gr-category, see [5] for details. The crux to determine all the pointed fusion categories is to give a complete list of the representatives of the 3-cohomology classes in $\mathrm{H}^3(G, \Bbbk^*)$ for all groups G. However, when G is a finite abelian group, the problem is solved in [11], and a list of the representatives of $\mathrm{H}^3(G, \Bbbk^*)$ can be given as follows.

Let $\mathbb{N}$ denote the set of nonnegative integers, $\mathbb{Z}$ the ring of integers, and $\mathbb{Z}_m$ the cyclic group of order m. Any finite abelian group G is of the form $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ with $m_j \in \mathbb{N}$ for $1 \leq j \leq n$. Denote by $\mathcal{A}$ the set of all $\mathbb{N}$-sequences

$$(c_1, \ldots, c_l, \ldots, c_n, c_{12}, \ldots, c_{ij}, \ldots, c_{n-1,n}, c_{123}, \ldots, c_{rst}, \ldots, c_{n-2,n-1,n}) \tag{2.3}$$

such that $0 \leq c_l < m_l$, $0 \leq c_{ij} < (m_i, m_j)$, $0 \leq c_{rst} < (m_r, m_s, m_t)$ for $1 \leq l \leq n$, $1 \leq i < j \leq n$, $1 \leq r < s < t \leq n$, where c_{ij} and c_{rst} are ordered in the lexicographic order of their indices. We denote by $\underline{\mathbf{c}}$ the sequence (2.3) in the following. Let g_i be a generator of $\mathbb{Z}_{m_i}, 1 \leq i \leq n$. For any $\underline{\mathbf{c}} \in \mathcal{A}$, define

$$\omega_{\underline{\mathbf{c}}}: \; G \times G \times G \longrightarrow \Bbbk^*$$

$$[g_1^{i_1}\cdots g_n^{i_n}, g_1^{j_1}\cdots g_n^{j_n}, g_1^{k_1}\cdots g_n^{k_n}] \mapsto \prod_{l=1}^{n} \zeta_{m_l}^{c_l i_l [\frac{j_l+k_l}{m_l}]} \prod_{1\le s<t\le n} \zeta_{m_t}^{c_{st} i_t [\frac{j_s+k_s}{m_s}]} \prod_{1\le r<s<t\le n} \zeta_{(m_r,m_s,m_t)}^{c_{rst} k_r j_s i_t}. \tag{2.4}$$

Here and below ζ_m stands for an m-th primitive root of unity.

Proposition 2.2. [11, Proposition 3.8] $\{\omega_{\underline{\mathbf{c}}} \mid \underline{\mathbf{c}} \in \mathcal{A}\}$ *forms a complete set of representatives of the normalized* 3*-cocycles on* G *up to* 3*-cohomology.*

2.2. **Twisted Yetter-Drinfeld module categories.** The Yetter-Drinfeld module category ${}^H_H\mathcal{YD}$ of a quasi-Hopf algebra H may be defined as the center $Z(H\text{-mod})$ of its module category H-mod and it is braided tensor equivalent to the module category of the quantum double $D(H)$ of H, see [13, 14] for more details. The Yetter-Drinfeld module category of a coquasi-Hopf algebra can be defined in a dual manner, see [11, 12].

In this paper we are mainly concerned with the Yetter-Drinfeld module category of the coquasi-Hopf algebra $(\Bbbk G, \Phi)$ of a finite abelian group G and a normalized 3-cocycle Φ on G for our purpose. To emphasize Φ, we denote the Yetter-Drinfeld category of $(\Bbbk G, \Phi)$ as ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$, and the objects in it are called **twisted Yetter-Drinfeld modules**. Define

$$\widetilde{\Phi}_g(x,y) = \frac{\Phi(g,x,y)\Phi(x,y,g)}{\Phi(x,g,y)} \tag{2.5}$$

for all $g, x, y \in G$. By direct computation one can show that $\widetilde{\Phi}_g$ is a 2-cocycle on G. The construction of category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ can be summarized as follows, the detailed computation can be found in [11, 12].

Proposition 2.3. *A vector space* V *is an object in* ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ *if and only if* $V = \oplus_{g\in G} V_g$ *with each* V_g *a projective* G*-representation with respect to the 2-cocycle* $\widetilde{\Phi}_g$*, namely*

$$e \triangleright (f \triangleright v) = \widetilde{\Phi}_g(e,f)(ef) \triangleright v. \tag{2.6}$$

The tensor product $V_g \otimes V_h$ *is determined by*

$$e \triangleright (X \otimes Y) = \widetilde{\Phi}_e(g,h) e \triangleright X \otimes e \triangleright Y, \ \ X \in V_g, \ \ Y \in V_h. \tag{2.7}$$

The associativity and the braiding constraints of ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ *are given respectively by*

$$a_{V_e,V_f,V_g}((X\otimes Y)\otimes Z) = \Phi(e,f,g)^{-1} X \otimes (Y \otimes Z) \tag{2.8}$$

$$R(X \otimes Y) = e \triangleright Y \otimes X \tag{2.9}$$

for all $X \in V_e, \ Y \in V_f, \ Z \in V_g$.

Remark 2.4. *For a simple twisted Yetter-Drinfeld module* V *in* ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$*, there exists some* $g \in G$ *such that* $V = V_g$ *and we define* $g_V := g$ *in this case. Recall that a 2-cocycle* φ *on* G *is called symmetric if* $\varphi(g,h) = \varphi(h,g)$ *for all* $h, g \in G$*. By* (2.6)*, it is not hard to show that a simple Yetter-Drinfeld module* V *with* $g_V = g$ *is* 1*-dimensional if and only if* $\widetilde{\Phi}_g$ *is symmetric.*

From the representatives of 3-cocycles on abelian groups given in Proposition 2.2, one can verify directly that

$$\widetilde{\Phi}_g\widetilde{\Phi}_h = \widetilde{\Phi}_{gh}, \ \forall g, h \in G. \tag{2.10}$$

The following proposition is fundamental and the proof follows from (2.7) and the fact that $S^2 = \mathrm{id}$.

Proposition 2.5. *Suppose V_g is $(G, \widetilde{\Phi}_g)$-representation, V_h is a $(G, \widetilde{\Phi}_h)$-representation, then $V_g \otimes V_h$ is a $(G, \widetilde{\Phi}_{gh})$-representation. In particular, the dual object V_g^* of V_g is a $(G, \widetilde{\Phi}_{g^{-1}})$-representation and $(V_g^*)^* = V_g$.*

A 3-cocycle Φ on G is called an **abelian 3-cocycle** if ${}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^\Phi$ is pointed, i.e. each simple object of ${}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^\Phi$ is 1-dimensional. Using the representatives of normalized 3-cocycles listed in Proposition 2.2, we can write out the representatives of abelian 3-cocycles of a finite abelian group.

Proposition 2.6. [11, Proposition 3.14] *Suppose $G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_n}$, e_i is a generator of $\mathbb{Z}_{m_i}$ for all $1 \le i \le n$, and Φ is an abelian 3-cocycle on G. Then up to cohomology Φ must be of the form*

$$\Phi(e_1^{i_1}\cdots e_n^{i_n}, e_1^{j_1}\cdots e_n^{j_n}, e_1^{k_1}\cdots e_N^{k_n}) = \prod_{l=1}^{n} \zeta_l^{c_l i_l [\frac{j_l+k_l}{m_l}]} \prod_{1\le s<t\le n} \zeta_{m_t}^{c_{st} i_t [\frac{j_s+k_s}{m_s}]}. \tag{2.11}$$

An object V of ${}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^\Phi$ is said to be **of diagonal type** if V is a direct sum of 1-dimensional objects. It is not hard to verify that each object of ${}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^\Phi$ is of diagonal type if and only if Φ is an abelian 3-cocycle on G.

2.3. **Quasi-version of bosonization.** The study of pointed coquasi-Hopf algebras may be reduced to that of Hopf algebras in twisted Yetter-Drinfeld categories. The related notions of algebras and Hopf algebras in a braided tensor category can be found in [13].

Assume that

$$\mathrm{M} = \bigoplus_{i\in\mathbb{N}} \mathrm{M}_i$$

is a coradically graded pointed coquasi-Hopf algebra over an abelian group G. So $\mathrm{M}_0 = (\Bbbk G, \Phi)$ for a 3-cocycle Φ on G. Let $\pi : \mathrm{M} \to \mathrm{M}_0$ be the canonical projection. Then M is a kG-bicomodule naturally via

$$\delta_L := (\pi \otimes \mathrm{id})\Delta, \quad \delta_R := (\mathrm{id} \otimes \pi)\Delta.$$

Thus there is a G-bigrading on M, that is,

$$\mathrm{M} = \bigoplus_{g,h\in G} {}^g\mathrm{M}^h$$

where ${}^g\mathrm{M}^h = \{m \in \mathrm{M} \mid \delta_L(m) = g \otimes m,\ \delta_R(m) = m \otimes h\}$. Define the coinvariant subalgebra of M by

$$R := \{m \in \mathrm{M} \mid (\mathrm{id} \otimes \pi)\Delta(m) = m \otimes 1\}.$$

Then $R = \oplus_{i \geq 0} R_i$ is a coradically graded Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ such that $R_0 = \Bbbk$.

Conversely, let $H = \oplus_{i \geq 0} H_i$ be a coradically graded Hopf algebra in ${}^G_G\mathcal{YD}^\Phi$ such that $H_0 = \Bbbk$. If $X \in H_n$, then we say that X has length n. Since H is a left G-comodule, there is a G-grading on H:

$$H = \bigoplus_{x \in G} {}^x H$$

where ${}^x H = \{X \in H | \delta_L(X) = x \otimes X\}$. Hence

$$H = \bigoplus_{g \in G} {}^g H = \bigoplus_{g \in G, n \in \mathbb{N}} {}^g H_n.$$

As a convention, homogeneous elements in H are denoted by capital letters, say $X, Y, Z, \ldots$, and the associated degrees are denoted by their lower cases, say $x, y, z, \ldots$. For any $X \in H$, we write its comultiplication as

$$\Delta_H(X) = X_{(1)} \otimes X_{(2)}.$$

Lemma 2.7. [12, Proposition 3.3] *Keep the notations as above. We define a coquasi-Hopf algebra on $H \otimes \Bbbk G$ as follows. The product is given by*

$$(X \otimes g)(Y \otimes h) = \frac{\Phi(xg, y, h)\Phi(x, y, g)}{\Phi(x, g, y)\Phi(xy, g, h)} X(g \triangleright Y) \otimes gh, \tag{2.12}$$

and the coproduct is determined by

$$\Delta(X \otimes g) = \Phi(x_{(1)}, x_{(2)}, g)^{-1}(X_{(1)} \otimes x_{(2)}g) \otimes (X_{(2)} \otimes g). \tag{2.13}$$

The quasi-antipode (S, α, β) is given by

$$S(X \otimes g) = \frac{\Phi(g^{-1}, g, g^{-1})}{\Phi(x^{-1}g^{-1}, xg, g^{-1})\Phi(x, g, g^{-1})}(1 \otimes x^{-1}g^{-1})(S_H(X) \otimes 1), \tag{2.14}$$

$$\alpha(1 \otimes g) = 1, \quad \alpha(X \otimes g) = 0, \tag{2.15}$$

$$\beta(1 \otimes g) = \Phi(g, g^{-1}, g)^{-1}, \quad \beta(X \otimes g) = 0, \tag{2.16}$$

here $g, h \in G$ and X, Y are homogeneous elements of length ≥ 1.

In the following, by $H \# \Bbbk G$ we denote the resulting coquasi-Hopf algebra defined on $H \otimes \Bbbk G$.

Lemma 2.8. [12, Proposition 3.4] *Let* M *be a coradically graded pointed coquasi-Hopf algebra over abelian group G and R the coinvariant subalgebra of* M*. Then we have $R \# \Bbbk G \cong \mathrm{M}$ as coquasi-Hopf algebras.*

2.4. **Nichols algebras and arithmetic root systems.** Nichols algebras are the analogue of the usual symmetric algebras in more general braided tensor categories. Here for our purpose we only give the definition of a Nichols algebra in twisted Yetter-Drinfeld module categories ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$.

Let V be a nonzero object in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. By $T_{\Phi}(V)$ we denote the tensor algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ generated freely by V. It is clear that $T_{\Phi}(V)$ is isomorphic to $\bigoplus_{n\geq 0} V^{\otimes \overrightarrow{n}}$ as a linear space, where $V^{\otimes \overrightarrow{n}}$ means $\underbrace{(\cdots((}_{n-1} V \otimes V) \otimes V) \cdots \otimes V)$. This induces a natural $\mathbb{N}$-graded structure on $T_{\Phi}(V)$. Define a comultiplication on $T_{\Phi}(V)$ by $\Delta(X) = X \otimes 1 + 1 \otimes X,\ \forall X \in V$, a counit by $\varepsilon(X) = 0$, and an antipode by $S(X) = -X$. These provide a graded Hopf algebra structure on $T_{\Phi}(V)$ in the braided tensor category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$.

Definition 2.9. *The Nichols algebra $B(V)$ of V is defined to be the quotient Hopf algebra $T_{\Phi}(V)/I$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, where I is the unique maximal graded Hopf ideal generated by homogeneous elements of degree greater than or equal to 2.*

A Nichols algebra $B(V)$ is called **of diagonal type** if V is a twisted Yetter-Drinfeld module of diagonal type. When Φ is an abelian 3-cocycle on G, any Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is of diagonal type. In the classification of finite-dimensional pointed Hopf algebras, one is mainly concerned with Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ with Φ trivial. Such a Yetter-Drinfeld module category is often written in the form ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. The Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ are called usual Nichols algebras in order to distinguish from those in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ with Φ nontrivial.

Arithmetic root systems are invariants of usual Nichols algebras of diagonal type with certain finiteness property. A complete classification of arithmetic root systems was given in [8] by Heckenberger. In [10, 11] arithmetic root systems are applied to classify finite-dimensional pointed coquasi-Hopf algebras of diagonal type.

Suppose $B(V)$ is a usual Nichols algebra of diagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. Then there is a basis $\{X_i | 1 \leq i \leq n\}$ of V called canonical basis such that $\Bbbk X_i$ is a simple Yetter-Drinfeld module for each $1 \leq i \leq n$. Suppose $\delta_L(X_i) = h_i \otimes X_i, 1 \leq i \leq n$. The structure constants of $B(V)$ are $\{q_{ij} | 1 \leq i, j \leq n\}$ such that $h_i \triangleright X_j = q_{ij} X_j$. Let $E = \{e_i | 1 \leq i \leq n\}$ be a canonical basis of $\mathbb{Z}^n$, and χ be a bicharacter of $\mathbb{Z}^n$ determined by $\chi(e_i, e_j) = q_{ij}$. As defined in [7, Sec.3], $\Delta^+(B(V))$ is the set of degrees of the (restricted) Poincare-Birkhoff-Witt generators counted with multiplicities and $\Delta(B(V)) := \Delta^+(B(V)) \bigcup -\Delta^+(B(V))$, which is called the root system of $B(V)$. Moreover, the triple $(\Delta = \Delta(B(V)), \chi, E)$ is called an arithmetic root system of $B(V)$ if the corresponding Weyl groupoid $W_{\chi,E}$ is full and finite (see [8, Sec.2,3]). In this case, we denote this arithmetic root system by $\Delta(B(V))_{\chi,E}$ for brevity. If there is another arithmetic root system $\Delta_{\chi',E'}$, and an isomorphism $\tau : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

$$\tau(E) = E', \quad \chi'(\tau(e), \tau(e)) = \chi(e, e),$$
$$\chi'(\tau(f), \tau(g))\chi'(\tau(g), \tau(f)) = \chi(f, g)\chi(g, f)$$

for all $e, f, g \in \mathbb{Z}^n$, then we say that $\Delta_{\chi,E}$ and $\Delta_{\chi',E'}$ are twist equivalent.

A generalized Dynkin diagram is an invariant of arithmetic root systems, and it can determine arithmetic root systems up to twist equivalence.

Definition 2.10. *The generalized Dynkin diagram of an arithmetic root system* $\Delta_{\chi,E}$ *is a nondirected graph* $\mathcal{D}_{\chi,E}$ *with the following properties:*

1) *There is a bijective map* ϕ *from* $I = \{1, 2, \ldots, n\}$ *to the set of vertices of* $\mathcal{D}_{\chi,E}$.
2) *For all* $1 \le i \le n$, *the vertex* $\phi(i)$ *is labelled by* q_{ii}.
3) *For all* $1 \le i, j \le n$, *the number* n_{ij} *of edges between* $\phi(i)$ *and* $\phi(j)$ *is either* 0 *or* 1. *If* $i = j$ *or* $q_{ij}q_{ji} = 1$ *then* $n_{ij} = 0$, *otherwise* $n_{ij} = 1$ *and the edge is labelled by* $\widetilde{q_{ij}} = q_{ij}q_{ji}$ *for all* $1 \le i < j \le n$.

An arithmetic root system $\Delta_{\chi,E}$ is called *connected* if and only if the corresponding generalized Dynkin diagram $\mathcal{D}_{\chi,E}$ is connected. All the generalized Dynkin diagrams of connected arithmetic root systems are listed in [8].

3. Nichols algebras in twisted Yetter-Drinfeld categories

In this section, we focus on the Nichols algebras in twisted Yetter-Drinfeld module categories ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ with Φ nonabelian. With a help of [11], an explicit description of the finite-dimensional Nichols algebra of a simple nondiagonal twisted Yetter-Drinfeld module is obtained.

3.1. **Some basic facts on Nichols algebras.** It is known that there is an $\mathbb{N}$-graded structure on $B(V)$. We will show that there is actually a $\mathbb{Z}^l$-graded structure on $B(V)$ for $V = \oplus_{i=1}^{l} V_i \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, where the V_i's are simple. Let $\{e_i : 1 \le i \le l\}$ be a set of free generators of $\mathbb{Z}^l$. Then we have the following proposition, which in fact is a generalization of [11, Proposition 4.2].

Proposition 3.1. *There is a* $\mathbb{Z}^l$*-grading on the Nichols algebra* $B(V) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ *by setting* $\deg V_i = e_i$.

Proof. Obviously, there is a $\mathbb{Z}^l$-grading on the tensor algebra $T_\Phi(V) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ by assigning $\deg V_i = e_i$, that is $\deg(X) = e_i$ for all $X \in V_i$, $1 \le i \le l$. Since $\Delta(X) = X \otimes 1 + 1 \otimes X$ for all $X \in V$, and Δ is a multiplicative, i.e., $\Delta(YZ) = \Delta(Y)\Delta(Z)$ for all $Y, Z \in T_\Phi(V)$. So the comultiplication Δ of $T_\Phi(V)$ preserves the $\mathbb{Z}^l$-grading. Let $I = \oplus_{i \ge 1} I_i$ be the maximal graded Hopf ideal generated by $\mathbb{N}$-homogeneous elements of degree greater than or equal to 2. To prove that $B(V)$ is $\mathbb{Z}^l$-graded, it suffices to prove that I is $\mathbb{Z}^l$-graded. This will be done by induction on the $\mathbb{N}$-degree.

Since $I = \oplus_{i \ge 1} I_i$ is generated by $\mathbb{N}$-homogeneous elements of degree greater than or equal to 2, it is obvious that $I_1 = 0$. Hence I_1 is $\mathbb{Z}^l$-graded.

Now suppose that $I^k := \oplus_{1 \le i \le k} I_i$ is $\mathbb{Z}^l$-graded. We shall prove that $I^{k+1} = \oplus_{1 \le i \le k+1} I_i$ is also $\mathbb{Z}^l$-graded. Let $X \in I_{k+1}$ and $X = X^1 + X^2 + \cdots + X^n$, with each X^i being $\mathbb{Z}^l$-homogenous and X^i and X^j having different $\mathbb{Z}^l$-degrees if $i \ne j$. Write $\Delta(X^i) = X^i \otimes 1 + 1 \otimes X^i + (X^i)_1 \otimes (X^i)_2$. Since $\Delta(X) = X \otimes 1 + 1 \otimes X + (X)_1 \otimes (X)_2$, where

$(X)_1\otimes(X)_2 \in T_\Phi(V)\otimes I^k+I^k\otimes T_\Phi(V)$, i.e., $\sum(X^i)_1\otimes(X^i)_2 \in T_\Phi(V)\otimes I^k+I^k\otimes T_\Phi(V)$. According to the inductive assumption, $T_\Phi(V)\otimes I^k+I^k\otimes T_\Phi(V)$ is a $\mathbb{Z}^l$-graded space. So each $(X^i)_1\otimes(X^i)_2 \in T_\Phi(V)\otimes I^k+I^k\otimes T_\Phi(V)$ as Δ preserves $\mathbb{Z}^l$-degrees. If there was an $X^i \notin I_{k+1}$, then $I+\langle X^i\rangle$ is a Hopf ideal properly containing I, which contradicts to the maximality of I. It follows that $X^i \in I_{k+1}$ for all $1 \le i \le n$ and hence I^{k+1} is also $\mathbb{Z}^l$-graded by the assumption on X. This completes the proof of the proposition. □

If $B(V)$ is a Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$, we say that G is the **base group** of $B(V)$. From Definition 2.9 we know that a Nichols algebra depends on both the base group G and the 3-cocycle Φ. But sometimes we are only concerned about the braided Hopf algebra structure of a Nichols algebra. Hence we need to omit or change the base group of the Nichols algebra in the sense of the following definition.

Definition 3.2. *Let $B(V)$ and $B(U)$ be Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ and ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^\Psi$ respectively with $\dim V = \dim U = l$. We say that $B(V)$ is isomorphic to $B(U)$ if there is a $\mathbb{Z}^l$-graded linear isomorphism $\mathcal{F} : B(V) \to B(U)$ which preserves the multiplication and comultiplication.*

Definition 3.3. *Let $V = \oplus_{i=1}^n V_i \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ be a Yetter-Drinfeld module, where V_i $(1 \le i \le n)$ are simple Yetter-Drinfeld modules. Let g_i be the corresponding degree of V_i. Then we call the subgroup $G' = \langle g_1, \cdots, g_n\rangle$, generated by $g_1, \dots, g_n$, the support group of V, which is denoted by G_V.*

Lemma 3.4. [11, Lemma 4.4] *Suppose $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ and $U \in {}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^\Psi$, where H is a finite abelian group. Let $G|_V$ and $H|_U$ be the support groups of V and U respectively. If there is a linear isomorphism $F : V \to U$ and a group epimorphism $f : G|_V \to H|_U$ such that:*

$$\delta \circ F = (f \otimes F) \circ \delta, \tag{3.1}$$

$$F(g \triangleright v) = f(g) \triangleright F(v), \tag{3.2}$$

$$\Phi|_{G|_V} = f^*\Psi|_{H|_U} \tag{3.3}$$

for any $g \in G|_V$, $v \in V$. Then $B(V)$ is isomorphic to $B(U)$.

Corollary 3.5. *Let $B(V)$ be a Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$, $H = G_V$ and $\Psi = \Phi|_{G_V}$. Then there is a Yetter-Drinfeld module U in ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^\Psi$ such that $B(V) \cong B(U)$.*

Proof. Let $U = V$ as linear space with module and comodule structures inherited from those of V. Then U is a Yetter-Drinfeld module in ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^\Psi$, and $B(V)$ is isomorphic to $B(U)$ by Lemma 3.4. □

Next we will introduce the twist of a Nichols algebra. Let $(V, \triangleright, \delta_L) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$, and let J be a 2-cochain of G. Then we can define a new action $\triangleright_J$ of G on V by

$$g \triangleright_J X = \frac{J(g, x)}{J(x, g)} g \triangleright X \tag{3.4}$$

for $X \in V$ and $g \in G$. We denote $(V, \triangleright_J, \delta_L)$ by V^J, and by definition we have $V^J \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi * \partial(J)}$. Moreover there is a tensor equivalence $(F_J, \varphi_0, \varphi_2) :\ {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi} \to {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi * \partial(J)}$ which takes V to V^J and

$$\varphi_2(U, V) : (U \otimes V)^J \to U^J \otimes V^J, \quad Y \otimes Z \mapsto J(y, z)^{-1} Y \otimes Z$$

for $Y \in U,\ Z \in V$.

Let $B(V)$ be a usual Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. Then it is clear that $B(V)^J$ is a Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\partial J}$ with multiplication $\circ$ determined by

$$X \circ Y = J(x, y) XY \tag{3.5}$$

for all homogenous elements $X, Y \in B(V)$, here $x = \deg X,\ y = \deg Y$ are the associated G-degrees as defined in Subsection 2.3. Using the same terminology as for coquasi-Hopf algebras , we call $B(V)$ and $B(V)^J$ twist equivalent. The following fact is obvious.

Lemma 3.6. [11, Lemma 2.12] *The twisting $B(V)^J$ of $B(V)$ is a Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\partial J}$ and $B(V)^J \cong B(V^J)$.*

3.2. **Nichols algebras of diagonal type.** In general, there are both Nichols algebras of diagonal type and of nondiagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ if Φ is nonabelian. Recall that for a simple twisted Yetter-Drinfeld module V in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, there exists a $g \in G$ such that $\delta_L(X) = g \otimes X$ for all $X \in V$ and in this case we write $g_V := g$.

Example 3.7. *Let $G = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle \times \langle g_4 \rangle = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} \times \mathbb{Z}_{m_4}$ such that $m_i | m_j$ if $1 \leq i < j \leq 4$, Φ a 3-cocycle on G given by*

$$\Phi(g_1^{i_1} \cdots g_4^{i_4}, g_1^{j_1} \cdots g_4^{j_4}, g_1^{k_1} \cdots g_4^{k_4}) = \zeta_{m_1}^{k_1 j_2 i_3}. \tag{3.6}$$

Let U and V be two simple twisted Yetter-Drinfeld modules in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ such that $g_U = g_1,\ g_V = g_4$. Then $\widetilde{\Phi}_{g_1}$ is not symmetric since $\widetilde{\Phi}_{g_1}(g_2, g_3) \neq \widetilde{\Phi}_{g_1}(g_3, g_2)$, and $\widetilde{\Phi}_{g_4}$ is symmetric. Hence by Remark 2.4, U is of nondiagonal type, while V is of diagonal type.

The following lemma says that the study of Nichols algebras of diagonal type can always be reduced to those in a suitable twisted Yetter-Drinfeld module category ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^{\Psi}$ such that Ψ is an abelian 3-cocycle on H.

Lemma 3.8. [11, Lemma 4.1] *Let $B(V)$ be a Nichols algebras of diagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. Then $\Phi|_{G_V}$ is an abelian 3-cocycle on G_V.*

Suppose $\mathbb{G} = \mathbb{Z}_{\mathbb{m}_1} \times \cdots \times \mathbb{Z}_{\mathbb{m}_n} = \langle \mathrm{g}_1 \rangle \times \cdots \langle \mathrm{g}_n \rangle$ and $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} = \langle g_1 \rangle \times \cdots \langle g_n \rangle$ where $\mathbb{m}_i = m_i^2$ for $1 \leq i \leq n$. Let

$$\pi :\ \Bbbk\mathbb{G} \to \Bbbk G, \quad \mathrm{g}_i \mapsto g_i, \quad 1 \leq i \leq n \tag{3.7}$$

be the canonical epimorphism. Then we have

Proposition 3.9. [11, Proposition 3.15] *Suppose that Φ is an abelian 3-cocycle on G. Then $\pi^*(\Phi)$ is a 3-coboundary on $\mathbb{G}$.*

Let δ_L and $\triangleright$ be the comodule and the module structure maps of $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. Define

$$\rho_L: V \to \Bbbk\mathbb{G} \otimes V, \quad \rho_L = (\iota \otimes \mathrm{id})\delta_L$$
$$\blacktriangleright: \Bbbk\mathbb{G} \otimes V \to V, \quad \mathrm{g} \blacktriangleright Z = \pi(\mathrm{g}) \triangleright Z$$

for all $\mathrm{g} \in \mathbb{G}$ and $Z \in V$. Then the following observation is immediate.

Lemma 3.10. *Defined in this way, $(V, \rho_L, \blacktriangleright)$, denoted simply by $\widetilde{V}$ in the following, is an object in ${}^{\Bbbk\mathbb{G}}_{\Bbbk\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi)}$.*

Proposition 3.11. *For any Nichols algebra $B(V) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, the Nichols algebra $B(\widetilde{V}) \in {}^{\Bbbk\mathbb{G}}_{\Bbbk\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi)}$ is isomorphic to $B(V)$. Moreover, if Φ is an abelian 3-cocycle on G, then $B(\widetilde{V})$ is twist equivalent to a usual Nichols algebra in ${}^{\Bbbk\mathbb{G}}_{\Bbbk\mathbb{G}}\mathcal{YD}$.*

Proof. The first statement is a direct consequence of Lemma 3.4. For the second, just note that $\pi^*(\Phi)$ is a 3-coboundary on G by Proposition 3.9. So there is a 2-cochain J on $\mathbb{G}$ such that $\partial J = \pi^*(\Phi)$. Therefore, $B(\widetilde{V})^{J^{-1}} \cong B(\widetilde{V}^{J^{-1}})$ is a Nichols algebra in ${}^{\Bbbk\mathbb{G}}_{\Bbbk\mathbb{G}}\mathcal{YD}$ according to Lemma 3.6. □

Thanks to the preceding proposition, each Nichols algebra of diagonal type $B(V)$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is twist equivalent to an ordinary Nichols algebra of diagonal type, thus has a PBW-type basis as well. So we can define root systems for Nichols algebras of diagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$.

Definition 3.12. *Suppose $B(V)$ is a rank n Nichols algebra of diagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. Let $\Delta^+(B(V))$ be the set of $\mathbb{Z}^n$-degrees of the Poincare-Birkhoff-Witt generators counted with multiplicities and let $\Delta(B(V)) := \Delta^+(B(V)) \bigcup -\Delta^+(B(V))$, which is called the root system of $B(V)$.*

It is obvious that if $B(V)$ is twist equivalent to an ordinary Nichols algebra $B(V')$, then $\Delta(B(V)) = \Delta(B(V'))$ since the twisting does not change the $\mathbb{Z}^n$-degrees of PBW generators.

Proposition 3.13. *Let $B(V)$ be a Nichols algebra of diagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, $\{X_i | 1 \leq i \leq n\}$ a canonical basis of V, $\delta_L(X_i) = g_i \otimes X_i, 1 \leq i \leq n$. Let $(q_{ij})_{n\times n}$ be the structure constants of V, i.e. $g_i \triangleright X_j = q_{ij}X_j$, $1 \leq i,j \leq n$. Let $E = \{e_1, \cdots, e_n\}$ be a basis of $\mathbb{Z}^n$, χ a bicharacter on $\mathbb{Z}^n$ such that*

$$\chi(e_i, e_j) = q_{ij}, \quad 1 \leq i,j \leq n.$$

Then $\Delta(B(V))$ is finite if and only if $\Delta(B(V))_{\chi,E}$ is an arithmetic root system.

Proof. If $\Delta(B(V))_{\chi,E}$ is an arithmetic root system, then clearly $\Delta(B(V))$ is finite.

Now suppose that $\Delta(B(V))$ is finite. Since $B(V)$ is of diagonal type, it is harmless to assume that Φ is an abelian 3-cocycle on G by Lemma 3.8. According to Proposition 3.11, there is a 2-cochain J on $\mathbb{G}$ such that $B(\widetilde{V})^J = B(\widetilde{V}^J)$ is an ordinary Nichols algebra. Let $V' = \widetilde{V}^J$, $\{q'_{ij}\}$ the structure constants of V', χ' a bicharacter of $\mathbb{Z}^n$ such that

$$\chi'(e_i, e_j) = q'_{ij}, \ 1 \leq i,j \leq n.$$

Hence $\Delta(B(V'))_{\chi',E}$ is an arithmetic root system since $B(V')$ is an ordinary Nichols algebra of diagonal type and $\Delta(B(V'))$ is finite. On the other hand, we have $q'_{ii} = q_{ii},\ q'_{ij}q'_{ji} = q_{ij}q_{ji},\ 1 \leq i,j \leq n$ by (3.4). So $\Delta(B(V'))_{\chi,E} = \Delta(B(V))_{\chi,E}$ is an arithmetic root system twist equivalent to $\Delta(B(V'))_{\chi',E}$. This implies that $\Delta(B(V))_{\chi,E}$ is an arithmetic root system. □

Let $R = \oplus_{i\geq 0} R[i]$ be a coradically graded Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. We call R connected if $R[0] = \Bbbk 1$. The following proposition is very important for our further investigation.

Proposition 3.14. *Suppose that $R = \oplus_{i\geq 0} R[i]$ is a finite-dimensional connected coradically graded Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ such that $\Phi|_{G_{R[1]}}$ is an abelian 3-cocycle on $G_{R[1]}$. Then $R = B(R[1])$ is a Nichols algebra.*

Proof. Since R is coradically graded, we have $G_{R[1]} = G_R$. Let $H = G_R$ and $\Psi = \Phi|_H$. It is obvious that R is also a connected coradically graded Hopf algebra in ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^{\Psi}$. Since Ψ is an abelian 3-cocycle on H, we have $R = B(R[1])$ by [11, Proposition 5.1]. □

3.3. **The Nichols algebras of simple twisted Yetter-Drinfeld modules.** In this subsection we focus on the Nichols algebras of nondiagonal Yetter-Drinfeld modules. Note that if Φ is an abelian 3-cocycle on G, then each object of ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is of diagonal type. So nondiagonal Yetter-Drinfeld modules appear in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ only if Φ is nonabelian. The following proposition is an immediate consequence of Propositions 2.2 and 2.6.

Proposition 3.15. *Suppose that G is a cyclic group $\mathbb{Z}_m$ or a direct product of two cyclic groups, say $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$, then all the 3-cocycles on G are abelian.*

Proposition 3.16. *Let G be a finite abelian group, Φ a 3-cocycle on G. Suppose that $B(V)$ is a Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, where V is a simple Yetter-Drinfeld module, or a direct sum of two simple Yetter-Drinfeld modules. Then $B(V)$ is isomorphic to a Nichols algebra of diagonal type $B(V')$ in ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^{\Psi}$, where $H = G_V$ and $\Psi = \Phi|_H$.*

Proof. By Proposition 2.3, G_V is either a cyclic group, or of the form $\mathbb{Z}_m \times \mathbb{Z}_n$. Hence Ψ is an abelian 3-cocycle of H by Proposition 3.15. According to Corollary 3.5, there is a Nichols algebra $B(V')$ in ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^{\Psi}$ such that $B(V) \cong B(V')$. Thus $B(V')$ is of diagonal type since Ψ is an abelian 3-cocycle on H. □

According to this proposition, we can apply the theory of Nichols algebras of diagonal type to study the Nichols algebras of simple twisted Yetter-Dinfeld modules, or of a direct sum of two simple twisted Yetter-Drinfeld modules.

Definition 3.17. *Let G be a finite group and α a 2-cocycle on G. An element $g \in G$ is called an α-element if $\alpha(g,h) = \alpha(h,g)$ for all $h \in G$.*

Proposition 3.18. *Let G be a finite abelian group, Φ a 3-cocycle on G. Suppose V is a simple Yetter-Drinfeld module of nondiagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, $g_V = g$. Then $B(V)$ is finite-dimensional if and only if V is one of the following two cases:*

(C1) *$g \triangleright v = -v$ for all $v \in V$;*
(C2) *$\dim(V) = 2$ and $g \triangleright v = \zeta_3 v$ for all $v \in V$, here ζ_3 is a 3-rd primitive root of unity.*

Proof. First of all, we study the simple Yetter-Drinfeld module V by considering its support group. According to Proposition 2.3, V is a simple $(G, \widetilde{\Phi}_g)$-representation. We claim that $g \triangleright v = \lambda v, \ \forall v \in V$ for some nonzero constant λ. Assume the order of g is n. Then

$$\underbrace{g \triangleright (g \triangleright (\cdots (g}_{n} \triangleright v) \cdots)) = \prod_{i=1}^{n-1} \widetilde{\Phi}_g(g, g^i) v, \quad \forall v \in V.$$

So the action of g on V is diagonal. On the other hand, for each $h \in G$, we have

$$\widetilde{\Phi}_g(g, h) = \frac{\Phi(g, g, h)\Phi(g, h, g)}{\Phi(g, g, h)} = \frac{\Phi(h, g, g)\Phi(g, h, g)}{\Phi(h, g, g)} = \widetilde{\Phi}_g(h, g).$$

Hence g is a $\widetilde{\Phi}_g$-element, and we have $g \triangleright (h \triangleright v) = \widetilde{\Phi}_g(g, h) gh \triangleright v = h \triangleright (g \triangleright v)$ for any $h \in G$ and $v \in V$. So the g-action on V is a morphism of $(G, \widetilde{\Phi}_g)$-representations, and the Schur's Lemma guarantees $g \triangleright v = \lambda v, \forall v \in V$ for some nonzero constant λ.

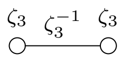
Let $H = \langle g \rangle$ and $\Psi = \Phi|_H$. Then by Corollary 3.5, $B(V)$ is isomorphic to a Nichols algebra $B(V')$ in ${}^{\Bbbk H}_{\Bbbk H}\mathcal{YD}^{\Psi}$. Thus in the following it is enough to consider V' and $B(V')$ instead. Let $\dim(V) = n$. The structure constants (q_{ij}) of V' are given by

$$q_{ij} = \lambda, \quad 1 \le i, j \le n,$$

since $g \triangleright v = \lambda v, \ \forall v \in V'$. Let $E = \{e_1, \cdots, e_n\}$ be a basis of $\mathbb{Z}^n$, χ a bicharacter on $\mathbb{Z}^n$ such that

$$\chi(e_i, e_j) = q_{ij} = \lambda, \ 1 \le i, j \le n.$$

If the simple Yetter-Drinfeld module V, and so V', satisfies either (C1) or (C2), then clearly $B(V')$, and so $B(V)$, is finite-dimensional, see e.g. [8]. Now we prove the converse. Suppose $B(V')$ is finite-dimensional. Then λ must be a root of 1 and $\lambda \neq 1$, since otherwise $B(V')$ is infinite-dimensional. By Proposition 3.13, $\Delta(B(V'))_{\chi,E}$ is an arithmetic root system. If $\lambda = -1$, then we have $q_{ii} = -1$ and $q_{ij}q_{ji} = 1$ for all $1 \le i \neq j \le n$. In this case, V' satisfies the condition C1 and so does V. Now assume $\lambda \neq -1$. Then the arithmetic root system associated to $B(V')$ is connected since $q_{ij}q_{ji} = \lambda^2 \neq 1$ for all $1 \le i \neq j \le n$. Hence the corresponding generalized Dynkin diagram has n vertices which are all labelled by λ, and there is an edge labelled by λ^2 between any two different vertices. By a careful check up on the generalized Dynkin diagrams listed in [8, Table 1-Table 4], one can easily conclude that the generalized Dynkin diagram of $\Delta(B(V'))_{\chi,E}$ must be as follows:



.

This forces $\lambda = \zeta_3$ and $n = 2$. Thus V', and so V, satisfies the condition C2. □

In the following, we give two examples of simple Yetter-Drinfeld modules of nondiagonal type satisfying conditions C1 and C2 respectively. The associated nondiagonal Nichols algebras are finite-dimensional.

Recall that, if φ is a 2-cocycle on an abelian group G, then a map $\rho\colon G \to \mathrm{GL}(V)$ is a (G, φ)-representation if and only if

$$\rho(1) = \mathrm{id}_V,\ \rho(g)\rho(h) = \frac{\varphi(g,h)}{\varphi(h,g)}\rho(h)\rho(g),\ \forall g, h \in G. \tag{3.8}$$

Example 3.19. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e_1\rangle \times \langle e_2\rangle \times \langle e_3\rangle$, Φ a 3-cocycle on G given by*

$$\Phi(e_1^{i_1}e_2^{i_2}e_3^{j_3}, e_1^{j_1}e_2^{j_2}e_3^{i_3}, e_1^{k_1}e_2^{k_2}e_3^{k_3}) = (-1)^{i_3 j_2 k_1}.$$

For a 2-dimensional $\Bbbk$-vector space V with a fixed basis $\{X_1, X_2\}$, define $\rho\colon G \to \mathrm{GL}(V)$ by

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad \rho(e_1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\quad \rho(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\rho(e_3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \rho(e_1e_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\quad \rho(e_1e_3) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$
$$\rho(e_2e_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad \rho(e_1e_2e_3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then one can verify that ρ satisfies (3.8), hence (V, ρ) is a $(G, \widetilde{\Phi}_{e_1})$-representation. According to Proposition 2.3, V is a simple Yetter-Drinfeld module in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ such that $g_V = e_1$ and $e_1 \triangleright v = -v$ for all $v \in V$.

Example 3.20. *Let $G = \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_6 = \langle e_1\rangle \times \langle e_2\rangle \times \langle e_3\rangle$, Φ a 3-cocycle on G given by*

$$\Phi(e_1^{i_1}e_2^{i_2}e_3^{j_3}, e_1^{j_1}e_2^{j_2}e_3^{i_3}, e_1^{k_1}e_2^{k_2}e_3^{k_3}) = (-1)^{i_3 j_2 k_1}.$$

Let V be a 2-dimensional $\Bbbk$-vector space with a fixed basis. Define a map $\rho\colon G \to \mathrm{GL}(V)$, with respect to the fixed basis of V, by

$$\rho(e_1^{i_1}e_2^{i_2}e_3^{i_3}) = (-1)^{i_2 i_3}\begin{pmatrix} \zeta_3^{i_1+i_2} & 0 \\ 0 & (-1)^{i_2}\zeta_3^{i_1+i_2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{i_3}.$$

Then one can verify that ρ satisfies (3.8), hence (V, ρ) is a $(G, \widetilde{\Phi}_{e_1})$-representation. Since $\widetilde{\Phi}_{e_1}(e_2, e_3) = -1 \neq \widetilde{\Phi}_{e_1}(e_3, e_2) = 1$, $\widetilde{\Phi}_{e_1}$ is not symmetric. So all the simple $(G, \widetilde{\Phi}_{g_1})$-representations have dimension ≥ 2 by Remark 2.4. This implies that (V, ρ) is a simple $(G, \widetilde{\Phi}_{e_1})$-representation. By Proposition 2.3, V is a simple Yetter-Drinfeld module in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ such that $g_V = e_1$ and $g_1 \triangleright v = \zeta_3 v$ for all $v \in V$.

4. Finite quasi-quantum groups over abelian groups of odd order

In this section we provide a complete classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras over abelian groups of odd order. This is also applied to the classification theory of pointed finite tensor categories. In particular, we give a partial answer to the following

Conjecture 4.1. [5, Conjecture 5.11.10.] *A pointed finite tensor category is tensor generated by objects of length* 2*.*

This conjecture is due to Etingof, Gelaki, Nikshych and Ostrik, hence will be called EGNO's conjecture in the following. It is a natural generalization of the well known Andruskiewitsch-Schneider conjecture [2, Conjecture 1.4]. Our main classification result on finite-dimensional pointed coquasi-Hopf algebras will induce the following

Theorem 4.2. *Suppose that $\mathcal{C}$ is a pointed finite tensor category with $G(\mathcal{C})$ an abelian group of odd order. Then $\mathcal{C}$ is tensor generated by objects of length* 2*.*

4.1. Some preparations.

In this subsection, we will give some important properties of twisted Yetter-Drinfeld modules, and prove that each finite-dimensional Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ must be of diagonal type if the order $|G|$ of G is odd. We need the following two propositions.

Proposition 4.3. *Suppose that V and W are two simple objects in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ such that $g_V = g_W$. Then $\dim(V) = \dim(W)$.*

Proof. By $\mathcal{C}$ we denote the tensor category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. Suppose $g_V = g_W = g$. Then V, W are $(G, \widetilde{\Phi}_g)$-representations. Let W^* be the dual object of W, which is a $(G, \widetilde{\Phi}_{g^{-1}})$-representation. Hence $V \otimes W^*$ is an ordinary representation of G by Proposition 2.5. Note that G is abelian, so simple G-representations are 1-dimensional. Therefore we may take a 1-dimensional subobject of $V \otimes W^*$, say K. It follows from

$$0 \neq \mathrm{Hom}_{\mathcal{C}}(K, V \otimes W^*) = \mathrm{Hom}_{\mathcal{C}}(K \otimes W, V) \tag{4.1}$$

that $\dim(V) = \dim(W)$ since V and W are simple objects. □

Proposition 4.4. *For each simple object V in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, we have $\dim(V) \mid |G|$.*

Proof. According to Proposition 2.3, V is a simple Yetter-Drinfeld module in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ with $g_V = g$ if and only if V is a simple $(G, \widetilde{\Phi}_g)$-representation. By $\Bbbk[G]_{\widetilde{\Phi}_g}$ we denote the twisted group algebra of G, i.e. the algebra with a basis $\{h | h \in G\}$ and product determined by

$$f \cdot h = \widetilde{\Phi}_g(f, h) fh, \quad \forall f, h \in G.$$

Note that the representation category of $\Bbbk[G]_{\widetilde{\Phi}_g}$ is equivalent to the category of projective representations of G with respect to $\widetilde{\Phi}_g$. Let $\{V^i | 1 \leq i \leq m\}$ be a set of iso-classes of simple representations of $\Bbbk[G]_{\widetilde{\Phi}_g}$. By Proposition 4.3, all simple representations of $\Bbbk[G]_{\widetilde{\Phi}_g}$ have the same dimension, which will be denoted by n. Since $\Bbbk[G]_{\widetilde{\Phi}_g}$ is a semisimple algebra, we have $mn^2 = \dim(\Bbbk[G]_{\widetilde{\Phi}_g}) = |G|$, hence $n \mid |G|$. □

Now we will prove two technical lemmas, which are necessary for our next exploration.

Lemma 4.5. *Suppose that* $\{g_1, g_2 \cdots, g_m\} \subset G$ *generates* G, *and* $\widetilde{\Phi}_{g_i}(g_j, g_k) = \widetilde{\Phi}_{g_i}(g_k, g_j)$ *for all* $1 \leq i, j, k \leq m$. *Then* Φ *is an abelian* 3*-cocycle on* G.

Proof. Suppose $G = \langle e_1 \rangle \times \cdots \times \langle e_n \rangle$ and $m_i = |e_i|$ for $1 \leq i \leq n$. By Proposition 2.2, we can assume that Φ is of the form (2.4). Let $\Phi = \Psi\Gamma$, where Ψ and Γ are 3-cocycles on G given by

$$\Psi(e_1^{i_1} \cdots e_n^{i_n}, e_1^{j_1} \cdots e_n^{j_n}, e_1^{k_1} \cdots e_n^{k_n}) = \prod_{l=1}^{n} \zeta_{m_l}^{c_l i_l [\frac{j_l+k_l}{m_l}]} \prod_{1 \leq s < t \leq n} \zeta_{m_t}^{c_{st} i_t [\frac{j_s+k_s}{m_s}]}, \tag{4.2}$$

$$\Gamma(e_1^{i_1} \cdots e_n^{i_n}, e_1^{j_1} \cdots e_n^{j_n}, e_1^{k_1} \cdots e_n^{k_n}) = \prod_{1 \leq r < s < t \leq n} \zeta_{(m_r, m_s, m_t)}^{c_{rst} k_r j_s i_t}. \tag{4.3}$$

Since Ψ is an abelian 3-cocycle on G, $\widetilde{\Psi}_{g_i}(g_j, g_k) = \widetilde{\Psi}_{g_i}(g_k, g_j)$ for all $1 \leq i, j, k \leq n$. Thus $\widetilde{\Phi}_{g_i}(g_j, g_k) = \widetilde{\Phi}_{g_i}(g_k, g_j)$ implies

$$\widetilde{\Gamma}_{g_i}(g_j, g_k) = \widetilde{\Gamma}_{g_i}(g_k, g_j) \tag{4.4}$$

for $1 \leq i, j, k \leq m$. From (4.3), it follows that

$$\begin{aligned} \widetilde{\Gamma}_{ef}(g, h) &= \widetilde{\Gamma}_e(g, h)\widetilde{\Gamma}_f(g, h), \\ \widetilde{\Gamma}_e(fg, h) &= \widetilde{\Gamma}_e(f, h)\widetilde{\Gamma}_e(g, h), \\ \widetilde{\Gamma}_e(f, gh) &= \widetilde{\Gamma}_e(f, g)\widetilde{\Gamma}_e(f, h) \end{aligned}$$

for all $e, f, g, h \in G$. So (4.4) implies

$$\widetilde{\Gamma}_f(g, h) = \widetilde{\Gamma}_f(h, g), \ \forall f, g, h \in G,$$

since $\{g_1, \cdots g_m\} \subset G$ generates G. Hence $c_{rst} = 0$ follows from

$$\widetilde{\Gamma}_{e_r}(e_s, e_t) = \widetilde{\Gamma}_{e_r}(e_t, e_s), \ 1 \leq r < s < t \leq n.$$

This implies Γ is trivial and $\Phi = \Psi$ is an abelian 3-cocycle on G. □

Lemma 4.6. *Let* g_1, g_2, g_3 *be elements in* G. *Then the following three identities*

$$\widetilde{\Phi}_{g_1}(g_2, g_3) = \widetilde{\Phi}_{g_1}(g_3, g_2), \tag{4.5}$$

$$\widetilde{\Phi}_{g_2}(g_1, g_3) = \widetilde{\Phi}_{g_2}(g_3, g_1), \tag{4.6}$$

$$\widetilde{\Phi}_{g_3}(g_1, g_2) = \widetilde{\Phi}_{g_3}(g_2, g_1) \tag{4.7}$$

are mutually equivalent.

Proof. Suppose $\widetilde{\Phi}_{g_1}(g_2, g_3) = \widetilde{\Phi}_{g_1}(g_3, g_2)$, that is

$$\frac{\Phi(g_1, g_2, g_3)\Phi(g_2, g_3, g_1)}{\Phi(g_2, g_1, g_3)} = \frac{\Phi(g_1, g_3, g_2)\Phi(g_3, g_2, g_1)}{\Phi(g_3, g_1, g_2)}. \tag{4.8}$$

Multiplying the identity (4.8) with the scalar $\frac{\Phi(g_2,g_1,g_3)\Phi(g_3,g_1,g_2)}{\Phi(g_1,g_2,g_3)\Phi(g_3,g_2,g_1)}$, we get

$$\widetilde{\Phi}_{g_2}(g_1,g_3)=\frac{\Phi(g_2,g_1,g_3)\Phi(g_1,g_3,g_2)}{\Phi(g_1,g_2,g_3)}=\frac{\Phi(g_2,g_3,g_1)\Phi(g_3,g_1,g_2)}{\Phi(g_3,g_2,g_1)}=\widetilde{\Phi}_{g_2}(g_3,g_1).$$

The equivalence between (4.7) and (4.5), (4.6) can be proved similarly. □

Proposition 4.7. *Let $V=\oplus_{i=1}^{n}V_i$ be an object of ${}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^{\Phi}$, where the V_i's are simple objects. Let $H=G_V$ be the support group of V. If $\Phi|_H$ is not an abelian 3-cocycle on H, then $n\geq 3$ and at least three summands, say V_{i_1},V_{i_2},V_{i_3} of V, are of nondiagonal type.*

Proof. Suppose $g_{V_i}=g_i$ for $1\leq i\leq n$, then $H=\langle g_1,\cdots,g_n\rangle$. Since $\Phi|_H$ is not an abelian 3-cocycle on H, there exist i,j,k such that $\widetilde{\Phi}_{g_i}(g_j,g_k)\neq\widetilde{\Phi}_{g_i}(g_k,g_j)$ according to Lemma 4.5. By Lemma 4.6, we also have

$$\begin{aligned}\widetilde{\Phi}_{g_j}(g_i,g_k) &\neq \widetilde{\Phi}_{g_j}(g_k,g_i),\\ \widetilde{\Phi}_{g_k}(g_i,g_j) &\neq \widetilde{\Phi}_{g_k}(g_j,g_i).\end{aligned}$$

Hence the 2-cocycles $\widetilde{\Phi}_{g_i},\widetilde{\Phi}_{g_j},\widetilde{\Phi}_{g_k}$ on G are not symmetric, this implies V_i,V_j,V_k are simple Yetter-Drinfeld modules of nondiagonal type. □

Now we can prove the following proposition, which says that if the order of G is odd, then each finite-dimensional Nichols algebra in ${}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^{\Phi}$ must be of diagonal type.

Proposition 4.8. *Let G be a finite abelian group of odd order and Φ be a 3-cocycle on G. Suppose that $V\in{}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^{\Phi}$ is not diagonal. Then $B(V)$ is infinite-dimensional.*

Proof. By assumption, there is a summand U of V such that U is a simple Yetter-Drinfeld module of nondiagonal type. Suppose that $g=g_U$. Then there exists some $\lambda\in\Bbbk^*$ such that

$$g\triangleright u=\lambda u,\quad \forall u\in U.$$

Since G is odd, we have $\dim(U)\neq 2$ by Proposition 4.4. Hence, U does not satisfy the condition C2 of Proposition 3.18. It is also obvious that $\lambda\neq -1$ since the order $|g|$ of g is odd. Thus U does not satisfy the condition C1 of Proposition 3.18 either. So $B(U)$ must be infinite-dimensional, therefore so is $B(V)$. □

Corollary 4.9. *Let G be a finite abelian group of odd order and Φ be a 3-cocycle on G. Suppose $V\in{}_{\Bbbk G}^{\Bbbk G}\mathcal{YD}^{\Phi}$ such that $\Phi|_{G_V}$ is not an abelian 3-cocycle on G_V. Then $B(V)$ is infinite-dimensional.*

Proof. According to Proposition 4.7, there exists a nondiagonal simple submodule U of V. By Proposition 4.8, $B(U)$ is infinite-dimensional, and so is $B(V)$. □

4.2. **A proof of Theorem 4.2.** In this subsection we will prove Theorem 4.2 in several steps. Recall that a comodule of a finite-dimensional coquasi-Hopf algebra M is called cofree if it is isomorphic to $M^{\oplus n}$ as comodules for an integer $n \geq 1$. It is well-known that any finite-dimensional module of an algebra is a quotient of a free module. Dually, any finite-dimensional comodule of a coalgebra is a subcomodule of a cofree comodule.

Proposition 4.10. *Suppose that M is a finite-dimensional pointed coquasi-Hopf algebra. Then M is generated by grouplike and skew-primitive elements if and only if* $\mathrm{comod}(M)$ *is tensor generated by objects of length* 2.

Proof. Suppose that M is generated by grouplike and skew-primitive elements. Let $G = G(M)$, and $\{X_i | 1 \leq i \leq n\}$ a maximal linear independent set of skew-primitive elements. It is obvious that $(gX)g^{-1}$ is a skew-primitive element if X is. So each element in M can be presented as a linear combination of elements of the form $g(\cdots(X_{i_1}X_{i_2})\cdots X_{i_m})$.

Let $\mathcal{A} = \{g(\cdots(X_{i_1}X_{i_2})\cdots X_{i_m}) | g \in G, (i_1, i_2, \cdots, i_m) \in \mathcal{I}\}$ be a minimal set that cogenerates the cofree comodule M. Let $V_i = \Bbbk\{g_i, X_i\}$, where g_i satisfies $\delta_L(X_i) = g_i \otimes X_i,\ 1 \leq i \leq n$. Let $V(g) = \Bbbk\{1, g\}$ for $g \in G$. Then it is obvious that $V_i, V(g)$ are subcomodules of M of length 2. Let

$$F : M \longrightarrow \oplus_{g\in G,(i_1,i_2,\cdots,i_m)\in\mathcal{I}} V(g) \otimes (\cdots(V_{i_1} \otimes V_{i_2}) \otimes \cdots \otimes V_{i_m})$$

be the linear map determined by

$$g(\cdots(X_{i_1}X_{i_2})\cdots X_{i_m}) \to g \otimes (\cdots(X_{i_1} \otimes X_{i_2}) \otimes \cdots \otimes X_{i_m}). \tag{4.9}$$

It is obvious that F is an injective comodule map. So we have proved the cofree comodule M is tensor generated by objects of length 2. This implies that each cofree comodule is tensor generated by objects of length 2. As any finite-dimensional comodule of M is a subcomodule of a cofree comodule, so $\mathrm{comod}(M)$ is tensor generated by objects of length 2.

Conversely, suppose $\mathrm{comod}(M)$ is tensor generated by objects of length 2. For each $g \in G$, by S_g we mean a simple object such that $\delta(v) = g \otimes v$ for all $v \in S_g$. Since $\mathrm{comod}(M)$ is pointed, each object of length 2 is an extension of S_g by S_h for some $g, h \in G$. So an object of length 2 must be of the form $S_g \oplus S_h$, or $V = \Bbbk\{h, X\}$ where X is a g-h-primitive element. Let $V(g), V_i, g \in G, 1 \leq i \leq n$ be objects in $\mathrm{comod}(M)$ defined as the first part of the proof. Then the cofree comodule M is a subquotient of object of the form

$$\oplus V(g) \otimes (\cdots(V_{i_1} \otimes V_{i_2}) \otimes \cdots \otimes V_{i_m})$$

according to the hypothesis. This implies that M is generated by grouplike and skew-primitive elements. □

Theorem 4.11. *Suppose that $R = \oplus_{i\geq 0} R[i]$ is a finite-dimensional connected coradically graded braided Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, where G is an abelian group of odd order and Φ is a 3-cocycle on G. Then $R = B(R[1])$.*

Proof. Since $R = \oplus_{i\geq 0} R[i]$ is a connected coradically graded braided Hopf algebra, $R[0] = \Bbbk 1$ and $R[1]$ is the set of primitive elements of R. So there exists a canonical injective linear map $\iota\colon B(R[1]) \to R$. Since R is finite-dimensional, $B(R[1])$ is finite-dimensional. By Corollary 4.9, we have that $\Phi|_{G_{R[1]}}$ is an abelian 3-cocycle on $G_{R[1]}$. Hence $R = B(R[1])$ according to Theorem 3.14. □

Proof of Theorem 4.2. Suppose that $\mathcal{C}$ is a pointed finite tensor category with $G(\mathcal{C})$ an abelian group of odd order. Then there exists a finite-dimensional pointed coquasi-Hopf algebra M such that $\mathcal{C}$ is tensor equivalent to comod(M). So we only need to prove that comod(M) is tensor generated by objects of length 2. By Proposition 4.10, this amounts to proving that M is generated by grouplike and skew-primitive elements. It is obvious that M is generated by grouplike and skew-primitive elements if and only if $\mathrm{gr}(M)$ is so. Let $G = G(M) = G(\mathcal{C})$, R the coinvariant subalgebra of $\mathrm{gr}(M)$. So R is a finite-dimensional coradically graded braided Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, where Φ is the associator of $\mathrm{gr}(M)$, which is actually a 3-cocycle on G. By Theorem 4.11, $R = B(R[1])$ is a Nichols algebra. Hence $\mathrm{gr}(M)$ is generated by grouplike and skew-primitive elements since $\mathrm{gr}(M) = R\#\Bbbk G(M)$ by Lemma 2.8. □

4.3. **The classification results.** With the help of Theorem 4.11, we achieve the classification of coradically graded pointed coquasi-Hopf algebras and that of pointed finite tensor categories over abelian groups of odd order.

We need some new notions to present the main result. Let $\Delta_{\chi,E}$ be an arithmetic root system. For each positive root $\alpha \in \Delta$, define $q_\alpha = \chi(\alpha,\alpha)$. Then the height of α is defined by

$$\mathrm{ht}(\alpha) = \begin{cases} |q_\alpha|, & \text{if } q_\alpha \neq 1 \text{ is a root of unity;} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.10}$$

A function $\chi : G \longrightarrow \Bbbk^*$ is called a **quasi-character** associated to a 2-cocycle ω on G if for all $f, g \in G$,

$$\chi(f)\chi(g) = \omega(f,g)\chi(fg), \quad \chi(1) = 1. \tag{4.11}$$

It is clear that there is a quasi-character associated to ω if and only if ω is symmetric. Recall that for a fixed 3-cocycle Φ on G, $\{\widetilde{\Phi}_g | g \in G\}$ gives 2-cocycles on G.

Definition 4.12. *Let $\chi_1, \cdots, \chi_n$ be quasi-characters of G associated to $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$ respectively. We say the series $(\chi_1, \cdots, \chi_n)$ is of finite type if there is an arithmetic root system $\Delta_{\chi,E}$ of rank n such that:*

- $\chi_i(g_j)\chi_j(g_i) = q_{ij}q_{ji}, \quad \chi_i(g_i) = q_{ii}$ *for all* $1 \leq i, j \leq n$. *Here* $q_{ij} = \chi(e_i, e_j)$ *for* $e_i, e_j \in E$.
- $\mathrm{ht}(\alpha) < \infty$ *for all* $\alpha \in \Delta$.

For a series of quasi-characters $(\chi_1, \cdots, \chi_n)$ of finite type associated to $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$, we can attach to it a twisted Yetter-Drinfeld module $V(\chi_1, \cdots, \chi_n)$ with a canonical basis $\{X_1, \cdots, X_n\}$ such that $g_i \triangleright X_j = \chi_j(g_i)X_j$ and $\delta_L(X_i) = g_i \otimes X_i$ for all $1 \leq i, j \leq n$.

Theorem 4.13. *Let G be a finite abelian group of odd order, Φ a 3-cocycle on G.*

(1) *If $(\chi_1, \cdots, \chi_n)$ is a series of quasi-characters of finite type associated to the 2-cocycles $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$, then $B(V(\chi_1, \cdots, \chi_n))$ is a finite-dimensional Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$.*

(2) *Suppose that $\mathcal{C}$ is a coradically graded pointed finite tensor category such that $G(\mathcal{C}) = G$ and the associator is Φ. Then there exists a series of quasi-characters $(\chi_1, \cdots, \chi_n)$ of finite type associated to $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$ such that*

$$\mathcal{C} \cong \operatorname{comod}(B(V(\chi_1, \cdots, \chi_n))\#\Bbbk G).$$

Proof. (1) Direct consequence of Proposition 3.13.

(2) As $\mathcal{C}$ is a coradically graded pointed finite tensor category, there is a finite-dimensional coradically graded pointed coquasi-Hopf algebra M over G such that $\mathcal{C} \cong \operatorname{comod}(M)$. Let R be the coinvariant subalgebra of M. By Theorem 4.11, $R = B(R[1])$ is a finite-dimensional Nichols algebra of diagonal type in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. So there exists a series of quasi-characters $\chi_1, \cdots, \chi_n$ of finite type associated to $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$ such that $R \cong B(V(\chi_1, \cdots, \chi_n))$. Hence we have $\mathcal{C} \cong \operatorname{comod}(M) \cong \operatorname{comod}(B(V(\chi_1, \cdots, \chi_n))\#\Bbbk G)$.

□

5. Further examples and problems

So far, we have clarified the Nichols algebras of simple twisted Yetter-Drinfeld modules over an arbitrary finite abelian group. The main idea is: *by considering the support group of a simple Yetter-Drinfeld module, one may transform a nondiagonal twisted Yetter-Drinfeld module to a diagonal one.* Then the previously developed related theories, see e.g. [7, 8, 10, 11], can be fully applied. Remarkably, this is already enough to help us classify the finite-dimensional coradically graded pointed coquasi-Hopf algebras over abelian groups of odd order.

To move on, we should study pointed finite tensor categories $\mathcal{C}$ over abelian groups with $2 \mid G(\mathcal{C})$. The critical step is a thorough investigation of the Nichols algebras of nonsimple nondiagonal twisted Yetter-Drinfeld modules. Thanks to Proposition 3.16, we may applied the same strategy to Nichols algebras of form $B(V_1 \oplus V_2)$, where V_1 and V_2 are nondiagonal simples. Thus, we will have a complete classification based on our previous work [11]. As this will not provide further valuable insights than what we have done in Sections 3 and 4, we do not include a detailed discussion about this. Instead, we provide several simple examples in the following to elucidate the process.

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle$, Φ a nonabelian 3-cocycle on G given by

$$\Phi(g_1^{i_1} g_2^{i_2} g_3^{j_3}, g_1^{j_1} g_2^{j_2} g_3^{i_3}, g_1^{k_1} g_2^{k_2} g_3^{k_3}) = (-1)^{i_3 j_2 k_1}.$$

Consider three simple objects V_1, V_2, V_3 in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ given as follows:

- $V_1 = \Bbbk\{X_1, X_2\}$, $g_{V_1} = g_1$, $g_1 \triangleright X_1 = -X_1$, $g_1 \triangleright X_2 = -X_2$, $g_2 \triangleright X_1 = X_1$, $g_2 \triangleright X_2 = -X_2$, $g_3 \triangleright X_1 = X_2$, $g_3 \triangleright X_2 = X_1$.
- $V_2 = \Bbbk\{Y_1, Y_2\}$, $g_{V_2} = g_2$, $g_1 \triangleright Y_1 = Y_1$, $g_1 \triangleright Y_2 = -Y_2$, $g_2 \triangleright Y_1 = -Y_1$, $g_2 \triangleright Y_2 = -Y_2$, $g_3 \triangleright Y_1 = Y_2$, $g_3 \triangleright Y_2 = Y_1$.
- $V_3 = \Bbbk\{Z_1, Z_2\}$, $g_{V_3} = g_3$, $g_1 \triangleright Z_1 = Z_2$, $g_1 \triangleright Z_2 = Z_1$, $g_2 \triangleright Z_1 = Z_1$, $g_2 \triangleright Z_2 = -Z_2$, $g_3 \triangleright Z_1 = -Z_1$, $g_3 \triangleright Z_2 = -Z_2$.

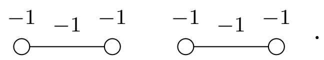
Note that V_1 is isomorphic to the twisted Yetter-Drinfeld module V given in Example 3.19. Similarly, one can verify that V_2, V_3 are indeed simple twisted Yetter-Drinfeld modules in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ with the help of Proposition 2.3. Note that V_1, V_2, V_3 satisfy the condition C1 of Proposition 3.18. Hence $B(V_1), B(V_2)$ and $B(V_3)$ are finite-dimensional Nichols algebras. In additon, we have the following observation.

Proposition 5.1. *The Nichols algebras $B(V_1 \oplus V_2), B(V_1 \oplus V_3)$ and $B(V_2 \oplus V_3)$ are finite-dimensional.*

Proof. Firstly we will show that $B(V_1 \oplus V_2)$ is finite-dimensional. Since the support group of $V_1 \oplus V_2$ is $G_1 := \langle g_1 \rangle \times \langle g_2 \rangle$, we have $\Phi|_{G_1}$ is an abelian 3-cocycle on G_1 by proposition 2.2. In the following we use the notations of Subsection 3.2. Let $\mathbb{G}_1 = \langle \mathbb{g}_1 \rangle \times \langle \mathbb{g}_2 \rangle$ and $\pi_1^* : \mathbb{G}_1 \to G_1$ be the group epimorphism given by $\pi_1^*(\mathbb{g}_1) = g_1, \pi_1^*(\mathbb{g}_2) = g_2$, and $V = V_1 \oplus V_2$. By Lemma 3.10 and Proposition 3.9, $B(V) \in {}^{\Bbbk \mathbb{G}_1}_{\Bbbk \mathbb{G}_1}\mathcal{YD}^{\pi_1^*(\Phi|_{G_1})}$. As a consequence, there is a 2-cochain J on $\mathbb{G}_1$ such that $B(V)^J = B(V^J) \in {}^{\Bbbk \mathbb{G}_1}_{\Bbbk \mathbb{G}_1}\mathcal{YD}$. According to (3.4), we have:

$$
\begin{aligned}
&\mathbb{g}_1 \triangleright_J X_1 = -X_1, \quad \mathbb{g}_1 \triangleright_J X_2 = -X_2, \\
&\mathbb{g}_1 \triangleright_J Y_1 = \frac{J(\mathbb{g}_1, \mathbb{g}_2)}{J(\mathbb{g}_2, \mathbb{g}_1)} Y_1, \quad \mathbb{g}_1 \triangleright_J Y_2 = -\frac{J(\mathbb{g}_1, \mathbb{g}_2)}{J(\mathbb{g}_2, \mathbb{g}_1)} Y_2, \\
&\mathbb{g}_2 \triangleright_J X_1 = \frac{J(\mathbb{g}_2, \mathbb{g}_1)}{J(\mathbb{g}_1, \mathbb{g}_2)} X_1, \quad \mathbb{g}_2 \triangleright_J X_2 = -\frac{J(\mathbb{g}_2, \mathbb{g}_1)}{J(\mathbb{g}_1, \mathbb{g}_2)} X_2, \\
&\mathbb{g}_2 \triangleright_j Y_1 = -Y_1, \quad \mathbb{g}_2 \triangleright_j Y_2 = -Y_2.
\end{aligned}
$$

So the generalized Dynkin diagram corresponding to $B(V^J)$ is



This implies that $B(V^J)$ is finite-dimensional according to [8, Table 1 of Section 3]. So $B(V_1 \oplus V_2)$ is finite-dimensional.

Next we will prove that $B(V_1 \oplus V_3)$ is finite-dimensional. Let $R_1 = X_1 + X_2$, $R_2 = X_1 - X_2$, $S_1 = Y_1 + Y_2$, $S_2 = Y_1 - Y_2$. Then we have:

$$
\begin{aligned}
&g_1 \triangleright R_1 = -R_1, \; g_1 \triangleright R_2 = -R_2, \; g_3 \triangleright R_1 = R_1, \; g_3 \triangleright R_2 = -R_2, \\
&g_1 \triangleright S_1 = S_1, \; g_1 \triangleright S_2 = -S_2, \; g_3 \triangleright S_1 = -S_1, \; g_3 \triangleright S_2 = -S_2.
\end{aligned}
$$

Let $G_2 = \langle g_1 \rangle \times \langle g_3 \rangle$. Then $B(V_1 \oplus V_3)$ is a Nichols algebra of diagonal type in ${}^{\Bbbk G_2}_{\Bbbk G_2}\mathcal{YD}^{\Phi|_{G_2}}$. The remaining steps of the proof are Similar to that of $B(V_1 \oplus V_2)$. One can show that $B(V_1 \oplus V_3)$ is twist equivalent to an ordinary Nichols algebra of diagonal type corresponding to the same generalized Dynkin diagram of $B(V_1 \oplus V_2)$. Hence $B(V_1 \oplus V_3)$ is finite-dimensional.

Similarly, one can show that the Nichols algebra $B(V_2 \oplus V_3)$ is finite-dimensional as the previous two cases. We complete the proof of the proposition. □

For more general situation, the main challenge is the study of Nichols algebra $B(V)$ where V has at least 3 nondiagonal simple Yetter-Drinfeld summands over its support group G_V. In our opinion, this is the truly new phenomenon for pointed finite tensor categories over finite abelian groups with nonabelian 3-cocycles as associators. The main difficulty lies in a lack of tools for the investigation of nondiagonal Nichols algebras in twisted Yetter-Drinfeld categories. Moreover, it is also much more difficult to prove EGNO's conjecture in this situation. However, for the Hopf case there is already a nice theory for nondiagonal Nichols algebras developed by Andruskiewitsch, Heckenberger and Schneider in [1, 9]. We wish to extend this theory to quasi-Hopf case and pursue a complete classification of pointed finite tensor categories over abelian groups in the future works.

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