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SEMIORTHOGONAL DECOMPOSITIONS OF THE CATEGORIES OF EQUIVARIANT COHERENT SHEAVES FOR SOME REFLECTION GROUPS

ALEXANDER POLISHCHUK AND MICHEL VAN DEN BERGH

ABSTRACT. We consider the derived category $D_G^b(V)$ of coherent sheaves on a complex vector space V equivariant with respect to an action of a finite reflection group G. In some cases, including Weyl groups of type A, B, G_2, F_4, as well as the groups $G(m, 1, n)$, we construct a semiorthogonal decomposition of this category, indexed by the conjugacy classes of G. The pieces of this decompositions are equivalent to the derived categories of coherent sheaves on the quotient-spaces $V^g/C(g)$, where $C(g)$ is the centralizer subgroup of $g \in G$. In the case of the Weyl groups the construction uses some key results about the Springer correspondence, due to Lusztig, along with some formality statement generalizing a result of Deligne in [21]. We also construct global analogs of some of these semiorthogonal decompositions involving derived categories of equivariant coherent sheaves on C^n, where C is a smooth curve.

INTRODUCTION

Let k be a field of characteristic zero. Let X be a smooth quasiprojective variety over k, equipped with an action of a finite group G. Then the Hochschild homology of the category of G-equivariant coherent sheaves on X has a decomposition (see [5])

$$(0.0.1) \qquad HH_*([X/G]) \simeq \bigoplus_{g \in G/\sim} HH_*(X^g)^{C(g)},$$

where $G/\sim$ is the set of conjugacy classes of G, $C(g)$ is the centralizer of g, $X^g \subset X$ is the invariant subvariety of g.

A similar decomposition exists for the equivariant K-theory groups, tensored with $\mathbb{Q}$ (see [4], [67]), and on the level of Chow motives (see [65]). Because of this we will loosely refer to (0.0.1) as the *motivic decomposition*.

The question we would like to pose is whether some of these decompositions can be lifted to semiorthogonal decompositions of the derived category $D_G^b(X)$ of G-equivariant coherent sheaves on X (we remind what is a semiorthogonal decomposition in §1.1). Of course, one needs some restriction on the action of G on X. For example, if X is a connected projective Calabi-Yau variety and the action of the group preserves the volume form then the category $D_G^b(X)$ does not have any non-trivial semiorthogonal decompositions. The following conjecture says that for a certain class of G-actions (0.0.1) does lift to a semiorthogonal decomposition of $D_G^b(X)$.

Conjecture A. *Assume that G acts on X effectively, and all the geometric quotients $X^g/C(g)$ are smooth for $g \in G$. Then there exists a semiorthogonal decomposition of the derived category $D_G^b(X)$ of G-equivariant coherent sheaves on X such that the pieces $\mathcal{C}_{[g]}$ of this decomposition are in bijection with conjugacy classes in G and $\mathcal{C}_{[g]} \simeq D^b(X^g/C(g))$.*

One of the motivations for the above conjecture is the observation that whenever X/G is smooth, there exists a semiorthogonal decomposition of $D_G^b(X)$ with $D^b(X/G)$ as one of the pieces. Indeed, it is easy to see that the pull-back functor $D^b(X/G) \to D_G^b(X)$ is fully faithful, so this follows from the fact that $D^b(X/G)$ is saturated (see [12]).

In the case when $\dim X = 1$ the required semiorthogonal decomposition was constructed in [57] (see Theorem 6.5.2 and Remark 6.5.3). Note that already in the case $\dim X = 1$, the assumption that G acts

effectively on X is important: in the paper [9] the authors give an example of a non-effective action of a finite group on an elliptic curve for which the conclusion of Conjecture A is false.

The primary class of actions for which we propose to study semiorthogonal decompositions of $D^b_G(X)$ is the case when G is a finite complex reflection group acting linearly on the corresponding affine space (one can think of this class of actions as the "local case" of Conjecture A). However, already for these actions the quotients $X^g/C(g)$ are not always smooth. In fact, such singular quotients appear for Weyl groups of type D and E. We show that the desired semiorthogonal decomposition exists for all the other Weyl groups.

Theorem B. *Assume $k = \mathbb{C}$. Conjecture A is true for Weyl groups of types A, B, G_2, F_4, and also for the complex reflection groups $G(m, 1, n)$. More presisely, in these cases all the quotients $X^g/C(g)$ are smooth, and the required semiorthogonal decompositions exist. For type A the corresponding semiorthogonal decomposition is numbered by partitions that should be ordered compatibly with the dominance order. For types D and E there exist $g \in G$ such that $X^g/C(g)$ is singular.*

Surprizingly, we could not find a direct construction of the semiorthogonal decomposition even for type A. Rather, we use a nontrivial connection to the category of constructible sheaves on the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ in the corresponding Lie algebra provided by the Springer correspondence. Namely, we use the isomorphism

$$\mathrm{Ext}^*_{\mathbf{G}}(\mathcal{A}, \mathcal{A}) \simeq W \ltimes S(\mathfrak{t}^*),$$

where $\mathbf{G}$ is a connected reductive group, $\mathfrak{t}$ is the Lie algebra of the maximal torus, W the Weyl group, and $\mathcal{A}$ is the *Springer sheaf*, which is a certain $\mathbf{G}$-equivariant perverse sheaf on $\mathcal{N}$, playing a key role in the Springer correspondence. The above isomorphism, explicitly stated in [38], is a consequence of Lusztig's identification of the $\mathbf{G} \times \mathbb{G}_m$-equivariant Ext-algebra with the graded Hecke algebra in [52].

Together with an appropriate formality statement, this leads to an equivalence of the subcategory of $\mathbf{G}$-equivariant constructible sheaves on $\mathcal{N}$, split-generated by $\mathcal{A}$, with DG-modules over the algebra $A_W = W \ltimes S(\mathfrak{t}^*)$. This equivalence of categories was first proved in [60] in a different way. Our proof is based on lifting the Frobenius action on $\mathrm{Ext}^*_{\mathbf{G}}(\mathcal{A}, \mathcal{A})$ to a DG-enhancement of the relevant equivariant derived category of ℓ-adic sheaves. We then show that properties of such a lifted Frobenius guarantee formality of the corresponding DG-algebra. We exploit this equivalence of categories to construct the required semiorthogonal decomposition: the pieces of the semiorthogonal decomposition have a natural geometric definition in terms of constructible sheaves on $\mathcal{N}$. In fact, many of the arguments we use (and some of the semiorthogonalities) are present already in the work of Syu Kato [38] on Kostka systems.

For the types D and E the assumption of smoothness of $X^g/C(g)$ fails for some group elements. In these cases we still obtain some semiorthogonal decompositions of $D^b_W(\mathfrak{t})$, which have a smaller number of pieces than the motivic decomposition (0.0.1).

We also prove Conjecture A for some nonlinear actions. The primary example is the action of the symmetric group S_n on C^n, the nth cartesian power of a smooth curve (see Theorem 6.3.1). The semiorthogonal decomposition we construct in this case refines the one constructed by Krug in [42, Cor. B, Sec. 5.7]. Our proof reduces to the case of the linear action of S_n treated in Theorem B. The important result which allows us to pass to the global case is the identification of the equivariant cohomology of Springer fibers in terms of a certain linear arrangement, proved in [27] and [43]. Unfortunately, this description requires the assumption of surjectivity of the restriction map $H^*(X, \mathbb{C}) \to H^*(X_a, \mathbb{C})$, where $X_a \subset X$ is the embedding of the Springer fiber into the flag variety. This is the reason why we are able to globalize our construction only in type A. However, we are then able to use the construction in type A to obtain a semiorthogonal decomposition for a global analog of type B, associated with a smooth curve equipped with a non-trivial involution. This is part of a more general picture associated with an arbitrary finite group G acting effectively on a smooth curve C. Then we have the induced action of $G^n \rtimes S_n$ on C^n. Combining the above decompositions for the actions of symmetric groups with the decomposition of $D^b_G(C)$ constructed in [18], we obtain a semiorthogonal decomposition of $D^b_{G^n \rtimes S_n}(C^n)$ which is sometimes even finer than the one in Conjecture A (see Theorem 6.5.4). In the case $G = \mathbb{Z}/m$ and $C = \mathbb{A}^1$ this gives a proof of Conjecture A for the complex reflection group $G(m, 1, n)$. In particular,

for $G = \mathbb{Z}/2$ and $C = \mathbb{A}^1$ we get a semiorthogonal decomposition for the action of the Weyl group of type B, which is different from the one obtained using the Springer correspondence.

There are other cases of complex reflection groups for which the assumptions of Conjecture A are satisfied (see Proposition 2.2.6). We leave the problem of constructing semiorthogonal decompositions in these cases for a future work.

It would be also interesting to study less refined semiorthogonal decompositions of $D_G^b(X)$ in the case when some of the quotients $X^g/C(g)$ are not smooth (like in the case of Weyl groups of type D and E). One can hope that pieces of the categorical decomposition that correspond to several motivic pieces provide noncommutative resolutions of singularities of $X^g/C(g)$.

The paper is organized as follows. In Section 1 we gather some preliminaries on semiorthogonal decompositions, Hochschild (co)homology, DG-modules, and equivariant sheaves. In Section 2 we study the canonical decomposition (0.0.1) of the equivariant Hochschild homology in the case when G is a reflection group acting on a complex vector space V. We show that in the case of real reflection groups (and for some complex reflection groups), it is a decomposition into indecomposable graded $\mathcal{O}(V)^G$-modules. Sections 3 and 4 contain two technical results needed to prove the key statement that the DG-algebra of endomorphisms of the Springer sheaf is formal (see Theorem 5.2.1). In Section 3 we show how to lift the Frobenius action on certain Ext^*-algebras in finite characteristic to an action on endomorphisms in the DG-version of the (equivariant) derived category of sheaves in characteristic zero. Then in Section 4, generalizing some known results of this kind, we show that the purity of the Frobenius action on the cohomology of a DG-algebra over $\mathbb{C}$ implies its formality. In fact, we show that this statement holds for DG-algebras over operads (see Theorem 4.1.1). In Section 5 we present the construction of semiorthogonal decompositions of the equivariant derived category $D_W^b(\mathfrak{t})$, where W is a Weyl group. After reminding the geometric setup of the Springer correspondence in §5.1 we prove the equivalence of the category of DG-modules over $W \ltimes S(\mathfrak{t}^*)$ with the full subcategory in the derived category of constructible sheaves on the nilpotent cone generated by the summands of the Springer sheaf (see §5.2). The main result of this section, Theorem 5.3.3, gives a semiorthogonal decomposition of $D_W^b(\mathfrak{t})$ into subcategories indexed by nilpotent orbits and describes these subcategories as derived categories of modules over certain algebras. In §5.4 we discuss the description (in some cases) of modules generating the subcategories of our semiorthogonal decomposition, in terms of some natural linear arrangements, which follows from a similar description of the equivariant cohomology of Springer fibers in [27] and [43]. In Section 6 we consider in more detail the case of the symmetric groups and of the complex reflection groups $G(m, 1, n)$. In §6.3 we consider the global type A setting, i.e., we construct a semiorthogonal decomposition of $D_{S_n}^b(C^n)$, where C is a smooth curve (see Theorem 6.3.1). In §6.5 we consider the case when C is equipped with an additional action of a finite group G and construct a semiorthogonal decomposition of $D_{G^n \rtimes S_n}^b(C^n)$. Finally, in Section 7 we consider in detail the semiorthogonal decompositions for the Weyl groups of type B and D, constructed using the Springer correspondence.

Note that the decompositions of Theorem B are obtained from Proposition 5.3.5 (types G_2 and F_4) and Corollary 6.5.7 (types A and B and the groups $G(m, 1, n)$). For type B we also have another construction in Theorem 7.1.3.

Conventions. Starting from Sec. 5 we work over $\mathbb{C}$. If X is a scheme of finite type over $\mathbb{C}$ then we denote by $X(\mathbb{C})$ the corresponding space of $\mathbb{C}$-points with the classical topology. By a *constructible* sheaf on $X(\mathbb{C})$ we mean a sheaf of $\mathbb{C}$-vector spaces, which is constructible with respect to an algebraic stratification. Outside sections §2.1 and §6.3-6.5, an equivariant sheaf means an object of the equivariant derived category of sheaves on $X(\mathbb{C})$ (defined in [10]), whereas in the specified sections we deal with equivariant coherent sheaves. When we discuss coherent sheaves we denote by $D^b(X)$ (resp., $D_G^b(X)$) the bounded derived category of (G-equivariant) coherent sheaves on X. For a ring R we denote by $D^f(R)$ the full subcategory in the derived category of R-modules, consisting of bounded complexes of finitely generated R-modules. For a group G acting on a ring R we denote by $G \ltimes R$ the corresponding crossed product ring. For a collection of subcategories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ in a triangulated category $\mathcal{C}$ we denote by $\langle \mathcal{C}_1, \ldots, \mathcal{C}_n \rangle$ the thick subcategory of $\mathcal{C}$ generated by $\mathcal{C}_1, \ldots, \mathcal{C}_n$.

1. Preliminaries

1.1. **Semiorthogonal decompositions.** Recall that a *semiorthogonal decomposition* of a triangulated category $\mathcal{A}$ is a pair of triangulated subcategories $\mathcal{B}, \mathcal{C}$ such that $\mathrm{Hom}(\mathcal{C}, \mathcal{B}) = 0$ and every object A of $\mathcal{A}$ fits into an exact triangle

$$C \to A \to B \to C[1]$$

with $B \in \mathcal{B}$ and $C \in \mathcal{C}$. The latter condition can be replaced by the condition that $\mathcal{A}$ is classically generated by $\mathcal{B}$ and $\mathcal{C}$, i.e., $\mathcal{A}$ is the smallest thick subcategory of $\mathcal{A}$ containing $\mathcal{B}$ and $\mathcal{C}$. We can also require that $\mathcal{A}$ is the smallest triangulated subcategory of $\mathcal{A}$ containing $\mathcal{B}$ and $\mathcal{C}$. The equivalence of all these conditions follows from [11, Lem. 1.20]. We write a semiorthogonal decomposition as $\mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle$.

We say that $\mathcal{B} \subset \mathcal{A}$ is *admissible* if there are semiorthogonal decompositions of $\mathcal{A}$ of the form $\langle \mathcal{B}, \mathcal{C} \rangle$, $\langle \mathcal{C}', \mathcal{B} \rangle$. In this case

$$\mathcal{C} = {}^{\perp}\mathcal{B} := \{ X \in \mathcal{A} \mid \mathrm{Hom}(X, B) = 0 \ \text{ for all } B \in \mathcal{B} \},$$

$$\mathcal{C}' = \mathcal{B}^{\perp} := \{ Y \in \mathcal{A} \mid \mathrm{Hom}(B, Y) = 0 \ \text{ for all } B \in \mathcal{B} \}.$$

The unique projection functor $\mathcal{A} \to {}^{\perp}\mathcal{B}$ (resp., $\mathcal{A} \to \mathcal{B}^{\perp}$) sending $\mathcal{B}$ to 0, is called *left (resp., right) orthogonalization with respect to $\mathcal{B}$.*

Let R be a commutative noetherian ring. We will say that $\mathcal{A}$ is R-finite if $\mathcal{A}$ is R-linear and for any two objects $A, B \in \mathcal{A}$ one has that $\bigoplus_i \mathrm{Hom}(A, B[i])$ is a finitely generated R-module.

We will use the following criterion for "intrinsic admissibility".

Lemma 1.1.1. *Assume that $\mathcal{B}$ is an R-finite enhanced triangulated category generated by objects $(M)_{1,\dots,n} \in \mathcal{B}$ such that $\mathrm{Hom}_{\mathcal{B}}(M_i, M_i[n]) = 0$ for $n \neq 0$, $\mathrm{Hom}_{\mathcal{B}}(M_i, M_j[n]) = 0$ for $i < j$, $n \in \mathbb{Z}$ and $\mathrm{End}_{\mathcal{B}}(M_i)$ has finite global dimension for all i. Assume we have a full inclusion $\mathcal{B} \subset \mathcal{A}$ of R-finite enhanced triangulated categories. Then $\mathcal{B}$ is admissible inside $\mathcal{A}$.*

Proof. The enhancement allows us to talk about RHom. Set $S = \mathrm{End}_{\mathcal{B}}(M_1)$. Let A be an object in $\mathcal{A}$. Then $\mathrm{RHom}_{\mathcal{A}}(M_1, A)$ has finitely generated cohomology over R, and hence it is a perfect complex of S-modules. Consider the following distinguished triangle

$$\mathrm{RHom}(M_1, A) \overset{L}{\otimes}_S M_1 \xrightarrow{\phi} A \to A' \to \dots$$

where ϕ is obtained in the usual way from the enhancement. We find

$$\mathrm{RHom}_{\mathcal{A}}(M_1, A') = 0,$$

and so we have a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{B}_1^{\perp}, \mathcal{B}_1 \rangle$, where $\mathcal{B}_1$ is generated by M_1 and $M_2, \dots, M_n \in \mathcal{B}_1^{\perp}$. Continuing in this way, we find a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{B}^{\perp}, \mathcal{B} \rangle$.

By considering the opposite categories $\mathcal{B}^{\circ} \subset \mathcal{A}^{\circ}$ we obtain a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{B}, {}^{\perp}\mathcal{B} \rangle$. $\qquad \square$

1.2. **Hochschild homology and cohomology.** For a DG-category $\mathcal{C}$ over a field k we denote by $HH_*(\mathcal{C})$ and $HH^*(\mathcal{C})$ the Hochschild homology and cohomology, respectively.

Recall that the Hochschild homology is functorial: a DG-functor $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ induces a map of Hochschild homology

$$\Phi_* : HH_*(\mathcal{C}_1) \to HH_*(\mathcal{C}_2).$$

On the other hand, if $\Phi : \mathcal{A} \subset \mathcal{C}$ is a DG-functor which is fully faithful on the cohomology level then there is a restriction homomorphism $\Phi^* : HH^*(\mathcal{C}) \to HH^*(\mathcal{A})$ (since in this case the diagonal bimodule for $\mathcal{C}$ restricts to the diagonal bimodule for $\mathcal{A}$). These two operations are compatible via the following identity:

$$(1.2.1) \qquad\qquad x \cap \Phi_*(y) = \Phi_*(\Phi^*(x) \cap y),$$

where $x \in HH^*(\mathcal{C})$, $y \in HH_*(\mathcal{A})$ and $\cap$ denotes the natural action of Hochschild cohomology on Hochschild homology.

Another result we need is that the Hochschild homology is additive with respect to semiorthogonal decompositions. Namely, if $\mathcal{C}$ is a DG-enhancement of a triangulated category and a pair of DG-subcategories $\mathcal{A}$, $\mathcal{B}$ in $\mathcal{C}$ induces a semiorthogonal decomposition $H^0\mathcal{C} = \langle H^0\mathcal{A}, H^0\mathcal{B} \rangle$ then the inclusion functors $\mathcal{A} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{C}$ induce a direct sum decomposition

$$(1.2.2) \qquad\qquad HH_*(\mathcal{C}) \simeq HH_*(\mathcal{A}) \oplus HH_*(\mathcal{B})$$

(see [45]). Together with the compatibility (1.2.1) this leads to the following result.

Lemma 1.2.1. *Let $\mathcal{C}$ be a DG-category admitting a semiorthogonal decomposition into the subcategories $\mathcal{C}_1, \ldots, \mathcal{C}_n$. Then there is a decomposition of $HH^*(\mathcal{C})$-modules*

$$HH_*(\mathcal{C}) \simeq \bigoplus_{i=1}^{n} HH_*(\mathcal{C}_i),$$

where the module structure on $HH_*(\mathcal{C}_i)$ is induced by the homomorphism $HH^*(\mathcal{C}) \to HH^*(\mathcal{C}_i)$.

1.3. **Graded modules versus DG-modules.** For an abelian category $\mathcal{A}$ let $\mathrm{Gr}(\mathcal{A})$ be the category of $\mathbb{Z}$-graded objects over $\mathcal{A}$ (with maps of degree 0), and let $C(\mathcal{A})$ be the corresponding category of complexes. The shift functor on $\mathrm{Gr}(\mathcal{A})$ is written as (?) and on $C(\mathcal{A})$ as [?]. Let Ab be the category of abelian groups. It is clear that the functor

$$(1.3.1) \qquad\qquad F : C(\mathrm{Gr}(\mathrm{Ab})) \to C(\mathrm{Ab}) : M \mapsto \bigoplus_i M_i[-i]$$

is compatible with quasi-isomorphisms, homological shifts and cones. Here M_i is the complex of abelian groups obtained by restricting M to grading degree i.

Let A be a $\mathbb{Z}$-graded ring. Let A^{dg} be the DG-algebra which is equal to A as graded algebra and which has zero differential. Similarly, if M is a graded A-module let M^{dg} be the corresponding DG-module over A^{dg} with zero differential.

We denote by $A-\mathrm{gr}$ and $A^{\mathrm{dg}}-\mathrm{mod}$ the categories of graded and DG-modules over A, respectively, and by $D(A-\mathrm{gr})$ and $D(A^{\mathrm{dg}})$ the corresponding derived categories.

The functor F from (1.3.1) specializes to a functor

$$F : C(A-\mathrm{gr}) \to A^{\mathrm{dg}}-\mathrm{mod} : M \mapsto \bigoplus_i M_i[-i]$$

still compatible with quasi-isomorphisms, shifts and cones since these properties do not depend on the A-structure. So one obtains an induced exact functor of triangulated categories

$$F : D(A-\mathrm{gr}) \to D(A^{\mathrm{dg}}).$$

This functor has the following properties
- $F(M) = M^{\mathrm{dg}}$ for $M \in A-\mathrm{gr}$;
- $F(A(i)[j]) = A^{\mathrm{dg}}[i+j]$, thus F preserves perfect complexes; and
- $F(M(i)[-i]) = FM$ for $M \in D(A-\mathrm{gr})$.

The last property suggests that $D(A^{\mathrm{dg}})$ is some kind of orbit category over $D(A-\mathrm{gr})$. This is indeed the case and follows from [41, §9.3]. For the benefit of the reader we provide a proof in our special case.

If $M, N \in D(A-\mathrm{gr})$ then the canonical morphisms in $C(\mathrm{Ab})$

$$F_n : \mathrm{RHom}_{A-\mathrm{gr}}(M, N(n))[-n] \to \mathrm{RHom}_{A^{\mathrm{dg}}}(FM, F(N(n)[-n])) = \mathrm{RHom}_{A^{\mathrm{dg}}}(FM, FN)$$

can be combined to a morphism

$$\tilde{F} : \bigoplus_n \mathrm{RHom}_{A-\mathrm{gr}}(M, N(n))[-n] \to \mathrm{RHom}_{A^{\mathrm{dg}}}(FM, FN)$$

Theorem 1.3.1. *Assume that M is perfect. Then $\tilde{F}$ is a quasi-isomorphism.*

Proof. By the usual argument involving the 5-lemma it is sufficient to prove this for $M = A(i)[j]$. Let N be a complex over $A-\mathrm{gr}$. Then we have

$$(1.3.2) \qquad \bigoplus_n \mathrm{RHom}_{A-\mathrm{gr}}(A(i)[j], N(n)[-n]) = \bigoplus_n N_{n-i}[-n-j]$$

Similarly

$$\mathrm{RHom}_{A^{\mathrm{dg}}}(FM, FN) = \mathrm{RHom}_{A^{\mathrm{dg}}}(A[i+j], FN) = (FN)[-i-j] = \bigoplus_m N_m[-m-i-j]$$

which is a reindexing of (1.3.2). $\qquad\square$

Corollary 1.3.2. *Assume that M is a perfect object in $D(A-\mathrm{gr})$. If FM generates $D(A^{\mathrm{dg}})$ then $M(i)_{i\in\mathbb{Z}}$ generates $D(A-\mathrm{gr})$ and M generates $D(A)$, the derived category of (ungraded) A-modules.*

Proof. Assume that $M(i)_{i\in\mathbb{Z}}$ does not generate $D(A-\mathrm{gr})$. Then there exists N in $D(A-\mathrm{gr})$ such that

$$\mathrm{RHom}_{A-\mathrm{gr}}(M(i), N) = 0$$

for all i. By (1.3.3) we conclude $\mathrm{RHom}_{A^{\mathrm{dg}}}(FM, FN) = 0$ and hence $FN = 0$. Then clearly $N = 0$.

Hence, $M(i)_{i\in\mathbb{Z}}$ generate $D(A-\mathrm{gr})$. The fact that M generates $D(A)$ follows from the fact that the essential image of $D(A-\mathrm{gr})$ generates $D(A)$. Indeed, the latter obviously contains the generator A of $D(A)$. $\qquad\square$

Theorem 1.3.3. *Let $M, N \in D(A-\mathrm{gr})$ and assume that M is perfect. Then*

$$(1.3.3) \qquad \bigoplus_n \mathrm{Ext}_A^{i-n}(M, N)_n = \mathrm{Ext}_{A^{\mathrm{dg}}}^i(FM, FN)$$

where the Ext on the left is for ordinary ungraded A-modules.

Proof. This follows by considering $H^i(\tilde{F})$ and invoking Theorem 1.3.1. $\qquad\square$

Below we will also need the following result similar to Corollary 1.3.2.

Lemma 1.3.4. *Let $M_1, \ldots, M_n$ be perfect objects in $D(A-\mathrm{gr})$ such that the corresponding ungraded objects $\overline{M}_1, \ldots, \overline{M}_n$ in $D(A)$ satisfy the assumptions of Lemma 1.1.1. Assume also that the objects $FM_1, \ldots, FM_n$ generate an admissible subcategory $\langle FM_1, \ldots, FM_n \rangle \subset D(A^{\mathrm{dg}})$. Assume now that for some $N \in D(A-\mathrm{gr})$ we have $FN \in \langle FM_1, \ldots, FM_n \rangle$. Then N, viewed as an object of $D(A)$, belongs to the subcategory $\langle M_1, \ldots, M_n \rangle \subset D(A)$.*

Proof. First, repeating the steps of the proof of Lemma 1.1.1 and observing that the corresponding Hom-complexes will be graded we can get an exact triangle in $D(A-\mathrm{gr})$

$$N' \to N \to C \to \ldots$$

with N' in the subcategory generated by $M_i(j)$ and C such that $\mathrm{Ext}_A^*(M_i, C) = 0$ for $i = 1, \ldots, n$. By Theorem 1.3.3, this implies that FC lies in the right orthogonal of $\langle FM_1, \ldots, FM_n \rangle$ in $D(A^{\mathrm{dg}})$. Applying F to the above exact triangle and using the fact that $FN \in \langle FM_1, \ldots, FM_n \rangle$, we derive that $FC = 0$, hence $C = 0$. $\qquad\square$

1.4. **Equivariant sheaves and equivariant cohomology.** We refer to [10] for general facts on equivariant derived categories.

For a G-equivariant sheaf (or complex of sheaves) F on X we denote by $R\Gamma_G(X, F)$ the equivariant push-forward of F to the point, which is an object of the derived category $D_G(pt)$ of G-equivariant sheaves on the point. The equivariant cohomology is obtained by passing to the cohomology of this object:

$$H_G^*(X, F) = H^* R\Gamma_G(X, F).$$

Similarly, for $F, F' \in D_G(X)$ we denote

$$\mathrm{RHom}_G(F, F') := R\Gamma_G(X, R\underline{\mathrm{Hom}}(F, F')),$$

$$\mathrm{Ext}_G^*(F, F') = H^* \mathrm{RHom}_G(F, F').$$

Lemma 1.4.1. *Let $H \subset G$ be a normal subgroup of finite index, and let X be a G-space.*
(i) For any G-equivariant sheaves $\mathcal{F}$ and $\mathcal{G}$ on X one has a natural isomorphism

$$\mathrm{RHom}_G(\mathcal{F}, \mathcal{G}) \simeq \mathrm{RHom}_H(\mathcal{F}, \mathcal{G})^{G/H}.$$

(ii) For any G/H-representation V we have a natural isomorphism

$$R\Gamma_G(X, V^\vee \otimes \underline{\mathbb{C}}) \simeq \mathrm{Hom}_{G/H}(V, R\Gamma_H(X, \underline{\mathbb{C}})).$$

This isomorphism is compatible with the action of $R\Gamma_G(pt) = R\Gamma_H(pt)^{G/H}$.

Proof. (i) Consider the G/H-torsor $\rho : EG \times_H X \to EG \times_G X$. We have a natural isomorphism

$$\rho_* R\underline{\mathrm{Hom}}(\mathcal{F}_H, \mathcal{G}_H)^{G/H} \simeq R\underline{\mathrm{Hom}}(\mathcal{F}_G, \mathcal{G}_G),$$

where $\mathcal{F}_H$ (resp., $\mathcal{F}_G$) is the sheaf on $EG \times_H X$ (resp., $EG \times_G X$) obtained from $\mathcal{F}$ by descent. Applying the functor $R\Gamma$ to both sides gives the required isomorphism.
(ii) This is a particular case of (i) for $\mathcal{F} = \underline{\mathbb{C}}_X$ and $\mathcal{G} = V^\vee \otimes \underline{\mathbb{C}}_X$. $\qquad\qquad\square$

Let $\mathbf{G}$ be a connected algebraic group. As in [26] (where the case of torus actions is considered) we say that a $\mathbf{G}$-space X is *equivariantly formal* if the Leray-Serre spectral sequence

$$E_2^{pq} = H^p(B\mathbf{G}, H^q(X, \mathbb{C})) \implies H_{\mathbf{G}}^{p+q}(X)$$

associated with the fibration $X \times_{\mathbf{G}} E\mathbf{G} \to B\mathbf{G}$ collapses. In this case one has a (non-canonical) isomorphism of $H_{\mathbf{G}}^*(pt)$-modules

$$H_{\mathbf{G}}^*(X, \mathbb{C}) \simeq H_{\mathbf{G}}^*(pt) \otimes H^*(X, \mathbb{C}).$$

For example, if $H^i(X, \mathbb{C})$ vanishes for all odd i then X is equivariantly formal.

Lemma 1.4.2. *Let $\mathbf{G}$ be a connected algebraic group, and let X be a $\mathbf{G}$-space. Assume that $H^i(X, \mathbb{C})$ vanishes for all odd i. Then one has a natural isomorphism*

$$H_{\mathbf{G}}^*(X, \mathbb{C}) \simeq H_{\mathbf{T}}^*(X, \mathbb{C})^W,$$

where $\mathbf{T} \subset \mathbf{G}$ is a maximal torus, $W = N(\mathbf{T})/\mathbf{T}$ is the corresponding Weyl group.

Proof. The action of $N(\mathbf{T})$ on the pair $(\mathbf{T}, X)$ induces an action of W on $H_{\mathbf{T}}^*(X, \mathbb{C})$ and on the corresponding Leray-Serre spectral sequence. Hence, from $\mathbf{T}$-equivariant formality we get an isomorphism

$$H_{\mathbf{T}}^*(X, \mathbb{C})^W \simeq (H_{\mathbf{T}}^*(pt) \otimes H^*(X, \mathbb{C}))^W \simeq H_{\mathbf{T}}^*(pt) \otimes H^*(X, \mathbb{C}),$$

where we used the fact that $N(\mathbf{T})$ acts trivially on $H^*(X, \mathbb{C})$ (since the action of $N(\mathbf{T})$ extends to an action of $\mathbf{G}$). On the other hand, by $\mathbf{G}$-equivariant formality we have an isomorphism

$$H_{\mathbf{G}}^*(X, \mathbb{C}) \simeq H_{\mathbf{G}}^*(pt) \otimes H^*(X, \mathbb{C}).$$

Since the natural morphism $H_{\mathbf{G}}^*(X, \mathbb{C}) \to H_{\mathbf{T}}^*(X, \mathbb{C})$ extends to a morphism of Leray-Serre spectral sequences, the required isomorphism follows from the standard isomorphism

$$H_{\mathbf{G}}^*(pt) \simeq H_{\mathbf{T}}^*(pt)^W.$$

$\qquad\qquad\square$

We also need to recall the equivariant version of Verdier duality following [10, Sec. 3]. For a linear algebraic group $\mathbf{G}$ acting on a variety X the Verdier duality is a contravariant functor D from the derived category $D_{\mathbf{G},c}(X)$ of equivariant constructible shaeves on X to itself defined by

$$D(F) = \mathrm{RHom}(F, D_X),$$

where $D_X = p^!\mathbb{C}$, where $p : X \to pt$ is the projection to the point. It satisfies the usual properties (see [10, Sec. 3.5, 3.6]). In particular, $D^2 = \mathrm{id}$, and for proper X and for $F \in D_{\mathbf{G},c}(X)$ we have an isomorphism

$$R\Gamma(X, D(F)) \simeq D(R\Gamma(X, F))$$

in the category $D_{\mathbf{G}}(pt)$. Consider the graded algebra $A = H_{\mathbf{G}}^*(pt)$, and let $D(dg - A - \mathrm{mod})$ denote the derived category of DG-modules over A, viewed as a DG-algebra with zero differential. Let $D^f(dg - A - \mathrm{mod}) \subset D(dg - A - \mathrm{mod})$ be the full subcategory of bounded complexes of finitely generated A-DG-modules. For connected $\mathbf{G}$, one has an equivalence

$$D^f(dg - A - \mathrm{mod}) \simeq D_{\mathbf{G},c}(pt)$$

such that the Verdier duality functor D corresponds to the duality $M \mapsto \mathrm{RHom}(M, A)$ on DG-modules over A (see [10, Thm. 12.7.2]).

2. Hochschild homology for some actions of finite groups

2.1. Canonical decomposition of the Hochschild homology.
In this section we work over a field k of characteristic zero.

Lemma 2.1.1. *Let X be a smooth quasiprojective variety with an action of a finite group G. Then there is a decomposition of $\mathcal{O}(X)^G$-modules*

$$(2.1.1) \qquad HH_*([X/G]) \simeq \bigoplus_{g \in G/\sim} HH_*(X^g)^{C(g)},$$

where $X^g \subset X$ is the fixed locus of $g \in G$, the $\mathcal{O}(X)^G$-module structure on the right is induced by the natural homomorphisms $\mathcal{O}(X)^G \to \mathcal{O}(X^g)^{C(g)}$. Here $HH_*([X/G])$ is the Hochschild homology of the category of G-equivariant perfect complexes on X. If in addition, X has a $\mathbb{G}_m$-action commuting with the action of G, such that X admits a $\mathbb{G}_m$-equivariant ample line bundle, then the relevant Hochschild homology groups get equipped with a natural additional grading, and (2.1.1) is a decomposition of graded $\mathcal{O}(X)^G$-modules.

Proof. The isomorphism (2.1.1) follows from [5, Prop. 4] which gives a quasi-isomorphism of the corresponding mixed complexes. More precisely, there is a quasi-isomorphism of mixed complexes of pairs

$$(2.1.2) \qquad C(\mathcal{C}_{ac}^b(X) \rtimes G, \mathcal{C}^b(X) \rtimes G) \to \left(\bigoplus_{g \in G} C(\mathcal{C}_{ac}^b(X^g), \mathcal{C}^b(X^g)) \right)_G,$$

where $\mathcal{C}^b(X)$ denotes the exact category of bounded complexes of vector bundles, $\mathcal{C}_{ac}^b(X)$ the subcategory of acyclic complexes, and for a category $\mathcal{C}$ with a G-action the category $\mathcal{C} \rtimes G$ is the full subcategory in $\mathcal{C}\#G$ consisting of objects $\bigoplus_{g \in G} g^*(O)$ with $O \in \mathcal{C}$. The mixed complex on the left of (2.1.2) also maps quasi-isomorphically to the one computing Hochschild homology of $[X/G]$. Assume now that X has an additional $\mathbb{G}_m$-action, such that there exists a $\mathbb{G}_m$-equivariant ample line bundle L on X. Now let $\mathcal{C}(L) \subset \mathcal{C}^b(X)$ be the full subcategory consisting of complexes with terms that are finite direct sums of line bundles L^n, $n \in \mathbb{Z}$, and let $\mathcal{C}_{ac}(L) \subset \mathcal{C}(L)$ be the subcategory of acyclic complexes. Then the mixed complex of the pair $(\mathcal{C}_{ac}(L), \mathcal{C}(L))$ has a natural grading coming from the $\mathbb{G}_m$-action, and the embedding functor $\mathcal{C}(L) \to \mathcal{C}^b(X)$ induces an equivalence of the corresponding derived categories (see e.g., [55, Thm. 4]). By [40, Thm. 2.4(b)], this implies that the morphism of mixed complexes

$$C(\mathcal{C}_{ac}(L) \rtimes G, \mathcal{C}(L) \rtimes G) \to C(\mathcal{C}_{ac}^b(X) \rtimes G, \mathcal{C}^b(X) \rtimes G)$$

is a quasi-isomorphism. The similar assertion holds for non-equivariant categories associated with X^g and the restriction $L|_{X^g}$. It remains to observe that the map (2.1.2) induces a similar morphism

$$C(\mathcal{C}_{ac}(L) \rtimes G, \mathcal{C}(L) \rtimes G) \to \left(\bigoplus_{g \in G} C(\mathcal{C}_{ac}(L|_{X^g}), \mathcal{C}(L|_{X^g})) \right)_G,$$

which is compatible with the gradings. $\qquad \square$

Proposition 2.1.2. *Assume that X is a smooth quasiprojective variety with an action of a finite group G, such that the quotient X/G is smooth. Then there is a natural isomorphism*

$$HH_*(X/G) \simeq HH_*(X)^G.$$

Proof. Let $\underline{HH}(X) := Rp_{1*}(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_\Delta)$ be the sheafified Hochschild homology, so that $HH_*(X) = R\Gamma(X, \underline{HH}(X))$. Then we have the Kostant-Hochschild-Rosenberg isomorphism in $D(X)$,

$$\underline{HH}(X) \simeq \bigoplus_i \Omega_X^i[i]$$

(see [53, Sec. 1.4]), and a similar isomorphism for $Y := X/G$. Let $\pi : X \to Y$ be the projection map. We claim that for each i the natural map induced by the pull-back,

$$\Omega_Y^i \to (\pi_* \Omega_X^i)^G,$$

is an isomorphism. Indeed, for affine X this is Brion's theorem in [13]. The general case follows since we can cover X by G-invariant affine open sets. Thus, we get an isomorphism

$$\underline{HH}(Y) \simeq (\pi_* \underline{HH}(X))^G,$$

and the result follows by applying the functor $R\Gamma(Y, \cdot)$ to both sides. $\qquad \square$

Thus, if for every $g \in G$ the variety $X^g/C(g)$ is smooth then the decomposition (2.1.1) can be rewritten as

$$HH_*(X/G) \simeq \bigoplus_{g \in G/\sim} HH_*(X^g/C(g)),$$

which looks like the decomposition associated with a semiorthogonal decomposition (see (1.2.2)). Below we will show that in some cases we can match this decomposition with the one obtained from a semiorthogonal decompositon of $D_G^b(X)$ (see Proposition 2.1.5 below).

We are interested in the case when a finite group G acts linearly on a vector space V. For $g \in G$ we denote by $C(g) \subset G$ the centralizer of g, and by $V^g \subset V$ the subspace of g-invariant vectors. Let us consider the following condition on such an action:

(⋆) **For every $g \in G$ the natural map $V^g/C(g) \to V/G$ is birational onto its image.**

Note that if R is a finitely generated commutative graded k-algebra then the category of finitely generated graded R-modules has finite-dimensional Hom-spaces and hence is *Krull-Schmidt*, i.e., every object has a direct sum decomposition into indecomposable objects, which are uniquely defined up to permutation, and the endomorphism ring of every indecomposable object is local (see [3]). This in particular applies to the category of finitely generated graded $\mathcal{O}(V)^G$-modules.

Proposition 2.1.3. *Assume that a finite group G acts linearly on a vector space V, satisfying the condition (⋆). The decomposition of graded $\mathcal{O}(V)^G$-modules*

(2.1.3) $$HH_0([V/G]) \simeq \bigoplus_{g \in G/\sim} \mathcal{O}(V^g))^{C(g)},$$

obtained from (2.1.1), is the (unique) decomposition into indecomposable graded $\mathcal{O}(V)^G$-modules.

Proof. We have to check that $\mathcal{O}(V^g)^{C(g)}$ is indecomposable as a graded $\mathcal{O}(V)^G$-module. By Lemma 2.1.4(i) below (applied to A being the image of the homomorphism $\mathcal{O}(V)^G \to \mathcal{O}(V^g)^{C(g)}$ and $B = \mathcal{O}(V^g)^{C(g)}$), we obtain

$$\mathrm{End}_{\mathcal{O}(V)^G}(\mathcal{O}(V^g)^{C(g)}) \simeq \mathcal{O}(V^g)^{C(g)}.$$

It follows that endomorphisms of $\mathcal{O}(V^g)^{C(g)}$ as a *graded* $\mathcal{O}(V)^G$-module reduce to k. This shows that $\mathcal{O}(V^g)^{C(g)}$ is indecomposable. The uniqueness follows from the Krull-Schmidt property. $\square$

Lemma 2.1.4. *(i) Let $A \to B$ be an embedding of commutative domains with the same fraction field. Then the natural map $B \to \mathrm{End}_A(B)$ is an isomorphism.*
(ii) For $A \to B_1$, $A \to B_2$ as in (i), if B_1 and B_2 are isomorphic as A-modules, then they are isomorphic as A-algebras.

Proof. (i) Let K be the common fraction field of A and of B. Then $B \otimes_A K = K$, so any endomorphism $B \to B$ of A-modules is induced by some K-linear map $K \to K$. Such map is a multiplication by $x \in K$. Since it sends 1 to an element of B, the assertion follows.
(ii) This follows immediately from (i). $\square$

Proposition 2.1.5. *Assume that a finite group G acts linearly on a vector space V, satisfying the condition $(\star)$. Suppose the derived category $D_G^b(V)$ of equivariant coherent sheaves on V has a semiorthogonal decomposition with the pieces $\mathbb{C}_i \simeq D^b(X_i)$, $i = 1, \ldots, r$, where X_i are smooth affine $\mathbb{G}_m$-varieties over V/G, and $r \geq c$, where c is the number of conjugacy classes in G (we are assume that the equivalences $\mathbb{C}_i \simeq D^b(X_i)$ are compatible with the $D^b(V/G)$-module structures). Then $r = c$, and there exists an ordering of the conjugacy classes in G, such that if $g_1, \ldots, g_r$ are representatives, then for each i we have an isomorphism $X_i \simeq V^{g_i}/C(g_i)$ over V/G. In particular, in this case all the varieties $V^{g_i}/C(g_i)$ are smooth.*

Proof. From such a semiorthogonal decomposition we get a decomposition of the Hochschild homology,

$$HH_0(V/G) = \bigoplus_{i=1}^r HH_0(X_i) = \bigoplus_{i=1}^r \mathcal{O}(X_i),$$

compatible with the $\mathcal{O}(V/G)$-action and with the grading (see Lemma 1.2.1). Now the Krull-Schmidt property and Proposition 2.1.3 implies that this decomposition should match (2.1.3) up to permutation. $\square$

2.2. The case of reflection groups.
In this section we reformulate propery $(\star)$ in a more convenient form and check it for some reflection groups.

Lemma 2.2.1. *For an element $g \in G$ let $H_g \subset G$ be the pointwise stabilizer of V^g. Then the following conditions are equivalent:*
(i) the map $V^g/C(g) \to V/G$ is birational onto its image;
(ii) $N_G(H_g) = C(g)H_g$;
(iii) for any $x \in N_G(H_g)$, the elements g and xgx^{-1} are conjugate in H_g.

Proof. Let us consider the following nonempty Zariski open subset in V^g:

$$U := V^g \setminus \cup_{g' \in G \setminus H_g} V^g \cap V^{g'}.$$

Then by definition, for every $v \in U$ the stabilizer subgroup of v is exactly H_g. Now suppose we have $v \in U$ and $x \in G$ such that $xv \in U$. Then the stabilizer of xv is xH_gx^{-1}, hence, $xH_gx^{-1} = H_g$. In particular, this shows the inclusion $C(g) \subset N_G(H_g)$.

Now let us show the equivalence of (i) and (ii). Using the above observation, we see that the map $V^g/C(g) \to V/G$ is birational onto its image if and only if for generic $v \in V^g$ and $x \in N_G(H_g)$, the points v and xv lie in the same $C(g)$-orbit. Since the stabilizer of such v is H_g, this is equivalent to $x \in C(g)H_g$.

On the other hand, the equivalence of (ii) and (iii) is clear: for $x \in N_G(H_g)$ the elements g and xgx^{-1} are conjugate in H_g if and only if $x \in H_g C(g) = C(g)H_g$. $\qquad \square$

The proof ot the following result was explained to us by Victor Ostrik.

Proposition 2.2.2. *Let W be a finite subgroup generated by reflections of a real vector space $V_{\mathbb{R}}$. Then the property ($\star$) holds for the action of W on $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$.*

Proof. For every $g \in G$ the subgroup $H_g \subset W$ is parabolic (i.e., conjugate to a standard subgroup $W_J \subset W$, where J is a subset of a simple reflections), and g is not contained in any smaller parabolic subgroup. In other words, the conjugacy class of g is cuspidal in H_g (see [24, Sec. 3.1]). Thus, the condition (iii) of Lemma 2.2.1 follows from [24, Thm. 3.2.11]. $\qquad \square$

Next, we are going to check the propery ($\star$) for some complex reflection groups.

Lemma 2.2.3. *Assume that a finite group G acts linearly on a complex vector space V over a field k, and for an element g the invariant subspace $V^g \subset V$ has codimension 1. Then the equivalent conditions of Lemma 2.2.1 hold for g.*

Proof. Note that the determinant homomorphism $\det : G \to k^*$ induces an embedding $H_g \hookrightarrow k^*$. Indeed, an element of H_g with trivial determinant is unipotent (since it fixes the hyperplane V^g) and of finite order, hence, it is trivial. Using this we can check the condition (iii) of Lemma 2.2.1: since $\det(xgx^{-1}) = \det(g)$, we get that for $x \in N_G(H_g)$ one has $xgx^{-1} = g$. $\qquad \square$

Recall that $G(m, k, n)$ for $k|m$ is the subgroup of $G(m, 1, n) := (\mu_m)^n \rtimes S_n$ formed by elements $(z_1, \ldots, z_n; \sigma)$, where z_i are mth roots of unity in $\mathbb{C}^*$ such that $(z_1 \ldots z_n)^{m/k} = 1$.

Proposition 2.2.4. *Property ($\star$) holds for*
(i) finite complex reflection groups of rank 2;
(ii) the groups $G(m, k, n)$ such that either m is a prime number or $k = 1$.

Proof. (i) The assertion of ($\star$) is clear if $V^g = 0$. In the case when V^g is 1-dimensional it holds by Lemma 2.2.3.
(ii) For $(z; \sigma) = (z_1, \ldots, z_n; \sigma) \in (\mu_m)^n \rtimes S_n$ let $(C_1, \ldots, C_s)$ be the orbits of σ on $\{1, \ldots, n\}$ (i.e., cycles in σ). Let us associate with every cycle C_j the element

$$(2.2.1) \qquad\qquad z(C_j) := \prod_{i \in C_j} z_i \in \mu_m.$$

It is easy to see that the map

$$(z; \sigma) \mapsto (|C_1|, z(C_1)), \ldots, (|C_s|, z(C_s))$$

gives a bijection between conjugacy classes in $(\mu_m)^n \rtimes S_n$ and colored partitions of n, i.e., partitions of n with an additional assignment of a color for each part, where the color is an element of μ_m. Given an element $(z; \sigma) \in G(m, k, n)$, let us order the orbits of σ so that $z(C_1) = \ldots = z(C_r) = 1$ and $z(C_j) \neq 1$ for $j > r$. Let us set

$$\Sigma = \cup_{j=1}^r C_j.$$

For notational convenience let us rename σ to σ^{-1}, so that our element is $(z; \sigma^{-1})$. Then the fixed subspace of $(z; \sigma^{-1})$ is

$$V^{(z; \sigma^{-1})} = \{(x_i) \in V \mid z_i x_{\sigma(i)} = x_i \ \text{ for } i \in C_j, j \le r; \ x_i = 0 \ \text{ for } i \notin \Sigma\}.$$

Let $U \subset V^{(z; \sigma^{-1})}$ be the open subset of points x such that $x_i \neq 0$ for $i \in \Sigma$ and $x_i^m \neq x_{i'}^m$ for $i \in C_j$, $i' \in C_{j'}$, with $j, j' \le r$, $j \neq j'$. It is enough to check that if $(t; \tau^{-1}) \cdot x = y$, where $x, y \in U$ and $(t; \tau^{-1}) \in G(m, k, n)$ then x and y belong to the same orbit of the centralizer of $(z; \sigma^{-1})$. First, we claim that τ induces a permutation $\overline{\tau}$ of $\{1, \ldots, r\}$ such that $\tau(C_j) = C_{\overline{\tau}(j)}$ for $j = 1, \ldots, r$. Indeed, let $i \in C_j$, where $j \le r$. Then using the equations $(t; \tau^{-1})x = y$ and $(z, \sigma^{-1})y = y$ we get

$$(2.2.2) \qquad\qquad t_i x_{\tau(i)} = y_i = z_i y_{\sigma(i)} = z_i t_{\sigma(i)} x_{\tau\sigma(i)},$$

hence $x_{\tau(i)} \neq 0$ and $x^m_{\tau(i)} = x^m_{\tau\sigma(i)}$. Since $x \in U$ this implies that $\tau(i)$ and $\tau\sigma(i)$ belong to the same subset Σ_k for some $k \leq r$. Thus, τ permutes the subsets $C_1, \ldots, C_r$, as we claimed. It follows that there exists a permutation τ_1, commuting with σ, acting trivially on the complement to Σ and such that $\tau_1(C_j) = \tau(C_j)$ for every $j \leq r$. Now let us distinguish two cases.

Case 1. $\Sigma = \{1, \ldots, n\}$. Then let us set

$$p_i = \frac{t_i x_{\tau(i)}}{x_{\tau_1}(i)}, \quad p = (p_1, \ldots, p_n) \in (\mathbb{C}^*)^n.$$

We claim that the element $(p; \tau_1^{-1}) \in (\mathbb{C}^*)^n \rtimes S_n$ belongs to $G(m, k, n)$, commutes with $(z; \sigma^{-1})$ and sends x to y. Indeed, since $x^m_{\tau(i)} = x^m_{\tau_1(i)}$ by the choice of τ_1, we have $p^m_i = 1$. Next $\prod_i p_i = \prod_i t_i$, so $(p; \tau_1^{-1})$ is in $G(m, k, n)$. The equation $(p; \tau_1^{-1})x = y = (t, \tau^{-1})x$ is equivalent to

$$p_i x_{\tau_1(i)} = t_i x_{\tau(i)}.$$

which holds by our choice of p_i. Finally, the fact that $(p; \tau_1^{-1})$ commutes with $(z; \sigma^{-1})$ is equivalent to the equations

$$\frac{p_{\sigma(i)}}{p_i} = \frac{z_{\tau_1(i)}}{z_i}.$$

By definition of p_i the left-hand side is

$$\frac{t_{\sigma(i)} x_{\tau\sigma(i)} x_{\tau_1(i)}}{t_i x_{\tau_1\sigma(i)} x_{\tau(i)}} = \frac{x_{\tau_1(i)}}{x_{\tau_1\sigma(i)}} = \frac{x_{\tau_1(i)}}{z_i x_{\sigma\tau_1(i)}} = \frac{z_{\tau_1(i)}}{z_i},$$

where in the first equality we used (2.2.2), and the last equality follows from $(z; \sigma^{-1})$-invariance of x.

Case 2. $\Sigma \neq \{1, \ldots, n\}$. In this case replacing $(z; \sigma^{-1})$ by a conjugate element in $G(m, k, n)$ we can assume that $z_i = 1$ for $i \in \Sigma$ (indeed, it is enough to use the conjugation by elements of $(\mu_m)^n \cap G(m, m, n)$). Then τ_1, viewed as an element of $G(m, k, n)$, commutes with $(z; \sigma^{-1})$. Thus, replacing x by $\tau_1(x)$, we can assume that $\tau(C_j) = C_j$ for $j \leq r$. Furthermore, we have $x_i = x_{i'}$, $y_i = y_{i'}$ for $i, i' \in C_j$ with $j \leq r$. Hence, $y_i = t_i x_i$ for $i \in \Sigma$. In the case $k = 1$ consider the element $t' \in (\mu_m)^n$ such that $t'_i = t_i$ for $i \in \Sigma$ and $t'_i = 1$ for $i \notin \Sigma$. Then t' commutes with (z, σ^{-1}) and $y = t'x$, which proves the assertion in this case. Now assume that m is prime. Let $\sigma^{-1} = \prod_j c_j$ be the cycle decomposition of σ^{-1}, so that the support of c_j is C_j. Let $\zeta = \prod_{i \in \Sigma} t_i \in \mu_m$. Since $z(C_{r+1}) \neq 1$, there exists an integer a such that $\zeta = z(C_{r+1})^a$. Let us define an element $p \in (\mu_m)^n$ by

$$p_i = \begin{cases} t_i, & i \in \Sigma, \\ z_i^{-a}, & i \in C_{r+1}, \\ 1 \text{ otherwise.} \end{cases}$$

Then $\prod_i p_i = 1$, so the element $(p; c_{r+1})$ is in $G(m, k, n)$. Also, $(p; c_{r+1})$ commutes with (z, σ^{-1}) and sends x to y. $\square$

Example 2.2.5. The property $(\star)$ often fails for groups $G(m, k, n)$ with non-prime m. Here is an example that works for both $G(4, 4, 5)$ and $G(4, 2, 5)$. Let $\sigma = (23)(45)$, $z = (1, 1, -1, 1, -1)$. Then the invariant subspace of $(z; \sigma)$ is the coordinate line spanned by e_1. Now the element $(\zeta_4, \zeta_4^{-1}, 1, 1, 1) \in (\mu_4)^5 \cap G(4, 4, 5)$ sends e_1 to $\zeta_4 e_1$. We claim that an element $(t; \tau) \in G(4, 2, 5)$, commuting with $(z; \sigma)$, can only map e_1 to $\pm e_1$. Indeed, the condition that $(t; \tau)$ is in the centralizer implies that $\tau(1) = 1$ and $t_3/t_2 = \pm 1$, $t_5/t_4 = \pm 1$. Now the condition $\prod t_i = \pm 1$ gives $t_1 t_2^2 t_4^2 = \pm 1$, hence $t_1 = \pm 1$.

Proposition 2.2.6. *Let G be the either a complex reflection group of rank 2, or a real reflection group of rank ≤ 3, or the group of type H_4. Then for every $g \in G$ the geometric quotient $V^g/C(g)$ is smooth.*

Proof. If $\dim V^g \leq 1$ or $g = 1$ then this is clear. Assume now that $\dim V = 3$ and $\dim V^g = 2$, so that g is a reflection. Let v be a unit vector orthogonal to V^g. Then every element of $C(g)$ sends v to $\pm v$. Let $G_v \subset G$ be the stabilizer of v. Then $G_v \subset C(g)$ and $C(g) = G_v \times \langle g \rangle$. Hence, $V^g/C(g) = V^g/G_v$. But G_v is generated by reflections, hence V^g/G_v is smooth.

In the case when G is of type H_4 we use the information about the normalizers of the parabolic subgroups from the work [30]. Recall that by Proposition 2.2.2, property $(\star)$ holds for G, hence by Lemma 2.2.1, $V^g/C(g) = V^g/N_G(H)$, where $H = H_g \subset G$ is the pointwise stabilizer of $V^g = V^H$. Thus, it is enough to check that $N_G(H)$ acts on V^H as a reflection group. But this follows from Theorem [30, Thm. 6] and the explicit description of $N_G(H)$ for the type H_4 (see [30, p. 79]). $\qquad\square$

Example 2.2.7. For groups $G(m, k, n)$ with $k \neq 1$ there often exist elements g with singular $V^g/C(g)$. For example for $G(3, 3, 3)$ we can take $g = (12)$, so that $V^g/C(g)$ is the quotient of the plane by the action of $\mathbb{Z}/3$ generated by $(x, y) \mapsto (\zeta_3 x, \zeta_3 y)$.

3. DG-MODELS AND FROBENIUS

3.1. **Lifting the Frobenius at the DG-level.** In this section we will prove the following "abstract nonsense" result.

Theorem 3.1.1. *Let X be a scheme of finite type over $\mathbb{C}$, and let $\mathcal{F}$ be a bounded complex of $\mathbb{C}$-vector spaces with constructible cohomology on $X(\mathbb{C})$ which "can be defined over a finite field" (see below). Then $\mathrm{RHom}_{\mathbb{C}}(\mathcal{F}, \mathcal{F})$ may be represented by a DG-algebra A which is in addition equipped with a DG-endomorphism F such that the action of F on $H^*(A)$ coincides with the action of the Frobenius endomorphism on $\mathrm{Ext}^*_\Lambda(\mathcal{F}_s, \mathcal{F}_s) \otimes_\Lambda \mathbb{C}$ where $\mathcal{F}_s$ which is a reduction of $\mathcal{F}$ to the algebraic closure of a finite field and $\Lambda \subset \mathbb{C}$ is a suitable coefficient ring which is a complete discrete valuation ring with finite residue field.*

The same result holds if $\mathcal{F}$ lies in the bounded Bernstein-Lunts equivariant derived category of $[X(\mathbb{C})/G(\mathbb{C})]$ for a linear algebraic group G acting on X.

We will say that $\mathcal{F} \in D^b_c(X(\mathbb{C}), \mathbb{C})$ or $\mathcal{F} \in D^b_c([X(\mathbb{C})/G(\mathbb{C})], \mathbb{C})$ is "defined over a finite field" if it can be obtained from an $\mathcal{F}_s$ defined over a finite field using the procedure "De $\mathbb{F}$ à $\mathbb{C}$" outlined in [7, §6] (this procedure will be reviewed below). Note that while in loc. cit. it is shown that any $\mathcal{F}$ is defined over the *algebraic closure of a finite field*, being actually defined over a finite field itself is a very subtle property[1]. In particular, this property is not stable under extension and so one must be careful with categorical constructions. In practice one usually completely sidesteps this problem by first defining the relevant objects over a finite field and then performing the lift to $\mathbb{C}$.

The main point of Theorem 3.1.1 is not the existence of the DG-algebra A, which in the topological case is obvious since we are working in a genuine derived category, but the fact that A may be chosen in such a way as to be equipped with a lift of the Frobenius endomorphism, as the latter only exists over a finite field.

To prove Theorem 3.1.1 we have to check that the procedure of reducing to finite characteristic as explained in [7, §6] (for schemes) can be lifted to the DG-level (i.e., it can be "enhanced"). We have to deal with the fact, however, that the l-adic derived category constructed in [7, 21] is not actually a subcategory of a genuine derived category, and hence it is not naturally enhanced. Luckily this defect has been repaired in recent years [6, 22, 47].

Another technical problem, which arises only in the G-equivariant case, is the following: a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ does not define a morphism between the corresponding lisse-étale topoi, as the pullback functor does not commute with finite limits (this was independently observed by Behrend and Gabber, see [54, Example 3.4] for an explicit counterexample between schemes!). This does not present a problem on the derived level since a functor $Lf^{-1} : D_c(\mathcal{Y}) \to D_c(\mathcal{X})$ has been constructed [54]. However it does present a problem on the DG-level. The functors connecting $\mathbb{C}$ and $\mathbb{F}$ are mostly given by inverse images, and since the DG-enhancements we use are based on injective resolutions, a non-exact inverse pullback functor is very inconvenient.

Luckily these problems disappear if we represent stacks by simplicial schemes equipped with the étale site. In that case inverse images may be computed levelwise, and hence it is obvious that they are exact.

---

[1]A simple perverse sheaf of "geometric" origin is defined over a finite field by [7, §6.2.4].

Since it is well-known that the derived category of constructible sheaves on an algebraic stack is equivalent to the derived category of contructible (cartesian) sheaves on the simplicial scheme associated to a smooth covering by a scheme, and since furthermore this equivalence extends to the l-adic case (see [47, Prop. 10.3]), our approach is equivalent to the usual one.

Let us finally mention that derived inverse images and derived direct images for a morphisms $f : \mathcal{X} \to \mathcal{Y}$ between algebraic stacks may indeed be computed on the corresponding simplicial schemes. For Lf^{-1} this is in fact true by the construction, and for Rf_* this is true for representable morphisms [6, Corollary 5.5.6].

Remark 3.1.2. A result like Theorem 3.1.1 for $\mathcal{F} = \underline{\mathbb{C}}$ was proved by Deligne in [21] using hypercoverings. Note that in this particular case Deligne constructs A in such a way that $A_{<0} = 0$, so that Sullivan's theory of minimal models applies to it. Our construction does not have this property, which is why need the more general Theorem 3.1.1.

Remark 3.1.3. Theorem 3.1.1 has an obvious analogue for a finite collection of complexes of $(\mathcal{F}_i)_{i=1,\ldots,n}$. In this case A becomes a DG-category with n objects. We leave the precise formulation to the reader.

The proof of Theorem 3.1.1 occupies the remainder of this section. After some preliminaries, we discuss in §3.5 the reduction to a finite field purely on the level of derived categories (essentially following [7, §6], but with appropriate changes to deal with the equivariant categories). Then in §3.6 we explain how to lift the entire picture to the DG-level, using standard DG-enhancements (see §3.4).

3.2. **The m-adic derived category of a topos.** Let X be a topos, and let $X^{\mathbb{N}}$ be the associated topos of $\mathbb{N}$-indexed[2] inverse systems over X. Let Λ be a complete discrete valuation ring with maximal ideal m.

There is a morphism of topoi

$$\pi : X^{\mathbb{N}} \to X$$

such that $\pi^{-1}\mathcal{F}$ is the constant inverse system $(\mathcal{F})^n$ and $\pi_*\mathcal{G} = \varprojlim_n \mathcal{G}^n$. Also, let $i_n : X \to X^{\mathbb{N}}$ be the morphism of topoi such that $i_n^{-1}\mathcal{F} = \mathcal{F}^n$.

Put $\Lambda^n = \Lambda/m^n$ and let $\Lambda^{\bullet} = (\Lambda^n)_n$ be the corresponding inverse system of rings. We consider $X^{\mathbb{N}}$ to be ringed by the constant sheaf $\Lambda^{\bullet}$.

Let $C(X^{\mathbb{N}}, \Lambda^{\bullet})$ denote the category of complexes of $\Lambda^{\bullet}$-modules on X, and let $D(X^{\mathbb{N}}, \Lambda^{\bullet})$ be the corresponding derived category.

The composition

$$L\pi^* R\pi_* : D(X^{\mathbb{N}}, \Lambda^{\bullet}) \to D(X^{\mathbb{N}}, \Lambda^{\bullet})$$

is called the *normalization functor* and is denoted by $\widehat{?}$. By construction there is a natural transformation

$$\mathcal{F} \to \widehat{\mathcal{F}}$$

An object $\mathcal{F}$ in $D(X^{\mathbb{N}}, \Lambda^{\bullet})$ is said to be *normalized* if $\mathcal{F} \to \widehat{\mathcal{F}}$ is an isomorphism. We write $D(X^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}}$ for the full subcategory of $D(X^{\mathbb{N}}, \Lambda^{\bullet})$ consisting of normalized complexes. Obviously $D(X^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}}$ is triangulated, and since it is a subcategory of a derived category, it is naturally enhanced, but it is not clear if it has any other desirable properties.

The good behaviour of $D(X^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}}$ depends crucially on the good behaviour of the normalization functor $\widehat{?}$, which in turn depends on the good behaviour of inverse limits. The good behaviour of inverse limits on sites depends on the local boundedness of cohomological dimension—see [47, §2] and in particular [47, Assumption 2.1.2]. Luckily these hypotheses are satisfied in the cases we consider (nice algebraic stacks and simplicial schemes, see [47] for the precise setting).

So from now on we assume that we are in the setting of [46, 47]. An important feature is that the normalization functor may be computed locally on stacks [47, Cor. 3.8] and levelwise on simplicial schemes (see the proof of [47, Prop. 10.2]).

---

[2]We follow the American convention that $0 \notin \mathbb{N}$.

Also, we have that a complex is normalized if and only for $n \geq m$ the canonical maps

$$(3.2.1) \qquad \mathcal{F}^n \overset{L}{\otimes}_{\Lambda^n} \Lambda^m \mapsto \mathcal{F}^m$$

are isomorphisms, where here and below we use the convention $\mathcal{F}^n = i_n^* \mathcal{F}$. For algebraic stacks this is [47, Prop. 3.5], and for simplicial schemes it follows from the fact that the normalization functor may be computed levelwise.

Now assume that c is the category of Cartesian constructible complexes in X (this notion is defined both in the stack and in the simplicial context). Then we define $D_c(X, \Lambda)$ as the full subcategory of $D(X^{\mathbb{N}}, \Lambda^\bullet)^{\mathrm{norm}}$, consisting of objects whose cohomology are AR-m-adic objects in c (see [35]).

It follows from [47] that $D_c(X, \Lambda)$ may be constructed differently as the category

$$(3.2.2) \qquad \mathsf{D}_c(X, \Lambda) \overset{\mathrm{def}}{=} D_c(X^{\mathbb{N}}, \Lambda^\bullet)/D_{\mathrm{null}}(X^{\mathbb{N}}, \Lambda^\bullet),$$

where $D_c(X^{\mathbb{N}}, \Lambda^\bullet)$ and $D_{\mathrm{null}}(X^{\mathbb{N}}, \Lambda^\bullet)$ are the full subcategories of $D(X^{\mathbb{N}}, \Lambda^\bullet)$ spanned by objects $\mathcal{F}$ whose cohomology are, respectively, locally AR-m-adic objects and locally AR-null objects in c (see [35]). There are inverse equivalences

$$(3.2.3) \qquad D_c(X, \Lambda) \underset{\widehat{?}}{\overset{can}{\rightleftarrows}} \mathsf{D}_c(X, \Lambda)$$

For algebraic stacks this is [47, Thm 3.9] and for simplicial schemes it is [47, Prop. 10.2].

It is clear that the standard truncation functors $\tau_{\leq n}$, $\tau_{\geq n}$ on $D(X^{\mathbb{N}}, \Lambda^\bullet)$ descend to $\mathsf{D}_c(X, \Lambda)$, and hence they define a t-structure on $\mathsf{D}_c(X, \Lambda)$ with the heart consisting of m-adic sheaves (see [35]) in c, and the same holds via (3.2.3) also for $D_c(X, \Lambda)$.

Using the t-structure introduced in the previous paragraph we now define the bounded l-adic derived category $\mathsf{D}_c^b(X, \Lambda) \subset \mathsf{D}_c(X, \Lambda)$, and via (3.2.3), also $D_c^b(X, \Lambda) \subset D_c(X, \Lambda)$. Note however, that $D_c^b(X, \Lambda)$ is not a subcategory of $D^b(X^{\mathbb{N}}, \Lambda^\bullet)$!

3.3. A spectral sequence for D-topoi.

The following result should be standard but we did not find the exact result in the literature.

Let D be a small category and let $\pi : X_\bullet \to D$ be a D-topos [2]. We write X_i for $\pi^{-1}(i)$ and $X_\bullet$ for the associated total topos. Thus, objects in $X_\bullet$ are collections of objects $(\mathcal{F}_i \in X_i)_i$ and maps $(\phi_\alpha : \alpha^{-1}\mathcal{F}_j \to \mathcal{F}_i)_\alpha$, for $\alpha : i \to j$, satisfying the standard cocycle condition.

We assume that $X_\bullet$ is ringed with a sheaf of rings $\Lambda_\bullet$ which is flat in the sense that the transition morphisms $\alpha : (X_i, \Lambda_i) \to (X_j, \Lambda_j)$ have the property that α^* is exact. We say that $\Lambda_\bullet$-module $\mathcal{F}$ is Cartesian if the ϕ_α's induce isomorphisms $\alpha^*\mathcal{F}_j \to \mathcal{F}_i$. We will denote the latter isomorphisms also by $(\phi_\alpha)_\alpha$. Note that $\Lambda_\bullet$ is automatically Cartesian.

Proposition 3.3.1. *Let $\mathcal{F}, \mathcal{G}$ be $\Lambda_\bullet$-modules with $\mathcal{F}$ being Cartesian. Then the assignment*

$$i \mapsto \mathrm{Ext}^j_{\Lambda_i}(\mathcal{F}_i, \mathcal{G}_i)$$

defines a contravariant functor $D \to \mathrm{Ab}$ (where Ab is the category of abelian groups). Furthermore, there is a convergent spectral sequence

$$E_2^{pq} = R^p \varprojlim_u \mathrm{Ext}^q_{\Lambda_u}(\mathcal{F}_u, \mathcal{G}_u) \Rightarrow \mathrm{Ext}^n_{\Lambda_\bullet}(\mathcal{F}, \mathcal{G})$$

Proof. Let $\alpha : v \to u$ be a morphism in D. We obtain a morphism

$$\mathrm{Ext}^j_{\Lambda_u}(\mathcal{F}_u, \mathcal{G}_u) \to \mathrm{Ext}^j_{\Lambda_v}(\alpha^*\mathcal{F}_u, \alpha^*\mathcal{G}_u) \cong \mathrm{Ext}^j_{\Lambda_v}(\mathcal{F}_v, \alpha^*\mathcal{G}_u) \to \mathrm{Ext}^j_{\Lambda_v}(\mathcal{F}_v, \mathcal{G}_v)$$

where the first morphism is induced by α^* and the latter two are induced by ϕ_α^{-1} and ϕ_α respectively. This yields the asserted functoriality. It is easy to see that

$$\mathrm{Hom}_\Lambda(\mathcal{F}, \mathcal{G}) = \varprojlim_u \mathrm{Hom}_{\Lambda_u}(\mathcal{F}_u, \mathcal{G}_u).$$

We will construct the spectral sequence as the Grothendieck spectral sequence for the above composition of functors. For this we have to show that $\mathcal{I}_u$ is injective for $\mathcal{I}$ injective and furthermore that $\mathrm{Hom}_{\Lambda_u}(\mathcal{F}_u, \mathcal{I}_u)$ is acyclic for $\varprojlim_u$.

As in [2, 46], we may assume that $\mathcal{I} = i_{v,*}I$ where $i_v : X_v \to X$ is the morphism of topoi $(\mathcal{F}_v)_v \mapsto \mathcal{F}_v$ and I is an injective Λ_v-module in X_v. We have the general formula

$$(i_{v,*}\mathcal{I})_u = \prod_{\alpha:v \to u} \alpha_* \mathcal{I}$$

(which does not require $\mathcal{I}$ to be injective). Since α_* has an exact left adjoint (by flatness), this implies that $(i_{v,*}\mathcal{I})_u$ is injective.

We also get

$$\begin{aligned}
\mathrm{Hom}_{\Lambda_u}(\mathcal{F}_u, \mathcal{I}_u) &= \mathrm{Hom}_{\Lambda_u}(\mathcal{F}_u, (i_{v,*}I)_u) \\
&= \prod_{\alpha:v \to u} \mathrm{Hom}_{\Lambda_u}(\mathcal{F}_u, \alpha_* I) \\
&= \prod_{\alpha:v \to u} \mathrm{Hom}_{\Lambda_v}(\alpha^* \mathcal{F}_u, I) \\
&= \prod_{\alpha:v \to u} \mathrm{Hom}_{\Lambda_v}(\mathcal{F}_v, I) \\
&= \mathrm{Set}\big(D(v,u), \mathrm{Hom}_{\Lambda_v}(\mathcal{F}_v, I)\big),
\end{aligned}$$

where $\mathrm{Set}(?,?)$ denotes morphisms in the category of sets. Thus, we have to show that

$$\mathrm{Set}\big(D(v,-), \mathrm{Hom}_{\Lambda_v}(\mathcal{F}_v, I)\big)$$

is acyclic for $\varprojlim$. Let $\mathrm{Inv}(D)$ denote the category of inverse systems of functors $D \to \mathrm{Ab}$. For W in $\mathrm{Inv}(D)$ we have

$$\mathrm{Hom}_{\mathrm{Inv}(D)}\big(W, \mathrm{Set}\big(D(v,-), \mathrm{Hom}_{\Lambda_v}(\mathcal{F}_v, I)\big)\big) = \mathrm{Hom}_{\Lambda_v}(W_v \otimes_{\mathbb{Z}} \mathcal{F}_v, I).$$

We claim that the derived functors of

$$W \mapsto \mathrm{Hom}_{\Lambda_v}(W_v \otimes_{\mathbb{Z}} \mathcal{F}_v, I)$$

are given by

$$\mathrm{Hom}_{\Lambda_v}(\mathrm{Tor}_i^{\mathbb{Z}}(W_v, \mathcal{F}_v), I)$$

This is clear if we replace W by a projective resolution and use the fact that projectives are summands of direct sums of objects of the form $\mathbb{Z}D(-, w)$, which are $\mathbb{Z}$-flat when evaluated at every v. Thus, we conclude that

$$\mathrm{Ext}_{\mathrm{Inv}(D)}^i\big(W, \mathrm{Set}(D(v,-), \mathrm{Hom}_{\Lambda_v}(\mathcal{F}_v, I))\big) = \mathrm{Hom}_{\Lambda_v}(\mathrm{Tor}_i^{\mathbb{Z}}(W_v, \mathcal{F}_v), I).$$

Let $\underline{\mathbb{Z}}$ be the constant inverse system on D with value $\mathbb{Z}$. We finally conclude that for $i > 0$ one has

$$\begin{aligned}
R^i \varprojlim \mathrm{Set}(D(v,-), \mathrm{Hom}_{\Lambda_v}(\mathcal{F}_v, I)) &= \mathrm{Ext}_{\mathrm{Inv}(D)}^i(\underline{\mathbb{Z}}, \mathrm{Set}(D(v,-), \mathrm{Hom}_{\Lambda_v}(\mathcal{F}_v, I))) \\
&= \mathrm{Hom}_{\Lambda_v}(\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}, \mathcal{F}_v), I) \\
&= 0 \quad \square
\end{aligned}$$

3.4. **DG-enhancements of derived categories and derived functors.**

3.4.1. *The "standard" DG enhancement of a derived category.* If $\mathcal{A}$ is an abelian category then the derived category $D(\mathcal{A})$ of $\mathcal{A}$ is the category of complexes $C(\mathcal{A})$ over $\mathcal{A}$ localized at quasi-isomorphisms.

If $\mathcal{A}$ is a Grothendieck category then we denote by $D^{\mathrm{dg}}(\mathcal{A})$ the "standard" DG-enhancement of $D(\mathcal{A})$ using fibrant resolutions. In other words, $\mathrm{Ob}(D^{\mathrm{dg}}(\mathcal{A})) = \mathrm{Ob}(D(\mathcal{A})) = \mathrm{Ob}(C(\mathcal{A}))$, and for every $C \in \mathrm{Ob}(C(\mathcal{A})$ we fix a quasi-isomorphism $C \to I_C$ such that I_C is fibrant for the standard injective model structure on $C(\mathcal{A})$. The Hom-complexes in $D^{\mathrm{dg}}(\mathcal{A})$ are then given by

$$\underline{\mathrm{Hom}}_{D^{\mathrm{dg}}(\mathcal{A})}(C, D) \stackrel{\mathrm{def}}{=} \underline{\mathrm{Hom}}_{C(\mathcal{A})}(I_C, I_D)$$

Although it is not essential we will assume that if C is left bounded then I_C is left bounded as well. In this case I_C is just a classical injective resolution.

3.4.2. *Co-quasi-functors.* The DG-functor is often too rigid a notion to compare different DG-categories. A suitable weakened notion is given by "quasi-functors" introduced in [39]. For technical reasons we will use the dual version which we call "co-quasi-functors".

If $\mathfrak{a}$, $\mathfrak{b}$ are DG-categories then a co-quasi-functor $M : \mathfrak{b} \to \mathfrak{a}$ is simply a quasi-functor [39] $\mathfrak{b}^\circ \to \mathfrak{a}^\circ$. To be more concrete: an $\mathfrak{a} - \mathfrak{b}$-bimodule M is a DG-bifunctor $M : \mathfrak{b}^\circ \times \mathfrak{a} \to C(\mathrm{Ab})$. We say that such an M is a co-quasi-functor $\mathfrak{b} \to \mathfrak{a}$ if for every $B \in \mathfrak{b}$ there is an $A \in \mathfrak{a}$, as well as a morphism of DG-functors $\mathfrak{a}(A, -) \to M(B, -)$ (by enriched Yoneda this is the same as an element of $x_B \in Z^0 M(B, A)$), such that for all $A' \in \mathfrak{a}$ the induced map $\mathfrak{a}(A, A') \to M(B, A')$ is a quasi-isomorphism. It will often be convenient to choose for every B a particular A, which we will then denote by $M(B)$.

A co-quasi-functor M induces an honest functor $H^0(\mathfrak{a}) \to H^0(\mathfrak{b})$ sending B to $M(B)$. We denote this functor by $H^0(M)$.

Example 3.4.1. Let F be a left exact functor between Grothendieck categories $\mathcal{C} \to \mathcal{D}$. Then the right derived functor

$$RF : D(\mathcal{C}) \to D(\mathcal{D})$$

can be lifted to a co-quasi-functor

$$RF^{\mathrm{dg}} : D^{\mathrm{dg}}(\mathcal{C}) \to D^{\mathrm{dg}}(\mathcal{D})$$

by setting

$$RF^{\mathrm{dg}}(C, D) = \underline{\mathrm{Hom}}_{C(\mathcal{D})}(FI_C, I_D)$$

We will need the following standard result [39].

Lemma 3.4.2. *Assume that $M : \mathfrak{b} \to \mathfrak{a}$ is a co-quasi-functor such that $H^0(M)$ is fully faithful. Then for every $B \in \mathfrak{b}$ one has that $\mathfrak{b}(B, B)$ and $\mathfrak{a}(M(B), M(B))$ are isomorphic in the homotopy category of DG-algebras.*

Proof. Put $A = M(B)$ and consider the DG-algebra

$$\Lambda = \begin{pmatrix} \mathfrak{a}(A, A) & M(B, A)[-1] \\ 0 & \mathfrak{b}(B, B) \end{pmatrix}$$

with the differential

$$\begin{pmatrix} f & s^{-1}m \\ 0 & g \end{pmatrix} \mapsto \begin{pmatrix} df & s^{-1}(x_B g - f x_B - dm) \\ 0 & dg \end{pmatrix}.$$

Then we see that Λ projects to $\mathfrak{a}(A, A)$ and $\mathfrak{b}(B, B)$, and it is easy to see that both maps are quasi-isomorphisms. $\square$

Remark 3.4.3. The notions introduced in this section have obvious k-linear versions in the case when k is a commutative base ring. We will use this without further comment.

3.5. **Passing to a finite field.** For a scheme Y of finite type over $\mathbb{C}$ and a noetherian ring R of finite global dimension we temporarily write $D_c^b(Y(\mathbb{C}), R)_{\mathrm{lit}}$ for the category of bounded complexes of sheaves of R-modules on $Y(\mathbb{C})$ with constructible homology, in order to avoid confusion with the l-adic derived category which will be introduced later.

3.5.1. *The Bernstein-Lunts derived category.* Let X be a scheme of finite type over $\mathbb{C}$ and let G be a linear algebraic group acting on X. The (topological) derived category $D_c^b([X(\mathbb{C})/G(\mathbb{C})], \mathbb{C})_{\mathrm{lit}}$ of $G(\mathbb{C})$-equivariant $\mathbb{C}$-linear sheaves on $X(\mathbb{C})$ was defined by Bernstein and Lunts in [10].

Let $(X(\mathbb{C})/G(\mathbb{C}))_\bullet$ be the standard simplicial space associated to the $G(\mathbb{C})$-action on $X(\mathbb{C})$. We view $(X(\mathbb{C})/G(\mathbb{C}))_\bullet$ as a topos indexed by Δ°, where $\Delta = \{[n] \mid n \in \mathbb{N}\}$ is the standard simplicial category. Let R be a noetherian ring of finite global dimension. The model of the $G(\mathbb{C})$-equivariant derived category we will use is

$$(3.5.1) \qquad D_c^b([X(\mathbb{C})/G(\mathbb{C})], R)_{\mathrm{lit}} \overset{\mathrm{def}}{=} D_{c,\mathrm{cart}}^b\big((X(\mathbb{C})/G(\mathbb{C}))_\bullet, R\big)_{\mathrm{lit}}$$

The right-hand side of (3.5.1) is the full subcategory of $D\big((X(\mathbb{C})/G(\mathbb{C}))_\bullet, R\big)_{\mathrm{lit}}$ consisting of bounded complexes with Cartesian, levelwise constructible homology.

We let Y be either X_{et} (the small étale site of X) or else $(X/G)_{\bullet,\mathrm{et}}$, the simplicial scheme of a linear algebraic group G acting on X. We write $Y(\mathbb{C})$ for the corresponding topological topoi $X(\mathbb{C})$ or $(X(\mathbb{C})/G(\mathbb{C}))_\bullet$. In the course of the proof the ground field will change.

3.5.2. *Changing the coefficient ring.* For use below we make the following definition. Let $R \to S$ be a morphism between commutative rings and let $\mathcal{A}$ be an R-linear category. Then the category $\mathcal{A}_S$ has the same objects as $\mathcal{A}$ but its Hom-spaces are defined by

$$\mathrm{Hom}_{\mathcal{A}_S}(\mathcal{F}, \mathcal{G}) = S \otimes_R \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$$

Note that if $\mathcal{A}$ is abelian or triangulated then this will usually not be the case for $\mathcal{A}_S$.

We recall the following fact.

Proposition 3.5.1.   (1) *Let S be commutative noetherian ring of finite global dimension which is a flat ring extension of R. Then the coefficient extension functor*

$$D_c^b(Y(\mathbb{C}), R)_{lit} \to D_c^b(Y(\mathbb{C}), S)_{lit} : \mathcal{F} \mapsto S \otimes_R \mathcal{F}$$

induces a fully faithful functor

$$D_c^b(Y(\mathbb{C}), R)_{lit,S} \to D_c^b(Y(\mathbb{C}), S)_{lit}.$$

(2) *We have*

$$D_c^b(Y(\mathbb{C}), \mathbb{C})_{lit} = \bigcup_S D_c^b(Y(\mathbb{C}), S)_{lit,\mathbb{C}},$$

where S runs through subrings of $\mathbb{C}$ which are finitely generated over $\mathbb{Z}$ and of finite global dimension.

(3) *We have*

$$D_c^b(Y(\mathbb{C}), \mathbb{C})_{lit} = \bigcup_S D_c^b(Y(\mathbb{C}), \Lambda)_{lit,\mathbb{C}},$$

where Λ runs through complete discrete valuation rings in $\mathbb{C}$ with finite residue field.

Proof. Assume first that Y is a scheme. By Hironaka's semi-algebraic triangulation theorem [29], the study of constructible sheaves on $X(\mathbb{C})$ can be reduced to the study of constructible sheaves on *finite* simplicial complexes, which is a purely combinatorial problem [37, §8.1]. From this (1) and (2) follow easily.

To deduce (3) from (2) one needs a bit of field theory. Let $S \subset \mathbb{C}$ be finitely generated over $\mathbb{Z}$. We claim that there is a complete discrete valuation ring with finite residue field Λ such that $S \subset \Lambda \subset \mathbb{C}$. This clearly implies (3).

First, note that from the fact that $\mathbb{Z}_l/\mathbb{Z}$ has infinite transcendence degree one easily constructs an injective map $S \to \Lambda'$, where Λ' is a complete discrete valuation ring with finite residue field. Indeed, write S as an integral extension of some $S_0 = \mathbb{Z}[X_1, \ldots, X_N]$. Then we may embed $S_0 \subset \mathbb{Z}_l$ as $\mathbb{Z}$-algebra. Extend the induced valuation on the fraction field of S_0 to the fraction field of S. This gives an injective map $S \to \Lambda_0$, where Λ_0 is a discrete valuation ring with the finite residue field. Now replace Λ_0 by its completion Λ'.

Thus, in particular Λ' is a finite extension of some $\mathbb{Z}_l$. Thus, the algebraic closure of the quotient field of Λ' is isomorphic to $\overline{\mathbb{Q}}_l$.

Let K be the quotient field of S. We now have the following diagram

$$\begin{array}{ccc} K & \longrightarrow & \mathbb{C} \\ \downarrow & & \\ \overline{\mathbb{Q}}_l & & \end{array}$$

Now $\mathbb{C}$ and $\overline{\mathbb{Q}}_l$ are both algebraically closed fields. Furthermore, since K is countable, $\mathbb{C}/K$ and $\overline{\mathbb{Q}}_l/K$ have the same transcendence degree. It follows by [33, Theorem 1.12] that there is an isomorphism $\tau : \overline{\mathbb{Q}}_l \xrightarrow{\cong} \mathbb{C}$ which is the identity on K. It now suffices to take $\Lambda = \tau(\Lambda')$.

Now consider the case $Y = [X/G]$. (1) follows immediately from Proposition 3.3.1 and the scheme case.

For (2) we note that a Cartesian constructible sheaf on $X(\mathbb{C})$ is the same as a constructible sheaf with a finite amount of descent data. Hence, this descent data can be descended to a finite extension of $\mathbb{Z}$. A similar argument applies to the extension data which build a complex from its cohomology. The proof of (3) does not have to be adapted. $\square$

3.5.3. *Passing to the topological m-adic derived category.* Now assume that we are in the setup of §3.2, so (Λ, m) is a complete discrete valuation ring. We assume in addition that its residue field is finite. Recall that we have a morphism of topoi $\pi : X^{\mathbb{N}} \to X$, where $X^{\mathbb{N}}$ is ringed by $\Lambda^{\bullet} = (\Lambda^n)_n$ with $\Lambda^n = \Lambda/m^n$.

Theorem 3.5.2. *The functors $L\pi^*$ and $R\pi_*$ define inverse equivalences between $D_c^b(Y(\mathbb{C}), \Lambda)_{lit}$ and $D_c^b(Y(\mathbb{C}), \Lambda)$.*

Proof. For schemes this is proved in [22], so let us consider the case $Y = (X/G)_{\bullet}$.

Note that an object in $D^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}}$ is first and foremost a system of complexes $(\mathcal{F}_m^n)_{n,m}$ of sheaves of Λ^n-modules on $G(\mathbb{C})^m \times X(\mathbb{C})$ equipped with various maps. The fact that $(\mathcal{F}_m^n)_{n,m}$ is normalized implies that the morphism

$$\mathcal{F}_{\bullet}^n \overset{L}{\otimes}_{\Lambda^n} \Lambda^{n'} \to \mathcal{F}_{\bullet}^{n'}$$

is an isomorphism (in the derived category). It is easy to see that this is equivalent to

$$\mathcal{F}_m^n \overset{L}{\otimes}_{\Lambda^n} \Lambda^{n'} \to \mathcal{F}_m^{n'}$$

being an isomorphism for each m. In other words, this is equivalent to $\mathcal{F}_m^{\bullet}$ being normalized as a complex of sheaves on the ringed topos $((G(\mathbb{C})^m \times X(\mathbb{C}))^{\mathbb{N}}, \Lambda^{\bullet})$ for each m.

Now let $X_1 = G^{m_1} \times X$, $X_2 = G^{m_2} \times X$, and let $f : [m_2] \to [m_1]$ be a morphism in Δ. It corresponds to a morphism of schemes $X_1 \to X_2$. The object $(\mathcal{F}_m^n)_{m,n}$ is Cartesian if and only if for each such f the map $f^* \mathcal{F}_{m_2}^{\bullet} \to \mathcal{F}_{m_1}^{\bullet}$ is an isomorphism for all n in $D^b(X_1(\mathbb{C})^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}}$.

We now claim that $R\pi_*$ and $L\pi^*$ define inverse equivalences

$$(3.5.2) \qquad D_c^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}} \longleftrightarrow D_c^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}, \Lambda)_{\mathrm{lit}}.$$

So we have to show that the unit and counit morphisms $\mathrm{id} \to R\pi_* L\pi^*$, $L\pi^* R\pi_* \to \mathrm{id}$ are isomorphisms. Since $R\pi_*$ and $L\pi^*$ can be computed levelwise, we are reduced to the scheme case which is known.

If f is as above then we have on the nose $f^* L\pi^* = L\pi^* f^*$. Hence, if $\mathcal{F} \in D_c^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}, \Lambda)_{\mathrm{lit}}$ has Cartesian cohomology then so does $L\pi^* \mathcal{F}$. Since $R\pi_*$ and $L\pi^*$ are inverses, we also have $f^* R\pi_* = R\pi_* f^*$, from which it follows that if $\mathcal{G} \in D_c^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}}$ has Cartesian cohomology then so does $R\pi_* \mathcal{G}$.

Thus, to show that (3.5.2) induces an equivalence of the subcategories

$$D_{c,\mathrm{cart}}^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}^{\mathbb{N}}, \Lambda^{\bullet})^{\mathrm{norm}} \simeq D_{c,\mathrm{cart}}^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}, \Lambda)_{\mathrm{lit}},$$

it remains to verify the compatibility with the condition that the cohomology of objects in $D_{c,\mathrm{cart}}^b((X(\mathbb{C})/G(\mathbb{C}))_{\bullet}, \Lambda)_{\mathrm{lit}}$ are Cartesian locally AR-m-adic objects. However, it is clear that if $\mathcal{F}$ is Cartesian then it is locally AR-m-adic if and only this is the case for $\mathcal{F}_1$. So we are again reduced to the scheme case which is known. $\square$

3.5.4. *Passing to the étale m-adic derived category.* In [47, Cor. 3.17] it is shown (for algebraic stacks) that $D_c^b(Y, \Lambda)$ coincides with the m-adic derived category as defined by Deligne in [21] (see also [7]). In other words, for $\mathcal{F}, \mathcal{G} \in D_c^b(Y, \Lambda)$ the obvious map

$$(3.5.3) \qquad \qquad \mathrm{Hom}_\Lambda(\mathcal{F}, \mathcal{G}) \to \varprojlim_n \mathrm{Hom}_{\Lambda^n}(\mathcal{F}^n, \mathcal{G}^n)$$

is an isomorphism.[3]

Now there is a morphism of topoi

$$\epsilon : Y(\mathbb{C}) \to Y_{\mathrm{et}}.$$

In the scheme case this is [7, p149]. In the G-equivariant case we define it levelwise.

It is easy to see that ϵ^* preserves normalized complexes with AR-m-adic cohomology and thus defines a functor

$$\epsilon^* : D_c^b(Y, \Lambda) \to D_c^b(Y(\mathbb{C}), \Lambda).$$

Proposition 3.5.3. [7, p149] *The functor ϵ^* defines an equivalence between $D_c^b(Y, \Lambda)$ and $D_c^b(Y(\mathbb{C}), \Lambda)$.*

Proof. Using (3.5.3) and its analogue for $Y(\mathbb{C})$ together with Proposition 3.3.1, it follows immediately that ϵ^* is fully faithful. To prove that it is essentially surjective it will be convenient to use $\mathsf{D}_c^b(Y, \Lambda)$ (see (3.2.2)). We have a commutative diagram

$$
\begin{array}{ccc}
D_c^b(Y_{\mathrm{et}}, \Lambda) & \xrightarrow[\cong]{\mathrm{can}} & \mathsf{D}_c^b(Y_{\mathrm{et}}, \Lambda) \\
{\scriptstyle \epsilon^*} \downarrow & & \downarrow {\scriptstyle \epsilon^*} \\
D_c^b(Y(\mathbb{C}), \Lambda) & \xrightarrow[\mathrm{can}]{\cong} & \mathsf{D}_c^b(Y(\mathbb{C}), \Lambda)
\end{array}
$$

So it is sufficient to prove that the right vertical arrow ϵ^* is essentially surjective. To this end it suffices to prove that if $\mathcal{F}$ is a constructible AR-m-adic object on $Y(\mathbb{C})$ then it can be lifted under ϵ^*. But this is clear since ϵ^* is an equivalence on constructible sheaves. $\qquad \square$

3.5.5. *Reduction to the algebraic closure of a finite field.* The following result is stated in [7, p155].

Lemma 3.5.4. *Suppose we have a finite diagram of schemes of finite type over a noetherian scheme S and on each scheme a finite number of constructible Λ^n-modules.*

If we perform a finite sequence of standard homological operations $R^* f_*$, $R^* f_!$, f^*, $R^* f^!$, $\mathcal{T}or_p$, $\mathcal{E}xt^p$ on the given constructible sheaves, then the result is constructible and compatible with the base change after replacing S with a suitable dense open subset.

Proof. (see [7]) The crucial case consists of $R^* f_*$ applied to a single constructible sheaf. This case is [20, Th. finitude 1.9]. $\qquad \square$

We recall a construction given in [7, p156]. Let $X/\mathbb{C}$ be of finite type. Let $\mathcal{T}$ be a stratification of X with smooth strata $T \in \mathcal{T}$. For each $T \in \mathcal{T}$ let $\mathcal{L}(T)$ be a finite family of locally constant sheaves (for the étale topology) of Λ^1-modules on T.

Lemma 3.5.5. *There exists A, a subring $\mathbb{C}$ of finite type over $\mathbb{Z}$, such that for $S = \mathrm{Spec}\, A$ there exist $(X_S, \mathcal{T}_S, \mathcal{L}_S)$ that give $(X, \mathcal{T}, \mathcal{L})$ through base extension, such that in addition the following properties hold.*

(1) *The strata $T \in \mathcal{T}_S$ are smooth over S with geometrically connected fibers.*
(2) *For F, G of the form $j_! L$ with $j : T \hookrightarrow X_S$ in $\mathcal{T}_S$ and $L \in \mathcal{L}(T)$, the sheaves $\mathcal{E}xt_{X_S}^q(F, G)$ are compatible with the base change in S.*
(3) *Let $a_S : X_S \to S$ be the structure morphism. With F, G as above, the sheaves $R^p a_* \mathcal{E}xt_{X_S}^q(F, G)$ are locally constant and compatible with the base change.*

---

[3]The term $\varprojlim_n \mathrm{Ext}^{-1}(\mathcal{F}^n, \mathcal{G}^n)$ in [47, Cor. 3.17] is zero since the groups $\mathrm{Ext}^{-1}(\mathcal{F}^n, \mathcal{G}^n)$ are finite in the current setting.

Proof. This is stated without proof in [7]. One may argue as follows. As the first step we may choose $S = \operatorname{Spec} A$, such that $(X_S, \mathcal{L}_S, \mathcal{T}_S)$ exist and the strata in $\mathcal{T}_S$ are smooth over S with geometrically connected fibers. Then we apply Lemma 3.5.4 to obtain constructibility and compatibility with the base change of (the finite number of) the sheaves $R^p a_* \mathcal{E}xt^q_{X_S}(F, G)$. Shrinking S further we may assume that the $R^p a_* \mathcal{E}xt^q_{X_S}(F, G)$ are locally constant. $\square$

Let A be as in the previous lemma. Pick $s \in S$. Then according to [7], there exists a strict Henselian discrete valuation ring $A \subset V \subset \mathbb{C}$, whose residue field is the algebraic closure of the residue field of s. We denote the closed point of $\operatorname{Spec} V$ also by s. Then we have a base extension diagram

$$X \xrightarrow{u} X_V \xleftarrow{i} X_s$$

Let $D^b_{\mathcal{T},\mathcal{L}}(X, \Lambda^n)$ be the full subcategory of $D^b_c(X, \Lambda^n)$ consisting of complexes K, such that $\operatorname{Tor}^\Lambda_j(\Lambda^1, H^i(K)|T)$, for $T \in \mathcal{T}$, is an extension of objects in $\mathcal{L}(T)$ for all i, j. We define $D^b_{\mathcal{T}_V,\mathcal{L}_V}(X_V, \Lambda^n)$, $D^b_{\mathcal{T}_s,\mathcal{L}_s}(X_s, \Lambda^n)$ in a similar way.

Lemma 3.5.6. *There are equivalences of categories*

$$D^b_{\mathcal{T},\mathcal{L}}(X, \Lambda^n) \xleftarrow{u^*} D^b_{\mathcal{T}_V,\mathcal{L}_V}(X_V, \Lambda^n) \xrightarrow{i^*} D^b_{\mathcal{T}_s,\mathcal{L}_s}(X_s, \Lambda^n).$$

Proof. (see [7]) Let $F = j_{1*}L_1$, $G = j_{2*}L_2$, where $j_i : T_i \to X_V$ in $\mathcal{T}_V$, for $i = 1, 2$, are the embeddings of strata, and $L_i \in \mathcal{L}(T_i)$ (these are sheaves of Λ^1-modules). We have to prove that $\operatorname{Ext}^p_{X_V,\Lambda^n}(F, G)$ does not change if we replace V by its generic or special point. Since F and G are sheaves of Λ^1-modules, we have

$$\operatorname{Ext}^p_{X_V,\Lambda^n}(F, G) = \operatorname{Ext}^p_{X_V,\Lambda^1}(F, \operatorname{RHom}_{\Lambda^n}(\Lambda^1, G)),$$

and $H^i(\operatorname{RHom}_{\Lambda^n}(\Lambda^1, G)) = G$ for all $i \geq 0$. Hence, it suffices to consider the case $n = 1$, by invoking the appropriate spectral sequence.

Now we have

$$\operatorname{RHom}_{X_V}(F, G) = R\Gamma(\operatorname{Spec} V, Ra_* R\mathcal{H}om_{X_V}(F, G))$$

Using Lemma 3.5.5(3), we are reduced to checking for a locally constant sheaf L on $(\operatorname{Spec} V)_{\text{et}}$ one has

$$H^i(\operatorname{Spec} \mathbb{C}, u^*L) = H^i(\operatorname{Spec} V, L) = H^i(s, i^*L).$$

But this is obvious, since V is stricly Henselian, and so $L = (\underline{\Lambda^1})^{\oplus n}$. $\square$

Let G be an algebraic group defined over $\mathbb{Z}$, acting on X. We keep the above setup, but we assume in addition that the strata in $\mathcal{T}$ are G-invariant, and the elements of $\mathcal{L}(T)$ are G-equivariant. We have the following result.

Lemma 3.5.7. *We can choose V in such a way that there are equivalences of categories*

$$D^b_{\mathcal{T},\mathcal{L}}(X/G, \Lambda) \xleftarrow{u^*} D^b_{\mathcal{T}_V,\mathcal{L}_V}(X_V/G_V, \Lambda) \xrightarrow{i^*} D^b_{\mathcal{T}_s,\mathcal{L}_s}(X_s/G_s, \Lambda).$$

Proof. We follow the above proof but replace a by the structure map $X \to \operatorname{Spec} \mathbb{C}/G$. Mimicking the the non-equivariant case we have to show that for a G-equivariant locally constant sheaf of Λ^1-modules L on $\operatorname{Spec} V$ one has

$$H^i(\operatorname{Spec} \mathbb{C}/G, u^*L) = H^i(\operatorname{Spec} V/G_V, L) = H^i(s/G_s, i^*L).$$

Note that if we forget the G-structure then L is a constant sheaf. Thus, using Proposition 3.3.1 we reduce to the scheme case. $\square$

3.5.6. *Passing to a finite field: summary.* For the benefit of the reader let us summarize the functors that were introduced in §3.5.2-3.5.5.

$$(3.5.4) \quad D_c^b(Y(\mathbb{C}),\mathbb{C})_{\text{lit}} \xleftarrow{\mathbb{C}\otimes_\Lambda -} D_c^b(Y(\mathbb{C}),\Lambda)_{\text{lit}} \xleftarrow[\cong]{R\pi_*} D_c^b(Y(\mathbb{C}),\Lambda) \xleftarrow[\cong]{\epsilon^*}$$
$$D_c^b(Y,\Lambda) \supset D_{\mathcal{T},\mathcal{L}}^b(Y,\Lambda) \xleftarrow[\cong]{u^*} D_{\mathcal{T}_V,\mathcal{L}_V}^b(Y_V,\Lambda) \xrightarrow[\cong]{i^*} D_{\mathcal{T}_s,\mathcal{L}_s}^b(Y_s,\Lambda)$$

After tensoring with $\mathbb{C}$ we then obtain functors

$$D_c^b(Y(\mathbb{C}),\mathbb{C})_{\text{lit}} \xleftarrow[f.f.]{\mathbb{C}\otimes_\Lambda -} D_c^b(Y(\mathbb{C}),\Lambda)_{\text{lit},\mathbb{C}} \xleftarrow[\cong]{R\pi_*} D_c^b(Y(\mathbb{C}),\Lambda)_\mathbb{C} \xleftarrow[\cong]{\epsilon^*}$$
$$D_c^b(Y,\Lambda)_\mathbb{C} \supset D_{\mathcal{T},\mathcal{L}}^b(Y,\Lambda)_\mathbb{C} \xleftarrow[\cong]{u^*} D_{\mathcal{T}_V,\mathcal{L}_V}^b(Y_V,\Lambda)_\mathbb{C} \xrightarrow[\cong]{i^*} D_{\mathcal{T}_s,\mathcal{L}_s}^b(Y_s,\Lambda)_\mathbb{C}$$

Furthermore, any object $\mathcal{F}$ in $D_c^b(Y(\mathbb{C}),\mathbb{C})$ is obtained from $\mathcal{F}_s$ in some $D_{\mathcal{T}_s,\mathcal{L}_s}^b(Y_s,\Lambda)$.

Assume $s = \operatorname{Spec} \bar{\mathbb{F}}_p$. There is some $s_0 = \operatorname{Spec} \mathbb{F}_q$ such that Y_s is obtained by the base extension from Y_0/s_0. Unfortunately one cannot draw a similar conclusion for $\mathcal{F}_s$. E.g., $\mathcal{F}_s$ might be a rank one Λ-local system on $\mathbb{A}_{\mathbb{F}_p}^1 - \{0\}$ with infinite monodromy.

For use below we make the following definition. Let $a : Y_0 \to s_0$, $p : s \to s_0$ be the structure maps. Then we define $D_c^b(Y_0,\Lambda)_F$ as the category that has the same objects as $D_c^b(Y_0,\Lambda)$ but whose Hom-spaces are given by

$$(3.5.5) \qquad\qquad \operatorname{Hom}_{D_c^b(Y_0,\Lambda)_F}(\mathcal{F},\mathcal{G}) = H^0 \operatorname{RHom}_Y(p^*\mathcal{F}, p^*\mathcal{G}).$$

The right hand side of (3.5.5) is equipped with a canonical Frobenius action, compatible with the Λ-module structure. Thus, we can think of $D_c^b(Y_0,\Lambda)_F$ as being enriched in $\operatorname{Mod}(\Lambda[F,F^{-1}])$ (the monoidal structure is given by tensoring over Λ), and more precisely, in the full subcategory of $\operatorname{Mod}(\Lambda[F,F^{-1}])$ consisting of modules that are finitely generated over Λ.

We have a functor

$$D_c^b(Y_0,\Lambda)_F \to D_c^b(Y,\Lambda) : \mathcal{F} \to p^*\mathcal{F}$$

which on the level of objects is given by the base extension and on the level of Hom-spaces amounts to forgetting the F-action.

Remark 3.5.8. The category $D_c^b(Y_0,\Lambda)_F$ is not the same as $D_c^b(Y_0,\Lambda)$. For example, the former is not triangulated.

3.6. **End of proof of Theorem 3.1.1.**

3.6.1. *Passing to the DG-context.* All the functors in (3.5.4) are induced from left derived functors between suitable abelian categories (the inverse images are exact, and so they are both left and right derived functors). Hence, they are represented by co-quasi-functors on the DG-level (as in Example 3.4.1). We may then extend the resulting bimodules to obtain co-quasi-functors

$$D_c^{b,\text{dg}}(Y(\mathbb{C}),\mathbb{C})_{\text{lit}} \leftarrow D_c^{b,\text{dg}}(Y(\mathbb{C}),\Lambda)_{\text{lit},\mathbb{C}} \leftarrow D_c^{b,\text{dg}}(Y(\mathbb{C}),\Lambda)_\mathbb{C} \leftarrow$$
$$D_c^{b,\text{dg}}(Y,\Lambda)_\mathbb{C} \supset D_{\mathcal{T},\mathcal{L}}^{b,\text{dg}}(Y,\Lambda)_\mathbb{C} \leftarrow D_{\mathcal{T}_V,\mathcal{L}_V}^{b,\text{dg}}(Y_V,\Lambda)_\mathbb{C} \to D_{\mathcal{T}_s,\mathcal{L}_s}^{b,\text{dg}}(Y_s,\Lambda)_\mathbb{C},$$

where all the arrows induce equivalences or fully faithful embeddings on the H^0-level. By Lemma 3.4.2, it follows that the DG-algebras $\underline{\operatorname{Hom}}_{D_c^{b,\text{dg}}(Y(\mathbb{C}),\mathbb{C})}(\mathcal{F},\mathcal{F})$ and $\underline{\operatorname{Hom}}_{D_c^{b,\text{dg}}(Y_s,\Lambda)_\mathbb{C}}(\mathcal{F}_s,\mathcal{F}_s)$ are isomorphic in the homotopy category of DG-algebras over $\mathbb{C}$.

Thus, it remains to construct a DG-algebra, quasi-isomorphic to $\underline{\operatorname{Hom}}_{D_c^{b,\text{dg}}(Y_s,\Lambda)_\mathbb{C}}(\mathcal{F}_s,\mathcal{F}_s)$, which carries a DG-endomorphism inducing the Frobenius on cohomology, assuming of course that $\mathcal{F}_s$ is indeed defined over a finite field. We do this in the next section.

3.6.2. *Lifting the Frobenius to the DG-level.* The technical problem we have to deal with is that p^* does not preserve injectives. So we have to follow a more complicated procedure.

For objects $\mathcal{F}_0, \mathcal{G}_0$ in $\mathrm{Sh}(Y_0^{\mathbb{N}}, \Lambda^{\bullet})$ we set

$$(3.6.1) \qquad \mathrm{Hom}_{\Lambda}(\mathcal{F}_0, \mathcal{G}_0)_F = \pi_* p^{\mathbb{N},*} a_*^{\mathbb{N}} \mathcal{H}om_{\mathrm{Sh}(Y_0^{\mathbb{N}}, \Lambda^{\bullet})}(\mathcal{F}_0, \mathcal{G}_0) \in \mathrm{Mod}(\Lambda[F, F^{-1}]),$$

where as before $a : Y_0 \to s_0$, $p : s \to s_0$ are the structure maps and π is the standard morphism of topoi $s^{\mathbb{N}} \to s$.

Note that $p^{\mathbb{N},*} a_*^{\mathbb{N}} \mathcal{H}om_{\mathrm{Sh}(Y_0^{\mathbb{N}}, \Lambda^{\bullet})}(\mathcal{F}_0, \mathcal{G}_0)$ is an object in $\mathrm{Sh}(s^{\mathbb{N}}, \Lambda^{\bullet})$, i.e., an inverse system of Λ-modules.

Lemma 3.6.1. *If $\mathcal{G}_0$ is injective then $p^{\mathbb{N},*} a_*^{\mathbb{N}} \mathcal{H}om_{\mathrm{Sh}(Y_0^{\mathbb{N}}, \Lambda^{\bullet})}(\mathcal{F}_0, \mathcal{G}_0)$ is an inverse system with surjective structure maps.*

Proof. By the smooth base change and the compatibility of étale cohomology with inverse limits [2, VII.5.8], we have $p^* a_* = a_* p^*$. Unfortunately if $\mathcal{F}$ is not constructible then $\mathcal{H}om$ does not commute with the base change. However, straight from the definitions we get

$$(a_*^{\mathbb{N}} p^{\mathbb{N},*} \mathcal{H}om_{\mathrm{Sh}(Y_0^{\mathbb{N}}, \Lambda^{\bullet})}(\mathcal{F}_0, \mathcal{G}_0))(s)^n = \varinjlim_{s'} \mathrm{Hom}_{\mathrm{Sh}(Y_{0,s'}^{\leq n}, \Lambda^{\bullet})}(\mathcal{F}_{0,s'}^{\leq n}, \mathcal{G}_{0,s'}^{\leq n}),$$

where s' runs through the factorizations $s \to s' \to s_0$ with s'/s_0 étale (=finite), and $Y^{\leq n}$ denotes the topos of inverse systems of length n.

Now if $\mathcal{G}$ is injective it is easy to see that $\mathcal{G}_{s'}$ is injective as well (since $Y_{s'}/Y$ is étale, there is an exact left adjoint, "extension by zero").

So we have to prove that if Y is a topos, $\mathcal{F}$ and $\mathcal{G}$ sheaves of $\Lambda^{\bullet}$-modules on $Y^{\mathbb{N}}$, with $\mathcal{G}$ injective, then

$$\mathrm{Hom}_{\mathrm{Sh}(Y^{\leq n+1}, \Lambda^{\bullet})}(\mathcal{F}^{\leq n+1}, \mathcal{G}^{\leq n+1}) \to \mathrm{Hom}_{\mathrm{Sh}(Y^{\leq n}, \Lambda^{\bullet})}(\mathcal{F}^{\leq n}, \mathcal{G}^{\leq n})$$

is surjective. This is an easy excercise. $\qquad \square$

We define a DG-model $D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_F$ for $D_c^b(Y_0, \Lambda)_F$ (see §3.5.6) as follows: $D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_F$ has the same objects as $D_c^b(Y_0, \Lambda)_F$, and it is enriched in $C(\Lambda[F, F^{-1}])$ by setting

$$\underline{\mathrm{Hom}}_{D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_F}(\mathcal{F}_0, \mathcal{G}_0) = \underline{\mathrm{Hom}}_{\Lambda}(I_{\mathcal{F}_0}, I_{\mathcal{G}_0})_F,$$

where the right-hand side is defined by (3.6.1), applied to injective resolutions. By the fact that $\underline{\mathrm{Hom}}_{\Lambda}(I_{\mathcal{F}_0}, I_{\mathcal{G}_0})$ is acyclic for $a_*^{\mathbb{N}}$ and by Lemma 3.6.1, we see that $\underline{\mathrm{Hom}}_{\Lambda}(I_{\mathcal{F}_0}, I_{\mathcal{G}_0})_F$ computes

$$R\pi_* p^{\mathbb{N},*} Ra_*^{\mathbb{N}} R\mathcal{H}om_{\mathrm{Sh}(Y_0^{\mathbb{N}}, \Lambda^{\bullet})}(\mathcal{F}_0, \mathcal{G}_0) = R\pi_* Ra_*^{\mathbb{N}} p^{\mathbb{N},*} R\mathcal{H}om_{\mathrm{Sh}(Y_0^{\mathbb{N}}, \Lambda^{\bullet})}(\mathcal{F}_0, \mathcal{G}_0)$$

$$= R\pi_* Ra_*^{\mathbb{N}} R\mathcal{H}om_{\mathrm{Sh}(Y_s^{\mathbb{N}}, \Lambda^{\bullet})}(p^* \mathcal{F}_0, p^* \mathcal{G}_0)$$

where we have used the fact that for constructible (cartesian in the G-equivariant case) sheaves $R\mathcal{H}om$ commutes with the base change. Since s is a point topos, a_* on Y_s is just taking global sections. Thus,

$$R\pi_* Ra_*^{\mathbb{N}} R\mathcal{H}om_{\mathrm{Sh}(Y_s^{\mathbb{N}}, \Lambda^{\bullet})}(p^* \mathcal{F}_0, p^* \mathcal{G}_0) = R\varprojlim_n \mathrm{RHom}_{\mathrm{Sh}(Y_s^{\leq n}, \Lambda^{\bullet})}(p^* \mathcal{F}_0^{\leq n}, p^* \mathcal{G}_0^{\leq n})$$

$$= \mathrm{RHom}_{D_c^b(Y_s, \Lambda)}(p^* \mathcal{F}_0, p^* \mathcal{G}_0),$$

so that $D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_F$ is indeed a DG-model for $D^b(Y_0, \Lambda)_F$.

Let $D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_F^0$, $D_c^b(Y_0, \Lambda)_F^0$ be obtained from $D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_F$, $D_c^b(Y_0, \Lambda)_F$ by forgetting the F-action. Then the fully faithful functor

$$D_c^b(Y_0, \Lambda)_F^0 \to D_c^b(Y, \Lambda) : \mathcal{F} \mapsto p^* \mathcal{F}$$

can be lifted to a co-quasi-functor

$$D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_F^0 \to D_c^{b,\mathrm{dg}}(Y, \Lambda)$$

as in Example 3.4.1. Furthermore, by Lemma 3.4.2, we deduce that for $\mathcal{F}_0 \in D_c^b(Y_0, \Lambda)$,

$$A \overset{\mathrm{def}}{=} \underline{\mathrm{Hom}}_{D_c^{b,\mathrm{dg}}(Y_0, \Lambda)_{F,\mathbb{C}}^0}(\mathcal{F}_0, \mathcal{F}_0)$$

and

$$\underline{\mathrm{Hom}}_{D_c^{b,\mathrm{dg}}(Y, \Lambda)_{\mathbb{C}}}(p^* \mathcal{F}_0, p^* \mathcal{F}_0)_{\mathbb{C}}$$

are isomorphic in the homotopy category of DG-algebras over $\mathbb{C}$. Since A is a ring object in $C(\mathbb{C}[F, F^{-1}])$, we are done.

4. FORMALITY FOR SOME ALGEBRAS OVER OPERADS

4.1. **Formulation of the main result on formality.** Throughout thi section k is a field. If $V = \bigoplus_n V_n$ is a $\mathbb{Z}$-graded k-vector space and F is a graded k-automorphism of V then we say that F acts *locally finitely* on V if every $v \in V$ is contained in a finite dimensional F-invariant graded subspace of V.

Assume now that F acts locally finitely on V and assume that k is algebraically closed. If $\alpha \in k$ then we denote by V_α the generalized eigenspace of V corresponding to α. In other words, $v \in V_\alpha$ iff there is some n such that $(F - \alpha)^n v = 0$. This yields an F-invariant direct sum decomposition $V = \bigoplus_{\alpha \in k^*} V_\alpha$ of graded vector spaces. We will think of this decomposition as a refinement of the given $\mathbb{Z}$-grading to a $\mathbb{Z} \times k^*$ grading.

We now assume in addition that $k = \mathbb{C}$. In that case we fix throughout some strictly positive real number ξ and we say that a pair (V, F), with V a graded k-vector space and F a k-automorphism of V, is *pure of weight m* (with respect to ξ) if F acts locally finitely on V and if $n \in \mathbb{Z}$, $\alpha \in k^*$ are such that $V_{n,\alpha} \neq 0$ then $|\alpha| = \xi^{n+m}$.

If V is a complex and F is an endomorphism of degree zero of V (i.e., V commutes with the differential) then we say that (V, F) is pure of weight m if $(F, H^*(V))$ is pure of weight m.

We now fix some k-DG-operad $\mathcal{O}$. Below we will prove the following result.

Theorem 4.1.1. *Assume that $k = \mathbb{C}$. Let $F : B \to B$ be an $\mathcal{O}$-algebra quasi-isomorphism such that (B, F) is pure of weight zero. Then B is formal. In other words, B is isomorphic to $(H^*(B), d = 0)$ in the* homotopy *category of $\mathcal{O}$-algebras. The latter is by definition the category of $\mathcal{O}$-algebras with quasi-isomorphisms inverted.*

Assume now that B itself has zero differential, and let N be a module over B equipped with a k-linear endomorphism $F : N \to N$ of complexes, such that for any $l \in \mathbb{N}$ and $\mu \in \mathcal{O}(l)$ we have $F(\mu(b_1, \ldots, b_{l-1}, n)) = \mu(F(b_1), \ldots, F(b_{l-1}), Fn)$ for $b_1, \ldots, b_{l-1} \in B$, $n \in N$. Assume in addition that (N, F) is pure (of some weight). Then N is formal. In other words, N is isomorphic in $D(B)$ to $(H^*(N), d = 0)$.

This theorem puts a number of partial results in the literature in the proper context—see e.g., [21, Cor. 5.3.7], [56], and [64, Thm 12.7]. Among other things these prior results assume that $B_n = 0$ for $n < 0$ and $B_0 = k$, since they depend on Sullivan's theory of minimal models.

Theorem 4.1.1 will be a consequence of the following result.

Theorem 4.1.2. *Assume that either k is a field of characteristic zero or that $\mathcal{O}$ is obtained from an asymmetric operad. Let $F : B \to B$ be an $\mathcal{O}$-algebra quasi-isomorphism such F acts locally finitely on $H^*(B)$. Then there is a quasi-isomorphism of $\mathcal{O}$-algebras $\alpha : A \to B$, where A is in addition equipped with a k-linear locally finite automorphism F', such that the diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \alpha\ } & B \\
{\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\
A & \xrightarrow[\ \alpha\ ]{} & B
\end{array}
$$

is commutative in the homotopy category of $\mathcal{O}$-algebras.

Similarly assume that B itself has zero differential. For a module P over B write $_F P$ for the module over B, which is the same as P as a complex but with the twisted B-action

$$_F \mu(b_1, \ldots, b_{l-1}, n) = \mu(F(b_1), \ldots, F(b_{l-1}), n),$$

where $\mu \in \mathcal{O}(l)$, $b_1, \ldots, b_{l-1} \in B$, $n \in N$.

Let N be a module over B equipped with a k-linear quasi-isomorphism $F : N \to {}_F N$ which acts locally finitely on $H^*(N)$. Then there is a quasi-isomorphism of B-modules $\alpha : M \to N$, where M is in addition

equipped with a locally finite k-isomorphism $F' : M \to {}_F M$, such that the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\;\alpha\;} & N \\
{\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\
{}_F M & \xrightarrow[\;\alpha\;]{} & {}_F N
\end{array}
$$

is commutative in $D(B)$.

We will prove Theorem 4.1.2 first in the characteristic zero case. In Section §4.6 we will give trivial modifications needed for the asymmetric operad case. Theorem 4.1.1 will be deduced from Theorem 4.1.2 in §4.7.

So, until further notice, k is a field of characteristic zero, A, B, C, D are $\mathcal{O}$-algebras, V, W are complexes over k, and v, w, a, b, c, d are elements of V, W, A, B, C, D, respectively.

Unspecified maps in diagrams below are assumed to be k-linear maps of complexes, unless they are between $\mathcal{O}$-algebras in which case they are assumed to be $\mathcal{O}$-algebra morphisms. Maps indexed by t are homotopies between $\mathcal{O}$-algebra morphisms (see §4.2 below for details). We denote by s the suspension functor on complexes.

Some maps between complexes serve as homotopies, or even as 2-homotopies, in which case they have degree -1,-2 respectively—this will be clear from the context.

4.2. **Homotopies between $\mathcal{O}$-algebra morphisms.** If $\alpha, \beta : A \to B$ are morphisms of $\mathcal{O}$-algebras then a homotopy between α and β is a pair (ϕ_t, h_t) where $\phi_t : A \to B[t]$ is an $\mathcal{O}$-algebra morphism such that $\phi_0 = \alpha$, $\phi_1 = \beta$, and $h_t : A \to B[t]$ is a ϕ_t-derivation of degree -1 satisfying

(4.2.1)
$$
\frac{\partial \phi_t}{\partial t} = dh_t \stackrel{\text{def}}{=} d_B \circ h_t + h_t \circ d_A.
$$

Note that (4.2.1) implies that ϕ_t is completely determined by h_t. Hence, it is not necessary to specify ϕ_t.

This definition of homotopy corresponds to a right homotopy in the sense of [59] associated to the path-object $B^I = B \otimes \Omega_{\mathbb{A}^1} = B[t] \oplus B[t]dt$. It is well-known and easy to see that if there exists a homotopy between α and β then α is equal to β in the homotopy category of $\mathcal{O}$-algebras (see e.g., [8, Prop. B.2]).

4.3. **Adding variables to kill cocycles.** We recall a construction from [28]. Let $i : s^{-1}V \to A$ be a map of complexes. Then as a graded $\mathcal{O}$-algebra $A\langle V, i\rangle$ is the coproduct of A and $T_{\mathcal{O}}V$ over k. We equip $A\langle V, i\rangle$ with a differential $\tilde{d}$ given by

$$
\tilde{d}a = d_A a,
$$
$$
\tilde{d}v = d_V v + c_i i(s^{-1}v),
$$

where $c_i : A \to A\langle V, i\rangle$ is the canonical map. Note that $c_i i = dh_i$ where $h_i(s^{-1}v) = v$. In particular, the composition

$$
s^{-1}V \xrightarrow{i} A \xrightarrow{c_i} A\langle V, i\rangle
$$

is zero on cohomology. Whence the title of this section.

A diagram of the form

$$
s^{-1}V \xrightarrow{i} A \xrightarrow{\alpha} B
$$

with h_α over the arc, $\alpha i = dh_\alpha$ can be transformed into a commutative diagram of $\mathcal{O}$-algebras

$$
A \xrightarrow{c_i} A\langle V, i\rangle \xrightarrow{\tilde{\alpha}} B
$$

with α over the arc, by setting $\tilde{\alpha}(v) = h_\alpha(s^{-1}v)$.

Suppose now we have the following diagram

(4.3.1)

such that besides the identities $dh_\alpha = \alpha i$, $dh_\gamma = \gamma j$ we have that $h_{2,t}$ is a homotopy of DG-algebras between $F_3\alpha$ and γF_2, and

$$(4.3.2) \qquad jF_1 - iF_2 = dh_1,$$

$$(4.3.3) \qquad h_\gamma F_1 - F_3 h_\alpha = \gamma h_1 + \left(\int_0^1 h_{2,t} dt \right) i + dp.$$

We can transform this into a diagram of the form

(4.3.4)

by setting

$$\tilde{F}_2(a) = F_2(a),$$
$$\tilde{F}_2(v) = F_1(v) - h_1(s^{-1}v),$$
$$\tilde{h}_{2,t}(a) = h_{2,t}(a),$$
$$\tilde{h}_{2,t}(v) = p(s^{-1}v),$$

where we have interpreted F_1 as a map $V \to W$ via $F_1(v) = sF_1(s^{-1}v)$.

4.4. **The proof for Theorem 4.1.2 for $\mathcal{O}$-algebras.** Theorem 4.1.2 is a consequence of the following slightly more specific theorem.

Theorem 4.4.1. *Let B be an $\mathcal{O}$-algebra and $F : B \to B$ an $\mathcal{O}$-algebra quasi-isomorphism which acts locally finitely on $H^*(B)$. Then there exists a diagram*

where α is a quasi-isomorphism, $F' : A \to A$ is a k-linear $\mathcal{O}$-algebra automorphism which acts locally finitely, and h_t is a homotopy of $\mathcal{O}$-algebras between $F\alpha$ and $\alpha F'$.

Proof. We will construct a sequence of $\mathcal{O}$-algebra morphisms

$$k = A_0 \xrightarrow{\beta_0} A_1 \xrightarrow{\beta_1} A_2 \xrightarrow{\beta_2} \cdots \quad B$$

with $\alpha_0, \alpha_1, \alpha_2$ arcs to B

where

(1) $\alpha_0 : k \to B$, $\beta_0 : k \to A$ are the unit maps;
(2) $\alpha_{i+1}\beta_i = \alpha_i$;
(3) α_i for $i > 0$ is surjective on the level of cohomology;
(4) the composition

$$(4.4.1) \qquad\qquad s^{-1}(\operatorname{cone}\alpha_i) \to A_i \xrightarrow{\beta_i} A_{i+1}$$

is zero on cohomology.

From this it will easily follow that the induced map

$$A \stackrel{\text{def}}{=} \varinjlim_i A_i \xrightarrow{\alpha \stackrel{\text{def}}{=} \varinjlim \alpha_i} B$$

is a quasi-isomorphism.

In addition, we will construct diagrams of the form

$$(4.4.2) \qquad\qquad \begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & B \\ {\scriptstyle F_i}\downarrow & {\scriptstyle h_{i,t}} & \downarrow{\scriptstyle F} \\ A_i & \xrightarrow{\alpha_i} & B \end{array}$$

such that F_i is a locally finite $\mathcal{O}$-algebra automorphism of A_i, and $h_{i,t}$ is a homotopy between $F\alpha_i$ and $\alpha_i F_i$ such that

$$h_{i+1,t}\alpha_i = h_{i,t}.$$

We proceed inductively. The induction starts with the given α_0, β_0 and $h_{0,t} = 0$. Next, assume we have a diagram like (4.4.2). Set $C_i = s^{-1}(\operatorname{cone}\alpha_i)$. If we identify C_i with the column vectors

$$\begin{pmatrix} A_i \\ s^{-1}B \end{pmatrix}$$

then the differential on C_i is given by

$$\begin{pmatrix} d_{A_i} & 0 \\ \alpha & d_{s^{-1}B} \end{pmatrix}.$$

We may now extend (4.4.2) to a diagram[4]

$$(4.4.3) \qquad\qquad \begin{array}{ccccc} & q & & & \\ C_i & \xrightarrow{p} & A_i & \xrightarrow{\alpha_i} & B \\ {\scriptstyle G_i}\downarrow & & {\scriptstyle F_i}\downarrow & {\scriptstyle h_{i,t}} & \downarrow{\scriptstyle F} \\ C_i & \xrightarrow{p} & A_i & \xrightarrow{\alpha_i} & B \\ & q & & & \end{array}$$

where p, q are the projection maps and

$$G_i = \begin{pmatrix} F_i & 0 \\ \int_0^1 h_{i,t}dt & F \end{pmatrix}.$$

---

[4]This is of course just the TR3 axiom but we need the precise nature of the maps.

We have

$$\alpha_i p = dq,$$
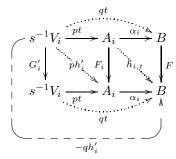
$$qG_i - Fq = \left(\int_0^1 h_{i,t}dt\right)p.$$

Set $V_i = sH^*(C_i)$, $G_i' = H^*(G_i)$. Next, we construct a diagram of complexes

(4.4.4)

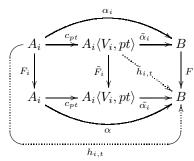where t is an arbitrary quasi-isomorphism $s^{-1}V_i \to C_i$, and $G_i't - tG_i = dh_i'$. We may combine (4.4.3) and (4.4.4) to obtain

with the same relations as in (4.3.1). Hence, we may transform this diagram into a diagram of the form (4.3.4):

Finally, we set

$$A_{i+1} = A_i\langle V_i, pt\rangle,$$
$$F_{i+1} = \tilde{F}_i,$$
$$h_{i+1} = \tilde{h}_i,$$
$$\beta_i = c_{pt}.$$

It remains to check several things.

(1) F_i acts locally finitely on A_i (and hence so does F' on A). We prove this by induction. The case $i = 0$ is clear. Now assume that F_i' acts locally finitely on A_i for some $i \geq 0$. Then by the long exact sequence for cohomology it follows that G_i' acts locally finitely on $H^*(C_i) = s^{-1}V_i$. Now if we equip $A_{i+1} = A_i\langle V_i, pt\rangle$ with the ascending filtration obtained from the grading on $A_i\langle V_i, pt\rangle$ defined by $|A_i| = 0$, $|V_i| = 1$, then we obtain $\operatorname{gr} A_{i+1} = A_i \coprod_k T_{\mathcal{O}}V_i$, and the induced action of $\operatorname{gr} F_{i+1} = \operatorname{gr}\tilde{F}_i$ is given by $F_i \coprod G_i'$. Since both F_i and G_i' act locally finitely, the same holds for $F_i \coprod G_i'$ and hence for F_{i+1}.

(2) F_i is an automorphism of A_i (and hence F' is an automorphism of A). This is again proved by induction. It is clearly true for A_0. Assume it is true for some A_i. By the same argument as above we find that $\operatorname{gr} F_{i+1}$ is an automorphism of $\operatorname{gr} A_{i+1}$. It follows easily that F_{i+1} is an automorphism of A_{i+1}.

(3) α_i for $i > 0$ is surjective on the level of cohomology. It suffices to prove this for A_1, which is a simple verification.

(4) The composition (4.4.1) is zero on cohomology. To see this note that it is sufficient to prove that the longer composition

$$(4.4.5) \qquad s^{-1}V_i \to s^{-1}(\operatorname{cone} \alpha_i) \to A_i \xrightarrow{\beta_i} A_{i+1}$$

is zero on cohomology since the first map is a quasi-isomorphism. The fact that (4.4.5) is zero on cohomology is clear since it is just the composition

$$s^{-1}V_i \xrightarrow{pt} A_i \to A_i\langle V, pt\rangle$$

which was discussed in §4.3. $\qquad\square$

4.5. Theorem 4.1.2 for modules.
The proof for modules is entirely analogous to the one for $\mathcal{O}$-algebras. So we will provide very few details.

The key ingredient is an analogue of $A\langle V, i\rangle$ (see §4.3). Let M be a module over A and let $i : s^{-1}V \to M$ be a map of complexes. Then as graded A-module $M\langle V, i\rangle$ is equal to $M \bigoplus A \otimes_k V$. We equip $M\langle V, i\rangle$ with the differential $\tilde{d}$ given by

$$\tilde{d}m = d_M m,$$
$$\tilde{d}v = d_V v + c_i i(s^{-1}v),$$

for $m \in M$, $v \in V$, where $c_i : M \to M\langle V, i\rangle$ is the canonical map. With this definition one may imitate the above proof for the case of $\mathcal{O}$-algebras. The only slight variation is that the diagram (4.4.2) now takes the form

$$(4.5.1) \qquad \begin{array}{ccc} M_i & \xrightarrow{\alpha_i} & N \\ {\scriptstyle F_i}\downarrow & {\scriptstyle h_i} \searrow & \downarrow{\scriptstyle F} \\ {}_F M_i & \xrightarrow{\alpha_i} & {}_F N \end{array}$$

where h_i is a homotopy between $F\alpha_i$ and $\alpha_i F_i$.

4.6. The asymmetric operad case.
Assume now that k is arbitrary but that $\mathcal{O}$ is obtained from an asymmetric operad which we denote by $\mathcal{O}$ as well. In this case the following different version of homotopy is more convenient.

If $\alpha, \beta : A \to B$ are morphisms of $\mathcal{O}$-algebras then a homotopy between α and β will be a (α, β)-derivation $h : A \to B$ of degree -1 such that $\beta - \alpha = dh$. This notion corresponds to a right homotopy associated the path object of upper triangular 2×2-matrices (see [40, §3.1]). The latter is characteristic free but can only be defined for asymmetric operads.

We may now adapt the proof of Theorem 4.1.2 to this new notion of homotopy. Since the homotopies are now constant, there is no t-parameter and consequently no integrals in the formulas. Everything else is the same.

4.7. Proof of Theorem 4.1.1.
Let (B, F) be as in the statement of Theorem 4.1.1. Let (A, F') be as in Theorem 4.1.1. By the construction of A, we see that the generalized eigenvalues μ of F' acting on A have the property $|\mu| = \zeta^n$ for some $n \in \mathbb{Z}$. Set

$$A_{n,m} = \bigoplus_{|\mu|=\zeta^m} A_{n,\mu}$$

It is easy to see that this defines a $\mathbb{Z}^2$-grading on A with the differential sending $A_{n,m}$ to $A_{n+1,m}$. The purity hypothesis implies that the cohomology of A is supported on the diagonal $\{(m, m) \mid m \in \mathbb{Z}\}$. To

conclude we may now use the standard subalgebra of A, quasi-isomorphic both to A and to its cohomology (see [61, Prop. 4]). Recall that this subalgebra A' is given by

$$A'_{ij} = \begin{cases} A_{ij} & i < j, \\ \ker(d : A_{ij} \to A_{i+1,j}), & i = j, \\ 0 & i > j. \end{cases}$$
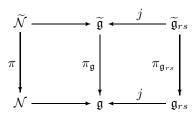
The argument for modules is similar.

5. Construction of modules via Springer correspondence

5.1. **Springer correspondence.** Starting from this section we always assume that $k = \mathbb{C}$. Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra. We denote by X the flag variety associated with $\mathbf{G}$, i.e., the variety of Borel subgroups in $\mathbf{G}$.

We start by recalling the construction of the Springer correspondence via perverse sheaves (see [49], [34, ch. 13]). Let

$$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$$

be the Springer resolution of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. Recall that this resolution fits into the diagram with cartesian squares

where $\widetilde{\mathfrak{g}} \subset \mathfrak{g} \times X$ is the set of pairs $(x, \mathbf{B})$ such that $x \in \mathrm{Lie}(\mathbf{B})$, $\mathfrak{g}_{rs} \subset \mathfrak{g}$ is the set of regular semisimple elements. The construction of the Springer correspondence is based on the study of the complex of constructible sheaves on $\mathcal{N}$, called the *Springer sheaf*,

$$\mathcal{A} := R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] \simeq R\pi_{\mathfrak{g},*} \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[\dim \mathcal{N}]|_{\mathcal{N}}.$$

We view $\mathcal{A}$ as an object of the equivariant derived category $D^b_{\mathbf{G}}(\mathcal{N})$.

The map $\pi_{\mathfrak{g}}$ is small, so we have an isomorphism of $\mathbf{G}$-equivariant perverse sheaves

$$R\pi_{\mathfrak{g},*} \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}] = j_{!*} \pi_{\mathfrak{g}_{rs},*} \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}_{rs}}[\dim \mathfrak{g}].$$

Now the natural W-action on fibers of $\pi_{\mathfrak{g}_{rs}}$ induces an action of W on $\pi_{\mathfrak{g},*} \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}$ and hence on $\mathcal{A}$ (obtained by restriction to $\mathcal{N}$). Restricting the sheaf $\mathcal{A}$ to a point $a \in \mathcal{N}$ we obtain the *Springer W-action* on $H^*(X_a, \mathbb{C})$, where

$$X_a = \pi^{-1}(a) \subset X$$

is the *Springer fiber*. Similarly, for a subgroup $K \subset \mathbf{G}_a$, where $\mathbf{G}_a \subset \mathbf{G}$ is the stabilizer of a in $\mathbf{G}$, we obtain a W-action on the equivariant cohomology $H^*_K(X_a, \mathbb{C})$.

The map π is semismall, so $\mathcal{A}$ is a ($\mathbf{G}$-equivariant) perverse sheaf. Using the decomposition theorem (see [7, Thm. 6.2.5]) one shows that $\mathcal{A}$ decomposes into a direct sum

(5.1.1) $$\mathcal{A} = \bigoplus_{\chi \in \mathrm{Irr}\, W} \chi \otimes S_{a,\xi},$$

where $S_{a,\xi}$ is an irreducible $\mathbf{G}$-equivariant perverse sheaf on $\mathcal{N}$ of the form

$$S_{a,\xi} = j_{a,!*} \mathcal{L}_\xi[\dim O_a],$$

where $a \in \mathcal{N}$, $O_a = \mathbf{G} \cdot a$, $j_a : O_a \to \mathcal{N}$ is the embedding, ξ is an irreducible representation of the component group $\mathbf{G}_a/\mathbf{G}_a^0$ (where $\mathbf{G}_a^0$ is the connected component of 1 in $\mathbf{G}_a$), and $\mathcal{L}_\xi$ is the local system on O_a associated with ξ. This gives a bijection between $\mathrm{Irr}\, W$ and certain pairs (O_a, ξ) (it is known that all nilpotent orbits occur). The irreducible representation $\chi(a, \xi)$ corresponding to (O_a, ξ) appears as

the ξ-isotypic component in the top degree cohomology of the Springer fiber X_a. Indeed, restricting the decomposition of $\mathcal{A}$ to a point $a \in \mathcal{N}$ we get

$$H^*(X_a, \mathbb{C})[\dim \mathcal{N} - \dim O_a] \simeq \mathcal{A}|_a[-\dim O_a] \simeq \bigoplus_\xi \chi(a, \xi) \otimes \xi \oplus \dots,$$

where the sum is over $\xi \in \operatorname{Irr} \mathbf{G}_a/\mathbf{G}_a^0$ such that $S_{a,\xi}$ occurs in the decomposition (5.1.1), the remaining summands correspond to Goresky-Macpherson extensions from orbits containing O_a in its closure, and hence, live in negative degrees. Thus, using the equality

(5.1.2) $$2 \dim X_a = \dim \mathcal{N} - \dim O_a$$

(see [34, Thm. 10.11]) we deduce the isomorphism of $W \times \mathbf{G}_a/\mathbf{G}_a^0$-representations

$$H^{2 \dim X_a}(X_a, \mathbb{C}) \simeq \bigoplus_\xi \chi(a, \xi) \otimes \xi.$$

Note that choosing a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ we obtain an isomorphism

(5.1.3) $$H_{\mathbf{G}}^*(\widetilde{\mathcal{N}}) \simeq H_{\mathbf{G}}^*(\mathbf{G} \times_{\mathbf{B}} \mathfrak{n}) \simeq H_{\mathbf{G}}^*(\mathbf{G}/\mathbf{B}) \simeq H_{\mathbf{B}}^*(pt) = H_{\mathbf{T}}^*(pt),$$

where $\mathfrak{n}$ is the nilpotent radical in $\operatorname{Lie}(\mathbf{B})$, $\mathbf{T} = \mathbf{B}/[\mathbf{B}, \mathbf{B}]$ is the maximal torus. Interpreting this cohomology as $\operatorname{Ext}_{\mathbf{G}}^*(\mathbb{C}_{\widetilde{\mathcal{N}}}, \mathbb{C}_{\widetilde{\mathcal{N}}})$ and combining it with the W-action on $\mathcal{A}$ we get a homomorphism

(5.1.4) $$A_W := W \ltimes H_{\mathbf{T}}^*(pt) \to \operatorname{Ext}_{\mathbf{G}}^\bullet(\mathcal{A}, \mathcal{A})$$

of graded algebras. The crucial fact for us is that this map is an isomorphism. This is explicitly stated in [38, Thm. 3.1], as a consequence of [52, Thm. 8.11], where the $\mathbf{G} \times \mathbb{G}_m$-equivariant Ext-algebra is identified with the graded Hecke algebra.

Thus, we have a graded action of A_W on $\mathcal{A}$, i.e., a homogeneous element $P \in A_W$ of degree d acts as a map $P : \mathcal{A} \to \mathcal{A}[d]$ in the $\mathbf{G}$-equivariant category. Now, given a point $a \in \mathcal{N}$, we get by restriction a graded action of A_W on

$$\mathcal{A}|_{O_a} \simeq R\pi_{O_a, *}\underline{\mathbb{C}}_{\pi^{-1}(O_a)}[\dim \mathcal{N}]$$

by $\mathbf{G}$-equivariant morphisms, where $\pi_{O_a} : \pi^{-1}(O_a) \to O_a$ is the natural projection. Using the identifications $O_a \simeq \mathbf{G}/\mathbf{G}_a$ and

(5.1.5) $$\pi^{-1}(O_a) \simeq \mathbf{G} \times_{\mathbf{G}_a} X_a,$$

we can identify $\mathbf{G}$-equivariant sheaves on $\pi^{-1}(O_a)$ (resp., O_a) with $\mathbf{G}_a$-equivariant sheaves on X_a (resp., pt), so that the $\mathbf{G}$-equivariant push-forward with respect to the projection π_{O_a} corresponds to taking the $\mathbf{G}_a$-equivariant cohomology. Thus, the above construction gives a structure of a graded A_W-module on the $\mathbf{G}_a$-equivariant cohomology of Springer fibers, $H_{\mathbf{G}_a}^*(X_a, \mathbb{C})$. More generally, if ξ is an irreducible representation of the component group $\mathbf{G}_a/\mathbf{G}_a^0$, then twisting $\mathcal{A}|_{O_a}$ by the corresponding local system $\mathcal{L}_\xi$, we get after the above identification a structure of a graded A_W-module on $(\xi \otimes H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C}))^{\mathbf{G}_a/\mathbf{G}_a^0}$ (here we used Lemma 1.4.1(ii)). We need some facts about this module structure.

Lemma 5.1.1. *The action of $H_{\mathbf{G}}^*(pt)$ on $(\xi \otimes H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C}))^{\mathbf{G}_a/\mathbf{G}_a^0}$, induced by the homomorphism $H_{\mathbf{G}}^*(pt) \to H_{\mathbf{G}_a}^*(pt) \simeq H_{\mathbf{G}_a^0}^*(pt)^{\mathbf{G}_a/\mathbf{G}_a^0}$ and by the natural action of $H_{\mathbf{G}_a^0}^*(pt)$ on the $\mathbf{G}_a^0$-equivariant cohomology, coincides with the restriction of the action of $H_{\mathbf{T}}^*(pt) \subset A_W$ via the natural embedding $H_{\mathbf{G}}^*(pt) \to H_{\mathbf{T}}^*(pt)$.*

Proof. Recall that the action of $H_{\mathbf{T}}^*(pt) \subset A_W$ on $(\xi \otimes H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C}))^{\mathbf{G}_a/\mathbf{G}_a^0}$ is given by the homomorphism $H_{\mathbf{T}}^*(pt) \simeq H_{\mathbf{G}}^*(X) \to H_{\mathbf{G}_a}^*(X_a)$ induced by the embedding $X_a \to X$. The commutative square of morphisms of quotient stacks

$$\begin{array}{ccc} [X_a/\mathbf{G}_a] & \longrightarrow & [X/G] \\ \downarrow & & \downarrow \\ [pt/\mathbf{G}_a] & \longrightarrow & [pt/G] \end{array}$$

leads to a commutative square of homomorphisms

$$
\begin{array}{ccc}
H^*_{\mathbf{G}_a}(X_a) & \longleftarrow & H^*_{\mathbf{G}}(X) \\
\uparrow & & \uparrow \\
H^*_{\mathbf{G}_a}(pt) & \longleftarrow & H^*_{\mathbf{G}}(pt)
\end{array}
$$

This reduces us to checking that the natural $H^*_{\mathbf{G}}(pt)$-action on $H^*_{\mathbf{G}}(X)$ is given by the homomorphism $H^*_{\mathbf{G}}(pt) \to H^*_{\mathbf{T}}(pt)$ via the isomorphism (5.1.3). To this end we note that $H^*_{\mathbf{G}}(X) = H^*_{\mathbf{G} \times \mathbf{B}}(\mathbf{G})$ and we have to calculate the $H^*_{\mathbf{G} \times \mathbf{B}}(pt)$-action on the latter space. Finally, we observe that $\mathbf{G} = (\mathbf{G} \times \mathbf{B})/\Delta(\mathbf{B})$, where $\Delta(\mathbf{B})$ is the diagonal copy of $\mathbf{B}$. Hence, there is an isomorphism

$$
H^*_{\mathbf{G} \times \mathbf{B}}(\mathbf{G}) \widetilde{\to} H^*_{\Delta(\mathbf{B})}(pt)
$$

compatible with the restriction homomoprhism $H^*_{\mathbf{G} \times \mathbf{B}}(pt) \to H^*_{\Delta(\mathbf{B})}(pt)$. Thus, the $H^*_{\mathbf{G} \times \mathbf{B}}(pt)$-action on $H^*_{\mathbf{G} \times \mathbf{B}}(\mathbf{G})$ factors through the latter homomorphism. $\qquad \square$

Lemma 5.1.2. *(i) For a subgroup $K \subset \mathbf{G}_a$ the restriction homomoprhism $H^*_K(X, \mathbb{C}) \to H^*_K(X_a, \mathbb{C})$ is compatible with the Springer W-actions.*
(ii) The natural isomorphism

$$
H^*_{\mathbf{T}}(\mathbf{G}/\mathbf{B}, \mathbb{C}) \xrightarrow{\ \sim\ } H^*_{\mathbf{T}}(\mathbf{G}/\mathbf{T}, \mathbb{C})
$$

is compatible with W-actions, where W acts on $H^*_{\mathbf{T}}(\mathbf{G}/\mathbf{B}, \mathbb{C})$ by Springer construction and it acts on $H^*_{\mathbf{T}}(\mathbf{G}/\mathbf{T}, \mathbb{C})$ via the geometric action on $\mathbf{G}/\mathbf{T}$.
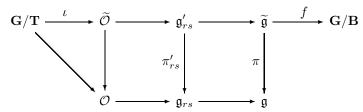
Proof. (i) We have a natural isomorphism

$$
(5.1.6) \qquad H^*_K(X, \mathbb{C}) \xrightarrow{\ \sim\ } H^*_K(\widetilde{\mathfrak{g}}, \mathbb{C}) \simeq H^*_K(\mathfrak{g}, R\pi_* \mathbb{C}),
$$

where the first arrow is the pull-back with respect to the projection $\widetilde{\mathfrak{g}} \to X$. It is easy to see that the restriction homomorphism $H^*_K(X, \mathbb{C}) \to H^*_K(X_a, \mathbb{C})$ is equal to the composition of (5.1.6) with the natural map

$$
(5.1.7) \qquad H^*_K(\mathfrak{g}, R\pi_* \mathbb{C}) \to i^*_a R\pi_* \mathbb{C} \simeq H^*_K(X_a, \mathbb{C}).
$$

In the case $a = 0$ this shows that the map (5.1.6) is W-equivariant (cr. [34, Lem. 13.6] for a similar argument). Since the map (5.1.7) is also W-equivariant, the assertion follows.
(ii) The same statement about non-equivariant cohomology is proved in [34, Prop. 13.7]. Recall that the proof is based on the commutative diagram

$$
\begin{array}{ccccccccc}
\mathbf{G}/\mathbf{T} & \xrightarrow{\ \iota\ } & \widetilde{\mathcal{O}} & \longrightarrow & \mathfrak{g}'_{rs} & \longrightarrow & \widetilde{\mathfrak{g}} & \xrightarrow{\ f\ } & \mathbf{G}/\mathbf{B} \\
& \searrow & \downarrow & & \downarrow{\scriptstyle \pi'_{rs}} & & \downarrow{\scriptstyle \pi} & & \\
& & \mathcal{O} & \longrightarrow & \mathfrak{g}_{rs} & \longrightarrow & \mathfrak{g} & &
\end{array}
$$

where $\mathcal{O} \subset \mathfrak{g}_{\mathbf{rs}}$ is the $\mathbf{G}$-orbit of some semisimple element in $\mathfrak{t}$, $\mathfrak{g}'_{rs} \subset \mathbf{G}/\mathbf{T} \times \mathfrak{g}_{rs}$ is the subvariety of $(g\mathbf{T}, x)$ such that $x \in \mathrm{Ad}(g)(\mathfrak{t})$, $\widetilde{\mathcal{O}} \subset \mathfrak{g}'_{rs}$ is the preimage of $\mathcal{O}$ under the natural projection π'_{rs}. The map ι is induced by the action of $\mathbf{G}$ on $\mathfrak{g}'_{rs}$. The only additional point that we have to make is that all the varieties in the above diagram are equipped with the action of $\mathbf{G}$ (and hence of $\mathbf{T} \subset \mathbf{G}$) and all the maps are $\mathbf{G}$-equivariant. Now the argument of [34, Prop. 13.7] can be repeated literally using the $\mathbf{T}$-equivariant cohomology. $\qquad \square$

Remark 5.1.3. The assertion of Lemma 5.1.2 for $K = \{1\}$ is proved also in [32, Thm. 1.1] using the original definition of the W-action due to Springer. We use Lusztig's definition of the W-action which differs from the original one by the twist with the sign character of W (see [31]).

5.2. **Construction of modules using Springer correspondence.** Let

$$A_W^{dg} := \mathrm{RHom}_{\mathbf{G}}^{\bullet}(\mathcal{A}, \mathcal{A})$$

be the DG-algebra of endomorphisms of $\mathcal{A}$ (where we use the DG-enhancement constructed in Section 3).

Theorem 5.2.1. *The DG-algebra A_W^{dg} is formal, so it is quasi-isomorphic to A_W.*

Proof. By Theorem 4.1.1, it is enough to verify the purity of the action of Frobenius on A_W. But the elements of W correspond to the endomorphisms of $\mathcal{A}$ defined over a finite field, so they commute with the Frobenius, while the algebra $H_{\mathbf{T}}^*(pt) \simeq H_{\mathbf{G}}^*(\mathbf{G}/\mathbf{B})$ is generated by the 1st Chern classes of line bundles defined over a finite field, which are pure classes in H^2. $\qquad\square$

We denote by $D(dg - A_W - \mathrm{mod})$ the category of DG-modules over A_W, viewed as a DG-algebra with zero differential. Let $\mathrm{Perf}(dg - A_W - \mathrm{mod}) \subset D(dg - A_W - \mathrm{mod})$ denote the subcategory of perfect DG-modules. Consider the duality functor

$$(5.2.1) \qquad \mathbb{D} : D(dg - A_W - \mathrm{mod})^{\circ} \to D(dg - A_W - \mathrm{mod}) : M \mapsto \mathrm{RHom}_{H_{\mathbf{G}}^*(pt)}(M, H_{\mathbf{G}}^*(pt)),$$

where we view $H_{\mathbf{G}}^*(pt) = H_{\mathbf{T}}^*(pt)^W$ as a central subalgebra of A_W. The derived functor RHom here can be defined using the bar-resolution as in [10, Sec. 10.12]. Note that for an A_W-module M the space $\mathrm{Hom}_{H_{\mathbf{G}}^*(pt)}(M, H_{\mathbf{G}}^*(pt))$ is equipped with a right A_W-module structure. We convert it into a left A_W-module structure using the isomorphism $A_W^{\circ} \to A_W$ sending w to w^{-1} and identical on $H_{\mathbf{T}}^*(pt)$. It is easy to see that the restriction of the functor (5.2.1) to $\mathrm{Perf}(dg - A_W - \mathrm{mod})$ is an involution.

We have a natural contravariant functor

$$(5.2.2) \qquad \mathrm{RHom}_{\mathbf{G}}(?, \mathcal{A}) : D_{\mathbf{G}}(\mathcal{N})^{\circ} \to D(A_W^{dg} - \mathrm{mod}).$$

Let $D_{\mathbf{G},\mathrm{Spr}}(\mathcal{N}) \subset D_{\mathbf{G}}(\mathcal{N})$ be the thick subcategory generated by $\mathcal{A}$. Then the functor (5.2.2) induces an equivalence of $D_{\mathbf{G},\mathrm{Spr}}(\mathcal{N})^{\circ}$ with the subcategory of $D(A_W^{dg} - \mathrm{mod})$ generated by A_W^{dg}. Recall that by Theorem 5.2.1, the DG-algebra is quasi-isomorphic to A_W. Thus, (5.2.2) induces an equivalence

$$(5.2.3) \qquad \Phi : D_{\mathbf{G},\mathrm{Spr}}(\mathcal{N})^{\circ} \xrightarrow{\sim} \mathrm{Perf}(dg - A_W - \mathrm{mod}) : F \mapsto \mathrm{RHom}_{\mathbf{G}}(F, \mathcal{A}).$$

Combining it with the duality (5.2.1) we also get an equivalence

$$\mathbb{D}\Phi : D_{\mathbf{G},\mathrm{Spr}}(\mathcal{N}) \xrightarrow{\sim} \mathrm{Perf}(dg - A_W - \mathrm{mod}).$$

Note that a similar equivalence is constructed in [60] in a somewhat different way.

Proposition 5.2.2. *(i) For any (O_a, ξ) appearing in the Springer correspondence, one has $j_{a,!}\mathcal{L}_{\xi} \in D_{\mathbf{G},\mathrm{Spr}}(\mathcal{N})$.*
(ii) Let S be the set of nilpotent orbits forming a stratification of a closed $\mathbf{G}$-invariant subset in $\mathcal{N}$. Let Spr_S be the set of all (a, ξ) appearing in the Springer correspondence such that $O_a \in S$, and let $D_{\mathbf{G},\mathrm{Spr}_S}(\mathcal{N})$ be the triangulated subcategory of $D_{\mathbf{G}}(\mathcal{N})$ generated by the simple perverse sheaves associated with Spr_S. Then the objects $(j_{a,!}\mathcal{L}_{\xi})_{(a,\xi)\in\mathrm{Spr}_S}$ generate $D_{\mathbf{G},\mathrm{Spr}_S}(\mathcal{N})$.

Proof. (i) This is deduced in [38] (see Claim B on p.19 and Theorem 3.1.8) from the orthogonality relation [51, 24.8c] for perverse sheaves arising in the generalized Springer correspondence of [50] from different cuspidal data.
(ii) If S consists of a single closed nilpotent orbit O_a then we have

$$j_{a,!}\mathcal{L}_{\xi}[\dim O_a] = j_{a,!*}\mathcal{L}_{\xi}[\dim O_a] = S_{a,\xi},$$

so the assertion holds by the definition of $D_{\mathbf{G},\mathrm{Spr}_S}(\mathcal{N})$. For general S we can assume by induction that the assertion holds for some $S' = S \backslash O_a$, such that the orbits corresponding to S' still form a stratification

of a closed subset. By part (i), $S_{a,\xi}$ lies in the subcategory generated by $j_{a,!}\mathcal{L}_\xi[\dim O_a]$ and $D_{\mathbf{G},\mathrm{Spr}_{S'}}(\mathcal{N})$. Hence, the assertion for S follows from the assertion for S'. $\qquad\square$

Definition 5.2.3. With each $\chi \in \mathrm{Irr}\, W$ we associate DG-modules over A_W^{dg} by setting

$$P_\chi^{dg} = P_{a,\xi}^{dg} := \mathrm{RHom}_{\mathbf{G}}(S_{a,\xi}, \mathcal{A}),$$

$$M_\chi^{dg} = M_{a,\xi}^{dg} := \mathrm{RHom}_{\mathbf{G}}^\bullet(j_{a,!}\mathcal{L}_\xi[\dim O_a], \mathcal{A}),$$

$$N_\chi^{dg} = N_{a,\xi}^{dg} := \mathbb{D}(M_{a,\xi}^{dg})[-\dim\mathcal{N}],$$

where (O_a, ξ) is the pair associated with χ by the Springer correspondence, $S_{a,\xi} = j_{a,!*}\mathcal{L}_\xi$ is the corresponding irreducible perverse sheaf. We also consider the corresponding graded A_W-modules

$$P_\chi = P_{a,\xi} := H^* P_\chi^{dg},\ \text{etc.}$$

From the decomposition of $\mathcal{A}$ we easily get the formula

$$P_\chi = \chi \otimes H_{\mathbf{T}}^*(pt)$$

which is the indecomposable projective A_W-module associated with χ.

Recall that $\mathbf{G}_a^0 \subset \mathbf{G}_a$ is the connected component of 1 in the stabilizer subgroup of a.

Proposition 5.2.4. *(i) One has natural isomorphisms of $H_{\mathbf{G}}^*(pt)$-modules*

(5.2.4)
$$M_{a,\xi} \simeq \left(\xi^\vee \otimes H_{\mathbf{G}_a^0}^*(X_a, D_{X_a}[-2\dim X_a])\right)^{\mathbf{G}_a/\mathbf{G}_a^0},$$
$$N_{a,\xi} \simeq \left(\xi \otimes H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C})\right)^{\mathbf{G}_a/\mathbf{G}_a^0}.$$

The second isomorphism is compatible with the A_W-action, where the action of A_W on the right is induced by the Springer W-action and by the restriction homomorphism

$$H_{\mathbf{T}}^*(pt) \simeq H_{\mathbf{G}}^*(X) \to H_{\mathbf{G}_a}^*(X_a, \mathbb{C}).$$

(ii) The spaces $H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C})$ and $H_{\mathbf{G}_a^0}^*(X_a, D_{X_a})$ are free $H_{\mathbf{G}_a^0}^*(pt)$-module, and the Frobenius action on $H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C})$ and $H_{\mathbf{G}_a^0}^*(X_a, D_{X_a})$ is pure.
(iii) The DG-modules $P_{a,\xi}^{dg}$, $M_{a,\xi}^{dg}$ and $N_{a,\xi}^{dg}$ are formal.
(iv) The $H_{\mathbf{G}}^*(pt)$-action on $M_{a,\xi}$ and $N_{a,\xi}$ (induced by the embedding $H_{\mathbf{G}}^*(pt) \subset A_W$) factors through the natural homomorphism $H_{\mathbf{G}}^*(pt) \to H_{\mathbf{G}_a}^*(pt)$. Furthermore, $M_{a,\xi}$ and $N_{a,\xi}$ are maximal Cohen-Macauley $H_{\mathbf{G}_a}^*(pt)$-modules. If in addition, the action of $\mathbf{G}_a/\mathbf{G}_a^0$ on the affine space $\mathrm{Spec}\, H_{\mathbf{G}_a^0}^*(pt)$ is generated by (pseudo)reflections, then $M_{a,\xi}$ and $N_{a,\xi}$ are projective modules over $H_{\mathbf{G}_a}^*(pt)$.

Proof. (i) We have

$$\mathrm{RHom}_{\mathbf{G}}(j_!\mathcal{L}_\xi, \mathcal{A}) \simeq \mathrm{RHom}_{\mathbf{G}}(j_!\mathcal{L}_\xi, \pi_*\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim\mathcal{N}]) \simeq \mathrm{RHom}_{\mathbf{G}}(\mathcal{L}_\xi, j^!\pi_* D_{\widetilde{\mathcal{N}}}[-\dim\mathcal{N}]).$$

Using the base change, we can rewrite this as

$$\mathrm{RHom}_{\mathbf{G}}(\mathcal{L}_\xi, \pi_{O_a,*} D_{\pi^{-1}(O_a)}[-\dim\mathcal{N}]) \simeq R\Gamma_{\mathbf{G}}(\pi^{-1}(O_a), \pi_{O_a}^* \mathcal{L}_\xi^\vee \otimes D_{\pi^{-1}(O_a)}[-\dim\mathcal{N}]),$$

where $\pi_{O_a} : \pi^{-1}(O_a) \to O_a$ is the restriction of π. Now using the isomorphism (5.1.5) we get an isomorphism

$$M_{a,\xi} = \mathrm{Ext}_{\mathbf{G}}^*(j_!\mathcal{L}_\xi, \mathcal{A})[-\dim O_a] \simeq H_{\mathbf{G}}^*(\pi^{-1}(O_a), \pi_{O_a}^* \mathcal{L}_\xi^\vee \otimes D_{\pi^{-1}(O_a)}[-\dim\mathcal{N} - \dim O_a]) \simeq$$

$$H_{\mathbf{G}_a}^*(X_a, \xi^\vee \otimes D_{X_a}[-\dim\mathcal{N} + \dim O_a]) \simeq (\xi^\vee \otimes H_{\mathbf{G}_a^0}^*(X_a, D_{X_a}[-2\dim X_a]))^{\mathbf{G}_a/\mathbf{G}_a^0},$$

where for the last isomorphism we used Lemma 1.4.1(ii) and the equality (5.1.2).

Now let us calculate $N_{a,\xi}$. First, we observe that the object $\mathcal{A} = R\pi_*\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim\mathcal{N}]$ is self-dual, i.e., we have an isomorphism $\mathbb{D}(\mathcal{A}) \simeq \mathcal{A}$. Next, we recall that for any constructible sheaf F we have an isomorphism

$$\mathbb{D}(F \otimes \mathcal{A}) \simeq R\underline{\mathrm{Hom}}(F, \mathbb{D}(\mathcal{A})).$$

Combining this with the self-duality of $\mathcal{A}$ we get for $F = j_! \mathcal{L}_\xi$,

$$\mathbb{D}(j_! \mathcal{L}_\xi \otimes \mathcal{A}) \simeq R\underline{\mathrm{Hom}}(j_! \mathcal{L}_\xi, \mathcal{A}).$$

Thus, we get

$$\mathbb{D}\, \mathrm{RHom}(j_! \mathcal{L}_\xi, \mathcal{A}) \simeq \mathbb{D} R\Gamma_{\mathbf{G}}(\mathcal{N}, R\underline{\mathrm{Hom}}(j_! \mathcal{L}_\xi, \mathcal{A})) \simeq R\Gamma_{c, \mathbf{G}}(\mathcal{N}, j_! \mathcal{L}_\xi \otimes \mathcal{A}).$$

Using the projection formula we can rewrite this as

$$R\Gamma_{c, \mathbf{G}}(\mathcal{N}, j_!(\mathcal{L}_\xi \otimes j^* \mathcal{A})) \simeq R\Gamma_{c, \mathbf{G}}(O_a, \mathcal{L}_\xi \otimes \pi_{O_a, *}\underline{\mathbb{C}}[\dim \mathcal{N}]) \simeq R\Gamma_{c, \mathbf{G}}(\pi^{-1}(O_a), \pi_{O_a}^* \mathcal{L}_\xi [\dim \mathcal{N}]),$$

hence, we get

$$N_{a,\xi} \simeq H_{c, \mathbf{G}}^*(\pi^{-1}(O_a), \pi_{O_a}^* \mathcal{L}_\xi) \simeq (\xi \otimes H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C})^{\mathbf{G}_a / \mathbf{G}_a^0}.$$

The compatibility of the constructed isomorphism with the action of $A_W = \mathrm{Ext}_{\mathbf{G}}^*(\mathcal{A}, \mathcal{A})$ follows from the functoriality of the isomorphisms we used together with the fact that the involution

(5.2.5) $$\mathrm{Ext}_{\mathbf{G}}^*(\mathcal{A}, \mathcal{A}) \to \mathrm{Ext}_{\mathbf{G}}^*(\mathbb{D}(\mathcal{A}), \mathbb{D}(\mathcal{A})) \simeq \mathrm{Ext}_{\mathbf{G}}^*(\mathcal{A}, \mathcal{A}),$$

coming from the self-duality of $\mathcal{A}$, sends $w \in W$ to w^{-1} and is the identity on $H_{\mathbf{T}}^*(pt) \simeq H_{\mathbf{G}}^*(X)$. Indeed, the fact about the W-action follows from the similar compatibility on the regular semisimple locus, where it can be easily checked.

(ii) The cohomology of X_a is concentrated in even degrees, hence X_a is $\mathbf{G}_a^0$-equivariantly formal (see §1.4), so we have

$$H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C}) \simeq B_a \otimes H^*(X_a, \mathbb{C}),$$

where $B_a = H_{\mathbf{G}_a^0}^*(pt)$. In particular, $H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C})$ is a free B_a-module, and the action of the Frobenius on it is pure, by the result of [63]. By Theorem 4.1.1, this implies the formality of $R\Gamma_{\mathbf{G}_a^0}(X_a, \mathbb{C})$ as an object of $D(dg - B_a - \mathrm{mod})$. By the equivariant Verdier duality on X_a (see Section 1.4), we deduce

$$R\Gamma_{\mathbf{G}_a^0}(X_a, D_{X_a}) \simeq \mathrm{RHom}_{B_a}(H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C}), B_a) \simeq B_a \otimes H^*(X_a, \mathbb{C})^*,$$

which implies our assertions about $H_{\mathbf{G}_a^0}^*(X_a, D_{X_a})$.

(iii) This follows from Theorem 4.1.1. The required purity for $P_{a,\xi}$ follows from the proof of Theorem 5.2.1, and for $M_{a,\xi}$ and $N_{a,\xi}$—from part (ii).

(iv) First, we note that by Lemma 5.1.1, the $H_{\mathbf{G}}^*(pt)$-module structure on $M_{a,\xi}$ and $N_{a,\xi}$ corresponds via the isomorphisms (5.2.4) to the action induced by the homomorphism

$$H_{\mathbf{G}}^*(pt) \to H_{\mathbf{G}_a}^*(pt) = H_{\mathbf{G}_a^0}^*(pt)^{\mathbf{G}_a / \mathbf{G}_a^0}.$$

and by the natural action of $H_{\mathbf{G}_a}^*(pt)$ on the equivariant cohomology. This implies the first assertion. The remaining statements follow from the fact that $H_{\mathbf{G}_a^0}^*(X_a, \mathbb{C})$ and $H_{\mathbf{G}_a^0}^*(X_a, D_{X_a})$ are free $H_{\mathbf{G}_a^0}^*(pt)$-modules (by part (ii)). To derive that $M_{a,\xi}$ and $N_{a,\xi}$ are Cohen-Macauley we use the fact that $H_{\mathbf{G}_a^0}^*(pt)$ is a polynomial ring together with the fact that the modules of covariants for finite groups are Cohen-Macauley (see [66, Thm. 4.1], [58, Sec. 5.1]). The assumption that $\mathbf{G}_a / \mathbf{G}_a^0$ acts on $\mathrm{Spec}\, H_{\mathbf{G}_a^0}^*(pt)$ as a complex reflection group implies that $H_{\mathbf{G}_a^0}^*(pt)$ is a projective module over $H_{\mathbf{G}_a}^*(pt) = H_{\mathbf{G}_a^0}^*(pt)^{\mathbf{G}_a / \mathbf{G}_a^0}$, hence, the same is true about the modules of covariants. $\square$

Let us consider the duality functor

$$D^b(A_W) \to D^b(A_W) : M \mapsto \mathrm{RHom}_{H_{\mathbf{G}}^*(pt)}(M, H_{\mathbf{G}}^*(pt)),$$

which is defined in the same way as (5.2.1), but on the derived category of ungraded A_W-modules.

Note that if in addition M is a graded module then $\mathrm{RHom}_{H_{\mathbf{G}}^*(pt)}(M, H_{\mathbf{G}}^*(pt))$ can be also defined as an object of the derived category of graded A_W-modules.

Corollary 5.2.5. *The A_W-modules $M_{a,\xi}$ and $N_{a,\xi}$, viewed as $H_{\mathbf{G}}^*(pt)$-modules, are Cohen-Macauley. One has*

$$\mathrm{RHom}_{H_{\mathbf{G}}^*(pt)}(M_{a,\xi}, H_{\mathbf{G}}^*(pt)) \simeq N_{a,\xi}(-i)[i],$$

where $i = \mathrm{rk}\, G - \mathrm{rk}\, \mathbf{G}_a^0$ (where $\mathrm{rk}\, \mathbf{G}_a^0$ is the rank of the reductive part of $\mathbf{G}_a^0$).

Proof. First, we claim that $H^*_{\mathbf{G}_a}(pt)$ is finitely generated as $H^*_{\mathbf{G}}(pt)$-module. Indeed, let $\mathbf{T}_a$ be a maximal torus in $\mathbf{G}^0_a$, and let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$, such that $\mathbf{T} \supset \mathbf{T}_a$. Note that $H^*_{\mathbf{T}_a}(pt)$ is a finitely generated $H^*_{\mathbf{G}}(pt)$-module, as a quotient of $H^*_{\mathbf{T}}(pt)$. Since $H^*_{\mathbf{G}}(pt)$ is Noetherian, our claim follows from the embedding

$$H^*_{\mathbf{G}_a}(pt) \subset H^*_{\mathbf{G}^0_a}(pt) \subset H^*_{\mathbf{T}_a}(pt).$$

By Proposition 5.2.4(iv), this implies that $M_{a,\xi}$ and $N_{a,\xi}$ are Cohen-Macauley $H^*_{\mathbf{G}}(pt)$-modules.

Next, consider the functor $F : D(A_W - \mathrm{gr}) \to D(dg - A_W - \mathrm{mod})$ (see §1.3). For any perfect complex $M \in D(A_W - \mathrm{gr})$ we have a natural isomorphism

$$F\left(\mathrm{RHom}_{H^*_{\mathbf{G}}(pt)}(M, H^*_{\mathbf{G}}(pt))\right) \simeq \mathrm{RHom}_{A_W}(FM, H^*_{\mathbf{G}}(pt)) \simeq \mathbb{D}(FM),$$

where $\mathbb{D}$ is the duality (5.2.1) on the derived category of DG-modules over A_W. It remains to apply this to $M = M_{a,\xi}$ and to use the fact that it is Cohen-Macauley over $H^*_{\mathbf{G}}(pt)$ with support of dimension $\dim \mathbf{T}_a$, so that $\mathrm{RHom}_{H^*_{\mathbf{G}}(pt)}(M_{a,\xi}, H^*_{\mathbf{G}}(pt))$ is concentrated in degree $i = \mathrm{rk}\, G - \dim \mathbf{G}^0_a$. $\square$

Example 5.2.6. For $a = 0$ we have $\mathbf{G}_a = \mathbf{G}^0_a = \mathbf{G}$, so the component group $\mathbf{G}_a/\mathbf{G}^0_a$ is trivial. The representation of W associated with $a = 0$ is given by the Springer action on the top degree cohomology of the flag variety X, which is the sign representation δ. Hence, we have

$$M_0 = P_0 \simeq \delta \otimes H^*_{\mathbf{G}}(X, \mathbb{C}) \simeq \delta \otimes H^*_{\mathbf{T}}(pt).$$

On the other hand,

$$N_0 = H^*_{\mathbf{G}}(X, \mathbb{C}) \simeq H^*_{\mathbf{T}}(pt).$$

Thus, the duality between M_0 and N_0 corresponds to an isomorphism of A_W-modules

$$\mathrm{RHom}_{H^*_{\mathbf{G}}(pt)}(H^*_{\mathbf{T}}(pt), H^*_{\mathbf{G}}(pt)) \simeq \delta \otimes H^*_{\mathbf{T}}(pt).$$

5.3. **Semiorthogonal decomposition.** By Proposition 5.2.4, the DG-modules $M^{dg}_{a,\xi}$ over $A^{dg}_W \simeq A_W$ are formal. The Springer correspondence gives us information about morphisms between these modules in the category $D(dg - A_W - \mathrm{mod})$. Using Theorem 1.3.3 we can convert it into the information on morphisms in $D(A_W - \mathrm{mod})$.

Proposition 5.3.1. *(i) Assume that $O_{a'}$ is not contained in the closure of O_a. Then for any $\mathcal{L}_\xi$ on O_a (resp., $\mathcal{L}_{\xi'}$ on $O_{a'}$) one has*

$$\mathrm{Ext}^*_{A_W}(M_{a,\xi}, M_{a',\xi'}) = \mathrm{Ext}^*_{A_W}(P_{a,\xi}, M_{a',\xi'}) = 0.$$

(ii) For fixed a and for $\xi \in \mathrm{Irr}\, \mathbf{G}_a/\mathbf{G}^0_a$ there is a natural morphism $P_{a,\xi} \to M_{a,\xi}$ such that for any $\xi' \in \mathrm{Irr}\, \mathbf{G}_a/\mathbf{G}^0_a$ the induced map

$$\mathrm{Ext}^*_{A_W}(M_{a,\xi}, M_{a,\xi'}) \to \mathrm{Ext}^*_{A_W}(P_{a,\xi}, M_{a,\xi'}) = \mathrm{Hom}_{A_W}(P_{a,\xi}, M_{a,\xi'})$$

is an isomorphism. In particular, $\mathrm{Ext}^{>0}_{A_W}(M_{a,\xi}, M_{a,\xi'}) = 0$. Furthermore, we have a natural isomorphism of algebras

$$\bigoplus_{\xi,\xi'} \mathrm{Hom}_{A_W}(M_{a,\xi}, M_{a,\xi'}) \simeq \bigoplus_{\xi,\xi'} \mathrm{Hom}_{\mathbf{G}_a/\mathbf{G}^0_a \ltimes B_a}(\xi' \otimes B_a, \xi \otimes B_a),$$

where ξ, ξ' run through irreducible representations of $\mathbf{G}_a/\mathbf{G}^0_a$ appearing in the Springer correspondence, and $B_a = H^*_{\mathbf{G}^0_a}(pt)$.

Proof. (i) Since $O_{a'}$ is not contained in the closure of O_a, we have

$$\mathrm{Ext}^*_{\mathbf{G}}(j_{a',!}\mathcal{L}_{\xi'}, j_{a,!}\mathcal{L}_\xi) = \mathrm{Ext}^*_{\mathbf{G}}(\mathcal{L}_{\xi'}, j^!_{a'}j_{a,!}\mathcal{L}_\xi) = 0.$$

By the equivalence (5.2.3), this implies the vanishing

$$\mathrm{Ext}^*_{dg-A_W-\mathrm{mod}}(M^{dg}_{a,\xi}, M^{dg}_{a',\xi}) = 0.$$

By Theorem 1.3.3, this implies the same semiorthogonality for A_W-modules $M_{a,\xi}$ and $M_{a',\xi'}$. The proof of the second vanishing is analogous using the vanishing $j^!_{a'}j_{a,!*} = 0$.

(ii) We have a morphism $P_{a,\xi} \to M_{a,\xi}$ induced by the natural map of sheaves $j_{a,!}\mathcal{L}_\xi \to j_{a,!*}\mathcal{L}_\xi$. Furthermore, we have the following commutative diagram in which the vertical arrows are induced by this map of sheaves and the horizontal arrows are adjunction isomorphisms:

$$\begin{array}{ccc}
\operatorname{Ext}^*_{\mathbf{G}}(j_{a,!}\mathcal{L}_{\xi'}, j_{a,!}\mathcal{L}_\xi) & \overset{\sim}{\longrightarrow} & \operatorname{Ext}^*_{\mathbf{G}}(\mathcal{L}_{\xi'}, j_a^! j_{a,!}\mathcal{L}_\xi) \\
\downarrow & & \sim \downarrow \\
\operatorname{Ext}^*_{\mathbf{G}}(j_{a,!}\mathcal{L}_{\xi'}, j_{a,!*}\mathcal{L}_\xi) & \overset{\sim}{\longrightarrow} & \operatorname{Ext}^*_{\mathbf{G}}(\mathcal{L}_{\xi'}, j_a^! j_{a,!*}\mathcal{L}_\xi)
\end{array}$$

Since $j_a^! j_{a,!*}\mathcal{L}_\xi \simeq j_a^! j_{a,!}\mathcal{L}_\xi$, by the equivalence (5.2.3), we deduce that the map

$$\operatorname{Ext}^*_{dg-A_W-\mathrm{mod}}(M_{a,\xi}^{dg}, M_{a',\xi}^{dg}) \to \operatorname{Ext}^*_{dg-A_W-\mathrm{mod}}(P_{a,\xi}^{dg}, M_{a',\xi}^{dg}),$$

induced by the map $P_{a,\xi}^{dg} \to M_{a,\xi}^{dg}$, is an isomorphism. By the formality of these modules (see Proposition 5.2.4) and by Theorem 1.3.3, we deduce that the similar map of Ext^*-spaces in the category $D(A_W)$ is an isomorphism. In particular, we deduce the vanishing of $\operatorname{Ext}^{>0}_{A_W}(M_{a,\xi}, M_{a,\xi'})$. Now the last assertion follows from the isomorphism

$$\operatorname{Ext}^*_{\mathbf{G}}(j_{a,!}\mathcal{L}_{\xi'}, j_{a,!}\mathcal{L}_\xi) = \operatorname{Ext}^*_{\mathbf{G}}(\mathcal{L}_{\xi'}, j_a^! j_{a,!}\mathcal{L}_\xi) =$$
$$\operatorname{Ext}^*_{\mathbf{G}}(\mathcal{L}_{\xi'}, \mathcal{L}_\xi) = H^*_{\mathbf{G}_a}(pt, (\xi')^\vee \otimes \xi) = \operatorname{Hom}_{\mathbf{G}_a/\mathbf{G}_a^0}(\xi', \xi \otimes H^*_{\mathbf{G}_a^0}(pt)),$$

where in the last equality we used Lemma 1.4.1(ii). $\qquad\qquad\square$

To proceed with constructing a semiorthogonal decomposition we need the following result based on some information about irreducible representations of the component groups $\mathbf{G}_a/\mathbf{G}_a^0$ appearing in the Springer representation. We also use the classification of nilpotent orbits for exceptional groups and the calculation of their centralizers (see [48, Ch. 22]). We specify the type of a nilpotent orbit by the class of the corresponding distinguished nilpotent in a Levi subgroup, as in [48, Sec. 3.3.6].

Lemma 5.3.2. *For every nilpotent $a \in \mathfrak{g}$ the algebra*

(5.3.1) $$E_a := \operatorname{End}_{\mathbf{G}_a/\mathbf{G}_a^0 \ltimes B_a}\Big(\bigoplus_{\xi: (a,\xi) \in \mathrm{Spr}} \xi \otimes B_a \Big),$$

where $B_a := H^*_{\mathbf{G}_a^0}(pt)$, has finite global dimension.

Proof. Let us write $E_a = E_a(\mathbf{G})$ to stress the dependence on $\mathbf{G}$. Note that if $\mathbf{G}$ is the quotient of $\widetilde{\mathbf{G}}$ by a finite central subgroup then we have an isomorphism of algebras $E_a(\mathbf{G}) \simeq E_a(\widetilde{\mathbf{G}})$. Indeed, in this case the natural map $H^*_{\widetilde{\mathbf{G}}_a^0}(pt) \to H^*_{\mathbf{G}_a^0}(pt)$ is an isomorphism, the corresponding homomorphism of the component groups $\rho : \widetilde{\mathbf{G}}_a/\widetilde{\mathbf{G}}_a^0 \to \mathbf{G}_a/\mathbf{G}_a^0$ is surjective, and the characters of $\widetilde{\mathbf{G}}_a/\widetilde{\mathbf{G}}_a^0$ appearing in the Springer correspondence for $\widetilde{\mathbf{G}}$ are obtained by composing the similar characters of $\mathbf{G}_a/\mathbf{G}_a^0$ with ρ. On the other hand, if $\mathbf{G}$ is the quotient of $\widetilde{\mathbf{G}}$ by a connected subgroup of the center then $E_a(\widetilde{\mathbf{G}}) \simeq E_a(\mathbf{G})[t_1, \ldots, t_k]$. Thus, it is enough to prove the assertion in the case when $\mathbf{G}$ is almost simple. Furthermore, we can replace $\mathbf{G}$ by any isogenous group.

If $\mathbf{G}_a$ is connected then the algebra E_a is isomorphic to B_a which is an algebra of polynomials, so it has finite global dimension. In particular, this is the case for all nilpotent orbits in type A (for the adjoint group).

On the other hand, if for some nilpotent orbit O_a all irreducible representations of $\mathbf{G}_a/\mathbf{G}_a^0$ appear in the Springer correspondence then

$$E_a = \operatorname{End}_{\mathbf{G}_a/\mathbf{G}_a^0 \ltimes B_a}(P_a), \quad \text{where} \quad P_a = \bigoplus_{\xi \in \mathrm{Irr}(\mathbf{G}_a/\mathbf{G}_a^0)} \xi \otimes B_a.$$

Since P_a is a projective generator of the category of $\mathbf{G}_a/\mathbf{G}_a^0 \ltimes B_a$ and since $\mathbf{G}_a/\mathbf{G}_a^0 \ltimes B_a$ has finite global dimension, it follows that E_a has finite global dimension.

Now, assume $\mathbf{G}$ is a symplectic or a special orthogonal group. In this case for any nilpotent a one has $\mathbf{G}_a/\mathbf{G}_a^0 \simeq (\mathbb{Z}/2)^k$. Using the explicit description of the Springer correspondence in this case (see [50]) we see that there exists a surjection $\pi : \mathbf{G}_a/\mathbf{G}_a^0 \to F$ such that the action on B_a factors through F and the set characters of $\mathbf{G}_a/\mathbf{G}_a^0$ appearing in the Springer representation is a union of cosets for the subgroup $\hat{\pi}(\widehat{F}) \subset \widehat{\mathbf{G}_a/\mathbf{G}_a^0}$ (see §7 for details in the orthogonal case, the symplectic case is similar). Note that if the characters ξ and ξ' of $\mathbf{G}_a/\mathbf{G}_a^0$ belong to different cosets then the projective modules $\xi \otimes B_a$ and $\xi' \otimes B_a$ are mutually orthogonal in $D(\mathbf{G}_a/\mathbf{G}_a^0 \ltimes B_a)$. Indeed, this immediately follows from the fact that ξ and ξ' have different restrictions to the subgroup $\ker(\pi) \subset \mathbf{G}_a/\mathbf{G}_a^0$ that acts trivially on B_a. Next, for a given coset $C = \xi_0 + \hat{\pi}(\widehat{F})$ we have

$$(5.3.2) \qquad \mathrm{End}\left(\bigoplus_{\xi \in C} \xi \otimes B_a\right) \simeq \mathrm{End}\left(\bigoplus_{\xi \in \hat{\pi}(\widehat{F})} \xi \otimes B_a\right) \simeq F \ltimes B_a,$$

where the first isomorphism is induced by the functor of tensoring with ξ_0. The algebra $F \ltimes B_a$ has finite global dimension, so we are done in this case.

To deal with exceptional groups we use the explicit calculations of the Springer correspondence for the corresponding adjoint groups (see [16, Sec. 13.3]). For types E_6 and E_7 all irreducible representations of the component groups $\mathbf{G}_a/\mathbf{G}_a^0$ appear in the Springer correspondence, so the assertion follows in these cases. For the remaining types G_2, F_4 and E_8 the only nilpotent orbits for which not all representations of the component group are present, are the orbits $G_2(a_1)$, $F_4(a_3)$ and $E_8(a_7)$. However, in all these cases the group $\mathbf{G}_a^0$ has trivial reductive part, so that $B_a = k$ and E_a is a semisimple algebra, hence of finite global dimension. $\square$

For a set S of nilpotent orbits forming a stratification of a closed subset in $\mathcal{N}$, let Spr_S, $D_{\mathbf{G},\mathrm{Spr}_S}$ be as in Proposition 5.2.2(ii). Let also $\mathcal{C}_S \subset D(A_W)$ be the thick subcategory generated by the projectives modules $P_{a,\xi}$ associated with the elements of Spr_S.

Theorem 5.3.3. *(i) For each nilpotent orbit O_a let $\mathcal{C}_a \subset D(A_W)$ be the thick subcategory generated by all the modules $M_{a,\xi}$, where $(a,\xi) \in \mathrm{Spr}$. Then $\mathcal{C}_a$ is admissible, $\mathcal{C}_a \simeq D^f(\mathrm{mod}\,{-}E_a)$, where E_a is given by (5.3.1), and we have*

$$\mathrm{Ext}^*_{A_W}(\mathcal{C}_a, \mathcal{C}_{a'}) = 0$$

whenever $O_{a'}$ is not in the closure of O_a.
(ii) Let S be a set of nilpotent orbits forming a stratification of a closed subset in $\mathcal{N}$. For any ordering $S = \{a_1, \ldots, a_n\}$ such that O_{a_i} is in the closure of O_{a_j} only if $i < j$, we have a semiorthogonal decomposition into admisible subcategories

$$\mathcal{C}_S = \langle \mathcal{C}_{a_n}, \ldots, \mathcal{C}_{a_1} \rangle.$$

Also, for any $(a,\xi) \in \mathrm{Spr}_S$ we have an exact triangle

$$(5.3.3) \qquad\qquad K \to P_{a,\xi} \to M_{a,\xi} \to \ldots$$

in $D(A_W)$ with $K \in \mathcal{C}_{S'}$, where S' is the set of orbits in $\overline{O_a} \setminus O_a$.
(iii) Let $\{a_1, \ldots, a_r\}$ be an ordering of all nilpotent orbits such that O_{a_i} is in the closure of O_{a_j} only if $i < j$. Then we have a semiorthogonal decomposition into admissible subcategories

$$D^f(A_W) = \langle \mathcal{D}_{a_1}, \ldots, \mathcal{D}_{a_r} \rangle,$$

where $\mathcal{D}_a$ is generated by all the modules $N_{a,\xi}$ with $(a,\xi) \in \mathrm{Spr}$. For each a we have an equivalence $\mathcal{D}_a \simeq D^f(\mathrm{mod}\,{-}E_a)$.

Proof. (i) Using Lemma 1.1.1, Proposition 5.3.1(ii) and Lemma 5.3.2, we see that $\mathcal{C}_a$ is admissible and is equivalent to $D^f(\mathrm{mod}\,{-}E_a)$. The required semiorthogonalities follow from Proposition 5.3.1(i).
(ii) Using part (i) we see that the subcategory

$$\widetilde{\mathcal{C}}_S := \langle \mathcal{C}_{a_n}, \ldots, \mathcal{C}_{a_1} \rangle \subset D(A_W)$$

is admissible. We have to check the equality of subcategories $\widetilde{\mathcal{C}}_S = \mathcal{C}_S$. Replacing S by the closure of a given orbit $O_a \in S$ we can assume that S is the set of all orbits in the closure of O_a (so $a = a_n$).

By induction, we can assume that the assertion holds for $S' = S \setminus \{O_a\}$. First, let us check that $P_{a,\xi}$ belongs to $\widetilde{\mathcal{C}}_S$ for all ξ. Indeed, by Lemma 1.3.4, it is enough to check the same statement in the derived category of DG-modules over A_W. Using an equivalence (5.2.3) this reduces to showing that the object $S_{a,\xi} \in D_{\mathbf{G}}(\mathcal{N})$ belongs to the subcategory generated by $(j_{a,!}\mathcal{L}_\xi)_{(a,\xi)\in \mathrm{Spr}_S}$. But this follows from Proposition 5.2.2(ii).

Next, the semiorthogonal decomposition

$$\widetilde{\mathcal{C}}_S = \langle \mathcal{C}_a, \mathcal{C}_{S'} \rangle$$

shows that $\mathcal{C}_{S'} = {}^\perp\mathcal{C}_a$, the left orthogonal to $\mathcal{C}_a$ in $\widetilde{\mathcal{C}}_S$. We already know that $P_{a,\xi} \in \widetilde{\mathcal{C}}_S$. Hence, the object K from the exact triangle (5.3.3) is also in $\widetilde{\mathcal{C}}_S$. Now by Proposition 5.3.1(ii), we have

$$\mathrm{Hom}(K, M_{a,\xi'}) = 0 \text{ for } (a, \xi') \in \mathrm{Spr}\,.$$

This implies that K is in ${}^\perp\mathcal{C}_a = \mathcal{C}_{S'}$.

Finally, using the triangles (5.3.3) we see that the category $\widetilde{\mathcal{C}}_S$, generated by $\mathcal{C}_{S'}$ and the modules $M_{a,\xi}$, is equal to $\mathcal{C}_S$.

(iii) This is obtained from (ii) using the duality $\mathrm{RHom}_{H^*_{\mathbf{G}}(pt)}(?, H^*_{\mathbf{G}}(pt))$ (see Corollary 5.2.5). $\qquad\square$

Remarks 5.3.4. 1. Kato proved that the natural morphism $P_\chi \to M_\chi$ is surjective and that

$$M_\chi = P_\chi/(\sum_{\chi'<\chi,\,f:P_{\chi'}\to P_\chi} \mathrm{Im}\,f)$$

(see the proof of [38, Cor. 3.6]).

2. Parts (i) and (ii) of Theorem 5.3.3 can be deduced without using the formality of A_W^{dg} and $M_{a,\xi}^{dg}$, but instead using some results from [36] (see [1, Sec. 4]).

Proposition 5.3.5. *Let $\mathbf{G}$ be the adjoint simple group of type G_2 or F_4. Then the semiorthogonal decomposition of Theorem 5.3.3 can be further refined to match the motivic decomposition (0.0.1).*

Proof. When $\mathbf{G}_a$ is not connected the subcategory $\mathcal{C}_a$ has a further semiorthogonal decomposition. In the case when $\mathbf{G}_a/\mathbf{G}_a^0$ acts trivially on B_a we just have an orthogonal decomposition numbered by irreducible representations of $\mathbf{G}_a/\mathbf{G}_a^0$. Let us go over other cases.

For type G_2 the only orbit that has a nontrivial component group $\mathbf{G}_a/\mathbf{G}_a^0$ is the orbit $G_2(a_1)$ for which $B_a = k$, so that we have further orthogonal decomposition of the corresponding subcategory $\mathcal{C}_a$ into the pieces numbered by the irreducible representations of $\mathbf{G}_a/\mathbf{G}_a^0 = S_3$.

For type F_4 the only orbits with nontrivial $\mathbf{G}_a/\mathbf{G}_a^0$ and nontrivial reductive part R_a of $\mathbf{G}_a^0$ are the orbits $\widetilde{A}_1$, A_2, B_2 and $C_3(a_1)$. In all of these cases $\mathbf{G}_a/\mathbf{G}_a^0 = S_2$. We claim that in all four cases the nontrivial generator $\mathbf{G}_a/\mathbf{G}_a^0$ acts on $\mathrm{Spec}\, H^*_{\mathbf{G}_a^0}(pt) = \mathrm{Spec}\, H^*_{R_a}(pt)$ as a reflection. For the orbit $C_3(a_1)$ the group R_a has type A_1, so this is automatic. For the orbits $\widetilde{A}_1$ and A_2 the group R_a has types A_3 and A_2, respectively. Note that $\mathbf{G}_a/\mathbf{G}_a^0$ acts on $H^*_{\mathbf{G}_a^0}(pt)$ via the natural homomorphism from $\mathbf{G}_a/\mathbf{G}_a^0$ to the group of outer automorphisms of $\mathbf{G}_a^0$. Hence, it is enough to check that the outer automorphism of R_a acts on $\mathrm{Spec}\, H^*_{R_a}(pt)$ by a reflection. But this action is induced by the automorphism $x \mapsto -x$ of the Cartan subalgebra, hence it preserves polynomial invariants of degree 2 and 4 and acts by -1 on the invariant of degree 3, and the assertion follows in this case. Finally, for the orbit B_2 the group R_a has type $A_1 + A_1$, so the algebra of invariant polynomials is generated by two quadratic polynomials on the Cartan subalgebra $\mathfrak{t}$ of the Lie algebra of R_a. Since the action of any involution on $\mathfrak{t}$ has at most one -1 eigenvalue on $S^2\mathfrak{t}$, our claim follows.

The fact that the pieces of the semiorthogonal decomposition match with those of the motivic decomposition (0.0.1) follows from Propositions 2.1.5 and 2.2.2. $\qquad\square$

Remark 5.3.6. For E_6 we get a semiorthogonal decomposition with one noncommutative block corresponding to the nilpotent orbit of type A_2. Indeed, one can see from the list of nilpotent orbits (see [48, Ch. 22]) that the only other orbit with nontrivial $\mathbf{G}_a/\mathbf{G}_a^0$ and nontrivial reductive part R_a of $\mathbf{G}_a^0$

is the orbit $D_4(a_1)$, for which the reductive part of the centralizer is the 2-dimensional torus, on which $\mathbf{G}_a/\mathbf{G}_a^0 \simeq S_3$ acts in a standard way. So we can refine the corresponding piece of the semiorthogonal decomposition of Theorem 5.3.3 using the same Theorem for the S_3-action. The appearance of the non-commutative block for the nilpotent orbit of type A_2 is due to the fact that in this case the component group $\mathbf{G}_a/\mathbf{G}_a^0 = \mathbb{Z}/2$ of the centralizer swaps the two copies of A_2 in the reductive part $A_2 + A_2$ of $\mathbf{G}_a^0$. To check this one can use the realization of $\mathbf{G} = E_6$ as the centralizer of a subgroup S of type A_2 in E_8 (see [48, Ch. 16]). Then the reductive part of the centralizer is obtained as $C_R(S)$, where R is the reductive part of the centralizer of our nilpotent element in E_8. As proved in [48, Ch. 15], R is the semidirect product of E_6 with $\mathbb{Z}/2$, acting on E_6 by the graph automorphism. Now we have to look at the subgroup A_2 in E_6, invariant under the graph automorphism, take its centralizer and see how the graph automorphism acts on it. This A_2 is generated by α_2 and by the maximal root, so the centralizer will have A_2's generated by α_1, α_3 and by α_5, α_6. It remains to observe that the graph automorphism swaps these two A_2's.

5.4. **Equivariant cohomology of the Springer fibers.** In this section we discuss a description of the equivariant cohomology of the Springer fibers as coordinate rings of some linear arrangements, due to Goresky-MacPherson (see [27, Thm. 3.1]) and Kumar-Procesi (see [43]). Note that [27] and [43] assume $\mathbf{G}$ to be semisimple, while we only assume it to be connected reductive, however, one can check that the constructions and arguments of [27] and [43] still go through in this case. In fact, we are mainly interested in the case $\mathbf{G} = \mathrm{GL}_n$ (due to the assumption (†) below).

We start with a nilpotent $a \in \mathfrak{g}$ which is a principal nilpotent in the Lie algebra of a Levi subgroup $\mathbf{L} \subset \mathbf{P}$ of some parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Define the torus $\mathbf{T}_a \subset \mathbf{L}$ as the connected component of 1 in the center of $\mathbf{L}$. Note that a is a distinguished nilpotent in $\mathbf{L}$, hence, $\mathbf{T}_a$ is a maximal torus in $\mathbf{G}_a^0$, the connected component of 1 in the centralizer of a in $\mathbf{G}$ (see [34, Sec. 4.7]). Let $\mathbf{T} \subset \mathbf{L}$ be a maximal torus, and let $\mathfrak{t}$ (resp., $\mathfrak{t}_a$) be the Lie algebra of $\mathbf{T}$ (resp., $\mathbf{T}_a$).

We define the *linear arrangement associated with* $(a, \mathbf{L})$ as above, to be the following union of linear subspaces in $\mathfrak{t}_a \times \mathfrak{t}$

$$(5.4.1) \qquad Z_a = \cup_{w \in W}(\mathrm{id} \times w)\Delta(\mathfrak{t}_a) \subset \mathfrak{t}_a \times \mathfrak{t},$$

where $\mathfrak{t}_a \subset \mathfrak{t}$ is the Lie algebra of $\mathbf{T}_a \subset \mathbf{T}$, $\Delta : \mathfrak{t}_a \to \mathfrak{t}_a \times \mathfrak{t}$ is the diagonal embedding, and $W = N_\mathbf{G}(\mathbf{T})/\mathbf{T}$ is the Weyl group of $\mathbf{G}$ acting on $\mathfrak{t}$. We view Z_a as a subscheme in the affine space $\mathfrak{t}_a \times \mathfrak{t}$, equipped with the reduced scheme structure, and denote by

$$\mathbb{C}[Z_a] = S(\mathfrak{t}_a^*) \otimes S(\mathfrak{t}^*)/I(Z_a)$$

the corresponding coordinate ring (where $I(Z_a)$ is the ideal of polynomials vanishing on Z_a). Note that Z_a is a cone, so the algebra $\mathbb{C}[Z_a]$ is naturally graded.

Let us make the following assumption:

(†) **the restriction homomorphism $H^*(X, \mathbb{C}) \to H^*(X_a, \mathbb{C})$ is surjective.**

For example, this assumption is satisfied in the case of type A, due to the work of Spaltenstein [62]. Under the assumption (†) there is an isomorphism of graded algebras (see [27, Thm. 3.1], [43])

$$(5.4.2) \qquad \mathbb{C}[Z_a] \xrightarrow{\sim} H^*_{\mathbf{T}_a}(X_a, \mathbb{C}),$$

where the standard grading on the ring of functions on Z_a is doubled. This isomorphism fits into the commutative diagram (which uniquely determines it)

$$(5.4.3)$$

$$\begin{array}{ccc} S(\mathfrak{t}_a^*) \otimes S(\mathfrak{t}^*) & \xrightarrow{\chi} & H^*_{\mathbf{T}_a}(X, \mathbb{C}) \\ \downarrow & & \downarrow \\ \mathbb{C}[Z_a] & \xrightarrow{\sim} & H^*_{\mathbf{T}_a}(X_a, \mathbb{C}), \end{array}$$

where the vertical arrows are induced by the embeddings $Z_a \hookrightarrow \mathfrak{t}_a \times \mathfrak{t}$ and $X_a \hookrightarrow X$, and χ is induced by the natural map

$$S(\mathfrak{t}^*) = H_{\mathbf{T}}^*(pt) \xrightarrow{\beta} H_{\mathbf{G}}^*(X) \to H_{\mathbf{T}_a}^*(X)$$

and by the standard action of $S(\mathfrak{t}_a^*) = H_{\mathbf{T}_a}^*(pt)$ on $H_{\mathbf{T}_a}^*(X)$. Here the map β is defined in [43] by $\beta(\lambda) = c_1(L_\lambda)$, where L_λ is the $\mathbf{G}$-equivariant line bundle on X associated with a character $\lambda : \mathbf{T} \to \mathbb{G}_m$, and c_1 denotes the $\mathbf{G}$-equivariant first Chern class. It is easy to see that β coincides with the natural isomorphism obtained using the identification $X = \mathbf{G}/\mathbf{B}$ (see (5.1.3)).

Recall that X_a is equivariantly formal (since its cohomology is concentrated in even degrees). Hence, by Lemma 1.4.2, we have

(5.4.4) $$H_{\mathbf{G}_a^0}^*(X_a) \simeq H_{\mathbf{T}_a}^*(X_a)^{W_a},$$

where W_a is the Weyl group attached to the maximal torus $\mathbf{T}_a \subset \mathbf{G}_a^0$.

Lemma 5.4.1. *Assume condition* (†) *holds.*
(i) The Springer action of W on $H_{\mathbf{T}_a}^*(X, \mathbb{C})$ corresponds under (5.4.2) *to the action of W on Z_a given by the action on the second component.*
(ii) The natural action of W_a on $H_{\mathbf{T}_a}^*(X_a, \mathbb{C})$ corresponds under (5.4.2) *to the action of W_a on Z_a given by the action on the first component.*

Proof. (i) Since the restriction homomorphism $H_{\mathbf{T}_a}^*(X, \mathbb{C}) \to H_{\mathbf{T}_a}^*(X_a, \mathbb{C})$ is compatible with the Springer W-action (see Lemma 5.1.2(i)), it is enough to check that the homomorphism χ from diagram (5.4.3) is compatible with the W-actions. In other words, it is enough to check that the natural isomomorphism

$$H_{\mathbf{T}}^*(pt) \simeq H_{\mathbf{G}}^*(\mathbf{G}/\mathbf{T}, \mathbb{C}) \simeq H_{\mathbf{G}}^*(\mathbf{G}/\mathbf{B}, \mathbb{C})$$

is compatible with W-actions. But this follows immediately from Lemma 5.1.2(ii).
(ii) It is enough to check that the surjective map

$$S(\mathfrak{t}_a^*) \otimes S(\mathfrak{t}^*) \to \mathbb{C}[Z_a] \xrightarrow{\sim} H_{\mathbf{T}_a}^*(X_a)$$

is W_a-equivariant, where W_a acts naturally on $S(\mathfrak{t}_a^*)$ and acts trivially on $S(\mathfrak{t}^*)$. But this follows from the fact that the restriction of χ to $S(\mathfrak{t}^*)$ is the homomorphism

$$S(\mathfrak{t}^*) \simeq H_{\mathbf{G}}^*(X) \to H_{\mathbf{G}_a^0}^*(X) \to H_{\mathbf{G}_a^0}^*(X_a) \simeq H_{\mathbf{T}_a}^*(X_a)^{W_a} \hookrightarrow H_{\mathbf{T}_a}^*(X_a).$$

$\square$

Remark 5.4.2. Kumar and Procesi in [43] sketch a proof of the compatibility with W-actions, similar to the one in Lemma 5.4.1(i), for the induced identification of the non-equivariant cohomology of the Springer fiber X_a.

Using the result of this Lemma and the isomorphisms (5.4.2) and (5.4.4), we get an isomorphism

(5.4.5) $$H_{\mathbf{G}_a^0}^*(X_a) \simeq \mathbb{C}[Z_a]^{W_a} \simeq \mathbb{C}[\overline{Z}_a],$$

compatible with W-actions, where

(5.4.6) $$\overline{Z}_a = Z_a/W_a$$

is the geometric quotient of Z_a with respect to the W_a-action induced by the action on the first factor of the product $\mathfrak{t}_a \times \mathfrak{t}$. We have a natural map induced by the projection $\mathfrak{t}_a \times \mathfrak{t} \to \mathfrak{t}$,

(5.4.7) $$\pi_a : \overline{Z}_a \to \mathbf{S}_a := W \cdot \mathfrak{t}_a \subset \mathfrak{t}.$$

In some cases the map π_a is a partial normalization of $\mathbf{S}_a$. In any case the singularities of $\overline{Z}_a$ are better behaved that those of $\mathbf{S}_a$, as the following result shows.

Lemma 5.4.3. *Assume that* (†) *holds. Then the schemes Z_a and $\overline{Z}_a$ are Cohen-Macaulay.*

Proof. By the equivariant formality of X_a, the equivariant cohomology $H^*_{\mathbf{T}_a}(X_a, \mathbb{C})$ (resp., $H^*_{\mathbf{G}^0_a}(X_a)$) is free of finite rank over $H^*_{\mathbf{T}_a}(pt)$ (resp., $H^*_{\mathbf{G}^0_a}(pt)$). It remains to use the isomorphism (5.4.2) (resp., (5.4.5)). $\square$

Remarks 5.4.4. 1. There exist other descriptions of the cohomology of Springer fibers, still under the assumption (†), see [19], [14], [44].
2. By Lemma 1.4.1(ii), we have

$$H^*_{\mathbf{G}_a}(X_a, \mathbb{C}) \simeq H^*_{\mathbf{G}^0_a}(X_a, \mathbb{C})^{\mathbf{G}_a/\mathbf{G}^0_a}.$$

Hence, from (5.4.5) we get in the above situation

$$H^*_{\mathbf{G}_a}(X_a, \mathbb{C}) \simeq \mathbb{C}[\overline{Z}_a]^{\mathbf{G}_a/\mathbf{G}^0_a}$$

provided (†) holds. It is easy to see using [27, Thm. 1.3] that the latter isomorphism holds under the weaker assumption that the homomorphism

(5.4.8) $$H^*_{\mathbf{G}_a}(X, \mathbb{C}) \to H^*_{\mathbf{G}_a}(X_a, \mathbb{C})$$

is surjective. For example, this holds if $\mathbf{G}_a/\mathbf{G}^0_a$ acts trivially on $H^*_{\mathbf{G}^0_a}(pt)$ and the image of the restriction homomorphism $H^*(X, \mathbb{C}) \to H^*(X_a, \mathbb{C})$ coincides with the subspace of $\mathbf{G}_a/\mathbf{G}^0_a$-invariants in $H^*(X_a, \mathbb{C})$.

6. Type A and the groups $G(m, 1, n)$

6.1. **Type A_n.** In this section we specialize to the case $\mathbf{G} = \mathrm{GL}_n$, so that $W = S_n$. In this case nilpotent orbits are parametrized by partitions of n. For a partition λ of n, the reductive part of the centralizer of the corresponding nilpotent element a_λ is $\mathbf{G}_\lambda = \prod_i (\mathrm{GL}_{r_i})^i)$, where r_i are multiplicities of parts of λ. In particular, $\mathbf{G}_\lambda$ is connected. The Springer correspondence associates with a nilpotent orbit O_λ the irreducible representation of S_n in the top degree cohomology of the corresponding Springer fiber $X_\lambda := X_{a_\lambda}$, which is known to be the usual representation L_λ of S_n corresponding to λ (e.g., $L_{(n)} = \mathbf{1}$ is the trivial representation, $L_{(1)^n} = \delta$, is the sign representation, where $(1)^n$ is 1 repeated n times). We use the *dominance partial ordering* $\leq$ on partitions of n defined by $\lambda \geq \mu$ if $\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i$ for all $i \geq 1$. Note that (n) is the biggest partition and $(1)^n$ is the smallest. It is well known (see [25]) that this order corresponds to the adjacency order of nilpotent orbits, i.e., $O_\mu \subset \overline{O_\lambda}$ if and only if $\mu \leq \lambda$. We will use the following property of this ordering.

Lemma 6.1.1. *Let $\lambda^{(1)}, \ldots, \lambda^{(s)}$ and $\mu^{(1)}, \ldots, \mu^{(s)}$ be partitions with*

$$|\lambda^{(1)}| + \ldots + |\lambda^{(s)}| = |\mu^{(1)}| + \ldots + |\mu^{(s)}| = n,$$

and let λ (resp., μ) be the partition of n obtaining by taking all the parts of the partitions $\lambda^{(1)}, \ldots, \lambda^{(s)}$ (resp., $\mu^{(1)}, \ldots, \mu^{(s)}$). Assume that $\lambda^{(i)} \geq \mu^{(i)}$ for $i = 1, \ldots, s$. Then $\lambda \geq \mu$. If in addition, $\lambda^{(i)} > \mu^{(i)}$ for at least one i then $\lambda > \mu$.

Proof. This becomes apparent from the following reformulation of the condition $\lambda \geq \mu$: for every $k \geq 1$ and for every collection of k parts of μ, $\mu_{i_1}, \ldots, \mu_{i_k}$ (where $i_1, \ldots, i_k$ are distinct), there exists a collection of k parts of λ, $\lambda_{j_1}, \ldots, \lambda_{j_k}$ with

$$\lambda_{j_1} + \ldots + \lambda_{j_k} \geq \mu_{i_1} + \ldots + \mu_{i_k}.$$

$\square$

We can identify the Lie algebra of a maximal torus $\mathfrak{t} \subset \mathfrak{g}$ with the standard n-dimensional representation of S_n, so that $H^*_{\mathbf{T}}(pt) = S(\mathfrak{t}^*) \simeq \mathbb{C}[x_1, \ldots, x_n]$ and

$$A := A_{S_n} = S_n \ltimes \mathbb{C}[x_1, \ldots, x_n],$$

where $\deg(x_1) = \ldots = \deg(x_n) = 2$, while the elements of S_n have degree 0.

For each partition λ, $|\lambda| = n$, the maximal torus $\mathbf{T}_\lambda$ in $\mathbf{G}_\lambda$ consists of diagonal matrices $\mathrm{diag}(t_1, \ldots, t_n) \in$ GL_n such that $t_1 = \ldots = t_{\lambda_1}$, $t_{\lambda_1+1} = \ldots = t_{\lambda_1+\lambda_2}$, etc. Its Lie algebra $\mathfrak{t}_\lambda \subset \mathfrak{t}$ coincides with $\mathfrak{t}^{w_\lambda}$, where $w_\lambda \in S_n$ is the permutation with the cycle type λ. The subgroup

$$S_\lambda = \prod_k S_{\lambda_k} \subset S_n$$

is the pointwise stabilizer of $\mathfrak{t}_\lambda$. Let $H_\lambda \subset S_n$ be the normalizer of S_λ:

$$H_\lambda = S_\lambda \rtimes W_\lambda,$$

where $W_\lambda = \prod_i S_{r_i}$ (r_i is the multiplicity of i in λ). Note that H_λ is exactly the set of $w \in S_n$ such that $w(\mathfrak{t}_\lambda) = \mathfrak{t}_\lambda$.

Thus, the module obtained from the Springer correspondence is supported precisely on $S_n \cdot \mathfrak{t}^{w_\lambda} \subset \mathfrak{t}$, where $w_\lambda \in S_n$ is the permutation with the cycle type λ.

The centralizer C_λ of w_λ in S_n is the semidirect product

$$C_\lambda = C_\lambda^0 \rtimes W_\lambda,$$

where

$$C_\lambda^0 = \prod_k \mathbb{Z}/\lambda_k, \quad W_\lambda = \prod_i S_{r_i}.$$

Note that C_λ^0 is the kernel of the homomorphism $C_\lambda \to \mathrm{Aut}(\mathfrak{t}_\lambda)$, while W_λ can be identified with the Weyl group of $\mathbf{G}_\lambda$.

For a partition λ, $|\lambda| = n$, let

$$P_\lambda = \mathcal{O}(\mathfrak{t}) \otimes L_\lambda.$$

be the standard A-projective modules, and let M_λ be the A-module obtained from the Springer correspondence (see Definition 5.2.3). Let us restate some of the results of Theorem 5.3.3 and of Propositions 5.3.1 and 5.2.4(iv) in the type A_n case.

Theorem 6.1.2. *(i) For each partition λ, $|\lambda| = n$, we have*

$$(6.1.1) \qquad \mathrm{RHom}_A(M_\lambda, M_\lambda) = \mathrm{Hom}_A(M_\lambda, M_\lambda) \cong R_\lambda := \mathcal{O}(\mathfrak{t}_\lambda)^{W_\lambda}$$

in a way which is compatible with the obvious $\mathcal{O}(\mathfrak{t})^{S_n}$-action on both sides. We also have

$$\mathrm{RHom}(N_\lambda, N_\lambda) \simeq R_\lambda.$$

Let

$$\mathcal{C}_\lambda = \langle M_\lambda \rangle \subset D^f(A)$$

be the thick subcategory generated by M_λ. Then $\mathcal{C}_\lambda$ is an admissible subcategory and we have an equivalence

$$\mathcal{C}_\lambda \simeq D^f(R_\lambda) = \mathrm{Perf}(R_\lambda)$$

given by the functor $\mathrm{RHom}(M_\lambda, ?)$. Similarly, the subcategory $\langle N_\lambda \rangle$ is admissible and is equivalent to $\mathrm{Perf}(R_\lambda)$. Furthermore, M_λ and N_λ are projective as R_λ-modules.
(ii) One has

$$\mathrm{RHom}_A(P_\mu, M_\lambda) = \mathrm{RHom}_A(M_\mu, M_\lambda) = \mathrm{RHom}_A(N_\lambda, N_\mu) = 0 \qquad \text{for } \mu \ngeq \lambda.$$

(iii) For any set S of partitions of n, closed under passing to smaller elements with respect to the dominance order, we have

$$\mathcal{C}_S := \langle P_\lambda \mid \lambda \in S \rangle = \langle M_\lambda \mid \lambda \in S \rangle.$$

For any ordering $S = \{\lambda_1 < \ldots < \lambda_k\}$ refining the dominance order, the subcategories $\mathcal{C}_{\lambda_k}, \ldots, \mathcal{C}_{\lambda_1}$ form a semiorthogonal decomposition of $\mathcal{C}_S$. Furthermore, for every λ we have an exact triangle

$$(6.1.2) \qquad K \to P_\lambda \to M_\lambda \to \ldots$$

in $D^f(A)$ such that $K \in \langle P_\mu \mid \mu < \lambda \rangle$.
(iv) Let $\lambda_1 < \ldots < \lambda_{p(n)}$ be an ordering refining the dominance order. Then we have a semiorthogonal decomposition

$$D^f(A) = \langle \langle N_{\lambda_1} \rangle, \ldots, \langle N_{\lambda_{p(n)}} \rangle \rangle.$$

The nilpotent a_λ is the principal nilpotent in the Levi subgroup $\mathbf{L} = \prod_k \mathrm{GL}_{\lambda_k}$, so we can associate the schemes Z_λ and $\overline{Z}_\lambda$ to it (see (5.4.1) and (5.4.6)). The linear spaces constituting Z_λ are in bijection with S_n/S_λ. Thus, the components of $\overline{Z}_\lambda = Z_\lambda/W_\lambda$ are numbered by S_n/H_λ. Hence, the components of $\overline{Z}_\lambda$ and of $S_n \cdot \mathfrak{t}_\lambda$ are in bijection. Furthermore, it is easy to see that the projection

$$\pi_\lambda : \overline{Z}_\lambda \to \mathbf{S}_\lambda := S_n \cdot \mathfrak{t}_\lambda \tag{6.1.3}$$

(see (5.4.7)) is finite, surjective and is an isomorphism over the general point of each component (where we equip $\mathbf{S}_\lambda$ with the reduced scheme structure).

The formula (5.4.5) leads to an explicit identification of the modules $N_\lambda = N_{a_\lambda}$ (see Definition 5.2.3). It is this identification that will allow us to globalize this picture in Section 6.3. Therefore, it will be convenient to restate our results in terms of the modules N_λ.

Proposition 6.1.3. *For each partition λ (with $|\lambda| = n$) we have an isomorphism of A-modules*

$$N_\lambda \simeq \mathbb{C}[\overline{Z}_\lambda],$$

where the grading on the right is doubled.

Proof. By Proposition 5.2.4,

$$N_\lambda \simeq H^*_{\mathbf{G}_\lambda}(X_\lambda, \mathbb{C}).$$

Combining this with (5.4.5) we get the required isomorphism. Note that the assumption (†) is satisfied in this case (see [62]). $\square$

We are going to show now how the isomorphism (5.4.5) can be used to deduce that the stratum $\mathbf{S}_\lambda$ is Cohen-Macauley in some cases.

Proposition 6.1.4. *If all parts in λ are equal then the map π_λ (see (6.1.3)) is an isomorphism. Thus, in this case we have an isomorphism*

$$H^*_{\mathbf{G}_\lambda}(X_\lambda) \simeq \mathbb{C}[\mathbf{S}_\lambda]$$

of graded algebras (where the grading on $\mathbb{C}[\mathbf{S}_\lambda]$ is doubled).

Proof. Note that $\overline{Z}_\lambda$ is a reduced closed subscheme of $\mathfrak{t}_\lambda/W_\lambda \times \mathbf{S}_\lambda \subset \mathfrak{t}_\lambda/W_\lambda \times \mathfrak{t}$. Thus, to show that π_λ is an isomorphism, it is enough to construct a morphism $\mathbf{S}_\lambda \to \mathfrak{t}_\lambda/W_\lambda$, such that the composition $\overline{Z}_\lambda \to \mathbf{S}_\lambda \to \mathfrak{t}_\lambda/W_\lambda$ is the natural projection. Let $\lambda = (m)^p$, where $n = mp$. Then $\mathfrak{t}_\lambda$ is the image of the embedding

$$i_m : \mathbb{A}^p \to \mathbb{A}^n : (y_1, \ldots, y_p) \mapsto (y_1, \ldots, y_1, y_2, \ldots, y_2, \ldots, y_p, \ldots, y_p)$$

where each y_i is repeated m times. Note that $W_\lambda = S_p$ so $\mathbb{C}[\mathfrak{t}_a/W_\lambda]$ is just the ring of symmetric functions in $(y_1, \ldots, y_p)$. Now we observe that if $e_i(\cdot)$ are elementary symmetric functions then $e_1(y), \ldots, e_p(y)$ can be expressed as some universal polynomials in $e_1(i_m(y)), \ldots, e_p(i_m(y))$. These universal polynomials give us the required map $\mathfrak{t} \to \mathfrak{t}_\lambda/W_\lambda$ which we then restrict to $\mathbf{S}_\lambda$.

Now the isomorphism with the equivariant cohomology of X_λ follows from (5.4.5). $\square$

Corollary 6.1.5. *If all parts in λ are equal then $\mathbf{S}_\lambda$ is Cohen-Macaulay.*

Proof. Combine Proposition 6.1.4 with Lemma 5.4.3. $\square$

Corollary 6.1.6. *For $\lambda = (m)^p$ there is an isomorphism of graded spaces*

$$\mathbb{C}[y_1, \ldots, y_p]^{S_p} \otimes H^{*/2}(X_\lambda, \mathbb{C}) \simeq \mathbb{C}[\mathbf{S}_\lambda]. \tag{6.1.4}$$

Remark 6.1.7. It was proved in [23] that $\mathbf{S}_\lambda$ is Cohen-Macaulay for all $\lambda = ((m)^s, (1)^r)$, where $r < m$. The above Corollary 6.1.5 gives a more direct proof in the case $r = 0$. The isomorphism (6.1.4) allows to compute the character of $\mathbb{C}[\mathbf{S}_\lambda]$, as a graded S_n-representation, still in the case $r = 0$. Namely, (6.1.4) reduces this to computing the character of $H^*(X_\lambda, \mathbb{C})$, which can be realized as the stalk of the Springer sheaf $\mathcal{A}$. Then its character can be computed using the decomposition (5.1.1) together with [49, Thm. 2]. The formula for the character of $\mathbb{C}[\mathbf{S}_\lambda]$ was also deduced in [23] by different methods.

6.2. **Induced modules in type A_n.** For each λ, a partition of n, let us consider the induced representations of S_n,

$$I_\lambda = \operatorname{Ind}_{H_\lambda}^{S_n} \mathcal{O}(\mathfrak{t}_\lambda).$$

We view I_λ as an A-module using the restriction homomorphism $\mathcal{O}(\mathfrak{t}) \to \mathcal{O}(\mathfrak{t}_\lambda)$, which is compatible with H_λ-action.

Lemma 6.2.1. *One has as $\mathcal{O}(\mathfrak{t})^{S_n}$-modules:*

$$\operatorname{Hom}_A(P_\lambda, I_\mu) = \begin{cases} 0 & \lambda \not\succeq \mu \\ \mathcal{O}(\mathfrak{t}_\mu)^{W_\mu} & \lambda = \mu. \end{cases}$$

Proof. We have

$$\operatorname{Hom}_A(P_\lambda, I_\mu) = \operatorname{Hom}_{S_n}(L_\lambda, I_\mu)$$

By Lemma 6.2.2 below, this is equal to

$$\operatorname{Hom}_{S_n}(L_\lambda, \operatorname{Ind}_{S_\mu}^{S_n} \mathbf{1}) \otimes \mathcal{O}(\mathfrak{t}_\mu)^{W_\mu}.$$

It follows from Pieri's formula that $\operatorname{Ind}_{S_\mu}^{S_n} \mathbf{1}$ is a direct sum of L_γ with $\gamma \geq \mu$, such that L_μ occurs with multiplicity one. This immediately implies our assertion. $\square$

We have used the following lemma.

Lemma 6.2.2. *We have*

$$I_\mu \cong \operatorname{Ind}_{S_\mu}^{S_n} \mathbf{1} \otimes \mathcal{O}(\mathfrak{t}_\mu)^{W_\mu}$$

as $S_n \otimes \mathcal{O}(\mathfrak{t})^{S_n}$-modules (with S_n acting trivially on $\mathcal{O}(\mathfrak{t}_\mu)^{W_\mu}$).

Proof. We observe that $W_\lambda = \prod_i S_{r_i}$ acts as a reflection group on $\mathfrak{t}_\lambda$. The harmonic decomposition yields

$$\mathcal{O}(\mathfrak{t}_\mu) = k[W_\mu] \otimes \mathcal{O}(\mathfrak{t}_\mu)^{W_\mu}$$

as W_μ-modules and hence as H_μ-modules. Applying $\operatorname{Ind}_{H_\mu}^{S_n}$ yields the result. $\square$

From now on we fix a total ordering $\prec$ on partitions of n which refines the dominance ordering.

Let $\mathcal{B}_\lambda$ be the thick subcategory of $D^f(A)$ generated by P_ϵ, $\epsilon \preceq \lambda$. In particular, $\mathcal{B}_{[1\cdots 1]} = D^f(A)$. By Theorem 6.1.2, we know that $\mathcal{B}_\lambda$ is admissible and that

$$\mathcal{B}_\lambda = \langle M_\epsilon \mid \epsilon \preceq \lambda \rangle.$$

Also, the subcategories $\langle M_\mu \rangle$, ordered in decreasing order of μ for $\prec$, form a semiorthogonal decomposition of $D^f(A)$. In particular, any subsegment of (M_μ) generates an admissible subcategory.

Let $\mathcal{A}_\lambda$ be the thick subcategory of $D^f(A)$ generated by I_μ, $\mu \succeq \lambda$.

Proposition 6.2.3. *One has*

$$\mathcal{A}_\lambda = \langle M_\mu \mid \mu \succeq \lambda \rangle.$$

In particular, $\mathcal{A}_\lambda$ is admissible and we have a semiorthogonal decomposition

$$D^f(A) = \langle \mathcal{A}_\lambda, \mathcal{B}_\gamma \rangle,$$

where γ is the predecessor to λ for $\prec$. Also, there exists an exact triangle

(6.2.1) $$M_\lambda \to I_\lambda \to C \to \dots$$

with $C \in \mathcal{A}_\tau$, where τ is the successor of λ. In other words, M_λ is obtained from I_λ by left orthogonalization with respect to the subcategory $\mathcal{A}_\tau$ generated by I_μ with $\mu \succ \lambda$.

Proof. By induction we may assume that $\mathcal{A}_\tau$ is generated by M_μ for $\mu \succeq \tau$. In particular, $\mathcal{A}_\tau$ is admissible.

Let M be the left orthogonalization of I_λ with respect to $\mathcal{A}_\tau$. Since M is right orthogonal to $\mathcal{B}_\gamma$ (by Lemma 6.2.1) and left orthogonal to M_μ, $\mu \succeq \tau$, we obtain that M is in the triangulated subcategory $\langle M_\lambda \rangle \subset D^f(A)$ generated by M_λ.

Recall that by Theorem 6.1.2,

$$(6.2.2) \qquad\qquad\qquad \langle M_\lambda \rangle \cong \mathrm{Perf}(R_\lambda)$$

where $R_\lambda = \mathrm{End}_A(M_\lambda) = \mathcal{O}(\mathfrak{t}_\lambda)^{W_\lambda}$, and the equivalence is given by $\mathrm{RHom}_A(M_\lambda, -)$. From the exact triangle (6.1.2) and from the vanishing of Theorem 6.1.2(ii), since $M \in \langle M_\lambda \rangle$, we deduce an isomorphism

$$\mathrm{RHom}_A(M_\lambda, M) = \mathrm{RHom}_A(P_\lambda, M).$$

Next, from Lemma 6.2.1 we deduce

$$\mathrm{RHom}_A(P_\lambda, M) = \mathrm{RHom}_A(P_\lambda, I_\lambda) = \mathcal{O}(\mathfrak{t}_\lambda)^{W_\lambda}.$$

A priori this isomorphism is only as $\mathcal{O}(\mathfrak{t})^{S_n}$-modules. However, it is sufficient to conclude that $\mathrm{RHom}_A(M_\lambda, M)$ is Cohen-Macaulay and hence a projective R_λ-module. Since its rank is equal to 1, we get that

$$\mathrm{RHom}_A(M_\lambda, M) \cong R_\lambda$$

as R_λ-modules. Since $M \in \langle M_\lambda \rangle$, using the equivalence (6.2.2) we deduce an isomorphism $M \cong M_\lambda$. We conclude that M_λ is in $\mathcal{A}_\lambda$, finishing the proof. $\qquad\square$

Corollary 6.2.4. *The subcategory $\mathcal{A}_\lambda$ consists of $C \in D^f(A)$ such that $H^i C$, as S_n-modules, have only L_μ with $\mu \succeq \lambda$ as summands.*

Corollary 6.2.5. *The morphism of modules $M_\lambda \to I_\lambda$ from (6.2.1) is injective and its localization over a dense open subset of $\mathrm{Spec}(R_\lambda)$ is an isomorphism.*

Proof. It is easy to check that if $\mathfrak{t}_\lambda \subset w\mathfrak{t}_\mu$ for some $w \in S_n$ then $\lambda \geq \mu$. This implies that all the modules I_μ with $\mu > \lambda$ are zero over the generic point of $\mathrm{Spec}(R_\lambda)$, so the triangle (6.2.1) implies that the morphism $M_\lambda \to I_\lambda$ is an isomorphism over the generic point of $\mathrm{Spec}(R_\lambda)$. Since M_λ is projective as R_λ-module, the injectivity follows. $\qquad\square$

Remarks 6.2.6. 1. One can check that in fact C from the exact triangle (6.2.1) lies in the subcategory $\langle M_\mu \mid \mu > \lambda \rangle$.

2. Kato's result that the natural morphism $P_\lambda \to M_\lambda$ is surjective (see [38, Cor. 3.6]) together with Corollary 6.2.5 imply that the module M_μ is exactly the image of the morphism of A-modules

$$P_\mu \to I_\mu$$

corresponding to $1 \in \mathcal{O}(\mathfrak{t}_\mu)^{W_\mu} = \mathrm{Hom}_A(P_\mu, I_\mu)$.

6.3. **Global version for type A_n.** Let C be a smooth curve over $\mathbb{C}$. For a partition λ of n consider the closed subvariety $C[\lambda] \subset C^n$ consisting of $(x_1, \ldots, x_n)$ such that

$$(6.3.1) \qquad\qquad x_1 = \ldots = x_{\lambda_1}, x_{\lambda_1+1} = \ldots = x_{\lambda_1+\lambda_2}, \ldots$$

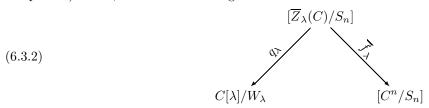($C[\lambda] \simeq C^k$ where k is the number of parts in λ). Now let

$$Z_\lambda(C) \subset C[\lambda] \times C^n$$

be the union over $w \in S_n/S_\lambda$ of the closed subvarieties $\{(x, wx) \mid x \in C[\lambda]\} \subset C[\lambda] \times C^n$ (we equip Z_λ with the reduced scheme structure). Let

$$\overline{Z}_\lambda(C) = Z_\lambda(C)/W_\lambda,$$

where $W_\lambda = \prod_i S_{r_i}$ acts on the first component. On the other hand, the projection $\overline{f}_\lambda : \overline{Z}_\lambda(C) \to C^n$ given by the projection to the second component is S_n-equivariant (where S_n acts on the second

component). Thus, we can use the diagram

$$[\overline{Z}_\lambda(C)/S_n]$$

(6.3.2)

$$\overline{q}_\lambda \qquad \overline{f}_\lambda$$

$$C[\lambda]/W_\lambda \qquad\qquad [C^n/S_n]$$

where $[X/G]$ denotes the stack quotient, to construct a functor

$$\mathcal{N}_\lambda = (\overline{f}_\lambda)_* \circ q_\lambda^* : D^b(C[\lambda]/W_\lambda) \to D^b_{S_n}(C^n).$$

For example, for $\lambda = (n)$, $C[\lambda]$ is the small diagonal in C^n, $\overline{Z}_\lambda(C) = Z_\lambda(C)$ is the graph of the diagonal embedding $C \to C^n$ and $\mathcal{N}_\lambda$ is simply the push-forward functor

$$\Delta_* : D^b(C) \to D^b_{S_n}(C^n).$$

For $\lambda = (1)^n = (1, \ldots, 1)$, $C[\lambda] = C^n$ and we get the pull-back functor with respect to the projection $[C^n/S_n] \to C^{[n]}$, where $C^{[n]}$ is the nth symmetric power of C. In general, one can think of the functors $\mathcal{N}_\lambda$ as modifications of the naive induction functors.

For $C = \mathbb{A}^1$ we can identify $C^n = \mathbb{A}^n$ with the Lie algebra of a maximal torus $\mathfrak{t}$ in GL_n, and $\overline{Z}_\lambda(\mathbb{A}^1)$ with the variety (5.4.6). Hence, by Proposition 6.1.3, we get

$$\mathcal{N}_\lambda(\mathcal{O}_{C[\lambda]/W_\lambda}) \simeq N_\lambda,$$

where N_λ are the $S_n \ltimes \mathcal{O}(\mathfrak{t})$-modules given by the equivariant cohomology of the Springer fibers. Therefore, Theorem 6.1.2 implies that for $C = \mathbb{A}^1$ the functors $\mathcal{N}_\lambda$ are fully faithful and their images satisfy the semiorthogonality with respect to the dominance order of partitions and form a semiorthogonal decomposition of the derived category of S_n-equivariant sheaves on $\mathbb{A}^n$ (by refining the dominance order to some total order). We claim that this picture extends to the general case of any smooth curve C.

Theorem 6.3.1. *For each λ, $|\lambda| = n$, the functor $\mathcal{N}_\lambda$ is fully faithful. One has*

$$\mathrm{Hom}_{D^b_{S_n}(C^n)}(\mathcal{N}_\lambda(\cdot), \mathcal{N}_\mu(\cdot)) = 0 \quad \text{for } \lambda \not\leq \mu.$$

For any ordering $\lambda_1 < \ldots < \lambda_p$ of partitions of n, refining the dominance order we have a semiorthogonal decomposition

$$D^b_{S_n}(C^n) = \langle \mathcal{N}_{\lambda_1}(D^b(C[\lambda_1]/W_{\lambda_1})), \ldots, \mathcal{N}_{\lambda_p}(D^b(C[\lambda_p]/W_{\lambda_p})) \rangle.$$

The idea of the proof is to reduce the statement to the local case considered before. For this we analyze the geometry of the diagram (6.3.2) over a formal neighborhood of a point $x \in S_n$. In fact, it is more convenient to work with the natural $W_\lambda \times S_n$-equivariant map

$$f_\lambda : Z_\lambda(C) \to C^n$$

inducing the map $\overline{f}_\lambda$ (where W_λ acts trivially on C^n). Note that the image of f_λ is $S_n \cdot C[\lambda]$, hence, it is enough to consider the case when $x = (x_1, \ldots, x_n) \in C[\lambda]$. However, x can be a non-generic point of $C[\lambda]$, i.e., in addition to the equalities (6.3.1) there may be some extra equalities $x_i = x_j$, so that the stabilizer subgroup St_x of x in S_n may be bigger than S_λ.

The following point of view on varieties $Z_\lambda(C)$ will be convenient. We replace the partition λ by any decomposition $\mathcal{P} = (L_p)_{p \in P}$ of the set $\{1, \ldots, n\}$ into (unordered) non-empty disjoint subsets: $\{1, \ldots, n\} = \sqcup_{p \in P} L_p$. The partition λ is recovered by considering the cardinalities $|L_p|$. The group S_n acts on such decompositions by $w\mathcal{P} = (wL_p)_{p \in P}$[5]. We denote by $H_\mathcal{P} \subset S_n$ the subgroup of w such that $w\mathcal{P} = \mathcal{P}$, and by $W_\mathcal{P}$ the quotient of $H_\mathcal{P}$ by the subgroup of permutations preserving each subset L_p.

Let $C[\mathcal{P}] \subset C^n$ be the subvariety consisting of (c_s) such that $c_s = c_{s'}$ whenever $s, s' \in L_p$ for some p (one has $C[\mathcal{P}] \simeq C^P$). Then we can define a subscheme $Z_\mathcal{P}(C) \subset C[\mathcal{P}] \times C^n$ as the union (with

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[5]The marking of the parts is not part of the data. For example, the decomposition $\mathcal{P} = (\{1, 2\}, \{3, 4\})$ is preserved by the permutation $(13)(24)$.

reduced scheme structure) of the graphs of the composed maps $C[\mathcal{P}] \xrightarrow{\iota} C^n \xrightarrow{w} C^n$, where ι is the natural embedding and w runs through S_n. It is equipped with a natural action of the group $W_{\mathcal{P}} \times S_n$. Let $f_{\mathcal{P}} : Z_{\mathcal{P}}(C) \to C^n$ denote the morphism induced by the projection to the second factor. For any permutation $w \in S_n$ we have a commutative diagram

$$
\begin{array}{ccc}
Z_{\mathcal{P}}(C) & \xrightarrow{w \times \mathrm{id}} & Z_{w\mathcal{P}}(C) \\
{\scriptstyle f_{\mathcal{P}}} \downarrow & & \downarrow {\scriptstyle f_{w\mathcal{P}}} \\
C^n & \xrightarrow{\mathrm{id}} & C^n
\end{array}
$$

where we use the isomorphism $C[\mathcal{P}] \to C[w\mathcal{P}]$ induced by w. This gives a natural isomorphism

$$(6.3.3) \qquad\qquad\qquad\qquad f_{\mathcal{P},*}\mathcal{O} \simeq f_{w\mathcal{P},*}\mathcal{O}$$

compatible with the S_n-action and with the action of $W_{\mathcal{P}} \simeq W_{w\mathcal{P}}$. Note that the above picture works similarly with $\{1, \ldots, n\}$ replaced by any finite set.

For each λ, a partition of n, we can consider the deomposition $\mathcal{P}_\lambda$ into the subsets $\{1, \ldots, \lambda_1\}$, $\{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}$, etc. Then we recover the previously defined schemes $C[\lambda] = C[\mathcal{P}_\lambda]$, $Z_\lambda(C) = Z_{\mathcal{P}_\lambda}(C)$, and we have $f_\lambda = f_{\mathcal{P}_\lambda}$, $H_\lambda = W_{\mathcal{P}_\lambda}$, $W_\lambda = W_{\mathcal{P}_\lambda}$. Thus, for any decomposition $\mathcal{P}$ we have an S_n-isomorphism $f_{\mathcal{P},*}\mathcal{O} \simeq f_{\lambda,*}\mathcal{O}$, where λ is the partition of n given by the sizes of the subsets forming $\mathcal{P}$.

We can view a decomposition $\mathcal{P} = (L_p)_{p \in P}$ as a surjective map

$$\psi : \{1, \ldots, n\} \to P,$$

defined by $L_p = \psi^{-1}(p)$ for each $p \in P$. We would like to investigate the map $f_{\mathcal{P}}$ near the fiber of a point $x \in C[\mathcal{P}]$. Extra equalities between the coordinates of a point x correspond to an equivalence relation on P, i.e., a surjective map $\phi : P \to Q$, where Q is a finite set. Note that the stabilizer subgroup of x, $\mathrm{St}_x \subset S_n$, is precisely the subgroup of permutations preserving the fibers of the composition $\phi \circ \psi$.

Now for each $q \in Q$ we have a decomposition $\mathcal{P}(q)$ of the set $S(q) := \psi^{-1}\phi^{-1}(q)$ corresponding to the surjective map

$$\psi|_{S(q)} : S(q) \to \phi^{-1}(q) = P(q),$$

so that the parts of $\mathcal{P}(q)$ are the fibers of ψ over $P(q)$. In other words, $\mathcal{P}(q)$ consists of all the parts of $\mathcal{P}$ corresponding to a given value of the coordinates of x. We will refer to the collection $(\mathcal{P}(q))_{q \in Q}$ as *decompositions associated with $\mathcal{P}$ and a point $x \in C[\mathcal{P}]$*.

The decomposition $P = \sqcup_{q \in Q} P(q)$ gives rise to the identification

$$(6.3.4) \qquad\qquad\qquad C[\mathcal{P}] = C^P = \prod_{q \in Q} C^{P(q)} = \prod_{q \in Q} C[\mathcal{P}(q)]$$

so that the point x corresponds to a collection $(x(q))_{q \in Q}$, where each $x(q)$ belongs to the small diagonal $C \subset C[\mathcal{P}(q)]$. Note also that we have a natural decomposition

$$(6.3.5) \qquad\qquad\qquad\qquad C^n = \prod_{q \in Q} C^{S(q)},$$

where $S(q) \subset \{1, \ldots, n\}$ are exactly the sets of indices with the same value of the corresponding coordinate of x.

Lemma 6.3.2. *For $w_1, w_2 \in S_n$ such that $w_1 x \in C[\mathcal{P}]$ and $w_2 x \in C[\mathcal{P}]$ the following conditions are equivalent:*
(i) $w_1 x$ and $w_2 x$ belong to the same $W_{\mathcal{P}}$-orbit;
(ii) $H_{\mathcal{P}} w_1 \mathrm{St}_x = H_{\mathcal{P}} w_2 \mathrm{St}_x$;
(iii) for each $q \in Q$ the decompositions of $S(q)$ induced by $w_1^{-1}\mathcal{P}$ and $w_2^{-1}\mathcal{P}$ differ by an automorphism of $S(q)$.

Proof. The equivalence of (i) and (ii) is clear since $W_\mathcal{P} w_i x = H_\mathcal{P} w_i x$ for $i = 1, 2$. Condition (iii) means that $g w_2^{-1} \mathcal{P} = w_1^{-1} \mathcal{P}$ for some $g \in \mathrm{St}_x$. This is equivalent to $w_1 g w_2^{-1} \mathcal{P} = \mathcal{P}$, i.e., $w_1 g w_2^{-1} \in H_\mathcal{P}$. Thus, we get an equivalence of (iii) and (ii). $\square$

Let us denote by $\hat{Z}_{\mathcal{P},(x,x)}$ the formal completion of $Z_\mathcal{P} = Z_\mathcal{P}(C)$ near the point $(x, x) \in Z_\mathcal{P} \subset C[\mathcal{P}] \times C^n$.

Lemma 6.3.3. *One has an isomorphism*

$$\hat{Z}_{\mathcal{P},(x,x)} \simeq \widehat{\prod}_{q \in Q} \hat{Z}_{\mathcal{P}(q),(x(q),x(q))},$$

where on the right we take the completion of the product, compatible with the decompositions (6.3.4) *and* (6.3.5). *In particular, it is compatible with the action of* $W_\mathcal{P} \cap \mathrm{St}_x = \prod_{q \in Q} W_{\mathcal{P}(q)}$.

Proof. By definition, the components of $Z_\mathcal{P}$ passing through the point (x, x) are numbered by $w \in \mathrm{St}_x$. Now the assertion follows from the decompositions (6.3.5) and (6.3.4), compatible via the embedding $C[\mathcal{P}] \to C^n$, together with the decompositions $\mathrm{St}_x = \prod_q S_{|S(q)|}$, $W_\mathcal{P} \cap \mathrm{St}_x = \prod_q W_{\mathcal{P}(q)}$. $\square$

We now return to the setting of Theorem 6.3.1. Let $\pi : C^n \to C^n/S_n$ be the natural morphism. Also, for a partition λ, $|\lambda| = n$, let

$$g_\lambda : C[\lambda]/W_\lambda \to C^n/S_n$$

be the morphism induced by the embedding of $C[\lambda]$ into C^n.

Lemma 6.3.4. *(i) For a pair of partitions of n, λ and μ, one has*

$$\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), \mathcal{N}_\mu(\mathcal{O}))^{S_n} = 0 \quad \text{for } \lambda \not\leq \mu,$$

where we view $\mathcal{N}_\lambda(\mathcal{O})$ and $\mathcal{N}_\mu(\mathcal{O})$ as sheaves on C^n equipped with S_n-action.
(ii) For each λ the natural map

$$g_{\lambda,*}\mathcal{O} \to \pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), \mathcal{N}_\lambda(\mathcal{O}))^{S_n}$$

is an isomorphism.
(iii) If $F \in D^b_{S_n}(C^n)$ satisfies

$$(\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), F))^{S_n} = 0$$

for all λ, $|\lambda| = n$, then $F = 0$.

Proof. Let us set $Z_\lambda = Z_\lambda(C)$, $Z_\mathcal{P} = Z_\mathcal{P}(C)$ for brevity.
(i) Let $x \in S_n$. Since $\pi^{-1}(\pi(x)) \simeq S_n/\mathrm{St}_x$, we can compute the completion of the sheaf in question at $\pi(x)$ as follows:

$$\left(\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), \mathcal{N}_\mu(\mathcal{O}))^{S_n}\right)^{\widehat{}}_{\pi(x)} \simeq R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O})^{\widehat{}}_x, \mathcal{N}_\mu(\mathcal{O})^{\widehat{}}_x)^{\mathrm{St}_x}.$$

Thus, it is enough to study the picture in the formal neighborhood of x in C^n (keeping track of the St_x-action). Note that

$$\mathcal{N}_\lambda(\mathcal{O}) = f_{\lambda,*}(\mathcal{O}_{Z_\lambda})^{W_\lambda}.$$

The fiber $f_\lambda^{-1}(x)$ consists of the points (y, x) such that $y \in S_n x \cap C[\lambda]$. Since the morphism f_λ is finite, we obtain

$$(6.3.6) \qquad \mathcal{N}_\lambda(\mathcal{O})^{\widehat{}}_x = \left(\bigoplus_{y \in S_n x \cap C[\lambda]} \hat{\mathcal{O}}_{Z_\lambda,(y,x)}\right)^{W_\lambda} = \bigoplus_{i=1}^N \left(\hat{\mathcal{O}}_{Z_\lambda,(y_i,x)}\right)^{W_\lambda \cap \mathrm{St}_{y_i}},$$

where $y_1, \ldots, y_N$ are representatives of W_λ-orbits on $S_n x \cap C[\lambda]$. For each y_i let us fix $w_i \in S_n$ such that $y_i = w_i x$. Then by (6.3.3), setting $\mathcal{P} = \mathcal{P}^\lambda$, we get an isomorphism

$$(6.3.7) \qquad \hat{\mathcal{O}}_{Z_\lambda,(y_i,x)} \simeq \hat{\mathcal{O}}_{Z_{w_i^{-1}\mathcal{P}},(x,x)}$$

of $\hat{\mathcal{O}}_{C^n_x}[\mathrm{St}_x]$-modules, compatible with the commuting action of $W_\lambda \cap \mathrm{St}_{y_i} \simeq W_{w_i^{-1}\mathcal{P}} \cap \mathrm{St}_x$.

Let

(6.3.8)
$$\{1,\ldots,n\} = \sqcup_{q\in Q} S(q)$$

be the decomposition corresponding to the subsets of equal coordinates of x. We have $x = (x(q))_{q\in Q}$ with $x(q) \in C \subset C^{S(q)}$. For each $i = 1,\ldots,N$, let $(\mathcal{P}_i(q))_{q\in Q}$ be the decompositions associated with $w_i^{-1}\mathcal{P}$ and the point $x \in C[w_i^{-1}\mathcal{P}]$, where $\mathcal{P}_i(q)$ is a decomposition of $S(q)$. The subgroup $\mathrm{St}_x \subset S_n$ consists of all permutations preserving each subset $S(q)$. Applying Lemma 6.3.3 we get

(6.3.9)
$$\hat{\mathcal{O}}_{Z_{w_i^{-1}\mathcal{P}},(x,x)} \simeq \hat{\bigotimes}_{q\in Q} \hat{\mathcal{O}}_{Z_{\mathcal{P}_i(q)},(x(q),x(q))}$$

compatible with $\hat{\mathcal{O}}_{C^n,x} = \hat{\bigotimes}_{q\in Q}\hat{\mathcal{O}}_{C^{S(q)},x(q)}$-module structure and with $\mathrm{St}_x = \prod S_{|S(q)|}$-action. The action of $W_{w_i^{-1}\mathcal{P}}\cap\mathrm{St}_x$ on the left-hand side of (6.3.9) corresponds to the action of $\prod_{q\in Q} W_{\mathcal{P}_i(q)}$ on the right-hand side. Also, by (6.3.3), we can replace $Z_{\mathcal{P}_i(q)}$ with $Z_{\lambda(i,q)}$ in the right-hand side of (6.3.9), where $\lambda(i,q)$ is the partition given by the sizes of the subsets forming $\mathcal{P}_i(q)$. Hence, combining isomorphisms (6.3.7) and (6.3.9) we get

$$\left(\hat{\mathcal{O}}_{Z_\lambda,(y_i,x)}\right)^{W_\lambda\cap\mathrm{St}_x} \simeq \hat{\bigotimes}_{q\in Q}\mathcal{N}_{\lambda(i,q)}(\mathcal{O})\hat{}_{x(q)}.$$

Thus, we can rewrite (6.3.6) as

(6.3.10)
$$\mathcal{N}_\lambda(\mathcal{O})\hat{}_x \simeq \bigoplus_{i=1}^N \hat{\bigotimes}_{q\in Q}\mathcal{N}_{\lambda(i,q)}(\mathcal{O})\hat{}_{x(q)}.$$

We have a similar decomposition for μ:

$$\mathcal{N}_\mu(\mathcal{O})\hat{}_x \simeq \bigoplus_{j=1}^M \hat{\bigotimes}_{q\in Q}\mathcal{N}_{\mu(j,q)}(\mathcal{O})\hat{}_{x(q)},$$

where $z_1,\ldots,z_M$ are representatives of W_μ-orbits on $S_n x \cap C[\mu]$. Linearizing the situation in the formal neighborhood of each $x(q) \in C \subset C^{I_q}$ and applying our local results (see Theorem 6.1.2), we see that $\mathrm{Ext}^*(\mathcal{N}_\lambda(\mathcal{O})\hat{}_x,\mathcal{N}_\mu(\mathcal{O})\hat{}_x)^{\mathrm{St}_x}$ can be nonzero only if for some i and some j one has $\lambda(y_i,q) \le \mu(z_j,q)$ for all $q \in Q$. By Lemma 6.1.1, this implies that $\lambda \le \mu$.

(ii) The commutative diagram

(6.3.11)
$$\begin{CD}
[\overline{Z}_\lambda/S_n] @>\overline{f}_\lambda>> [C^n/S_n] \\
@Vq_\lambda VV @VV\pi V \\
C[\lambda]/W_\lambda @>g_\lambda>> C^n/S_n
\end{CD}$$

gives rise to a homomorphism of sheaves of algebras

$$\pi^* g_{\lambda,*}\mathcal{O} \to \overline{f}_{\lambda,*}\mathcal{O}.$$

which corresponds by adjunction to the natural homomorphism

$$g_{\lambda,*}\mathcal{O} \to \pi_*\overline{f}_{\lambda,*}\mathcal{O} \simeq g_{\lambda,*}q_{\lambda,*}\mathcal{O}.$$

Hence, we have the induced homomorphism

$$\pi^* g_{\lambda,*}\mathcal{O} \to R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}),\mathcal{N}_\lambda(\mathcal{O})),$$

or by adjunction,

$$g_{\lambda,*}\mathcal{O} \to \pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}),\mathcal{N}_\lambda(\mathcal{O}))^{S_n}.$$

To show that this is an isomorphism, we can pass to completions of the stalks at a point $\pi(x) \in C^n/S_n$ for some $x \in C[\lambda]$.

First, note that the fiber $g_\lambda^{-1}(\pi(x))$ consists of W_λ-orbits on $S_n x \cap C[\lambda]$, so that

$$(g_{\lambda,*}\mathcal{O})^{\widehat{}}_{\pi(x)} \simeq \bigoplus_{i=1}^{N} \hat{\mathcal{O}}^{W_\lambda}_{C[\lambda],y_i} \simeq \bigoplus_{i=1}^{N} \hat{\mathcal{O}}^{W_{w_i^{-1}\mathcal{P}}}_{C[w_i^{-1}\mathcal{P}],x}.$$

Next, we claim that for each $i \neq i'$ there exists $q \in Q$ such that

$$\mathrm{Ext}^*(\mathcal{N}_{\lambda(i,q)}(\mathcal{O})^{\widehat{}}_{x(q)}, \mathcal{N}_{\lambda(i',q)}(\mathcal{O})^{\widehat{}}_{x(q)})^{S_{|S(q)|}} = 0.$$

Indeed, otherwise by Theorem 6.1.2, we would get $\lambda(i,q) \leq \lambda(i',q)$ for each q. Since merging the partitions $(\lambda(i,q))_{q \in Q}$ (resp., $(\lambda(i',q))_{q \in Q}$) gives the same partition λ, this can happen only if $\lambda(i,q) = \lambda(i',q)$ for all $q \in Q$ (using the last assertion of Lemma 6.1.1). But this is impossible by Lemma 6.3.2, since y_i and $y_{i'}$ are not in the same W_λ-orbit. Thus, using (6.3.10), we get an isomorphism

$$(\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), \mathcal{N}_\lambda(\mathcal{O}))^{\widehat{}}_x)^{S_n} = \bigoplus_{i=1}^{N} R\,\mathrm{End}(\hat{\bigotimes}_{q \in Q} \mathcal{N}_{\lambda(i,q)}(\mathcal{O})^{\widehat{}}_{x(q)})^{\mathrm{St}_x},$$

and the assertion follows from the isomorphisms

$$\hat{\mathcal{O}}^{W_{\lambda(i,q)}}_{C[\lambda(i,q)],x(q)} \to R\,\mathrm{End}(\mathcal{N}_{\lambda(i,q)}(\mathcal{O})^{\widehat{}}_{x(q)})^{S_{|S(q)|}},$$

which follow from Theorem 6.1.2.

(iii) It is enough to prove that for such F one has $\hat{F}_x = F_x \otimes \hat{\mathcal{O}}_{C^n,x} = 0$ for every $x \in C^n$. Let us consider the decomposition (6.3.8) into subsets of equal coordinates of x, so that $x = (x(q))_{q \in Q}$, with $x(q) \in C \subset C^{S(q)}$. Now let us choose any collection $(\mathcal{P}(q))_{q \in Q}$ of decompositions of $S(q)$. Let $\mathcal{P}$ be the decomposition of $\{1, \ldots, n\}$ obtained by merging all $\mathcal{P}(q)$. Then $\mathcal{P} = w\mathcal{P}^\lambda$ for some $w \in S_n$ and some partition λ, $|\lambda| = n$. Hence, by assumption,

(6.3.12) $$\qquad (\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), F)^{\widehat{}}_x)^{S_n} = R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O})^{\widehat{}}_x, \hat{F}_x)^{\mathrm{St}_x} = 0.$$

By (6.3.3), we have

$$\mathcal{N}_\lambda(\mathcal{O})^{\widehat{}}_x = (f_{\mathcal{P},*}\mathcal{O}_{Z_\mathcal{P}})^{\widehat{}}_x.$$

Arguing as in part (i) we get a decomposition of $\hat{\mathcal{O}}_{C^n,x}[\mathrm{St}_x]$-modules

$$(f_{\mathcal{P},*}\mathcal{O}_{Z_\mathcal{P}})^{\widehat{}}_x = \bigoplus_{i=1}^{N} \left(\hat{\mathcal{O}}_{Z_\mathcal{P},(y_i,x)}\right)^{W_\mathcal{P} \cap \mathrm{St}_{y_i}},$$

where y_i are representatives of $W_\mathcal{P}$-orbits on $S_n x \cap C[\mathcal{P}]$, and we can assume that $y_1 = x$. Thus, the vanishing (6.3.12) implies that

$$\mathrm{Hom}_{\hat{\mathcal{O}}_{S_n,x}}\left(\left(\hat{\mathcal{O}}_{Z_\mathcal{P},(x,x)}\right)^{W_\mathcal{P} \cap \mathrm{St}_x}, \hat{F}_x\right)^{\mathrm{St}_x} = 0.$$

But, as we have shown in part (i),

$$\left(\hat{\mathcal{O}}_{Z_\mathcal{P},(x,x)}\right)^{W_\mathcal{P} \cap \mathrm{St}_x} \simeq \hat{\bigotimes}_{q \in Q} \mathcal{N}_{\lambda(i,q)}(\mathcal{O})^{\widehat{}}_{x(q)}.$$

By construction, $(\lambda(i,q))_{q \in Q}$ independently run over all partitions of $|S(q)|$. Hence, the assertion follows from the local Theorem 6.1.2. $\qquad\square$

Proof of Theorem 6.3.1. Let L be an ample line bundle on C^n/S_n. Then for every λ, $g_\lambda^* L$ is an ample line bundle on $C[\lambda]/W_\lambda$ (since g_λ is a finite morphism). Therefore, the line bundles $(g_\lambda^* L^m)_{m \in \mathbb{Z}}$ strongly generate the derived category $D^b(C[\lambda]/W_\lambda)$. Therefore, to check the semiorthogonality of images of $\mathcal{N}_\lambda$ and $\mathcal{N}_\mu$, it is enough to check that

$$\mathrm{RHom}_{D^b_{S_n}(C^n)}(\mathcal{N}_\lambda(g_\lambda^* L^m), \mathcal{N}_\mu(g_\mu^* L^k)) = 0 \quad \text{for } \lambda \not\leq \mu$$

and for any $m, k \in \mathbb{Z}$. Next, we observe that due to commutativity of diagram (6.3.11), one has

$$\mathcal{N}_\lambda(g_\lambda^* L^m) = \overline{f}_{\lambda,*} q_\lambda^* g_\lambda^* L^m \simeq \overline{f}_{\lambda,*}(\overline{f}_\lambda^* \pi^* L^m) \simeq \overline{f}_{\lambda,*}(\mathcal{O}) \otimes \pi^* L^m = \mathcal{N}_\lambda(\mathcal{O}) \otimes \pi^* L^m.$$

Thus,

$$\mathrm{RHom}_{D^b_{S_n}(C^n)}(\mathcal{N}_\lambda(g_\lambda^* L^m), \mathcal{N}_\mu(g_\mu^* L^k)) \simeq \mathrm{RHom}_{D^b_{S_n}(C^n)}(\mathcal{N}_\lambda(\mathcal{O}), \mathcal{N}_\mu(\mathcal{O}) \otimes \pi^* L^{k-m})) \simeq$$
$$R\Gamma(C^n, R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), \mathcal{N}_\mu(\mathcal{O})) \otimes \pi^* L^{k-m})^{S_n} \simeq$$
$$R\Gamma(C^n/S_n, (\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), \mathcal{N}_\mu(\mathcal{O})))^{S_n} \otimes L^{k-m}),$$

and the required vanishing follows from Lemma 6.3.4(i).

The fact that $\mathcal{N}_\lambda$ is fully faithful follows from Lemma 6.3.4(ii) in a similar fashion. Finally, once we know this, we derive that the images of $\mathcal{N}_\lambda$ for all λ generate an admissible subcategory of $D^b_{S_n}(C^n)$, so it is enough to check that its right orthogonal is zero. Suppose $F \in D^b_{S_n}(C^n)$ is such that $\mathrm{RHom}_{D^b_{S_n}(C_n)}(\mathcal{N}_\lambda(L^m), F) = 0$ for all $m \in \mathbb{Z}$ and all partitions λ. Then, as before, we can rewrite this as the vanishing

$$R\Gamma(C^n/S_n, (\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), F))^{S_n} \otimes L^m) = 0$$

for all m and λ. Since L is ample, this implies the vanishing of $(\pi_* R\underline{\mathrm{Hom}}(\mathcal{N}_\lambda(\mathcal{O}), F))^{S_n}$, and the assertion follows from Lemma 6.3.4(iii). $\square$

6.4. Semiorthogonal decompositions for actions of $(\mathbb{Z}/2)^n$ and $(\mathbb{Z}/2)^n \rtimes S_n$.

Let X be a smooth variety equipped with an involution $i : X \to X$ such that the fixed locus of i is a smooth divisor $X^i \subset X$, or equivalently, the corresponding geometric quotient $Y = X/(\mathbb{Z}/2)$ is smooth. In this case the derived category $D^b_{\mathbb{Z}/2}(X)$ of $\mathbb{Z}/2$-equivariant coherent sheaves on X has a natural semiorthogonal decomposition (see [18, Thm. 5.1])

$$(6.4.1) \qquad D^b_{\mathbb{Z}/2}(X) = \langle \pi^* D^b(Y), i_* D^b(X^i) \rangle,$$

where $\pi : X \to Y$ is the projection, the image of the functor π^* (resp. i_*) is equipped with the natural (resp., trivial) $\mathbb{Z}/2$-action.

More generally, if we have a collection $(X_a, i_a)_{1 \leq a \leq n}$ of varieties with involutions as above, then $X_1 \times \ldots \times X_n$ has a natural $(\mathbb{Z}/2)^n$-action and the corresponding derived category of equivariant sheaves has a semiorthogonal decomposition with pieces of the form

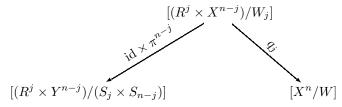$$D^b(\prod_{a \in S} X_a \times \prod_{a \notin S} X_a^{i_a}),$$

where S ranges over all subsets of $\{1, \ldots, n\}$.

In the case when $(X_a, i_a) = (X, i)$ for all $a = 1, \ldots, n$ we get an action of the bigger group

$$W = W_{B_n} = (\mathbb{Z}/2)^n \rtimes S_n$$

on X^n, where $(\mathbb{Z}/2)^n$ acts as above and S_n acts by permuting the factors. Now we can lump together some pieces of the above semiorhogonal decomposition to get a semiorthogonal decomposition of the category $D^b_W(C^n)$.

Set $R = X^i$, and for each j, $0 \leq j \leq n$, consider the natural diagram

$$
\begin{array}{ccc}
 & [(R^j \times X^{n-j})/W_j] & \\
 \swarrow^{\mathrm{id} \times \pi^{n-j}} & & \searrow^{q_j} \\
[(R^j \times Y^{n-j})/(S_j \times S_{n-j})] & & [X^n/W]
\end{array}
$$

where $W_j = (\mathbb{Z}/2)^n \rtimes (S_j \times S_{n-j}) \subset W$, and q_j is induced by the natural closed embedding $R^j \times X^{n-j} \hookrightarrow X^n$. Let

$$\Phi_j = (q_j)_*(\mathrm{id} \times \pi^{n-j})^* : D^b_{S_j \times S_{n-j}}(R^j \times Y^{n-j}) \to D^b_W(X^n)$$

be the corresponding functor between the derived categories.

Proposition 6.4.1. *The functors Φ_j are fully faithful, and we have a semiorthogonal decomposition*

$$D_W^b(X^n) = \langle \Phi_0(D_{S_n}^b(Y^n)), \ldots, \Phi_j(D_{S_j \times S_{n-j}}^b(R^j \times Y^{n-j})), \ldots, \Phi_n(D_{S_n}^b(R^n)) \rangle.$$

Let us denote the normal subgroup $(\mathbb{Z}/2)^n \subset W$ by H. The regular representation of H decomposes as a W-module into the direct sum of W-representations as follows:

$$k[H] = \bigoplus_{j=0}^n V_j,$$

where V_j is the direct sum of all characters of H that are nontrivial on exactly j basis elements in $(\mathbb{Z}/2)^n$ (so V_0 is the trivial representation of W). Hence, the multiplication map

$$k[H] \otimes k[S_n] \xrightarrow{\;\sim\;} k[W]$$

induces an isomorphism of W-modules

(6.4.2) $$\bigoplus_{j=0}^n V_j \otimes k[S_n] \xrightarrow{\;\sim\;} k[W].$$

Proof of Proposition 6.4.1. For an $S_j \times S_{n-j}$-equivariant coherent sheaf $\mathcal{F}$ on $R^j \times Y^{n-j}$ one has

$$\Phi_j(\mathcal{F}) = \bigoplus_{J, |J|=j} (i_J)_* (\mathrm{id} \times \pi^{n-j})^* \mathcal{F},$$

where J ranges over subsets of $\{1, \ldots, n\}$ of cardinality j,

$$i_J : R^j \times X^{n-j} \to X^n$$

is the embedding sending the factors R^j to the components numbered by the subset J and preserving the order of components in R^j and X^{n-j}.

Let $\mathcal{G}$ be a $S_l \times S_{n-l}$-equivariant coherent sheaf on $R^l \times Y^{n-l}$. We have to check that $\mathrm{Ext}_W^*(\Phi_j(\mathcal{F}), \Phi_l(\mathcal{G})) = 0$ for $l < j$ and that for $l = j$ the natural map

$$\mathrm{Ext}_{S_j \times S_{n-j}}^*(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}_W^*(\Phi_j(\mathcal{F}), \Phi_l(\mathcal{G})) = 0$$

is an isomorphism. The idea is to use repeatedly the semiorthogonality of the subcategories in (6.4.1) for appropriate involutions. More precisely, we have

$$\mathrm{Ext}_W^*(\Phi_j(\mathcal{F}), \Phi_l(\mathcal{G})) = \mathrm{Ext}_{W_l}^* \Big(\bigoplus_{J, |J|=j} (i_J)_* (\mathrm{id} \times \pi^{n-j})^* \mathcal{F}, (i_L)_* (\mathrm{id} \times \pi^{n-l})^* \mathcal{G} \Big),$$

where $L = \{1, \ldots, l\}$. We claim that if $l < j$ then all such Ext's vanish even after replacing W_l with $(\mathbb{Z}/2)^n$. Indeed, it suffices to check the vanishing

(6.4.3) $$\mathrm{Ext}_{(\mathbb{Z}/2)^n}^* \big((i_J)_* (\mathrm{id} \times \pi^{n-j})^* \mathcal{F}, (i_L)_* (\mathrm{id} \times \pi^{n-l})^* \mathcal{G} \big) = 0$$

for any J such that $|J| = j$. To this end we observe that there exists an element $s \in J \setminus L$. Then the above vanishing immediately follows by considering the semiorthogonality (6.4.1) for the $\mathbb{Z}/2$-action on the s-th component of the product X^n.

A similar argument shows that the vanishing (6.4.3) holds for $l = k$ and $J \neq L$. Thus, in the case $l = j$ we obtain

$$\mathrm{Ext}_{(\mathbb{Z}/2)^n}^* \Big(\bigoplus_{J, |J|=j} (i_J)_* (\mathrm{id} \times \pi^{n-j})^* \mathcal{F}, (i_L)_* (\mathrm{id} \times \pi^{n-j})^* \mathcal{G} \Big) =$$
$$\mathrm{Ext}_{(\mathbb{Z}/2)^n}^* \big((i_L)_* (\mathrm{id} \times \pi^{n-j})^* \mathcal{F}, (i_L)_* (\mathrm{id} \times \pi^{n-j})^* \mathcal{G} \big) =$$
$$\mathrm{Ext}^* (\mathcal{F}, \mathcal{G}),$$

where the last equality is obtained from the repeated application of the fully faithfulness in (6.4.1). Passing to $S_j \times S_{n-j}$-invariants we deduce our claim.

Next, we need to show that each subcategory $\operatorname{im}(\Phi_j)$ is admissible. It suffices to show that the left and right adjoint functors to Φ_j exist. But this immediately follows from the existence of left and right adjoints for the functors $(q_j)_*$ and $(\pi^{n-j})^*$.

Finally, let us check that the subcategories $\mathcal{C}_j = \Phi_j(D^b_{S_j \times S_{n-j}}(R^j \times Y^{n-j}))$, $j = 0, \dots, n$, generate $D^b_W(X^n)$. Let L be an ample S_n-equivariant line bundle on Y^n. Since the powers of the W-equivariant line bundle $(\pi^n)^* L$ generate the non-equivariant category $D^b(X^n)$, it is enough to check that the W-equivariant bundles $(\pi^n)^* L^i \otimes k[W]$ belong to the subcategory generated by $(\mathcal{C}_j)$. Using the decomposition (6.4.2) we can write

$$(\pi^n)^* L^i \otimes k[W] = \bigoplus_j (\pi^n)^* (L^i \otimes k[S_n]) \otimes V_j.$$

Thus, it is enough to prove that for any S_n-equivariant bundle E and any $k = 0, \dots, n$, the W-equivariant bundle $(\pi^n)^* E \otimes V_k$ belongs to the subcategory generated by $(\mathcal{C}_j)$. Furthermore, since tensoring with the W-bundles of the form $(\pi^n)^* E$ preserves all the subcategories $\mathcal{C}_j$, we only have to consider the bundles $\mathcal{O}_{X^n} \otimes V_k$. Let us start with the exact sequence of $\mathbb{Z}/2$-equivariant sheaves on X,

$$0 \to \mathcal{O}_X \otimes \xi \to \mathcal{O}_X(R) \otimes \xi \to \mathcal{O}_R \to 0.$$

where ξ is a nontrivial character of $\mathbb{Z}/2$. Since $\mathbb{Z}/2$ acts trivially on the fibers of the $\mathbb{Z}/2$-equivariant line bundle $\mathcal{O}_X(R) \otimes \xi$ at fixed points, there exists a line bundle L on Y such that $\mathcal{O}_X(R) \otimes \xi \simeq \pi^* L$. Thus, the above sequence can be rewritten as

(6.4.4) $$0 \to \mathcal{O}_X \otimes \xi \to \pi^* L \to \mathcal{O}_R \to 0.$$

Let $\xi_1, \dots, \xi_n$ be the natural basis of characters of $(\mathbb{Z}/2)^n$. Taking the exterior tensor product of k sequences (6.4.4) and pulling back via the projection $X^n \to X^k$ we obtain a resolution for $\mathcal{O}_{X^n} \otimes \xi_1 \dots \xi_k$ by $(\mathbb{Z}/2)^n$-equivariant sheaves of the form

$$0 \to \mathcal{O}_{X^n} \otimes \xi_1 \dots \xi_k \to \mathcal{F}_0 \to \mathcal{F}_1 \to \dots \to \mathcal{F}_k \to 0,$$

where

$$\mathcal{F}_j = \bigoplus_{J \subset \{1, \dots, k\}, |J| = j} (i_J)_* (\operatorname{id} \times \pi^{n-j})^* \left(\mathcal{O}_{R^j} \boxtimes L^{\boxtimes k - j} \boxtimes \mathcal{O}_{Y^{n-k}} \right),$$

where $i_J : R^j \times X^{n-j} \to X^n$ are the embeddings considered before. Finally, taking direct sum of such resolutions over all subsets in $\{1, \dots, n\}$ of cardinality k we get a resolution of $\mathcal{O}_{X^n} \otimes V_k$ by W-equivariant sheaves of the form

$$\tilde{\mathcal{F}}_j = \Phi_j \left(\bigoplus_{K \subset \{1, \dots, n-j\}, |K| = k-j} \mathcal{O}_{R^j} \boxtimes L^{\boxtimes K} \right)$$
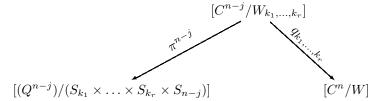
where $L^{\boxtimes K}$ is the line bundle on Y^{n-j} obtained as the tensor product of pull-backs of L in the components corresponding to K.   □

6.5. **Decompositions for certain actions of finite groups on powers of curves.** Let C be a smooth connected curve over $\mathbb{C}$ equipped with a non-trivial involution $i : C \to C$, let $\pi : C \to Q$ be the corresponding double covering, where $Q = C/(\mathbb{Z}/2)$, and let $R = C^i$ be the set of ramification points of π. Then as above the group $W = (\mathbb{Z}/2)^n \rtimes S_n$ acts naturally on C^n. Now we can combine the decomposition of Proposition 6.4.1 with the semiorthogonal decompositions of $D^b_{S_j \times S_{n-j}}(R^j \times Q^{n-j})$ constructed in Section 6.3. More precisely, for each $j = 0, \dots, n$ we have to combine such a decomposition with the natural orthogonal decomposition coming from the decomposition of R^j into S_j-orbits. The resulting semiorthogonal decomposition of $D^b_W(C^n)$ is described in Theorem 6.5.1 below. It will be further generalized in Theorem 6.5.4.

Let $R = p_1, \dots, p_r$. For each collection of non-negative integers $k_1, \dots, k_r$ such that $k_1 + \dots + k_r = j \leq n$ let us consider the functor

$$\Phi_{k_1, \dots, k_r} = (q_{k_1, \dots, k_r})_* (\pi^{n-j})^* : D^b_{S_{k_1} \times \dots \times S_{k_r} \times S_{n-j}}(Q^{n-j}) \to D^b_W(C^n)$$

associated with the correspondence

$$[C^{n-j}/W_{k_1,\ldots,k_r}]$$

$$\pi^{n-j} \qquad\qquad q_{k_1,\ldots,k_r}$$

$$[(Q^{n-j})/(S_{k_1} \times \ldots \times S_{k_r} \times S_{n-j})] \qquad\qquad [C^n/W]$$

where $W_{k_1,\ldots,k_r} = (\mathbb{Z}/2)^n \rtimes (S_{k_1} \times \ldots \times S_{k_r} \times S_{n-j}) \subset W$, and $q_{k_1,\ldots,k_r}$ is induced by the closed embedding

$$C^{n-j} \to C^n : (x_1,\ldots,x_{n-j}) \mapsto (p_1^{k_1},\ldots,p_r^{k_r}, x_1,\ldots,x_{n-j}).$$

Theorem 6.5.1. *The functors $\Phi_{k_1,\ldots,k_r}$, for $k_1 + \ldots + k_r \leq n$, are fully faithful and the subcategories $\operatorname{im}(\Phi_{k_1,\ldots,k_r}) \subset D_W^b(C^n)$ are admissible. We have*

$$\operatorname{Ext}^*(\operatorname{im}(\Phi_{k_1,\ldots,k_r}), \operatorname{im}(\Phi_{k_1',\ldots,k_r'})) = 0$$

unless $k_1 \leq k_1', \ldots, k_r \leq k_r'$. We have a semiorthogonal decomposition of $D^b(C^n)$ into these subcategories ordered compatibly with the above semiorthogonalities. Furthermore, for each $(k_1,\ldots,k_r)$ we have a further semiorthogonal decomposition of

$$\operatorname{im}(\Phi_{k_1,\ldots,k_r}) \simeq D_{S_{k_1} \times \ldots \times S_{k_r} \times S_{n-j}}^b(Q^{n-j}),$$

where $j = k_1 + \ldots + k_r$, with pieces numbered by collections of partitions $\mu^{(1)},\ldots,\mu^{(r)},\nu$, where $|\mu^{(i)}| = k_i$, $|\nu| = n - j$, such that the corresponding sheaves are supported on the stratum in Q^{n-j} associated with ν.

Note that in the case when R is one point the semiorthogonal decomposition of the above theorem is numbered by pairs of partitions (μ,ν) such that $|\mu| + |\nu| = n$. Thus, in the case when $C = \mathbb{A}^1$ and $i(x) = -x$ we get a semiorthogonal decomposition for the standard representation of the Weyl group of type B_n on its vector representation.

More generally, assume that we have a smooth connected curve C with an effective action of a finite group G, so that the quotient $Q = C/G$ is still smooth. Let $R = D_1 \sqcup \ldots \sqcup D_r$ be the decomposition into G-orbits of the ramification locus of the projection $\pi : C \to Q$ (note that D_i's are also fibers of π). For each $i = 1,\ldots,r$, pick a point $p_i \in D_i$. Note that the stabilizer subgroup $G_i \subset G$ of p_i is a cyclic group of some order m_i (since we are in characteristic zero). The following result was proved in [57, Thm. 1.2]:

Theorem 6.5.2. *For each i the collection of G-equivariant sheaves*

$$(6.5.1) \qquad\qquad \mathcal{O}_{(m_i-1)D_i},\ldots,\mathcal{O}_{2D_i},\mathcal{O}_{D_i}$$

is exceptional in $D_G(C)$. Let $\mathcal{B}_i \subset D_G(C)$ be the subcategory generated by the collection (6.5.1). Then the subcategories $\mathcal{B}_1,\ldots,\mathcal{B}_r$ are mutually orthogonal and we have a semiorthogonal decomposition

$$(6.5.2) \qquad\qquad D_G^b(C) = \langle \pi^* D^b(Q), \mathcal{B}_1,\ldots,\mathcal{B}_r \rangle.$$

Remark 6.5.3. It is not hard to see that the pieces of the semiorthogonal decomposition of Theorem 6.5.2 (where we decompose each $\mathcal{B}_i$ into the subcategories generated by the individual exceptional sheaves) matches with the motivic decomposition (0.0.1), and thus confirms Conjecture A in this case. Since all the motivic pieces except for Q are points, this is equivalent to the identity

$$\sum_{g \in G \setminus \{1\}/\sim} |C(g) \backslash C^g| = \sum_{i=1}^r (m_i - 1).$$

One way to prove this is to compare the corresponding decomposition of Hochschild homology of $[C/G]$ with the general decomposition (2.1.1). A more direct matching is obtained by comparing contributions to both sides corresponding to each special fiber D_i of π. More precisely if $H = G_i$, the stabilizer subgroup of one of the ramification points p_i then the equality between these contributions has form

$$(6.5.3) \qquad\qquad \sum_{h \in H \setminus \{1\}/\sim_G} |C(h) \backslash (G/H)^h| = |H| - 1.$$

Here $\sim_G$ denotes the equivalence relation "being conjugate in G", and $(G/H)^h$ denotes the h-invariant G/H (which is in bijection with $C^h \cap D_i$). Now we observe that the identity (6.5.3) holds for any abelian subgroup H in a finite group G and follows from from the bijection

$$C(h)\backslash(G/H)^h \to H \cap \operatorname{Ad}(G)(h) : C(h)gH \mapsto g^{-1}hg,$$

where $\operatorname{Ad}(G)(h)$ is the conjugacy class of h in G.

Now let us consider the induced action of the group

$$\mathbb{G} := G^n \rtimes S_n$$

on C^n. We are going to construct a semiorthogonal decomposition of the corresponding equivariant category $D^b_{\mathbb{G}}(C^n)$, which reduces to that of Theorem 6.5.1 in the case $G = \mathbb{Z}/2$.

We start by constructing a coarser decomposition with pieces numbered by collections of r partitions $(\lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $\ell(\lambda^{(1)}) + \ldots + \ell(\lambda^{(r)}) \leq n$ and nonzero parts of $\lambda^{(i)}$ are restricted by $1 \leq \lambda^{(i)}_j \leq m_i - 1$ (here we denote by $\ell(\lambda)$ the number of nonzero parts in a partition λ). Recall that for a partition λ we denote $W_\lambda = S_{r_1} \times S_{r_2} \times \ldots \subset S_{\ell(\lambda)}$, where $r_1, r_2, \ldots$ are the multiplicities of parts in λ. For each such $(\lambda^{(1)}, \ldots, \lambda^{(r)})$ with $\ell(\lambda^{(1)}) + \ldots + \ell(\lambda^{(r)}) = j$ we will define a functor

$$\Phi_{\lambda^{(1)}, \ldots, \lambda^{(r)}} : D^b_{W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j}}(Q^{n-j}) \to D^b_{\mathbb{G}}(C^n).$$

Note that here the group $W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j}$ acts on Q^{n-j} via the projection to S_{n-j}. The relevant quotient stacks are connected by the diagram

$$[C^n/\mathbb{G}_{\lambda^{(1)}, \ldots, \lambda^{(r)}}]$$

$$\pi^{n-j}\,\mathrm{pr}_{n-j} \swarrow \qquad\qquad \searrow q$$

$$[(Q^{n-j})/(W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j})] \qquad\qquad [C^n/\mathbb{G}]$$

where

$$\mathbb{G}_{\lambda^{(1)}, \ldots, \lambda^{(r)}} = G^n \rtimes (W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j}) \subset \mathbb{G},$$

$\mathrm{pr}_{n-j} : C^n \to C^{n-j}$ is the projection onto the last $n - j$ coordinates and q is the natural projection. For each i let us consider the coherent sheaf

$$K_{\lambda^{(i)}} := \mathcal{O}_{(m_i - \lambda^{(i)}_1)D_i} \boxtimes \mathcal{O}_{(m_i - \lambda^{(i)}_2)D_i} \boxtimes \ldots$$

on $C^{\ell(\lambda^{(i)})}$. Note that it has a natural $G^{\ell(\lambda^{(i)})} \rtimes W_{\lambda^{(i)}}$-equivariant structure. Now we define a $G^j \rtimes (W_{\lambda^{(1)}} \times \ldots \times W_{\lambda^{(r)}})$-equivariant sheaf K on C^j by

$$K_{\lambda^{(1)}, \ldots, \lambda^{(r)}} = K_{\lambda^{(1)}} \boxtimes \ldots \boxtimes K_{\lambda^{(r)}}$$

and set

$$\Phi_{\lambda^{(1)}, \ldots, \lambda^{(r)}}(\mathcal{F}) = q_*(K_{\lambda^{(1)}, \ldots, \lambda^{(r)}} \boxtimes (\pi^{n-j})^* \mathcal{F}).$$

For a pair of partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \ldots)$ we write $\lambda \subset \mu$ if $\lambda_j \leq \mu_j$ for each j.

Theorem 6.5.4. *The functors $\Phi_{\lambda^{(1)}, \ldots, \lambda^{(r)}}$ are fully faithful, and their images are admissible subcategories. We have*

(6.5.4) $$\operatorname{Ext}^*_{\mathbb{G}}(\operatorname{im}(\Phi_{\lambda^{(1)}, \ldots, \lambda^{(r)}}), \operatorname{im}(\Phi_{\mu^{(1)}, \ldots, \mu^{(r)}})) = 0$$

unless $\lambda^{(i)} \subset \mu^{(i)}$ for $i = 1, \ldots, r$. Furthermore, we have a semiorthogonal decomposition of $D^b_{\mathbb{G}}(C^n)$ into these subcategories, ordered compatibly with the above semiorthogonalities. Each subcategory $\operatorname{im}(\Phi_{\lambda^{(1)}, \ldots, \lambda^{(r)}})$ has a further semiorthogonal decomposition into pieces numbered by irreducible representations of $W_{\lambda^{(1)}} \times \ldots \times W_{\lambda^{(r)}} \times S_{n-j}$ (where $j = \ell(\lambda^{(1)}) + \ldots + \ell(\lambda^{(r)})$).

We start with some auxiliary statements. Suppose a group H acts on a space X and $H' \subset H$ is a subgroup. Then the push-forward with respect to the natural projection $[X/H'] \to [X/H]$ gives a natural functor between the categories of equivariant sheaves that we denote

$$\text{Ind}_{H'}^H : D_{H'}^b(X) \to D_H^b(X).$$

Note that in the definition of the functors $\Phi_{\lambda^{(1)},...,\lambda^{(r)}}(\mathcal{F})$ we have $q_* = \text{Ind}_{\mathbb{G}_{\lambda^{(1)},...,\lambda^{(r)}}}^{\mathbb{G}}$.

Lemma 6.5.5. *Let* $a_1, \ldots, a_m$ *be nonnegative integers such that* $a_1 + \ldots + a_m = n$. *For each* $i = 1, \ldots, m$, *let* $a_{i1}, a_{i2}, \ldots$ *be nonnegative integers such that* $a_{i1} + a_{i2} + \ldots = a_i$. *Set* $H = G^n \rtimes (S_{a_1} \times \ldots \times S_{a_m})$. *Then*

$$\text{Ind}_H^{\mathbb{G}} \left(\text{Ind}_{G^{a_1} \rtimes (S_{a_{11}} \times S_{a_{12}} \times \ldots)}^H (\mathcal{F}_1) \boxtimes \ldots \boxtimes \text{Ind}_{G^{a_m} \rtimes (S_{a_{m1}} \times S_{a_{m2}} \times \ldots)}^H (\mathcal{F}_m) \right) \simeq$$

$$\text{Ind}_{G^n \rtimes (S_{a_{11}} \times S_{a_{12}} \times \ldots \times S_{a_{m1}} \times S_{a_{m2}} \times \ldots)}^{\mathbb{G}} (\mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_m).$$

Proof. This immediately follows from the fact that for a chain of subgroups $H'' \subset H' \subset H$, where H acts on a space X, one has $\text{Ind}_{H'}^H \circ \text{Ind}_{H''}^{H'} \simeq \text{Ind}_{H''}^H$. $\qquad\square$

Lemma 6.5.6. *Let* E *be a* G-*equivariant vector bundle on* C. *Then the* $\mathbb{G}$-*equivariant vector bundle* $E^{\boxtimes n}$ *belongs to the triangulated subcategory generated by the images of the functors* $\Phi_{\lambda^{(1)},...,\lambda^{(r)}}$.

Proof. First, we have a canonical exact sequence in $\text{Coh}_G(C)$

$$0 \to E \to \pi^* E' \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is some sheaf in the subcategory generated by $\mathcal{B}_1, \ldots, \mathcal{B}_r$ (see Theorem 6.5.2). By taking the nth exterior tensor power, we get a resolution of $E^{\boxtimes n}$ with terms of the form

$$\text{Ind}_{G^n \rtimes (S_k \times S_{n-k})}^{\mathbb{G}} \left(\mathcal{F}^{\boxtimes k} \boxtimes \pi^*(E')^{\boxtimes n-k} \right)$$

Hence, by Lemma 6.5.5, it is enough to prove that the assertion is true with E replaced by $\mathcal{F}$. Next, we claim that if $\mathcal{F}$ fits into an exact sequence of G-equivariant sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

and the assertion holds for $\mathcal{F}'$ and $\mathcal{F}''$ then it holds for $\mathcal{F}$. Indeed, we have an sequence of the form

$$0 \to (\mathcal{F}')^{\boxtimes n} \to \mathcal{F}^{\boxtimes n} \to \ldots$$

where other terms are induced by exterior tensor powers involving $< n$ factors $\mathcal{F}$ (and some factors $\mathcal{F}''$). Hence, our claim follows, using the induction in n and Lemma 6.5.5.

Thus, the statement reduces to the case when $\mathcal{F}$ is one of the simple G-equivariant sheaves supported on one G-orbit D_i. Note that for $p = 1, \ldots, m_i - 1$, we have

$$\mathcal{O}_{pD_i}^{\boxtimes s} \in \text{im}(\Phi_{0,\ldots,0,(p)^s,0,\ldots,0)}),$$

where we take $\lambda^{(i)} = (p)^s$ (p repeated s times) and the remaining partitions are zero. Also,

$$\mathcal{O}_{mD_i} = \pi^* \mathcal{O}_{\pi(p_i)},$$

so that $\mathcal{O}_{mD_i}^{\boxtimes s}$ is in the image of $(\pi^s)^*$. Thus, the assertion holds for the exterior tensor powers of the sheaves $\mathcal{O}_{pD_i}$, with $p = 1, \ldots, m_i$. Let $\xi : G_i \to \mathbb{G}_m$ be the character of the stabilizer subgroup corresponding to the action on the fiber of ω_C at p_i. Then the simple G-equivariant sheaves supported on D_i are of the form $\xi^j \otimes \mathcal{O}_{D_i}$, $j = 0, \ldots, m_i - 1$. Now the assertion for the exterior tensor powers of these sheaves follows as before by induction on j, using the exact sequences of G-equivariant sheaves

$$0 \to \xi \otimes \mathcal{O}_{D_i} \to \mathcal{O}_{2D_i} \to \mathcal{O}_{D_i} \to 0,$$

$$0 \to \xi^2 \otimes \mathcal{O}_{D_i} \to \mathcal{O}_{3D_i} \to \mathcal{O}_{2D_i} \to 0,$$

etc. $\qquad\square$

Proof of Theorem 6.5.4. For
$$\mathcal{F} \in D^b_{W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j}}(Q^{n-j}) \text{ and } \mathcal{G} \in D^b_{W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-k}}(Q^{n-k}),$$
where $j = \ell(\lambda^{(1)}) + \ldots + \ell(\lambda^{(r)})$, $k = \ell(\mu^{(1)}) + \ldots + \ell(\mu^{(r)})$, we have
$$\text{Ext}^*_{\mathbb{G}}\left(\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}}(\mathcal{F}), \Phi_{\mu^{(1)},\ldots,\mu^{(r)}}(\mathcal{G})\right) =$$
$$\text{Ext}^*_{\mathbb{G}_{\mu^{(1)},\ldots,\mu^{(r)}}}\left(\text{Ind}^{\mathbb{G}}_{\mathbb{G}_{\lambda^{(1)},\ldots,\lambda^{(r)}}}(K_{\lambda^{(1)},\ldots,\lambda^{(r)}} \boxtimes (\pi^{n-j})^*\mathcal{F}), K_{\mu^{(1)},\ldots,\mu^{(r)}} \boxtimes (\pi^{n-k})^*\mathcal{G}\right).$$
As in the proof of Proposition 6.4.1, considering only the action of G^n and using the semiorthogonalities of Theorem 6.5.2, we see that the above Ext^* vanishes unless $\lambda^{(1)} \subset \mu^{(1)}, \ldots, \lambda^{(r)} \subset \mu^{(r)}$ (recall that each part p in $\lambda^{(i)}$ corresponds to the sheaf $\mathcal{O}_{(m_i-p)D_i}$ on the corresponding factor). Similarly, in the case when $\lambda^{(i)} = \mu^{(i)}$, using only G^n-invariance in the above Ext^* we get $\text{Ext}^*(\mathcal{F},\mathcal{G})$ computed in $D^b(Q^{n-j})$. Taking the invariants with respect to $W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j}$ gives Ext^* in the correct equivariant category, so we deduce that the functor $\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ is fully faithful. Note that each category $D^b_{W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j}}(Q^{n-j})$ is saturated, so it remains to show that the images of the functors $\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ generate the category $D^b_{\mathbb{G}}(C^n)$.

Let L be an ample line bundle on Q. Then the line bundles $(\pi^*L^i)^{\boxtimes n}$ generate the non-equivariant category $D^b(C^n)$. Hence, to check that the subcategories $\text{im}(\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}})$ generate $D^b_{\mathbb{G}}(C^n)$, it is enough to check that the $\mathbb{G}$-equivariant bundles $(\pi^*L^i)^{\boxtimes n} \otimes k[\mathbb{G}]$ belong to the subcategory generated by $(\text{im}(\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}}))$.

Let $V_1, \ldots, V_d$ be all non-isomorphic irreducible representations of G. For non-negative integers $m_1, \ldots, m_d$, such that $m_1 + \ldots + m_d = n$, consider the representation $V_1^{\otimes m_1} \otimes \ldots \otimes V_d^{\otimes m_d}$ of $G^n \rtimes (S_{m_1} \times \ldots \times S_{m_d})$, and define a $\mathbb{G}$-representation by
$$V_{m_1,\ldots,m_d} = \text{Ind}^{\mathbb{G}}_{G^n \rtimes (S_{m_1} \times \ldots \times S_{m_d})}(V_1^{\otimes m_1} \otimes \ldots \otimes V_d^{\otimes m_d}).$$

Note that if V is any G-representation then $V^{\otimes n}$ decomposes as a $\mathbb{G}$-representation into a direct sum of representations $V_{m_1,\ldots,m_d}$ taken with some multiplicities. On the other hand, we have an isomorphism of $\mathbb{G}$-representations
$$k[G^n] \otimes k[S_n] \simeq k[\mathbb{G}],$$
where $\mathbb{G}$ acts on $k[S_n]$ via the projection $\mathbb{G} \to S_n$, and on G^n by conjugation. Furthermore, $k[G^n] \simeq k[G]^{\otimes n}$, where G acts on $k[G]$ by conjugation. Hence, every irreducible $\mathbb{G}$-representation is a direct summand in a representation of the form $V_{m_1,\ldots,m_d} \otimes k[S_n]$.

Thus, it is enough to prove that the $\mathbb{G}$-equivariant bundles
$$(\pi^*L^i)^{\boxtimes n} \otimes V_{m_1,\ldots,m_d} \otimes k[S_n]$$
belong to the subcategory generated by $(\text{im}(\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}}))$. It is easy to see that tensoring with $k[S_n]$ preserve each subcategory $\text{im}(\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}})$, so we are reduced to proving the same assertion for $(\pi^*L^i)^{\boxtimes n} \otimes V_{m_1,\ldots,m_d}$. We have
$$(\pi^*L^i)^{\boxtimes n} \otimes V_{m_1,\ldots,m_d} = \text{Ind}^{\mathbb{G}}_{G^n \rtimes (S_{m_1} \times \ldots \times S_{m_d})}\left((\pi^*L^i \otimes V_1)^{\boxtimes m_1} \otimes \ldots \otimes (\pi^*L^i \otimes V_d)^{\boxtimes m_d}\right).$$
By Lemma 6.5.6, for each $j = 1, \ldots, d$ the object
$$(\pi^*L^i \otimes V_j)^{\boxtimes m_j} \in D^b_{G^{m_i} \rtimes S_{m_i}}(C^{m_i})$$
belongs to the subcategory generated by $(\text{im}(\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}}))$ (with n replaced by m_i). It remains to apply Lemma 6.5.5.

We have an orthogonal decomposition of
$$\text{im}(\Phi_{\lambda^{(1)},\ldots,\lambda^{(r)}}) = D^b_{W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}} \times S_{n-j}}(Q^{n-j})$$
numbered by irreducible representations of $W_{\lambda^{(1)}} \times \ldots W_{\lambda^{(r)}}$ with each piece equivalent to $D^b_{S_{n-j}}(Q^{n-j})$. The latter category has a semiorthogonal decomposition constructed in Section 6.3. $\qquad\square$

Note that in the case $G = \mu_m$ (roots of unity of order m) the group $\mathbb{G}$ above becomes the reflection group $G(m,1,n)$, and in the case when $C = \mathbb{A}^1$ and μ_m acts on $\mathbb{A}^1$ through the embedding $\mu_m \subset \mathbb{G}_m$,

the action of $G(m, 1, n)$ on $C^n = \mathbb{A}^n$ becomes the standard vector representation of $G(m, 1, n)$. We can check that in this case the pieces of the semiorthogonal decomposition of Theorem 6.5.4 match with the pieces of the motivic decomposition (0.0.1).

Corollary 6.5.7. *Conjecture A holds for the standard action of $G(m, 1, n)$ on $\mathbb{A}^n$.*

Proof. The matching of pieces of the semiorthogonal decomposition of Theorem 6.5.4 (the finer of the two decompositions considered in this Theorem) with the varieties of $V^g / C(g)$ follows from Propositions 2.1.5 and 2.2.4. □

Remark 6.5.8. Here is an explicit bijection between the pieces of the semiorthogonal decomposition in the above Corollary and conjugacy classes of $G(m, 1, n)$. In our case $r = 1$, so the pieces of the rougher decomposition are numbered by partitions λ such that $\ell(\lambda) = j \leq n$, and λ has all parts $\leq m - 1$. The pieces of the finer decomposition are numbered by further choosing partitions $\mu^{(1)}, \ldots, \mu^{(m-1)}$ with $|\mu^{(a)}|$ equal to the multiplicity of the part a in λ, and in addition a partition $\mu^{(0)}$ such that $|\mu^{(0)}| = n - j$. On the other hand, conjugacy classes in $G(m, 1, n)$ are also numbered by such collections of partitions $(\mu(0), \ldots, \mu^{(m-1)})$: $\mu^{(i)}$ records the cycles C in the cycle type of an element with $z(C) = \zeta_m^i$, where ζ_m is a fixed primitive mth root of unity (see (2.2.1)). The corresponding semiorthogonal piece is the same as the piece corresponding to the partition $\mu^{(0)}$ in the semiorthogonal decomposition of $D^b_{S_{n-j}}(\mathbb{A}^{n-j})$, so it is $D^b(\mathbb{A}^{\ell(\mu^{(0)})})$.

7. Types B and D via Springer correspondence

7.1. Type B_n.
We are using the explicit description of the Springer correspondence given in [50] (see also [16]).

Recall that representations of W_{B_n} are parametrized by the set $\Pi_n^{(2)}$ of pairs of partitions (ξ, ν) such that $|\xi| + |\nu| = n$. We write such a pair of partitions in the form $\xi = (\xi_0 \leq \xi_1 \leq \ldots \leq \xi_r)$, $\nu = (\nu_0 \leq \nu_1 \leq \ldots \leq \nu_{r-1})$, where we complete one of the paritions with zero parts, and associate with it the symbol

$$
(7.1.1) \qquad \begin{pmatrix} \xi_0 & & \xi_1 + 2 & & \xi_2 + 4 & & \ldots & & \xi_r + 2r \\ & \nu_0 & & \nu_1 + 2 & & \ldots & \nu_{r-1} + 2(r-1) & & \end{pmatrix}
$$

In [50] this symbol is described as a pair (R, R') of sets of numbers, where R and R' are the first and the second rows in (7.1.1), respectively.

On the other hand, the nilpotent orbits in $\mathrm{SO}(2n + 1)$ are parametrized by the set Π_{2n+1}^+ of partitions of $2n + 1$ such that each even part appears with even multiplicity. We denote by O_λ the nilpotent orbit corresponding to $\lambda \in \Pi_{2n+1}^+$. For each m let r_m be the multiplicity with which m appears in λ. Then the reductive part of the centralizer $\mathbf{G}_\lambda$ of an element in O_λ is

$$
(7.1.2) \qquad R\mathbf{G}_\lambda = S\left(\prod_{m \text{ even}} \Delta_m \mathrm{Sp}(r_m) \times \prod_{m \text{ odd}} \Delta_m \mathrm{O}(r_m) \right),
$$

where Δ_m denotes the diagonal embedding into product of m identical copies, $S(?)$ denotes passing to matrices of determinant 1. To get the reductive part of $\mathbf{G}_\lambda^0$ one has to replace each $\mathrm{O}(r_m)$ by $\mathrm{SO}(r_m)$ (see [17, Sec. 6.1]). Let

$$
\mathcal{I}_\lambda = \{m \mid m \text{ is odd and } r_m > 0\}.
$$

Then from (7.1.2) we see that the component group F_λ of $\mathbf{G}_\lambda$ is given by

$$
(7.1.3) \qquad F_\lambda = F(\mathcal{I}_\lambda) \subset (\mathbb{Z}/2)^{\mathcal{I}_\lambda},
$$

where for a finite set S we define the group $F(S)$ as the kernel of the addition map

$$
(\mathbb{Z}/2)^S \to \mathbb{Z}/2 : \sum n_s e_s \mapsto \sum n_s.
$$

Note that the algebra

$$
(7.1.4) \qquad B_\lambda := H^*_{\mathbf{G}_\lambda^0}(pt) = \bigotimes_m B_{\lambda, m}
$$

is the tensor product of the algebras of polynomials on the maximal tori, invariant with respect to the Weyl group, for the groups $\mathrm{Sp}(r_m)$ and $\mathrm{SO}(r_m)$, where m ranges over parts in λ. Furthermore, we can explicitly determine the action of F_λ on B_λ.

Let $\mathcal{I}_\lambda^+ \subset \mathcal{I}_\lambda$ (resp., $\mathcal{I}_\lambda^- \subset \mathcal{I}_\lambda$) be the subset of odd m appearing in λ with even (resp., odd) multiplicity. Note that if r is odd then we can find an involution in $O(r) \setminus SO(r)$ acting trivially on the Lie algebra of the maximal torus in $SO(r)$. On the other hand, in the case of even r any involution in $O(r) \setminus SO(r)$ gives rise to an outer involution of $SO(r)$. Furthermore, in this case, setting $r = 2k$, we can choose the basic W-invariant polynomials to be $f_1, \ldots, f_{k-1}, f_k$, where for $i < k$, $f_i = e_i(x_1^2, \ldots, x_k^2)$ is the ith elementary symmetric function in the squares of the coordinates, and $f_k = x_1 x_2 \ldots x_k$. Then the outer involution preserves $f_1, \ldots, f_{k-1}$ and sends f_k to $-f_k$. This leads to the following

Lemma 7.1.1. *The group F_λ acts on B_λ through the natural projection*

$$(7.1.5) \qquad\qquad \pi_\lambda : F_\lambda \to F_\lambda' := (\mathbb{Z}/2)^{\mathcal{I}_\lambda^+}.$$

The action of F_λ' on $B_{\lambda,m}$ is trivial if either m is even or r_m is odd. For $m \in \mathcal{I}_\lambda^+$ the action of F_λ' on $B_{\lambda,m} = \mathbb{C}[f_1, \ldots, f_{m/2}]$ factors through the projection to the corresponding $\mathbb{Z}/2$-factor and sends $f_{m/2}$ to $-f_{m/2}$ while leaving other coordinates invariant. Thus, we have

$$\mathrm{Spec}(B_\lambda) \simeq \mathbb{A}^l \times \mathbb{A}^N,$$

where $F_\lambda' = (\mathbb{Z}/2)^l$ acts by the A_1^l-type action on $\mathbb{A}^l$ (and acts trivially on $\mathbb{A}^N$).

We define a map

$$\mathrm{Spr}_1 : \Pi_{2n+1}^+ \to \Pi_n^{(2)}$$

as follows: for $\lambda = (\lambda_0 \leq \lambda_1 \leq \ldots)$ consider $\lambda_i' = \lambda_i + i$. Then let

$$\widetilde{\xi}_0 < \widetilde{\xi}_1 < \ldots \quad (\text{resp. } \widetilde{\nu}_0 < \widetilde{\nu}_1 < \ldots)$$

be all the odd (resp., even) numbers among (λ_i'). We set $\xi_i' = (\widetilde{\xi}_i - 1)/2$, $\nu_i' = \widetilde{\nu}_i/2$, and define $\mathrm{Spr}_1(\lambda) = (\xi, \nu)$, where $\xi_i = \xi_i' - i$, $\nu_i = \nu_i' - i$ (there can be some zero parts in ξ and ν).

The map Spr_1 describes the image of a pair of the form $(O_\lambda, \mathbf{1})$ under the Springer correspondence, where $\mathbf{1}$ is the trivial character of the component group F_λ of the centralizer $\mathbf{G}_\lambda$ of an element in O_λ. The pairs (ξ, ν) in the image of Spr_1 are characterized by the inequalities

$$\xi_0 \leq \nu_1 \leq \xi_1 + 2 \leq \nu_2 + 2 \leq \xi_2 + 4 \leq \ldots,$$

i.e., the entries in the symbol (7.1.1) are (weakly) increasing from left to right (regardless of the row).

Let (R_λ, R_λ') be the row sets of the symbol of $\mathrm{Spr}_1(\lambda)$. The maximal intervals of integers in $(R_\lambda \cup R_\lambda') \setminus (R_\lambda \cap R_\lambda')$ are in natural bijection with the odd numbers appearing as parts in λ (where the leftmost interval corresponds to the smallest odd part, etc.). Furthermore, the length of the interval is equal to the multiplicity of the corresponding odd part in λ.

Two symbols associated with pairs of partitions are called similar if they contain the same entries with the same multiplicities. Each similarity class contains the unique symbol in the image of Spr_1. Let $\mathcal{S}_\lambda \subset \Pi_n^{(2)}$ be the set of all pairs of partitions whose symbol is similar to the symbol of $\mathrm{Spr}_1(\lambda)$. The set of symbols of $\mathcal{S}_\lambda$ has the following explicit description. Recall that $\mathcal{I}_\lambda$ is the set of odd numbers m appearing as parts in λ. For each $m \in \mathcal{I}_\lambda$ let $I_m \subset (R_\lambda \cup R_\lambda') \setminus (R_\lambda \cap R_\lambda')$ be the corresponding interval. Let us define a function ℓ on $\mathcal{I}_\lambda$ by

$$\ell(m) = |I_m \cap R_\lambda| - |I_m \cap R_\lambda'|.$$

Note that $\ell(m) = 0$ precisely when $m \in \mathcal{I}_\lambda^+$, and $\ell(m) = \pm 1$ for $m \in \mathcal{I}_\lambda^-$. Consider the subset

$$\Sigma_\lambda = \{(c_m) \in (\mathbb{Z}/2)^{\mathcal{I}_\lambda} \mid \sum_{m : c_m \neq 0} \ell(m) = 0\} \subset (\mathbb{Z}/2)^{\mathcal{I}_\lambda}.$$

Then there is a bijection $\Sigma_\lambda \to \mathcal{S}_\lambda$ which associates with $(c_m) \in \Sigma_\lambda$ the symbol (R, R') obtained from (R_λ, R_λ') by swapping $I_m \cap R_\lambda$ and $I_m \cap R_\lambda'$ for each m such that $c_m \neq 0$ (i.e., $I_m \cap R_\lambda$ goes into the second row and $I_m \cap R_\lambda'$ goes into the first row). The condition $(c_m) \in \Sigma_\lambda$ ensures that $|R| = |R'| + 1$.

Let $\mathrm{Spr}_\lambda \subset \widehat{F_\lambda}$ be the set of characters ξ of F_λ such that (O_λ, ξ) appears in the Springer correspondence for $\mathrm{SO}(2n+1)$. We have a bijection

$$\Sigma_\lambda \xrightarrow{\ \sim\ } \mathrm{Spr}_\lambda$$

defined as follows: we view an element $(c_m) \in \Sigma_\lambda$ as a character ξ of $(\mathbb{Z}/2)^{\mathcal{I}_\lambda}$ and then restrict it to $F_\lambda = F(\mathcal{I}_\lambda)$. The image of this character $\xi \in \widehat{F_\lambda}$ under the Springer correspondence is precisely the pair of partitions corresponding to the element of $\mathcal{S}_\lambda$ associated with (c_m).

Note that $|\mathcal{I}_\lambda^-|$ is odd (since λ is a partition of $2n+1$), so $\mathcal{I}_\lambda^+$ is always a proper subset of $\mathcal{I}_\lambda$, hence, the homomorphism π_λ (see (7.1.5)) is surjective.

Proposition 7.1.2. *Let* $|\mathcal{I}_\lambda^-| = 2k+1$. *Then* Spr_λ *is the union of* $\binom{2k+1}{k}$ *cosets of the subgroup*

$$\widehat{\pi}_\lambda(\widehat{F'_\lambda}) \subset \widehat{F_\lambda}.$$

Proof. It is enough to check that the set Σ_λ is the union of $\binom{2k+1}{k}$ cosets of the subgroup $(\mathbb{Z}/2)^{\mathcal{I}_\lambda^+} \subset (\mathbb{Z}/2)^{\mathcal{I}_\lambda}$. The set $\mathcal{I}_\lambda^-$ is partitioned into two subsets $\mathcal{I}_\lambda^-(\pm 1)$ depending on the value $\ell(m) = \pm 1$. Since $|R_\lambda| = |R'_\lambda| + 1$, we have $|\mathcal{I}_\lambda^-(1)| = k+1$ and $|\mathcal{I}_\lambda^-(-1)| = k$. The set Σ_λ is the union of $(\mathbb{Z}/2)^{\mathcal{I}_\lambda^+}$-cosets corresponding to arbitrary choices of pairs of subsets $P(1) \subset \mathcal{I}_\lambda^-(1)$, $P(-1) \subset \mathcal{I}_\lambda^-(-1)$, such that $|P(1)| = |P(-1)|$. Associating with the pair $(P(1), P(-1))$ the set $P(1) \cup (\mathcal{I}_\lambda^-(-1) \setminus P(-1))$ we get a bijection with the set of all subsets of $\mathcal{I}_\lambda^-$ of cardinality k. $\qquad\square$

For a partition $\lambda \in \Pi_{2n+1}^+$, let $\mathfrak{t}_{\mathbf{G}_\lambda} \subset \mathfrak{t}$ be the Lie algebra of the maximal torus of $R\mathbf{G}_\lambda$ embedded into the Lie algebra $\mathfrak{t}$ of the maximal torus of $\mathbf{G}$ in a standard way (where $R\mathbf{G}_\lambda$ is given by (7.1.2)). On the other hand, for every partition μ with $|\mu| \le n$ let us consider the subspace

$$(7.1.6) \qquad \mathfrak{t}_\mu := \{t \in \mathfrak{t} \mid t_1 = \ldots = t_{\mu_1},\ t_{\mu_1+1} = \ldots = t_{\mu_1+\mu_2},\ \ldots,\ t_{|\mu|+1} = \ldots = t_n = 0\},$$

where we use the standard identification of $\mathfrak{t}$ with the n-dimensional space. Let also $W_{B,\mu}$ be the reflection group of type $B_{k_1} \times \ldots \times B_{k_s}$, where $k_1, \ldots, k_s$ are multiplicities of parts in μ, acting on the space $\mathfrak{t}_\mu$ by permutations of groups of coordinates corresponding to equal parts in μ and by changing signs (simultaneously in each group corresponding to a part of μ).

For $\lambda \in \Pi_{2n+1}^+$ let λ^{red} denote the partition in which each i appears with multiplicity $\lfloor r_i/2 \rfloor$, where r_i is the multiplicity with which i appears in λ. Then using (7.1.2) one can easily check that

$$(7.1.7) \qquad\qquad\qquad \mathfrak{t}_{\mathbf{G}_\lambda} = \mathfrak{t}_{\lambda^{\mathrm{red}}}.$$

Theorem 7.1.3. *(i) Let A_W be the algebra* (5.1.4) *defined for the Weyl group of type B_n. For each $\lambda \in \Pi_{2n+1}^+$ let $\mathcal{C}_\lambda \subset D(A_W)$ be the thick subcategory generated by modules $M_{\lambda,\xi}$ where ξ varies over Spr_λ. Then we have a semiorthogonal decomposition*

$$D^b(A_W) = \langle \mathcal{C}_{\lambda_N}, \ldots, \mathcal{C}_{\lambda_1} \rangle,$$

where we order the nilpotent orbits $O_{\lambda_1}, \ldots, O_{\lambda_N}$ in such a way that O_{λ_i} is in the closure of O_{λ_j} only if $i < j$.

(ii) Fix $\lambda \in \Pi_{2n+1}^+$, and let $|\mathcal{I}_\lambda^-| = 2k+1$. We have a decomposition of $\mathcal{C}_\lambda$ into $\binom{2k+1}{k}$ mutually orthogonal subcategories $\mathcal{C}_\lambda(C)$, where C runs over the $\widehat{\pi}_\lambda(\widehat{F'_\lambda})$-cosets comprising Spr_λ, and $\mathcal{C}_\lambda(C)$ is generated by all the modules $M_{\lambda,\xi}$ with $\xi \in C$. Furthermore, for each such coset $C = \xi_0 + \widehat{\pi}_\lambda(\widehat{F'_\lambda})$ we have an equivalence

$$(7.1.8) \qquad\qquad\qquad \mathcal{C}_\lambda(C) \simeq D^f(F'_\lambda \ltimes B_\lambda),$$

where B_λ is given by (7.1.4), *sending the module $M_{\lambda,\xi}$, with $\xi = \xi_0 + \widehat{\pi}_\lambda(\eta)$, to the projective module $\eta \otimes B_\lambda$.*

(iii) For each λ and $C = \xi_0 + \widehat{\pi}_\lambda(\widehat{F'_\lambda})$ as in (ii) there exists a semiorthogonal decomposition of $\mathcal{C}_\lambda(C)$ into subcategories generated by certain A_W-modules $M'_{\lambda,\xi}$, where $\xi \in C$ are arranged in some order, such that $\mathrm{Ext}^{>0}_{A_W}(M'_{\lambda,\xi}, M'_{\lambda,\xi}) = 0$, and

$$\mathrm{End}_{A_W}(M'_{\lambda,\xi_0+\widehat{\pi}_\lambda(\eta)}) \simeq \mathbb{C}[\mathfrak{t}_\mu]^{W_{B,\mu}},$$

where the partition μ is defined as follows. Let $m_1, \ldots, m_s \in \mathcal{I}_\lambda^+$ be all the elements such that the restriction of η to the corresponding factor in F_λ' is nontrivial. Then μ is obtained from λ^{red} by reducing by 1 the multiplicities of each of the parts $m_1, \ldots, m_s$. Furthermore, the support of the module $M'_{\lambda, \xi_0 + \widehat{\pi}_\lambda(\eta)}$ is equal to $W \cdot \mathfrak{t}_\mu$.

Proof. (i) This follows from Theorem 5.3.3.

(ii) By Proposition 5.3.1(ii), there is fully faithful functor

$$\mathcal{C}_\lambda \to D(F_\lambda \ltimes B_\lambda),$$

sending $M_{\lambda, \xi}$ to the projective module $\xi \otimes B_\lambda$. Recall that for $\lambda \in \Pi_{2n+1}^+$ the set Spr_λ is the union of $\binom{2k+1}{k} \, \widehat{\pi}_\lambda(\widehat{F_\lambda'})$-cosets (see Proposition 7.1.2). If C and C' are different $\widehat{\pi}_\lambda(\widehat{F_\lambda'})$-cosets then the projective $F_\lambda \ltimes B_\lambda$-modules $\xi \otimes B_a$ and $\xi' \otimes B_a$ are mutually orthogonal for any $\xi \in C$, $\xi' \in C'$ (see the proof of Lemma 5.3.2). This implies that the subcategories $\mathcal{C}_\lambda(C)$ and $\mathcal{C}_\lambda(C')$ are orthogonal for $C \neq C'$. Furthermore, for a coset $C = \xi_0 + \widehat{\pi}_\lambda(\widehat{F_\lambda'})$ we have an equivalence of $\mathcal{C}_\lambda(C)$ with the subcategory of $D(F_\lambda \ltimes B_\lambda)$ generated by $\xi \otimes B_\lambda$ with $\xi \in C$. Hence using the isomorphism (5.3.2) with $F = F_\lambda'$ (which involves tensoring with ξ_0) we get the required equivalence

$$\mathcal{C}_\lambda(C) \simeq D_f^b(F_\lambda' \ltimes B_\lambda).$$

(iii) Recall that by Lemma 7.1.1, the action of F_λ' on B_λ is of type A_1^l (on some of the coordinates). Thus, the required semiorthogonal decomposition arises from the equivalence (7.1.8) and the semiorthogonal decomposition for actions of type A_1^l (see §6.4). Furthermore, we have

$$\mathrm{End}_{A_W}(M'_{\lambda, \xi_0 + \widehat{\pi}_\lambda(\eta)}) \simeq (B_\lambda/(g_1, \ldots, g_s))^{F_\lambda'},$$

where $g_i = f_{m_i/2} \in B_{\lambda, m_i}$ (see Lemma 7.1.1), It remains to observe that for each factor $\mathrm{SO}(r)$ in $\mathbf{G}_\lambda^0$ with even r the action of the Weyl group of $\mathrm{SO}(r)$ on its maximal ($r/2$-dimensional) torus together with the action of the corresponding factor in F_λ' generate the action which is equivalent to the action of $W_{B_{r/2}}$. Also, we note that the embedding of the hyperplane given by vanishing of one coordinate induces an isomorphism of $B_{\lambda, m_i}/(g_i)$ with the algebra of invariant polynomials of type B_{r-1}, where r is the multiplicity of m_i in λ.

The assertion about the support follows from Proposition 5.2.4(iv) and from (7.1.7). □

Remark 7.1.4. The semiorthogonal decomposition of $D^b(A_W)$ obtained from the above theorem is very different from the one obtained in §6.5. For example, the modules $M'_{\lambda, \xi}$ supported at the origin appear for all λ which has only odd parts appearing with multiplicities 1 or 2, whereas in the decomposition of §6.5 all such modules are consecutive. It is likely that one decomposition is obtained from another by a sequence of mutations.

By Propositions 2.1.5 and 2.2.2, the pieces of the semiorthogonal decomposition of the above theorem (where we replace each $\mathcal{C}_\lambda$ with its natural semiorthogonal decomposition) match with the pieces in the corresponding motivic decomposition (0.0.1). We are going to present an independent proof of this using the explicit description of the Springer correspondence in this case. It turns out that this matching reflects in a certain combinatorial identity related to the Jacobi triple identity (see Proposition 7.1.6 and Corollary 7.1.7 below).

Recall that in the motivic decomposition we look at subspaces in $\mathfrak{t}$ of the form $\mathfrak{t}^w$, where $w \in W$ runs through representatives of conjugacy classes in W, and at the action of the centralizer $C(w)$ on $\mathfrak{t}^w$. The conjugacy classes in W are given by pairs of partitions (μ, μ') such that $|\mu| + |\mu'| = n$, where μ (resp., μ') records positive (resp., negative) cycle type (see [15]). The corresponding subspace $\mathfrak{t}^w$ is precisely $\mathfrak{t}_\mu$ (see (7.1.6)). The centralizer $C(w)$ of w acts on $\mathfrak{t}^w = \mathfrak{t}_\mu$ through the quotient $C(w) \to W_{B,\lambda}$ which leads to the $B_{k_1} \times \ldots \times B_{k_s}$-type action on $\mathfrak{t}_\mu$. Thus, the corresponding piece in the motivic decomposition of $[\mathfrak{t}/W]$ is

(7.1.9) $$\mathfrak{t}_\mu/W_{B, \mu} = \prod_i \mathbb{A}^{k_i}/W_{B_{k_i}},$$

where k_i is the multiplicity of i in μ. Note that this motive is determined by the single partition μ. By Theorem 7.1.3(iii), the motive associated with each module $M'_{\lambda,\xi}$ is of the same type for a certain partition μ associated with (λ, ξ). Thus, the fact that the motivic pieces from the semiorthogonal decomposition of Theorem 7.1.3 match the parts of the decomposition (0.0.1), amounts to the equality between the number of times that each partition μ with $|\mu| \leq n$ arises from pairs (λ, ξ) occuring in the Springer correspondence and the number of partitions of $n - |\mu|$.

Let $u_1, u_2, \ldots$ be independent variables. We associate with every partition μ the monomial

$$(7.1.10) \qquad\qquad u(\mu) = u_1^{k_1} u_2^{k_2} \ldots .$$

where k_i is the multiplicity of i in μ. The formal sum (with multiplicities) of the monomials $u(\mu)$ associated with all conjugacy classes (μ, μ') in W appears as the coefficient of x^n in the generating function

$$F^W(x, u_1, u_2, \ldots) = \prod_{n \geq 1}(1 - u_n x^n)^{-1}(1 - x^n)^{-1} = \sum_{\mu, \mu'} u(\mu) x^{|\mu| + |\mu'|}.$$

Lemma 7.1.5. *For $\lambda \in \Pi_{2n+1}^+$ consider the polynomial*

$$P_\lambda(u_1, u_2 \ldots) := \sum_{\xi \in \mathrm{Spr}_\lambda} u(\lambda, \xi),$$

where $u(\lambda, \xi) = u(\mu)$ is the monomial associated with the partition μ obtained from (λ, ξ) as in Theorem 7.1.3(iii). Then P_λ is equal to the coefficient of x^{2n+1} in the series $S(F^{\mathcal{N}})$, where

$$F^{\mathcal{N}}(x, u_1, u_2, \ldots) = \prod_{m \geq 1}(1 - u_m x^{2m})^{-1} \cdot \prod_{m \text{ odd}}(1 + tx^m + x^{2m}),$$

where t is one more independent variable, and S is the $\mathbb{Z}[u_1, u_2, \ldots]$-linear operator sending t^{2k+1} to $\binom{2k+1}{k}$, x^{2k} to 0 and x^{2k+1} to itself.

Proof. Let $\Pi^+ = \cup_n \Pi_{2n+1}^+$ be the set of all partitions λ with odd $|\lambda|$, in which even parts appear with even multiplicity. We want to prove the identity

$$(7.1.11) \qquad\qquad \sum_{\lambda \in \Pi^+} P_\lambda(u_1, u_2, \ldots) x^{|\lambda|} = S(F^{\mathcal{N}}).$$

The description of μ in Theorem 7.1.3(iii) easily leads to the equality

$$P_\lambda(u_1, u_2, \ldots) = S\left(\prod_{i \text{ even}} u_i^{r_i/2} \cdot \prod_{i \text{ odd}, r_i \text{ odd}} (t u_i^{(r_i-1)/2}) \cdot \prod_{i \text{ odd}, r_i \text{ even}} (u_i^{r_i/2} + u_i^{r_i/2-1})\right).$$

Note that the presence of t and the use of the operator S correspond to the multiplicity $\binom{2k+1}{k}$ in Theorem 7.1.3(ii). Thus, the left-hand side of (7.1.11) is equal to $S(F)$, where

$$F = \prod_{i \text{ even}}(1 + u_i x^{2i} + u_i^2 x^{4i} + u_i^3 x^{6i} + \ldots) \cdot$$

$$\prod_{i \text{ odd}}\left(1 + t[x^i + u_i x^{3i} + u_i^2 x^{5i} + \ldots] + (1 + u_i)[x^{2i} + u_i x^{4i} + u_i^2 x^{6i} + \ldots]\right).$$

Simplifying, one immediately gets that $F^{\mathcal{N}} = F$. $\qquad\square$

Proposition 7.1.6. *One has*

$$S(F^{\mathcal{N}})(x, u_1, u_2, \ldots) = x F^W(x^2, u_1, u_2, \ldots).$$

Hence, the multiplicity with which each motivic piece (7.1.9) arises from a module in the semiorthogonal decomposition of Theorem 7.1.3 is equal to the multiplicity with which it appears in the decomposition (0.0.1).

Proof. We know that the stated equality of the generating functions holds after the substitution $u_1 = u_2 = \ldots = 1$. Indeed, this follows from the fact that the cardinality of the set of pairs (O, ξ) appearing in the Springer correspondence is the same as the number of conjugacy classes in W. Set

$$F_0^{\mathcal{N}}(x, u_1, u_2, \ldots) = \prod_{n \text{ odd}} (1 + tx^n + x^{2n}) = F^{\mathcal{N}}|_{u_1 = u_2 = \ldots = 0}.$$

Note that

$$(7.1.12) \qquad S(F^{\mathcal{N}}) = \prod_{n \geq 1} (1 - u_n x^{2n})^{-1} \cdot S(F_0^{\mathcal{N}}),$$

since the first factor does not depend on t and contains only even powers of x. Hence,

$$S(F^{\mathcal{N}})|_{u_1 = u_2 = \ldots = 1} = \prod_{n \geq 1} (1 - x^{2n})^{-1} \cdot S(F_0^{\mathcal{N}}).$$

Thus, the identity

$$S(\mathcal{F}^{\mathcal{N}})|_{u_1 = u_2 = \ldots = 1} = x F^W(x^2, 1, 1, \ldots)$$

becomes

$$\prod_{n \geq 1} (1 - x^{2n})^{-1} \cdot S(F_0^{\mathcal{N}}) = x \cdot \prod_{n \geq 1} (1 - x^{2n})^{-2}.$$

Therefore, we have

$$S(F_0^{\mathcal{N}}) = x \cdot \prod_{n \geq 1} (1 - x^{2n})^{-1}.$$

Plugging this into (7.1.12) we get the result. $\qquad \square$

Corollary 7.1.7. *One has*

$$(7.1.13) \qquad S\left(\prod_{n \text{ odd}} (1 + tx^n + x^{2n}) \right) = x \cdot \prod_{n \geq 1} (1 - x^{2n})^{-1}.$$

The following independent proof of (7.1.13) was found by Ben Young. Start with the Jacobi triple identity

$$\prod_{m \geq 1} (1 - x^{2m})(1 + x^{2m-1} y^2)(1 + x^{2m-1} y^{-2}) = \sum_{n \in \mathbb{Z}} x^{n^2} y^{2n}.$$

Now set $y = \sqrt{q}$ and divide both sides by $\prod_m (1 - x^{2m})$:

$$\prod_{m \geq 1} (1 + x^{2m-1} q)(1 + x^{2m-1} q^{-1}) = \sum_{n \in \mathbb{Z}} x^{n^2} q^n \prod_{m \geq 1} (1 - x^{2m})^{-1}.$$

Thus, we get

$$(7.1.14) \qquad \prod_{m \text{ odd}} (1 + (q + q^{-1})x^m + x^{2m}) = \sum_{n \in \mathbb{Z}} x^{n^2} q^n \prod_{m \geq 1} (1 - x^{2m})^{-1}$$

Now we observe that the left-hand side of (7.1.13) is exactly the coefficient of q^1 in the q-expansion of the left-hand side of (7.1.14).

Remark 7.1.8. The identity of Corollary 7.1.7 has the following combinatorial interpretation. Consider partitions of $2n + 1$ with the following restrictions: all parts are odd, multiplicity of each part is ≤ 2. If there are $2k + 1$ parts of multiplicity 1 then this partition should be counted with weight $\binom{2k+1}{k}$. Then the sum of weights of all such partitions is $p(n)$, the usual partition number.

7.2. Type D_n.

Now let us consider the Springer correspondence for the group $\mathbf{G} = \mathrm{SO}(2n)$. The picture is going to be very similar to the one for B_n but with some important differences, which lead to the presence of "noncommutative" pieces in the corresponding semiorthogonal decomposition.

Representations of W_{D_n} are parametrized by the set $\Pi_n^{(2)}/S_2$ of unordered pairs of partitions (ξ, ν) such that $|\xi| + |\nu| = n$, except that there are two irreducible representations associated with the pair when $\xi = \nu$. Similarly to the case of type B_n we associate with (ξ, ν) its symbol, which is an *unordered* pair of sets (R, R'), setting

$$R = \{\xi_0, \xi_1 + 2, \dots, \xi_r + 2r\}, R' = \{\eta_0, \eta_1 + 2, \dots, \eta_r + 2r\},$$

where $\xi = (\xi_0 \le \xi_1 \le \dots \le \xi_r)$, $\eta = (\eta_0 \le \eta_1 \le \dots \le \eta_r)$ (if necessary we extend one of the partitions by 0s).

The nilpotent orbits in $\mathrm{SO}(2n)$ are parametrized by the set Π_{2n}^+ of partitions λ of $2n$ such that each even part appears with even multiplicity, except that there are two nilpotent orbits associated with λ in the case when all parts of λ are even. The reductive part of the centralizer $\mathbf{G}_\lambda$ of an element of the orbit O_λ and its component group F_λ are still given by (7.1.2) and (7.1.3), where $\mathcal{I}_\lambda$ is still the set of odd m appearing as parts in λ.

Similarly to type B_n, there is a map

$$\mathrm{Spr}_1 : \Pi_{2n}^+ \to \Pi_n^{(2)}/S_2$$

defined as follows. For $\lambda = (\lambda_0 \le \lambda_1 \le \dots)$ define the numbers $\xi_1 \le \dots \le \xi_r$ and $\eta_1 \le \dots \le \eta_r$ in the same way as in the case of type B_n. It is easy to see that $\lambda \in \Pi_{2n}^+$ has all parts even if and only if $\mathrm{Spr}_1(\lambda)$ has $\xi = \eta$. In this case there is some rule which of the two nilpotent orbits associated with λ corresponds to which of the two irreducible representations associated with $\mathrm{Spr}_1(\lambda)$, but the precise form of the correspondence is not important for us.

Let (R_λ, R'_λ) be the symbol associated with $\mathrm{Spr}_1(\lambda)$. Assume that $R_\lambda \ne R'_\lambda$, i.e., λ has at least one odd part. As in the case of type B_n the maximal intervals of integers in $(R_\lambda \cup R'_\lambda) \setminus (R_\lambda \cap R'_\lambda)$ are in bijection with odd numbers i appearing as parts of λ, and the length of the interval is equal to the multiplicity r_i. We define similarity classes of symbols of pairs of partitions in $\Pi_n^{(2)}/S_2$ as before, so that each similarity class contains the unique symbol in the image of Spr_1.

In the case when $\mathcal{I}_\lambda = \emptyset$, i.e., all parts of λ are even, the component group F_λ is trivial, and the Springer correspondence matches two nilpotent orbits corresponding to λ with two representations of W_{D_n} associated with $\mathrm{Spr}_1(\lambda)$ (the precise rule is not important for us).

Now assume that $\mathcal{I}_\lambda \ne \emptyset$, and let Σ_λ have the same meaning as in the case of type B_n. Note that we have $\sum_{m \in \mathcal{I}_\lambda} \ell(m) = 0$, so Σ_λ is a coset for the subgroup $\langle \mathbf{e} \rangle \subset (\mathbb{Z}/2)^{\mathcal{I}_\lambda}$ generated by the element $\mathbf{e} = (1, \dots, 1) \in (\mathbb{Z}/2)^{\mathcal{I}_\lambda}$. Then we have a bijection

$$\Sigma_\lambda/\langle \mathbf{e} \rangle \xrightarrow{\sim} \mathcal{S}_\lambda$$

to the set $\mathcal{S}_\lambda$ of symbols (R, R') similar to (R_λ, R'_λ), defined as in the case of type B_n (the action of $\mathbf{e}$ has the effect of swapping R and R', so it acts trivially on $\mathcal{S}_\lambda$ since we view (R, R') as unordered pairs). We still can identify the component group F_λ with $F(\mathcal{I}_\lambda)$, and view elements of $(c_m) \in \Sigma_\lambda/\langle \mathbf{e} \rangle$ as characters of F_λ (by definition, $\mathbf{e}$ restricts to the trivial character of F_λ). This gives a bijection $\Sigma_\lambda/\langle \mathbf{e} \rangle \xrightarrow{\sim} \mathrm{Spr}_\lambda$ that computes the Springer correspondence for the nilpotent orbit associated with λ.

Since λ is a partition of $2n$, we have $|\mathcal{I}_\lambda^-| = 2k$ for some k. Assume first that $k > 0$. Then the natural projection

$$\pi_\lambda : F_\lambda \to F'_\lambda := (\mathbb{Z}/2)^{\mathcal{I}_\lambda^+}.$$

is surjective. We claim that in this case similarly to Proposition 7.1.2 the set of characters of F_λ that appear in the Springer correspondence is the union of $\binom{2k}{k}/2$ cosets of the subgroup $\widehat{\pi_\lambda}(\widehat{F'_\lambda}) \subset \widehat{F_\lambda}$. Indeed, it is enough to check that Σ_λ is the union of $\binom{2k}{k}$ cosets of the subgroup $(\mathbb{Z}/2)^{\mathcal{I}_\lambda^+}$. Using the notation from Proposition 7.1.2 we have $|\mathcal{I}_\lambda^-(1)| = |\mathcal{I}_\lambda^-(-1)| = k$. The number in question is obtained as the number of choices of pairs of subsets $P(1) \subset \mathcal{I}_\lambda^-(1)$, $P(-1) \subset \mathcal{I}_\lambda^-(-1)$ with $|P(1)| = |P(-1)|$, which is equal to $\binom{2k}{k}$.

Next, assume that $k = 0$. Then $\Sigma_\lambda = (\mathbb{Z}/2)^{\mathcal{I}_\lambda}$ and the restriction map

$$\Sigma_\lambda/\langle \mathbf{e} \rangle \to \widehat{F_\lambda}$$

is an isomorphism, so in this case all characters of the component group appear in the Springer correspondence. Note also that in this case the map $\pi_\lambda : F_\lambda \to F'_\lambda$ is the natural embedding of the index 2 subgroup $F(\mathcal{I}_\lambda) \subset (\mathbb{Z}/2)^{\mathcal{I}_\lambda}$.

Now we can formulate the analog of Theorem 7.1.3 for type D_n. As before, for each partition μ with $|\mu| \leq n$ we consider the subspace $\mathfrak{t}_\mu \subset \mathfrak{t}$ given by (7.1.6). Let $W_{D,\mu}$ be the group acting on $\mathfrak{t}_\mu$ by permutations of groups of coordinates corresponding to equal parts in μ, and by changing signs of an even number of coordinates $t_1, \ldots, t_n$ (simultaneously in each group of coordinates corresponding to a part in μ). Note that if $|\mu| < n$ then $W_{D,\mu} = W_{B,\mu}$, since in this case $t_n = 0$ and hence arbitrary changes of signs are allowed.

Below we will use for $\lambda \in \Pi_{2n}^+$ the notations B_λ, λ^{red}, etc., introduced in §7.1 for $\lambda \in \Pi_{2n+1}^+$. We also denote by Nilp_{2n} the set of nilpotent orbits, so that we have a surjective map $\mathrm{Nilp}_{2n} \to \Pi_{2n}^+$ with fibers of cardinality 2 over λ that have only even parts.

Theorem 7.2.1. *(i) Let A_W be the algebra (5.1.4) defined for the Weyl group of type D_n. For each $\widetilde{\lambda} \in \mathrm{Nilp}_{2n}$ over $\lambda \in \Pi_{2n}^+$ let $\mathcal{C}_{\widetilde{\lambda}} \subset D(A_W)$ be the thick subcategory generated by modules $M_{\lambda,\xi}$, where ξ varies over Spr_λ. Then we have a semiorthogonal decomposition*

$$D^b(A_W) = \langle \mathcal{C}_{\widetilde{\lambda}_N}, \ldots, \mathcal{C}_{\widetilde{\lambda}_1} \rangle,$$

where we order the nilpotent orbits $O_{\widetilde{\lambda}_1}, \ldots, O_{\widetilde{\lambda}_N}$ in such a way that $O_{\widetilde{\lambda}_i}$ is in the closure of $O_{\widetilde{\lambda}_j}$ only if $i < j$. The category $\mathcal{C}_{\widetilde{\lambda}}$ depends only on $\lambda \in \Pi_{2n}^+$, up to equivalence, so we denote it by $\mathcal{C}_\lambda$.
(ii) Fix $\lambda \in \Pi_{2n}^+$, and let $|\mathcal{I}_\lambda^-| = 2k$. Assume first that $k > 0$. Then we have a decomposition of $\mathcal{C}_\lambda$ into $\binom{2k}{k}/2$ mutually orthogonal subcategories $\mathcal{C}_\lambda(C)$, where C runs over the $\widehat{\pi}_\lambda(\widehat{F'_\lambda})$-cosets comprising Spr_λ, and $\mathcal{C}_\lambda(C)$ is generated by all the modules $M_{\lambda,\xi}$ with $\xi \in C$. Furthermore, for each such coset $C = \xi_0 + \widehat{\pi}_\lambda(\widehat{F'_\lambda})$ we have an equivalence

$$\mathcal{C}_\lambda(C) \simeq D^f(F'_\lambda \ltimes B_\lambda).$$

sending the module $M_{\lambda,\xi}$, with $\xi = \xi_0 + \widehat{\pi}_\lambda(\eta)$ to the projective module $\eta \otimes B_\lambda$.
(ii)' Now let $\lambda \in \Pi_{2n}^+$ be such that $\mathcal{I}_\lambda^- = \emptyset$. Then we have

$$\mathcal{C}_\lambda \simeq D_{F(\mathcal{I}_\lambda)}^b(B_\lambda) \simeq D^b([\mathfrak{t}_\mu/W_{D,\mu}]),$$

where $\mu = \lambda^{\mathrm{red}}$ (note that in this case all multiplicities of parts in λ are even, so $|\mu| = n$).
(iii) For each λ with $|\mathcal{I}_\lambda^-| > 0$ and $C = \xi_0 + \widehat{\pi}_\lambda(\widehat{F'_\lambda})$ as in (ii), there exists a semiorthogonal decomposition of $\mathcal{C}_\lambda(C)$ into subcategories generated by certain A_W-modules $M'_{\lambda,\xi}$, where $\xi \in C$ are arranged in some order, such that $\mathrm{Ext}_{A_W}^{>0}(M'_{\lambda,\xi}, M'_{\lambda,\xi}) = 0$, and

$$\mathrm{End}_{A_W}(M'_{\lambda,\xi_0+\widehat{\pi}_\lambda(\eta)}) \simeq \mathbb{C}[\mathfrak{t}_\mu]^{W_{B,\mu}},$$

where the partition μ is defined as follows. Let $m_1, \ldots, m_s \in \mathcal{I}_\lambda^+$ be all the elements such that the restriction of η to the corresponding factor in F'_λ is nontrivial. Then μ is obtained from λ^{red} by reducing by 1 the multiplicities of each of the parts $m_1, \ldots, m_s$. Furthermore, the support of the module $M'_{\lambda,\xi_0+\widehat{\pi}_\lambda(\eta)}$ is equal to $S_n \cdot \mathfrak{t}_\mu$.

The proof is absolutely analogous to that of Theorem 7.1.3, where in the case of $\lambda \in \Pi_{2n}^+$ with $\mathcal{I}_\lambda^- = \emptyset$ we use the fact that all characters of F_λ appear in the Springer correspondence to check the applicability of Theorem 5.3.3. Note also that in (iii) we have $|\mu| < n$ so that $W_{D,\mu} = W_{B,\mu}$.

Thus, we obtain a semiorthogonal decomposition of $D^b(A_W)$ for type D_n into two kinds of subcategories: 1) the subcategories generated by the modules $M'_{\lambda,\xi}$ where λ is such that $|\mathcal{I}_\lambda^-| > 0$; 2) the subcategories $\mathcal{C}_\lambda$ for λ such that $|\mathcal{I}_\lambda^-| = 0$. We will refer to pieces of type 2) as *noncommutative pieces* in our semiorthogonal decomposition (since these subcategories are not equivalent to derived categories of any smooth variety).

Now we are going to match this decomposition with the motivic decomposition (0.0.1), where the subcategories of the second kind absorb several pieces of the motivic decomposition.

Recall that conjugacy classes in $W = W_{D_n}$ are parametrized by (ordered) pairs of partitions (μ, μ') such that $|\mu| + |\mu'| = n$ and such that μ' has even number of parts, except that in the case when $\mu' = \emptyset$ and all parts of μ are even there are two corresponding conjugacy classes. As in the case of type B_n, the corresponding subspace $\mathfrak{t}^w$ is $\mathfrak{t}_\mu$. The centralizer $C(w)$ acts on $\mathfrak{t}^w = \mathfrak{t}_\mu$ through the quotient $W_{D,\mu}$. In the case $|\mu| < n$ (i.e., $\mu' \neq \emptyset$) then $W_{D,\mu} = W_{B,\mu}$, so the quotient is smooth:

$$\mathfrak{t}_\mu / W_{D,\mu} \simeq \prod_i \mathbb{A}^{k_i} / W_{B_{k_i}},$$

where k_i is the multiplicity of i in λ. If $\mu' = \emptyset$ and there is only one odd part i_0 (appearing with some multiplicity k_{i_0}) in μ then $W_{D,\mu}$ acts by type B action on each group of coordinates corresponding to equal even parts and by type D action on coordinates corresponing to the part i_0. Thus, in this case the quotient is still smooth:

$$\mathfrak{t}_\mu / W_{D,\lambda} \simeq \prod_{i \text{ even}} \mathbb{A}^{k_i} / W_{B_{k_i}} \times \mathbb{A}^{k_{i_0}} / W_{D_{k_{i_0}}}.$$

Finally, if $\mu' = \emptyset$ and the odd parts appearing in μ are $i_1, \ldots, i_s$, where $s > 1$, appearing with multiplicities $k_{i_1}, \ldots, k_{i_s}$, then $W_{D,\mu}$ acts by changing signs in even number of coordinates corresponding to odd parts (in addition to permuting coordinates corresponding to equal parts). In this case the quotient is singular, and we have an isomorphism of stacks

$$[\mathfrak{t}_\mu / W_{D,\mu}] \simeq \prod_{i \text{ even}} \mathbb{A}^{k_i} / W_{B_{k_i}} \times \left[\left(\prod_{m=1}^{s} (\mathbb{A}^{k_{i_m}} / W_{D_{k_{i_m}}}) \right) / F_s \right],$$

where $F_s = F(\{1, \ldots, s\}) \subset (\mathbb{Z}/2)^s$ acts on each space $\mathbb{A}^{k_{i_m}} / W_{D_{k_{i_m}}}$ via the projection to the mth factor $\mathbb{Z}/2$, using the outer involution of $\mathbb{A}^{k_{i_m}} / W_{D_{k_{i_m}}}$. Note that by choosing basic invariant polynomials as in Lemma 7.1.1, we get an isomorphism

$$\left[\left(\prod_{m=1}^{s} (\mathbb{A}^{k_{i_m}} / W_{D_{k_{i_m}}}) \right) / Z_s \right] \simeq \prod_{m=1}^{s} \mathbb{A}^{k_{i_m} - 1} \times [\mathbb{A}^s / Z_s].$$

In any case the corresponding piece in the motivic decomposition is $\mathfrak{t}_\mu / W_{D,\mu}$, to which we associate as before the monomial $u(\mu)$ (see (7.1.10)). In the case when $\mu' = \emptyset$ and all parts of μ are even our convention is that this monomial stands for the two pieces in the decomposition (corresponding to the two conjugacy classes associated with $(\mu, \emptyset)$). The formal sum of the obtained monomials over all conjugacy classes in W is the coefficient of x^n in

$$F^W(x, u_1, u_2, \ldots) = \prod_{n \geq 1} (1 - u_n x^n)^{-1} \cdot P^+,$$

where

$$P^+(x) = \frac{1}{2} \left(\prod_{n \geq 1} (1 - x^n)^{-1} + \prod_{n \geq 1} (1 + x^n)^{-1} \right)$$

is the generating function for partitions with even number of parts.

Now let us count contributions to pieces of the motivic decomposition from the semiorthogonal decomposition of Theorem 7.2.1. Let λ be a partition in Π_{2n}^+ and let r_i denote the multiplicity of i in λ. We distinguish two cases: 1) λ has at least one odd part appearing with odd multiplicity; 2) all odd parts in λ appear with even multiplicity. In case 1) the counting is similar to the case of type B_n, so the coresponding contribution is

$$P_\lambda = S' \left(\prod_{i \text{ even}} u_i^{r_i/2} \cdot \prod_{i \text{ odd}, r_i \text{ odd}} (t u_i^{(r_i-1)/2}) \cdot \prod_{i \text{ odd}, r_i \text{ even}} (u_i^{r_i/2} + u_i^{r_i/2-1}) \right),$$

where S' is the $\mathbb{Z}[u_1, u_2, \ldots]$-linear operator sending t^{2k+1} to 0, t^{2k} to $\binom{2k}{k}/2$ for $k > 0$ and 1 to 1. In case 2) we have to replace a noncommutative piece in the semiorthogonal decomposition by the corresponding motivic pieces

$$(1 + D_2 + D_4 + \ldots)(\prod_i u_i^{r_i/2}).$$

Here for a monomial M in u_i we set

$$D_p(M) = \left(\sum_{2m_1 - 1 < \ldots < 2m_p - 1} \frac{M}{u_{2m_1 - 1} \ldots u_{2m_p - 1}} \right)_{\text{reg}},$$

where we throw away summands that are not polynomials. Finally, in the case when all parts of λ are even we have two corresponding nilpotent orbits and the corresponding stratum is $u(\lambda^{\text{red}}) = \prod_i u_i^{r_i/2}$. The strata appearing in this way are those corresponding to pairs $(\mu, \emptyset)$, where all parts of μ are even, which are exactly the strata that have to be taken twice in the motivic decomposition (since there are two associated conjugacy classes).

The resulting contribution from all $\lambda \in \Pi_{2n}^+$ is equal to the coefficient of x^{2n} in

$$F^{\mathcal{N}}(x, u_1, u_2, \ldots) = \prod_{m \text{ even}} (1 - u_m x^{2m})^{-1} \cdot [S'(F_1) + F_2],$$

where $S'(F_1)$ and F_2 correspond to cases 1) and 2) above:

$$F_2 = \prod_{m \text{ odd}} (1 - u_m x^{2m})^{-1} \cdot \frac{1}{2}(P^{\text{dist,odd}}(x^2) + P^{\text{dist,odd}}(-x^2)) \quad \text{with}$$

$$P^{\text{dist,odd}}(x) = \prod_{m \text{ odd}} (1 + x^m);$$

$$F_1 = \widetilde{F}_1(x, t) - \widetilde{F}_1(x, 0), \quad \text{where}$$

$$\widetilde{F}_1 = \prod_{m \text{ odd}} (1 + tx^m[1 - u_m x^{2m}]^{-1} + (1 + u_m)x^{2m}[1 - u_m x^{2m}]^{-1}) = \prod_{m \text{ odd}} (1 - u_m x^{2m})(1 + tx^m + x^{2m}).$$

Thus, we can rewrite

$$F_1 = \prod_{m \text{ odd}} (1 - u_m x^{2m}) \cdot \left(\prod_{m \text{ odd}} (1 + tx^m + x^{2m}) - P^{\text{dist,odd}}(x^2) \right),$$

$$F^{\mathcal{N}} = \prod_{n \geq 1} (1 - u_n x^{2n})^{-1} \cdot \left[S'\left(\prod_{m \text{ odd}} (1 + tx^m + x^{2m}) \right) - \frac{1}{2}P^{\text{dist,odd}}(x^2) + \frac{1}{2}P^{\text{dist,odd}}(-x^2) \right].$$

Now we claim that there is an equality

(7.2.1) $$F^{\mathcal{N}}(x, u_1, u_2, \ldots) = F^W(x^2, u_1, u_2, \ldots).$$

The equality of the corresponding coefficients of x^{2n} shows that we get the right number of pieces in the motivic decomposition from our semiorthogonal decomposition. Both sides in the equality have the factor $\prod_n (1 - u_n x^{2n})^{-1}$, and the remaining parts do not depend on u_i. Hence, (7.2.1) is equivalent to the identity obtained from it by setting $u_1 = u_2 = \ldots = 1$, which holds due to the Springer correspondence. The obtained combinatorial identity is

(7.2.2) $$S'\left(\prod_{m \text{ odd}} (1 + tx^m + x^{2m}) \right) - \frac{1}{2}P^{\text{dist,odd}}(x^2) + \frac{1}{2}P^{\text{dist,odd}}(-x^2) = P^+(x^2).$$

Using the well known identity

$$\prod_{n \geq 1} (1 + x^n) = \prod_{m \text{ odd}} (1 - x^m)^{-1}$$

we obtain

$$P^{\text{dist,odd}}(-x) = \prod_{m \text{ odd}} (1 - x^m) = \prod_{n \geq 1} (1 + x^n)^{-1}.$$

Thus,

$$P^+(x) - \frac{1}{2}P^{\text{dist,odd}}(-x) = \frac{1}{2}\prod_{n\geq 1}(1-x^{2n})^{-1},$$

and the idenity (7.2.2) is equivalent to

$$S'\left(\prod_{m \text{ odd}}(1+tx^m+x^{2m})\right) - \frac{1}{2}P^{\text{dist,odd}}(x^2) = \frac{1}{2}\prod_{n\geq 1}(1-x^{2n})^{-1}.$$

This in turn is equivalent to

$$\left[\prod_{m \text{ odd}}(1+(q+q^{-1})x^m+x^{2m})\right]_{q^0} = \prod_{n\geq 1}(1-x^{2n})^{-1},$$

which follows from (7.1.14) by taking the constant coefficient of the expansion in powers of q.

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