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# A METRIC INEQUALITY CHARACTERIZING THE LORENZ DOMINANCE ORDER

by

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## ABSTRACT

A metric on the space of real N-vectors $\mathbf{R}^N$ is defined which has the property to characterise the Lorenz dominance order $X \prec Y$ for $X,Y \in \mathbf{R}^{+N}$. The metric d is derived from the Euclidean norm $\|X^*\|_2$ on $X^*$ where $X^*$ denotes the vector $(X_j)$,

$$X_j = \sum_{i=1}^{j} x_i$$

where $X = (x_i)$. The key element in the characterization of $X \prec Y$ is the inequality

$$d(X^{**}, Y^{**}) \leq d(X_\pi^{**}, Y_\pi^{**})$$

for every elementary permutation $\pi$ of $\{1,...,N\}$, where $X_\pi^{**} = (X^{**})_\pi$, i.e. $\pi$ applied to $X^{**}$ and where a permutation is called elementary if two consecutive coordinates are interchanged.

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## I. INTRODUCTION

Let $X,Y \in \mathbf{R}^N$, where $N \in \mathbf{N}$. The Lorenz dominance order is defined between vectors $X = (x_1,\ldots,x_N)$, $Y = (y_1,\ldots,y_N)$ for which one has

(i) $x_i, y_i \geq 0, \forall i = 1,\ldots,N$ (1)

(ii) $X,Y$ decreasing (2)

If this is the case we say that X is dominated by Y, denoted $X \prec Y$ if

(a) $$\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i, \qquad \forall j = 1,\ldots,N \tag{3}$$

(b) $$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i \tag{4}$$

If only (a) is valid we say that X is weakly dominated by Y and denoted by $X \prec_w Y$.

The Lorenz dominance order was introduced by M.O. Lorenz in 1905 in order to measure income inequality (Lorenz [4] ) and has been studied heavily since then. The notation and terminology are due to Hardy, Littlewood and Polya [3] . We refer the reader to the important book of Marshall and Olkin who present this subject in a clear and illustrative way and in which many references are given (see Marshall and Olkin [5] ). The topic and this book have become so important that there has been created a "follow-up" service on results on the Lorenz dominance order : details appear in Bondar [2] [(1)]

To the good of my knowledge there has never been given a metric theory of the Lorenz dominance order. Yet, such a theory could be helpfull in the comparison

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(1) The author is indebted to Prof. Dr. R. Rousseau for drawing his attention to this article.
We point out a small mistake in this paper : inequality (8.1b) : $(a_1,a_2,a_3) \prec (s,s/2,s/2)$ for isosceles triangles is always true if $a_1 > a_2 = a_3$ but if $a_1 = a_2 > a_3$ there is the conditon $a_1 \leq 3/2\, a_3$, as can readily be seen.

of vectors $X \in \mathbb{R}^N$ as above, according to certain order $(\prec)$ properties. Such comparisons have been provided e.g. in Batagelj [1] between similarity measures and between dissimilarity measures. Such measures compare two vectors X,Y as above according to their "degree" of similarity or dissimilarity. Therefore it seems more basic to search for a metric theory of the Lorenz order itself rather than of derived measures as described above. This is done in this paper.

We managed to reach a metric characterization of the Lorenz dominance order in the following (rather surprising) way. Let $N \in \mathbb{N}$ and $X,Y \in \mathbb{R}^N$. Denote $X^* = (X_j)_{j \in N}$, where

$$X_j = \sum_{i=1}^{j} x_i \tag{5}$$

i.e. the coordinates that form the Lorenz curve of X (cf. Marshall and Olkin [5] ). Denote

$$|||X|||_2 = ||X^*||_2 = \sqrt{\sum_{j=1}^{N} X_j^2} \tag{6}$$

and, for every $X,Y \in \mathbb{R}^N$

$$d(X,Y) = |||X-Y|||_2 \tag{7}$$

Denote, for every permutation $\pi$ of $\{1,\ldots,N\}$

$$X_\pi = (x_{\pi(1)}, \ldots, x_{\pi(N)}) \tag{8}$$

Finally denote $(X^*)^* = X^{**}$ and

$$X_\pi^{**} = (X^{**})_\pi \tag{9}$$

(and <u>not</u> $(X_\pi)^{**}$ or any other interpretation).

Amongst other characterzations of Lorenz dominance orders we present the following theorem :

**Theorem :**

Let $X, Y \in \mathbb{R}^{+N}$ be such that X, Y are decreasing and that

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$$

Then the following assertions are equivalent :

(i) $X \prec Y$

(ii) (a) $$\sum_{k=1}^{\ell} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i \leq \sum_{k=1}^{\ell} \sum_{j=1}^{k} \sum_{i=1}^{j} y_i , \quad \forall \ell = 1, \ldots, N \tag{10}$$

(b) $$d(X^{**}, Y^{**}) \leq d(X_\pi^{**}, Y_\pi^{**}) \tag{11}$$

for every elementary permutation $\pi$ of $\{1,\ldots,N\}$.

A permutation $\pi$ is called elementary if $\exists j_o \in \{1,\ldots,N\}$ such that

$\pi(j_o+1) = j_o$

$\pi(j_o) = j_o+1$

$\pi(i) = i, \forall i \neq j_o, j_o+1$

In this case we denote $\pi = \pi_{j_o}$.

If $j_o = N$ then $j_o + 1 = N + 1 = 1 \pmod N$ and we use 1 instead of N+1. This boils down to changing the index set $\{1,\ldots,N\}$ to

$\{1 \pmod N,\ldots,N \pmod N = 0 \pmod N\}$

but we keep the notation $\{1,\ldots N\}$ for simplicity. There are N elementary permutations.

In the sequel we prove the above result and several other ones (involving "$X \prec Y$ or $Y \prec X$" or the same assertions with $\prec$ replaced by $\prec_w$). We furthermore show that (ii) (a) $\not\Rightarrow$ (i) and (ii) (b) $\not\Rightarrow$ (i) so that both properties (ii) (a) and (ii) (b) are essential (note that (i) $\Rightarrow$ (ii) (a), trivially). Note also that (ii) (a) $\Longleftrightarrow X^{**} \prec_w Y^{**}$.

We investigate the inequality

$$d(X^{**}, Y^{**}) < d\left(X_\pi^{**}, Y_\pi^{**}\right) \tag{12}$$

and give necessary and sufficient conditions on X, Y in order that (12) is valid for some or all elementary permutations $\pi$ of $\{1,\ldots,N\}$.

We also show that in (11) no * can be deleted so that we cannot replace (11) by

$$d(X^{*}, Y^{*}) \leq d\left(X_\pi^{*}, Y_\pi^{*}\right) \tag{13}$$

or

$$d(X, Y) \leq d(X_\pi, Y_\pi) \tag{14}$$

and we also show that (13) or (14) with the inequality signs reversed cannot replace (11).

Formula (11) is remarkable since it involves quadruple summations and, as said, no summation can be deleted!

## II. CHARACTERIZATION THEOREMS FOR THE LORENZ DOMINANCE ORDER

Let $X,Y \in \mathbb{R}^{+N}$, where $N \in \mathbb{N}$. Hence X and Y are N-vectors with positve coordinates : $X = (x_1,...,x_N)$, $Y = (y_1,...,y_N)$. We further assume that X and Y are decreasing. We denote

$$X_j = \sum_{i=1}^{j} x_i \quad , \quad j = 1, \ldots, N \tag{15}$$

$$Y_j = \sum_{i=1}^{j} y_i \quad , \quad j = 1, \ldots, N \tag{16}$$

$$\xi_k = \sum_{j=1}^{k} X_j \quad , \quad k = 1, \ldots, N \tag{17}$$

$$\eta_k = \sum_{j=1}^{k} Y_j \quad , \quad k = 1, \ldots, N \tag{18}$$

and

$$A_\ell = \sum_{k=1}^{\ell} \xi_k \quad , \quad \ell = 1, \ldots, N \tag{19}$$

$$B_\ell = \sum_{k=1}^{\ell} \eta_k \quad , \quad \ell = 1, \ldots, N \tag{20}$$

Note that $X^{***} = (A_1,...,A_N)$ and $Y^{***} = (B_1,...,B_N)$. Note also that (using (10)) : $X^{**}_\pi = (\xi_{\pi(1)},...,\xi_{\pi(N)})$ and $Y^{**}_\pi = (\eta_{\pi(1)},...,\eta_{\pi(N)})$, for every permtation $\pi$ of $\{1,...,N\}$.

We have the following theorem :

**Theorem II.1 :**

Suppose X an Y are as above and that $X \prec_w Y$. Then

$$d(X^{**}, Y^{**}) \leq d(X_\pi^{**}, Y_\pi^{**}) \tag{21}$$

for every elementary permutation $\pi$.

**Proof :**

Since $\pi$ is an elementary permutation, there exists $j_o \in \{1,\ldots,N\}$ such that $\pi = \pi_{j_o}$.

Note that we take $j_0 + 1 = 1$ in case $j_o = N$ as explained in the introduction.

**A. Let $j_o \in \{1,\ldots,N\text{-}1\}$**

We have :

$$d^2(X^{**}, Y^{**}) = \sum_{\ell=1}^{N} (A_\ell - B_\ell)^2$$

and

$$d^2(X_{\pi_{j_o}}^{**}, Y_{\pi_{j_o}}^{**}) = \sum_{\ell=1}^{j_o-1} (A_\ell - B_\ell)^2 +$$

$$\left(\sum_{k=1}^{j_o-1} \xi_k + \xi_{j_o+1} - \sum_{k=1}^{j_o-1} \eta_k - \eta_{j_o+1}\right)^2 + \sum_{\ell=j_o+1}^{N} (A_\ell - B_\ell)^2 \quad .$$

Here we denote

$$\sum_{k=1}^{j_o-1} = 0$$

for $j_o = 1$, hence making the above equality valid.

$$\left(\sum_{k=1}^{j_o-1} \xi_k + \xi_{j_o+1} - \sum_{k=1}^{j_o-1} \eta_k - \eta_{j_o+1}\right)^2 \geq \sum_{\ell=j_o+1}^{N} (A_{j_o} - B_{j_o})^2$$

Using (19) and (20) and decomposing into factors yields :

$$(\xi_{j_o+1} - \eta_{j_o+1} - \xi_{j_o} + \eta_{j_o}) \cdot \left(\sum_{k=1}^{j_o-1} \xi_k - \sum_{k=1}^{j_o-1} \eta_k + \sum_{k=1}^{j_o+1} \xi_k - \sum_{k=1}^{j_o+1} \eta_k\right) \geq 0 \qquad (22)$$

Now $X \prec_w Y$ implies

$$X_j \leq Y_j \ , \qquad \forall j = 1, \ldots, N$$

$$\rightarrow \ \xi_k \leq \eta_k \ , \qquad \forall k = 1, \ldots, N$$

This implies that the second factor in (22) is negative. The first factor is also negative since $X \prec_w Y$ implies

$$\xi_{j_o+1} - \xi_{j_o} = X_{j_o+1} \leq Y_{j_o+1} = \eta_{j_o+1} - \eta_{j_o}$$

This proves (21).

**B. Let $j_o = N$**

We have now

$$d^2 \left(X^{**}_{\pi_N}, Y^{**}_{\pi_N}\right) = (\xi_N - \eta_N)^2 + \sum_{\ell=2}^{N-1} \left(\xi_N + \sum_{k=2}^{\ell} \xi_k - \eta_N - \sum_{k=2}^{\ell} \eta_k\right)^2 + (A_N - B_N)^2 \text{ D}$$

enote

$$\sum_{k=2}^{\ell} = 0$$

for $\ell = 1$. Then (21) is valid if and only if

$$\sum_{\ell=1}^{N-1}\left(\xi_N+\sum_{k=2}^{\ell}\xi_k-\eta_N-\sum_{k=2}^{\ell}\eta_k\right)^2\geq\sum_{\ell=1}^{N-1}\left(\sum_{k=1}^{\ell}\xi_k-\sum_{k=1}^{\ell}\eta_k\right)^2 .$$

Decomposing in factors we find the condition

$$\sum_{\ell=1}^{N-1}\left[(\xi_N-\eta_N-\xi_1+\eta_1).\left(\sum_{k=2}^{\ell}\xi_k+\xi_N+\sum_{k=1}^{\ell}\xi_k-\sum_{k=2}^{\ell}\eta_k-\eta_N-\sum_{k=1}^{\ell}\eta_k\right)\right]\geq 0 . \tag{23}$$

$X \prec_w Y$ implies $X_j \leq Y_j$, $\forall j = 1,\ldots,N$ and hence $\xi_k \leq \eta_k$, $\forall k = 1,\ldots,N$. Hence the second factor is negative. Furthermore

$$\xi_N-\eta_N-\xi_1+\eta_1=\sum_{j=2}^{N}X_j-\sum_{j=2}^{N}Y_j\leq 0 .$$

This proves (21) for all elementary permutations. □

We note that (22) ≡ 0 for $j_o = N - 1$ and $X_N = Y_N$ (e.g. $X \prec Y$) :

$$X_{j_o+1}-Y_{j_o+1}=X_N-Y_N=0$$

Hence

$$d(X^{**},Y^{**}) = d\left(X^{**}_{\pi_{N-1}},Y^{**}_{\pi_{N-1}}\right) \tag{24}$$

for all vectors X, Y such that $X \prec Y$.

This is not so if $X \prec_w Y$ and certainly not so for the other elementary permutations $\pi_1,\pi_2,\ldots,\pi_{N-2},\pi_N$. Ths is shown in the next theorems.

**Theorem II.2** :

Under the conditions of theorem II.1 we have that

$$d(X^{**},Y^{**}) < d\left(X^{**}_{\pi_{j_o}},Y^{**}_{\pi_{j_o}}\right) \tag{25}$$

($j_o = 1,2,\dots,N-1$) if and only if

$$X_{j_o+1} \neq Y_{j_o+1} \tag{26}$$

**Proof :**

(25) is true iff

$$(\xi_{j_o+1} - \eta_{j_o+1} - \xi_{j_o} + \eta_{j_o}) \cdot \left(\sum_{k=1}^{j_o-1} \xi_k - \sum_{k=1}^{j_o-1} \eta_k + \sum_{k=1}^{j_o+1} \xi_k - \sum_{k=1}^{j_o+1} \eta_k\right) > 0$$

This follows from the proof of theorem II.1.

Now the second factor is negative since $X \prec_w Y$. So this condition is equivalent with

$$\xi_{j_o+1} - \xi_{j_o} - (\eta_{j_o+1} - \eta_{j_o}) < 0$$

Hence

$$X_{j_o+1} < Y_{j_o+1}$$

(equivalently $X_{j_o+1} \neq Y_{j_o+1}$ sine $X \prec_w Y$). □

**Theorem II.3 :**

Under the conditions of theorem II.1, we have that

$$d(X^{**}, Y^{**}) < d\left(X^{**}_{\pi_N}, Y^{**}_{\pi_N}\right) \tag{27}$$

if and only if there exist a $j \in \{2,\dots,N\}$ such that $X_j \neq Y_j$.

**Proof :**

In (23) all terms are positive since $X \prec_w Y$.

In order for (23) to hold with a > sign (this is equivalent with (27)) it is necessary and sufficient that there exist a $\ell \in \{1,\dots,N-1\}$ such that

$$(\xi_N - \eta_N - \xi_1 + \eta_1) \cdot \left( \sum_{k=2}^{\ell} \xi_k + \xi_N + \sum_{k=1}^{\ell} \xi_k - \sum_{k=2}^{\ell} \eta_k - \eta_N - \sum_{k=1}^{\ell} \eta_k \right) > 0$$

(recall that $\sum_{k=2}^{\ell} = 0$ for $\ell = 1$). This condition is equivalent with

$$\xi_N - \xi_1 < \eta_N - \eta_1 \tag{28}$$

and

$$2 \sum_{k=2}^{\ell} (\xi_k - \eta_k) + \xi_N - \eta_N + \xi_1 - \eta_1 < 0 \tag{29}$$

(since $\leq$ is certainly valid, since $X \prec_w Y$).
Equivalently :

$$\sum_{j=2}^{N} X_j < \sum_{j=2}^{N} Y_j \tag{30}$$

and (29). (30) is valid if and only if there exists a $j \in \{2,\dots,N\}$ such that $X_j \neq Y_j$ (since $X \prec_w Y$), in which case also (29) is satisfied at least if $\ell \geq j$ (and for $\ell < j$ is (29) zero if j is the first such index). In any case, the sum $\sum_{\ell=1}^{N-1}$ as appearing in (23) is strictly positive if and only if $X_j \neq Y_j$ for a certain $j \in \{2,\dots,N\}$, proving the equivalence asserted in the theorem. □

**Corollary II.4** :
$X \prec_w Y$, $X \neq Y$ and

$$d(X^{**}, Y^{**}) = d(X_\pi^{**}, Y_\pi^{**}) \tag{1}$$

for <u>all</u> elementary permutations is only possible if $X_j = Y_j$ for all $j = 2,\dots,N$, and this condition is necessary and sufficient.

**Proof :**

This follows readily from theorems II.2 and II.3. □

More concretely this means that only vectors $X \prec_w Y$ are allowed (in order to have (31)) such that $X = Y$ or, if $X \neq Y$, $x_1 < y_1$, $X_2 = x_1 + x_2 = y_1 + y_2 = Y_2, \ldots, X_N = Y_N$. We give an example : $N = 3$,

$$X = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad Y = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$$

Then $X \prec Y$, $X \neq Y$. Furthermore

$$X^{***} = \left(\frac{1}{3}, \frac{4}{3}, \frac{10}{3}\right), \quad Y^{***} = \left(\frac{1}{2}, \frac{10}{6}, \frac{23}{6}\right)$$

$$\left(X^{**}_{\pi_1}\right)^* = \left(1, \frac{4}{3}, \frac{10}{3}\right), \quad \left(Y^{**}_{\pi_1}\right)^* = \left(\frac{7}{6}, \frac{10}{6}, \frac{23}{6}\right)$$

$$\left(X^{**}_{\pi_2}\right)^* = \left(\frac{1}{3}, \frac{7}{3}, \frac{10}{3}\right), \quad \left(Y^{**}_{\pi_2}\right)^* = \left(\frac{1}{2}, \frac{16}{6}, \frac{23}{6}\right)$$

$$\left(X^{**}_{\pi_3}\right)^* = \left(2, 3, \frac{10}{3}\right), \quad \left(Y^{**}_{\pi_3}\right)^* = \left(\frac{13}{6}, \frac{20}{6}, \frac{23}{6}\right)$$

We find

$$d(X^{**}, Y^{**}) = d\left(X^{**}_{\pi_1}, Y^{**}_{\pi_1}\right)$$

$$= d\left(X^{**}_{\pi_2}, Y^{**}_{\pi_2}\right) = d\left(X^{**}_{\pi_3}, Y^{**}_{\pi_3}\right)$$

$$= \sqrt{\frac{7}{18}}$$

We continue our study of relation (21).

**Theorem II.5 :**

Theorem II.1 is also valid if $Y \prec_w X$

**First proof :**

This is trivial from the symmetry of the distance d. □

**Second proof :**

Repeat the proof of theorem II.1. Both factors in (22) and (23) are now positive, giving again a positive product. □

Hence we have the result that $X \prec_w Y$ or $Y \prec_w X$ implies

$$d(X^{**},\ Y^{**}) \le d(X_\pi^{**},\ Y_\pi^{**}) \tag{21}$$

for every elementary permutation $\pi$ of $\{1,\ldots,N\}$. The converse is not true as will be proved in the next section. However, if we consider the trivial implications

$$X \prec_w Y \Rightarrow A_\ell \le B_\ell\ , \qquad \forall \ell = 1,\ldots,N$$
$$Y \prec_w X \Rightarrow A_\ell \ge B_\ell\ , \qquad \forall \ell = 1,\ldots,N$$

we can formulate necessary and sufficient conditions for $\prec$ and $\prec_w$.

**Theorem II.6 :**

Let X, Y be two decreasing vectors with positive coordinates. Then the following assertions are equivalent :

(i) $X \prec_w Y$ or $Y \prec_w X$

(ii) (a) $A_\ell - B_\ell$ have the same sign, $\forall \ell = 1,\ldots,N$

(b) $d(X^{**}, Y^{**}) \le d(X_\pi^{**}, Y_\pi^{**})$, for every elementary permutation $\pi$ of $\{1,\ldots,N\}$.

**Proof :**

**(i) ⇒ (ii)** follows from theorems II.1 and II.5 and the above remark.

**(ii) ⇒ (i)**

(ii) (b) implies, using (22) which is equivalent with (21),

$$(\xi_{j_o+1} - \eta_{j_o+1} - \xi_{j_o} + \eta_{j_o})\cdot\left(\sum_{k=1}^{j_o-1} \xi_k - \sum_{k=1}^{j_o-1} \eta_k + \sum_{k=1}^{j_o+1} \xi_k - \sum_{k=1}^{j_o+1} \eta_k\right) \geq 0$$

for every $j_o = 1,\ldots,N-1$. Hence

$$(\xi_{j_o+1} - \eta_{j_o+1} - \xi_{j_o} + \eta_{j_o})(A_{j_o-1} - B_{j_o-1} + A_{j_o+1} - B_{j_o+1}) \geq 0 \tag{32}$$

Suppose (ii) (a) with the positive sign. Then the second factor in (32) is positive implying the positivity of the first factor. Hence

$$X_{j_o+1} \geq Y_{j_o+1}$$

$\forall j_o = 1,\ldots,N-1$. Hence $X_i \geq Y_i$, $\forall i = 2,\ldots,N$.
Furthermore $X_1 = A_1 \geq B_1 = Y_1$. Hence $Y \prec_w X$. The same proof for $X \prec_w Y$ under the condition that $A_\ell \leq B_\ell$, $\forall \ell = 1,\ldots,N$. □

**Important remark II.7 :**
The above theorem is also true when we only use the elementary permutations $\pi_1,\ldots,\pi_{N-1}$. This follows from the above proof since we only used (22) and not (23).
Furthermore, if

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$$

then the above theorem is also true when we only use $\pi_1,\ldots,\pi_{N-2}$. This follows from the above and (24).

In completely the same way we have the following theorems (we skip the trivial proofs). Also remark II.7 is valid.

**Theorem II.8** :

Let X, Y be two decreasing vectors. Then the following assertions are equivalent :

(i) $X \prec_w Y$

(ii) (a) $A_\ell \leq B_\ell$ , $\forall \ell = 1,\dots,N$

(b) $d(X^{**}, Y^{**}) \leq d(X_\pi^{**}, Y_\pi^{**})$, for every elementary permutation $\pi$ of $\{1,\dots,N\}$.

**Theorem II.9** :

Let X, Y be two decreasing vectors with positive coordinates, such that

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$$

Then the following assertions are equivalent :

(i) $X \prec Y$ or $Y \prec X$

(ii) (a) $A_\ell - B_\ell$ have the same sign, $\forall \ell = 1,\dots,N$

(b) $d(X^{**}, Y^{**}) \leq d(X_\pi^{**}, Y_\pi^{**})$ for every elementary permutation $\pi$ of $\{1,\dots,N\}$.

**Theorem II.10** :

Let X, Y be two decreasing vectors with positive coordinates such that

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$$

Then the following assertions are equivalent :

(i) $X \prec Y$

(ii) (a) $A_\ell \leq B_\ell$ , $\forall \ell = 1,\dots,N$

(b) $d(X^{**}, Y^{**}) \leq d(X_\pi^{**}, Y_\pi^{**})$ for every elementary permutation $\pi$ of $\{1,\dots,N\}$.

The above four theorems can also be formulated in the following form (we only present this form for the last theorem).

**Corollary II.11** :

Let X, Y be two decreasing vectors with positive coordinates such that

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$$

and such that

$$A_\ell = \sum_{k=1}^{\ell} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i \leq B_\ell = \sum_{k=1}^{\ell} \sum_{j=1}^{k} \sum_{i=1}^{j} y_i \tag{33}$$

Then the following assertions are equivalent :

(i) $X \prec Y$

(ii) $d(X^{**}, Y^{**}) \leq d(X_\pi^{**}, Y_\pi^{**})$ for every elementary permutation $\pi$ of $\{1,\ldots,N\}$.

Let us look again at theorem II.6 (but the remarks go as well for the next 3 theorems). We will show in the next section that (ii) (a) $\not\Rightarrow$ (i) and that (ii) (b) $\not\Rightarrow$ (i), so that both conditions in (ii) are necessary.

Furthermore, in the next section, we show that (ii) (b) cannot be replaced by

$$d(X^*, Y^*) \leq d(X_\pi^*, Y_\pi^*) \tag{34}$$

or

$$d(X,Y) \leq d(X_\pi, Y_\pi) \tag{35}$$

We will also show that (34) nor (35), with $\leq$ replaced by $\geq$ are true.

Hence, the obtaind results are the best possible.

We also remark that all the above theorms are also true when $\|\cdot\|_2$ in (7) is replaced by any $\|\cdot\|_p$-norm with $1 \leq p \leq \infty$. This can readily be verified.

## III. THE CHARACTERIZATION THEOREMS ARE THE BEST POSSIBLE

We show that all conditions in the theorems II.6, II.8, II.9 and II.10 are needed.

**II.1. (ii)(a) $\not\Rightarrow$ (i)**, i.e. there exist decreasing vectors X, Y with positive coordinates for which $A_\ell - B_\ell$ have the same sign for all $\ell = 1,\dots,N$ and yet $X \not\prec_w Y$ and $Y \not\prec_w X$.

**Proof** :

Take

$$N = 3, \quad X = \left(\frac{5}{12}, \frac{9}{24}, \frac{5}{24}\right), \quad Y = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

Then $X \not\prec_w Y$ and $Y \not\prec_w X$ and $X_3 = Y_3 = 1$. But

$$A_1 - B_1 = -\frac{1}{12}, \quad A_2 - B_2 = -\frac{3}{24} \text{ and } A_3 - B_3 = -\frac{1}{6}$$

This example works to prove (ii) (a) $\not\Rightarrow$ (i) in all four theorems.

**II.2. (ii)(b) $\not\Rightarrow$ (i)**, i.e. there exist decreasing vectors X, Y with positive coordinates for which

$$d(X^{**}, Y^{**}) \leq d(X^{**}_\pi, Y^{**}_\pi)$$

for every elementary permutation $\pi$ of $\{1,..,N\}$ and such that $X \not\prec_w Y$ and $Y \not\prec_w X$.

**Proof** :

From the theorems, we already know that not all $A_\ell - B_\ell$ ($\ell = 1,\dots,N$) can have the same sign. We start with this. We take

$$N = 3, \quad X = \left(\frac{11}{24}, \frac{10}{24}, \frac{3}{24}\right), \quad Y = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

It is clear that $X \not\prec_w Y$ and $Y \not\prec_w X$ and $X_3 = Y_3 = 1$. Note that

$$A_1 - B_1 = -\frac{1}{24} \text{ and } A_2 - B_2 = \frac{1}{24}$$

We show that

$$d(X^{**}, Y^{**}) \leq d(X^{**}_\pi, Y^{**}_\pi)$$

for every elementary permutation $\pi$ of {1,2,3}. By remark II.7 it is enough to check $\pi_1$ only. We have

$$d^2(X^{**}, Y^{**}) = \frac{11}{(24)^2}$$

$$< \frac{14}{(24)^2} = d^2\left(X^{**}_{\pi_1}, Y^{**}_{\pi_1}\right)$$

Verify that

$$d^2\left(X^{**}_{\pi_3}, Y^{**}_{\pi_3}\right) = \frac{29}{(24)^2} > d^2(X^{**}, Y^{**})$$

and

$$d^2\left(X^{**}_{\pi_2}, Y^{**}_{\pi_2}\right) = \frac{11}{(24)^2} > d^2(X^{**}, Y^{**})$$

as it should (cf. (24)).

**II.3.** The theorems II.6, II.8, II.9 and II.10 ar wrong when $X^{**}$, $Y^{**}$ is replaced by $X^*$, $Y^*$ or by X, Y in (ii)(b).

**Proof :**

Take

$$N = 3, \quad X = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and } Y = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

Then $X \prec Y$. Take $\pi_1$. Then, with $X^*_{\pi_1} =: (X^*)_{\pi_1}$ and the same for Y, we have

$$d^2(X_{\pi_1}, Y_{\pi_1}) = \frac{2}{(12)^2} < d^2(X, Y) = \frac{5}{(12)^2} \tag{36}$$

$$d^2(X^*_{\pi_1}, Y^*_{\pi_1}) = \frac{19}{(12)^2} < d^2(X^*, Y^*) = \frac{20}{(12)^2} \tag{37}$$

(verify that

$$d^2(X^{**}_{\pi_1}, Y^{**}_{\pi_1}) = \frac{98}{(12)^2} > d^2(X^{**}, Y^{**}) = \frac{93}{(12)^2}$$

as it should).

**II.4.** From (36) and (37) one might have the conjecture that the above four theorems are also valid for (ii)(b) replaced by

$$d(X_\pi, Y_\pi) \le d(X,Y)$$

or

$$d(X^*_\pi, Y^*_\pi) \le d(X^*,Y^*)$$

This is not true.

**Proof :**

Take

$$N = 3,\ X = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right),\ Y = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

Then $X \prec Y$ and

$$d^2(X, Y) = \frac{5}{36} < d^2(X_{\pi_1}, X_{\pi_1}) = \frac{22}{36}$$

and

$$d^2(X^*, Y^*) = \frac{19}{36} < d^2\left(X^*_{\pi_1}, Y^*_{\pi_1}\right) = \frac{22}{36} .$$

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