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Rigorous homogenisation of reactive transport and flow fully coupled to an evolving microstructure

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We consider the homogenisation of a coupled Stokes flow and advection–reaction–diffusion problem in a perforated domain with an evolving microstructure of size ε. Reactions at the boundaries of the microscopic interfaces lead to the formation of a solid layer, which is assumed to be radially symmetric with a variable, a priori unknown thickness. This results in a growth or shrinkage of the solid phase and, thus, the domain evolution is not known a priori but induced by the advection–reaction–diffusion process. The achievements of this work are the existence and uniqueness of a weak microscopic solution and the rigorous derivation of an effective model for $\varepsilon \to 0$, based on ε-uniform a priori estimates. As a result of the passage to the limit, the processes on the macroscale are described by an advection–reaction–diffusion problem coupled to Darcy's equation with effective coefficients (porosity, diffusivity and permeability) depending on local cell problems. These local problems are formulated on cells that depend on the macroscopic position and evolve in time. In particular, the evolution of these cells depends on the macroscopic concentration. Thus, the cell problems (respectively the effective coefficients) are coupled to the macroscopic unknowns and vice versa, leading to a strongly coupled micro–macro model. Homogenisation results have been obtained recently in Ref. 31, 72 for the case of reactive–diffusive transport coupled with microscopic domain evolution, but in the absence of advective transport. We extend these models by including the advective transport, which is driven by the Stokes equations in the a priori unknown evolving pore domain.

Keywords: Homogenisation; free boundary, evolving microstructure; advection–reaction–diffusion process; Stokes flow; two-scale convergence.

AMS Subject Classification: 35B27, 35K57, 35R35, 76M50, 76S05, 80A19

1. Introduction

Reaction–diffusion problems, coupled with adsorption–desorption effects and possibly with flow, are encountered as mathematical models for numerous real-world processes. With respect to the processes like adsorption–desorption, which take place at the boundary of the domain and not inside, such problems can be divided into two categories. Whenever adsorption–desorption effects have no or minimal impact on the local geometry, the adsorbed species is modelled by an unknown defined at the boundary. This

*Present address: Department of Mathematics, Technical University of Dortmund, Vogelspothsweg 87, 44227 Dortmund, Germany leads to so-called fixed-domain models. Alternatively, the adsorption–desorption effects may lead to the formation of a layer having a non-negligible thickness. This thickness is not known a priori, but depends on the other unknowns of the model. Therefore, it may vary in time in an a priori unknown manner, and the fluid–solid interface is a free boundary. In this case, one deals with evolving domains.

Additional challenges are encountered whenever these processes take place in a complex domain. From an application point of view, one may consider a porous medium, consisting of many solid grains surrounded by voids (the pores). The flow, advective transport, reaction and diffusion take place inside the pores, and adsorption and desorption at the grain boundaries/pore walls. At this scale, the (microscopic) mathematical model is a coupled system with fluid flow and reaction–diffusion taking place in a complex, perforated domain (the pore space) and ordinary differential equations defined at the grain surface. Examples in this sense are related to mineral precipitation and dissolution[10,66,62], biofilm growth[60], colloid deposition[22] and water diffusion into absorbent particles[25,64].

An illustrative example of this phenomenon is precipitation and dissolution taking place in a porous medium, where the void space is filled with a fluid, e.g. water. The dissolved chemical species can be diffusively and advectively transported. These chemical species may react at the grain boundaries, yielding a solid layer of e.g. salt, this being the precipitation process. Conversely, the reverse process of dissolution is also possible, reducing the salt layer and releasing chemical species into the fluid.

The complexity of the domain limits the practical usage of the microscopic models. A natural step is to approximate such models by homogenised ones, which provides the macroscopic behaviour of the system. Extending the idea mentioned above to the case of complex media, two different types of models can be identified: the microscopic domain is either fixed or evolving. In the former case, the adsorbed species (e.g. the salt) is modelled by an unknown concentration defined at the (complex) boundary. In the latter, the adsorbed species forms a (time-dependent) layer at the fluid–solid interface, leading to microscopic free boundaries.

In this work, we address the latter case of a free boundary under the assumption that the (microscopic) pore space is modelled by spherical obstacles, which are ε-periodically distributed in the macroscopic domain, with time-dependent local radii. The reactive transport and flow in the locally evolving pore space is modelled by an advection–reaction–diffusion equation, where the advective transport velocity is modelled by the Stokes equations. The radii of the obstacles grow or shrink according to a concentration-dependent reaction rate at their interfaces.

Whereas the literature on the analysis and homogenisation for fixed-domain models is extremely vast, the one devoted to evolving domains is rather limited and the rigorous upscaling of fully coupled reactive transport in evolving porous media is still a challenging task. The locally different evolution of the microstructure requires a transformation-based homogenisation approach, which reduces the problem to an – albeit complex – fixed-domain problem. This approach is visualised for the geometry under consideration in Figure 1. As the evolution occurs on the microscale, this introduces terms in the (transformed) advection–reaction–diffusion equation and Stokes system to be upscaled with only weak convergence properties, similar as with slow diffusion terms, which have been studied extensively for fixed-domain problems.

Restricting the model to a radially symmetric growth or shrinkage of the microstructure in each pore makes this detail of the problem one-dimensional, which gives just enough regularity to enhance the convergence based on the a priori estimates and to allow the passage to the limit in the transformed terms. Additionally, we restrict the reaction rate at the interface in a way avoiding clogging or complete disappearance of the precipitate layer. While the latter is justified by physical assumptions (for example, precipitation and dissolution take place at surfaces of solids), the assumption of no clogging is motivated from a mathematical point of view, since otherwise we can only expect local-in-time solutions for the model. After passing to the limit $\varepsilon \to 0$ in the fixed configuration, the limit problem can be transformed back to the original evolving situation leading to an advection–reaction–diffusion problem coupled to Darcy's equation with effective coefficients (porosity, diffusivity and permeability) depending on local cell problems.

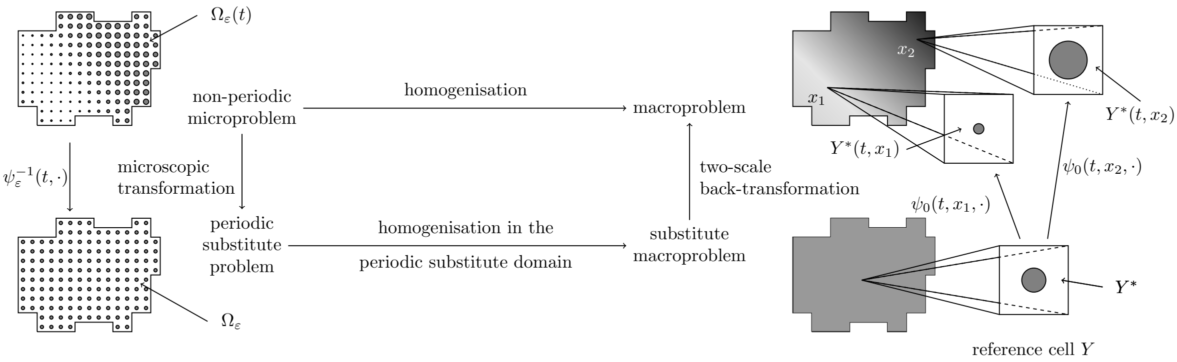

Fig. 1. Transformation-based homogenisation approach. The non-periodic microproblem is transformed to a periodic substitute problem, for which the homogenisation limit can be taken. The resulting substitute macroproblem is then transformed back. This approach is equivalent to the direct homogenisation of the original non-periodic problem[71] but is simplified by the periodic geometry.

The highlights of the paper are as follows:

- Existence and uniqueness of a global-in-time weak solution for a coupled Stokes and advection–reaction–diffusion system in an evolving microdomain
- Essential bounds for the concentration uniformly with respect to ε
- General (strong) two-scale compactness results based on ε-uniform a priori estimates associated to problems in evolving microdomains
- Coupled Darcy and advection–reaction–diffusion system as the homogenised limit, with effective coefficients depending on cell solutions on evolving micro-cells

Literature overview After transformation we are dealing with reaction–diffusion–advection and Stokes equations in fixed domains. Of course, the coefficients in these equations depend on the unknown of the system which makes the problem highly nonlinear. However, several results from periodic homogenisation theory can be transferred to our problem so, in what follows, we mention some of them related to our problem. In the case of a fixed domain, rigorous homogenisation results for reactive transport with nonlinear interface and bulk terms are presented in Ref. 29, 27 and in Ref. 35, 20, 13, 47 for different scalings of the interface terms. The homogenisation of non-linear models involving slowly diffusing quantities (diffusion of order ε^2) are for instance considered in Ref. 7, 38, 55 and with (fast) diffusion on internal boundaries in Ref. 32, 33, 34. In Ref. 39, a precipitation–dissolution model involving a multi-valued dissolution rate is homogenised. For the case of a fixed domain, rigorous homogenisation results for Stokes flow are presented in Ref. 65, 2, 76, 4, 8, see also Ref. 74 for a more detailed overview. Related (advection–) diffusion problems were homogenised rigorously in Ref. 19, 47, 46.

For the case of an evolving geometry, the moving boundaries are modelled in Ref. 66, 10, 58, 11, 62, 36, 12 (for dissolution and precipitation processes), Ref. 60, 61 (for biofilm formation) Ref. 57 (for colloid dynamics) and Ref. 56 (for a general framework) by means of a level-set function or approximated by phase-field approaches. However, the models are only upscaled formally by asymptotic two-scale expansions. A convergence proof for a given evolving level-set function has recently been presented in Ref. 26. On the other hand, the mathematically rigorous homogenisation of microscopic models defined in an evolving geometry is extremely challenging and the literature on the subject is very scarce. In particular, reactive transport and flow fully coupled to an evolving microstructure have not been considered before. To cope with the evolving microstructure rigorously, one possibility is to transform the equations to a fixed, periodic substitute domain and to pass to the limit there, as illustrated in Figure 1. This approach, which we follow also in this article, was presented in Ref. 51 and applied for (advection–)reaction–diffusion processes Ref. 30, 52, 53, 54 and for an elasticity problem in Ref. 21. The transformation approach was used for the homogenisation of quasi-stationary Stokes flow in Ref. 73 and for instationary Stokes flow in Ref. 74. In the latter two works, a given generic microscopic domain evolution was considered and the fluid velocity at the moving interface was given. In Ref. 71, the transformation approach was justified by showing that the homogenisation process commutes with the transformation, i.e. transforming the problem on a periodic reference domain, performing the homogenisation there and transforming the problem back actually gives the homogenisation result for the original equation. Moreover, two-scale transformation rules for the homogenisation correctors are presented in Ref. 71 enabling a two-scale back-transformation, which provides a completely transformation-independent homogenisation result. The transformation rules are extended for the Stokes problem in Ref. 73. These results on the homogenisation of problems with evolving microstructure by means of the transformation-based approach are summarised and presented in more detail in Ref. 70.

In Ref. 31, 72, the homogenisation of reactive transport for a priori not given evolving microstructure, which depends on the unknowns itself, was considered. The geometrical setting is the same as in the present work. More precisely, we assume that, around each grain, the microstructure evolution is radially symmetric and the free boundary can be described locally by the (uniform) thickness of the layer. However, we consider here additional advective transport, where the fluid velocity is modelled by the Stokes equations. This requires not only the inclusion of the Stokes equations in a non-linear manner, but we also have to deal with the new advective term in the advection–reaction–diffusion equation. Since the coefficents of this advective term arise as solution of the Stokes equations, it provides low regularity and complicates the homogenisation.

Content of the paper In what follows, let the time interval $(0,T)$ be given by $T > 0$. The heterogeneous medium $\Omega \subset \mathbb{R}^n$ (for $n \in \{2,3,4\}$) consists of the ε-scaled pore space Ω_ε which is obtained by removing the solid grains from Ω. The centres of the grains are distributed periodically, where the distance between two neighbouring centres is of order ε and, thus, the pore space $\Omega_\varepsilon(t)$ depends on ε and the time t for $t \in [0,T]$. In the pore space $\Omega_\varepsilon(t)$, we consider an advection–reaction–diffusion equation for the solute species u_ε and the Stokes equations for the fluid velocity v_ε and the fluid pressure p_ε. We assume that the periodically distributed solid grains are spheres with different radii R_ε of the order of the microscopic length scale ε, which do not intersect. The grains evolve independently of each other and we assume that the evolution is radially symmetric and, thus, well-described by the radii. The radii are given as a solution of ordinary differential equations (ODEs), which depend on the solute concentration on the surface of the grains and, therefore, on the unknowns of the system.

To establish existence of a microscopic solution and for deriving the macroscopic model, we transform the problem onto a substitute domain which is periodic and constant with respect to time. Since the evolution of the domain is coupled with the solution of the system, the transformation mapping depends on the unknowns of the model and, thus, the resulting transformed equations become highly non-linear with coefficients depending on the transformation. To show the existence and uniqueness of a weak solution of the microscopic model (on the fixed domain Ω_ε) we use Schäfer's fixed-point theorem, where we construct the fixed-point operator as the composition of the following three operators. The first provides the domain evolution for given concentration (solves the ODE for R_ε), the second provides a solution of the Stokes equation for given domain evolution, and the third operator gives the solution of the nonlinear advection–reaction–diffusion equation for given fluid flow and domain evolution. In order to obtain the continuity of these operators, a very subtle choice for the corresponding spaces is required. Moreover, the uniqueness of the solution for the full microscopic problem is obtained by energy methods, which require additional Lipschitz continuity for the operators.

To pass to the limit $\varepsilon \to 0$ and derive the macroscopic model, we use the concept of two-scale convergence, respectively the unfolding method. In order to deal with the nonlinear terms, we need the strong two-scale convergence. In particular, we require such strong compactness results for the concentration u_ε and the nonlinear coefficients arising from the domain transformation. These coefficients depend on the transformation mapping, and since the evolution of the microscopic domain can be described by the radii R_ε, the strong two-scale convergence of the coefficients can be reduced to the strong (two-scale) convergence of R_ε. Let us point out a crucial difficulty in the derivation of the strong two-scale convergence of u_ε. Due to the nonlinear structure of the problem, we are not able to obtain ε-uniform a priori bounds for the time derivative $\partial_t u_\varepsilon$, which together with an L^2-energy bound for u_ε and its gradient would guarantee the strong two-scale convergence of u_ε, see Ref. 28, 46. In our case, we only obtain an ε-uniform bound for $\partial_t(J_\varepsilon u_\varepsilon)$, where J_ε denotes the Jacobi-determinant of the transformation mapping. Now, for the strong convergence of u_ε, the idea is to use a Kolmogorov–Simon-type compactness result, for which we have to control shifts with respect to space and time, and for the latter we use the bound on $\partial_t(J_\varepsilon u_\varepsilon)$. Such a result was already obtained in Ref. 31. In this paper, we present a similar strong two-scale compactness result with a simplified proof and under weaker a priori bounds. We point out that in Ref. 72 the strong convergence of u_ε for the pure reaction–diffusion was obtained by showing an estimate for the time-shifts directly from the weak variational equation. We emphasise that, in contrast to the results in Ref. 31, 72, due to the additional advective term in our microscopic problem, we need an ε-uniform essential bound for the concentration u_ε in order to control the time derivative $\partial_t(J_\varepsilon u_\varepsilon)$. In view of the application, this essential boundedness is natural.

For the strong convergence of the radii R_ε, the standard Kolmogorov compactness theorem is applied. For this, we use the ε-uniform energy bounds for u_ε and its gradient, and that R_ε depends Lipschitz-continuously on u_ε. Such a result (for the same a priori estimates) was also obtained in Ref. 31, but under slightly stronger assumptions on the initial value for R_ε. In Ref. 72, the strong convergence was obtained by comparing the microscopic ODE for R_ε with the (expected) macroscopic ODE. For this, the strong convergence of u_ε was necessary. We emphasise that, in our approach, this strong convergence is necessary only to identify the limit problem for the macroscopic radii, and not to obtain the strong convergence of R_ε. With the strong two-scale convergence of the coefficients and of the concentration u_ε, and the weak two-scale convergence of the fluid velocity v_ε, we are able to pass to the limit in the microscopic reaction–advection–diffusion equation. Finally, to obtain the full macromodel, we have to derive Darcy's law for the Stokes system with the pulled-back coefficients. For the latter, again using the strong convergence of R_ε, we obtain the strong two-scale convergence, and therefore we can pass to the limit $\varepsilon \to 0$. We mention that a similar homogenisation result was obtained in Ref. 73, but for an a priori known evolving microstructure. However, in the present paper, we use a slightly different technique. In fact, for the derivation of the so-called two-pressure Stokes problem (with pulled-back coefficients), we use a two-pressure decomposition as in Ref. 5. Further, we first formulate the Darcy law with its cell problems on the fixed reference elements, and finally transform it back to the evolving reference element.

The resulting effective equations consist of a homogenised advection–reaction–diffusion equation describing the species transport, an ODE (depending on the macroscopic variable in the sense of a parameter) describing the local reference cell evolution and a Darcy law for evolving microstructure. The porosity as well as the effective diffusivity and permeability can be obtained by means of the solutions of cell problems, which take the local microstructure into account. Since we perform the homogenisation on the fixed microscopic domain Ω_ε with the pulled-back coefficients depending on the chosen transformation, we obtain, in a first step, cell problems on a fixed reference cell with coefficients depending on a local transformation. By transforming these equations to the actual local space- and time-dependent cell domain, we obtain a completely transformation-independent homogenisation result. In particular, the cell problems and the effective coefficients depend on the macroscopic quantities via the local cell geometry. This leads to a strongly coupled micro–macro system.

The paper is organised as follows: In Section 2, we present the microscopic model on the evolving domain $\Omega_\varepsilon(t)$ and transform it onto the fixed substitute domain Ω_ε. In Section 3, we formulate the macroscopic model and state the main results of the paper. The proof of existence and uniqueness of a microscopic solution are given in Section 4. Afterwards, we derive in Section 5 the ε-uniform a priori estimates, which form the basis for the compactness results necessary to pass to the limit $\varepsilon \to 0$. These compactness results are given in Section 6, where we also derive the macroscopic model with the effective coefficients depending on cell problems formulated on fixed reference elements and with coefficients depending on the transformation. Moreover, we transform the effective diffusivity, Darcy velocity and permeability on the space- and time-dependent cells, which yields a transformation-independent homogenisation result. In Appendix A, we discuss the properties of the transformation and, in Appendix B, we state some further technical auxiliary results. In Appendix C, we give a brief introduction to two-scale convergence and to the unfolding operator, and state some basic properties.

Basic notations Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}^{m\times l}$ for $n, m, l \in \mathbb{N}$. We denote the derivative of f by ∂f. Specifically for $l = 1$ ($f\colon \Omega \to \mathbb{R}^m$), we identify ∂f with its Jacobian matrix given by $(\partial f(x))_{ij} := \partial_{x_j} f_i(x) = \partial_j f_i(x)$ for $i = 1, \ldots, n$, and $j = 1, \ldots, m$. Further, we define the gradient of $f\colon \Omega \to \mathbb{R}^m$ as the transpose of the Jacobian matrix, $\nabla f := \partial f^\top$. For a matrix-valued function $A\colon \Omega \to \mathbb{R}^{n\times n}$, we define its divergence $\nabla\cdot A\colon \Omega\to\mathbb{R}^n$ for $i = 1,\dots,n$ by

$$[\nabla\cdot A]_i := \sum_{j=1}^n \partial_j A_{ji}.$$

Thus, the operator $\nabla\cdot$ applies to the columns of A. In particular, for a vector field $v : \Omega \to \mathbb{R}^n$, we define the Laplace operator by $\Delta v := \nabla\cdot(\nabla v) = (\nabla\cdot(\nabla v_i))_{i=1}^n$. With $\phi : \Omega \to \mathbb{R}$, we have the following identities which will be used frequently throughout the paper:

$$\nabla\cdot(Av) = [\nabla\cdot A]\cdot v + A : \nabla v, \tag{1.1}$$

$$\nabla\cdot(\phi A) = \phi\nabla\cdot A + A^\top\nabla\phi, \tag{1.2}$$

with the Frobenius product $B : C := \mathrm{tr}(B^\top C) = \sum_{i,j=1}^n B_{ij}C_{ij}$ for $B, C \in \mathbb{R}^{n\times n}$.

Now, let $\psi : (0,T)\times\Omega \to \widetilde{\Omega} \subset \mathbb{R}^n$ such that $\psi(t,\cdot)$ is a diffeomorphism from Ω to $\widetilde{\Omega}$. We define its Jacobian determinant $J := \det(\nabla\psi)$ and write $A(x) := J(x)\,(\partial\psi(x))^{-1}$ for $x \in \Omega$. Then, the Piola identity (see Ref. 44)

$$\nabla\cdot A = 0 \tag{1.3}$$

holds. Together with the Jacobi formula, $\partial_t \det B = \mathrm{tr}(\mathrm{adj}(B)\partial_t B)$ for the derivative of the determinant of a matrix-valued function $t \mapsto B(t)$, we get for $V := A\partial_t\psi$

$$\partial_t J = \nabla\cdot V. \tag{1.4}$$

Let $n, d \in \mathbb{N}$, then for $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain, we denote by $L^p(\Omega)^d$, $W^{1,p}(\Omega)^d$ the standard Lebesgue and Sobolev spaces with $p \in [1,\infty]$. In particular, for $p = 2$, we write $H^1(\Omega)^d := W^{1,2}(\Omega)^d$. With S a subset of $\partial\Omega$, we let $H^1_S(\Omega)$ denote the $H^1(\Omega)$ functions vanishing on S (in the sense of traces). For the norms, we omit the upper index d, for example we write $\|\cdot\|_{L^p(\Omega)}$ instead of $\|\cdot\|_{L^p(\Omega)^d}$. For a separable Banach space X and $p \in [1,\infty]$, we denote the usual Bochner spaces by $L^p(\Omega, X)$ or $L^p((0,T), X)$ when time is involved. For the dual space of X, we use the notation X'. Further, we consider the following periodic function spaces. Let $Y := (0,1)^n$ be the unit cell in $\mathbb{R}^n$, then $C^\infty_{\mathrm{per}}(Y)$ is the space of smooth functions on $\mathbb{R}^n$, which are Y-periodic, and $W^{1,p}_{\mathrm{per}}(Y)$ is the closure of $C^\infty_{\mathrm{per}}(Y)$ with respect to the norm on $W^{1,p}(Y)$. Again, we use the notation $H^1_{\mathrm{per}}(Y) := W^{1,2}_{\mathrm{per}}(Y)$. Finally, for a subset $Y^* \subset Y$ with $\partial Y \subset \partial Y^*$, we denote by $H^1_{\mathrm{per}}(Y^*)$ the space of functions from $H^1_{\mathrm{per}}(Y)$ restricted to Y^*.

2. The microscopic model

In this section, we introduce the microscopic model, which is at first formulated on a time-dependent evolving microscopic domain $\Omega_\varepsilon(t)$. More precisely, the transport and fluid equations are formulated in the current configuration (Eulerian coordinates) $\Omega_\varepsilon(t)$. The heterogeneous domain $\Omega_\varepsilon(t)$ is given as a rectangle periodically perforated by solid grains of spherical shape, where the radii of the solids differ and change in time, and are described by $R_\varepsilon(t,x)$. The evolution of the radii depends on the (local) concentration at the surface of the grains. As a first step of our approach, we then transform the model to a fixed (time-independent) domain with periodically distributed solid grains of identical radii (Lagrangian coordinates), which corresponds to the microscopic transformation illustrated on the left-hand side of Figure 1.

2.1. *The microscopic model in the evolving domain*

Let $T > 0$ and, for $n \in \{2,3,4\}$, we consider a rectangle $\Omega = (a,b) \subset \mathbb{R}^n$ with $a, b \in \mathbb{Z}^n$ and $a_i < b_i$ for all $i \in \{1,\dots,n\}$ and we write $[x]$ for the (component-wise) integer part of $x \in \Omega$ and $\{x\} := x - [x]$. Further, let $\varepsilon > 0$ be a sequence tending to zero such that $\varepsilon^{-1} \in \mathbb{N}$. We denote the unit cube by $Y := (0,1)^n$ and we define

$$K_\varepsilon := \{k \in \mathbb{Z}^n \,:\, \varepsilon(k+Y) \subset \Omega\}\,.$$

We describe by R_ε the radii of the solid grains in the evolving microdomain, where the evolution of R_ε is given by the ordinary differential equation (2.2e) below. More precisely, let $R_{\min}$ and $R_{\max}$ be constants such that $0 < R_{\min} < R_{\max} < 0.5$ and let $R_\varepsilon : (0,T) \times \Omega \to [\varepsilon R_{\min}, \varepsilon R_{\max}]$ be constant on every microcell $\varepsilon(Y+k)$ with $k \in K_\varepsilon$. We define the evolving microdomain $\Omega_\varepsilon(t)$ by

$$\Omega_\varepsilon(t) := \Omega \setminus \bigcup_{k \in K_\varepsilon} \overline{B_{R_\varepsilon,k}(t)},$$

where

$$B_{R_\varepsilon,k}(t) := B_{R_\varepsilon(t,x)}(\varepsilon(\mathfrak{m}+k))$$

with $x \in \varepsilon(k+Y)$ and $\mathfrak{m}$ the center of the cube Y. Further, we denote the evolving surface of the solid grains by

$$\Gamma_\varepsilon(t) := \partial\Omega_\varepsilon(t) \setminus \partial\Omega\,, \qquad \Gamma_\varepsilon\,(t,x) := \partial B_{R_\varepsilon,k}(t) \text{ for } x \in \varepsilon(k+Y)\,.$$

The non-cylindrical domains are now given by

$$Q_\varepsilon^T := \bigcup_{t \in (0,T)} \{t\} \times \Omega_\varepsilon(t) \quad \text{and} \quad H_\varepsilon^T := \bigcup_{t \in (0,T)} \{t\} \times \Gamma_\varepsilon(t). \tag{2.1}$$

Finally, we let ν_ε denote the unit normal of $\Gamma_\varepsilon(t)$ pointing out of $\Omega_\varepsilon(t)$.

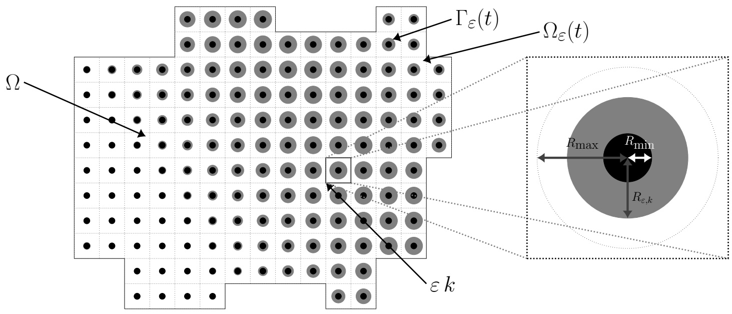

Fig. 2. Visualisation of the ε-scaled geometry $\Omega_\varepsilon(t)$ for some fixed time t. Note that the domain Ω is not a rectangle in this illustration; in the analysis, this assumption is only made to avoid further technical notations in the definition of the microscopic geometry and can be relaxed. The essential assumption is that the whole domain Ω is covered by micro-cells of the form $\varepsilon(Y+k)$.

We consider the following microscopic problem:

Problem P$_\varepsilon$. Find $u_\varepsilon, R_\varepsilon, p_\varepsilon : Q_\varepsilon^T \to \mathbb{R}$ and $v_\varepsilon : Q_\varepsilon^T \to \mathbb{R}^n$ such that u_ε solves the transport equation

$$\partial_t u_\varepsilon - \nabla\cdot\left(D\nabla u_\varepsilon - v_\varepsilon u_\varepsilon\right) = f(u_\varepsilon) \qquad \text{in } Q_\varepsilon^T, \tag{2.2a}$$

$$-(D\nabla u_\varepsilon - v_\varepsilon u_\varepsilon)\cdot\nu_\varepsilon = -\partial_t R_\varepsilon u_\varepsilon + \rho g(u_\varepsilon, \varepsilon^{-1}R_\varepsilon) \qquad \text{on } H_\varepsilon^T, \tag{2.2b}$$

$$u_\varepsilon = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{2.2c}$$

$$u_\varepsilon(0) = u_\varepsilon^{\text{in}} \qquad \text{in } \Omega_\varepsilon(0), \tag{2.2d}$$

the radii R_ε of the balls in the microcells solve the ODE

$$\partial_t R_\varepsilon = \varepsilon \fint_{\Gamma_\varepsilon\left(t,\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)} g\left(u_\varepsilon(t,z), \frac{R_\varepsilon(t,z)}{\varepsilon}\right) d\sigma_z \qquad \text{in } Q_\varepsilon^T, \tag{2.2e}$$

$$R_\varepsilon(0) = R_\varepsilon^{\text{in}} \qquad \text{in } \Omega_\varepsilon(0), \tag{2.2f}$$

and the fluid velocity v_ε and the fluid pressure p_ε are solutions of the Stokes problem

$$-\varepsilon^2\,\nabla\cdot(e(v_\varepsilon)) + \nabla p_\varepsilon = h_\varepsilon \qquad \text{in } Q_\varepsilon^T, \tag{2.2g}$$

$$\nabla\cdot v_\varepsilon = 0 \qquad \text{in } Q_\varepsilon^T, \tag{2.2h}$$

$$v_\varepsilon = -\partial_t R_\varepsilon \nu_\varepsilon \qquad \text{on } H_\varepsilon^T \tag{2.2i}$$

$$\left(-\varepsilon^2 e(v_\varepsilon) + p_\varepsilon I\right)\nu = p_{\text{b}}\nu \qquad \text{on } (0,T)\times\partial\Omega, \tag{2.2j}$$

with the symmetric gradient

$$e(v_\varepsilon) := \frac{1}{2}\left(\nabla v_\varepsilon + \nabla v_\varepsilon^T\right).$$

This system describes advection, reaction and diffusion of the concentration u_ε in the pore fluid having velocity v_ε and pressure p_ε, where f describes bulk reaction rate and g the adsorption/desorption rate at the mineral layer surface. The evolution of the microstructure is encoded in the evolution of the solid grain radii R_ε. Moreover, initial data are denoted by a superscript "in" and p_b is the exterior pressure. Details on the data, the corresponding assumptions and their implications are discussed at the end of Section 2.3, in Assumptions (A1) – (A7) in particular. E.g. Assumption (A1) on g ensures that clogging of the pores cannot occur.

Observe that the speed $|v_\varepsilon|$ in (2.2i) equals $|\partial_t R_\varepsilon|$, which is analogous to assuming that the molar density of the solid mineral layer is negligible when dissolved in the fluid phase. In particular, the free boundary and the fluid move with the same velocity. We refer to Ref. 50, 67 for details. Further, by assuming that the growth is radially symmetric, mass conservation only holds in an averaged sense. More precisely, the mass is conserved at the level of every cell but not pointwisely, as $\partial_t R_\varepsilon$ is defined in (2.2e) as the average of the total effect of the surface reaction over the grain. Therefore, (2.2b) and (2.2e) can be interpreted as a cell-averaged Rankine–Hugoniot condition.

Furthermore, note that we have chosen the viscosity in the Stokes problem equal to $\frac{1}{2}$ for sake of shorter notation. Since the model considered here is dimensionless, this is not a restriction but the result of the choice of the scaling. We emphasise that the physically relevant cases are $n = 2$ and $n = 3$. However, since our results are also valid for $n = 4$, we also include this case. In fact, from a mathematical point of view, the advective term in the weak formulation turns out to be well-defined owing the continuous embedding from H^1 into L^4 for $n \leqslant 4$.

2.2. *Microscopic problem on the fixed domain*

We reformulate **Problem $\mathbf{P}_\varepsilon$** on an in-time cylindrical and ε-periodic domain by transforming (2.2) via a family of diffeomorphisms $\psi_\varepsilon(t,\cdot) : \Omega_\varepsilon \to \Omega_\varepsilon(t)$ for a periodic reference domain Ω_ε. The domain Ω_ε is given by a reference radius, which we choose to be equal $R_{\max}$, via

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{k\in K_\varepsilon} \overline{B_{\varepsilon R_{\max}}(\varepsilon(\mathfrak{m}+k)}. \tag{2.3}$$

We construct ψ_ε explicitly by the corresponding displacement mapping $\breve{\psi}_\varepsilon(t,x) = \psi_\varepsilon(t,x) - x$. Therefore, we construct the following generic displacement in the reference cell similarly to Ref. 31: let $\chi \in C^\infty([0,\infty);[0,1])$ be monotonically decreasing with $\chi(s) = 1$ for $s \leqslant R_{\max}$ and $\chi(s) = 0$ for $s \geqslant (R_{\max} + 0.5)/2$, then

$$\breve{\psi}(R;y) := (R - R_{\max})\tfrac{y-\mathfrak{m}}{\|y-\mathfrak{m}\|}\chi(\|y-\mathfrak{m}\|).$$

Then, the mapping

$$y \mapsto \psi(R;y) := y + \breve{\psi}(R;y) = y + (R - R_{\max})\tfrac{y-\mathfrak{m}}{\|y-\mathfrak{m}\|}\chi(\|y-\mathfrak{m}\|), \tag{2.4}$$

is a diffeomorphism from $Y^*_{R_{\max}}$ onto Y^*_R for every $R \in [R_{\min}, R_{\max}]$, where $Y^*_R := (0,1)^n \setminus \overline{B_{R_{\max}}(\mathfrak{m})}$ for $R \in [R_{\min}, R_{\max}]$ and $Y^* := Y^*_{R_{\max}}$.

By ε-scaling the generic displacement, we obtain

$$\breve{\psi_\varepsilon}(t,x) := \varepsilon\breve{\psi}\left(\varepsilon^{-1}R_\varepsilon(t,x); \{\tfrac{x}{\varepsilon}\}\right) \tag{2.5}$$

for $R_\varepsilon(t,\cdot)$ constant on $\varepsilon(k + Y^*)$. We note that $\breve{\psi}$ is zero in a neighbourhood of the boundary of Y and, thus, $\breve{\psi_\varepsilon}$ is also smooth at the cell interfaces $\varepsilon(k + \partial Y)$ for $k \in K_\varepsilon$, even though $R_\varepsilon(t,x)$ and $\{\frac{x}{\varepsilon}\}$ contain jumps at the interface of the cells. From the ε-scaled displacements, we obtain the ε-scaled diffeomorphisms

$$\psi_\varepsilon(t,x) = x + \varepsilon\breve{\psi}\left(\varepsilon^{-1}R_\varepsilon(t,x); \{\tfrac{x}{\varepsilon}\}\right). \tag{2.6}$$

We use the following notation for the Jacobians of ψ_ε, its adjugate and the transformed diffusion coefficient:

$$F_\varepsilon = \partial\psi_\varepsilon = \nabla\psi_\varepsilon^\top, \tag{2.7}$$
$$J_\varepsilon = \det(F_\varepsilon), \tag{2.8}$$
$$A_\varepsilon = J_\varepsilon F_\varepsilon^{-1}, \tag{2.9}$$
$$D_\varepsilon = J_\varepsilon F_\varepsilon^{-1} D F_\varepsilon^{-T}. \tag{2.10}$$

Having such a well-posed mapping between the radii R_ε and the domain transformation ψ_ε at hand, the microscopic model in the fixed domain reads as follows:
Reaction–diffusion–advection equation:

$$\begin{aligned}
\partial_t(J_\varepsilon u_\varepsilon) - \nabla\cdot(D_\varepsilon\nabla u_\varepsilon - u_\varepsilon A_\varepsilon(v_\varepsilon - \partial_t\psi_\varepsilon)) &= J_\varepsilon f(u_\varepsilon) && \text{in } (0,T)\times\Omega_\varepsilon,\\
-(D_\varepsilon\nabla u_\varepsilon - u_\varepsilon A_\varepsilon(v_\varepsilon - \partial_t\psi_\varepsilon))\cdot\nu &= J_\varepsilon\rho g(u_\varepsilon, \varepsilon^{-1}R_\varepsilon) && \text{on } (0,T)\times\Gamma_\varepsilon,\\
u_\varepsilon &= 0 && \text{on } (0,T)\times\partial\Omega,\\
u_\varepsilon(0) &= u_\varepsilon^{\text{in}} && \text{in } \Omega_\varepsilon.
\end{aligned} \tag{2.11a}$$

Evolution of the radii:

$$\partial_t R_\varepsilon = \varepsilon \fint_{\Gamma_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)} g\left(u_\varepsilon(t,z), \frac{R_\varepsilon(t,z)}{\varepsilon}\right) \mathrm{d}\sigma_z \qquad \text{in } (0,T)\times\Omega_\varepsilon, \tag{2.11b}$$
$$R_\varepsilon(0) = R_\varepsilon^{\text{in}} \qquad \text{in } \Omega_\varepsilon, \tag{2.11c}$$

with $\Gamma_\varepsilon(z) := \varepsilon\Gamma + z$ for $z \in \mathbb{R}^n$.

Stokes equations:

$$\begin{aligned}
-\varepsilon^2\,\nabla\cdot(A_\varepsilon e_\varepsilon(v_\varepsilon)) + A_\varepsilon^\top\nabla p_\varepsilon &= J_\varepsilon h_\varepsilon && \text{in } (0,T)\times\Omega_\varepsilon,\\
\nabla\cdot(A_\varepsilon v_\varepsilon) &= 0 && \text{in } (0,T)\times\Omega_\varepsilon,\\
v_\varepsilon &= -\partial_t R_\varepsilon\nu && \text{on } (0,T)\times\Gamma_\varepsilon,\\
\left(-\varepsilon^2 e_\varepsilon(v_\varepsilon) + p_\varepsilon I\right) A_\varepsilon^\top\nu &= p_{\mathrm{b}} A_\varepsilon^\top\nu && \text{on } (0,T)\times\partial\Omega,
\end{aligned} \tag{2.11d}$$

where

$$e_\varepsilon(v) := \frac{1}{2}\left(F_\varepsilon^{-\top}\nabla v + \nabla v^\top F_\varepsilon^{-1}\right). \tag{2.12}$$

Remark 2.1.

(i) With the Piola identity (1.3), the divergence equation for the Stokes system can be written as

$$A_\varepsilon : \nabla v_\varepsilon = \nabla\cdot(A_\varepsilon v_\varepsilon) = 0.$$

(ii) The boundary condition for the fluid velocity v_ε on Γ_ε can be expressed as

$$-\partial_t R_\varepsilon\nu = \partial_t\psi_\varepsilon \qquad \text{on } (0,T)\times\Gamma_\varepsilon.$$

(iii) Since $\psi_\varepsilon(t,x) = x$ in a neighbourhood of $\partial\Omega$, we obtain for the stress boundary condition for the fluid

$$\left(-\varepsilon^2 e(v_\varepsilon) + p_\varepsilon I\right)\nu = p_{\mathrm{b}}\nu \qquad \text{on } \partial\Omega.$$

However, in what follows, we also use the transformed stress boundary condition including ψ_ε, which also covers more general geometries. For example, in Ref. 73 an a priori known evolving microscopic domain was considered with perforations intersecting the outer boundary $\partial\Omega$. In this case, one has to consider $p_{\mathrm{b}}\circ\psi_\varepsilon$.

(iv) From (1.4), we obtain

$$\nabla\cdot(A_\varepsilon w_\varepsilon) = -\nabla\cdot(A_\varepsilon\partial_t\psi_\varepsilon) = -\partial_t J_\varepsilon.$$

In order to formulate the Stokes equations as a problem with a no-slip condition on Γ_ε, we subtract the Dirichlet boundary values $-\partial_t R_\varepsilon \nu = \partial_t \psi_\varepsilon$ from the solution v_ε. More precisely, we consider $w_\varepsilon := v_\varepsilon - \partial_t \psi_\varepsilon$, and further subtract the normal stress boundary values from the pressure p_ε. With $q_\varepsilon := p_\varepsilon - p_{\mathrm{b}}$ we get the following Stokes system (we use again $\psi_\varepsilon(t,x) = x$ in a neighborhood of $\partial\Omega$):

$$\begin{aligned}
-\varepsilon^2 \nabla\cdot(A_\varepsilon e_\varepsilon(w_\varepsilon)) + A_\varepsilon^\top \nabla q_\varepsilon &= J_\varepsilon h_\varepsilon + \varepsilon^2 \nabla\cdot(A_\varepsilon e_\varepsilon(\partial_t \psi_\varepsilon)) - A_\varepsilon^\top \nabla p_{\mathrm{b}} && \text{in } (0,T)\times\Omega_\varepsilon,\\
\nabla\cdot(A_\varepsilon w_\varepsilon) &= -\nabla\cdot(A_\varepsilon \partial_t \psi_\varepsilon) && \text{in } (0,T)\times\Omega_\varepsilon,\\
w_\varepsilon &= 0 && \text{on } (0,T)\times\Gamma_\varepsilon,\\
\left(-\varepsilon^2 e_\varepsilon(w_\varepsilon) + q_\varepsilon I\right) A_\varepsilon^\top \nu &= 0 && \text{on } (0,T)\times\partial\Omega.
\end{aligned} \tag{2.13}$$

2.3. *The weak formulation*

The following weak form of (2.11) with subtracted Dirichlet boundary values for the velocity is derived classically by multiplication with test functions and integration (by parts):

Definition 2.1. We call $(u_\varepsilon, R_\varepsilon, w_\varepsilon, q_\varepsilon)$ a weak solution of the micromodel (2.11a)–(2.11c) and (2.13) if

$$\begin{aligned}
&u_\varepsilon \in L^2((0,T), H^1_{\partial\Omega}(\Omega_\varepsilon)) \quad \text{with } \partial_t(J_\varepsilon u_\varepsilon) \in L^2((0,T), H^1_{\partial\Omega}(\Omega_\varepsilon)'),\\
&R_\varepsilon \in W^{1,\infty}((0,T), L^2(\Omega_\varepsilon)) \cap L^\infty((0,T)\times\Omega_\varepsilon) \quad \text{with } \partial_t R_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon),\\
&w_\varepsilon \in L^2((0,T), H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n),\\
&q_\varepsilon \in L^2((0,T), L^2(\Omega_\varepsilon)),
\end{aligned}$$

such that, for every $\phi_\varepsilon \in H^1_{\partial\Omega}(\Omega_\varepsilon)$, there holds almost everywhere in $(0,T)$

$$\begin{aligned}
\langle \partial_t(J_\varepsilon u_\varepsilon), \phi_\varepsilon\rangle_{H^1_{\partial\Omega}(\Omega_\varepsilon)', H^1_{\partial\Omega}(\Omega_\varepsilon)} &+ \int_{\Omega_\varepsilon} [D_\varepsilon \nabla u_\varepsilon - u_\varepsilon A_\varepsilon w_\varepsilon]\cdot\nabla\phi_\varepsilon \,\mathrm{d}x\\
&= \int_{\Omega_\varepsilon} J_\varepsilon f(u_\varepsilon)\phi_\varepsilon dx - \rho\int_{\Gamma_\varepsilon} J_\varepsilon g(u_\varepsilon, \varepsilon^{-1}R_\varepsilon)\phi_\varepsilon \,\mathrm{d}\sigma
\end{aligned} \tag{2.14a}$$

and there holds almost everywhere in $(0,T)$ that

$$\partial_t R_\varepsilon = \varepsilon \fint_{\Gamma_\varepsilon(\varepsilon[\frac{x}{\varepsilon}]_Y)} g\left(u_\varepsilon(t,z), \frac{R_\varepsilon(t,z)}{\varepsilon}\right) \mathrm{d}\sigma_z \text{ in } \quad (0,T)\times\Omega_\varepsilon. \tag{2.14b}$$

Moreover, for all $\eta_\varepsilon \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n$, there holds almost everywhere in $(0,T)$

$$\begin{aligned}
\varepsilon^2 \int_{\Omega_\varepsilon} J_\varepsilon e_\varepsilon(w_\varepsilon) : e_\varepsilon(\eta_\varepsilon)\,\mathrm{d}x &- \int_{\Omega_\varepsilon} q_\varepsilon \nabla\cdot(A_\varepsilon\eta_\varepsilon)\,\mathrm{d}x\\
= -\varepsilon^2 \int_{\Omega_\varepsilon} J_\varepsilon e_\varepsilon(\partial_t\psi_\varepsilon) : e_\varepsilon(\eta_\varepsilon)\,\mathrm{d}x &+ \int_{\Omega_\varepsilon} J_\varepsilon h_\varepsilon\cdot\eta_\varepsilon\,\mathrm{d}x - \int_{\Omega_\varepsilon} A_\varepsilon^\top \nabla p_{\mathrm{b}}\cdot\eta_\varepsilon\,\mathrm{d}\sigma,
\end{aligned} \tag{2.14c}$$

and, almost everywhere in $(0,T)\times\Omega_\varepsilon$, we have

$$\nabla\cdot(A_\varepsilon w_\varepsilon) = -\partial_t J_\varepsilon. \tag{2.14d}$$

Further, we require the initial condition $(u_\varepsilon, R_\varepsilon)(0) = (u_\varepsilon^{\mathrm{in}}, R_\varepsilon^{\mathrm{in}})$.

Remark 2.2.

(i) The initial condition for u_ε makes sense since the regularity of u_ε, J_ε, and $\partial_t(J_\varepsilon u_\varepsilon)$ imply $u_\varepsilon \in C^0([0,T], L^2(\Omega_\varepsilon))$. In fact, as a consequence of the product rule, we obtain $\partial_t u_\varepsilon \in L^2((0,T), H^1_{\partial\Omega}(\Omega_\varepsilon))$, which implies the continuity of u_ε as a mapping from $[0,T]$ to $L^2(\Omega_\varepsilon)$. However, in the context of homogenisation, it is typically not appropriate to work with $\partial_t u_\varepsilon$ since this function usually does not fulfill a uniform *a priori* bound with respect to ε.

(ii) From the transformation of surface integrals, see for example Theorem 1.7-1 in Ref. 15, we first obtain a boundary term of the form

$$-\rho\int_{\Gamma_\varepsilon} J_\varepsilon |\nabla\psi_\varepsilon^{-T}\nu| g(u_\varepsilon, \varepsilon^{-1}R_\varepsilon)\phi\,\mathrm{d}\sigma$$

in (2.14a). However, in our case, we have $|\nabla\psi_\varepsilon^{-T}\nu| = 1$ owing to the choice of the transformation in (2.6). For more details, we refer to Appendix A of Ref. 31.

(iii) For the derivation of *a priori* estimates for the microscopic solutions, it might be helpful to consider a time-integrated version of the transport equation (2.14a) and choose time-dependent test functions $\phi_\varepsilon(t,x)$. However, under the regularity assumptions in Definition 2.1, u_ε is not an admissible test function since the advective term is not well-defined. In fact, the Hölder inequality is not applicable owing to low regularity with respect to the time variable. In the existence proof, where we work with the Galerkin method and, therefore, with more regular approximations, this causes no problems since the advective term cancels out after an integration by parts (where we have to use the boundary and divergence conditions for u_ε and w_ε), see equation (5.5) in the proof of the *a priori* estimates.

(iv) To keep the notation of the weak solution simpler, we only require enough regularity such that the variational equations are well-defined (pointwisely in time). We will show, see Lemma 5.1 and 5.2, that the microscopic solution is in fact more regular. More precisely, we additionally have

$$\begin{aligned} u_\varepsilon &\in L^\infty((0,T)\times\Omega_\varepsilon),\\ w_\varepsilon &\in L^\infty((0,T),H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n),\\ q_\varepsilon &\in L^\infty((0,T),L^2(\Omega_\varepsilon)). \end{aligned}$$

(v) Note that the transformation of Problem P_ε would yield $h_\varepsilon(t,x) := h(t,\psi_\varepsilon(t,x))$. For sake of simplicity, we assume that h_ε is a given sequence and we neglect its dependence on the transformation ψ_ε. We also assume that f_ε does not depend explicitly on t and x. In this, way we can keep the notation simpler and the focus on the relevant aspects of the paper. Of course, more general assumptions are possible and the following analysis also holds for $h_\varepsilon(t,x) := h(t,\psi_\varepsilon(t,x))$, where $h \in L^\infty((0,T),C^{0,1}(\Omega))$. Thereby, the Lipschitz regularity with respect to Ω is only necessary for the proof of Lemma 4.4. The two-scale convergence of $h(t,\psi_\varepsilon(t,x)) \to h(t,x)$ was shown in Theorem 3.8 of Ref. 71. One can argue analogously for f_ε.

In what follows, we summarise the assumptions on the data, for which we use the notion of two-scale convergence, the definition of which can be found in Section Appendix C of the appendix.

Assumptions on the data:

(A1) We assume that $g \in C^{0,1}(\mathbb{R}^2)$ is bounded. More precisely, there exists a constant $C_g > 0$ such that

$$|g| \leqslant C_g, \tag{2.15}$$

and a constant $L_g \in \mathbb{R}$ such that for every $u_1, u_2, r_1, r_2 \in \mathbb{R}$ the Lipschitz estimate

$$|g(u_1,r_1) - g(u_2,r_2)| \leqslant L_g(|u_1-u_2| + |r_1-r_2|) \tag{2.16}$$

holds. Moreover, we assume that

$$g(u,r) \geqslant 0 \text{ if } r \leqslant R_{\min}, \tag{2.17}$$

$$g(u,r) \leqslant 0 \text{ if } r \geqslant R_{\max} \tag{2.18}$$

for $u \in \mathbb{R}$ in order to ensure that the radii do not become too small or too large, i.e. the obstacles do not vanish or touch each other (or the boundary of Y).

(A2) For the microscopic initial data $u_\varepsilon^{\mathrm{in}} \in L^\infty(\Omega_\varepsilon)$, a $C > 0$ exists such that

$$\|u_\varepsilon^{\mathrm{in}}\|_{L^\infty(\Omega_\varepsilon)} \leqslant C$$

uniformly with respect to ε. Further, there exists $u^{\mathrm{in}} \in L^2(\Omega)$ such that $u_\varepsilon^{\mathrm{in}} \xrightarrow{2,2} u^{\mathrm{in}}$.

(A3) The initial radii $R_\varepsilon^{\mathrm{in}} \in L^\infty(\Omega)$ are constant on every cell $\varepsilon(Y+\mathbf{k})$, $\mathbf{k} \in K_\varepsilon$. Furthermore, there exist $R_{\min}, R_{\max} \in \mathbb{R}$, $0 < R_{\min} < R_{\max} < \frac{1}{2}$, such that almost everywhere in Ω

$$\varepsilon R_{\min} \leqslant R_\varepsilon^{\mathrm{in}} \leqslant \varepsilon R_{\max}.$$

We also assume that

$$\varepsilon^{-1}R_\varepsilon^{\mathrm{in}} \to R_0^{\mathrm{in}} \qquad \text{in } L^2(\Omega).$$

(A4) For the reaction rate in the transport equation, we assume $f \in C^{0,1}(\mathbb{R})$. In particular, there holds the following growth condition:

$$|f(u)| \leqslant C(1+|u|) \tag{2.19}$$

for a constant $C > 0$ and for all $u \in \mathbb{R}$.

(A5) For the force term in the Stokes equations, we assume $h_\varepsilon \in L^\infty((0,T), L^2(\Omega_\varepsilon))^n$ and $h_\varepsilon \xrightarrow{2,p,2} h_0$ with $h_0 \in L^\infty((0,T), L^2(\Omega))^n$ for all $p \in [1,\infty)$.

(A6) For the external pressure, we assume $p_\mathrm{b} \in L^\infty((0,T), H^{\frac{1}{2}}(\partial\Omega))$. In particular, there exists an extension (for which we use the same symbol) $p_\mathrm{b} \in L^\infty((0,T), H^1(\Omega))$.

(A7) The diffusion coefficient $D \in \mathbb{R}^{n\times n}$ is symmetric and positive definite.

Typically, the rates used in the modelling of adsorption/desorption processes are Lipschitz continuous and bounded, which is motivated by (multi-) enzyme kinetics like the Michaelis–Menten kinetics. The assumption in (2.17) is natural when considering salt dissolution and precipitation or biofilm growth in a porous medium. For example, in the former case (2.17) is analogous to saying that dissolution cannot happen in the absence of a salt layer along the grains (see e.g. Ref. 66, 67). Furthermore, (2.18) can be justified if additional effects are taken into account. Referring to the same application, if the salt layers along two neighbouring grains grow too close to each other, this narrowing of the space occupied by the fluid typically gives rise to very high velocities and, consequently, to high shearing stresses at the salt boundaries, as well as to an increased local fluid pressure. The former can lead to salt detachment, while the latter to increased dissolution effects. However, such effects are not included explicitly in our model but are implicitly taken into account by (2.18), which, from a mathematical point of view, we use to guarantee the global existence in time. Geometrically, (2.17) and (2.18) allow the microscopic geometry to evolve locally without changing its topology.

3. Main results and macroscopic model

Let us summarise the main results of the paper in two theorems, which will be proved in the subsequent sections. The first is the existence and uniqueness of a weak microscopic solution of the problem (2.11) with subtracted Dirichlet boundary values for the velocity:

Theorem 3.1. *There exists a unique weak solution of the microscopic problem* (2.11a)–(2.11c) *and* (2.13) *in the sense of Definition 2.1.*

In a second step, we show suitable compactness results for the microscopic solutions and pass to the limit $\varepsilon \to 0$. We show that the limit functions are a weak solution of a macroscopic model with homogenised coefficients defined as follows. The triple (u_0, R_0, p_0) with the macroscopic concentration $u_0 : (0,T)\times\Omega \to \mathbb{R}$, the macroscopic function of local radii $R_0 : (0,T)\times\Omega \to [R_{\min}, R_{\max}]$, and the Darcy pressure $p_0 : (0,T)\times\Omega \to \mathbb{R}$ solves the following problem: We define the evolving reference element and surface

$$Y^*(t,x) := Y^*_{R_0(t,x)} = Y \backslash \overline{B_{R_0(t,x)}(\mathfrak{m})}, \qquad \Gamma(t,x) := \partial B_{R_0(t,x)}(\mathfrak{m}), \tag{3.1}$$

which allows to define the porosity θ via

$$\theta(t,x) := |Y^*(t,x)|. \tag{3.2}$$

Between the porosity θ and the specific surface $\Gamma(t,x)$, we have the relation $\partial_t\theta(t,x) = |\Gamma(t,x)|\partial_t R_0$. The transport equation describing the evolution of u_0 is given by

$$\partial_t(\theta u_0) - \nabla\cdot(D^*\nabla u_0 - u_0 v^*) = \theta f(u_0) + \partial_t\theta\rho \qquad \text{in } (0,T)\times\Omega, \tag{3.3a}$$

$$u_0 = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{3.3b}$$

$$(\theta u_0)(0) = \theta^{\mathrm{in}} u^{\mathrm{in}} \qquad \text{in } \Omega, \tag{3.3c}$$

with the homogenised diffusion coefficient $D^* : (0,T) \times \Omega \to \mathbb{R}^{n \times n}$ defined in (6.32) respectively (6.36), and the Darcy velocity $v^* : (0,T) \times \Omega \to \mathbb{R}^n$ given in (3.3h). The evolution of the local radii in every macroscopic point, R_0, satisfies

$$\partial_t R_0 = g(u_0, R_0) \qquad \text{in } (0,T) \times \Omega, \tag{3.3d}$$

$$R_0(0) = R^{\text{in}} \qquad \text{in } \Omega, \tag{3.3e}$$

and the Darcy pressure $p_0 : (0,T) \times \Omega \to \mathbb{R}$ is given by

$$-\nabla \cdot (K^*(h_0 - \nabla p_0)) = -\partial_t \theta \qquad \text{in } (0,T) \times \Omega, \tag{3.3f}$$

$$p_0 = p_{\text{b}} \qquad \text{on } (0,T) \times \partial\Omega, \tag{3.3g}$$

with the permeability tensor $K^* : (0,T) \times \Omega \to \mathbb{R}^{n \times n}$ defined in (6.28) respectively (6.34). We emphasise that the following relation between the Darcy velocity v^* and the Darcy pressure p_0 holds:

$$v^* = K^*(h_0 - \nabla p_0). \tag{3.3h}$$

We note that, in our special situation where the evolving reference elements are perforated with balls, the homogenised diffusion coefficient D^* and the permeability tensor K^* are in fact scalar-valued, see Lemma Appendix B.2 in the appendix.

A weak solution of the macroscopic model (3.3) is defined as follows:

Definition 3.1. We call (u_0, R_0, p_0) a weak solution of the macroscopic model (3.3) if

$$\begin{aligned} u_0 &\in L^2((0,T), H^1_{\partial\Omega}(\Omega)) \quad \text{ with } \partial_t(\theta u_0) \in L^2((0,T), H^1_0(\Omega)'), \\ R_0 &\in W^{1,\infty}((0,T), L^2(\Omega)) \cap L^\infty((0,T) \times \Omega) \quad \text{ with } \partial_t R_0 \in L^\infty((0,T) \times \Omega), \\ p_0 - p_{\text{b}} &\in L^\infty((0,T), H^1_0(\Omega)), \end{aligned}$$

with $(\theta u_0)(0) = \theta^{\text{in}} u^{\text{in}}$ (see Remark 3.1) for θ given by (3.2) such that the following variational equations are fulfilled almost everywhere in $(0,T)$: For all $\phi \in H^1_0(\Omega)$, we have

$$\langle \partial_t(\theta u_0), \phi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + \int_\Omega [D^* \nabla u_0 - u_0 v^*] \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega (\theta f + \partial_t \theta \rho) \phi \, \mathrm{d}x, \tag{3.4}$$

while for all $\xi \in H^1_0(\Omega)$, we have

$$\int_\Omega K^*(h - \nabla p_0) \cdot \nabla \xi \, \mathrm{d}x = \int_\Omega -\partial_t \theta \xi \, \mathrm{d}x.$$

Furthermore,

$$\partial_t R_0 = g(u_0, R_0) \tag{3.5}$$

almost everywhere in $(0,T)$.

Remark 3.1. We immediately obtain that a weak solution in the sense of Definition 3.1 fulfills $\theta u_0 \in C^0([0,T], L^2(\Omega))$ and, therefore, the initial condition $(\theta u_0)(0) = \theta^{\text{in}} u^{\text{in}}$ makes sense. For $R_0^{\text{in}} \in H^1(\Omega)$, the spatial regularity of u_0 and R_0^{in} can be transfered onto the ODE (2.11b) and we obtain $R_0 \in C([0,T]; H^1(\Omega) \cap L^\infty(\Omega))$ and, thus, $\theta, \theta^{-1} \in C([0,T]; H^1(\Omega) \cap L^\infty(\Omega))$. Moreover, the uniform $L^\infty((0,T) \times \Omega)$-bound of u_ε yields $u_0 \in L^\infty((0,T) \times \Omega)$ and, thus, $\nabla(\theta u_0) = \nabla \theta u_0 + \theta \nabla u_0 \in L^2((0,T) \times \Omega)$. Then, we can conclude $\theta u_0 \in C([0,T]; L^2(\Omega))$ and, thus, $u_0 = \theta^{-1} \theta u_0 \in C([0,T]; L^2(\Omega))$. Then, the initial condition (3.3c) can be replaced by $u_0(0) = u^{\text{in}}$.

The second main result of the paper is the convergence of the microscopic solutions to solutions of the macroscopic model just introduced.

Theorem 3.2. *Let $(u_\varepsilon, w_\varepsilon, p_\varepsilon, R_\varepsilon)$ be the unique weak solution of the microscopic problem* (2.11)*. Then, up to a subsequence, these sequences converge to the limit functions (u_0, w_0, p_0, R_0) in the sense of Proposition 6.2, which is a weak solution of the macroscopic model* (3.3) *in the sense of Definition 3.1.*

4. Existence and uniqueness of the microscopic model

In this section, we show the existence and uniqueness of the weak solution $(u_\varepsilon, w_\varepsilon, p_\varepsilon, R_\varepsilon)$ of the microscopic model (2.14a)–(2.14c). For the existence result, we employ a fixed-point argument. Thereby, we show not only the continuity of the operator, which is necessary for the existence of a solution, but also its Lipschitz continuity. Thus, we can conclude the uniqueness of the fixed point and, thereby, the uniqueness of the solution of (2.14a)–(2.14c). To control the coordinate transformation, we use results from Appendix A and to derive the Lipschitz continuity of the solution of the Stokes equations, we use an abstract result on the continuity of saddle-point problems given in Appendix B.

4.1. *Construction of the fixed-point operator and its properties*

In order to show the existence of a weak solution of the microscopic model (2.11) in the sense of Definition 2.1, we apply a fixed-point argument as follows. For $\beta \in \left(\frac{1}{2}, 1\right)$, we define the operator $\mathcal{F}_1 : L^2((0,T), H^\beta(\Omega_\varepsilon)) \to L^\infty((0,T) \times \Omega_\varepsilon) \cap H^1((0,T), L^\infty(\Omega_\varepsilon))$ by $\mathcal{F}_1(\widehat{u}_\varepsilon) := R_\varepsilon$, where R_ε solves the problem

$$\begin{aligned} \partial_t R_\varepsilon &= \varepsilon \fint_{\Gamma_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)} g\left(\widehat{u}_\varepsilon(t,z), \frac{R_\varepsilon(t,z)}{\varepsilon}\right) \mathrm{d}\sigma_z && \text{in } (0,T) \times \Omega_\varepsilon, && (4.1a)\\ R_\varepsilon(0) &= R_\varepsilon^{\mathrm{in}} && \text{in } \Omega_\varepsilon. && (4.1b)\end{aligned}$$

In Lemma 4.1, we will see that the operator is well-defined. In particular, R_ε is constant on every microcell $\varepsilon(Y^* + k)$ for $k \in K_\varepsilon$ and the assumptions on g ensure that the values of R_ε are in $[R_{\min}, R_{\max}]$ and its time derivative is essentially bounded.

For the linearisation of the Stokes problem, we first introduce the set

$$\begin{aligned} Y_\varepsilon := \big\{ & R_\varepsilon \in L^\infty((0,T) \times \Omega_\varepsilon) \cap H^1((0,T), L^\infty(\Omega_\varepsilon)) \,: \\ & \|\partial_t R_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant \varepsilon C_g, R_\varepsilon|_{\varepsilon(Y^*+k)} \text{ constant for every } k \in K_\varepsilon, \\ & \varepsilon^{-1} R_\varepsilon(t,x) \in [R_{\min}, R_{\max}] \text{ for almost every } (t,x) \in (0,T) \times \Omega_\varepsilon \big\}. \end{aligned} \tag{4.2}$$

Note that solutions of (4.1) are actually elements of Y_ε as shown in Lemma 4.1 below. Now, we define $\mathcal{F}_2 : Y_\varepsilon \to L^2((0,T); H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n)$ as the solution of the linearised Stokes problem, i.e. $\mathcal{F}_2(\widehat{R}_\varepsilon) := w_\varepsilon$, where $(w_\varepsilon, q_\varepsilon) \in L^2((0,T); H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n) \times L^2((0,T); L^2(\Omega_\varepsilon))$ is the unique weak solution of

$$\begin{aligned} -\varepsilon^2 \nabla\cdot(\widehat{A}_\varepsilon \widehat{e}_\varepsilon(w_\varepsilon)) + \widehat{A}_\varepsilon^\top \nabla q_\varepsilon &= \widehat{J}_\varepsilon \widehat{h}_\varepsilon + \varepsilon^2 \nabla\cdot(\widehat{e}_\varepsilon(\partial_t \psi_\varepsilon)) - \widehat{A}_\varepsilon^\top \nabla p_{\mathrm{b}} && \text{in } (0,T) \times \Omega_\varepsilon, \\ \nabla \cdot (\widehat{A}_\varepsilon w_\varepsilon) &= -\nabla \cdot (\widehat{A}_\varepsilon \partial_t \widehat{\psi}_\varepsilon) && \text{in } (0,T) \times \Omega_\varepsilon, \\ w_\varepsilon &= 0 && \text{on } (0,T) \times \Gamma_\varepsilon, \\ \left(-\varepsilon^2 \widehat{e}_\varepsilon(w_\varepsilon) + q_\varepsilon I\right) \widehat{A}_\varepsilon^\top \nu &= 0 && \text{on } (0,T) \times \partial\Omega, \end{aligned} \tag{4.3}$$

where $\widehat{e}_\varepsilon, \widehat{A}_\varepsilon, \widehat{F}_\varepsilon, \widehat{J}_\varepsilon$ and $\widehat{h}_\varepsilon$ are defined in the same way as $e_\varepsilon, A_\varepsilon, F_\varepsilon, J_\varepsilon$ and h_ε but with $\widehat{R}_\varepsilon$ instead of R_ε. We emphasise that the operator is well-defined since $\widehat{R}_\varepsilon$ is constant on every micro-cell and, therefore, the trace of $\partial_t \widehat{R}_\varepsilon$ on Γ_ε exists and is an element of $L^\infty((0,T) \times \Gamma_\varepsilon)$. Moreover, we note that the velocity fields which solve (4.3) are elements of Z_ε, where

$$Z_\varepsilon := \{w_\varepsilon \in L^2((0,T), H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n)) \,:\, \nabla \cdot (A_\varepsilon(w_\varepsilon + \partial_t \psi_\varepsilon)) = 0, \|w_\varepsilon\|_{L^\infty((0,T);H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n)} \leqslant C_h\}, \tag{4.4}$$

where C_h depends on C_g and h_ε. The existence of such an upper bound C_h is given by Lemma 4.3 below.

Finally, we define the operator

$$\mathcal{F}_3 : Y_\varepsilon \times Z_\varepsilon \times L^2((0,T), H^\beta(\Omega_\varepsilon)) \to L^2((0,T), H^\beta(\Omega_\varepsilon))$$

via $\mathcal{F}_3(\widehat{R}_\varepsilon, \widehat{w}_\varepsilon, \widehat{u}_\varepsilon) := u_\varepsilon$, where u_ε is a weak solution of the linearised transport problem

$$\begin{aligned} \partial_t(\widehat{J}_\varepsilon u_\varepsilon) - \nabla \cdot \left(\widehat{D}_\varepsilon \nabla u_\varepsilon - u_\varepsilon \widehat{A}_\varepsilon \widehat{w}_\varepsilon\right) &= \widehat{J}_\varepsilon f(\widehat{u}_\varepsilon) && \text{in } (0,T) \times \Omega_\varepsilon, \\ -(\widehat{D}_\varepsilon \nabla u_\varepsilon - u_\varepsilon \widehat{A}_\varepsilon \widehat{w}_\varepsilon) \cdot \nu &= \rho \widehat{J}_\varepsilon g(\widehat{u}_\varepsilon, \varepsilon^{-1} \widehat{R}_\varepsilon) && \text{on } (0,T) \times \Gamma_\varepsilon, \\ u_\varepsilon &= 0 && \text{on } (0,T) \times \partial\Omega, \\ u_\varepsilon(0) &= u_\varepsilon^{\mathrm{in}} && \text{in } \Omega_\varepsilon, \end{aligned} \tag{4.5}$$

which is analogously defined as in (2.14a). Here, the data $\widehat{J}_\varepsilon$, $\widehat{D}_\varepsilon$ and $\widehat{w}_\varepsilon$ are again defined in the same way as J_ε, D_ε and w_ε with $\widehat{R}_\varepsilon$ instead of R_ε. Now, we are able to define the fixed-point operator for the whole system:

$$\mathcal{F} : L^2((0,T), H^\beta(\Omega_\varepsilon)) \to L^2((0,T), H^\beta(\Omega_\varepsilon)), \qquad \mathcal{F}(\widehat{u}_\varepsilon) \coloneqq \mathcal{F}_3\left(\mathcal{F}_1(\widehat{u}_\varepsilon), \mathcal{F}_2\left(\mathcal{F}_1(\widehat{u}_\varepsilon)\right), \widehat{u}_\varepsilon\right) \,,$$

which can be visualised as follows:

$$\begin{tikzcd}[row sep=small]
 & & Y_\varepsilon & \\
 & & \times & \\
L^2((0,T);H^\beta(\Omega_\varepsilon)) \arrow[r, "\mathcal{F}_1", "\widehat{u}_\varepsilon \longmapsto R_\varepsilon"'] \arrow[ddrr, "\mathrm{id}"'] & Y_\varepsilon \arrow[uur, "\mathrm{id}"] \arrow[r, "\mathcal{F}_2", "\widehat{R}_\varepsilon \longmapsto w_\varepsilon"'] & Z_\varepsilon & \left.\vphantom{\Bigg|}\right\} \xrightarrow[(\widehat{R}_\varepsilon, \widehat{w}_\varepsilon, \widehat{u}_\varepsilon) \mapsto u_\varepsilon]{\mathcal{F}_3} L^2((0,T);H^\beta(\Omega_\varepsilon))\,. \\
 & & \times & \\
 & & L^2((0,T);H^\beta(\Omega_\varepsilon)) &
\end{tikzcd}$$

In what follows, we show that the operators $\mathcal{F}_i$, $i = 1,2,3$, are well-defined (Lemma 4.1, 4.3 and 4.5, respectively), and, thus, also the fixed-point operator $\mathcal{F}$. Moreover, the Lipschitz continuity of $\mathcal{F}_i$, $i = 1,2,3$, is addressed in Lemma 4.2, 4.4 and Remark 4.1, respectively, which allows us to deduce the uniqueness of a fixed point of $\mathcal{F}$ and, thus, the assertion of Proposition 4.2.

Lemma 4.1. *The operator $\mathcal{F}_1$ is well-defined and the range of $\mathcal{F}_1$ is in Y_ε.*

Proof. Let $\widehat{u}_\varepsilon \in L^2((0,T);H^\beta(\Omega_\varepsilon))$. Due to the trace embedding $H^\beta(\Omega_\varepsilon) \hookrightarrow L^2(\Gamma_\varepsilon)$, the right-hand side of (4.1a) is well-defined. Then, for every $x \in \varepsilon(k+Y)$ (with $k \in K_\varepsilon$), the mapping

$$(t,r) \mapsto \varepsilon \fint_{\Gamma_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)} g\left(\widehat{u}_\varepsilon(t,z), \varepsilon^{-1} r\right) \,\mathrm{d}\sigma_z$$

is constant with respect to x, globally Lipschitz continuous with respect to r and measurable with respect to t. Hence, Carathéodory's existence theorem[18] yields the existence and uniqueness of a solution $R_\varepsilon(\cdot_t, x) \in W^{1,1}(0,T)$ of (4.1a)–(4.1b) for every $x \in \Omega_\varepsilon$, and $R_\varepsilon(\cdot_t, x)$ is constant with respect to x on every microcell $\varepsilon(k+Y)$ (here, we also used that $R_\varepsilon^{\mathrm{in}}$ is constant on every microcell). Then, we immediately obtain together with the essential boundedness of g that $R_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon) \cap H^1((0,T), L^2(\Omega_\varepsilon))$ and also $\partial_t R_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon)$. More precisely, we have for almost every $(t,x) \in (0,T)\times\Omega_\varepsilon$

$$|\partial_t R_\varepsilon(t,x)| \leqslant \varepsilon \|g\|_{L^\infty(\mathbb{R}^2)} \leqslant \varepsilon C_g. \tag{4.6}$$

Moreover, (2.17)–(2.18) ensure $\varepsilon^{-1} R_\varepsilon(t,x) \in [R_{\min}, R_{\max}]$ for every $t \in (0,T)$ and almost every $x \in \Omega_\varepsilon$. Note also that, for every $t \in (0,T)$, the range $R_\varepsilon(t,\cdot)$ lies in a finite-dimensional subspace of $L^\infty(\Omega_\varepsilon)$ and, therefore, R_ε is measurable as a function from $(0,T)$ to $L^\infty(\Omega_\varepsilon)$. □

The following Lemma provides a Lipschitz-type estimate for the operator $\mathcal{F}_1$.

Lemma 4.2. *Let $u_\varepsilon^j \in L^2((0,T)\times\Gamma_\varepsilon)$, $j = 1,2$, and R_ε^j be given by* (4.1a) *(resp.* (2.11b)*) for $\widehat{u}_\varepsilon = u_\varepsilon^j$ (not necessarily with identical initial conditions).*

Then, if $R_\varepsilon^1(0) = R_\varepsilon^2(0)$ it holds for almost every $t \in (0,T)$ that

$$\begin{aligned}
\|R_\varepsilon^1 - R_\varepsilon^2\|_{L^\infty((0,t)\times\Omega)} &\leqslant C_\varepsilon \|u_\varepsilon^1 - u_\varepsilon^2\|_{L^1((0,t)\times\Gamma_\varepsilon)}, \\
\|\partial_t R_\varepsilon^1 - \partial_t R_\varepsilon^2\|_{L^2((0,t),L^\infty(\Omega))} &\leqslant C_\varepsilon \|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2((0,t),L^1(\Gamma_\varepsilon))}
\end{aligned}$$

for a constant C_ε (depending on ε). In particular, for $u_\varepsilon^j \in L^2((0,T), H^1(\Omega_\varepsilon))$, we obtain for every $\mu > 0$ the existence of a constant $C(\mu) > 0$ such that the right-hand sides in the inequalities above can be further estimated by

$$\|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2((0,t),L^1(\Gamma_\varepsilon))} \leqslant \frac{C(\mu)}{\varepsilon} \|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2((0,t)\times\Omega_\varepsilon)} + \mu \|\nabla u_\varepsilon^1 - \nabla u_\varepsilon^2\|_{L^2((0,t)\times\Omega_\varepsilon)}.$$

We also have the ε-uniform Lipschitz estimate (for different initial conditions)

$$\begin{aligned}
&\varepsilon^{-1} \|R_\varepsilon^1 - R_\varepsilon^2\|_{L^\infty((0,T),L^2(\Omega))} \\
&\qquad \leqslant C\left(\sqrt{\varepsilon}\|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2((0,T)\times\Gamma_\varepsilon)} + \varepsilon^{-1}\|R_\varepsilon^1(0) - R_\varepsilon^2(0)\|_{L^2(\Omega)}\right)
\end{aligned} \tag{4.7}$$

for a constant $C > 0$ independent of ε, which implies that $\mathcal{F}_1$ is Lipschitz continuous.

Proof. A proof for the first two estimates can be found in Lemma 14 of Ref. 31 and inequality (4.7) follows by similar arguments as in Proposition 4 of Ref. 31. However, for the sake of completeness, we sketch the proof. We use the notation $\delta R_\varepsilon := R_\varepsilon^1 - R_\varepsilon^2$ and $\delta u_\varepsilon := u_\varepsilon^1 - u_\varepsilon^2$. Using (4.1a), we obtain with the Lipschitz continuity of g almost everywhere in $(0,T)\times\Omega$

$$\pm\partial_t \delta R_\varepsilon \leqslant C\left(\varepsilon^{2-n}\|\delta u_\varepsilon\|_{L^1\left(\Gamma_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)\right)} + |\delta R_\varepsilon|\right). \tag{4.8}$$

Integrating with respect to time and using the Gronwall inequality, the first two inequalities in the statement follow. For (4.7), we integrate (4.8) with respect to time. After squaring and integrating over Ω, we get for almost every $t \in (0,T)$

$$\begin{aligned}
\int_\Omega |\delta R_\varepsilon(t,x)|^2\,\mathrm{d}x &\leqslant C\int_0^t\int_\Omega\left(\varepsilon^{3-n}\|\delta u_\varepsilon\|^2_{L^2\left(\Gamma_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)\right)} + |\delta R_\varepsilon|^2\right)\mathrm{d}x\,\mathrm{d}s + C\|\delta R_\varepsilon(0)\|^2_{L^2(\Omega)}\\
&= C\sum_{k\in K_\varepsilon}\left\{\int_0^t\int_{\varepsilon(Y+k)}\varepsilon^{3-n}\|\delta u_\varepsilon\|^2_{L^2(\varepsilon(\Gamma+k))}\,\mathrm{d}x\,\mathrm{d}s\right\} + C\int_0^t\|\delta R_\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\|\delta R_\varepsilon(0)\|^2_{L^2(\Omega)}\\
&\leqslant C\left(\varepsilon^3\|\delta u_\varepsilon\|^2_{L^2((0,T)\times\Gamma_\varepsilon)} + \|\delta R_\varepsilon\|^2_{L^2((0,t)\times\Omega)} + \|\delta R_\varepsilon(0)\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

Now, the desired result follows from Gronwall's inequality. □

Next, we analyse the operator $\mathcal{F}_2$ and show that it is, in particular, Lipschitz continuous.

Lemma 4.3. *The operator $\mathcal{F}_2$ is well-defined and the range of $\mathcal{F}_2(R_\varepsilon)$ is in Z_ε.*

Proof. In Theorem 3.1 of Ref. 73, the existence of a unique solution $(w_\varepsilon(t), q_\varepsilon(t)) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n \times L^2(\Omega_\varepsilon)$ of (4.3) for a.e. $t \in (0,T)$ was shown under the assumptions (A.3)–(A.6) on $\widehat{\psi}_\varepsilon(t)$ and the assumptions $\widehat{h}_\varepsilon(t) \in L^2(\Omega)^n$ and $\partial_t\psi_\varepsilon(t) \in H^1(\Omega_\varepsilon)^n$. Moreover, this pointwise argumentation was transferred to the existence and uniqueness of a solution $(w_\varepsilon, q_\varepsilon) \in L^\infty((0,t); H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n) \times L^\infty((0,T); L^2(\Omega_\varepsilon))$ of (4.3), where the $L^\infty(0,T)$-boundedness of the solution arises from the $L^\infty(0,T)$-boundedness of $\widehat{h}_\varepsilon$ (see (A5)) and $\partial_t\psi_\varepsilon$, which is given by Lemma Appendix A.2.

To conclude $\mathcal{F}_2(R_\varepsilon) \in Z_\varepsilon$, we note that the divergence condition is obviously fulfilled. The existence of an upper bound C_h such that $\|\mathcal{F}_2(R_\varepsilon)\|_{L^\infty((0,T);H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n)} \leqslant C_h$ for all $R_\varepsilon \in Y_\varepsilon$ is shown in Theorem 3.1 of Ref. 73 as well. □

In what follows, for two given functions ϕ^1 and ϕ^2, we denote its difference by

$$\delta\phi := \phi^1 - \phi^2.$$

Lemma 4.4. *The operator $\mathcal{F}_2$ is locally Lipschitz continuous. More precisely, for every ball $B \subset Y_\varepsilon$, there exists a constant $L_{B,\varepsilon} > 0$ such that*

$$\begin{aligned}
&\|\mathcal{F}_2(\widehat{R}_\varepsilon^1) - \mathcal{F}_2(\widehat{R}_\varepsilon^1)\|_{L^2((0,t)\times\Omega_\varepsilon)} + \left\|\nabla(\mathcal{F}_2(\widehat{R}_\varepsilon^1) - \mathcal{F}_2(\widehat{R}_\varepsilon^1))\right\|_{L^2((0,t)\times\Omega_\varepsilon)}\\
&\quad\leqslant L_{B,\varepsilon}\left(\left\|\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2\right\|_{L^\infty((0,t),L^2(\Omega_\varepsilon))} + \left\|\partial_t\widehat{R}_\varepsilon^1 - \partial_t\widehat{R}_\varepsilon^2\right\|_{L^2((0,t)\times\Omega_\varepsilon)}\right)
\end{aligned} \tag{4.9}$$

for all $\widehat{R}_\varepsilon^1, \widehat{R}_\varepsilon^2 \in B$ and every $t \in (0,T)$.

Proof. We prove this lemma by applying Lemma Appendix B.4, which states that the solution of saddle-point problems depends Lipschitz-continuously on the right-hand side and the operators of the saddle-point problem. Within this proof, we use the shorthand notation $V = H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ with the norm $\|v\|_V := \|\nabla v\|_{L^2(\Omega_\varepsilon)}$ and $Q = L^2(\Omega_\varepsilon)$. Then, we define almost everywhere in $(0,T)$ the following bilinear and linear forms related to the linearised Stokes problem for given radii $\widehat{R}_\varepsilon$

$$
\begin{aligned}
a_{\widehat{R}_\varepsilon}(v,w) &:= \tfrac{1}{2}\varepsilon^2(\widehat{e}_\varepsilon(v), \widehat{A}_\varepsilon^\top \nabla w)_Q = \tfrac{1}{2}\varepsilon^2(\widehat{F}_\varepsilon^\top \nabla v + (\widehat{F}_\varepsilon^\top \nabla v)^\top, \widehat{A}_\varepsilon^\top \nabla w)_Q,\\
b_{\widehat{R}_\varepsilon}(v,q) &:= -(q, \nabla\cdot(\widehat{A}_\varepsilon v))_Q = -(q, \widehat{A}_\varepsilon : \nabla v)_Q,\\
f_{\widehat{R}_\varepsilon}(v) &:= (\widehat{J}_\varepsilon h_\varepsilon, v)_Q - a_{\widehat{R}_\varepsilon}(\partial_t \widehat{\psi}_\varepsilon, v) - (\widehat{A}_\varepsilon \nabla p_{\mathrm{b}}, v)_Q,\\
g_{\widehat{R}_\varepsilon}(q) &:= b(\partial_t\widehat{\psi}_\varepsilon, q) = (q, \nabla\cdot(\widehat{A}_\varepsilon \partial_t \widehat{\psi}_\varepsilon))_Q = (q, \partial_t \widehat{J}_\varepsilon)_Q
\end{aligned}
$$

for $v, w \in V$ and $q \in Q$. We note that $\widehat{R}_\varepsilon^1, \widehat{R}_\varepsilon^2 \in Y_\varepsilon$ and, thus, we obtain with the Hölder inequality the bounds for $\widehat{F}_\varepsilon$ and A_ε from Lemma Appendix A.2 and the Lipschitz estimates of Lemma Appendix A.3

$$
\begin{aligned}
|a_{\widehat{R}_\varepsilon^1}(v,w) - a_{\widehat{R}_\varepsilon^2}(v,w)| =& \tfrac{1}{2}\varepsilon^2(\delta\widehat{F}_\varepsilon \nabla v + (\delta\widehat{F}_\varepsilon \nabla v)^\top, \widehat{A}_\varepsilon^1 \nabla w)_Q\\
&+ \tfrac{1}{2}\varepsilon^2(\widehat{F}_\varepsilon^{2\top}\nabla v + (\widehat{F}_\varepsilon^{2\top}\nabla v)^\top, \delta\widehat{A}_\varepsilon^\top \nabla w)_Q\\
\leqslant& C_\varepsilon \left(\left\|\delta\widehat{F}_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} + \left\|\delta\widehat{A}_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)}\right) \|v\|_V \|w\|_V .\\
\leqslant& C_\varepsilon \|\delta\widehat{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}
\tag{4.10}
$$

By employing the boundedness of $\widehat{A}_\varepsilon$ and ∂A_ε from Lemma Appendix A.2 and the Lipschitz estimates of Lemma Appendix A.3,

$$
\begin{aligned}
|b_{\widehat{R}_\varepsilon^1}(v,q) - b_{\widehat{R}_\varepsilon^2}(v,q)| = |(q, \delta\widehat{A}_\varepsilon : \nabla v)_Q| &\leqslant \|\delta\widehat{A}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \|q\|_Q \|v\|_V\\
&\leqslant C_\varepsilon \|\delta R_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \|q\|_Q \|v\|_V
\end{aligned}
\tag{4.11}
$$

for all $v \in V$ and $q \in Q$.

Taking into account the Hölder inequality, the Lipschitz continuity of $\widehat{h}_\varepsilon$, the boundedness of $\widehat{A}_\varepsilon$, $\widehat{F}_\varepsilon$ and $\widehat{h}_\varepsilon^1$, (4.10), the Poincaré inequality for $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ and Lemma Appendix A.2, we obtain

$$
\begin{aligned}
|f_{\widehat{R}_\varepsilon^1}(v) - f_{\widehat{R}_\varepsilon^2}(v)| \leqslant& |(\delta\widehat{J}_\varepsilon \widehat{h}_\varepsilon, v)_{L^2((0,t)\times\Omega_\varepsilon)}| + |a_{\widehat{R}_\varepsilon^1}(\delta\partial_t\widehat{\psi}_\varepsilon, v)|\\
&+ |a_{\widehat{R}_\varepsilon^1}(\partial_t\widehat{\psi}_\varepsilon^2, v) - a_{\widehat{R}_\varepsilon^2}(\partial_t\widehat{\psi}_\varepsilon^2, v)| | + |(\delta\widehat{A}_\varepsilon \nabla p_{\mathrm{b}}, v)_Q|\\
\leqslant& C \left\|\delta\widehat{J}_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} \|v\|_Q + C\left\|\delta\partial_t\widehat{\psi}_\varepsilon\right\|_V \|v\|_V\\
&+ C_\varepsilon \|\delta\widehat{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left\|\partial_t\widehat{\psi}_\varepsilon^2\right\|_V \|v\|_V + C_\varepsilon \left\|\delta\widehat{A}_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} \|v\|_Q\\
\leqslant& C_\varepsilon \left[\left(1 + \|\partial_t\widehat{R}_\varepsilon^2\|_{L^2(\Omega_\varepsilon)}\right) \|\delta\widehat{R}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \|\delta\partial_t\widehat{R}_\varepsilon\|_Q \right] \|v\|_V .
\end{aligned}
$$

From Lemma Appendix A.3, we get

$$
|g_{\widehat{R}_\varepsilon^1}(q) - g_{\widehat{R}_\varepsilon^2}(q)| = |(q, \delta\partial_t\widehat{J}_\varepsilon)_Q| \leqslant \|q\|_Q \|\delta\partial_t J_\varepsilon\|_Q \leqslant C_\varepsilon(\|\delta\partial_t\widehat{R}_\varepsilon\|_Q + \|\delta\widehat{R}_\varepsilon\|_Q)\|q\|_Q.
$$

Altogether, it holds that

$$
\begin{aligned}
&\left\|a_{\widehat{R}_\varepsilon^1} - a_{\widehat{R}_\varepsilon^2}\right\|_{\mathrm{Bil}} + \left\|b_{\widehat{R}_\varepsilon^1} - b_{\widehat{R}_\varepsilon^2}\right\|_{\mathrm{Bil}} \leqslant C_\varepsilon \left\|\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2\right\|_{L^2(\Omega_\varepsilon)},\\
&\left\|f_{\widehat{R}_\varepsilon^1} - f_{\widehat{R}_\varepsilon^2}\right\|_{V'} + \left\|g_{\widehat{R}_\varepsilon^1} - g_{\widehat{R}_\varepsilon^2}\right\|_{Q'} \leqslant C_\varepsilon M \left\|\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2\right\|_{L^2(\Omega_\varepsilon)} + C_\varepsilon \left\|\partial_t(\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2)\right\|_{L^2(\Omega_\varepsilon)}
\end{aligned}
\tag{4.12}
$$

for $M = 1 + \|\partial_t\widehat{R}_\varepsilon^1\|_{L^2(\Omega_\varepsilon)} + \|\partial_t\widehat{R}_\varepsilon^2\|_{L^2(\Omega_\varepsilon)}$ with the norms defined in Lemma Appendix B.4. By similar estimates, it can be easily observed that

$$
\begin{aligned}
\left\|a_{\widehat{R}_\varepsilon}\right\|_{\mathrm{Bil}} + \left\|b_{\widehat{R}_\varepsilon}\right\|_{\mathrm{Bil}} &\leqslant C_\varepsilon,\\
\left\|f_{\widehat{R}_\varepsilon}\right\|_{V'} + \left\|g_{\widehat{R}_\varepsilon}\right\|_{Q'} &\leqslant C_\varepsilon M
\end{aligned}
$$

for $\widehat{R}_\varepsilon^1, \widehat{R}_\varepsilon^2 \in B$. Moreover, (A.8) and (A.9) yield constants $\alpha_0, \beta_0 > 0$ such that

$$
a_{\widehat{R}_\varepsilon}(v,v) \geqslant \alpha_0 \|v\|_V^2
$$

for all $v \in V$ and

$$\inf_{q \in Q} \sup_{v \in V} \frac{b_{\widehat{R}_\varepsilon}(v, q)}{\|v\|_V \|q\|_Q} \geqslant \beta_0.$$

Thus, $a_{\widehat{R}_\varepsilon}, b_{\widehat{R}_\varepsilon}, f_{\widehat{R}_\varepsilon}$ and $g_{\widehat{R}_\varepsilon}$ fulfill the assumptions of Lemma Appendix B.4 and we can transfer the Lipschitz continuity of the data (see (4.12)) onto the solution which yields almost everywhere in $(0, T)$

$$\begin{aligned}
&\|\mathcal{F}_2(\widehat{R}_\varepsilon^1) - \mathcal{F}_2(\widehat{R}_\varepsilon^1)\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla(\mathcal{F}_2(\widehat{R}_\varepsilon^1) - \mathcal{F}_2(\widehat{R}_\varepsilon^1))\right\|_{L^2(\Omega_\varepsilon)} \\
&\qquad \leqslant LC_\varepsilon M \left\|\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2\right\|_{L^2(\Omega_\varepsilon)} + C_\varepsilon \left\|\partial_t(\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2)\right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

Integrating with respect to time from 0 to $t \in (0, T)$ and employing the Hölder inequality yields

$$\begin{aligned}
&\|\mathcal{F}_2(\widehat{R}_\varepsilon^1) - \mathcal{F}_2(\widehat{R}_\varepsilon^1)\|_{L^2((0,t)\times\Omega_\varepsilon)} + \left\|\nabla(\mathcal{F}_2(\widehat{R}_\varepsilon^1) - \mathcal{F}_2(\widehat{R}_\varepsilon^1))\right\|_{L^2((0,T)\times\Omega_\varepsilon)} \\
&\qquad \leqslant C_\varepsilon \|M\|_{L^2(0,T)} \left\|\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2\right\|_{L^\infty((0,T);L^2(\Omega_\varepsilon))} + C_\varepsilon \left\|\partial_t(\widehat{R}_\varepsilon^1 - \widehat{R}_\varepsilon^2)\right\|_{L^2((0,T)\times\Omega_\varepsilon)}
\end{aligned}$$

and, thus, (4.9). □

Lemma 4.5. *The operator $\mathcal{F}_3$ is well-defined and continuous.*

Proof. For the reaction–diffusion–advection equation in (4.5), existence of a weak solution can be established for example by using the Galerkin method, see Chapter 23 of Ref. 75, which is based on standard energy estimates. Similar estimates will be shown with explicit dependence on ε in Section 5, so we skip the details.

For the continuity of the operator, we consider sequences $(\widehat{R}_\varepsilon^k, \widehat{w}_\varepsilon^k) \in Y_\varepsilon \times Z_\varepsilon$, see (4.2) and (4.4), and $\widehat{u}_\varepsilon^k \in L^2((0,T), H^\beta(\Omega_\varepsilon))$, such that for $k \to \infty$

$$\begin{aligned}
\widehat{R}_\varepsilon^k &\to \widehat{R}_\varepsilon && \text{in } L^\infty((0,T)\times\Omega_\varepsilon) \cap H^1((0,T), L^\infty(\Omega_\varepsilon)), \\
\widehat{w}_\varepsilon^k &\to \widehat{w}_\varepsilon && \text{in } L^2((0,T), H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^n), \\
\widehat{u}_\varepsilon^k &\to \widehat{u}_\varepsilon && \text{in } L^2((0,T), H^\beta(\Omega_\varepsilon)).
\end{aligned}$$

From Lemma Appendix A.3, we also get the strong convergence of $\widehat{J}_\varepsilon^k$, $\widehat{D}_\varepsilon^k$, and $\widehat{A}_\varepsilon^k$ in $L^\infty((0,T)\times\Omega_\varepsilon)$ to $\widehat{J}_\varepsilon$, $\widehat{D}_\varepsilon$, and $\widehat{A}_\varepsilon$. From the a priori estimates in Lemmata 5.1 and 5.2 below, we obtain for $u_\varepsilon^k := \mathcal{F}_3(\widehat{R}_\varepsilon^k, \widehat{w}_\varepsilon^k)$ and a constant $C_\varepsilon > 0$ (depending on ε) the estimate

$$\|\partial_t u_\varepsilon^k\|_{L^2((0,T),H^1_{\partial\Omega}(\Omega_\varepsilon)')} + \|u_\varepsilon^k\|_{L^2((0,T),H^1(\Omega_\varepsilon))} \leqslant C_\varepsilon.$$

Hence, there exists $u_\varepsilon \in L^2((0,T), H^1_{\partial\Omega}(\Omega_\varepsilon)) \cap H^1((0,T), H^1_{\partial\Omega}(\Omega_\varepsilon)')$ such that, up to a subsequence, u_ε^k converges weakly to u_ε in $L^2((0,T), H^1(\Omega_\varepsilon))$ and strongly in $L^2((0,T), H^\beta(\Omega_\varepsilon))$ by the Aubin–Lions lemma and, by continuity of the trace operator, also in the space $L^2((0,T)\times\Gamma_\varepsilon)$. Further, $\partial_t u_\varepsilon^k$ converges weakly to $\partial_t u_\varepsilon$ in $L^2((0,T), H^1_{\partial\Omega}(\Omega_\varepsilon)')$. Testing the weak formulation of the linearised problem (4.5) with $\phi(x)\psi(t)$ with $\phi \in H^1_{\partial\Omega}(\Omega_\varepsilon)$ and $\psi \in C_0^\infty([0,T))$, we get

$$\begin{aligned}
&-\int_0^T \langle \widehat{J}_\varepsilon^k u_\varepsilon^k, \phi\rangle_{H^1_{\partial\Omega}(\Omega_\varepsilon)', H^1_{\partial\Omega}(\Omega_\varepsilon)} \psi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} [\widehat{D}_\varepsilon^k \nabla u_\varepsilon^k - u_\varepsilon^k \widehat{A}_\varepsilon^k \widehat{w}_\varepsilon^k] \cdot \nabla\phi\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega_\varepsilon} \widehat{J}_\varepsilon^k f(\widehat{u}_\varepsilon^k)\phi\psi \,\mathrm{d}x\,\mathrm{d}t - \rho \int_0^T \int_{\Gamma_\varepsilon} \widehat{J}_\varepsilon^k g\left(\widehat{u}_\varepsilon^k, \varepsilon^{-1}\widehat{R}_\varepsilon^k\right)\phi\psi \,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad + \int_{\Omega_\varepsilon} \widehat{J}_\varepsilon^k(0) u_\varepsilon^{\mathrm{in}} \phi\psi(0) \,\mathrm{d}x.
\end{aligned}$$

Passing to the limit $k \to \infty$, we obtain

$$\begin{aligned}
&-\int_0^T \langle \widehat{J}_\varepsilon u_\varepsilon, \phi\rangle_{H^1_{\partial\Omega}(\Omega_\varepsilon)', H^1_{\partial\Omega}(\Omega_\varepsilon)} \psi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} [\widehat{D}_\varepsilon \nabla u_\varepsilon - u_\varepsilon \widehat{A}_\varepsilon \widehat{w}_\varepsilon] \cdot \nabla\phi\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega_\varepsilon} \widehat{J}_\varepsilon f(\widehat{u}_\varepsilon)\phi\psi \,\mathrm{d}x\,\mathrm{d}t - \rho \int_0^T \int_{\Gamma_\varepsilon} \widehat{J}_\varepsilon g\left(\widehat{u}_\varepsilon, \varepsilon^{-1}\widehat{R}_\varepsilon\right)\phi\psi \,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad + \int_{\Omega_\varepsilon} \widehat{J}_\varepsilon(0) u_\varepsilon^{\mathrm{in}} \phi\psi(0) \,\mathrm{d}x.
\end{aligned}$$

A density argument shows $\mathcal{F}_3(\widehat{R}_\varepsilon, \widehat{w}_\varepsilon, \widehat{u}_\varepsilon) = u_\varepsilon$. The initial condition $u_\varepsilon(0) = u_\varepsilon^{\mathrm{in}}$ can be obtained by a similar argument as above and integrating by parts in time. This implies the continuity of $\mathcal{F}_3$. □

Remark 4.1. The operator $\mathcal{F}_3$ is even Lipschitz continuous. This can be shown by similar arguments as in the proof of Theorem 3 of Ref. 31 and the proof of Proposition 4.2 below. In both proofs, the difference of two fixed points of $\mathcal{F}$ are estimated. However, it is straightforward to generalise these arguments to show the Lipschitz continuity. Here, we restrict ourselves to the continuity of $\mathcal{F}_3$, which is enough to guarantee the existence of a solution. For uniqueness of the weak solution, we can use the results from the proof of Theorem 3 of Ref. 31 and, therefore, avoid repeating the same arguments, which shortens the uniqueness proof.

4.2. *Existence of a solution for the micoscopic model*

Proposition 4.1. *There exists a weak solution* $(u_\varepsilon, R_\varepsilon, w_\varepsilon, q_\varepsilon)$ *of the microscopic problem* (2.11a)–(2.11c) *and* (2.13) *in the sense of Definition 2.1.*

Proof. We apply Schäfer's fixed-point theorem (see e.g. Ref. 23) to the operator $\mathcal{F}$. For this, we have to show that $\mathcal{F}$ is compact and continuous and that the set

$$\mathcal{X} := \left\{ u_\varepsilon \in L^2((0,T), H^\beta(\Omega_\varepsilon)) \,:\, u_\varepsilon = \lambda \mathcal{F}(u_\varepsilon) \text{ for some } 0 \leqslant \lambda \leqslant 1 \right\}$$

is bounded. Let $(\widehat{u}_\varepsilon^k)_{k\in\mathbb{N}} \subset L^2((0,T), H^\beta(\Omega_\varepsilon))$ be bounded and $u_\varepsilon^k := \mathcal{F}(\widehat{u}_\varepsilon^k)$. Standard energy estimates for parabolic equations provide

$$\|\partial_t u_\varepsilon^k\|_{L^2((0,T),H^1_{\partial\Omega}(\Omega_\varepsilon)')} + \|u_\varepsilon^k\|_{L^2((0,T),H^1(\Omega_\varepsilon))} \leqslant C_\varepsilon$$

for a constant $C_\varepsilon > 0$ independent of k (but depending on ε). For more details on this estimate, we refer to the proof of Lemma 5.1 in Section 5 below, where we show uniform a priori estimates with respect to ε. The Aubin–Lions lemma, see Ref. 41, implies the existence of a convergent subsequence in $L^2((0,T), H^\beta(\Omega_\varepsilon))$ and, therefore, $\mathcal{F}$ is compact.

The continuity of the operator $\mathcal{F}$ follows from the continuity of the operators $\mathcal{F}_i$, $i = 1,2,3$, see Lemmata 4.1, 4.4, and 4.5.

It remains to show the boundedness of $\mathcal{X}$. We only have to consider $u_\varepsilon = \lambda\mathcal{F}(u_\varepsilon)$ for $u_\varepsilon \in L^2((0,T), H^\beta(\Omega_\varepsilon))$ and $\lambda \in (0,1]$. Hence, we have to replace u_ε in the microscopic model (2.11) with $\frac{u_\varepsilon}{\lambda}$. The associated weak formulation of this equation is obtained by multiplying the right-hand side of (2.14a) by λ. Then, the boundedness of u_ε (uniform with respect to λ) can be obtained again by standard energy estimates. Now, Schäfer's fixed-point theorem implies the existence of a fixed point and, therefore, the existence of a microscopic solution. □

4.3. *Uniqueness of a solution for the microscopic model*

Now, we show the uniqueness of the weak microscopic solution. In Theorem 3 of Ref. 31, the uniqueness was established for the problem without convection. Hence, here we generalise this result, where the crucial new ingredient is the Lipschitz estimate from Lemma 4.4.

Proposition 4.2. *The weak solution* $(u_\varepsilon, R_\varepsilon, w_\varepsilon, q_\varepsilon)$ *of the microscopic problem* (2.11) *obtained by Proposition 4.1 is unique.*

Proof.

Consider two microscopic solutions $(u_\varepsilon^j, R_\varepsilon^j, w_\varepsilon^j, q_\varepsilon^j)$ for $j = 1,2$. Since every problem associated to the operators $\mathcal{F}_i$, $i = 1,2,3$, admits a unique solution, it follows that u_ε^j are fixed points of $\mathcal{F}$. Hence, we have $u_\varepsilon^j = \mathcal{F}(u_\varepsilon^j)$, $R_\varepsilon^j = \mathcal{F}_1(u_\varepsilon^j)$ and $w_\varepsilon^j = \mathcal{F}_2(R_\varepsilon^j)$. The triple $(u_\varepsilon, w_\varepsilon, R_\varepsilon)$ fulfills the a priori estimates of Section 5. This can be shown by the same arguments as in the proofs of Lemmata 5.1 and 5.2. In particular, the estimates from Lemmata Appendix A.2, Appendix A.3 and 4.2 are valid, as well as the inequality (4.9).

Now, as in the proof of Theorem 3 of Ref. 31 (see the last inequality in the proof in particular) with only slight modifications for the nonlinear right-hand sides with f and g in the transport equation, we subtract the weak equations for u_ε^1 and u_ε^2 and test with δu_ε to obtain for almost every $t \in (0,T)$ and a constant $c > 0$ the estimate

$$\begin{aligned} &\|\delta u_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} + c\|\nabla \delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\qquad\qquad \leqslant C_\varepsilon \|\delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} + (\delta(u_\varepsilon A_\varepsilon w_\varepsilon), \nabla \delta u_\varepsilon)_{(0,t)\times\Omega_\varepsilon}, \end{aligned} \tag{4.13}$$

where $C_\varepsilon > 0$ is a constant depending on ε (C_ε is a generic constant in what follows) and A_ε^j is defined as in (2.9). It remains to estimate the convective term on the right-hand side:

$$\begin{aligned} &\big(\delta(u_\varepsilon A_\varepsilon w_\varepsilon), \nabla \delta u_\varepsilon\big)_{(0,t)\times\Omega_\varepsilon} \\ &\quad = \big(u_\varepsilon^1 A_\varepsilon^1 \delta w_\varepsilon, \nabla \delta u_\varepsilon\big)_{(0,t)\times\Omega_\varepsilon} + \big(u_\varepsilon^1 \delta A_\varepsilon w_\varepsilon^2, \nabla \delta u_\varepsilon\big)_{(0,t)\times\Omega_\varepsilon} + \big(\delta u_\varepsilon A_\varepsilon^2 w_\varepsilon^2, \nabla \delta u_\varepsilon\big)_{(0,t)\times\Omega_\varepsilon} \\ &\quad =: I_1 + I_2 + I_3. \end{aligned}$$

In what follows, we use the essential bound for A_ε^1, see Lemma Appendix A.2, and the essential bound for u_ε^1 from Lemma 5.2 in Section 5. Note that these bounds do not require a uniqueness result. We estimate I_1 by

$$\begin{aligned} I_1 &\leqslant C_\varepsilon \|u_\varepsilon^1\|_{L^\infty((0,t)\times\Omega_\varepsilon))} \|\delta w_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \|\nabla \delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C_\varepsilon \|\delta w_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \|\nabla \delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}. \end{aligned}$$

Using estimate (4.9) from Lemma 4.4, as well as Lemma 4.2, we obtain for arbitrary $\mu > 0$ the existence of $C_\varepsilon(\mu)$ such that

$$\begin{aligned} I_1 &\leqslant C_\varepsilon \left(\|\delta R_\varepsilon\|_{L^\infty((0,t)\times\Omega_\varepsilon)} + \|\delta \partial_t R_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \right) \|\nabla \delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C_\varepsilon \|\delta u_\varepsilon\|_{L^2((0,t),L^1(\Gamma_\varepsilon))} \|\nabla \delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C_\varepsilon(\mu) \|\delta u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \mu \|\delta \nabla u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}. \end{aligned}$$

Next, using again Lemma Appendix A.3 and 4.2 and the a priori bounds for u_ε^1 and w_ε^2, we get with the Hölder inequality

$$\begin{aligned} I_2 &\leqslant C_\varepsilon \|u_\varepsilon^1\|_{L^\infty((0,t)\times\Omega_\varepsilon))} \|\delta A_\varepsilon\|_{L^\infty((0,t)\times\Omega_\varepsilon)} \|w_\varepsilon^2\|_{L^\infty((0,t),L^2(\Omega_\varepsilon))} \|\nabla \delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C_\varepsilon \|\delta u_\varepsilon\|_{L^1((0,t)\times\Gamma_\varepsilon)} \|\nabla \delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C_\varepsilon(\mu) \|\delta u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \mu \|\delta \nabla u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}. \end{aligned}$$

Finally, for I_3, we use integration by parts, the zero boundary condition of u_ε on $\partial\Omega$ resp. w_ε on Γ_ε as well as $\partial_t J_\varepsilon = \nabla \cdot (A_\varepsilon w_\varepsilon)$ and Lemma Appendix A.2 to obtain

$$I_3 = \frac{1}{2} (\delta u_\varepsilon \partial_t J_\varepsilon^2, \delta u_\varepsilon)_{(0,t)\times\Omega_\varepsilon} \leqslant C_\varepsilon \|\delta u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}.$$

Altogether, we obtain (perhaps after some redefinition of μ)

$$\begin{aligned} &\big(\delta(u_\varepsilon A_\varepsilon w_\varepsilon), \nabla \delta u_\varepsilon\big)_{(0,t)\times\Omega_\varepsilon)} \\ &\quad \leqslant C_\varepsilon(\mu) \left(\|\delta u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|\delta \widehat{u}_\varepsilon\|^2_{L^2((0,t),H^\beta(\Omega_\varepsilon))} \right) + \mu \|\delta \nabla u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}. \end{aligned}$$

Plugging this estimate into (4.13) and choosing μ small enough, such that we can absorb the gradient term on the right-hand side by the left-hand side, we deduce

$$\|\delta u_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} + c\|\delta \nabla u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant C_\varepsilon \|\delta u_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}.$$

Application of the Gronwall inequality leads to the desired result. □

5. A priori estimates for the microscopic solution

In this section, we show uniform a priori estimates for the weak solution $(u_\varepsilon, w_\varepsilon, R_\varepsilon)$ of the microscopic problem (2.11). These estimates are the basis for the two-scale compactness results stated in Proposition 6.1 and Proposition 6.2 of Section 6.1 below, which allow us to pass to the limit $\varepsilon \to 0$ in the weak formulation (2.14). The proof follows the following scheme: The uniform bound for R_ε is a direct consequence of the essential boundedness of g. Therefore, R_ε and $\partial_t R_\varepsilon$ can be estimated independently of u_ε. The bounds for R_ε allow to control the velocity w_ε and the pressure p_ε. Finally, we are able to estimate the concentration u_ε.

Lemma 5.1. *The weak solution $(u_\varepsilon, R_\varepsilon, w_\varepsilon, q_\varepsilon)$ of the microscopic problem* (2.11) *fulfills the following a priori estimates*

$$\|R_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} + \|\partial_t R_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant C\varepsilon, \tag{5.1}$$

$$\|w_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))} + \varepsilon\|\nabla w_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))} + \|q_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))} \leqslant C, \tag{5.2}$$

$$\varepsilon\|\partial_t u_\varepsilon\|_{L^2((0,T),H^1(\Omega_\varepsilon)')} + \|u_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))} + \|\nabla u_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C \tag{5.3}$$

$$\|\partial_t(J_\varepsilon u_\varepsilon)\|_{L^2((0,T),H^1_{\partial\Omega}(\Omega_\varepsilon)')} + \|J_\varepsilon u_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))} + \varepsilon\|\nabla(J_\varepsilon u_\varepsilon)\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C \tag{5.4}$$

Proof. *Bound for R_ε:* We start with the bounds for the radii R_ε. This follows immediately from the essential boundedness of g: For almost every $(t,x) \in (0,T)\times\Omega_\varepsilon$, we have

$$|\partial_t R_\varepsilon(t,x)| \leqslant \varepsilon\|g\|_{L^\infty(\mathbb{R}^2)} \leqslant \varepsilon C_g.$$

This also implies the L^∞-bound for R_ε (which was already shown in Lemma 4.1).
Bound for w_ε and q_ε: Due to the bound (5.1) for R_ε shown above, we can apply Lemma Appendix A.2. Now, the results follow directly from Theorem 3.1 of Ref. 73, where the existence and uniqueness of a solution as well as a priori estimates for the Stokes problem for given evolving microstructure are shown.
Bound for u_ε: Here, we use similar arguments as in the proof of Lemma 6 Ref. 72, so we skip some details. The main new part is the convective term. We divide the proof into two steps, showing the estimate for $\|u_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))}$ and $\|\nabla u_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)}$ first (which then directly also implies the estimate for $\varepsilon\|\nabla(J_\varepsilon u_\varepsilon)\|_{L^2((0,T)\times\Omega_\varepsilon)}$):
Step A: We begin by testing the weak equation for u_ε (2.14a) with u_ε to obtain almost everywhere in $(0,T)$

$$\begin{aligned}&\langle\partial_t(J_\varepsilon u_\varepsilon), u_\varepsilon\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)} + \int_{\Omega_\varepsilon} D_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon\,\mathrm{d}x =\\ &\int_{\Omega_\varepsilon} u_\varepsilon A_\varepsilon w_\varepsilon\cdot\nabla u_\varepsilon\,\mathrm{d}x + \int_{\Omega_\varepsilon} J_\varepsilon f(u_\varepsilon)u_\varepsilon\,\mathrm{d}x - \rho\int_{\Gamma_\varepsilon} g\left(u_\varepsilon,\varepsilon^{-1}R_\varepsilon\right)J_\varepsilon u_\varepsilon\,\mathrm{d}\sigma.\end{aligned}$$

Using the uniform coercivity of D_ε, see (A.7), we get for almost every $t\in(0,T)$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{J_\varepsilon}u_\varepsilon(t)\right\|^2_{L^2(\Omega_\varepsilon)} + \alpha\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\\ &\quad\leqslant -\frac{1}{2}\int_{\Omega_\varepsilon}\partial_t J_\varepsilon|u_\varepsilon|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} u_\varepsilon A_\varepsilon w_\varepsilon\cdot\nabla u_\varepsilon\,\mathrm{d}x\\ &\qquad+\int_{\Omega_\varepsilon} J_\varepsilon f(u_\varepsilon)u_\varepsilon\,\mathrm{d}x - \rho\int_{\Gamma_\varepsilon} g\left(u_\varepsilon,\varepsilon^{-1}R_\varepsilon\right)J_\varepsilon u_\varepsilon\,\mathrm{d}\sigma.\end{aligned}$$

Using the uniform bounds for R_ε, J_ε (see Lemma Appendix A.2), the growth condition on f from assumption (A4) and the boundedness of g from assumption (2.15) together with the trace inequality, we obtain for arbitrary $\mu > 0$ a constant $C_\mu > 0$ such that

$$\begin{aligned}&\int_{\Omega_\varepsilon} J_\varepsilon f_\varepsilon u_\varepsilon\,\mathrm{d}x - \rho\int_{\Gamma_\varepsilon} g\left(u_\varepsilon,\varepsilon^{-1}R_\varepsilon\right)J_\varepsilon u_\varepsilon\,\mathrm{d}\sigma\\ &\qquad\leqslant C_\mu\left(1+\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\right) + \mu\varepsilon^2\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

For the convective term, we use integration by parts and $\nabla \cdot (A_\varepsilon w_\varepsilon) = -\partial_t J_\varepsilon$ (see Remark 2.1) to obtain

$$-\frac{1}{2}\int_{\Omega_\varepsilon} \partial_t J_\varepsilon |u_\varepsilon|^2 \,\mathrm{d}x + \int_{\Omega_\varepsilon} u_\varepsilon A_\varepsilon w_\varepsilon \cdot \nabla u_\varepsilon \,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega_\varepsilon} |u_\varepsilon|^2 A_\varepsilon w_\varepsilon \cdot \nu \,\mathrm{d}\sigma = 0. \tag{5.5}$$

For the last equality, we used $w_\varepsilon = 0$ on Γ_ε and $u_\varepsilon = 0$ on $\partial\Omega$. Choosing $\mu > 0$ small enough, integration by parts and Gronwall's inequality imply

$$\|u_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))} + \|\nabla u_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C.$$

Now, from the properties of J_ε and ∇J_ε in Lemma Appendix A.2, we get

$$\varepsilon\|\nabla(J_\varepsilon u_\varepsilon)\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C.$$

Step B: Now, we show the estimate for $\partial_t(J_\varepsilon u_\varepsilon)$. For this, we use the uniform bound for u_ε in $L^\infty((0,T)\times\Omega_\varepsilon)$, which is shown in Lemma 5.2 below, for the proof of which we only need the regularity of the weak solution but no ε-uniform a priori bounds. However, we emphasise that for the proof of the existence of a weak microscopic solution we used the Galerkin method, see Lemma 4.5, and for the Galerkin approximation we have no (uniform) L^∞-bound (the proof of Lemma 5.2 is not valid for the Galerkin approximation). We also obtain the regularity for $\partial_t(J_\varepsilon u_\varepsilon)$ without the L^∞-bound for u_ε, see Remark 5.1 below for more details.

We test (2.14a) with $\phi \in H^1(\Omega_\varepsilon)$ such that $\|\phi\|_{H^1(\Omega_\varepsilon)} \leqslant 1$. We only estimate the convective term. The estimates for the other terms follow straightforwardly. The L^∞-bound for u_ε is obtained from Lemma 5.2 using the essential bound for A_ε (see Lemma Appendix A.2) and the Hölder inequality for almost every $t \in (0,T)$. More concretely, we obtain $u_\varepsilon(t) \in L^\infty(\Omega_\varepsilon)$ and

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)} \leqslant \|u_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant C$$

for almost every $t \in (0,T)$. We emphasise that, in general, the mapping $t \mapsto \|u_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)}$, in other words $u_\varepsilon : (0,T) \to L^\infty(\Omega_\varepsilon)$, is not measurable. Furthermore, we obtain for almost every $t \in (0,T)$

$$\int_{\Omega_\varepsilon} (u_\varepsilon A_\varepsilon w_\varepsilon \cdot \nabla\phi)(t,x) \,\mathrm{d}x \leqslant C\|u_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon)}\|w_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}\|\nabla\phi\|_{L^2(\Omega_\varepsilon)} \leqslant C.$$

Here, we also used the L^∞-bound with respect to time of w_ε. Hence, we obtain for almost every $t \in (0,T)$ that

$$|\langle \partial_t(J_\varepsilon u_\varepsilon)(t), \phi\rangle| \leqslant C\left(1 + \|\nabla u_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}\right)$$

and, therefore,

$$\|\partial_t(J_\varepsilon u_\varepsilon)(t)\|_{H^1_{\partial\Omega}(\Omega_\varepsilon)'} \leqslant C\left(1 + \|\nabla u_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}\right).$$

Using the a priori bound for ∇u_ε, we further obtain

$$\|\partial_t(J_\varepsilon u_\varepsilon)\|_{L^2((0,T),H^1_{\partial\Omega}(\Omega_\varepsilon)')} \leqslant C.$$

Now, the estimate for $\partial_t u_\varepsilon$ follows from the product rule and the bound for $\varepsilon\nabla J_\varepsilon$ in Lemma Appendix A.1. A similar argument is valid for $J_\varepsilon u_\varepsilon$ and $\nabla(J_\varepsilon u_\varepsilon)$. Thus, the proof is complete except for the verification of Lemma 5.2 which is undertaken below. □

Remark 5.1. As indicated in Step B in the proof above, to obtain the ε-uniform bound for $\partial_t(J_\varepsilon u_\varepsilon)$ in Lemma 5.1, we need the uniform L^∞-bound for u_ε from Lemma 5.2 below. However, in the Galerkin method in the proof of Lemma 4.5 (where we are not allowed to use Lemma 5.2), we can argue in the following way to obtain a bound for $\partial_t(J_\varepsilon u_\varepsilon)$ (not uniformly with respect to ε): From the *a priori* bounds already obtained for w_ε and u_ε in Step A of the proof above, we get

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} u_\varepsilon A_\varepsilon w_\varepsilon \cdot \nabla\phi \,\mathrm{d}x\right| &\leqslant C_\varepsilon\|u_\varepsilon\|_{L^4(\Omega_\varepsilon)}\|w_\varepsilon\|_{L^4(\Omega_\varepsilon)}\|\nabla\phi\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant C_\varepsilon\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\|w_\varepsilon\|_{H^1(\Omega_\varepsilon)}\end{aligned}$$

for a constant C_ε depending on ε. Now, arguing as in the proof above, we obtain with the $L^\infty((0,T),H^1(\Omega_\varepsilon)^n)$-bound for w_ε and the $L^2((0,T),H^1(\Omega_\varepsilon))$-bound for u_ε the estimate

$$\|\partial_t(J_\varepsilon u_\varepsilon)\|_{L^2((0,T),H^1_{\partial\Omega}(\Omega_\varepsilon)')} \leqslant C_\varepsilon.$$

We now show the essential bound for u_ε, which holds uniformly with respect to the homogenisation parameter ε. Further, under some additional assumptions on the data, the non-negativity of the concentration u_ε also follows.

Lemma 5.2. *The solution u_ε is uniformly essentially bounded. More precisely, there exists a constant $C > 0$ (independent of ε) such that*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)} \leqslant C. \tag{5.6}$$

Under the additional assumptions on the data

$$f(u)(u)^- \leqslant |(u)^-|^2, \qquad g(u,R)(u)^- \geqslant 0, \qquad u_\varepsilon^{\text{in}} \geqslant 0 \tag{5.7}$$

for every $R, u \in \mathbb{R}$, where $(\cdot)^- = \min\{\cdot, 0\}$, the solution u_ε is non-negative.

Proof. Following the classic strategy for proving such results, we define for $\omega \geqslant 0$ the function $W := \mathrm{e}^{-\omega t}u_\varepsilon$ and for $k \geqslant 1$ the functions $W_k := W - k$ and $W_k^+ := (W-k)^+$, where $(\cdot)^+ := \max(\cdot, 0)$. Choosing $\mathrm{e}^{-\omega t}W_k^+$ as a test function in (2.14a), we obtain

$$\begin{aligned}
&\int_0^t \langle \partial_t(J_\varepsilon u_\varepsilon), \mathrm{e}^{-\omega s}W_k^+\rangle \,\mathrm{d}s + (D_\varepsilon \nabla u_\varepsilon, \mathrm{e}^{-\omega s}\nabla W_k^+)_{(0,t)\times\Omega_\varepsilon} \\
&\quad = \left(A_\varepsilon w_\varepsilon u_\varepsilon, \mathrm{e}^{-\omega s}\nabla W_k^+\right)_{(0,t)\times\Omega_\varepsilon} + (J_\varepsilon f(u_\varepsilon), \mathrm{e}^{-\omega s}W_k^+)_{(0,t)\times\Omega_\varepsilon} \\
&\qquad + \rho\left(J_\varepsilon g(u_\varepsilon, \varepsilon^{-1}R_\varepsilon), \mathrm{e}^{-\omega s}W_k^+\right)_{(0,t)\times\Gamma_\varepsilon}.
\end{aligned} \tag{5.8}$$

Using the identity

$$\mathrm{e}^{-\omega t}\partial_t(J_\varepsilon u_\varepsilon) = \partial_t\left[J_\varepsilon\left(\mathrm{e}^{-\omega t}u_\varepsilon - k\right)\right] + \omega \mathrm{e}^{-\omega t}J_\varepsilon u_\varepsilon + k\partial_t J_\varepsilon$$

in $L^2((0,T);H^1_{\partial\Omega}(\Omega_\varepsilon)')$ and similar arguments as in Lemma 3.2 of Ref. 69, we get (for k large enough)

$$\begin{aligned}
\int_0^t \langle \partial_t(J_\varepsilon u_\varepsilon), \mathrm{e}^{-\omega s}W_k^+\rangle \,\mathrm{d}s &= \int_0^t \langle \mathrm{e}^{-\omega s}\partial_t(J_\varepsilon u_\varepsilon), W_k^+\rangle \,\mathrm{d}s \\
&= \int_0^t \langle \partial_t\left[J_\varepsilon W_k\right], W_k^+\rangle \,\mathrm{d}s + \left(\omega \mathrm{e}^{-\omega s}J_\varepsilon u_\varepsilon, W_k^+\right)_{(0,t)\times\Omega_\varepsilon} + \left(k\partial_t J_\varepsilon, W_k^+\right)_{(0,t)\times\Omega_\varepsilon} \\
&= \frac{1}{2}\left\|\sqrt{J_\varepsilon}W_k^+(t)\right\|^2_{L^2(\Omega_\varepsilon)} - \frac{1}{2}\underbrace{\left\|\sqrt{J_\varepsilon}W_k^+(0)\right\|^2_{L^2(\Omega_\varepsilon)}}_{=0} + \frac{1}{2}\int_0^t\int_{\Omega_\varepsilon} \partial_t J_\varepsilon W_k W_k^+ \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \left(\omega \mathrm{e}^{-\omega s}J_\varepsilon u_\varepsilon, W_k^+\right)_{(0,t)\times\Omega_\varepsilon} + \left(k\partial_t J_\varepsilon, W_k^+\right)_{(0,t)\times\Omega_\varepsilon}.
\end{aligned}$$

For the second term in (5.8), we obtain with the ε-uniform coercivity of D_ε from (A.7)

$$\begin{aligned}
(D_\varepsilon \nabla u_\varepsilon, \mathrm{e}^{-\omega s}\nabla W_k^+)_{(0,t)\times\Omega_\varepsilon} &= (D_\varepsilon \nabla W_k, \nabla W_k^+)_{(0,t)\times\Omega_\varepsilon} = (D_\varepsilon \nabla W_k^+, \nabla W_k^+)_{(0,t)\times\Omega_\varepsilon} \\
&\geqslant c_0\|\nabla W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\sqrt{J_\varepsilon}W_k^+(t)\right\|^2_{L^2(\Omega_\varepsilon)} + c_0\|\nabla W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\
&\leqslant \left(A_\varepsilon w_\varepsilon u_\varepsilon, \mathrm{e}^{-\omega s}\nabla W_k^+\right)_{(0,t)\times\Omega_\varepsilon} + (J_\varepsilon f(u_\varepsilon), \mathrm{e}^{-\omega s}W_k^+)_{(0,t)\times\Omega_\varepsilon} \\
&\quad + \rho\left(J_\varepsilon g(u_\varepsilon, \varepsilon^{-1}R_\varepsilon), \mathrm{e}^{-\omega s}W_k^+\right)_{(0,t)\times\Gamma_\varepsilon} - \frac{1}{2}\int_0^t\int_{\Omega_\varepsilon} \partial_t J_\varepsilon W_k W_k^+ \,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \left(\omega J_\varepsilon W, W_k^+\right)_{(0,t)\times\Omega_\varepsilon} - \left(k\partial_t J_\varepsilon, W_k^+\right)_{(0,t)\times\Omega_\varepsilon} =: \sum_{l=1}^{6} I_\varepsilon^l.
\end{aligned} \tag{5.9}$$

For the convective term I_ε^1, we have

$$\begin{aligned} I_\varepsilon^1 &= \left(A_\varepsilon w_\varepsilon W, \nabla W_k^+\right)_{(0,t)\times\Omega_\varepsilon} \\ &= \left(A_\varepsilon w_\varepsilon W_k^+, \nabla W_k^+\right)_{(0,t)\times\Omega_\varepsilon} + k\left(A_\varepsilon w_\varepsilon, \nabla W_k^+\right)_{(0,t)\times\Omega_\varepsilon} \\ &=: A_\varepsilon^1 + A_\varepsilon^2. \end{aligned}$$

For the first term, we use $\nabla\cdot(A_\varepsilon w_\varepsilon) = \partial_t J_\varepsilon$ almost everywhere in Ω_ε and $A_\varepsilon w_\varepsilon \cdot \nu = 0$ almost everywhere on Γ_ε. Moreover, we use the zero boundary condition of u_ε on $\partial\Omega$, which is then also valid for W_k^+, to obtain with integration by parts

$$\begin{aligned} A_\varepsilon^1 &= \frac{1}{2}\left(A_\varepsilon w_\varepsilon, \nabla (W_k^+)^2\right)_{(0,t)\times\Omega_\varepsilon} = -\frac{1}{2}\left(\nabla\cdot(A_\varepsilon w_\varepsilon), (W_k^+)^2\right)_{(0,t)\times\Omega_\varepsilon} \\ &= \frac{1}{2}\left(\partial_t J_\varepsilon, (W_k^+)^2\right)_{(0,t)\times\Omega_\varepsilon} = -I_\varepsilon^4. \end{aligned}$$

For the term A_ε^2, we obtain with similar arguments

$$A_\varepsilon^2 = k(\partial_t J_\varepsilon, W_k^+)_{(0,t)\times\Omega_\varepsilon} = -I_\varepsilon^6.$$

Now, we note that $f(u) \leqslant C(1+u) \leqslant C(1+We^{\omega s})$ for $u \geqslant 0$ and $W_k^+ = 0$ for $u \leqslant 0$ almost everywhere. Thus, we obtain

$$|f(u)\mathrm{e}^{-\omega s}W_k^+| \leqslant C(1+We^{\omega s})\mathrm{e}^{-\omega s}W_k^+ \leqslant C(\mathrm{e}^{-\omega s} + W_k^+ + k)W_k^+ \leqslant C(1+W_k^+ + k)W_k^+ . \tag{5.10}$$

Using additionally the essential boundedness of J_ε as well as the Hölder and Young inequalities, we obtain for I_ε^2

$$\begin{aligned} |I_\varepsilon^2| &\leqslant \int\limits_{(0,t)\times\Omega_\varepsilon} C(1+W_k^+ + k)W_k^+ \,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C(1+k)\|W_k^+\|_{L^1((0,t)\times\Omega_\varepsilon)} + C\|W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C(1+k^2)\int_0^t\int_{\{W(t)>k\}} \mathrm{d}x\,\mathrm{d}s + C\|W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)}. \end{aligned}$$

For the boundary term I_ε^3, we obtain with the essential boundedness of J_ε and g as well as the scaled trace inequality for $\mu > 0$ (with $\varepsilon \leqslant 1$)

$$\begin{aligned} |I_\varepsilon^3| &\leqslant \varepsilon C C_g \|\mathrm{e}^{-\omega s}W_k^+\|_{L^1((0,t)\times\Gamma_\varepsilon)} \leqslant C(\mu)\|W_k^+\|_{L^1((0,t)\times\Omega_\varepsilon)} + \varepsilon\mu\|\nabla W_k^+\|_{L^1((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C(\mu)\int_0^t\int_{\{W(t)>k\}} \mathrm{d}x\,\mathrm{d}s + C(\mu)\|W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \mu\|\nabla W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)}. \end{aligned}$$

For I_ε^5, we get

$$\begin{aligned} I_\varepsilon^5 &= -\omega(J_\varepsilon W_k^+, W_k^+)_{(0,t)\times\Omega_\varepsilon} + k(J_\varepsilon, W_k^+)_{(0,t)\times\Omega_\varepsilon} \\ &\leqslant -\omega J_{\min}\|W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + C\left(k^2\int_0^t\int_{\{W(t)>k\}} \mathrm{d}x\,\mathrm{d}s + \|W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)}\right). \end{aligned}$$

Combining all the results, we obtain

$$\begin{aligned} &\frac{1}{2}\left\|\sqrt{J_\varepsilon}W_k^+(t)\right\|^2_{L^2(\Omega_\varepsilon)} + c_0\|\nabla W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant C(\mu)k^2\int_0^t\int_{\{W(t)>k\}} \mathrm{d}x\,\mathrm{d}s + \left[C(\mu) - \omega J_{\min}\right]\|W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \mu\|\nabla W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)}. \end{aligned}$$

For μ small enough, the term including the gradient on the right-hand side can be absorbed by the left-hand side. Hence, choosing $\omega > \frac{C(\mu)}{J_{\min}}$ in the inequality above (for fixed δ), we obtain for almost every $t \in (0,T)$

$$\|W_k^+\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla W_k^+\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \leqslant Ck^2\int_0^t\int_{\{W(t)>k\}} \mathrm{d}x\,\mathrm{d}s. \tag{5.11}$$

Now, the result follows from II Theorem 6.1 of Ref. 40 implying

$$u_\varepsilon \leqslant C$$

for a constant $C > 0$ independent of ε. Repeating these arguments with $-u_\varepsilon$ instead of u_ε, we obtain

$$-u_\varepsilon \leqslant C,$$

which gives the essential boundedness of u_ε uniformly with respect to ε.

It remains to show the non-negativity. For this, we choose $(u_\varepsilon)^- := \min(u_\varepsilon, 0)$ as a test function in (2.14a) and obtain (by a similar calculation as above for W_k^+)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u_\varepsilon)^-\|^2_{L^2(\Omega_\varepsilon)} + c_0\|\nabla(u_\varepsilon)_-\|^2_{L^2(\Omega_\varepsilon)}&\\ &\leqslant \int_{\Omega_\varepsilon} J_\varepsilon f(u_\varepsilon)(u_\varepsilon)^- \,\mathrm{d}x - \rho\int_{\Gamma_\varepsilon} J_\varepsilon g\left(u_\varepsilon, \varepsilon^{-1}R_\varepsilon\right)(u_\varepsilon)^- \,\mathrm{d}\sigma\\ &\leqslant C\|(u_\varepsilon)^-\|^2_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Now, the Gronwall inequality implies the non-negativity of the microscopic concentration u_ε. □

6. Derivation of the macroscopic model

In this section, we show that the microscopic solution $(u_\varepsilon, w_\varepsilon, p_\varepsilon, R_\varepsilon)$ converges in a suitable sense to the solution of the macroscopic model (3.3) formulated in Section 3 for $\varepsilon \to 0$. In a first step, we establish compactness results obtained from the a priori estimates in Section 5 and, in a second step, we pass to the limit $\varepsilon \to 0$ in the weak formulation of the microscopic problem in Definition 2.1, which corresponds to the passage to the limit illustrated on the bottom of Figure 1.

6.1. *Compactness results for the microscopic solutions*

Based on the a priori estimates from Section 5, we derive compactness results for the microscopic solutions. As a suitable convergence concept for the treatment of problems including oscillating coefficients and domains, we use the two-scale convergence, see Appendix C.1. Since we are dealing with functions on perforated domains, we have to extend these functions to the whole domain. While for the fluid velocity w_ε it is natural to use the zero extension, the situation is more complicated for the pressure p_ε and the concentration u_ε. For the pressure, we use an extension from Tartar given in Ref. 59, see also Ref. 42. For the concentration, we would lose spatial regularity for the zero extension and, therefore, we use an extension operator from Ref. 17, see also Ref. 1 for more general domains. Namely, there exists an extension operator $E_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega)$ such that

$$\begin{aligned}\|E_\varepsilon\phi_\varepsilon\|_{L^2(\Omega)} &\leqslant C\|\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)},\\ \|E_\varepsilon\phi_\varepsilon\|_{H^1(\Omega)} &\leqslant C\|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon)}\end{aligned}$$

for all $\phi_\varepsilon \in H^1(\Omega_\varepsilon)$ with a constant $C > 0$ independent of ε. For time dependent functions, we apply this operator pointwisely in t. We write in short

$$\widetilde{u}_\varepsilon := E_\varepsilon u_\varepsilon \in L^2((0,T), H^1(\Omega)) \tag{6.1}$$

and we obtain

$$\|\widetilde{u}_\varepsilon\|_{L^2((0,T),H^1(\Omega))} + \|\widetilde{u}_\varepsilon\|_{L^\infty((0,T),L^2(\Omega))} \leqslant C. \tag{6.2}$$

However, we emphasise that this extension operator in general does not preserve the regularity for the time derivative and, therefore, we lose control for the time variable of the microscopic sequence $\widetilde{u}_\varepsilon$, which is important to obtain strong two-scale compactness results. The strong two-scale convergence of u_ε was shown for the same a priori estimates in Proposition 5 of Ref. 31 and with additional estimates for the differences of shifts with respect to time in Ref. 72. Here, we simplify the proof of Ref. 31 by showing the following general strong two-scale compactness result. The Proposition will then be applied to J_ε and u_ε, so we keep the same notation, but we slightly generalise the assumptions for the sake of the proposition.

For the sake of readability, and in view of the context in which the proposition is applied, we state the result in terms of functions J_ε and u_ε, but the assumptions are slightly more general than needed here.

Proposition 6.1. *Let $n \geqslant 2$ and $V_\varepsilon \subset H^1(\Omega_\varepsilon)$ dense in $L^2(\Omega_\varepsilon)$. Let $u_\varepsilon \in L^2((0,T),V_\varepsilon)$ and $J_\varepsilon \in L^\infty((0,T)\times\Omega_\varepsilon)$ with $\partial_t(J_\varepsilon u_\varepsilon) \in L^2((0,T),V_\varepsilon')$ with respect to the usual Gelfand triple $V_\varepsilon \subset L^2(\Omega_\varepsilon) \subset V_\varepsilon'$. Further, let $J_\varepsilon \geqslant c_0 > 0$ for a constant $c_0 > 0$ independent of ε and*

$$\|\partial_t(J_\varepsilon u_\varepsilon)\|_{L^2((0,T),V_\varepsilon')} + \|u_\varepsilon\|_{L^2((0,T),H^1(\Omega_\varepsilon))} \leqslant C.$$

Moreover, assume there exists $\kappa(h)$ with $\kappa(h) \to 0$ for $h \to 0$ and $h > 0$ such that

$$\|J_\varepsilon(\cdot + h) - J_\varepsilon\|_{L^\infty((0,T-h),L^s(\Omega_\varepsilon))} \leqslant \kappa(h) \tag{6.3}$$

with $s > 1$ arbitrary for $n = 2$ and $s = \frac{n}{2}$ for $n \geqslant 3$. Then, there exists $\kappa_0(h)$ with $\kappa_0(h) \to 0$ for $h \to 0$ such that

$$\|u_\varepsilon(\cdot + h) - u_\varepsilon\|_{L^2((0,T-h),L^2(\Omega_\varepsilon))} \leqslant \kappa_0(h).$$

In particular, there exists $u_0 \in L^2((0,T)\times\Omega)$ (even in $L^2((0,T),H^1(\Omega))$) such that, up to a subsequence, $\chi_{\Omega_\varepsilon} u_\varepsilon$ converges strongly in the two-scale sense to $\chi_{Y^*} u_0$.

Proof. For a function $\phi : (0,T)\times U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$, we define the shift with respect to time for $0 < h \ll 1$ and $t \in (0, T-h)$ by

$$\delta_h\phi(t) := \delta_h\phi(t, \cdot_x) := \phi(t+h) - \phi(t).$$

Since $J_\varepsilon \geqslant c_0 > 0$ almost everywhere in $(0,T)\times\Omega_\varepsilon$, it holds for almost every $t \in (0,T-h)$ that

$$c_0\|\delta_h u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \leqslant \int_{\Omega_\varepsilon} J_\varepsilon(t+h)(\delta_h u_\varepsilon(t))^2\,\mathrm{d}x.$$

By an elementary calculation, we get

$$J_\varepsilon(t+h)(\delta_h u_\varepsilon(t))^2 = \delta_h(J_\varepsilon u_\varepsilon)(t)\delta_h u_\varepsilon(t) - \delta_h J_\varepsilon(t) u_\varepsilon(t)\delta_h u_\varepsilon(t),$$

from which we obtain

$$\begin{aligned} c_0\|\delta_h u_\varepsilon\|^2_{L^2((0,T-h)\times\Omega_\varepsilon)} \leqslant & \int_0^{T-h}\int_{\Omega_\varepsilon} \delta_h(J_\varepsilon u_\varepsilon)(t)\delta_h u_\varepsilon(t)\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_0^{T-h}\int_{\Omega_\varepsilon} \delta_h J_\varepsilon(t) u_\varepsilon(t)\delta_h u_\varepsilon(t)\,\mathrm{d}x\,\mathrm{d}t \\ =: & \, A^1_{\varepsilon,h} + A^2_{\varepsilon,h}. \end{aligned}$$

Let us estimate the two terms separately. We start with the second term and restrict ourselves to the case $n \geqslant 3$ (the case $n = 2$ follows by similar arguments). Using the Sobolev embedding $H^1(\Omega_\varepsilon) \hookrightarrow L^{2^*}(\Omega_\varepsilon)$ with $2^* := \frac{2n}{n-2}$, we obtain $u_\varepsilon \in L^2((0,T), L^{2^*}(\Omega_\varepsilon))$ such that

$$\|u_\varepsilon\|_{L^2((0,T),L^{p^*}(\Omega_\varepsilon))} \leqslant C\|u_\varepsilon\|_{L^2((0,T),H^1(\Omega_\varepsilon))} \leqslant C$$

with a constant $C > 0$ independent of ε. More precisely, this constant is independent of ε owing to the existence of the extension operator E_ε defined at the beginning of Section 6.1. Now, the Hölder inequality implies

$$\begin{aligned} |A^2_{\varepsilon,h}| &\leqslant \int_0^{T-h} \|\delta_h J_\varepsilon(t)\|_{L^{\frac{n}{2}}(\Omega_\varepsilon)} \|u_\varepsilon(t)\|_{L^{2^*}(\Omega_\varepsilon)} \left(\|u_\varepsilon(t+h)\|_{L^{2^*}(\Omega_\varepsilon)} + \|u_\varepsilon(t)\|_{L^{2^*}(\Omega_\varepsilon)}\right)\mathrm{d}t \\ &\leqslant 2\|u_\varepsilon\|^2_{L^2((0,T),L^{2^*}(\Omega_\varepsilon))}\|\delta_h J_\varepsilon\|_{L^\infty((0,T),L^{\frac{n}{2}}(\Omega_\varepsilon))} \\ &\leqslant C\kappa(h). \end{aligned}$$

For the first term $A^1_{\varepsilon,h}$, we have (using the natural embedding in the Gelfand triple $V_\varepsilon \hookrightarrow L^2(\Omega_\varepsilon) \hookrightarrow V'_\varepsilon$)

$$\begin{aligned} A^1_{\varepsilon,h} &= \int_0^{T-h} \langle \delta_h (J_\varepsilon u_\varepsilon)(t), \delta_h u_\varepsilon(t) \rangle_{V'_\varepsilon, V_\varepsilon} \,\mathrm{d}t \\ &\leqslant \underbrace{\|\delta_h (J_\varepsilon u_\varepsilon)\|_{L^2((0,T-h),V'_\varepsilon)}}_{\leqslant h \|\partial_t (J_\varepsilon u_\varepsilon)\|_{L^2((0,T),V'_\varepsilon)}} \|\delta_h u_\varepsilon(t)\|_{L^2((0,T),H^1(\Omega_\varepsilon))} \\ &\leqslant Ch. \end{aligned}$$

Altogether, we obtain

$$\|\delta_h u_\varepsilon\|_{L^2((0,T-h)\times\Omega_\varepsilon)} \leqslant C\sqrt{h + \kappa(h)} =: \kappa_0(h)$$

and, obviously, $\kappa_0(h) \to 0$ for $h \to 0$.

It remains to show the strong two-scale convergence of u_ε. The proof follows similar ideas as in Lemma 10 of Ref. 28. Using the extension $\widetilde{u}_\varepsilon$ of u_ε defined in (6.1), we obtain with (6.2)

$$\left\| \int_{t_1}^{t_2} \widetilde{u}_\varepsilon \,\mathrm{d}t \right\|_{H^1(\Omega)} \leqslant C \|\widetilde{u}_\varepsilon\|_{L^2((0,T),H^1(\Omega))} \leqslant C.$$

Then, noting the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we have that $\int_{t_1}^{t_2} \widetilde{u}_\varepsilon \,\mathrm{d}t$ is relatively compact in $L^2(\Omega)$. Further, it holds that (the extension operator is pointwisely defined in time)

$$\|\delta_h \widetilde{u}_\varepsilon\|_{L^2((0,T-h),L^2(\Omega))} \leqslant C \|\delta_h u_\varepsilon\|_{L^2((0,T-h),L^2(\Omega_\varepsilon))} \leqslant C\kappa_0(h) \xrightarrow{h\to 0} 0.$$

Therefore, by the Kolmogorov–Simon compactness theorem, see Theorem 1 of Ref. 63, $\widetilde{u}_\varepsilon$ converges strongly in $L^2((0,T)\times\Omega)$. This gives the desired result. □

Remark 6.1.

(i) Proposition 6.1 is not restricted to the case of radially symmetric perforations and remains valid for arbitrarily perforated domains Ω_ε as long as there exists a linear extension operator $E^*_\varepsilon \colon H^1(\Omega_\varepsilon) \to H^1(\Omega)$ such that

$$\|E^*_\varepsilon u_\varepsilon\|_{H^1(\Omega)} \leqslant C \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \qquad \text{for all } u_\varepsilon \in H^1(\Omega_\varepsilon).$$

Such an operator exists for example for periodically perforated connected Lipschitz domains, see Ref. 1 for more details. In particular, the result is valid for $\Omega_\varepsilon = \Omega$ and gives (besides the strong two-scale convergence) the strong convergence in $L^2((0,T)\times\Omega)$ in this case.

(ii) Under the additional assumption $u_\varepsilon \in L^\infty((0,T),L^2(\Omega_\varepsilon))$ with

$$\|u_\varepsilon\|_{L^\infty((0,T),L^2(\Omega_\varepsilon))} \leqslant C$$

for a constant $C > 0$ independent of ε, we can replace (6.3) with

$$\|J_\varepsilon(\cdot + h) - J_\varepsilon\|_{L^2((0,T-h),L^\infty(\Omega_\varepsilon))} \leqslant \kappa(h)$$

and obtain the same result (using slightly different arguments for the term $A^2_{\varepsilon,h}$ in the proof).

In the following proposition, we give the compactness results for the microscopic solutions. The results for $(u_\varepsilon, R_\varepsilon)$ have already been established in Ref. 31 for the same a priori estimates as in Lemma 5.1 and the Lipschitz estimate for R_ε from Lemma 4.2 (with slightly stronger regularity assumptions on $R_\varepsilon(0)$). Similar compactness results were also obtained in Ref. 72, where an additional estimate for the differences of the shifts of u_ε with respect to time was shown for the strong convergence of u_ε, and the strong convergence of R_ε was shown by a direct comparison with the macroscopic radii R_0 using the microscopic and macroscopic ODE.

For u_ε, we use the extension from (6.1). Further, since $w_\varepsilon \in L^\infty((0,T);H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon))$, we can extend it by zero to a function $w_\varepsilon \in L^\infty((0,T);H^1(\Omega))$ in order to state the convergence. Moreover, to obtain not only a weak but also strong convergence result for the pressure, we extend it as follows (see Ref. 42):

$$Q_\varepsilon(t,x) := \begin{cases} q_\varepsilon(t,x) & \text{if } x \in \Omega_\varepsilon, \\ \frac{1}{|Y^*|} \int\limits_{k+\varepsilon Y^*} q_\varepsilon(t,z)\,\mathrm{d}z & \text{if } x \in k + \varepsilon(Y\backslash Y^*) \text{ for } k \in K_\varepsilon. \end{cases} \tag{6.4}$$

Note that this extension coincides with the extension from Tartar in Ref. 59.

Proposition 6.2. *The microscopic solutions $(u_\varepsilon, w_\varepsilon, p_\varepsilon, R_\varepsilon)$ fulfill the following convergence results up to a subsequence:*

(i) There exist $u_0 \in L^2((0,T), H_0^1(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ and $u_1 \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(Y^*)/\mathbb{R})$ such that

$$\begin{aligned} \widetilde{u}_\varepsilon &\to u_0 \quad \text{in } L^2((0,T)\times\Omega),\\ \chi_{\Omega_\varepsilon}\nabla u_\varepsilon &\xrightarrow{2,2} \chi_{Y^*}\left(\nabla u_0 + \nabla_y u_1\right),\\ \chi_{\Omega_\varepsilon} u_\varepsilon &\xrightarrow{2,2} \chi_{Y^*} u_0,\\ u_\varepsilon|_{\Gamma_\varepsilon} &\xrightarrow{2,2} u_0 \quad \text{on } \Gamma_\varepsilon. \end{aligned}$$

(ii) There exists $R_0 \in H^1((0,T), L^2(\Omega)) \cap L^\infty((0,T)\times\Omega)$ with $\partial_t R_0 \in L^\infty((0,T)\times\Omega)$ such that for all $p \in [1,\infty)$

$$\varepsilon^{-1}R_\varepsilon \to R_0 \qquad \text{in } L^p((0,T)\times\Omega), \tag{6.5}$$
$$\varepsilon^{-1}\partial_t R_\varepsilon \to \partial_t R_0 \qquad \text{in } L^p((0,T)\times\Omega). \tag{6.6}$$

These convergences are also valid in the strong two-scale sense in Ω and Γ_ε. Further, it holds almost everywhere in $(0,T)\times\Omega$ that

$$\partial_t R_0 = g(u_0, R_0).$$

(iii) There exists $w_0 \in L^\infty((0,T); L^2(\Omega; H^1_\#(Y)))$ with

$$w_0 = 0 \qquad \text{in } (0,T)\times\Omega\times(Y\backslash Y^*), \tag{6.7}$$
$$\nabla\cdot_y(A_0 w_0) = 0 \qquad \text{in } (0,T)\times\Omega\times Y, \tag{6.8}$$
$$\nabla\cdot_x\left(\int_{Y^*} A_0 w_0\right) = -\partial_t\theta \qquad \text{in } (0,T)\times\Omega\times Y, \tag{6.9}$$

such that for the zero extension of w_ε to Ω it holds for every $p \in [1,\infty)$ that

$$w_\varepsilon \xrightarrow{2,p,2} w_0, \tag{6.10}$$
$$\varepsilon\nabla w_\varepsilon \xrightarrow{2,p,2} \nabla_y w_0, \tag{6.11}$$
$$\varepsilon e_\varepsilon(w_\varepsilon) \xrightarrow{2,p,2} e_0(w_0), \tag{6.12}$$

where $e_0(w_0) = \frac{1}{2}(F_0^{-\top}\nabla_y w_0 + \nabla_y w_0^\top F_0^{-1})$ and F_0 is defined in (6.16) below. Moreover, there exists $Q_0 \in L^\infty((0,T); H_0^1(\Omega))$ such that for every $p \in [1,\infty)$

$$Q_\varepsilon \to Q_0 \qquad \text{in } L^p((0,T); L^2(\Omega)), \tag{6.13}$$
$$Q_\varepsilon + p_{\mathrm{b}} =: P_\varepsilon \to P_0 := Q_0 + p_{\mathrm{b}} \qquad \text{in } L^p((0,T), L^2(\Omega)). \tag{6.14}$$

Proof. We start with (i). The existence of u_0 and u_1 and the weak two-scale convergence results follow from the a priori estimates in Lemma 5.1 and the two-scale compactness results from Lemma Appendix C.1 in the appendix. Using again the a priori estimates from Lemma 5.1, the strong convergence of $\widetilde{u}_\varepsilon$ follows from Proposition 6.1 provided that (6.3) holds true. However, this is a direct consequence of Lemma Appendix A.3 (with $R^1_\varepsilon = R_\varepsilon$ and $R^2_\varepsilon := R_\varepsilon(\cdot_t + h, \cdot_x))$ and the Lipschitz continuity of R_ε with respect to time ($\varepsilon^{-1}\partial_t R_\varepsilon$ is essentially bounded, see (5.1)).

Next, we consider the convergence results in (ii) for R_ε. The first result (6.5) was shown in Proposition 4 of Ref. 31, and in the strong two-scale sense on the surface Γ_ε in Corollary 7 Ref. 31 but with slightly stronger regularity assumptions on the initial values $R_\varepsilon^{\mathrm{in}}$ and R_0^{in}. In particular, it was assumed that $R_0^{\mathrm{in}} \in H^1(\Omega)$. Here, we dropped this assumption and only assume (as in Ref. 72) the strong convergence of $\varepsilon^{-1}R_\varepsilon^{\mathrm{in}}$ in $L^2(\Omega)$. However, since the proof still follows the same lines as in Proposition 4 Ref. 31, we only sketch the main ideas. We use the standard Kolmogorov compactness result, see for example Ref. 9, and since R_ε is bounded in $L^2(\Omega)$ it is enough to control the shifts in time and space. In what follows, we extend R_ε and u_ε by zero outside Ω, which is possible due to the zero boundary condition of u_ε on $\partial\Omega$. (It is also possible to consider shifts within Ω but then the argumentation becomes more technical.) The essential bound of $\partial_t R_\varepsilon$ implies for all $h > 0$ and $t \in (0, T - h)$

$$\varepsilon^{-1}\|R_\varepsilon(\cdot_t + h, \cdot_x) - R_\varepsilon\|_{L^\infty((0,T-h)\times\Omega)} \leqslant Ch.$$

For the spatial variable, it is enough to consider shifts with εl for $l \in \mathbb{Z}^n$ (see also (42) in Ref. 31). From the Lipschitz estimate (4.7) in Lemma 4.2, we obtain, using the trace inequality (see Lemma Appendix B.3 in the appendix), the strong convergence of $\varepsilon^{-1}R_\varepsilon^{\text{in}}$ in $L^2(\Omega)$ (which guarantees a bound for the shifts), the mean value theorem (u_ε is extended via $\widetilde{u}_\varepsilon$ to the whole domain Ω) and the a priori bound for ∇u_ε already obtained above,

$$\varepsilon^{-1}\|R_\varepsilon(\cdot_t, \cdot_x + \varepsilon l) - R_\varepsilon\|_{L^\infty((0,T),L^2(\Omega))} \leqslant C\left(\varepsilon + \kappa(|\varepsilon l|)\right)$$

with $\kappa(|\varepsilon l|) \to 0$ for $|\varepsilon l| \to 0$. Hence, the Kolmogorov compactness result implies the desired result.

The convergence of the time derivative $\partial_t R_\varepsilon$ was shown in Proposition 6 of Ref. 31 and Theorem 17 of Ref. 72, but here we give a much simpler compactness argument using the unfolding operator. More precisely, on page 27 of Ref. 72, the strong two-scale convergence of the radii and their time derivatives was shown by considering the difference between the microscopic and macroscopic radii R_ε and R_0 and using their respective equations (the equation for R_0 can be obtained a priori by using formal arguments without using compactness of the radii R_ε). Using the unfolding operator $\mathcal{T}_\varepsilon$ (see Appendix C.2), we obtain from the ODE (2.14b)

$$\varepsilon^{-1}\partial_t R_\varepsilon(t,x) = \frac{1}{|\Gamma|}\int_\Gamma g\left(\mathcal{T}_\varepsilon(u_\varepsilon)(t,x,z), \mathcal{T}_\varepsilon(\varepsilon^{-1}R_\varepsilon)(t,x,z)\right) \mathrm{d}\sigma_z$$

for almost every $(t,x) \in (0,T)\times\Omega$. Using the strong two-scale convergence of u_ε and $\varepsilon^{-1}R_\varepsilon$ on Γ_ε, we obtain

$$\begin{aligned} \mathcal{T}_\varepsilon(\varepsilon^{-1}R_\varepsilon) &\to R_0 && \text{in } L^p((0,T)\times\Omega\times\Gamma), \\ \mathcal{T}_\varepsilon(u_\varepsilon) &\to u_0 && \text{in } L^2((0,T)\times\Omega\times\Gamma). \end{aligned}$$

Hence, up to a subsequence, these convergences also hold pointwise almost everywhere in $(0,T)\times\Omega\times\Gamma$. Then, for almost every $(t,x) \in (0,T)\times\Omega\times\Gamma$, we get

$$\varepsilon^{-1}\partial_t R_\varepsilon(t,x) \to \frac{1}{|\Gamma|}\int_\Gamma g(u_0(t,x), R_0(t,x))\,\mathrm{d}\sigma_z = g(u_0(t,x), R_0(t,x)). \tag{6.15}$$

Lebesgue's dominated-convergence theorem now implies the convergence in $L^p((0,T)\times\Omega)$. It remains to show the convergence of the traces of $\partial_t R_\varepsilon$. Since $\partial_t R_\varepsilon$ is constant on every microscopic cell, we also have that $\mathcal{T}_\varepsilon(\partial_t R_\varepsilon)(t,x,\cdot_y)$ is constant for almost every (t,x). Hence, using the trace inequality, we obtain (with $\nabla_y(\mathcal{T}_\varepsilon(\varepsilon^{-1}\partial_t R_\varepsilon) - g(u_0,R_0)) = 0$)

$$\|\mathcal{T}_\varepsilon(\varepsilon^{-1}\partial_t R_\varepsilon) - g(u_0,R_0)\|_{L^p((0,T)\times\Omega\times\Gamma)} \leqslant C\|\mathcal{T}_\varepsilon(\varepsilon^{-1}\partial_t R_\varepsilon) - g(u_0,R_0)\|_{L^p((0,T)\times\Omega\times Y*)} \to 0.$$

Last, we consider the convergence results in (iii). The compactness results (6.10) and (6.11) with the properties (6.7), (6.8), (6.9) for the limit function w_0 are shown in Lemma 4.9 of Ref. 73 and Corollary 5.2 of Ref. 73. Moreover, (6.12) can be concluded from (6.11) and the strong two-scale convergence of F_ε^{-1}, which is given by Lemma 6.2 below, by a standard argument. (Use the unfolding operator and the fact that a product of a weakly L^2-convergent and a strongly L^2-convergent sequence, which is additionally uniformly essentially bounded, converges weakly in L^2-sense.) The strong convergence of $Q_\varepsilon(t)$ in $L^2(\Omega)$ to $Q_0(t)$ was shown in Lemma 4.8 of Ref. 73 for almost every $t \in (0,T)$ similarly as in Ref. 2 (where different boundary conditions were considered). The uniform pointwise upper bound for $\|q_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}$ for almost every $t \in (0,T)$ leads to a uniform pointwise upper bound for the extension $\|Q_\varepsilon(t)\|_{L^2(\Omega)}$ for almost every $t \in (0,T)$. Thus, Lebesgue's dominated convergence theorem provides the strong convergence $Q_\varepsilon \to Q_0$ in $L^p((0,T);L^2(\Omega))$ for every $p \in [1,\infty)$. We note that the a priori estimates and the compactness results of Ref. 73 are formulated pointwisely with respect to time but can easily be adapted to the time-dependent two-scale convergence, since the time variable only acts as a parameter, see also Ref. 16 for results on the parameter-dependent unfolding operator. □

As an immediate consequence of the strong convergence results for R_ε and $\partial_t R_\varepsilon$, we obtain strong two-scale compactness for the coefficients in the transformed equations. Therefore, we define the limit transformation ψ_0 and its Jacobians with respect to the microscopic variable by (see also (2.4) for the definition of ψ)

$$\begin{aligned}\check{\psi}_0(t,x,y) &:= \check{\psi}(R_0(t,x),y) = (R_0(t,x)-R_{\max})\tfrac{y-m}{\|y-m\|}\chi(\|y-\mathfrak{m}\|),\\ \psi_0(t,x,y) &:= y+\check{\psi}_0(t,x,y) = y+(R_0(t,x)-R_{\max})\tfrac{y-m}{\|y-m\|}\chi(\|y-\mathfrak{m}\|).\end{aligned}$$

Thus, $\psi_0(t,x,\cdot)$ is a diffeomorphism from Y^* onto $Y^*(t,x)$ for a.e. $(t,x)\in(0,T)\times\Omega$ with inverse $\psi_0^{-1}(t,x,y)=\psi^{-1}(R_0(t,x),y)$ for $y\in Y^*(t,x)$. Moreover, we define analogously to the ε-scaled setting

$$F_0 := \nabla_y\psi_0^\top, \qquad J_0 := \det(F_0), \qquad A_0 := J_0F_0^{-1}, \qquad D_0 := J_0F_0^{-1}DF_0^{-\top}. \tag{6.16}$$

In the following lemma, we show the strong two-scale convergence for the derivatives of ψ_ε. Since the coefficients $F_\varepsilon, A_\varepsilon, D_\varepsilon$ and J_ε in the weak form (2.14) can be expressed as rational functions of the derivatives of ψ_ε, we can conclude the strong two-scale convergence for them in Lemma 6.2 afterwards.

Lemma 6.1. *Let R_ε be the subsequence from Proposition 6.2 and ψ_ε be given by (2.5) and $\check{\psi}_\varepsilon$ be given by (2.6). Then, it holds*

$$\begin{aligned}\varepsilon^{|\alpha|-1}\partial_{x_\alpha}\check{\psi}_\varepsilon &\xrightarrow{2,p} \partial_{y_\alpha}\check{\psi}_0,\\ \varepsilon^{|\alpha|}\partial_{x_\alpha}\nabla\psi_\varepsilon &\xrightarrow{2,p} \partial_{y_\alpha}\nabla_y\psi_0,\\ \varepsilon^{|\alpha|-1}\partial_t\partial_{x_\alpha}\psi_\varepsilon = \varepsilon^{|\alpha|-1}\partial_t\partial_{x_\alpha}\check{\psi}_\varepsilon &\xrightarrow{2,p} \partial_t\partial_{y_\alpha}\check{\psi}_0 = \partial_t\partial_{y_\alpha}\psi_0\end{aligned}$$

for all $\alpha\in\mathbb{N}_0^n$.

Proof. We follow the argumentation of Ref. 72. We note that $R_\varepsilon \xrightarrow{2,p} R_0$ and $\partial_t R_\varepsilon \xrightarrow{2,p} \partial_t R_0$ yield the pointwise convergences $\mathcal{T}_\varepsilon R_\varepsilon(t,x)\to R_0(t,x)$ and $\partial_t\mathcal{T}_\varepsilon R_\varepsilon(t,x)\to\partial_t R_0(t,x)$ for a.e. $(t,x)\in(0,T)\times\Omega$ for a subsequence. By employing the continuity of $r\mapsto\partial_{y_\alpha}\check{\psi}(r,y)$ and $r\mapsto\partial_{y_\alpha}\partial_t\check{\psi}(r,y)$ for $\alpha\in\mathbb{N}_0^n$, we obtain

$$\begin{aligned}\varepsilon^{|\alpha|-1}\mathcal{T}_\varepsilon\partial_{x_\alpha}\check{\psi}_\varepsilon(t,x,y) &= \varepsilon^{-1}\partial_{y_\alpha}\mathcal{T}_\varepsilon\check{\psi}_\varepsilon(t,x,y) = \partial_{y_\alpha}(\check{\psi}(\mathcal{T}_\varepsilon R_\varepsilon(t,x),\{[x]+\varepsilon y\}))\\ &= \partial_{y_\alpha}\check{\psi}(\mathcal{T}_\varepsilon R_\varepsilon(t,x),y)\to\partial_{y_\alpha}\check{\psi}(R_0(t,x),y)=\partial_{y_\alpha}\check{\psi}_0(t,x,y)\end{aligned}$$

and

$$\begin{aligned}\varepsilon^{|\alpha|-1}\mathcal{T}_\varepsilon\partial_t\partial_{x_\alpha}\check{\psi}_\varepsilon(t,x,y) &= \varepsilon^{-1}\partial_{y_\alpha}\mathcal{T}_\varepsilon\partial_t\check{\psi}_\varepsilon(t,x,y) = \partial_{y_\alpha}\partial_t(\check{\psi}(\mathcal{T}_\varepsilon R_\varepsilon(t,x),\{[x]+\varepsilon y\}))\\ &= \partial_{y_\alpha}\partial_t(\check{\psi}(\mathcal{T}_\varepsilon R_\varepsilon(t,x),y)) = \partial_{y_\alpha}\partial_R\check{\psi}(\mathcal{T}_\varepsilon R_\varepsilon(t,x),y)\partial_t\mathcal{T}_\varepsilon R_\varepsilon(t,x)\\ &\to \partial_{y_\alpha}\partial_R\check{\psi}(R_0(t,x),y)\partial_t R_0(t,x) = \partial_{y_\alpha}\partial_t\check{\psi}_0(t,x,y)\end{aligned}$$

for a.e. $(t,x,y)\in(0,T)\times\Omega\times Y$. Then, we obtain $\varepsilon^{|\alpha|-1}\mathcal{T}_\varepsilon\partial_{x_\alpha}\check{\psi}_\varepsilon\to\partial_{y_\alpha}\check{\psi}_0$ and $\varepsilon^{|\alpha|-1}\mathcal{T}_\varepsilon\partial_{x_\alpha}\partial_t\check{\psi}_\varepsilon\to\partial_{y_\alpha}\partial_t\check{\psi}_0$ by Lebesgue's convergence theorem, which yields $\varepsilon^{|\alpha|-1}\partial_{x_\alpha}\check{\psi}_\varepsilon\xrightarrow{2,p}\partial_{y_\alpha}\check{\psi}_0$ and $\varepsilon^{|\alpha|-1}\partial_{x_\alpha}\partial_t\check{\psi}_\varepsilon\xrightarrow{2,p}\partial_{y_\alpha}\partial_t\check{\psi}_0$. Since this argumentation holds for every arbitrary subsequence, it holds for the whole sequence. Since $F_\varepsilon=\partial_x\check{\psi}_\varepsilon+\mathbb{1}$ and $F_0=\partial_y\check{\psi}_0+\mathbb{1}$, we can transfer the convergence to F_ε. Moreover, it can be easily observed that $\partial_t\psi_\varepsilon=\partial_t\check{\psi}_\varepsilon$ as well as $\partial_t\psi_0=\partial_t\check{\psi}_0$. □

Lemma 6.2. *Let R_ε be the subsequence from Proposition 6.2. Then, it holds for every $p\in[1,\infty)$ that*

$$\begin{aligned}&F_\varepsilon\xrightarrow{2,p}F_0,\quad J_\varepsilon\xrightarrow{2,p}J_0,\quad J_\varepsilon^{-1}\xrightarrow{2,p}J_0^{-1},\quad F_\varepsilon^{-1}\xrightarrow{2,p}F_0^{-1},\\ &D_\varepsilon\xrightarrow{2,p}D_0,\quad A_\varepsilon\xrightarrow{2,p}A_0,\quad A_\varepsilon^{-1}\xrightarrow{2,p}A_0^{-1},\quad \varepsilon\partial_x A_\varepsilon\xrightarrow{2,p}\partial_y A_0,\\ &\varepsilon\partial_x A_\varepsilon^{-1}\xrightarrow{2,p}\partial_y A_0^{-1},\quad \partial_t J_\varepsilon\xrightarrow{2,p}\partial_t J_0,\quad \varepsilon^{-1}\partial_x\partial_t\psi_\varepsilon\xrightarrow{2,p}\partial_t\partial_y\psi_0.\end{aligned}\tag{6.17}$$

The convergence of J_ε is also valid on the surface Γ_ε.

Proof. The convergences of $F_\varepsilon, J_\varepsilon$ and D_ε were shown in Corollary 8 of Ref. 31 using the structure of ψ_ε and the strong convergence of R_ε. In Lemma 3.3 of Ref. 71, Lemma 10 of Ref. 72 and Lemma 4.6 of

Ref. 73, a more abstract argumentation was used in order to provide additionally the strong convergences for J_ε^{-1}, F_ε^{-1}, A_ε, A_ε^{-1}, $\varepsilon\partial_x A_\varepsilon$ and $\varepsilon\partial_x A_\varepsilon^{-1}$ from the first two convergences of Lemma 6.1. The strong two-scale convergence of J_ε follows with the two-scale convergence of $\varepsilon^{-1}\partial_t\partial_x\psi_\varepsilon$ by the same argumentation. For sake of completeness, we recap this argumentation briefly. First, we note that the convergence of $F_\varepsilon = \nabla\psi_\varepsilon^\top$ is given by Lemma 6.1. Since $J_\varepsilon = \det(F_\varepsilon)$ is a polynomial with respect to the entries of F_ε, the strong two-scale convergence can be transferred on J_ε. Having the bound for $J_\varepsilon \geqslant c_J$ from below, we can transfer the strong two-scale convergence of J_ε to J_ε^{-1}.

In the next step, we notice that all other terms on the left-hand side in (6.17) can be represented as polynomials with respect to the entries of $\nabla\psi_\varepsilon$, $\varepsilon\partial_x\partial_x\psi_\varepsilon$, $\partial_t\nabla\psi_\varepsilon$, $\varepsilon\partial_t\partial_x\partial_x\psi_\varepsilon$ and J_ε^{-1}. Their convergences are provided by Lemma 6.1 and, thus, can be transferred to these polynomials, which yields (6.17).

The strong two-scale convergence of J_ε on the surface Γ_ε follows with the same argumentation and using the unfolding operator for periodic surfaces in the proof of Lemma 6.1. □

Finally, we have the following weak compactness result for the time derivative of $J_\varepsilon u_\varepsilon$:

Corollary 6.1. *Let $\widehat{J_\varepsilon u_\varepsilon}$ denote the zero extension of $J_\varepsilon u_\varepsilon$. Then, it holds up to a subsequence that*

$$\partial_t\left(\widehat{J_\varepsilon u_\varepsilon}\right) \rightharpoonup \partial_t(\theta u_0) \quad \text{weakly in } L^2((0,T), H^{-1}(\Omega)), \tag{6.18}$$

where the porosity θ is defined in (3.2).

Proof. This proof was shown on p. 133–134 in Ref. 31 for slightly different boundary conditions and can be easily adapted to our setting. We emphasise that in our notations we have

$$\theta(t,x) = |Y^*(t,x)| = \int_{Y^*(t,x)} \mathrm{d}y = \int_{Y^*} J_0(t,x,y)\,\mathrm{d}y.$$ □

6.2. *Proof of Theorem 3.2*

Now, we are able to prove our main result Theorem 3.2, i. e. pass to the limit $\varepsilon \to 0$ in the weak formulation of the microscopic model. The ODE for R_0 in (3.3d) – (3.3e) follows directly from Proposition 6.2(ii). The Darcy law in (3.3f) – (3.3g) was derived in Ref. 73 for a given evolution of the microscopic domain (with the same properties of ψ_ε). Hence, the results are also valid in our setting. However, for the sake of completeness, we will sketch here the essential steps and also use a slightly different proof with an orthogonality argument, see Lemma Appendix B.1 in the appendix, to obtain the two-pressure Stokes model, which was also used by Allaire in Ref. 37. This alternative proof is of interest in itself and provides a method for dealing with Stokes problems in evolving domains. Further, throughout the proof, we work on the fixed reference element and additionally formulate the cell problems and the Darcy velocity on the fixed reference cell Y^*. Finally, the derivation of the macroscopic transport equation (3.3a) – (3.3c) follows similar lines as for pure diffusion without convection in Section 5.2 of Ref. 31 and, therefore, we will focus only on the convective term in our proof below.

We start with the derivation of Darcy's law, where we mainly follow the ideas of Allaire in Ref. 37, and first obtain a generalised two-pressure Stokes model for evolving microdomains. For this, we introduce the space

$$\begin{aligned}\mathcal{V} := \Big\{\eta \in & L^2(\Omega, H^1_{\mathrm{per},0}(Y^*)^n) \ : \\ & \nabla_y\cdot\eta = 0 \text{ in } \Omega\times Y^*,\ \nabla_x\cdot\int_{Y^*}\eta\,\mathrm{d}y = 0 \text{ in } \Omega\Big\}\end{aligned} \tag{6.19}$$

with $H^1_{\mathrm{per},0}(Y^*) := \left\{v \in H^1_{\mathrm{per}}(Y^*) \ : \ v = 0 \text{ on } \Gamma\right\}$. We also define the subspace

$$\mathcal{V}^\infty := \mathcal{V} \cap C^\infty(\overline{\Omega}, H^1_{\mathrm{per},0}(Y^*)^n).$$

The space $\mathcal{V}^\infty$ is a dense subspace of $\mathcal{V}$. For $\eta \in \mathcal{V}^\infty$ and $\psi \in C_0^\infty(0,T)$, we choose

$$\eta_\varepsilon(t,x) := A_\varepsilon^{-1}\psi(t)\eta\left(x, \frac{x}{\varepsilon}\right) \tag{6.20}$$

as a test function in (2.14c) to obtain after integration with respect to time and using $A_\varepsilon : \nabla\eta_\varepsilon = \psi\nabla\cdot\left(\eta\left(\cdot_x, \frac{\cdot_x}{\varepsilon}\right)\right) = \psi\nabla_x\cdot\eta\left(\cdot_x, \frac{\cdot_x}{\varepsilon}\right)$, where the notation is used to clarify that we insert the same variable in both arguments,

$$\begin{aligned}\varepsilon^2\int_0^T\int_{\Omega_\varepsilon} J_\varepsilon e_\varepsilon(w_\varepsilon) : e_\varepsilon(\eta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega_\varepsilon} q_\varepsilon\nabla_x\cdot\eta\left(x,\frac{x}{\varepsilon}\right)\psi\,\mathrm{d}x\,\mathrm{d}t\\ = -\varepsilon^2\int_0^T\int_{\Omega_\varepsilon} J_\varepsilon e_\varepsilon(\partial_t\psi_\varepsilon) : e_\varepsilon(\eta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon}\left[J_\varepsilon A_\varepsilon^{-\top}h_\varepsilon - \nabla p_\mathrm{b}\right]\cdot\eta\left(x,\frac{x}{\varepsilon}\right)\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{6.21}$$

From the strong two-scale convergence results in Lemma 6.2 and the strong two-scale convergence of $\eta\left(\cdot,\frac{\cdot}{\varepsilon}\right)$, which follows from the oscillation lemma Lemma 1.3 of Ref. 3, we obtain that the unfolded sequence $\mathcal{T}_\varepsilon(\varepsilon e_\varepsilon(\eta_\varepsilon))$ converges strongly in $L^s((0,T)\times\Omega\times Y^*)$ to $e_0(A_0^{-1}\eta)\psi$ for $s\in[1,2)$ and, therefore, (up to a subsequence) also pointwise almost everywhere in $(0,T)\times\Omega\times Y^*$. Further, it holds due to the essential bounds for F_ε^{-1} and A_ε^{-1} (and its derivative) almost everywhere in $(0,T)\times\Omega\times Y^*$

$$|\mathcal{T}_\varepsilon(\varepsilon e_\varepsilon(\eta_\varepsilon))| \leqslant C\left(\left|\varepsilon\mathcal{T}_\varepsilon\left(\nabla_x\eta\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right)\right| + \left|\mathcal{T}_\varepsilon\left(\nabla_y\eta\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right)\right|\right).$$

Since the right-hand side converges strongly in $L^2((0,T)\times\Omega\times Y^*)$, Lebesgue's dominated-convergence theorem implies $\varepsilon e_\varepsilon(\eta_\varepsilon) \xrightarrow{2,2} e_0(A_0^{-1}\eta)\psi$. Further, Lemma Appendix A.2 implies that

$$\|e_\varepsilon(\partial_t\psi_\varepsilon)\|_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C$$

for a constant $C > 0$ independent of ε. Hence, the first term on the right-hand side in (6.21) is of order ε and vanishes for $\varepsilon\to 0$. Using the compactness results from Proposition 6.2 and Lemma 6.2, we can pass to the limit $\varepsilon\to 0$ in (6.21) and obtain almost everywhere in $(0,T)$

$$\int_\Omega\int_{Y^*} J_0e_0(w_0) : e_0(A_0^{-1}\eta)\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega\int_{Y^*}\left[J_0A_0^{-\top}h_0 - \nabla p_\mathrm{b}\right]\cdot\eta\,\mathrm{d}y\,\mathrm{d}x = 0.$$

By density this result is valid for all $\eta\in\mathcal{V}$. The left-hand side defines a functional on $L^2(\Omega, H^1_{\mathrm{per},0}(Y^*)^n)$. Due to Lemma Appendix B.1 in the appendix, there exist $P_0\in L^\infty((0,T), H^1_0(\Omega))$ and $q_1\in L^\infty((0,T)\times\Omega\times Y^*)/\mathbb{R}$ such that for all $\eta\in L^2(\Omega, H^1_{\mathrm{per},0}(Y^*)^n)$ it holds almost everywhere in $(0,T)$ that

$$\begin{aligned}\int_\Omega\int_{Y^*} J_0e_0(w_0) : e_0(A_0^{-1}\eta)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega\int_{Y^*}\nabla P_0\cdot\eta\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega\int_{Y^*} q_1\nabla_y\cdot\eta\,\mathrm{d}y\,\mathrm{d}x\\ = \int_\Omega\int_{Y^*} J_0h_0\cdot A_0^{-1}\eta\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega\int_{Y^*}\nabla p_\mathrm{b}\cdot\eta\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{6.22}$$

Repeating the arguments above, where we choose as the test function in (6.20) the function η as $\eta\in C_0^\infty(\Omega, C^\infty_{\mathrm{per},0}(Y^*)^n)$ with $\nabla_y\cdot\eta = 0$, and subtracting the resulting equation from (6.22), we obtain by a density argument for all $\eta\in L^2(\Omega, H^1_{\mathrm{per},0}(Y^*)^n)$ with $\nabla_y\cdot\eta = 0$ almost everywhere in $(0,T)$

$$\int_\Omega\int_{Y^*}\nabla P_0\cdot\eta\,\mathrm{d}y + \int_\Omega\int_{Y^*} Q_0\nabla_x\cdot\eta\,\mathrm{d}y\,\mathrm{d}x = 0.$$

Using Lemma 2.10 of Ref. 3, we obtain $P_0 = Q_0 = p_0 - p_\mathrm{b}$ (where p_0 is the limit of p_ε, see also Proposition 6.2) and, in particular, $p_0\in L^2((0,T), H^1(\Omega))$. Altogether, replacing η with $A_0\eta$ (which is still an admissible test function) in (6.22), we get almost everywhere in $(0,T)$

$$\begin{aligned}\int_\Omega\int_{Y^*} J_0e_0(w_0) : e_0(\eta)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega\int_{Y^*} A_0^\top\nabla p_0\cdot\eta\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega\int_{Y^*} q_1\nabla_y\cdot(A_0\eta)\,\mathrm{d}y\,\mathrm{d}x\\ = \int_\Omega\int_{Y^*} J_0h_0\cdot\eta\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{6.23}$$

In other words, the triple (w_0, p_0, q_1) is the unique weak solution of the two-pressure Stokes model

$$\begin{aligned}
-\nabla_y \cdot (A_0 e_0(w_0)) + A_0^\top \nabla_x p_0 + A_0^\top \nabla_y q_1 &= J_0 h_0 && \text{in } (0,T)\times\Omega\times Y^*,\\
-\nabla_y \cdot (A_0 w_0) &= 0 && \text{in } (0,T)\times\Omega\times Y^*,\\
w_0 &= 0 && \text{on } (0,T)\times\Omega\times\Gamma,\\
-\nabla_x \cdot \int_{Y^*} A_0 w_0 dy &= 0 && \text{in } (0,T)\times\Omega,\\
p_0 &= p_{\mathrm{b}} && \text{on } (0,T)\times\partial\Omega,\\
w_0,\ q_1 \text{ is } Y\text{-periodic.}
\end{aligned} \tag{6.24}$$

An elementary calculation shows that for all $\xi \in \mathbb{R}^n$ it holds almost everywhere in $(0,T)\times\Omega\times Y^*$ that

$$J_0 \xi = A_0^\top \left[\nabla_y \left((\psi_0 - y)\cdot\xi\right)\right] + A_0^\top \xi. \tag{6.25}$$

Hence, we can write the first equation in (6.24) as

$$-\nabla_y \cdot (A_0 e_0(w_0)) + A_0^\top \nabla_y \left[q_1 - (\psi_0 - y)\cdot h_0\right] = A_0^\top \left[h_0 - \nabla_x p_0\right].$$

We emphasise that this substitution holds also for the weak formulation, where we note that $(\psi_0 - y)\cdot h_0$, $A_0^\top$ and the test functions are Y-periodic and that the test functions are zero on Γ. This leads to the following cell problems on the fixed reference element Y^*: We denote for $i = 1,\dots,n$ by $(w_i, \pi_i) \in L^\infty((0,T)\times\Omega, H^1_{\mathrm{per},0}(Y^*)^n) \times L^\infty((0,T)\times\Omega; L^2(Y^*)/\mathbb{R})$ the unique weak solutions of the cell problems

$$\begin{aligned}
-\nabla_y \cdot (A_0 e_0(w_i)) + A_0^\top \nabla_y \pi_i &= A_0^\top e_i && \text{in } (0,T)\times\Omega\times Y^*,\\
\nabla_y \cdot (A_0 w_i) &= 0 && \text{in } (0,T)\times\Omega\times Y^*,\\
w_i,\ \pi_i \text{ are } Y\text{-periodic.}
\end{aligned} \tag{6.26}$$

Since the two-pressure Stokes problem (6.24) admits a unique weak solution, we obtain from the linearity of the problem the following result:

Corollary 6.2. *It holds that*

$$\begin{aligned}
w_0 &= \sum_{i=1}^n (h_0 - \nabla_x p_0)_i w_i,\\
q_1 &= (\psi_0 - y)\cdot h_0 + \sum_{i=1}^n (h_0 - \nabla_x p_0)_i \pi_i
\end{aligned}$$

and the Darcy-velocity v^* is given for almost every $(t,x) \in (0,T)\times\Omega$ by

$$v^*(t,x) := \int_{Y^*} A_0 w_0 \,\mathrm{d}y = K^*(h_0 - \nabla p_0) \tag{6.27}$$

with the permeability tensor $K^* \in L^\infty((0,T)\times\Omega)^{n\times n}$ defined by (for $i,j = 1,\dots,n$)

$$K^*_{ij} = \int_{Y^*} J_0 e_0(w_i) : e_0(w_j)\,\mathrm{d}y = \int_{Y^*} A_0^\top e_i \cdot w_j \,\mathrm{d}y. \tag{6.28}$$

In particular, it holds that $\nabla\cdot v^* = -\partial_t\theta$ with the porosity θ defined in (3.2) *and, therefore, p_0 satisfies the Darcy law* (3.3f) – (3.3g)*.*

Next, we consider the transport equation and derive (3.4). As a test function in the variational equation (2.14a) we choose $\phi(t,x) = \phi_0(t,x) + \varepsilon\phi_1\left(t,x,\frac{x}{\varepsilon}\right)$ with $\phi_0 \in C_0^\infty(\Omega)$ and $\phi_1 \in C_0^\infty((0,T), C_0^\infty(\Omega, C^\infty_{\mathrm{per}}(Y)))$ to obtain after integration with respect to time:

$$\begin{aligned}
&\int_0^T \left\langle \partial_t(J_\varepsilon u_\varepsilon), \phi_0 + \varepsilon\phi_1\left(t,\cdot_x,\frac{\cdot_x}{\varepsilon}\right)\right\rangle_{H^1_{\partial\Omega}(\Omega_\varepsilon)', H^1_{\partial\Omega}(\Omega_\varepsilon)} \mathrm{d}t\\
&\quad + \int_0^T\int_{\Omega_\varepsilon} \left[D_\varepsilon \nabla u_\varepsilon - u_\varepsilon A_\varepsilon w_\varepsilon\right]\cdot\left[\nabla\phi_0 + \varepsilon\nabla_x\phi_1\left(t,x,\frac{x}{\varepsilon}\right) + \nabla_y\phi_1\left(t,x,\frac{x}{\varepsilon}\right)\right]\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T\int_{\Omega_\varepsilon} J_\varepsilon F_\varepsilon \left[\phi_0 + \varepsilon\phi_1\left(t,x,\frac{x}{\varepsilon}\right)\right]\mathrm{d}x\,\mathrm{d}t - \rho\int_0^T\int_{\Gamma_\varepsilon} J_\varepsilon g\left(u_\varepsilon, \varepsilon^{-1}R_\varepsilon\right)\left[\phi_0 + \varepsilon\phi_1\left(t,x,\frac{x}{\varepsilon}\right)\right]\mathrm{d}\sigma\,\mathrm{d}t.
\end{aligned} \tag{6.29}$$

Using the convergence results from Proposition 6.2, Lemma 6.2 and Corollary 6.1, we can pass to the limit $\varepsilon \to 0$. For more details on the treatment of the non-convective terms, we refer to Section 5.2 of Ref. 31 and Theorem 17 of Ref. 72, in particular for the boundary term.

Using integration by parts with respect to the microscopic variable y and using $\nabla_y \cdot (A_0 w_0) = 0$ (see Proposition 6.2(iii)), we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} u_\varepsilon A_\varepsilon w_\varepsilon \cdot \left[\nabla\phi_0 + \varepsilon\nabla_x\phi_1\left(t,x,\frac{x}{\varepsilon}\right) + \nabla_y\phi_1\left(t,x,\frac{x}{\varepsilon}\right)\right] \mathrm{d}x\,\mathrm{d}t \\
&\qquad \overset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_\Omega \int_{Y^*} u_0(t,x)A_0(t,x,y)w_0(t,x,y)\cdot[\nabla\phi_0(t,x) + \nabla_y\phi_1(t,x,y)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega u_0(t,x)v^*(t,x)\nabla\phi_0(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{Y^*} u_0(t,x)\underbrace{\nabla_y\cdot(A_0w_0)}_{=0}\phi_1(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega u_0 v^*\nabla\phi_0\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Hence, for $\varepsilon \to 0$, we obtain from (6.29) the limit equation (neglecting the arguments of the functions for a simpler notation)

$$\begin{aligned}
&\int_0^T \langle\partial_t(\theta u_0),\phi_0\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_0^T\int_\Omega u_0v^*\phi_0\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega\int_{Y^*} D_0\left[\nabla u_0 + \nabla_y u_1\right]\cdot[\nabla\phi_0 + \nabla_y\phi_1]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \theta f\phi_0 - \rho|\Gamma(t,x)|\partial_t R_0\phi_0\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{6.30}$$

with the porosity θ defined in (3.2). In the last term, we can replace $\partial_t\theta = -|\Gamma(t,x)|\partial_t R_0$. By density, this equation is valid for all $\phi_0 \in L^2((0,T),H_0^1(\Omega))$ and $\phi_1 \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(Y^*))$. We emphasise that we also used the $L^\infty((0,T)\times\Omega)$-bound for u_0 here. By standard arguments from homogenisation theory, see for example Ref. 3, we obtain by choosing $\phi_0 = 0$ that u_1 admits a representation of the form

$$u_1(t,x,y) = \sum_{i=1}^n \partial_{x_i}u_0(t,x)\chi_i(t,x,y)$$

for almost every $(t,x,y) \in (0,T)\times\Omega\times Y^*$, where $\chi_i \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(Y^*)/\mathbb{R})$ are the unique weak solutions of the cell problems

$$\begin{aligned}
-\nabla_y\cdot(D_0(\nabla_y\chi_i + e_i)) &= 0 && \text{in } (0,T)\times\Omega\times Y^*, \\
-D_0(\nabla_y\chi_i + e_i)\cdot\nu &= 0 && \text{on } (0,T)\times\Omega\times\Gamma, \\
\chi_i \text{ is } Y\text{-periodic}, \int_{Y^*}\chi_i\,\mathrm{d}y &= 0.
\end{aligned} \tag{6.31}$$

We define the homogenised diffusion coefficient $D^* \in L^\infty((0,T)\times\Omega)^{n\times n}$ for $i,j = 1,\dots,n$ and almost every $(t,x)\in(0,T)\times\Omega)$ by

$$D^*_{ij}(t,x) := \int_{Y^*} D_0(t,x)[\nabla_y\chi_i(t,x,y) + e_i]\cdot[\nabla_y\chi_j(t,x,y) + e_j]\,\mathrm{d}y. \tag{6.32}$$

Note that the right-hand side depends on the transformation ψ_0 and, therefore, on the radii R_0.

6.3. *Transformation of the cell problems back to the evolving domain*

Instead of the cell problems (6.26), we can consider the cell problems in the evolving domain, which corresponds to the two-scale back-transformation illustrated on the right-hand side of Figure 1. This relation was already obtained in Ref. 73 and this cell problem on the evolving micro-cell was already derived by formal methods in Ref. 66. We have the following formulation: Find $(\tilde{w}_i, \tilde{\pi}_i) \in L^\infty((0,T)\times$

$\Omega; H^1_{\mathrm{per},0}(Y^*(t,x))) \times L^\infty((0,T)\times\Omega; L^2(Y^*(t,x))/\mathbb{R})$ such that

$$\begin{aligned} -\nabla_y \cdot (e(\tilde{w}_i)) + \nabla_y \tilde{\pi}_i &= e_i && \text{in} \quad Y^*(t,x), \\ \nabla_y \cdot \tilde{w}_i &= 0 && \text{in} \quad Y^*(t,x), \\ \tilde{w}_i,\ \tilde{\pi}_i \text{ are } Y\text{-periodic}, \int_{Y^*(t,x)} &\tilde{\pi}(t,x,y)\,\mathrm{d}y = 0 \end{aligned} \tag{6.33}$$

for almost every $(t,x) \in (0,T)\times\Omega$. (The Bochner spaces of vector-valued functions with range depending on the domain of definition are defined via restriction, see for example Ref. 45). Using (6.25), we observe that $\tilde{w}_i(t,x,y) = w_i(t,x,\psi_0^{-1}(t,x,y))$, $\tilde{\pi}_i(t,x,y) = \pi_i(t,x,\psi_0^{-1}(t,x,y)) - (\psi_0^{-1}(t,x,y)-y)$ up to a constant depending on t and x, where (w_i,π_i) is the solution of (6.26). Having this relation, the permeability tensor (6.28) can be transformed accordingly, which yields

$$K^*(t,x)_{ij} = \int\limits_{Y^*(t,x)} e(\tilde{w}_i) : e(\tilde{w}_j)\,\mathrm{d}y = \int\limits_{Y^*(t,x)} e_i \cdot \tilde{w}_j\,\mathrm{d}y, \tag{6.34}$$

where the second equality is obtained with (6.33). With the periodicity and the divergence property, it can be shown that for $\varphi \in H^1_{\mathrm{per},0}(Y^*)$ with $\nabla\cdot\varphi = 0$ it holds that

$$\int\limits_{Y^*(t,x)} (\nabla\tilde{w}_i)^\top : \nabla\varphi\,\mathrm{d}y = 0.$$

Thus, in (6.33), the symmetric gradients $e_0(\tilde{w}_i)$ can be replaced by $\frac{1}{2}\nabla\tilde{w}_i$, and therefore K^* is given by

$$K^*(t,x)_{ij} = \tfrac{1}{2} \int\limits_{Y^*(t,x)} \nabla\tilde{w}_i : \nabla\tilde{w}_j\,\mathrm{d}y. \tag{6.35}$$

By the same argumentation, the symmetric gradient can be replaced in (6.24), (6.26) and (6.28).

Following Ref. 71, we obtain that the representation of D^* is independent of the transformation ψ_0 included in the diffusion coefficient D_0 as an integral over the evolving reference element $Y^*(t,x)$. More precisely, we have

$$D^*_{ij}(t,x) = \int_{Y^*(t,x)} D[\tilde{\chi}_i(t,x,y) + e_i]\cdot[\tilde{\chi}_j(t,x,y)+e_j]\,\mathrm{d}y, \tag{6.36}$$

where $\tilde{\chi}_i \in L^\infty((0,T)\times\Omega, H^1_{\mathrm{per}}(Y^*(t,x))/\mathbb{R})$ are the solutions of the following cell problems on the evolving reference element $Y^*(t,x)$ for almost every $(t,x)\in(0,T)\times\Omega$:

$$\begin{aligned} -\nabla_y \cdot (D(\nabla_y\tilde{\chi}_i + e_i)) &= 0 && \text{in } Y^*(t,x), \\ -D(\nabla_y\tilde{\chi}_i + e_i)\cdot\nu &= 0 && \text{on } \Gamma(t,x), \\ \tilde{\chi}_i \text{ is } Y\text{-periodic}, \int_{Y^*} &\tilde{\chi}_i(t,x,y)\,\mathrm{d}y = 0. \end{aligned} \tag{6.37}$$

Between χ_i and $\tilde{\chi}_i$, the following relation holds for almost every $(t,x,y)\in(0,T)\times\Omega\times Y^*$ (up to a constant depending on t and x):

$$\tilde{\chi}_i\,(t,x,\psi_0(t,x,y)) = \chi_i(t,x,y) + (y-\psi_0(t,x,y))\cdot e_i.$$

Now, we proceed with the macroscopic equation. Choosing $\phi_1 = 0$ in (6.30) and using the definition of D^*, we obtain by a standard calculation

$$\begin{aligned} &\int_0^T \langle\partial_t(\theta u_0),\phi_0\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} - \int_0^T\int_\Omega u_0 v^*\phi_0\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega D^*\nabla u_0\cdot\nabla\phi_0\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \theta f\phi_0 + \rho\partial_t\theta\phi_0\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{6.38}$$

for all $\phi_0 \in L^2((0,T),H^1_0(\Omega))$, which is precisely the variational equation for u_0 in the weak formulation of the macroscopic model in Definition 3.1. It remains to establish the initial condition $u_0(0) = u^{\mathrm{in}}$, see also Remark 3.1, where it suffices to show $(\theta u_0)(0) = \theta^{\mathrm{in}}u^{\mathrm{in}}$. This follows by integration by parts with respect to time in (6.29), choosing $\phi_1 = 0$ and passing to the limit $\varepsilon\to 0$ and comparing the resulting equation with (6.38). For more details, we refer again to Section 5.2 of Ref. 31. Note that the homogenised equations in Ref. 31 contain additional terms on the right- and left-hand side which cancel each other. In fact, the origins of these terms in the ε-scaled model already cancel each other, see also Ref. 72, and they do not appear in the homogenisation process and the ε-scaled weak form here. Consequently, the homogenised equations of Ref. 31 and Ref. 72, in which advection is not considered, coincide, whereas (6.38) contains the additional advective transport term. This completes the proof of Theorem 3.2.

7. Conclusions and outlook

Developing a rigorous homogenisation approach, we derived an effective reactive transport and flow model consisting of an advection–reaction–diffusion problem coupled to Darcy's equation with effective coefficients (porosity, diffusivity and permeability) defined via cell problems, which depend on the macroscopic position and evolve in time. In particular, the evolution of these cells depends on the macroscopic concentration. Thus, the cell problems (respectively the effective coefficients) are coupled to the macroscopic unknowns and vice versa, leading to a strongly coupled micro–macro model. This limit system is obtained by upscaling a coupled Stokes flow and advection–reaction–diffusion problem in a perforated domain with an evolving microstructure of size ε, in which reactions at the boundaries of the microscopic interfaces lead to the formation of a solid layer, which is modelled to be radially symmetric with a variable a priori unknown thickness induced by the advection–reaction–diffusion process.

The approach strongly relies on the regularisation induced by the restriction to a radially symmetric solid layer. While the recovery of well-posed limit problems seems to be out of reach for a fully general evolution of the microstructure, slightly more general models might still be manageable within this framework and this is a subject of further investigation. E.g. if the growth/shrinkage of the solid layer is restricted to occur in normal direction only, this can be handled by the same methods in principle at the price of having to restrict to local results with respect to time. A critical aspect is that the local-in-time existence interval remains independent of ε. However, this can be obtained from a uniform bound on the growth function g in Assumption (A1).

Appendix A. Properties of the domain transformation

In this section, we summarise some technical details on the transformation ψ_ε. Most of the following results were shown in Ref. 72 or Ref. 31 for a similar transformation. For the sake of completeness, we sketch the proofs.

More precisely, we show several ε-scaled a priori estimates for the transformation ψ_ε, which was defined in (2.6), as well as its Lipschitz-continuous dependence on the radii. We employ these properties for proving the existence and uniqueness of a microscopic solution of (2.14) and for the derivation of ε-independent a priori estimates of the solution. Moreover, in Lemma 6.2, we show that several two-scale convergence results for the transformation ψ_ε follow if R_ε converges. We will use this convergence in order to pass to the two-scale homogenisation limit. The direct two-scale limit problem as well as the homogenised problem contain coefficients which depend on the limit of ψ_ε for $\varepsilon \to 0$. Nevertheless, we show in the Subsection 6.3 how the cell problems and effective tensors can be transformed back on reference cells, the domains of which depend on their macroscopic position and time. Thus, the homogenised problem and the cell problems become independent of the transformations ψ_ε and its limit. Having this in mind, the transformation ψ_ε is reduced to a tool for proving existence and uniqueness of the microscopic problem, as well as the homogenisation.

In order to enhance the understanding of the ε-scaled diffeomorphisms ψ_ε, we note that (2.6) and (2.4) yield

$$\begin{aligned}\psi_\varepsilon(t,x) &= x + \varepsilon\psi\left(\varepsilon^{-1}R_\varepsilon(t,x); \{\tfrac{x}{\varepsilon}\}\right) - \{\tfrac{x}{\varepsilon}\} \\ &= \varepsilon[\tfrac{x}{\varepsilon}] + \varepsilon\psi\left(\varepsilon^{-1}R_\varepsilon(t,x); \{\tfrac{x}{\varepsilon}\}\right).\end{aligned} \tag{A.1}$$

Thus, ψ_ε is nothing but a union of diffeomorphisms $\psi\left(\varepsilon^{-1}R_\varepsilon(t,x);\cdot\right)$ for every ε-scaled cell. Hence, the bijectivity of $\psi\left(\varepsilon^{-1}R_\varepsilon(t,x);\cdot\right)$ from Y^* onto $Y^*_{\varepsilon^{-1}R_\varepsilon(t,x)}$ can be transfered into the bijectivity of $\psi_\varepsilon(t,\cdot)$ from Ω_ε onto $\Omega_\varepsilon(t)$.

Moreover, the ε-scaled estimates for ψ_ε are obtained from corresponding results for $\psi\left(\varepsilon^{-1}R_\varepsilon(t,x);\cdot\right)$ which are independent of R_ε.

Lemma Appendix A.1 (Cell transformation). *Let $\chi \in C^\infty([0,\infty);[0,1])$ be monotonically decreasing with $\chi(\lambda)=1$ for $\lambda \leqslant R_{\max}$ and $\chi(\lambda)=0$ for $\lambda \geqslant (R_{\max}+0.5)/2$. Then, $\psi(R,\cdot)$ as defined in (2.4) is a diffeomorphism from Y^* onto Y^*_R for every $R \in [R_{\min},R_{\max}]$ such that*

$$\begin{aligned}
\psi(R;Y^*) &= Y^*_R && \text{for every } R\in[R_{\min},R_{\max}],\\
\psi(R;x) &= x && \text{for every } R\in[R_{\min},R_{\max}], x\in Y\backslash Y^*_{(R_{\max}+0.5)/2},\\
\|\psi(R;\cdot)\|_{C^2(\overline{Y})} &\leqslant C && \text{for every } R\in[R_{\min},R_{\max}],\\
\det(\partial_x\psi(r;x)) &\geqslant c_J && \text{for every } R\in[R_{\min},R_{\max}], x\in Y^*,\\
R\mapsto\psi(R;\cdot) & && \in C^\infty([R_{\min},R_{\max}],C^2(Y^*)^n)).
\end{aligned}$$

Proof. After translation, we can assume without loss of generality that $\mathfrak{m}=0$ and $Y^*=\left(-\frac{1}{2},\frac{1}{2}\right)^n\backslash\overline{B_{R_{\max}}(0)}$ (and, similarly, we redefine Y^*_R). First, we consider ψ as a function from $\mathbb{R}^n\backslash\{0\}$ into $\mathbb{R}^n$. With $\alpha(\lambda) := \lambda + (R-R_{\max})\chi(\lambda)$, for $\lambda\geqslant 0$, we can write ψ in the following way for $y\in\mathbb{R}^n\backslash\{0\}$

$$\psi(y) = \alpha(|y|)\frac{y}{|y|}. \tag{A.2}$$

Since χ is decreasing and $R-R_{\max}\leqslant 0$, we obtain that α is strictly increasing. Hence, with $\alpha(R_{\max}) = R$ and $\lim_{\lambda\to\infty}\alpha(\lambda)=\infty$, we get that $\alpha:[R_{\max},\infty)\to[R,\infty)$ is bijective. This implies that $\psi:\mathbb{R}^n\backslash B_{R_{\max}}(0)\to\mathbb{R}^n\backslash B_R(0)$ is bijective. In fact, for $z\in\mathbb{R}^n$ with $|z|\geqslant R$, there exists $\lambda_z\geqslant R_{\max}$ such that $\alpha(\lambda_z)=|z|$. For $y:=\lambda_z\frac{z}{|z|}$, we get from (A.2) that $\psi(y)=z$ and, therefore, ψ is surjective. For $y_1,y_2\in\mathbb{R}^n\backslash B_{R_{\max}}(0)$ with $\psi(y_1)=\psi(y_2)$, we obtain (taking the absolute values), using again (A.2), that $\alpha(|y_1|)=\alpha(|y_2|)$, and the bijectivity of α gives $|y_1|=|y_2|$. In particular, this implies $y_1=y_2$, so $\psi:\mathbb{R}^n\backslash B_{R_{\max}}(0)\to\mathbb{R}^n\backslash B_R(0)$ is injective. Finally, since $\psi(y)=y$ for $y\geqslant\frac{1}{2}(R_{\max}+0.5)$, we obtain that $\psi: Y^*\to Y^*_R$ is bijective.

Moreover, since ψ is smooth with respect to both arguments, $\|\psi(R;\cdot)\|_{C^2(\overline{Y})}$ is uniformly bounded with respect to the parameter R in the compact set $[R_{\max},R_{\min}]$. The essential uniform boundedness of $\det(\partial_x\psi(r;x))$ away from 0 follows from the continuity of $\det(\partial_x\psi)$ on the compact set $[R_{\min},R_{\max}]\times\overline{Y^*}$ and the pointwise positivity in one point, which can be shown by an explicit calculation. □

We note that the construction of $\psi(R;\cdot)$ by (2.4) does in fact not give a diffeomorphism from Y onto Y. In Ref. 72, a diffeomorphism with the same properties from Y onto Y was constructed. Further, the corresponding cell displacement $\breve{\psi}(R,y)$ vanishes at ∂Y and, thus, $\breve{\psi}(R;\cdot)\in C^\infty_{\mathrm{per}}(\overline{Y^*};\mathbb{R}^n)$ for all $R\in[R_{\min},R_{\max}]$. Next, we transfer the radius-independent estimates of Lemma Appendix A.1 into ε-uniform estimates for ψ_ε with the following lemma.

Lemma Appendix A.2. *Let $R_\varepsilon\in W^{1,\infty}((0,T),L^2(\Omega_\varepsilon))\cap L^\infty((0,T)\times\Omega_\varepsilon)$ be constant on every micro cell with $\|\partial_t R_\varepsilon\|_{L^\infty((0,T)\times\Omega_\varepsilon)}\leqslant C_g$ and $\varepsilon^{-1}R_\varepsilon\in[R_{\min},R_{\max}]$. Let ψ_ε be defined via (2.6) with its Jacobian matrix $F_\varepsilon=\partial\psi_\varepsilon=\nabla\psi_\varepsilon^\top$ and Jacobian determinant $J=\det(F_\varepsilon)$. Moreover, let $D_\varepsilon, A_\varepsilon$ be given by (2.10), (2.9). Then, there exists constants $C,\alpha>0$ such that almost everwhere in $(0,T)$*

$$\varepsilon^{-1}\|\psi_\varepsilon-\mathrm{id}_{\Omega_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)}+\|F_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}+\|A_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\leqslant C, \tag{A.3}$$

$$\varepsilon\|\partial_x F_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}+\varepsilon\|D_x J_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}+\varepsilon\|\partial_x A_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\leqslant C, \tag{A.4}$$

$$\varepsilon^{-1}\|\partial_t\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}+\|\partial_x\partial_t\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}+\|\partial_t J_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\leqslant C, \tag{A.5}$$

$$J_\varepsilon(t,x)\geqslant c_J, \tag{A.6}$$

$$\zeta^\top D_\varepsilon\zeta=\zeta^\top J_\varepsilon F_\varepsilon^{-1}DF_\varepsilon^{-\top}\zeta\geqslant\alpha\|\zeta\|^2 \text{ for all } \zeta\in\mathbb{R}^n. \tag{A.7}$$

Moreover, we obtain for every $v\in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$

$$(\varepsilon^2 e_\varepsilon(v), A_\varepsilon^\top\nabla v)\geqslant\alpha\|\nabla v\|^2_{L^2(\Omega_\varepsilon)}, \tag{A.8}$$

$$\inf_{q\in L^2(\Omega_\varepsilon)} \sup_{v\in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)} \frac{(q,\nabla\cdot(A_\varepsilon v))_{\Omega_\varepsilon}}{\varepsilon\left\|\nabla v\right\|_{L^2(\Omega_\varepsilon)}\left\|q\right\|_{L^2(\Omega_\varepsilon)}} \geqslant \beta. \tag{A.9}$$

Proof. The estimates (A.3)–(A.6) are proven in Ref. 31, 72. They follow by elementary calculations via scaling the estimates of Lemma Appendix A.1, which we carry out for (A.6) as an example,

$$J_\varepsilon(x) = \det(\partial_x\psi_\varepsilon(t,x))) = \det\left(\varepsilon\partial_x\psi\left(\varepsilon^{-1}R_\varepsilon(t,x);\{\tfrac{x}{\varepsilon}\}\right)\right) = \det\left(\partial_y\psi\left(R_\varepsilon(t,x);\{\tfrac{x}{\varepsilon}\}\right)\right) \geqslant c_J\,.$$

The coercivity (A.7) is shown as part of the proof of Proposition 4.1 in Ref. 71. It follows from (A.3) and (A.6). The uniform coercivity (A.8) is shown in Lemma 3.6 of Ref. 73 using a uniform Korn-type inequality for two-scale transformations. The inf–sup condition (A.9) is shown in the Proof of Theorem 3.1 of Ref. 73 by means of an ε-scaled right-inverse of the divergence operator, which is based on the extension operator developed in Ref. 65, 2. □

As already mentioned above, these estimates and particularly the explicit dependence on ε are for technical reasons and crucial for the a priori estimates and the homogenisation. Let us point out some properties in more detail. The inequality (A.3) shows that the transformation ψ_ε is close to the identity and oscillations are small but leads to macroscopic effects for the derivatives of ψ_ε. Further, (A.6) means that no volume is "compressed" to one single point. Finally, (A.7) and (A.8) give coercivity for the transport equation and the Stokes equations in the fixed coordinates. The following inequalities are used to establish the uniqueness of the micro-model in particular and, for this, the explicit dependence on ε is not necessary. Only for J_ε, an ε-uniform bound is necessary for the strong two-scale convergence. However, for the sake of completeness, we formulate the precise dependence on ε in all inequalities.

Lemma Appendix A.3. *Let $R^1_\varepsilon, R^2_\varepsilon$ fulfill the assumptions of Lemma Appendix A.2 and let ψ^i_ε, $i \in \{1,2\}$, be given by* (2.6) *and $F^i_\varepsilon, A^i_\varepsilon, D^i_\varepsilon$, accordingly. Then, the following estimates hold almost everywhere in $(0,T)$:*

$$\begin{aligned}
&\varepsilon^{-1}\left\|\psi^1_\varepsilon-\psi^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} + \left\|F^1_\varepsilon - F^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\left\|R^1_\varepsilon - R^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)}\,,\\
&\left\|A^1_\varepsilon - A^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} + \left\|D^1_\varepsilon - D^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} + \left\|J^1_\varepsilon - J^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\left\|R^1_\varepsilon - R^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)}\,,\\
&\varepsilon\left\|\partial_x(A^1_\varepsilon - A^2_\varepsilon)\right\|_{L^\infty(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\left\|R^1_\varepsilon - R^2_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)}\,,\\
&\varepsilon^{-1}\left\|\partial_t\psi^1_\varepsilon - \partial_t\psi^2_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\left\|R^1_\varepsilon - R^2_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-1}C\left\|\partial_t(R^1_\varepsilon - R^2_\varepsilon)\right\|_{L^2(\Omega_\varepsilon)}\,,\\
&\left\|\nabla\partial_t\psi^1_\varepsilon - \nabla\partial_t\psi^2_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\left\|R^1_\varepsilon - R^2_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-1}C\left\|\partial_t(R^1_\varepsilon - R^2_\varepsilon)\right\|_{L^2(\Omega_\varepsilon)}\,,\\
&\left\|\partial_t J^1_\varepsilon - \partial_t J^2_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \leqslant \varepsilon^{-1}C\left\|R^1_\varepsilon - R^2_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-1}C\left\|\partial_t(R^1_\varepsilon - R^2_\varepsilon)\right\|_{L^2(\Omega_\varepsilon)}\,.
\end{aligned}$$

Proof. We note that Lemma Appendix A.1 provides $R \mapsto \psi(R;\cdot) \in C^\infty([R_{\min}, R_{\max}], C^2(Y^*)^n))$. Thus, ψ as well as all its derivatives depend Lipschitz-continuously on R. After rescaling, this can be carried over to ψ_ε via (A.1), which yields the Lipschitz estimates above. For the Lipschitz estimate of D_ε, the Lipschitz estimate of F^{-1}_ε is used in addition. This, estimate can be obtained by the Lipschitz estimate of F_ε and the uniform bound $J_\varepsilon \geqslant c_J$ from below. We note that, for the Lipschitz estimate of $A_\varepsilon = J_\varepsilon F^{-1}_\varepsilon$, this is not necessary since A_ε is the adjugate of F_ε, and, thus, is nothing but a polynomial with respect to the entries of F_ε. □

Appendix B. Auxiliary results

In this section, we summarise some technical results important for the treatment of our problem. We start with a two-pressure decomposition:

Lemma Appendix B.1. *Let the space $\mathcal{V}$ be defined as in* (6.19)*. We denote its annihilator in $L^2(\Omega, H^1_{\mathrm{per},0}(Y^*)^n)'$ by $\mathcal{V}^\perp$. Then, it holds that*

$$\mathcal{V}^\perp = \left\{\nabla_x p_0 + \nabla_y p_1 \in L^2(\Omega, H^1_{\mathrm{per},0}(Y^*)^n)' \,:\, p_0 \in H^1_0(\Omega),\, p_1 \in L^2(\Omega\times Y^*)/\mathbb{R}\right\}.$$

Proof. Similar results can be found in Lemma 1.5 Ref. 37 and Section 14 of Ref. 14. For a detailed proof (for thin domains, which can be easily transferred to our situation), we refer to Lemma 5.3 and Remark 5.5 of Ref. 24. $\square$

Next, we show that, in the case of symmetrical inclusions in the perforated reference cell Y^* (in our case simply balls), we get a simplified structure for the homogenised diffusion coefficient and permeability tensor. More precisely, in the special situation when the perforations are balls, we get that the homogenised diffusion coefficient D^* and the permeability tensor K^* in Section 3 are scalars, which we formulate in the following lemma for the permeability tensor (the proof follows similar lines for the homogenised diffusion coefficient). Here, we consider $Y^* := \left(-\frac{1}{2}, \frac{1}{2}\right)^n \backslash B_r(0)$ with $r \in \left(0, \frac{1}{2}\right)$, and define the permeability tensor $K^* \in \mathbb{R}^{n\times n}$ for $i, j = 1, \ldots n$ via

$$K_{ij} := \int_{Y^*} w_j \cdot e_i \,\mathrm{d}y, \tag{B.1}$$

where (w_j, π_j) solve

$$\begin{aligned} -\Delta_y w_j + \nabla_y \pi_j &= e_j && \text{in } Y^*, \\ \nabla_y \cdot w_j &= 0 && \text{in } Y^*, \\ w_j &= 0 && \text{on } \Gamma, \\ w_j \text{ is } Y\text{-periodic.} \end{aligned} \tag{B.2}$$

The following result is well-known in the literature, but we were not able to find a proof which is why we give its short proof for the sake of completeness.

Lemma Appendix B.2. *The permeability tensor K^* fulfills $K_{ij} = 0$ for $i \neq j$ and $K_{ii} = K_{jj}$ for all $i, j = 1, \ldots, n$. In other words, there exists $k > 0$ such that $K = kE_n$.*

Proof. Let $R \in \mathbb{R}^{n\times n}$ be an orthonormal matrix. More precisely, we assume $R = R^\top = R^{-1}$. Later, we will choose R as a reflection at a hyperplane through the origin. We define $\tilde{w}_j(y) := Rw_j(Ry)$ and $\tilde{\pi}_j(y) := \pi_j(Ry)$. Hence, an elementary calculation shows that $(\tilde{w}_j, \tilde{\pi}_j)$ solves (B.2) with e_j replaced by Re_j.

Now, for fixed $j \in \{1, \ldots, n\}$, we choose R with $Re_j = -e_j$ and $Re_i = e_i$ for $i \neq j$, i.e. $Ry = (y_1, \ldots, y_{j-1}, -y_j, y_{j+1}, \ldots, y_n)$. Then, $(-\tilde{w}_j, -\tilde{\pi}_j)$ is also a solution of (B.2) and, therefore, $\tilde{w}_j = -w_j$. We get after transformation and using the definition of $\tilde{w}_j$ (using $|\det(R)| = 1$ and $R^\top = R = R^{-1}$):

$$K_{ij} = \int_{Y^*} w_j \cdot e_i \,\mathrm{d}y = \int_{Y^*} R\tilde{w}_j \cdot e_i \,\mathrm{d}y = -\int_{Y^*} w_j \cdot Re_i \,\mathrm{d}y.$$

This implies that K is a diagonal matrix. In order to identify the diagonal elements $K_{ii} = K_{jj}$ for $i \neq j$, we choose $Re_j = e_i$, $Re_i = e_j$ and $Re_l = e_l$ for $l \in \{1, \ldots, n\}\backslash\{i, j\}$. The same argumentation as above yields $\tilde{w}_j$ is a solution of (B.2) with right-hand side e_i. Therefore, we obtain $\tilde{w}_j = w_i$, which yields

$$K_{jj} = \int_{Y^*} w_j \cdot e_j \,\mathrm{d}y = \int_{Y^*} R\tilde{w}_j \cdot e_j \,\mathrm{d}y = \int_{Y^*} w_i \cdot Re_j dy = \int_{Y^*} w_i \cdot e_i \,\mathrm{d}y = K_{ii}.$$

Hence, we get the desired result with $k := K_{ii}$ (with $i \in \{1, \ldots, n\}$ arbitrary). $\square$

We have the following well-known trace inequality for heterogeneous domains Ω_ε, which can be obtained by a standard decomposition argument.

Lemma Appendix B.3. *Let $p \in [1, \infty)$. For every $\mu > 0$, there exists a constant $C(\mu) > 0$ independent of ε such that*

$$\varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^p(\Gamma_\varepsilon)} \leqslant C(\mu)\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \mu\varepsilon\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}$$

holds for all $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$.

Finally, we show that the solution operator for a saddle-point problem is locally Lipschitz continuous. We formulate this result for an abstract saddle-point problem.

Lemma Appendix B.4 (Lipschitz continuity of saddle-point problems). *Let V, Q be Banach spaces and*

$$A := \{a \in \mathrm{Bil}(V,V) \mid \|a\|_{\mathrm{Bil}} < \infty \text{ and } a(v,v) \geqslant \alpha \|v\|_V^2 \text{ for an } \alpha > 0 \text{ and all } v \in V\}, \tag{B.3}$$

$$B := \{b \in \mathrm{Bil}(Q,V) \mid \|b\|_{\mathrm{Bil}} < \infty \text{ and } \inf_{q \in Q\setminus\{0\}} \sup_{v \in V\setminus\{0\}} b(q,v)/(\|q\|_Q \|v\|_V) \geqslant \beta \text{ for a } \beta > 0\}, \tag{B.4}$$

where for $a \in A$ and $b \in B$

$$\|a\|_{\mathrm{Bil}} := \sup_{v,w \in V\setminus\{0\}} \frac{|a(v,w)|}{\|v\|_V \|w\|_V}, \tag{B.5}$$

$$\|b\|_{\mathrm{Bil}} := \sup_{q \in Q\setminus\{0\},\, v \in V\setminus\{0\}} \frac{|b(q,v)|}{\|v\|_V \|q\|_Q}. \tag{B.6}$$

Let $\mathcal{L} : A \times B \times V' \times Q' \to V \times Q$ be the solution operator of the following saddle-point problem with $\mathcal{L}(a,b,f,g) := (v,p)$, where $(v,p) \in V \times Q$ is the unique solution of

$$a(v,\cdot) + b(p,\cdot) = f \qquad \text{in } V', \tag{B.7}$$

$$b(\cdot,v) = g \qquad \text{in } Q'. \tag{B.8}$$

Let $\alpha_0, \beta_0, H, M > 0$ and

$$\begin{aligned}
A_H^{\alpha_0} &:= \{a \in A \mid \|a\|_{\mathrm{Bil}} \leqslant H \text{ and } a(v,v) \geqslant \alpha_0 \|v\|_V^2 \text{ for all } v \in V\},\\
B^{\beta_0} &:= \{b \in B \mid \inf_{q \in Q\setminus\{0\}} \sup_{v \in V\setminus\{0\}} b(q,v)/(\|q\|_Q \|v\|_V) \geqslant \beta_0\},\\
V_M' &:= \{v' \in V' \mid \|v'\|_{V'} \leqslant M\},\\
Q_M' &:= \{q' \in Q' \mid \|q'\|_{Q'} \leqslant M\}.
\end{aligned}$$

Then, $\mathcal{L}$ is Lipschitz continuous on $A_H^{\alpha_0} \times B^{\beta_0} \times V_M' \times Q_M'$ and the Lipschitz constant depends only on α_0, β_0, H, i.e. there exists $L(\alpha_0, \beta_0, H)$ such that

$$\begin{aligned}
&\|\mathcal{L}(a_1,b_1,f_1,g_1) - \mathcal{L}(a_2,b_2,f_2,g_2)\|_{V\times Q}\\
&\leqslant L(\alpha_0,\beta_0,H)\Big(M\left(\|a_1-a_2\|_{\mathrm{Bil}} + \|b_1-b_2\|_{\mathrm{Bil}}\right) + \Big(\|f_1-f_2\|_{V'} + \|g_1-g_2\|_{Q'}\Big)\Big).
\end{aligned}$$

Since the coercivity constant and the inf–sup constant depend continuously on the bilinear form, $\mathcal{L}$ is locally Lipschitz continuous on $A \times B \times V' \times Q'$ in particular.

Proof. For $i \in \{1,2\}$, let $(a_i, b_i, f_i, g_i) \in A_H^{\alpha_0} \times B^{\beta_0} \times V_M' \times Q_M'$ and let $(v_i, p_i) \in V \times Q$ be the unique solution of the corresponding saddle-point problem (B.7)–(B.8), i.e.

$$a_i(v_i,\cdot) + b_i(p_i,\cdot) = f_i \qquad \text{in } V', \tag{B.9}$$

$$b_i(\cdot,v_i) = g_i \qquad \text{in } Q'. \tag{B.10}$$

By applying the coercivity of a_1 and the linearity of the bilinear forms and the right-hand sides, we obtain

$$\begin{aligned}
\alpha_0 \|\delta v\|_V^2 \leqslant & a_1(\delta v, \delta v)\\
&= -\delta b(p_2,\delta v) - \delta g(\delta p) + \delta b(\delta p, v_2) + \delta f(\delta v) - \delta a(v_2,\delta v)\\
&\leqslant \|\delta b\|_{\mathrm{Bil}} \|p_2\|_Q \|\delta v\|_V + \|\delta g\|_{Q'} \|\delta p\|_Q + \|\delta b\|_{\mathrm{Bil}} \|\delta p\|_Q \|v_2\|_V\\
&\quad + \|\delta f\|_{V'} \|\delta v\|_V + \|\delta a\|_{\mathrm{Bil}} \|v_2\|_V \|\delta v\|_V.
\end{aligned}$$

We note that standard existence theory provides the following bounds for the solution (v_i, p_i) of (B.9)–(B.10)

$$\|v_i\|_V \leqslant \frac{1}{\alpha_0} \|f_i\|_{V'} + \frac{2\|a_i\|_{\mathrm{Bil}}}{\alpha_0 \beta_0} \|g_i\|_{Q'} \leqslant CM, \tag{B.11}$$

$$\|p_i\|_Q \leqslant \frac{2\|a_i\|_{\mathrm{Bil}}}{\alpha_0 \beta_0} \|f_i\|_{V'} + \frac{2\|a_i\|_{\mathrm{Bil}}^2}{\alpha_0 \beta_0^2} \|g_i\|_{Q'} \leqslant CM, \tag{B.12}$$

where C depends on H but not on M. Then, the boundedness of the bilinear forms and the right-hand sides yields the boundedness of the solutions and we obtain

$$\alpha_0 \|\delta v\|_V^2 \leqslant CM\Big(\|\delta b\|_{\mathrm{Bil}} \|\delta v\|_V + \|\delta b\|_{\mathrm{Bil}} \|\delta p\|_Q + \|\delta a\|_{\mathrm{Bil}} \|\delta v\|_V \Big) \\ +C\Big(\|\delta g\|_{Q'} \|\delta p\|_Q + \|\delta f\|_{V'} \|\delta v\|_V \Big). \tag{B.13}$$

Moreover, employing the inf–sup estimate for b_1 yields

$$\beta_0 \|\delta p\|_Q \leqslant \|b_1(\cdot, \delta p)\|_{V'} = \|-a_1(\delta v, \cdot) - (\delta a)(v_2, \cdot) + \delta f - \delta b(\cdot, p_2)\|_{V'} \\ \leqslant \|a_1\|_{\mathrm{Bil}} \|\delta v\|_V + \|\delta a\|_{\mathrm{Bil}} \|v_2\|_V + \|\delta f\|_{V'} + \|\delta b\|_{\mathrm{Bil}} \|p_2\|_Q .$$

Using the boundedness of the data and of the solutions, we obtain

$$\beta_0 \|\delta p\|_Q \leqslant C\left(\|\delta v\|_V + \|\delta f\|_{V'}\right) + CM\left(\|\delta a\|_{\mathrm{Bil}} + \|\delta b\|_{\mathrm{Bil}}\right). \tag{B.14}$$

Inserting (B.14) in (B.13) and applying the Young inequality yields for arbitrary $\lambda > 0$

$$\begin{aligned} \|\delta v\|_V^2 &\leqslant CM\Big(\|\delta b\|_{\mathrm{Bil}} \|\delta v\|_V + \|\delta a\|_{\mathrm{Bil}} \|\delta v\|_V \Big) \\ &\quad +CM \|\delta b\|_{\mathrm{Bil}} \left(\|\delta v\|_V + \|\delta f\|_{V'}\right) + CM^2 \|\delta b\|_{\mathrm{Bil}} \left(\|\delta a\|_{\mathrm{Bil}} + \|\delta b\|_{\mathrm{Bil}}\right) \\ &\quad +C \|\delta g\|_{Q'} \left(\|\delta v\|_V + \|\delta f\|_{V'}\right) + CM \|\delta g\|_{Q'} \left(\|\delta a\|_{\mathrm{Bil}} + \|\delta b\|_{\mathrm{Bil}}\right) \\ &\quad + \|\delta f\|_{V'} \|\delta v\|_V \\ &\leqslant C_\lambda M^2 (\|\delta b\|_{\mathrm{Bil}}^2 + \|\delta a\|_{\mathrm{Bil}}^2) + \lambda \|\delta v\|_V^2 + C_\lambda \|\delta f\|_{V'}^2 + C_\lambda \|\delta g\|_{Q'}^2 . \end{aligned}$$

By choosing λ small enough and taking the square root, we obtain

$$\|\delta v\|_V \leqslant CM(\|\delta b\|_{\mathrm{Bil}} + \|\delta a\|_{\mathrm{Bil}}) + C \|\delta f\|_{V'} + C \|\delta g\|_{Q'} . \tag{B.15}$$

Inserting (B.15) into (B.14) yields

$$\|\delta p\|_Q \leqslant CM(\|\delta b\|_{\mathrm{Bil}} + \|\delta a\|_{\mathrm{Bil}}) + C \|\delta f\|_{V'} + C \|\delta g\|_{Q'} . \tag{B.16}$$

Thus, (B.15)–(B.16) imply the local Lipschitz continuity of $\mathcal{L}$. □

Appendix C. Two-scale convergence and unfolding

In this section, we briefly summarise the notion of two-scale convergence and the concept of the unfolding operator. These methods provide the basic techniques to pass to the limit $\varepsilon \to 0$ in the microscopic problem.

C.1. *Two-scale convergence*

We start with the definition of two-scale convergence, which was first introduced and analysed in Ref. 49 and Ref. 3, see also Ref. 43. Here, we formulate two-scale convergence for time-dependent functions. However, the aforementioned references can be easily extended to such problems since time acts as a parameter. For the initial data, we use the usual two-scale convergence for time-independent functions, which can be defined in the same way.

Definition Appendix C.1. A sequence $u_\varepsilon \in L^q((0,T), L^p(\Omega))$ for $p, q \in [1, \infty)$ is said to converge in the two-scale sense (in L^qL^p) to the limit function $u_0 \in L^q((0,T); L^p(\Omega \times Y))$ if the following relation holds

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u_\varepsilon(t,x) \phi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_\Omega \int_Y u_0(t,x,y)\phi(t,x,y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t$$

for every $\phi \in L^{q'}((0,T); L^{p'}(\Omega, C^0_{\mathrm{per}}(Y)))$. In this case, we write $u_\varepsilon \xrightarrow{2,q,p} u_0$. For $q = p$, we use the notation $u_\varepsilon \xrightarrow{2,p} u_0$.

A two-scale convergent sequence u_ε convergences strongly in the two-scale sense to u_0 (in L^qL^p) if

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^q((0,T), L^p(\Omega))} = \|u_0\|_{L^q((0,T), L^p(\Omega \times Y))},$$

and we write $u_\varepsilon \xrightarrow{2,q,p} u_0$ in this case. For $q = p$, we write $u_\varepsilon \xrightarrow{2,p} u_0$.

In Ref. 6, 48, the method of two-scale convergence was extended to oscillating surfaces:

Definition Appendix C.2. A sequence of functions $u_\varepsilon \in L^q((0,T), L^p(\Gamma_\varepsilon))$ for $p,q \in [1,\infty)$ is said to converge in the two-scale sense on the surface Γ_ε (in L^qL^p) to a limit $u_0 \in L^q((0,T), L^p(\Omega \times \Gamma))$ if

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} u_\varepsilon(t,x)\phi\left(t,x,\frac{x}{\varepsilon}\right) \mathrm{d}\sigma \,\mathrm{d}t = \int_0^T \int_\Omega \int_\Gamma u_0(t,x,y)\phi(t,x,y) \,\mathrm{d}\sigma_y \,\mathrm{d}x \,\mathrm{d}t$$

holds for every $\phi \in C\left([0,T] \times \overline{\Omega}, C^0_{per}(\Gamma)\right)$. We write $u_\varepsilon \xrightarrow{2,q,p} u_0$ on Γ_ε in this case. For $q = p$, we use the notation $u_\varepsilon \xrightarrow{2,p} u_0$ on Γ_ε.

We say that a two-scale convergent sequence u_ε converges strongly in the two-scale sense on Γ_ε (in L^pL^q) if

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^q((0,T),L^p(\Gamma_\varepsilon))} = \|u_0\|_{L^q((0,T),L^p(\Omega\times\Gamma))}$$

additionally holds and we write $u_\varepsilon \xrightarrow{2,q,p} u_0$ in this case. For $q = p$, we write $u_\varepsilon \xrightarrow{2,p} u_0$ on Γ_ε.

We have the following compactness results (see e.g. Ref. 3, 43, 48, with slight modification for time-dependent functions):

Lemma Appendix C.1. *For every $p,q \in (1,\infty)$ we have:*

(i) For every bounded sequence $u_\varepsilon \in L^q((0,T), L^p(\Omega))$, there exists $u_0 \in L^q((0,T), L^p(\Omega \times Y))$ such that, up to a subsequence,

$$u_\varepsilon \xrightarrow{2,q,p} u_0.$$

(ii) For every bounded sequence $u_\varepsilon \in L^q((0,T), W^{1,p}(\Omega))$, there exist $u_0 \in L^q((0,T), L^p(\Omega))$ and $u_1 \in L^q((0,T), L^p(\Omega, W^{1,p}_{\mathrm{per}}(Y)/\mathbb{R}))$ such that, up to a subsequence,

$$u_\varepsilon \xrightarrow{2,q,p} u_0,$$
$$\nabla u_\varepsilon \xrightarrow{2,q,p} \nabla_x u_0 + \nabla_y u_1.$$

(iii) For every sequence $u_\varepsilon \in L^q((0,T), W^{1,p}(\Omega))$ with u_ε and $\varepsilon\nabla u_\varepsilon$ bounded in $L^q((0,T), W^{1,p}(\Omega))$, there exists $u_0 \in L^q((0,T), L^p(\Omega, W^{1,p}_{\mathrm{per}}(Y)))$ such that, up to a subsequence, it holds that

$$u_\varepsilon \xrightarrow{2,q,p} u_0,$$
$$\varepsilon\nabla u_\varepsilon \xrightarrow{2,q,p} \nabla_y u_0.$$

(iv) For every sequence $u_\varepsilon \in L^q((0,T), L^p(\Gamma_\varepsilon))$ with

$$\varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^q((0,T),L^p(\Gamma_\varepsilon))} \leqslant C,$$

there exists $u_0 \in L^q((0,T), L^p(\Omega \times \Gamma))$ such that, up to a subsequence,

$$u_\varepsilon \xrightarrow{2,q,p} u_0 \quad \text{on } \Gamma_\varepsilon.$$

C.2. *The unfolding operator*

When dealing with nonlinear problems it is helpful to work with the unfolding method, which allows a characterisation of weak and strong two-scale convergence, see Lemma Appendix C.3 below. We refer to Ref. 16 for a detailed investigation of the unfolding operator and its properties. For a perforated domain (here we also allow the case $Y^* = Y$, i. e. $\Omega_\varepsilon = \Omega$), we define the unfolding operator for $p,q \in [1,\infty]$ by

$$\mathcal{T}_\varepsilon : L^q((0,T), L^p(\Omega_\varepsilon)) \to L^q((0,T), L^p(\Omega \times Y^*)),$$
$$\mathcal{T}_\varepsilon(u_\varepsilon)(t,x,y) = u_\varepsilon\left(t, \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right).$$

In the same way, we define the boundary unfolding operator for the oscillating surface Γ_ε via

$$\mathcal{T}_\varepsilon : L^q((0,T), L^p(\Gamma_\varepsilon)) \to L^q((0,T), L^p(\Omega \times \Gamma)),$$
$$\mathcal{T}_\varepsilon(u_\varepsilon)(t,x,y) = u_\varepsilon\left(t, \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right).$$

We emphasise that we use the same notation for the unfolding operator on Ω_ε and the boundary unfolding operator on Γ_ε. We summarise some basic properties of the unfolding operator, see Ref. 16, generalised in an obvious way to time-dependent functions with different integrability p and q:

Lemma Appendix C.2. *Let $q, p \in [1,\infty]$.*

(i) For $u_\varepsilon \in L^q((0,T), L^p(\Omega_\varepsilon))$, it holds that
$$\|\mathcal{T}_\varepsilon(u_\varepsilon)\|_{L^p((0,T),L^p(\Omega\times Y^*))} = \|u_\varepsilon\|_{L^q((0,T),L^p(\Omega_\varepsilon))}.$$
(ii) For $u_\varepsilon \in L^q((0,T), W^{1,p}(\Omega_\varepsilon))$, it holds that
$$\nabla_y \mathcal{T}_\varepsilon(u_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x u_\varepsilon).$$
(iii) For $u_\varepsilon \in L^q((0,T), L^p(\Gamma_\varepsilon))$, it holds that
$$\|\mathcal{T}_\varepsilon(u_\varepsilon)\|_{L^q((0,T),L^p(\Omega\times\Gamma))} = \varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^q((0,T),L^p(\Gamma_\varepsilon))}.$$

The following lemma gives a relation between the unfolding operator and two-scale convergence. Its proof is quite standard and we refer the reader to Ref. 7 and Ref. 16 for more details (see also Proposition 2.5 of Ref. 68).

Lemma Appendix C.3. *Let $p, q \in (1,\infty)$.*

(a) For a sequence $u_\varepsilon \in L^q((0,T), L^p(\Omega))$, the following statements are equivalent:
- *(i) $u_\varepsilon \xrightharpoonup{2,q,p} u_0$ ($u_\varepsilon \xrightarrow{2,q,p} u_0$)*
- *(ii) $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0$ ($\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$) in $L^q((0,T), L^p(\Omega \times Y))$.*

(b) For a sequence $u_\varepsilon \in L^q((0,T), L^p(\Gamma_\varepsilon))$, the following statements are equivalent:
- *(i) $u_\varepsilon \xrightharpoonup{2,q,p} u_0$ ($u_\varepsilon \xrightarrow{2,q,p} u_0$) on Γ_ε,*
- *(ii) $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0$ ($\mathcal{T}_\varepsilon(u_\varepsilon) \to u_0$) in $L^q((0,T), L^p(\Omega \times \Gamma))$.*

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