Uniqueness result for an age-dependent reaction-diffusion problem

Peer-reviewed author version

ANH-KHOA, Vo; The Hung, Tran & Lesnic, Daniel (2021) Uniqueness result for an age-dependent reaction-diffusion problem. In: APPLICABLE ANALYSIS, 100 (13), p. 2873-2890.

DOI: 10.1080/00036811.2019.1698730
Handle: http://hdl.handle.net/1942/30581

UNIQUENESS RESULT FOR AN AGE-DEPENDENT REACTION-DIFFUSION PROBLEM

VO ANH KHOA, TRAN THE HUNG, AND DANIEL LESNIC

Abstract. This paper is concerned with an age-structured model in population dynamics. We investigate the uniqueness of solution for this type of nonlinear reaction-diffusion problem when the source term depends on the density, indicating the presence of, for example, mortality and reaction processes. Our result shows that in a spatial environment, if two population densities obey the same evolution equation and possess the same terminal data of time and age, then their distributions must coincide therein.

1. Introduction

In population dynamics, there are several factors interestingly contributing to the complicated and nontrivial spatio-temporal spread patterns of diseases. Especially demographic-disease ages and spatial movement naturally interweave in many ways. Their correlation and interaction are expected to participate in mathematical modeling and analysis, and can lead to models with distinct levels of complexity.

As far as we know, disease management measures are often age-dependent and their effectiveness may also be dependent on the mobility status of the involved species, such as larvicides and insecticide sprays for mosquito-borne diseases (e.g. the invasion and spread of West Nile virus in North America in [9]) used to control the mosquito population at different levels of their maturity. We recall the structured population model in [14] for the total number of matured individuals in a single species population at the demographic age, which reads as

$$u_t + u_a = D\Delta u - d_0 u. \tag{1.1}$$

Here, t denotes the time, a denotes age, u represents the population density, D and d_0 are, respectively, the diffusion and death rates of the immature individual, under the assumption that the maturation rate is regulated by the birth process and the dynamics of the individual during the maturation phase.

A generalized approach for this mosquito-borne disease model is given by the coupled McKendrick von-Foerster equations (see also [12, 15, 16]) where the reservoir as the host and the age-structured host population are observed. Following [9, Section 4.4], the spatial spread of vector-borne diseases reaction-diffusion model involving age-structure is

$$\begin{cases} u_t + u_a = D_1(a)\,\Delta u - u\left(d_1(a) + \chi(a)\,\mathrm{m}(t,x)\right), \\ v_t + v_a = D_2(a)\,\Delta v - v d_2(a) + u\chi(a)\,\mathrm{m}(t,x). \end{cases} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35A07, 35K57, 92D25, 45Q05.

Key words and phrases. Backward age-dependent reaction-diffusion, Uniqueness, Population dynamics.

Here, u and v play roles as susceptible and infected hosts, respectively, while $D_i(a)$ are the age-dependent diffusivities by natural means, $\chi(a)$ is the age-dependent transmission coefficient, $d_i(a)$ are the age-dependent death rates of such hosts for $i = 1, 2$ and m can be regarded as the number of infected adult mosquitoes.

We note that in one spatial dimension, the system (1.2) was also analyzed in [2] with ("PDE-model") or without ("ODE-model") diffusion. In that paper, the Mathematical Biology interest was in establishing the positivity of the solution for the ODE-model and studying the travelling wave solution and its spreading for the PDE-model. However, in our paper we do not address the existence of solution (which indeed we assume), but the equally important uniqueness of solution. For this latter objective, we do not need to assume any sign restriction on the solution and we also consider the semi-linear version of the PDE-model of (1.1) in any dimension with Dirichlet or nonlinear boundary conditions, as given by (1.4)–(1.6) below. Moreover, we consider the backward and ill-posed situation where the initial conditions at $t = 0$ and $a = 0$ are not known, but instead we measure the solution at the later times $t = T$ and $a = a_\dagger$, as studied in [17].

Let therefore $\Omega \subset \mathbb{R}^n$ for $n \geq 1$ be a bounded, open and connected domain with sufficiently smooth boundary and $\mathcal{A}$ be the linear second-order differential operator

$$\mathcal{A}u(t,a,x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(d_{ij}(t,a,x)\frac{\partial u(t,a,x)}{\partial x_j}\right), \tag{1.3}$$

accounting e.g. for the anisotropic diffusion and possibly taxis processes.

Setting $Q^{\Omega}_{T,a_\dagger} := (0,T) \times (0,a_\dagger) \times \Omega$ for $T, a_\dagger > 0$, we consider the problem:

$$\begin{cases} u_t + u_a - \mathcal{A}u = F(t,a,x;u) & \text{in } Q^{\Omega}_{T,a_\dagger}, \\ u(T,a,x) = u_0(a,x) & \text{in } (0,a_\dagger) \times \Omega, \\ u(t,a_\dagger,x) = u_1(t,x) & \text{in } (0,T) \times \Omega, \end{cases} \tag{1.4}$$

where F is a given source term, endowed either with the homogeneous Dirichlet boundary condition

$$u(t,a,x) = 0 \quad \text{on } (0,T) \times (0,a_\dagger) \times \partial\Omega, \tag{1.5}$$

or with the Robin-type boundary condition

$$-\mathrm{d}(t,a,x)\nabla u(t,a,x) \cdot \mathrm{n} = S(u) \quad \text{on } (0,T) \times (0,a_\dagger) \times \partial\Omega. \tag{1.6}$$

Here, n denotes the outward unit normal vector to the boundary $\partial\Omega$, $\mathrm{d} = (d_{ij})_{i,j=\overline{1,n}}$ is the diffusion tensor and S is some given function. Equation (1.1) is a particular form of the PDE in (1.4).

In this paper, we prove that under certain assumptions on the source term F and surface reaction S, the solution to the system (1.4) and (1.5) or (1.6) is unique, if it exists. The present source term F can model several types of polynomial reactions (e.g. logistic and von Bertalanffy) and even exponential growth (Arrhenius laws). It can also be modelled as

$$F(t,a,x;u) = -\tilde{\mu}(t;u)\,u(t,a,x),$$

where $\tilde{\mu} > 0$ is called the time- and density-dependent mortality modulus. This mortality-related functional usually arises in the Lotka-von Foerster model, where the simple modulus is

$$\tilde{\mu}(t;u) = \int_0^{a_\dagger} \int_\Omega u(t,a,x)\, dx da,$$

provided the mortality process is also controlled by the total population at time t during the whole age and environment.

It is worth noting that when age can be viewed as a temporal time, the first equation in (1.4) is usually referred to as an ultra-parabolic equation with (t,a) viewed as a multi-dimensional time. In literature, the backward problem (1.4) and (1.5) has been explored in [17] for an abstract linear class of ultra-parabolic problems. Such problems are basically ill-posed in the sense that the solution does not depend continuously on the data no matter how smooth it is. As a matter of fact, a regularization has to be designed. Therefore, as an inception, our work draws a way to prove the Carleman estimate for time-reversed reaction-diffusion problems with age structures, which eventually leads to the conditional stability of the inverse problem (cf. [5]). The starting point of an evolution equation involving multi-time variables is from [7, 8]. It is shown in [7] that the forward problem for (1.4) and (1.5) is well-posed in the space of Hölder continuous functions. As a byproduct, the result therein provides the representation of solution, based on the semi-group theory along the upper and lower triangles dividing the rectangle of times. The evolution of the system with two different times were discussed by an argument where some diffusion processes with memory take place. In the framework of stochastic optimal control, the reader can be referred to e.g. [10, 11], for related contributions to this problem.

The rest of this paper is organized as follows: Section 2 is devoted to introduce notation and conventions throughout this paper. In addition, we provide technical assumptions on parameters and coefficients involved in (1.4). In Section 3, we deliberately present two subsections where the Dirichlet and Robin-type boundary conditions are considered, respectively. Our main results are reported in Theorem 3.2 and Theorem 3.5, whilst their cores of proof are based on Lemma 3.1 and Lemma 3.4, respectively. Essentially, the main technique here is inspired from [1, Chapter 6]. This approach was used to treat the backward parabolic operator within the study of the large-time behavior of solutions to a linear class of initial-boundary value parabolic equations. It is also helpful in the analysis of regularization for inverse and ill-posed problems. In this work, we extend the uniqueness result particularly to a class of semi-linear age-dependent reaction-diffusion equations.

2. Preliminaries

In parallel with using the notation $Q^{\Omega}_{T,a_\dagger}$, the same meaning is also given to the notation $Q_{T,a_\dagger} = (0,T) \times (0,a_\dagger)$ and $Q^{\Omega}_{(t_1,t_2)\times(a_1,a_2)} = (t_1,t_2) \times (a_1,a_2) \times \Omega$, etc. throughout this paper. Moreover, depending on the context, by $|\cdot|$ we denote either the volume measure of a domain or the absolute value of a function.

To this end, for any domain D we also set $\|\cdot\|_D$ to be the norm in $L^2(D)$ and the same indication goes to the inner product $\langle \cdot,\cdot \rangle_D$.

For the sake of our analysis in this work, we need the following set of assumptions:

(A_1) The diffusion tensor $\mathrm{d} = (d_{ij})_{1\le i,j\le n} \in \left[C^1\left(\overline{Q^{\Omega}_{T,a_\dagger}}\right)\right]^{n\times n}$ is symmetric and there exist positive constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}\,|\xi|^2 \le \sum_{i,j=1}^{n} d_{ij}\,(t,a,x)\,\xi_i\xi_j \le \overline{c}\,|\xi|^2 \quad \text{for any } \xi\in\mathbb{R}^n \text{ and for all } (t,a,x)\in Q^{\Omega}_{T,a_\dagger}.$$

Denote by $\overline{M}$ a positive constant such that

$$|\partial_t d_{ij}\,(t,a,x)| + |\partial_a d_{ij}\,(t,a,x)| \le \overline{M} \quad \text{for all } (t,a,x)\in \overline{Q^{\Omega}_{T,a_\dagger}}.$$

(A_2) The source term $F : \overline{Q^{\Omega}_{T,a_\dagger}} \times L^2\left(Q^{\Omega}_{T,a_\dagger}\right) \to \mathbb{R}$ is $(0,1] \ni \alpha$-Hölder continuous with respect to the density argument, i.e. there exists $L_F > 0$ such that for all $(t,a,x)\in\overline{Q^{\Omega}_{T,a_\dagger}}$,

$$|F\,(t,a,x;u_1) - F\,(t,a,x;u_2)| \le L_F\,|u_1-u_2|^{\alpha} \text{ for all } u_1,u_2\in L^2\left(Q^{\Omega}_{T,a_\dagger}\right).$$

(A_3) For the surface reaction term $S : L^2(\partial\Omega)\to\mathbb{R}$ there exists $\overline{m}>0$ such that

$$\langle(\partial_t+\partial_a)\,(S\,(u_1)-S\,(u_2))\,,u_1-u_2\rangle_{\partial\Omega} \le \overline{m}\,\|u_1-u_2\|^2_{\partial\Omega} \quad \text{for all } u_1,u_2\in L^2\,(\partial\Omega)\,.$$

(A_4) The surface reaction term $S : L^2(\partial\Omega)\to\mathbb{R}$ is monotone, i.e.

$$\langle S\,(u_1)-S\,(u_2)\,,u_1-u_2\rangle_{\partial\Omega}\ge 0 \quad \text{for all } u_1,u_2\in L^2\,(\partial\Omega)\,.$$

(A_5) The surface reaction term $S : L^2(\partial\Omega)\to\mathbb{R}$ is $(0,1]\ni\beta$-Hölder continuous, i.e. there exists $L_S>0$ such that

$$|S\,(u_1)-S\,(u_2)| \le L_S\,|u_1-u_2|^{\beta} \quad \text{for all } u_1,u_2\in L^2\,(\partial\Omega)\,.$$

Lemma 2.1. *(cf. [4]) For any $\gamma>0$ and $\alpha_0\in(0,1]$, the following inequality holds*

$$X^{\alpha_0} \le \alpha_0\gamma^{\alpha_0-1}X + (1-\alpha_0)\,\gamma^{\alpha_0} \quad \textit{for all } X\ge 0. \tag{2.1}$$

Lemma 2.2. *(cf. [3]) There exists a positive constant C_0 such that*

$$\|u\|^2_{\partial\Omega} \le C_0\left(\|u\|^2_{\Omega} + \|\nabla u\|^2_{\Omega}\right) \quad \textit{for any } u\in H^1(\Omega). \tag{2.2}$$

Remark 2.3. In this work, the trace estimate (2.2) is employed only when the Robin boundary condition (1.6) is considered. The constant C_0 in (2.2) can be identified from the trace inequality

$$\|u\|^2_{\partial\Omega} \le c\,(\varepsilon,\Omega)\,\|u\|^2_{\Omega} + \varepsilon\,\|\nabla u\|^2_{\Omega} \quad \text{for any } u\in H^1\,(\Omega) \text{ and } \varepsilon>0. \tag{2.3}$$

This inequality is roughly dependent on the geometry of Ω. As a typical example from [3, Theorem 2], if Ω is star-shaped, $c(\varepsilon,\Omega)$ is of the order of $\mathcal{O}(1+\varepsilon^{-1})$. When u has zero mean on $\partial\Omega$, i.e. $\int_{\partial\Omega} u d\sigma_x = 0$, then we have the stronger Poincaré inequality $\|u\|^2_{\partial\Omega} \le \tilde{C}_0\,\|\nabla u\|^2_{\Omega}$, for some constant $\tilde{C}_0>0$.

Remark 2.4. First, the assumptions (A_3)–(A_5) are obviously not needed when considering the Dirichlet boundary condition (1.5). Secondly, the monotonicity assumption (A_4) and the Hölder continuity assumption (A_5) are obviously satisfied when $S(u) = C(u-u_\infty)$ (for constant $C\ge 0$ and u_∞ given function) is a linear function. Furthermore, assumption (A_3) can be removed in this linear case when Lemma 3.4 can be brought back to Lemma 3.1 below. Physically, in the linear case, the resulting convective Robin boundary condition expresses how proportional the prescribed flux of population is to the difference of the population density and the population density u_∞ of the surroundings at the habitat boundary [13]. In the nonlinear case, the Robin-type boundary condition (1.6) could model radiation and is mathematically more challenging. In our case, this requires the price of assuming (A_3) which expresses that at the boundary, the time- and age-varying evolution of the surface reaction rate should be bounded in some density-dependent way.

3. Main results

3.1. **Dirichlet boundary condition** (1.5)**.** Let us set the function space

$$W_{T,a_\dagger}^{\Omega} := C\left(\overline{Q_{T,a_\dagger}}; H_0^1(\Omega)\right) \cap L^2\left(Q_{T,a_\dagger}; H^2(\Omega)\right) \cap C^1\left(Q_{T,a_\dagger}; H_0^1(\Omega)\right). \tag{3.1}$$

Assume that there exist two solutions u_1 and u_2 of (1.4) and (1.5) that belong to $W_{T,a_\dagger}^{\Omega}$. Then their difference $w = u_1 - u_2 \in W_{T,a_\dagger}^{\Omega}$ satisfies

$$\begin{cases} w_t + w_a - \mathcal{A}w = F(t,a,x;u_1) - F(t,a,x;u_2) & \text{in } Q_{T,a_\dagger}^{\Omega}, \\ w(t,a,x) = 0 & \text{on } Q_{T,a_\dagger}^{\partial\Omega}, \\ w(T,a,x) = 0 & \text{in } (0,a_\dagger)\times\Omega, \\ w(t,a_\dagger,x) = 0 & \text{in } (0,T)\times\Omega. \end{cases} \tag{3.2}$$

Let us denote

$$P_{T,a_\dagger}^{\Omega} := \left\{ u \in W_{T,a_\dagger}^{\Omega} : u|_{\partial\Omega} = 0, u|_{t\in\{0,T\}} = 0, u|_{a\in\{0,a_\dagger\}} = 0 \right\}$$

and $\lambda(t,a) := t - T + a - a_\dagger - \eta < 0$ for some $\eta > 0$. In the following, since λ is a negative quantity, when raised to an arbitrary real power $r \in \mathbb{R}$, we convene to assign it to the well-defined quantity $\left(\lambda^2\right)^{r/2}$.

Lemma 3.1. *Assume (A_1) holds. Then for any $v \in P_{T,a_\dagger}^{\Omega}$, $m \in \mathbb{N}^*$ and $k \in \mathbb{R}_+^*$, we have*

$$\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}v - v_t - v_a\right)\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 \geq \frac{4m}{k}\left\|\lambda^{-\frac{m}{k}-1}v\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 - \overline{M}n\left\|\lambda^{-\frac{m}{k}}\nabla v\right\|_{Q_{T,a_\dagger}^{\Omega}}^2. \tag{3.3}$$

Moreover, if $0 < T + a_\dagger \leq \mu$ for a sufficiently small constant $\mu > 0$, there exists a positive K such that

$$K\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}v - v_t - v_a\right)\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 \geq \left\|\lambda^{-\frac{m}{k}-1}v\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 + \frac{1}{2}\left\|\lambda^{-\frac{m}{k}}\nabla v\right\|_{Q_{T,a_\dagger}^{\Omega}}^2, \tag{3.4}$$

for sufficiently large m.

Proof. Let $z = \lambda^{-\frac{m}{k}} v \in P_{T,a_\dagger}^{\Omega}$. Then

$$\lambda^{-\frac{m}{k}}\left(\mathcal{A}v - v_t - v_a\right) = \mathcal{A}z - (z_t + z_a) - \frac{2m}{k}\lambda^{-1}z. \tag{3.5}$$

By the definition of inner product in $L^2\left(Q_{T,a_\dagger}^{\Omega}\right)$, we obtain

$$\begin{aligned} \left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}v - v_t - v_a\right)\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 &= \left\|z_t + z_a\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 - 2\left\langle z_t + z_a, \mathcal{A}z\right\rangle_{Q_{T,a_\dagger}^{\Omega}} \\ &\quad + \frac{4m}{k}\left\langle \lambda^{-1}z, z_t + z_a\right\rangle_{Q_{T,a_\dagger}^{\Omega}} + \left\|\mathcal{A}z - \frac{2m}{k}\lambda^{-1}z\right\|_{Q_{T,a_\dagger}^{\Omega}}^2. \end{aligned} \tag{3.6}$$

For the second term in right-hand side (RHS) of (3.6), using integration by parts

$$
\begin{aligned}
-2\left\langle z_t+z_a,\mathcal{A}z\right\rangle_{Q^{\Omega}_{T,a_\dagger}} &= 2\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}\left(\partial_t\partial_{x_i}z+\partial_a\partial_{x_i}z\right)d_{ij}\partial_{x_j}zdxdadt\\
&= 2\int_0^T\int_0^{a_\dagger}\int_\Omega(\partial_t+\partial_a)\left(\sum_{1\le i,j\le n}\partial_{x_i}zd_{ij}\partial_{x_j}z\right)dxdadt\\
&- 2\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}\partial_{x_i}z(\partial_t+\partial_a)\left(d_{ij}\partial_{x_j}z\right)dxdadt\\
&= -2\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}\partial_{x_i}z(\partial_t+\partial_a)\left(d_{ij}\right)\partial_{x_j}zdxdadt\\
&- 2\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}\partial_{x_i}zd_{ij}(\partial_t+\partial_a)\partial_{x_j}zdxdadt,
\end{aligned}
$$

where we have used that $z\in P^{\Omega}_{T,a_\dagger}$. Since $d_{ij}=d_{ji}$ (cf. (A_1)), from the first and fifth rows we obtain

$$
\begin{aligned}
&2\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}(\partial_t+\partial_a)\left(\partial_{x_i}z\right)d_{ij}\partial_{x_j}zdxdadt\\
&= -\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}\partial_{x_i}z\left(\partial_t+\partial_a\right)\left(d_{ij}\right)\partial_{x_j}zdxdadt.
\end{aligned}
$$

Thus, we obtain that

$$
-2\left\langle z_t+z_a,\mathcal{A}z\right\rangle_{Q^{\Omega}_{T,a_\dagger}} = -\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}\partial_{x_i}z\left(\partial_t+\partial_a\right)\left(d_{ij}\right)\partial_{x_j}zdxdadt. \tag{3.7}
$$

In the same manner, the third term in the RHS of (3.6) is

$$
\begin{aligned}
\frac{4m}{k}\left\langle\lambda^{-1}z,z_t+z_a\right\rangle_{Q^{\Omega}_{T,a_\dagger}} &= \frac{2m}{k}\int_0^T\int_0^{a_\dagger}\int_\Omega(\partial_t+\partial_a)\left(\lambda^{-1}z^2\right)dxdadt\\
&+\frac{4m}{k}\int_0^T\int_0^{a_\dagger}\int_\Omega\lambda^{-2}z^2dxdadt = \frac{4m}{k}\left\|\lambda^{-1}z\right\|^2_{Q^{\Omega}_{T,a_\dagger}}.
\end{aligned} \tag{3.8}
$$

Using the upper bound $\overline{M}$ defined in (A_1), from (3.7) and (3.8), we obtain

$$
\begin{aligned}
&\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}v-v_t-v_a\right)\right\|^2_{Q^{\Omega}_{T,a_\dagger}} \ge \frac{4m}{k}\left\|\lambda^{-1}z\right\|^2_{Q^{\Omega}_{T,a_\dagger}}\\
&-\overline{M}\int_0^T\int_0^{a_\dagger}\int_\Omega\sum_{1\le i,j\le n}\left|\partial_{x_i}z\partial_{x_j}z\right|dxdadt \ge \frac{4m}{k}\left\|\lambda^{-1}z\right\|^2_{Q^{\Omega}_{T,a_\dagger}}-\overline{M}n\left\|\nabla z\right\|^2_{Q^{\Omega}_{T,a_\dagger}}.
\end{aligned}
$$

Substituting $z=\lambda^{-\frac{m}{k}}v$, we end the proof of (3.3).

Using (A_1), integration by parts and $|\lambda(t,a)|\le T+a_\dagger+\eta$ we have

$$
\begin{aligned}
-\left\langle\lambda^{-\frac{2m}{k}}v,\mathcal{A}v-v_t-v_a\right\rangle_{Q^{\Omega}_{T,a_\dagger}} &\ge -\frac{2m}{k}\left(T+a_\dagger+\eta\right)\left\|\lambda^{-\frac{m}{k}-1}v\right\|^2_{Q^{\Omega}_{T,a_\dagger}}\\
&+\underline{c}\left\|\lambda^{-\frac{m}{k}}\nabla v\right\|^2_{Q^{\Omega}_{T,a_\dagger}}
\end{aligned} \tag{3.9}
$$

It then follows from (3.3) that (3.9) can be estimated by

$$-\left\langle \lambda^{-\frac{2m}{k}} v, \mathcal{A}v - v_t - v_a \right\rangle_{Q^{\Omega}_{T,a_\dagger}} \geq -\frac{(T + a_\dagger + \eta)}{2} \left(\left\| \lambda^{-\frac{m}{k}} (\mathcal{A}v - v_t - v_a) \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \right.$$

$$\left. + \overline{M}_1 \left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \right) + \underline{c} \left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{T,a_\dagger}}, \tag{3.10}$$

where we have denoted $\overline{M}_1 = n\overline{M}$. Using (3.3) and Young's inequality, we have

$$\begin{aligned} -\left\langle \lambda^{-\frac{2m}{k}} v, \mathcal{A}v - v_t - v_a \right\rangle_{Q^{\Omega}_{T,a_\dagger}} &\leq \frac{1}{2} \left\| \lambda^{-\frac{m}{k}-1} v \right\|^2_{Q^{\Omega}_{T,a_\dagger}} + \frac{1}{2} \left\| \lambda^{-\frac{m}{k}+1} (\mathcal{A}v - v_t - v_a) \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\ &\leq \frac{k}{8m} \left(\left\| \lambda^{-\frac{m}{k}} (\mathcal{A}v - v_t - v_a) \right\|^2_{Q^{\Omega}_{T,a_\dagger}} + \overline{M}_1 \left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \right) \\ &\quad + \frac{(T + a_\dagger + \eta)^2}{2} \left\| \lambda^{-\frac{m}{k}} (\mathcal{A}v - v_t - v_a) \right\|^2_{Q^{\Omega}_{T,a_\dagger}}. \end{aligned}$$

Combining this with (3.10) then reads as

$$\begin{aligned} &\underline{c} \left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\ &\leq \left(\frac{k}{8m} + \frac{(T + a_\dagger + \eta)}{2} + \frac{(T + a_\dagger + \eta)^2}{2} \right) \left\| \lambda^{-\frac{m}{k}} (\mathcal{A}v - v_t - v_a) \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\ &+ \left(\frac{(T + a_\dagger + \eta) \overline{M}_1}{2} + \frac{k\overline{M}_1}{8m} \right) \left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{T,a_\dagger}}. \end{aligned}$$

Accordingly, if $\mu_0 \geq \mu \geq T + a_\dagger > 0$, $\eta_0 \geq \eta > 0$ and m_0 are such that

$$2 \left(\frac{(\mu_0 + \eta_0) \overline{M}_1}{2} + \frac{k\overline{M}_1}{8m_0} \right) \leq \underline{c}, \tag{3.11}$$

then for any $m \geq m_0$, we obtain

$$\left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \leq C_1 \left\| \lambda^{-\frac{m}{k}} (\mathcal{A}v - v_t - v_a) \right\|^2_{Q^{\Omega}_{T,a_\dagger}}, \tag{3.12}$$

where $C_1 := \frac{2}{\underline{c}} \left(\frac{k}{8m} + \frac{(\mu+\eta)}{2} + \frac{(\mu+\eta)^2}{2} \right)$. Combining (3.12) and (3.3), we conclude that

$$\begin{aligned} &\left\| \lambda^{-\frac{m}{k}-1} v \right\|^2_{Q^{\Omega}_{T,a_\dagger}} + \frac{1}{2} \left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\ &\leq \left[\frac{k}{4m} + C_1 \left(\frac{1}{2} + \frac{k\overline{M}_1}{4m} \right) \right] \left\| \lambda^{-\frac{m}{k}} (\mathcal{A}v - v_t - v_a) \right\|^2_{Q^{\Omega}_{T,a_\dagger}}. \end{aligned}$$

Then denoting $C_2 = \frac{2}{\underline{c}} \left(\frac{k}{8m_0} + \frac{(\mu_0+\eta_0)}{2} + \frac{(\mu_0+\eta_0)^2}{2} \right)$ and choosing

$$K := \frac{k}{4m_0} + C_2 \left(\frac{1}{2} + \frac{k\overline{M}_1}{4m_0} \right), \tag{3.13}$$

we conclude that (3.4) holds for $m \geq m_0$. □

Let us now choose m_0 sufficiently large, and μ_0 and η_0 sufficiently small such that K given by (3.13) satisfies

$$0 < K \leq \frac{1}{4\alpha L_F^2}. \tag{3.14}$$

Let also $0 < \eta_1 \le \min\{1, \eta_0\}$ and choose

$$\eta = \eta_1,\ \mu_0' = \eta^{\frac{m\left(\frac{1}{\alpha}-1\right)}{m\left(\frac{1}{\alpha}-1\right)+k}} 2^{\frac{k}{2m\left(\frac{1}{\alpha}-1\right)+2k}} - \eta > 0. \tag{3.15}$$

We consider here $T + a_\dagger \le \mu_0'$ since the uniqueness result for the latter case $T + a_\dagger > \mu_0'$ is a direct consequence from the former case. In that case, the time and age intervals can be divided into many countable subsets whose lengths are not larger than μ_0', which brings us back to the former case.

Let $0 < t_1 < t_2 < T$ and $0 < a_1 < a_2 < a_\dagger$ and take $\kappa : \overline{Q_{T,a_\dagger}} \to \mathbb{R}$ such that it is twice continuously differentiable in $(t_1, t_2) \times (a_1, a_2)$ and

$$\kappa(t,a) = \begin{cases} 0 & \text{if } (t,a) \in Q_{T,a_\dagger} \backslash \left((t_1,t_2)\times(a_1,a_2) \cup [t_2,T]\times[a_2,a_\dagger]\right), \\ 1 & \text{if } (t,a) \in [t_2,T]\times[a_2,a_\dagger]. \end{cases} \tag{3.16}$$

Let $v = \kappa w$, where w is the solution of (3.2), then notice that $v \in P^{\Omega}_{T,a_\dagger}$. Starting from the estimate (3.4), we have that for sufficiently large $m \ge m_0$

$$\begin{aligned} K \left\| \lambda^{-\frac{m}{k}} \left(\mathcal{A}v - v_t - v_a\right) \right\|^2_{Q^{\Omega}_{(t_1,t_2)\times(a_1,a_2)}} + K \left\| \lambda^{-\frac{m}{k}} \left(\mathcal{A}w - w_t - w_a\right) \right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \\ \ge \left\| \lambda^{-\frac{m}{k}-1} w \right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}} + \frac{1}{2} \left\| \lambda^{-\frac{m}{k}} \nabla v \right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}. \end{aligned} \tag{3.17}$$

From (3.2) and (A_2), we get $\lambda^{-\frac{2m}{k}} \left(\mathcal{A}w - w_t - w_a\right)^2 \le \lambda^{-\frac{2m}{k}} L_F^2 |w|^{2\alpha}$, then applying Lemma 2.1 with $X = |w|^2 > 0$, $\alpha_0 = \alpha$ and

$$\gamma = \left[(t_2 - t_1)(a_2 - a_1)\right]^{-\frac{1}{\alpha}} \lambda^{\frac{2m}{k\alpha}} \|\lambda^{-\frac{m}{k}}\|^{\frac{2}{\alpha}}_{(t_1,t_2)\times(a_1,a_2)} > 0, \tag{3.18}$$

we have

$$\begin{aligned} \lambda^{-\frac{2m}{k}} \left(\mathcal{A}w - w_t - w_a\right)^2 \le \alpha L_F^2 \frac{\lambda^{-\frac{2m}{k\alpha}} \|\lambda^{-\frac{m}{k}}\|^{\frac{2}{\alpha}(\alpha-1)}_{(t_1,t_2)\times(a_1,a_2)}}{\left[(t_2-t_1)(a_2-a_1)\right]^{1-\frac{1}{\alpha}}} w^2 \\ + (1-\alpha) L_F^2 \frac{\|\lambda^{-\frac{m}{k}}\|^2_{(t_1,t_2)\times(a_1,a_2)}}{(t_2-t_1)(a_2-a_1)}. \end{aligned}$$

Integrating both sides over Ω, we have

$$\begin{aligned} \left\| \lambda^{-\frac{m}{k}} \left(\mathcal{A}w - w_t - w_a\right) \right\|^2_{\Omega} \le \alpha L_F^2 \frac{\lambda^{-\frac{2m}{k\alpha}+\frac{2m}{k}} \|\lambda^{-\frac{m}{k}}\|^{\frac{2}{\alpha}(\alpha-1)}_{(t_1,t_2)\times(a_1,a_2)}}{\left[(t_2-t_1)(a_2-a_1)\right]^{1-\frac{1}{\alpha}}} \lambda^{-\frac{2m}{k}} \|w\|^2_{\Omega} \\ + (1-\alpha)|\Omega| L_F^2 \frac{\|\lambda^{-\frac{m}{k}}\|^2_{(t_1,t_2)\times(a_1,a_2)}}{(t_2-t_1)(a_2-a_1)}. \end{aligned} \tag{3.19}$$

Notice that for $\alpha \in (0,1]$, it holds $\lambda^{-\frac{2m}{k\alpha}+\frac{2m}{k}} = \lambda^{\frac{2m}{k}\left(1-\frac{1}{\alpha}\right)} \le \eta^{\frac{2m}{k}\left(1-\frac{1}{\alpha}\right)}$. Using this bound and integrating both sides of (3.19) over $(t_2,T)\times(a_2,a_\dagger)$, we obtain

$$\begin{aligned}
&\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^\Omega_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \\
&\le \alpha L_F^2 \frac{\left\|\lambda^{-\frac{m}{k}}\right\|^{2\left(1-\frac{1}{\alpha}\right)}_{(t_1,t_2)\times(a_1,a_2)}}{\left(t_2-t_1\right)^{1-\frac{1}{\alpha}}\left(a_2-a_1\right)^{1-\frac{1}{\alpha}}} \left\|\lambda^{-\frac{m}{k}} w\right\|^2_{Q^\Omega_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \eta^{\frac{2m}{k}\left(1-\frac{1}{\alpha}\right)} \\
&+ (1-\alpha)\left|\Omega\right| \frac{\left(T-t_2\right)\left(a_\dagger - a_2\right)}{\left(t_2-t_1\right)\left(a_2-a_1\right)} L_F^2 \left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)}. \qquad (3.20)
\end{aligned}$$

Using that $|\lambda| \le \mu_0' + \eta$, we have

$$\left\|\lambda^{-\frac{m}{k}}\right\|^{2\left(1-\frac{1}{\alpha}\right)}_{(t_1,t_2)\times(a_1,a_2)} \le \left(t_2-t_1\right)^{1-\frac{1}{\alpha}}\left(a_2-a_1\right)^{1-\frac{1}{\alpha}}\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\alpha}-1\right)}, \qquad (3.21)$$

$$\left\|\lambda^{-\frac{m}{k}} w\right\|^2_{Q^\Omega_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \le \left(\mu_0'+\eta\right)^2 \left\|\lambda^{-\frac{m}{k}-1} w\right\|^2_{Q^\Omega_{(t_2,T)\times\left(a_2,a_\dagger\right)}}. \qquad (3.22)$$

Plugging (3.20)-(3.22) into (3.17) and by the choice (3.15), we obtain

$$\begin{aligned}
&K\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}v - v_t - v_a\right)\right\|^2_{Q^\Omega_{(t_1,t_2)\times(a_1,a_2)}} \qquad (3.23)\\
&+K\left|\Omega\right| \frac{\left(T-t_2\right)\left(a_\dagger-a_2\right)}{\left(t_2-t_1\right)\left(a_2-a_1\right)} L_F^2 \left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)} \ge \frac{1}{2}\left\|\lambda^{-\frac{m}{k}-1} w\right\|^2_{Q^\Omega_{(t_2,T)\times\left(a_2,a_\dagger\right)}}.
\end{aligned}$$

For any $t_2 < t_3 < T$ and $a_2 < a_3 < a_\dagger$, (3.23) can be further estimated by

$$\begin{aligned}
&K\left(T+a_\dagger+\eta-t_2-a_2\right)^{-\frac{2m}{k}}\left(\left\|\mathcal{A}v - v_t - v_a\right\|^2_{Q^\Omega_{(t_1,t_2)\times(a_1,a_2)}} + Ta_\dagger\left|\Omega\right| L_F^2\right) \\
&\ge \frac{\left(T+a_\dagger+\eta-t_3-a_3\right)^{-\frac{2m}{k}-2}}{2}\left\|w\right\|^2_{Q^\Omega_{(t_3,T)\times\left(a_3,a_\dagger\right)}}. \qquad (3.24)
\end{aligned}$$

Observing that

$$\left(\frac{T+a_\dagger+\eta-t_2-a_2}{T+a_\dagger+\eta-t_3-a_3}\right)^{-\frac{2m}{k}} \to 0 \quad \text{as } m\to\infty, \qquad (3.25)$$

we obtain from (3.24) that $w \equiv 0$ for $(t,a,x) \in Q^\Omega_{(t_3,T)\times(a_3,a_\dagger)}$ and thus for $(t,a,x) \in Q^\Omega_{T,a_\dagger}$ since $0 < t_1 < t_2 < t_3$ and $0 < a_1 < a_2 < a_3$ can be taken arbitrarily small.

Hence, we state the following uniqueness theorem.

Theorem 3.2. *Assume* (A_1) *and* (A_2) *hold. Then, the problem* (1.4) *with the Dirichlet boundary condition* (1.5) *admits no more than one solution in* $W^\Omega_{T,a_\dagger}$.

Remark 3.3. The presence of $k \in \mathbb{R}^*_+$ in the context of Lemma 3.1 means that m/k can be considered as a real positive number. As a corollary, the uniqueness result holds when the source term F is globally Lipschitz, i.e. $\alpha = 1$. When F is locally Lipschitz, it can also be reduced to the global case if we set

$$\widehat{W}^\Omega_{T,a_\dagger} = C\left(\overline{Q_{T,a_\dagger}}; H_0^1(\Omega)\cap L^\infty(\Omega)\right) \cap L^2\left(Q_{T,a_\dagger}; H^2(\Omega)\right) \cap C^1\left(Q_{T,a_\dagger}; H_0^1(\Omega)\right).$$

This space indicates that the population density is essentially bounded in space and uniformly bounded in time and age.

Concerning the existence of the function κ in (3.16), we can rely on the application of partitions of unity (cf. [6, Proposition 2.25]). It says that if $\mathcal{M}$ is a smooth manifold with or without boundary, then for any closed subset $A \subseteq \mathcal{M}$ and any open subset U containing A, there exists a smooth bump function for A supported in U. Accordingly, the existence of κ is guaranteed by taking,

$$A = [t_2, T] \times [a_2, a_\dagger], \ \mathcal{M} = \overline{Q_{T,a_\dagger}} \text{ and } U = \{(t_1, t_2) \times (a_1, a_2)\} \cup A.$$

3.2. **Robin-type boundary condition** (1.6)**.** For the Robin boundary condition (1.6), we consider $n \geq 2$ in order to apply the trace estimate (2.2) in a meaningful way. Similar to the previous subsection, to prove uniqueness we consider u_1 and u_2 as two solutions (in some appropriate spaces) of (1.4) and (1.6) and then denote $w = u_1 - u_2$, which satisfies

$$\begin{cases} w_t + w_a - \mathcal{A}w = F(t, a, x; u_1) - F(t, a, x; u_2) & \text{in } Q_{T,a_\dagger}^{\Omega}, \\ -\mathrm{d}(t, a, x) \nabla w(t, a, x) \cdot \mathrm{n} = S(u_1) - S(u_2) & \text{on } Q_{T,a_\dagger}^{\partial\Omega}, \\ w(T, a, x) = 0 & \text{in } (0, a_\dagger) \times \Omega, \\ w(t, a_\dagger, x) = 0 & \text{in } (0, T) \times \Omega. \end{cases} \tag{3.26}$$

We set the following function spaces:

$$\widetilde{W}_{T,a_\dagger}^{\Omega} := C\left(\overline{Q_{T,a_\dagger}}; H^1(\Omega)\right) \cap L^2\left(Q_{T,a_\dagger}; H^2(\Omega)\right) \cap C^1\left(Q_{T,a_\dagger}; H^1(\Omega)\right), \tag{3.27}$$

$$\widetilde{P}_{T,a_\dagger}^{\Omega} := \left\{ u \in \widetilde{W}_{T,a_\dagger}^{\Omega} : u|_{t\in\{0,T\}} = 0, \, u|_{a\in\{0,a_\dagger\}} = 0 \right\}. \tag{3.28}$$

The space $\widetilde{P}_{T,a_\dagger}^{\Omega}$ does not now contain any boundary information on $\partial\Omega$ as in $P_{T,a_\dagger}^{\Omega}$ due to the Robin-type boundary condition (1.6) instead of the Dirichlet boundary condition (1.5). Thus, Lemma 3.1 cannot be applied with any function in $\widetilde{P}_{T,a_\dagger}^{\Omega}$. However, we are able to formulate the following lemma with $w \in \widetilde{P}_{T,a_\dagger}^{\Omega}$ directly as a solution of (3.26). Here, we also set $\lambda(t, a) := t - T + a - a_\dagger - \eta$ for $\eta > 0$.

Lemma 3.4. *Let $\beta \in (0, 1)$ and assume (A_1) and (A_3)–(A_5) hold. Let $w \in \widetilde{P}_{T,a_\dagger}^{\Omega}$ be a solution of the problem (3.26). Then, for any $m \in \mathbb{N}^*$, $k \in \mathbb{R}_+^*$ and $0 < t_1 < t_2 < T$, $0 < a_1 < a_2 < a_\dagger$, we have*

$$\begin{aligned} &\left\|\lambda^{-\frac{m}{k}}(\mathcal{A}w - w_t - w_a)\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 \\ &\geq \frac{4m}{k}\left\|\lambda^{-\frac{m}{k}-1}w\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 - \overline{M}n\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 - 2\overline{m}\left\|\lambda^{-\frac{m}{k}}w\right\|_{Q_{T,a_\dagger}^{\partial\Omega}}^2 \\ &- \frac{\eta^2}{4C_0}\left\|\lambda^{-\frac{m}{k}-1}w\right\|_{Q_{T,a_\dagger}^{\partial\Omega}}^2 - 16C_0\mathbf{A}, \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} \mathbf{A} = \beta L_S^2 \frac{\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)+2}\left\|\lambda^{-\frac{m}{k}}\right\|_{(t_1,t_2)\times(a_1,a_2)}^{\frac{2}{\beta}(\beta-1)}}{\left[(t_2 - t_1)(a_2 - a_1)\right]^{1-\frac{1}{\beta}}}\left\|\lambda^{-\frac{m}{k}}w\right\|_{Q_{T,a_\dagger}^{\partial\Omega}}^2 \\ + \left(\frac{m}{k\eta^{\beta+1}}\right)^{\frac{2}{1-\beta}}(1-\beta)\left|\partial\Omega\right| L_S^2 \frac{Ta_\dagger\left\|\lambda^{-\frac{m}{k}}\right\|_{(t_1,t_2)\times(a_1,a_2)}^2}{(t_2 - t_1)(a_2 - a_1)}. \end{aligned} \tag{3.30}$$

Also, if $0 < T + a_\dagger \leq \mu$ for sufficiently small $\mu > 0$, there exists $K > 0$ such that

$$K\left\|\lambda^{-\frac{m}{k}}(\mathcal{A}w - w_t - w_a)\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 + K\mathbf{A} \geq \left\|\lambda^{-\frac{m}{k}-1}w\right\|_{Q_{T,a_\dagger}^{\Omega}}^2 + \frac{1}{2}\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|_{Q_{T,a_\dagger}^{\Omega}}^2, \tag{3.31}$$

for sufficiently large m.

Proof. We can adapt the proof of Lemma 3.1 to prove the estimates (3.29) and (3.31). Omitting some calculus, let $z = \lambda^{-\frac{m}{k}} w \in \widetilde{P}^{\Omega}_{T,a_\dagger}$, then (as in (3.6))

$$\begin{aligned} \left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^{\Omega}_{T,a_\dagger}} &= \left\|z_t + z_a\right\|^2_{Q^{\Omega}_{T,a_\dagger}} - 2\left\langle z_t + z_a, \mathcal{A}z\right\rangle_{Q^{\Omega}_{T,a_\dagger}} \\ &+ \frac{4m}{k}\left\langle \lambda^{-1}z, z_t + z_a\right\rangle_{Q^{\Omega}_{T,a_\dagger}} + \left\|\mathcal{A}z - \frac{2m}{k}\lambda^{-1}z\right\|^2_{Q^{\Omega}_{T,a_\dagger}}. \end{aligned} \tag{3.32}$$

Using the Robin boundary condition from (3.26), we obtain

$$\begin{aligned} -2\left\langle z_t + z_a, \mathcal{A}z\right\rangle_{Q^{\Omega}_{T,a_\dagger}} &= \underbrace{-2\int_0^T\int_0^{a_\dagger}\int_{\partial\Omega}\left(z_t + z_a\right)\mathrm{d}\nabla z\cdot \mathrm{n} d\sigma_x dadt}_{:=\mathcal{I}_1} \\ &\underbrace{+ 2\int_0^T\int_0^{a_\dagger}\int_{\Omega}\sum_{1\le i,j\le n}\left(\partial_t\partial_{x_i}z + \partial_a\partial_{x_i}z\right)d_{ij}\partial_{x_j}z dxdadt}_{:=\mathcal{I}_2}. \end{aligned} \tag{3.33}$$

As in the derivation of (3.7) and using the upper bound $\overline{M}$ in (A_1), we obtain

$$\begin{aligned} \mathcal{I}_2 = -\int_0^T\int_0^{a_\dagger}\int_{\Omega}\sum_{1\le i,j\le n}\partial_{x_i}z\left(\partial_t + \partial_a\right)\left(d_{ij}\right)\partial_{x_j}z dxdadt \\ \ge -\overline{M}_1\left\|\nabla z\right\|^2_{Q^{\Omega}_{T,a_\dagger}}. \end{aligned} \tag{3.34}$$

To estimate $\mathcal{I}_1$, recall the Robin boundary condition from (3.26) to get

$$\begin{aligned} \mathcal{I}_1 &= 2\int_0^T\int_0^{a_\dagger}\int_{\partial\Omega}\left(z_t + z_a\right)\lambda^{-\frac{m}{k}}\left(S\left(u_1\right) - S\left(u_2\right)\right)d\sigma_x dadt \\ &= \underbrace{2\int_0^T\int_0^{a_\dagger}\int_{\partial\Omega}\left(\partial_t + \partial_a\right)\left[\lambda^{-\frac{m}{k}}z\left(S\left(u_1\right) - S\left(u_2\right)\right)\right]d\sigma_x dadt}_{:=\mathcal{I}_3} \\ &\underbrace{-2\int_0^T\int_0^{a_\dagger}\int_{\partial\Omega}\lambda^{-\frac{m}{k}}z\left(\partial_t + \partial_a\right)\left(S\left(u_1\right) - S\left(u_2\right)\right)d\sigma_x dadt}_{:=\mathcal{I}_4} \\ &\underbrace{+\frac{4m}{k}\int_0^T\int_0^{a_\dagger}\int_{\partial\Omega}\lambda^{-\frac{m}{k}-1}z\left(S\left(u_1\right) - S\left(u_2\right)\right)d\sigma_x dadt}_{:=\mathcal{I}_5}. \end{aligned} \tag{3.35}$$

Notice that due to the zero conditions in the definition (3.28) of $\widetilde{P}^{\Omega}_{T,a_\dagger}$, the first term $\mathcal{I}_3$ of (3.35) vanishes. Using (A_3), we estimate $\mathcal{I}_4$ by

$$\begin{aligned} \mathcal{I}_4 &= -2\int_0^T\int_0^{a_\dagger}\int_{\partial\Omega}\lambda^{-\frac{2m}{k}}\left(u_1 - u_2\right)\left(\partial_t + \partial_a\right)\left(S\left(u_1\right) - S\left(u_2\right)\right)d\sigma_x dadt \\ &\ge -2\overline{m}\left\|\lambda^{-\frac{m}{k}}w\right\|^2_{Q^{\partial\Omega}_{T,a_\dagger}}. \end{aligned} \tag{3.36}$$

By the back-substitution $z = \lambda^{-\frac{m}{k}}w$, the term $\mathcal{I}_5$ can be estimated by

(3.37)

$$\mathcal{I}_5 \geq -\frac{2m}{k}\left(\frac{k\eta^2}{8mC_0}\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\partial\Omega}_{T,a_\dagger}} + \frac{8mC_0}{k\eta^2}\left\|\lambda^{-\frac{m}{k}}\left(S\left(u_1\right)-S\left(u_2\right)\right)\right\|^2_{Q^{\partial\Omega}_{T,a_\dagger}}\right).$$

Apply (A_5) and the inequality (2.1) for $X = |w|^2 \geq 0$, $\alpha_0 = \beta \in (0,1)$ and

$$\gamma = \left[\left(t_2-t_1\right)\left(a_2-a_1\right)\right]^{-\frac{1}{\beta}}\left(\frac{\eta^2 k}{m}\right)^{\frac{2}{\beta-1}}\lambda^{\frac{2m}{k\beta}}\left\|\lambda^{-\frac{m}{k}}\right\|^{\frac{2}{\beta}}_{(t_1,t_2)\times(a_1,a_2)} > 0,$$

for $0 < t_1 < t_2 < T, 0 < a_1 < a_2 < a_\dagger$, to get (using that $|\lambda| > \eta$), as in (3.19),

$$\frac{m^2}{k^2\eta^2}\left\|\lambda^{-\frac{m}{k}}\left(S\left(u_1\right)-S\left(u_2\right)\right)\right\|^2_{Q^{\partial\Omega}_{T,a_\dagger}} \leq \mathbf{A}. \tag{3.38}$$

As in (3.8) since $z \in \widetilde{P}^{\Omega}_{T,a_\dagger}$, we observe that

$$\frac{4m}{k}\left\langle\lambda^{-1}z, z_t + z_a\right\rangle_{Q^{\Omega}_{T,a_\dagger}} = \frac{4m}{k}\left\|\lambda^{-1}z\right\|^2_{Q^{\Omega}_{T,a_\dagger}}. \tag{3.39}$$

We complete the proof of (3.29) by grouping together (3.32), (3.34), (3.36)-(3.39). It remains to prove the estimate (3.31). Using Green's formula we have

$$\begin{aligned}-\left\langle\lambda^{-\frac{2m}{k}}w, \mathcal{A}w - w_t - w_a\right\rangle_{Q^{\Omega}_{T,a_\dagger}} &= \underbrace{\int_0^T\int_0^{a_\dagger}\int_{\partial\Omega}\lambda^{-\frac{2m}{k}}w\left(S\left(u_1\right)-S\left(u_2\right)\right)d\sigma_x da dt}_{:=\mathcal{I}_6} \\ &+ \underbrace{\int_0^T\int_0^{a_\dagger}\int_{\Omega}\lambda^{-\frac{2m}{k}}\left(\mathrm{d}\nabla w\right)\cdot\nabla w dx da dt}_{:=\mathcal{I}_7} \\ &+ \underbrace{\int_0^T\int_0^{a_\dagger}\int_{\Omega}\lambda^{-\frac{2m}{k}}w\left(w_t+w_a\right)dx da dt}_{:=\mathcal{I}_8}.\end{aligned} \tag{3.40}$$

Since $|\lambda(t,a)| \leq T + a_\dagger + \eta$ for all $(t,a) \in \overline{Q_{T,a_\dagger}}$, $\mathcal{I}_8$ can be estimated using integration by parts, while for $\mathcal{I}_6$ and $\mathcal{I}_7$ we use (A_1) and (A_4) to obtain

$$\begin{aligned}-\left\langle\lambda^{-\frac{2m}{k}}w, \mathcal{A}w - w_t - w_a\right\rangle_{Q^{\Omega}_{T,a_\dagger}} &\geq \underline{c}\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\ &- \frac{2m}{k}\left(T + a_\dagger + \eta\right)\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{T,a_\dagger}}.\end{aligned} \tag{3.41}$$

Using Young's inequality

$$\begin{aligned}&\frac{1}{2}\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} + \frac{1}{2}\left\|\lambda^{-\frac{m}{k}+1}\left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\ &\geq -\left\langle\lambda^{-\frac{2m}{k}}w, \mathcal{A}w - w_t - w_a\right\rangle_{Q^{\Omega}_{T,a_\dagger}},\end{aligned}$$

then (3.41) yields (using also that $|\lambda| \leq T + a_\dagger + \eta$)

$$\begin{aligned}
(3.42)\qquad & K_2 \left\|\lambda^{-\frac{m}{k}-1} w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} + \frac{(T+a_\dagger+\eta)^2}{2} \left\|\lambda^{-\frac{m}{k}} \left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\
& \geq \underline{c} \left\|\lambda^{-\frac{m}{k}} \nabla w\right\|^2_{Q^{\Omega}_{T,a_\dagger}}
\end{aligned}$$

where $K_2 := \frac{1}{2} + \frac{2m}{k}\left(T + a_\dagger + \eta\right)$. Now using (2.2) in the third and fourth terms of the RHS of (3.29), we obtain (using also that $|\lambda| > \eta$ and assuming $\eta < 1$)

$$\begin{aligned}
(3.43)\qquad & \left\|\lambda^{-\frac{m}{k}} \left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\
& \geq K_1 \left\|\lambda^{-\frac{m}{k}-1} w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} - \left(\overline{M}_1 + 2\overline{m}C_0 + \frac{1}{4}\right) \left\|\lambda^{-\frac{m}{k}} \nabla w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} - 16C_0\mathbf{A},
\end{aligned}$$

where $K_1 := \frac{4m}{k} - 2\overline{m}C_0\left(T + a_\dagger + \eta\right)^2 - \frac{\eta^2}{4}$. Multiplying (3.43) by $K_2K_1^{-1}$ and applying (3.42), we obtain

$$(3.44)\qquad \left[\underline{c} - K_2K_1^{-1}\left(\overline{M} + 2C_0\overline{m} + \frac{1}{4}\right)\right] \left\|\lambda^{-\frac{m}{k}} \nabla w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} \leq \mathcal{I}_9 + \mathcal{I}_{10},$$

which is in line with the estimate (3.12). In (3.44), we have denoted

$$\begin{aligned}
\mathcal{I}_9 &:= \left[K_2K_1^{-1} + \frac{(T+a_\dagger+\eta)^2}{2}\right] \left\|\lambda^{-\frac{m}{k}} \left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^{\Omega}_{T,a_\dagger}}, \\
\mathcal{I}_{10} &:= 16K_2K_1^{-1}C_0\mathbf{A}.
\end{aligned}$$

Since $K_2K_1^{-1} \to \frac{1}{2}(T + a_\dagger + \eta)$ as $m \to \infty$, if $0 < T + a_\dagger \leq \mu$ for sufficiently small $\mu > 0$, then can choose $\mu_0 \geq \mu \geq T + a_\dagger > 0$, $\eta_0 \geq \eta > 0$ and $m_0 \leq m$ such that $K_1 > 0$ and $2K_2K_1^{-1}\left(\overline{M}_1 + 2C_0\overline{m} + \frac{1}{4}\right) \leq \underline{c}$. Then, for any $m \geq m_0$ denoting

$$C_3 := \frac{2}{\underline{c}} \max\left\{K_2K_1^{-1} + \frac{(\mu+\eta)^2}{2}, 16K_2K_1^{-1}C_0\right\},$$

from (3.43) and (3.44) we obtain

$$\begin{aligned}
(3.45)\qquad & \left\|\lambda^{-\frac{m}{k}-1} w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} + \left[1 - K_1^{-1}\left(\overline{M}_1 + 2C_0\overline{m} + \frac{1}{4}\right)\right] \left\|\lambda^{-\frac{m}{k}} \nabla w\right\|^2_{Q^{\Omega}_{T,a_\dagger}} \\
& \leq \left(K_1^{-1} + C_3\right)\left(\left\|\lambda^{-\frac{m}{k}} \left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^{\Omega}_{T,a_\dagger}} + \mathbf{A}\right)
\end{aligned}$$

We can now take m_0 sufficiently large, and μ_0 and η_0 sufficiently small such that $K_1^{-1}\left(\overline{M}_1 + 2C_0\overline{m} + \frac{1}{4}\right) \leq \frac{1}{2}$ for any $m \geq m_0$. Then, we conclude that (3.31) holds for $m \geq m_0$ by choosing $K := K_3^{-1} + C_4$, where

$$\begin{aligned}
K_3 &:= \frac{4m_0}{k} - \frac{\eta^2}{4} - 2\overline{m}(\mu_0 + \eta_0)^2C_0, \quad K_4 := \frac{1}{2} + \frac{2m_0}{k}\left(\mu_0 + \eta_0\right), \\
C_4 &:= \frac{2}{\underline{c}} \max\left\{K_4K_3^{-1} + \frac{(\mu_0+\eta_0)^2}{2}, 16K_4K_3^{-1}C_0\right\},
\end{aligned}$$

using that $K_4K_3^{-1} \geq K_2K_1^{-1}$ since the function $K_2K_1^{-1}$ is monotonically decreasing, as a function of m. □

Remark that in deriving (3.29) and (3.31), we have never used that w satisfies the first equation in (3.26). Let us now define

$$\overline{\kappa}(t,a)=\begin{cases}0 & \text{if } (t,a)\in Q_{T,a_\dagger}\setminus\left(\{(t_1,t_2)\times(a_1,a_2)\}\cup[t_2,T]\times[a_2,a_\dagger]\right),\\ 1 & \text{if } (t,a)\in\{(t_1,t_2)\times(a_1,a_2)\}\cup[t_2,T]\times[a_2,a_\dagger],\end{cases}$$

and set $v:=\overline{\kappa}w\in\widetilde{P}^{\Omega}_{T,a_\dagger}$. Then, from the second equation of (3.26) we have

$$\begin{aligned}&-\mathrm{d}(t,a,x)\nabla v(t,a,x)\cdot\mathrm{n}\\&=\begin{cases}S(u_1)-S(u_2) \text{ on } \left((t_1,t_2)\times(a_1,a_2)\cup[t_2,T]\times[a_2,a_\dagger]\right)\times\partial\Omega,\\ 0 \text{ on } \left(Q_{T,a_\dagger}\setminus\left((t_1,t_2)\times(a_1,a_2)\cup[t_2,T]\times[a_2,a_\dagger]\right)\right)\times\partial\Omega.\end{cases}\end{aligned}$$

Then, it can be remarked that the inequality (3.31) in Lemma 3.4 holds not only for w but also for $v=\overline{\kappa}w$ since in (3.40) we only need to use that $\mathcal{I}_6\geq 0$, with the rest of the argument remaining the same. Then, from (3.31) applied to v we have (since $v=w$ in $((t_1,t_2)\times(a_1,a_2)\cup[t_2,T]\times[a_2,a_\dagger])\times\Omega$ and $v=0$ otherwise)

(3.46)

$$\begin{aligned}&K\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}w-w_t-w_a\right)\right\|^2_{Q^{\Omega}_{(t_1,t_2)\times(a_1,a_2)}}+K\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}w-w_t-w_a\right)\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}\\&+K\mathbf{A}_1+K\mathbf{A}_2\geq\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}+\frac{1}{2}\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}},\end{aligned}$$

where

$$\begin{aligned}\mathbf{A}_1&=\beta L_S^2\frac{\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)+2}\left\|\lambda^{-\frac{m}{k}}\right\|^{\frac{2}{\beta}(\beta-1)}_{(t_1,t_2)\times(a_1,a_2)}}{\left[(t_2-t_1)(a_2-a_1)\right]^{1-\frac{1}{\beta}}}\left\|\lambda^{-\frac{m}{k}}w\right\|^2_{Q^{\partial\Omega}_{(t_1,t_2)\times(a_1,a_2)}}\\&\quad+\left(\frac{m}{k\eta^{\beta+1}}\right)^{\frac{2}{1-\beta}}(1-\beta)\left|\partial\Omega\right|L_S^2\frac{Ta_\dagger\left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)}}{(t_2-t_1)(a_2-a_1)},\end{aligned}$$

$$\mathbf{A}_2=\beta L_S^2\frac{\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)+2}\left\|\lambda^{-\frac{m}{k}}\right\|^{\frac{2}{\beta}(\beta-1)}_{(t_1,t_2)\times(a_1,a_2)}}{\left[(t_2-t_1)(a_2-a_1)\right]^{1-\frac{1}{\beta}}}\left\|\lambda^{-\frac{m}{k}}w\right\|^2_{Q^{\partial\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}.$$

Since $|\lambda|\leq T+a_\dagger+\eta\leq\mu_0'+\eta$, we have that

$$\begin{aligned}\mathbf{A}_1&\leq\beta L_S^2\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\beta}-1\right)+2}\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)+2}\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\partial\Omega}_{(t_1,t_2)\times(a_1,a_2)}}\\&\quad+\left(\frac{m}{k\eta^{\beta+1}}\right)^{\frac{2}{1-\beta}}(1-\beta)\left|\partial\Omega\right|L_S^2\frac{Ta_\dagger\left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)}}{(t_2-t_1)(a_2-a_1)}.\end{aligned}\tag{3.47}$$

Also, as in (3.21), with α replaced by β, using (2.2) and $|\lambda|\leq\mu_0'+\eta$, we have

$$\begin{aligned}\mathbf{A}_2&\leq\beta L_S^2\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\beta}-1\right)+2}\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)+2}C_0\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}\\&\quad+\beta L_S^2\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\beta}-1\right)}\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)+2}C_0\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}.\end{aligned}\tag{3.48}$$

From (3.20)-(3.22), we have that

$$\begin{aligned}
&\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}w-w_t-w_a\right)\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \\
&\le \alpha L_F^2 \frac{\eta^{\frac{2m}{k}\left(1-\frac{1}{\alpha}\right)}\left\|\lambda^{-\frac{m}{k}}\right\|^{\frac{2}{\alpha}(\alpha-1)}_{(t_1,t_2)\times(a_1,a_2)}}{\left[(t_2-t_1)(a_2-a_1)\right]^{1-\frac{1}{\alpha}}}\left\|\lambda^{-\frac{m}{k}}w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \\
&+(1-\alpha)\left|\Omega\right| L_F^2 \frac{Ta_\dagger\left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)}}{(t_2-t_1)(a_2-a_1)} \\
&\le \alpha L_F^2\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\alpha}-1\right)+2}\eta^{\frac{2m}{k}\left(1-\frac{1}{\alpha}\right)}\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \\
&+(1-\alpha)\left|\Omega\right| L_F^2 \frac{Ta_\dagger\left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)}}{(t_2-t_1)(a_2-a_1)}.
\end{aligned} \tag{3.49}$$

Combining (3.47)-(3.48), for sufficiently large $m \ge m_0$ it follows from (3.46) (noticing also $\eta^2 \le (\mu_0'+\eta)^2$) that

$$\begin{aligned}
&K\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}w-w_t-w_a\right)\right\|^2_{Q^{\Omega}_{(t_1,t_2)\times(a_1,a_2)}} \\
&+K\beta L_S^2\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\beta}-1\right)+2}\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)}\Bigg(\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\partial\Omega}_{(t_1,t_2)\times(a_1,a_2)}} \\
&+C_0\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}\Bigg) \\
&+K\Big[\alpha L_F^2\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\alpha}-1\right)+2}\eta^{\frac{2m}{k}\left(1-\frac{1}{\alpha}\right)} \\
&+C_0\beta L_S^2\left(\mu_0'+\eta\right)^{\frac{2m}{k}\left(\frac{1}{\beta}-1\right)+2}\eta^{\frac{2m}{k}\left(1-\frac{1}{\beta}\right)}\Big]\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}} \\
&+K\frac{Ta_\dagger}{(t_2-t_1)(a_2-a_1)}\Bigg[(1-\alpha)\left|\Omega\right| L_F^2 \\
&+(1-\beta)\left|\partial\Omega\right| L_S^2\left(\frac{m}{k\eta^{\beta+1}}\right)^{\frac{2}{1-\beta}}\Bigg]\left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)} \\
&\ge \left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}+\frac{1}{2}\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}.
\end{aligned} \tag{3.50}$$

Let $0<\eta_1\le\min\{1,\eta_0\}$, take $\eta=\eta_1$ and

$$\mu_0' = \begin{cases} \eta^{\frac{m\left(\frac{1}{\beta}-1\right)}{m\left(\frac{1}{\beta}-1\right)+k}}\,2^{\frac{k}{2m\left(\frac{1}{\beta}-1\right)+2k}}-\eta & \text{if } \alpha\ge\beta, \\ \eta^{\frac{m\left(\frac{1}{\alpha}-1\right)}{m\left(\frac{1}{\alpha}-1\right)+k}}\,2^{\frac{k}{2m\left(\frac{1}{\alpha}-1\right)+2k}}-\eta & \text{if } \alpha<\beta. \end{cases} \tag{3.51}$$

Also, for sufficiently large m_0, and μ_0, η_0 sufficiently small, choose

$$0<K\le\frac{1}{8}\min\left\{\frac{1}{\alpha L_F^2},\frac{1}{C_0\beta L_S^2},\frac{1}{\beta L_S^2}\right\}. \tag{3.52}$$

With the choice (3.51), we have that $(\mu_0' + \eta)^{\frac{2m}{k}\left(\frac{1}{\chi}-1\right)+2}\eta^{\frac{2m}{k}\left(1-\frac{1}{\chi}\right)} = 2$, where $\chi \in \{\alpha, \beta\}$. Further, with the choice (3.52) we have that $2K\left(\alpha L_F^2 + C_0\beta L_S^2\right) \le \frac{1}{2}$ and then (3.50) yields

$$K\left\|\lambda^{-\frac{m}{k}}\left(\mathcal{A}w - w_t - w_a\right)\right\|^2_{Q^{\Omega}_{(t_1,t_2)\times(a_1,a_2)}} + \frac{1}{4}\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\partial\Omega}_{(t_1,t_2)\times(a_1,a_2)}}$$
$$+ \frac{KTa_\dagger}{(t_2-t_1)(a_2-a_1)}\left[|\Omega| L_F^2 + |\partial\Omega| L_S^2\left(\frac{m}{k\eta^{\beta+1}}\right)^{\frac{2}{1-\beta}}\right]\left\|\lambda^{-\frac{m}{k}}\right\|^2_{(t_1,t_2)\times(a_1,a_2)}$$
$$\ge \frac{1}{2}\left\|\lambda^{-\frac{m}{k}-1}w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}} + \frac{1}{4}\left\|\lambda^{-\frac{m}{k}}\nabla w\right\|^2_{Q^{\Omega}_{(t_2,T)\times\left(a_2,a_\dagger\right)}}.$$

Observe that to complete the uniqueness result, we only need to mimic the way (3.24) was obtained from (3.23), and the limit

$$\left(\frac{m}{k}\right)^{\frac{2}{1-\beta}}\left(\frac{T+a_\dagger+\eta-t_2-a_2}{T+a_\dagger+\eta-t_3-a_3}\right)^{-\frac{2m}{k}} \to 0 \quad \text{as } m\to\infty$$

for any $t_2 < t_3 < T$ and $a_2 < a_3 < a_\dagger$, and thus $w \equiv 0$ for $(t,a,x) \in Q^{\Omega}_{T,a_\dagger}$.

In case $\beta = 1$, $\mathcal{I}_5$ in (3.35) can be directly estimated from (3.37), using the Lipschitz continuity in (A_5), without the need of employing the inequality (2.1) to obtain (3.38). The rest of details in obtaining $w \equiv 0$ in case $\beta = 1$ are skipped.

In conclusion, we can state the following uniqueness theorem.

Theorem 3.5. *Assume* (A_1)–(A_5) *hold. Then, the problem* (1.4) *satisfying the Robin boundary condition* (1.6) *admits no more than one solution in* $\widetilde{W}^{\Omega}_{T,a_\dagger}$.

Acknowledgments

This work is in commemoration of the first death anniversary of V.A.K's father. V.A.K thanks Prof. Nguyen Huy Tuan for introducing him the ultra-parabolic problem.

References

[1] A. Friedman. *Partial Differential Equations of Parabolic Type*. R.E. Krieger Publishing Company, 1983.
[2] S.A. Gourley, R. Liu, and J. Wu. Some vector borne diseases with structured host populations: extinction and spatial spread. *SIAM Journal on Applied Mathematics*, 67(2):408–433, 2007.
[3] C.O. Horgan. Eigenvalue estimates and the trace theorem. *Journal of Mathematical Analysis and Applications*, 69:231–242, 1979.
[4] F.C. Jiang and F.W. Meng. Explicit bounds on some new nonlinear integral inequality with delay. *Journal of Computational and Applied Mathematics*, 205:479–486, 2007.
[5] M.V. Klibanov and A. Yagola. Convergent numerical methods for parabolic equations with reversed time via a new Carleman estimate. *Inverse Problems*, 2019. to appear.
[6] J.M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics, Volume 218. Springer-Verlag, New York, 2003.
[7] L. Lorenzi. An abstract ultraparabolic integrodifferential equation. *Matematiche*, 58(2):401–435, 1998.
[8] L. Lorenzi. An identification problem for an ultraparabolic integrodifferential equation. *Journal of Mathematical Analysis and Applications*, 234:417–456, 1999.
[9] P. Magal and S. Ruan. *Structured Population Models in Biology and Epidemiology*. Mathematical Biosciences Subseries, Volume 1936. Springer-Verlag, Berlin, 2008.
[10] M.D. Marcozzi. On the valuation of Asian options by variational methods. *SIAM Journal on Scientific Computing*, 24:1124–1140, 2003.

[11] M.D. Marcozzi. Extrapolation discontinuous Galerkin method for ultraparabolic equations. *Journal of Computational and Applied Mathematics*, 224:679–687, 2009.

[12] A.G. McKendrick. Applications of mathematics to medical problems. *Proceedings of the Edinburgh Mathematical Society*, 44:98–130, 1926.

[13] A. Okubo and S.A. Levin. *Diffusion and Ecological Problems: Modern Perspectives*. Springer, New York, 2001.

[14] J.W.H. So, J. Wu, and X. Zou. A reaction diffusion model for a single species with age structure, I. Travelling wave fronts on unbounded domains. *Proceedings of the Royal Society A: Mathematical, Physical & Engineering Sciences*, 457:1841–1853, 2001.

[15] H. von Foerster. *Some remarks on changing populations, In: The Kinetics of Cellular Proliferation*. 382–407. Grune & Stratton, New York, 1959.

[16] M. Witten. Modeling cellular systems and aging processes. I. Mathematics of cell system models: A review. *Mechanisms of Ageing and Development*, 17:53–94, 1981.

[17] F. Zouyed and F. Rebbani. A modified quasi-boundary value method for an ultraparabolic ill-posed problem. *Journal of Inverse and Ill-Posed Problems*, 22(4):449–466, 2014.

Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, North Carolina 28223, USA

Faculty of Sciences, Hasselt University, Campus Diepenbeek, Belgium
E-mail address: vakhoa.hcmus@gmail.com

Faculty of Solid State Physics, Gdansk University of Technology, Poland
E-mail address: thehung.tran@mathmods.eu

Department of Applied Mathematics, University of Leeds, United Kingdom
E-mail address: amt5ld@maths.leeds.ac.uk