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A Simulation Optimisation Approach for Inventory Management Decision Support based on Incomplete Information

Proefschrift voorgelegd tot het behalen van de graad van Doctor in de Toegepaste Economische Wetenschappen aan de Universiteit Hasselt te verdedigen door

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Chapter 1

Introduction

Logistics systems appear with uncertainties in demand, in lead times, in transportation times, in availability of resources and in quality. Dealing with uncertainty is an important issue in supply chain modelling and in analysis of supply chain behaviour and performance. Some of the uncertainty is due to suppliers but some is also attributable to factors as customers or economic conditions. Management decisions have to take these uncertainties in consideration.

Four clusters of sources of uncertainty can be distinguished from literature: supplier performance, customer demand, manufacturing process and environmental conditions.

Instead of studying the whole logistics system, this research focusses on uncertainty in *inventory systems*, an important part of logistics systems. Two specific aspects of uncertainty in inventory management systems are examined in more detail.

Firstly, in literature, it is mostly assumed that uncertainties in parameters, such as the demand during lead time or the lead time itself, can be described by a probability distribution. However, information about the functional form of the probability distribution is often limited in practice. For example, it might be that only the range, or the first moments, or the mode of the probability distribution is known. This limited information is a problem as the shape of the distribution is important in terms of performance of inventory control.

Secondly, a special case of limited information, intermittent demand is discussed.

In literature, less attention has been paid to irregular demand. This type of demand is characterised by a high level of variability, but may be also of the intermittent type, i.e. demand peaks follow several periods of zero or low demand. In practice, items with intermittent demand include service or spare parts and high-priced capital goods. A common example of such goods are spare parts for airline fleets. The intermittent character of demand makes forecasting difficult. However, the high cost of modern aircraft and the expense of such repairable spares constitute a large part of the total investment of many airline operators. These parts are critical to operations and their unavailability can lead to excessive down time costs.

Therefore, the *overall objective* of this thesis is twofold: on the one hand we want to describe the demand process under the condition of limited information, on the other hand we want to develop a framework for inventory management decision support for intermittent demand.

To develop the framework, we first investigate, using a simulation model, the performance of several forecasting methods and their impact on inventory management policies for intermittent demand, a special type of demand where information on the demand process is often limited. We aim to determine optimal parameter settings using several simulation optimization techniques. Finally, we investigate the impact of uncertainty in the supply side on previous conclusions and recommendations.

These main objectives can also be found in Figure 1.1. In this figure, an outline of the thesis is presented. In a first phase, the research topic is explored. The results of this phase are documented in Chapters 1 and 2. The main part of this thesis (Chapters 3 to 7) is divided in two subparts, as indicated in Figure 1.1. In the final phase, the main research findings, conclusions and recommendations for further research are presented in Chapter 8.

By fulfilling these main objectives, this thesis offers following key contributions: (1) identify characteristics as demand shape and unimodality under the condition of limited information on demand, (2) determine the optimal inventory level given a desired performance level under the condition of limited information on demand, (3) propose a best strategy in combining inventory decision making and demand forecasting for intermittent demand and (4) describe the impact of uncertainty in the supply side on the best strategy for intermittent demand.

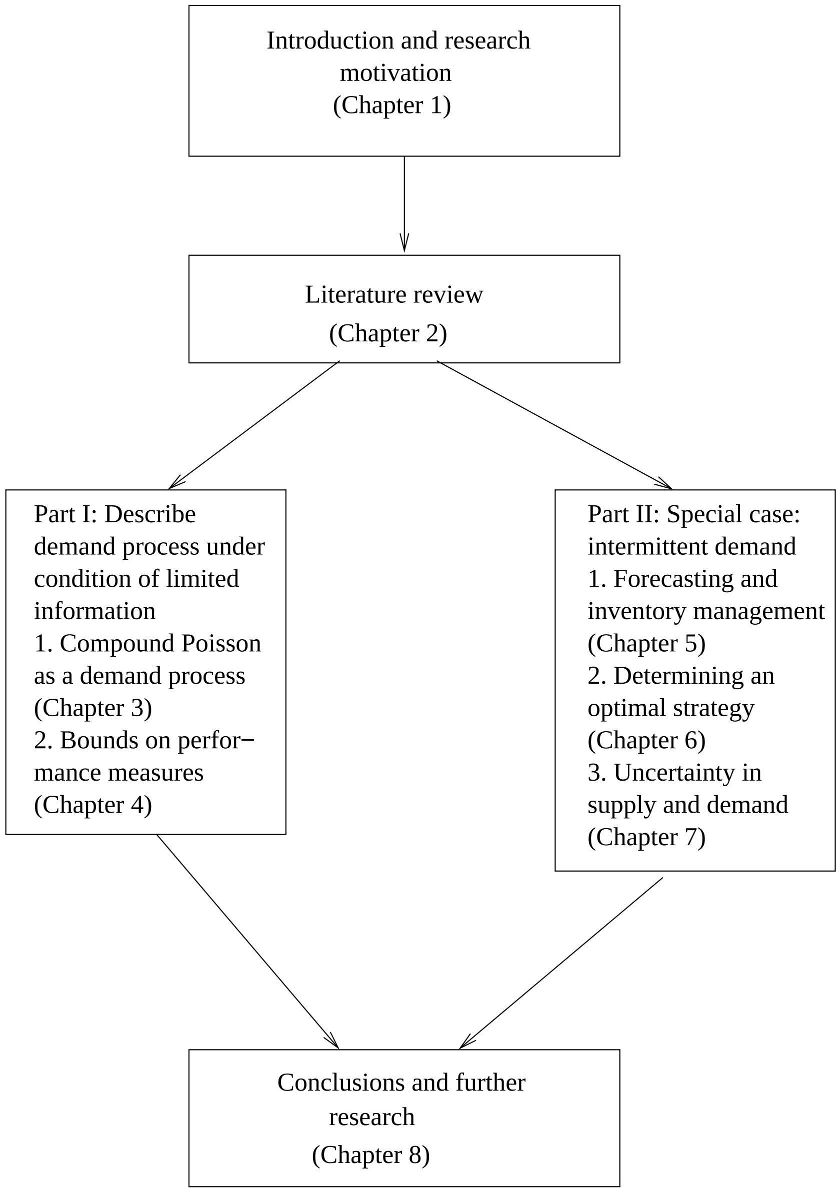

Figure 1.1: Outline of the thesis

The first two key contributions stem from the first objective, describing the demand process under the condition of limited information. Key contribution 3 and 4 follow from the second objective, developing a framework for inventory decision making for intermittent demand.

Chapter 2 presents a literature review, in which uncertainty in inventory systems is treated. Three main topics can be distinguished: the use of probability distributions for demand; irregular and intermittent demand; and uncertainty in the supply side.

Most studies assume that the probability distribution of demand during lead time is known. In chapter 3, the demand process under the condition of limited information is described. In this chapter, as in chapters 4, 5 and 6, a deterministic lead time is assumed. In chapter 3, we relax the assumption that the distribution of the lead time demand is completely known and merely assume that the first two moments are known and finite. We assume that the same mean and standard deviation can be obtained by various patterns regarding demand frequency and size. In this chapter, it is assumed that the frequency of demand is modelled by a Poisson process. But, for the demand size, various types of distributions are investigated. Each experiment leads to a single point on a two-dimensional chart representing an asymmetry measure and a kurtosis measure. Well-known are the Pearson two-dimensional charts, indicating a range of distributions in terms of an asymmetry characteristic and a kurtosis characteristic. In this way, it is possible to identify characteristics as asymmetry and unimodality. The findings of this chapter are validated by generating a large number of compound Poisson random numbers and fit a distribution to the data.

In chapter 4, bounds on performance measures of inventory systems are calculated when only limited information on lead-time demand is available. Upper and lower bounds are determined for performance measures, given the inventory level. Two performance measures are considered: the expected number of stock-out units and the probability of a stock-out. Based on these results, the decision-maker can decide on the optimal inventory level given a desired maximum number of stock-out units or a desired maximum stock-out probability.

A special type of demand, for which information on the demand process is often limited, is intermittent demand. Intermittent demand is the type of demand that does not occur in every period and, if it appears, it shows high variability. Items with intermittent demand include service spare parts and high-priced capital goods, such as heavy machinery. In literature, only few studies deal with this type of demand and the intermittent nature makes forecasting difficult. In chapters 5 and 6, we investigate several forecasting methods for intermittent demand and their impact on inventory management policies. In chapter 5, a simulation model is used in order to investigate the effects on costs and performance. In chapter 6, the parameters of the simulation model are optimized using several simulation optimization techniques, in order to obtain the best strategy in combining inventory decision making and demand forecasting. It is assumed throughout both chapters that the lead time is deterministic and that there are no disruptions in the supply.

When dealing with intermittent demand, often only few items are ordered every period and there is a high variability in order sizes between orders, which means that the administrative order and follow-up cost and the transport cost may be relatively high compared to the value of the product. In such a situation it might be economically interesting to place orders of several products in the assortment simultaneously, known in the literature as joint replenishment. Therefore, in chapter 5, we elaborate the simulation model to include joint replenishment policies.

In chapter 7, the interaction of uncertainty in the supply side with the earlier defined uncertainty in demand will be investigated. Therefore the uncertainty in supply will be incorporated in the simulation model developed in chapter 5. A new best strategy in combining inventory decision making and demand forecasting is determined for the situation in which the supplier is unreliable.

In chapter 8, we summarize the main findings of this research. General conclusions are given and some directions for further research are suggested.

The research in this thesis is mainly theory-driven but the outcome is also useful in practice. Bartezzaghi, Verganti, and Zotteri (1999a) demonstrate the significant impact of demand shape on inventories. However, a lot of authors assume that demand in a certain period of time is continuous and follows a Normal distribution. When characteristics of the demand distribution are identified under the condition of limited information, it is shown that the Normal distribution is valid only in special cases. The use of the determination of the optimal inventory level given a desired performance level under the condition of limited information is shown in chapter 4 by an example. The framework for inventory management decision support for intermittent demand leads to two policies, depending on the environmental factors, that are best used when facing demand of the intermittent type. A classification tree is constructed to decide which of the two strategies is best in a specific situation. Calculations also show that a good classification is necessary because there is a considerable increase in the costs of the inventory system when using the other strategy.

Chapter 2

Literature review

This chapter reports on the general literature review that has been carried out to explore the research topic (Figure 2.1). The literature review starts with an overview of sources of uncertainty in inventory systems in section 2.1. In the previous chapter, the main objectives for this thesis were determined: describe the demand process under the condition of limited information; investigate, using a simulation model, the performance of several forecasting methods and their impact on inventory management policies for intermittent demand, a special type of demand where information on the demand process is often limited; and investigate the impact of uncertainty in the supply side on previous conclusions. For each of these objectives, the relevant literature for this thesis is considered in sections 2.2, 2.3 and 2.4.

2.1 Uncertainty in inventory systems

The importance and impact of uncertainty in inventory systems is widely discussed in literature. Davis (1993) identifies three sources of uncertainty: supplier performance, manufacturing processes and customer demand. Schwarz and Weng (2000) study the effect of lead time uncertainty on safety stock in a supply chain. The bullwhip effect is a very disturbing characteristic in supply chains. One of the main causes of the bullwhip effect is the use of inadequate forecasting methods, which do not correctly quantify the degree of uncertainty in the market demand (Chen, Drezner, Ryan, and Simchi-Levi 2000). Wilding (1998) also defines demand amplification as one of the effects that increases the degree of uncertainty in supply chains. Van der Vorst, Beulens, De Wit, and Van Beek (1998) define four clusters of sources of uncertainty:

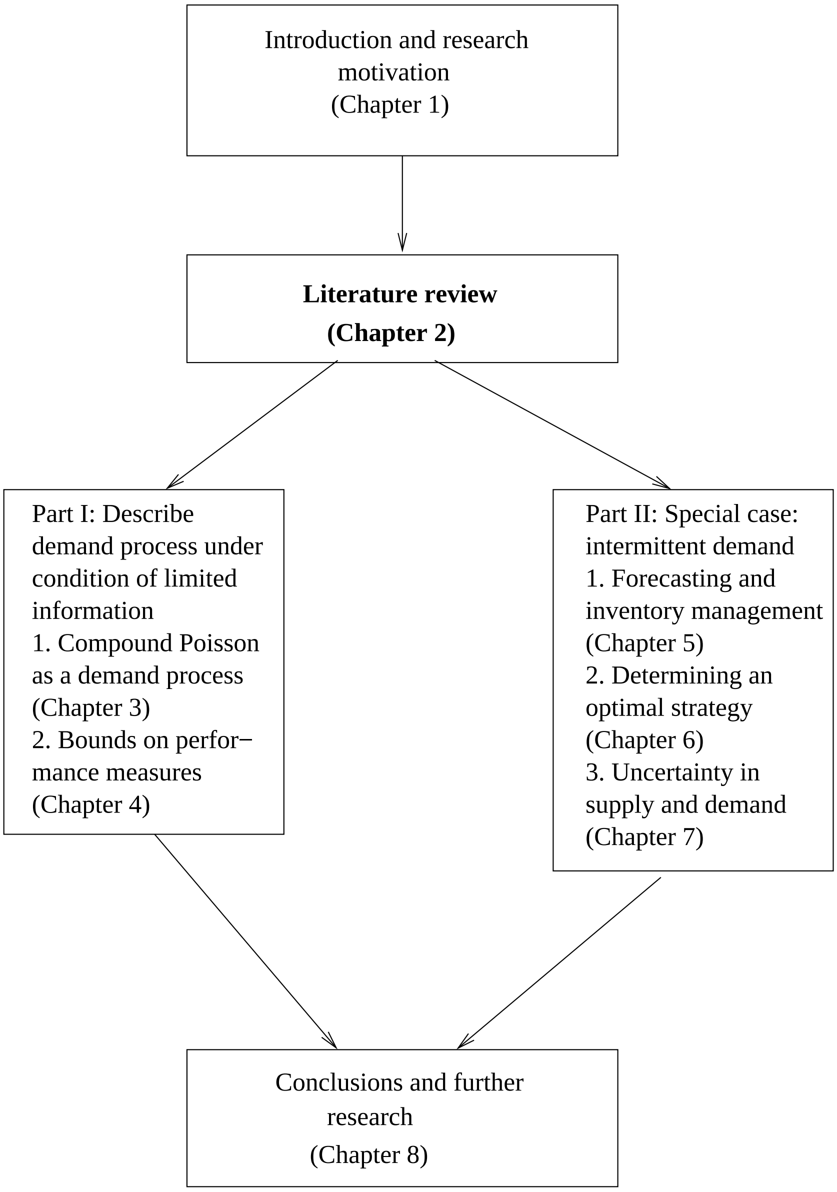

Figure 2.1: Outline of the thesis - Chapter 2

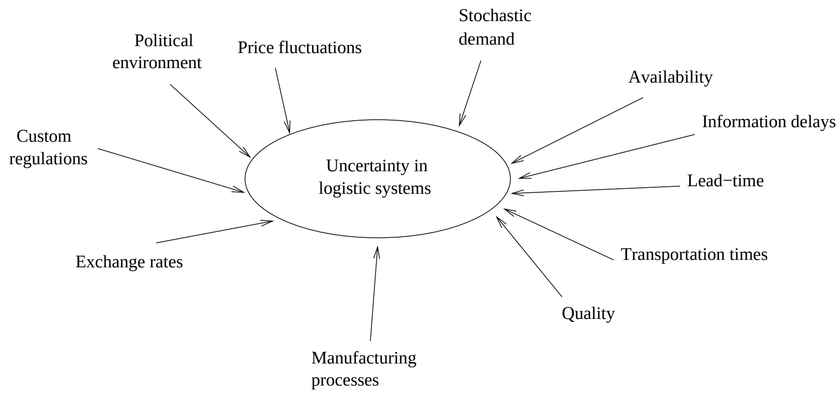

Figure 2.2: Sources of uncertainty in inventory systems

order forecast horizon, input data, administrative and decision processes, and inherent uncertainties. Vidal and Goetschalckx (2000) identify exchange rate fluctuations, variable transportation times, stochastic demand, variability of market prices and political instability as most important sources of uncertainty.

Based on the literature, we distinguish four clusters of sources of uncertainty: demand, supplier performance, manufacturing process and environmental conditions.

The first cluster of sources of uncertainty refers to *demand uncertainty*. This source of uncertainty has received most attention in literature when dealing with uncertainty in inventory systems. A second cluster relates to *supplier performance*. This cluster comprises uncertainty in lead time, in transportation time, in availability of resources, in quality and in information delays. The third cluster of sources of uncertainty is related to the *manufacturing process*. Familiar examples of uncertainty in this cluster are machine breakdowns and manufacturing yield and stochastic costs. The last cluster of sources of uncertainty comprises *environmental conditions*. Price fluctuations, customs regulations, exchange rates and the political environment are all sources of uncertainty in logistics systems due to environmental conditions.

Figure 2.2 presents an overview of sources of uncertainty in inventory systems.

2.2 Probability distribution of demand during lead time

In re-order point models for inventory management the probability distribution of demand is an important characteristic. Most textbooks assume that the demand for an item is formed by a large number of smaller demands from individual customers. As a result, the authors assume that the resulting demand size in a certain period of time is continuous and follows a Normal distribution. For fast moving items a Normal distribution is appropriate. Silver and Peterson (1985) recommend the Normal distribution for items with average lead time demand higher than 10. Using the Normal distribution for a demand size distribution can be questioned because (1) the distribution is defined both on the positive and negative axes; and (2) it is symmetrical. While the Normal distribution could be approximately correct in many cases, it is conceptually not. It cannot be used in computer simulation as negative demand may be generated at random. When of relevance, one rather should look for a probability distribution, which is defined only for non-negative values and allows for skewness.

In the literature on inventory control, many times reference is made to the Gamma distribution. It is defined only on non-negative values and, according to the parameters of its distribution, ranges from a monotonic decreasing function, through unimodal distributions skewed to the right, to Normal distributions. The Gamma distribution is attractive because of the ease it can deal with fixed lead times and how the situations can be extended to probabilistic lead times (Burgin 1975). For items with low demand, Silver and Peterson (1985) propose the Laplace or Poisson distributions. The Poisson distribution has been found to provide a reasonable fit when the demand is very low (only a few pieces per year).

But when demand frequency is not too high, an alternative approach is offered by the use of seperate distributions for the demand occurrence and for the demand size. Dunsmuir and Snyder (1989) estimate the gap between successive demands and assume the positive periodically demand to be Gamma distributed.

Models have been developed using the Poisson distribution for the demand occurrence. When the demand size is described by an arbitrary probability distribution and the demand occurrence process is described as a Poisson process, the total demand during a finite time period can be described by a compound Poisson distribution.

A case study by Vereecke and Verstraeten (1994) shows that demand variance often is a multiple of the average demand, showing that the Poisson distribution is not a good approximation of the demand size. They propose a construction called the 'Package Poisson', where the average demand is expressed in numbers of packages of fixed size. The size of the package is defined by using empirical data on both the average and variance of the demand in terms of units. Other studies make use of a Poisson distribution for demand occurrence and a geometric distribution for the demand size (Hadley and Whitin 1963).

Petrovic (2001) discusses the use of probability distributions in inventory management. A probability distribution is usually derived from evidence recorded in the past. This requires a valid hypothesis that evidence collected is complete and unbiased, and that the stochastic mechanism generating the data recorded continues unchanged. However, there are situations where the requirements are not satisfied. For example, there may be a lack of evidence available or lack of confidence in evidence or simply evidence may not exist, as in the case of launching a new product (Petrovic and Petrovic 2001). In any specific problem the selection of a definite probability distribution is made on the basis of a number of factors, such as the sequence of past demands, judgements about trends, etc. For various reasons, however, these factors may be insufficient to estimate the future probability distribution (Scarf 1958).

This thesis deals with the case where the demand distribution during lead time is not completely known. This situation is realistic either with products which have been introduced recently to the market or with slow moving products. In both cases insufficient data are available to decide on the functional form of the demand distribution function. Limited but not full information might exist like the range of the demand, its expected value, its variance and maybe some knowledge about unimodality of the distribution. In this thesis, the demand process is described under the condition of limited information. First we identify characteristics as demand shape and unimodality under the condition of limited information on demand. Next, we determine the optimal inventory level given a desired performance level under the condition of limited information on demand.

2.3 Intermittent demand

In literature, less attention has been paid to irregular demand. This type of demand is characterised by a high level of variability, but may be also of the intermittent type, i.e. demand peaks follow several periods of zero or low demand. Items with intermittent demand include service or spare parts and high-priced capital goods. Bartezzaghi, Verganti, and Zotteri (1999b) consider five characteristics that cause demand to be of the intermittent type: the numerousness of potential customers, the heterogenity of customers, the frequency of customers requests, the variety of customers requests and the correlation between customers requests.

Demand that is intermittent is often also 'lumpy', meaning that there is also a great variability among the nonzero values (Willemain, Smart, and Schwarz 2004). A lumpy demand is characterised by a high level of variability, which can be measured by the coefficient of variation. However, a lumpy demand is not only variable but may also be nervous thus entailing great differences between successive demand observations, and sporadic, that is, demand peaks follow several periods of zero or low demand (Ward 1978).

Demand forecasting is one of the most crucial issues in inventory management (Willemain, Smart, and Schwarz 2004) but for intermittent demand, forecasting is difficult and errors in prediction may be costly in terms of obsolescent stock or unmet demand (Syntetos and Boylan 2005). The literature that proposes forecasting solutions to this demand uncertainty problem, is relatively small (Ghobbar and Friend 2003). Single exponential smoothing and Croston's method are the most frequently used methods for forecasting intermittent demands (Croston 1972; Willemain, Smart, Shockor, and DeSautels 1994). Croston's method builds estimates taking into account both demand size and the interval between demand occurrences (Croston 1972). Willemain, Smart, Shockor, and DeSautels (1994) conclude that Croston's method is significantly superior to exponential smoothing under intermittent demand conditions. Johnston and Boylan (1996) observe an improvement in forecast performance using Croston's method compared to Holt's method. Bartezzaghi, Verganti, and Zotteri (1999b) find in their experimental simulation that Holt's method appears applicable with low levels of lumpiness. Ghobbar and Friend (2003) compare 13 different forecasting methods when faced with intermittent demand. Weighted moving averages, Holt's method and Croston's method are found superior. Verganti (1997) proposes order overplanning as a technique to forecast intermittent demand. Another forecasting method is early sales, which exploits information from actual orders that have already been received for future delivery (Bartezzaghi, Verganti, and Zotteri 1999b).

Vereecke and Verstraeten (1994) propose an inventory management model using the Package Poisson distribution and calculate the reorder points using an iterative procedure and the probability of no stock-out as service level. Dunsmuir and Snyder (1989) develop an inventory model for intermittent demand which includes a component to explicitly model the probability of positive demand and a Gamma distribution for the size of those demands. The service level used is the proportion of requests met directly from stock. Hollier, Mak, and Lai (2002) develop a mathematical model for the analysis of optimal replenishment policies for intermittent demand items. They introduce a maximum quantity such that customer demands with sizes exceeding this quantity are filtered out of the inventory system and treated as special orders to be satisfied by special deliveries. Sani and Kingsman (1997) indicate a periodic review inventory control system with a re-order level and a replenishment level to be the best for the management of intermittent demand items. Haddock, Natarajan, and Nagar (1994) present a simple and practical heuristic to the slow-moving items problem. Snyder (2002) presents a parametric bootstrap approach that integrates demand forecasting with inventory control. Levén and Segerstedt (2004) create a procedure that can handle both fast-moving items with regular demand and slow-moving items.

In this thesis, several forecasting methods and inventory management policies for intermittent demand are compared. A simulation model is built and optimised to find a best strategy in combining inventory decision making and demand forecasting for intermittent demand.

2.4 Supplier's reliability

Traditionally, attention focuses on uncertainty in customer demand (Petrovic, Roy, and Petrovic 1998) and the implicit assumption is made that the availability of supply is uninterrupted (Parlar 1997).

Inventory managers are often faced with the challenge of incorporating the issue of supplier's reliability into their stocking decisions. The term "supplier reliability" may refer to a number of attributes ranging from availability in responding to replenishment orders, when needed, to variability in delivery lead times as well as the quality of delivered goods. The availability of a supplier can be negatively impacted by a variety of factors including equipment breakdowns, material shortages, capacity constraints, price inflations, strikes, embargos and political crises thereby disrupting the supply process (Mohebbi 2004).

Silver (1981) appears to be one of the first authors to indicate the need for models dealing with supply uncertainty. Nahmias (1993) also discusses the importance of incorporating this type of uncertainty in inventory models.

Models incorporating supply uncertainties are a useful tool for inventory managers (Parlar and Perry 1995). The significance of modelling the issue of supply interruption is due to the severity of its potential negative impact on the performance of supply chains in every competitive business market (Mohebbi 2004).

A lot of literature on uncertainty in supply deals with uncertainty in lead times. We focus on the literature that specifically deals with uncertain lead times in inventory systems. Common topics in this field include order splitting, dual sourcing and order crossover. Since these topics are not the focus of this thesis, they are not considered here.

Kaplan (1970) studies a dynamic inventory model with stochastic lead times and is concerned with characterizing optimal policies for a dynamic inventory problem. Nahmias (1979) constructs simple approximations for three realistic versions of the classical lead time lost-sales inventory problem. One of the versions is the random lead time lost-sales inventory problem.

Bagchi, Hayya, and Chu (1986) investigate the impact of lead time variability on stockouts and stockout risk in a reorder-point reorder-quantity system with i.i.d. demands. They make calculations for accurate safety stock levels. Song (1994) studies a continuous-time single-item inventory model where demands form a compound Poisson process and lead times are stochastic. The focus of this study is on the behavior of the optimal safety stock level in response to stochastically larger or more variable lead times. The results show that a stochastically larger lead time results in a higher optimal safety stock level but does not necessarily lead to a higher optimal average cost. On the other hand, a more variable lead time always leads to a higher long-run average cost for any fixed safety stock policy.

Several authors question the assumption of normality for the distributions of lead time, daily demand and lead time demand. Chopra, Reinhardt, and Dada (2004) also focus on the relationship between lead time uncertainty and safety stock when using exact demand during the lead time instead of the normal approximation. Their results indicate that decreasing the lead time uncertainty increases the required safety stock and that one should focus on decreasing lead times rather than lead time variability in order to decrease inventories. Lau (1989) assumes, instead of normality for the distributions of an inventory item's lead time, daily demand and lead time demand, these distributions can be characterized and estimated in terms of their first four moments. Numerical illustrations show that if a lead time demand is nonnormal, reorder points and service levels can differ significantly from the approximations obtained by incorrectly assuming that the lead time demand is normal. Keaton (1995) states that there is little likelihood that demand over a random lead time is normally distributed. The paper shows that the continuous Gamma distribution is ideal for modeling the demand of slow moving items, and, with appropriate scaling of the units of measure, can easily be adapted for fast moving items as well.

Several papers deal with the optimization of the inventory policy. Sphicas (1982) investigates the solution to an inventory model with constant noninterchangeable demand and random lead times. Bounds are developed for the optimal values of the decision variables. Ehrhardt (1984) analyzes a stochastic lead time inventory model. Conditions for the optimality of base-stock policies and (s, S) policies are established for both finite and infinite planning horizons. Friedman (1988) solves the special case of the EOQ model with stochastic lead times. Eppen and Martin (1988) consider the problem of setting safety stock when both the demand in a period and the lead time are random variables. They consider a procedure for setting the reorder point when both demand during a period and the lead time are random variables. Bashyam and Fu (1998) use a simulation-based procedure to optimize an *(s, S)* inventory system with stochastic lead times, where orders are allowed to cross in time. Schwarz and Weng (2000) develop a model to study the joint effect of lead time and demand uncertainties on safety stocks in a JIT supply chain. They derive expressions for distributor safety stocks. Bollapragada, Rao, and Zhang (2004) consider a two-echelon serial inventory system with demand and supply uncertainty, non zero lead times and a minimum customer service level requirement. They present solution approaches to determine optimal base-stock levels.

A lot of research has been done on supply disruptions in production-inventory systems. In today's business environment, a supplier can deliver the orders in small quantities until a certain level of inventory is reached. So the production-inventory models may also be useful when dealing with supply interruptions during a certain period.

Mohebbi (2004) distinguishes two major classes in literature that treat the analysis of supply disruptions within the context of production/inventory control models. The first class consists of studies that examine the supply disruption problem in a production-storage setting comprised of a failure-prone production facility that supplies a single product at a constant known rate, while operating, into an immediate storage facility which faces a deterministic or stochastic demand process. One of the earlier publications on the issue of supply uncertainty is an article by Meyer, Rothkopf, and Smith (1979). They investigate a system consisting of a production facility subject to random failure and repair processes. They develop expressions for the average inventory level and the fraction of time demand is met but no optimisation analysis is provided. Chao, Chapel, Clark, Morris, Sandlind, and Grimes (1989) used stochastic dynamic programming to find optimal inventory policies for electric utility companies which may face market disruptions.

Other work on production-inventory systems with deterministic demand and supply disruptions includes Bielecki and Kumar (1988), Groenevelt, Pintelon, and Seidmann (1992a), Groenevelt, Pintelon, and Seidmann (1992b) and Moinzadeh and Aggarwal (1997). Bielecki and Kumar (1988) show that there exist ranges of parameter values describing an unreliable manufacturing system for which zero-inventory policies are optimal even when there is uncertainty in manufacturing capacity. Groenevelt, Pintelon, and Seidmann (1992a) have formulated the cost function of the production lot sizing problem when machine time-to-failure follows a general distribution. However, a major limitation of the model is the assumption of negligible repair times. In a follow-up paper, Groenevelt, Pintelon, and Seidmann (1992b) extend the model so that repair times follow a general distribution and a certain fraction of the items produced is diverted into a safety stock while the remaining fraction is used to meet demand. Moinzadeh and Aggarwal (1997) study an unreliable bottleneck production/inventory system with a constant production and demand rate that is subject to random disruptions. The authors assume that the repair times are constant, the time between breakdowns is exponential and excess demand is back-ordered.

Liu and Cao (1999) investigate a production-inventory model under the assumptions that demand for the product is governed by a compound Poisson process, and the machine is subjected to random failures.

Hopp, Pati, and Jones (1989) consider a continuous flow production process subject to failures with an intermediate buffer. They gave a procedure for determining the optimal buffer size and control parameter.

Posner and Berg (1989) incorporated the element of machine imperfection into a relatively simplified production situation: one machine, constant production rate and a compound Poisson demand process for the product. They derive a closed form solution for the steady-state distribution of the inventory level and computed system performance measures. Berg, Posner, and Zhao (1994) extend the model of Posner and Berg (1989) in which only one machine is considered. They studied the performance of production systems that consist of a number of machines, each producing the same type of item and obtained the stationary distribution of the inventory process for different assumptions on the random behaviour of the production, demand and reliability processes.

The second class of studies includes those that focus on an inventory system with an unreliable supplier whose status alternates randomly between two possible states: "available" or "unavailable". A substantial portion of research work in this category corresponds to incorporating the supply disruption phenomenon into classical EOQ-type inventory models under various characterizations of the probability distributions. Parlar and Berkin (1991) derive expressions for optimal order quantity when the reorder point is fixed at zero, the lead time is zero and demand is a known constant. Parlar and Perry (1995) examine a similar problem in which orders may be placed before the on-hand stock level reaches zero. In addition, the inventory manager incurs a fixed cost for every attempt to determine the supplier's state. Weiss and Rosenthal (1992) find the optimal ordering policy in presence of supply or demand disruptions where the start of such disruptions is known a priori. However, the length of the disruption is assumed random. Notice that each of the studies cited above utilizes an EOQ-type framework: it assumes deterministic demand and constant lead times.

Several studies consider the randomness of the demand process in inventory systems with unreliable suppliers and zero lead times. Özekici and Parlar (1999) consider periodic-review inventory models with unreliable suppliers where the demand, supply and cost parameters change with respect to a randomly changing environment. Arreola-Risa and DeCroix (1998) explore the management of (s, S) inventory systems for stochastic demand, where the product's supply is randomly disrupted for periods of random duration, and demand orders that arrive when the inventory system is temporarily out of stock become a mix of backorders and lost sales. Parlar, Wang, and Gerchak (1995) analyse a finite-horizon periodic-review (s, S) inventory model with backlogging and a Markovian supply and ordering cost structure.

Only a few studies exist that address the supply interruption problem within the context of an inventory system with stochastic demand and non-zero lead time. Gupta (1996) presents an exact analytical cost-minimization treatment of a continuous-review lost-sales *(s, Q)* inventory system with Poisson demand and constant lead time in which the supplier's on/off periods are exponentially distributed, and the number of outstanding orders at any time is at most one. Parlar (1997) develops a heuristic cost-minimization model for the supply interruption problem in a continuous-review *(s, Q)* inventory system with random demand, random lead time, and backorders where the duration of the on-period follows an Erlang distribution and the off-period is general. More recently, Mohebbi (2003) presents an exact cost-minimization model for a continuous-review lost sales inventory system in which demands occur according to a compound Poisson process and lead times follow an Erlang distribution. He uses independent non-identical exponential distributions to describe the supplier's on/off periods and applies an *(s, Q)*-type control policy while allowing for a maximum of one order to be outstanding at any time. Mohebbi (2004) considers a similar problem but extended the earlier analysis to a larger family of lead time distributions, using a fairly general stochastic process to describe the supplier's availability.

This thesis deals with the context of an inventory system with intermittent demand and non-zero lead time. A best strategy in combining inventory decision making and demand forecasting for intermittent demand with uncertainty in the supply side is determined. The focus of this chapter is on uncertainty in availability. The supplier alternates randomly between an available and an unavailable state. When the supplier is available, the order is delivered after the usual lead time. If the supplier is unavailable when the order arrives, the order is executed when the supplier turns available again.

2.5 Concluding remarks

In this chapter, the literature is addressed on the research topics which are relevant for this thesis.

After an overview of sources of uncertainty in inventory systems is given, the use of probability distributions is discussed. In literature, most studies assume that uncertainties in parameters can be described by a probability distribution. Several distributions that are used for describing lead time demand are considered. However, there are situations where there is not sufficient data to decide on the functional form of the distribution. Next, intermittent demand, a special type of demand where information on the demand process is often limited, is discussed. The literature on forecasting and inventory management models for intermittent demand is reviewed. Finally, uncertainty in supply is discussed. A lot of research on uncertainty in supply deals with uncertain lead times. Furthermore, supply disruptions in production-inventory systems are considered.

Chapter 3

Identify shape characteristics of demand under the condition of limited information

3.1 Introduction

Inventory management systems are mostly based on the characteristics of the demand distribution during lead-time by means of two characteristics: mean and standard deviation. However, Bartezzaghi, Verganti, and Zotteri (1999a) show a significant impact of the demand shape on inventories. They compare the inventories needed to achieve a predefined service level for six different shapes of the demand distribution. The coefficient of variation is a constant in their experiments. The analysis shows that the demand shape is not a secondary factor in the determination of the inventories and that the impact of different demand shapes on inventories is comparable to the effect of doubling the coefficient of variation. For example, if, given the coefficient of variation, the demand shape changes from a uniform distribution to a bi-modal distribution, inventories increase more than 100%.

In this chapter, there is only limited information of the demand process, i.e. only the first two moments are known. Bartezzaghi, Verganti, and Zotteri (1999a) indicate that it is nevertheless important to know the shape of the distribution. A procedure is described to determine shape characteristics when only the first two moments of the distribution of demand during lead time are known, using a compound Poisson distribution and the Pearson chart. As indicated in Figure 3.1, this chapter[1] is part of Part I, in which the demand process is described under the condition of limited information.

The assumption is made that the same mean and standard deviation for demand can be obtained by various patterns regarding demand frequency and size. The frequency of demand is modelled by a Poisson process. But, for the demand size, we investigate various types of distributions. Like this, total demand during the lead time follows a compound Poisson distribution.

A compound Poisson distribution is the probability distribution of a Poisson-distributed number of independent identically-distributed random variables. More precisely, suppose i.e., N is a random variable whose distribution is a Poisson distribution with expected value λ, and D_1, D_2, D_3,... are identically distributed random variables that are mutually independent and also independent of N. Then the probability distribution of the sum

$$X = \sum_{n=1}^{N} D_n \tag{3.1}$$

is a compound Poisson distribution. The mean and variance of the compound Poisson distribution are:

$$\mu = \lambda \mu_D \tag{3.2}$$

and

$$\sigma^2 = \lambda(\sigma_D^2 + \mu_D^2) \tag{3.3}$$

with μ_D and σ_D^2 the mean and variance of the distribution of the demand size. The central moments of the compound Poisson distribution are:

$$\mu_2' = \lambda \mu_{D,2}, \tag{3.4}$$

$$\mu_3' = \lambda \mu_{D,3} \tag{3.5}$$

[1]This chapter is based on Janssens and Ramaekers (2003).

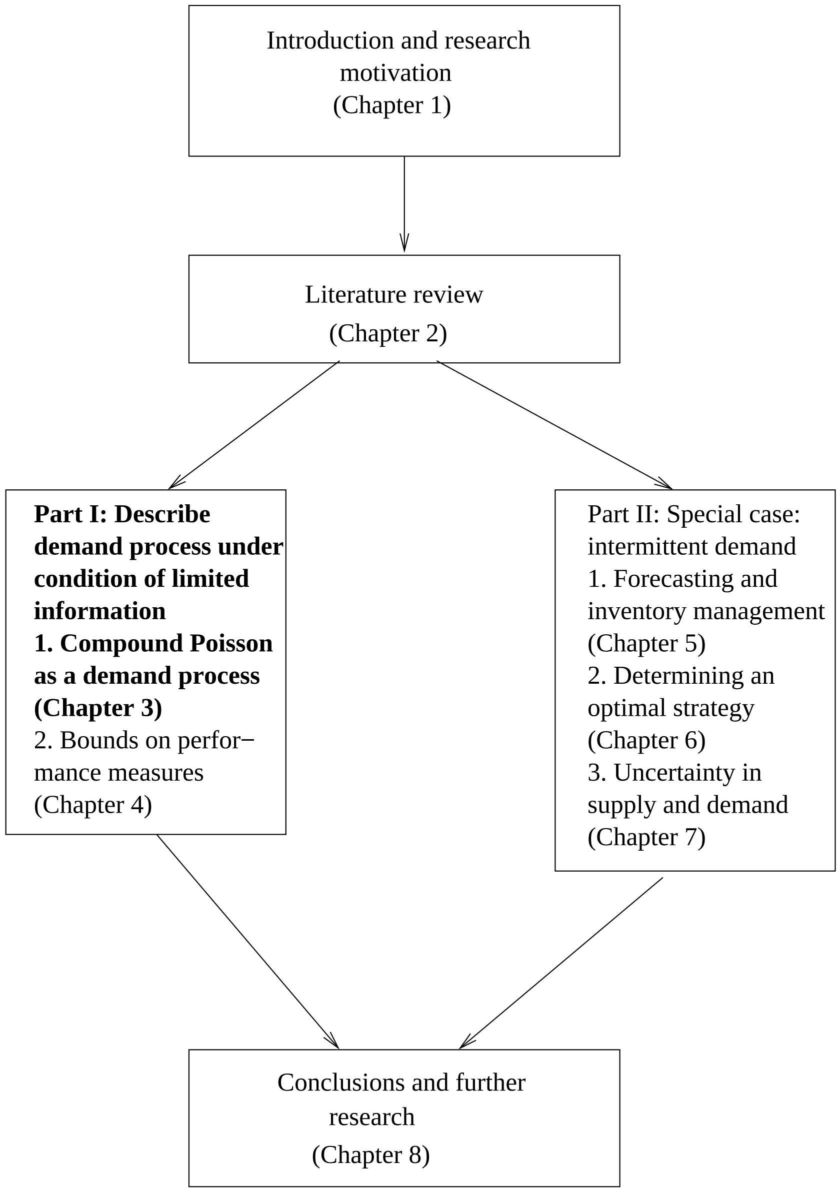

Figure 3.1: Outline of the thesis - Chapter 3

and

$$\mu'_4 = \lambda\mu_{D,4} + 3\lambda^2(\mu_{D,2})^2 \tag{3.6}$$

with $\mu_{D,2}$, $\mu_{D,3}$ and $\mu_{D,4}$ the second, third and fourth raw moment of the demand size distribution.

If the demand size distribution is a discrete distribution, Panjer and Willmot (1992) proof that

$$f_X(x) = \frac{\lambda}{x}\sum_{y=1}^{x} y f_D(y) f_X(x-y), \qquad x = 1, 2, 3... \tag{3.7}$$

$$\text{with } f_X(0) = e^{-\lambda}$$

can be used to define the compound Poisson distribution. If the demand size distribution is continuous, we present a method to obtain distribution characteristics as asymmetry and kurtosis for the compound Poisson distribution in this chapter.

When only the first two moments of the distribution of demand during lead time are known, as is assumed in this chapter, Bartezzaghi, Verganti, and Zotteri (1999a) indicate it is nevertheless important to know distribution characteristics as asymmetry and kurtosis. When a compound Poisson distribution is used for the demand process, it is possible to calculate, for a given mean and variance of the demand distribution, a range of possible λ-values and for each λ, the parameters of the demand size distribution. These outcomes can then be used to calculate the central moments of the distribution of demand during lead time, using formulas 3.4, 3.5 and 3.6. These moments can then be used to determine a single point on the Pearson chart, indicating a shape of the distribution of demand during lead time. In the next paragraphs, the Pearson chart is discussed.

In statistical studies a wide class of distributions, called the Pearson family of distributions is used, allowing for a wide variability of characteristics as asymmetry and kurtosis. Very well-known are the Pearson two-dimensional charts, indicating a range of distributions in terms of an asymmetry characteristic β_1 and a kurtosis characteristic β_2. The Pearson two-dimensional chart is shown in Figure 3.2.

The Pearson distribution is a family of continuous probability distributions, first published by Karl Pearson (Pearson 1895) and subsequently extended by him. The

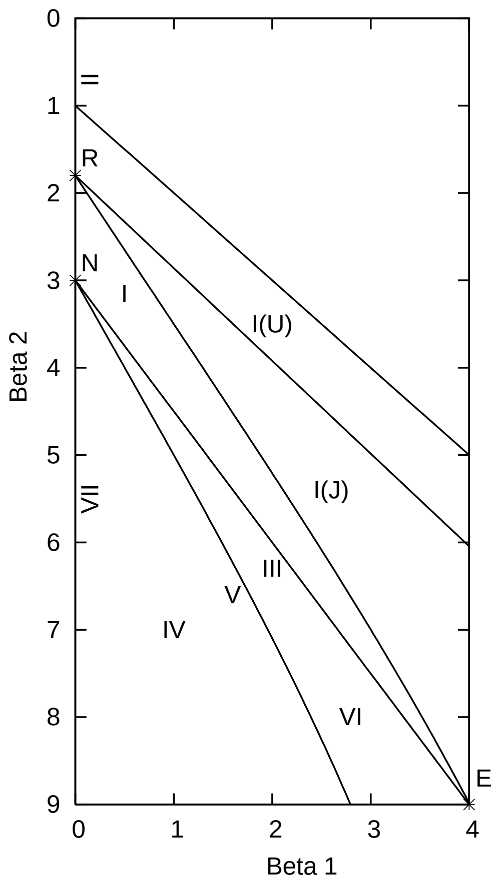

Figure 3.2: Pearson chart

following types of distributions can be distinguished on the graph:

- type I: Beta-distribution. Three areas of type I can be distinguished based on the shape of the Beta-distribution: a unimodal Beta-distribution (I), a J-shaped Beta-distribution (I(J)) and a U-shaped Beta-distribution (I(U));
- type II-line: special case of type I, restricted to symmetric distributions;
- type III-line: Gamma or Chi-square distribution;
- type IV: Cauchy distribution;
- type V-line: reciprocal of Gamma or Chi-square distribution;
- type VI: F-distribution;
- type VII-line: Student's t distribution;
- Normal point N: (0,3);
- Rectangular point R: (0,1.8);
- Exponential point E: (4,9).

The asymmetry and kurtosis measures , which we require for the Pearson chart are defined as

$$\beta_1 = \frac{{\mu'_3}^2}{{\mu'_2}^3} \tag{3.8}$$

and

$$\beta_2 = \frac{\mu'_4}{{\mu'_2}^2}. \tag{3.9}$$

To calculate β_1 and β_2, the necessary information are the moments μ'_2, μ'_3 and μ'_4, which are, for the compound Poisson distribution, calculated using formulas 3.4, 3.5 and 3.6. Like this, any experiment, starting from a given mean and variance of the distribution of demand during lead time (i.e. limited information on demand) and a type of distribution for the demand size, leads to a single point on the Pearson chart and it is possible to determine characteristics as asymmetry and kurtosis of the lead time demand distribution.

The chapter is organized as follows: in section 3.2 1-parameter distributions are considered; section 3.3 deals with the 2-parameter distributions; 3-parameter distributions are discussed in section 3.4; section 3.5 discusses the results; section 3.6 describes the validation of the results of the previous sections and in section 3.7 conclusions are formulated.

3.2 1-parameter distributions

3.2.1 Constant demand size

In case the demand size is constant with size a, the central moments of the compound Poisson distribution are:

$$\mu_2' = \lambda a^2 \tag{3.10}$$

$$\mu_3' = \lambda a^3 \tag{3.11}$$

$$\mu_4' = \lambda a^4 + 3\lambda^2 [a^2]^2 \tag{3.12}$$

When the mean and variance of the aggregated demand are given as M and V, then a can be expressed as:

$$a = \frac{M}{\lambda} = \sqrt{\frac{V}{\lambda}}. \tag{3.13}$$

This leads to the condition that

$$\lambda = \frac{M^2}{V}. \tag{3.14}$$

3.2.2 Uniform demand size in [0,b]

When the demand size follows a uniform distribution with range $[0, b]$, the central moments of the compound Poisson distribution are:

$$\mu_2' = \lambda \frac{b^2}{3} \tag{3.15}$$

$$\mu_3' = \lambda \frac{b^3}{4} \tag{3.16}$$

$$\mu_4' = \lambda \frac{b^4}{5} + 3\lambda^2 \left[\frac{b^2}{3}\right]^2 \tag{3.17}$$

When the mean and variance of the aggregated demand are given as M and V, then b can be expressed as:

$$b = \frac{2M}{\lambda} = \sqrt{\frac{3V}{\lambda}}. \tag{3.18}$$

This gives us the condition

$$\lambda = \frac{4}{3}\frac{M^2}{V}. \tag{3.19}$$

3.2.3 Exponential demand size

In case the demand size is exponentially distributed with parameter α, the central moments of the compound Poisson distribution are:

$$\mu_2' = \lambda \frac{2}{\alpha^2} \tag{3.20}$$

$$\mu_3' = \lambda \frac{6}{\alpha^3} \tag{3.21}$$

$$\mu_4' = \lambda \frac{24}{\alpha^4} + 3\lambda^2 \left[\frac{2}{\alpha^2}\right]^2 \tag{3.22}$$

When the mean and variance of the aggregated demand are given as M and V, then α can be expressed as:

$$\alpha = \frac{\lambda}{M} = \sqrt{\frac{2\lambda}{V}}. \tag{3.23}$$

This leads to the condition that

$$\lambda = \frac{2M^2}{V}. \tag{3.24}$$

3.2.4 Two-point distributions

When the distribution of the demand size has more than one parameter, e.g. the demand size follows a two-point distribution, no unique solution for the compound Poisson distribution with given mean and variance can be found. In case the distribution of the demand size is a two-point distribution, the mean and variance of the compound Poisson distribution are:

$$\mu = \lambda \mu_D \tag{3.25}$$

$$\sigma^2 = \lambda(\sigma_D^2 + \mu_D^2) = \lambda \mu_{D,2} \tag{3.26}$$

with μ_D and σ_D^2 the mean and variance and $\mu_{D,2}$ the second raw moment of the distribution of the demand size with density function:

$$f(x) = \begin{cases} q_r & \text{if } x = r; \\ q_p & \text{if } x = p; \\ 0 & \text{else.} \end{cases} \tag{3.27}$$

The mean and variance of the two-point distribution are equal to:

$$\mu_D = q_r r + q_p p \tag{3.28}$$

$$\sigma_D^2 = q_r r^2 + q_p p^2 - \mu_D^2 \tag{3.29}$$

When the mean and variance of the aggregated demand are given as M and V, then the mean and the second raw moment of the distribution of the demand size can be expressed as:

$$\mu_{D,1} = \frac{M}{\lambda} \tag{3.30}$$

$$\mu_{D,2} = \frac{V}{\lambda} \tag{3.31}$$

If r is known, p, q_r and q_p can be calculated as:

$$p = \frac{\mu_{D,2} - \mu_{D,1} r}{\mu_{D,1} - r} \tag{3.32}$$

$$q_r = \frac{\mu_{D,1} - p}{r - p} \tag{3.33}$$

$$q_p = \frac{\mu_{D,1} - r}{p - r} \tag{3.34}$$

The following example shows that several solutions for the compound Poisson distribution with given mean and variance are possible. Let $M = 50$, $V = 400$ and $\lambda = 10$, then $\mu_{D,1} = 5$ and $\mu_{D,2} = 40$. If we set $r_1 = 2$, then

$$p_1 = 10$$
$$q_{r_1} = \frac{5}{8}$$
$$q_{p_1} = \frac{3}{8}.$$

If $r_2 = 4$, then

$$\begin{aligned} p_2 &= 20 \\ q_{r_2} &= \frac{15}{16} \\ q_{p_2} &= \frac{1}{16}. \end{aligned}$$

We get two different distributions of the demand size that lead to different compound Poisson distributions but with the same mean and variance. In section 3.3 and 3.4, two- and three-parameter distributions will be dealt with.

3.3 2-parameter distributions

3.3.1 Uniform demand size

In the case the demand size follows a uniform distribution with density function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b; \\ 0 & \text{else,} \end{cases} \tag{3.35}$$

the second, third and fourth central moment of the compound Poisson distribution are:

$$\mu_2' = \lambda \frac{b^3 - a^3}{3(b-a)} \tag{3.36}$$

$$\mu_3' = \lambda \frac{b^4 - a^4}{4(b-a)} \tag{3.37}$$

$$\mu_4' = \lambda \frac{b^5 - a^5}{5(b-a)} + 3\lambda^2 \left[\frac{b^3 - a^3}{3(b-a)} \right]^2. \tag{3.38}$$

When the mean and variance of the aggregated demand are given as M and V, then the lower and upper bound of the uniform distribution can be expressed in their values as:

$$a = \frac{\frac{2M}{\lambda} - \sqrt{\frac{12V}{\lambda} - \frac{12M^2}{\lambda^2}}}{2} \tag{3.39}$$

$$b = \frac{\frac{2M}{\lambda} + \sqrt{\frac{12V}{\lambda} - \frac{12M^2}{\lambda^2}}}{2}. \tag{3.40}$$

From the knowledge that both a and $b \geq 0$ and that $M^2 \leq E[X^2]$, a valid lower and upper bound for the value of λ can be determined as:

$$\frac{M^2}{V} \leq \lambda \leq \frac{4M^2}{3V}. \tag{3.41}$$

3.3.2 Triangular demand size with mode = b

In the case the demand size follows a triangular distribution in $[a, b]$ with mode=b and thus density function

$$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)^2} & \text{if } a \leq x \leq b; \\ 0 & \text{else,} \end{cases} \tag{3.42}$$

this leads to:

$$\mu_2' = \lambda \frac{a^2 + 3b^2 + 2ab}{6} \tag{3.43}$$

$$\mu_3' = \lambda \frac{a^3 + 4b^3 + 2a^2b + 3ab^2}{10} \tag{3.44}$$

$$\mu_4' = \lambda \frac{a^4 + 5b^4 + 2a^3b^+3a^2b^2 + 4ab^3}{15} + 3\lambda^2 \left[\frac{a^2 + 3b^2 + 2ab}{6}\right]^2. \tag{3.45}$$

When the mean and variance of the aggregated demand are given as M and V, then a and b can be expressed in their values as:

$$a = \frac{M - 2\sqrt{2\lambda V - 2M^2}}{\lambda} \tag{3.46}$$

$$b = \frac{M + \sqrt{2\lambda V - 2M^2}}{\lambda}. \tag{3.47}$$

For similar reasons as above, a valid lower and upper bound for the value of λ can be determined as:

$$\frac{M^2}{V} \leq \lambda \leq \frac{9M^2}{8V}. \tag{3.48}$$

3.3.3 Triangular demand size with mode = a

In the case the demand size follows a triangular distribution in $[a, b]$ with mode=a and thus density function

$$f(x) = \begin{cases} \frac{2(b-x)}{(b-a)^2} & \text{if } a \leq x \leq b; \\ 0 & \text{else,} \end{cases} \tag{3.49}$$

this leads to:

$$\mu_2' = \lambda \frac{3a^2 + b^2 + 2ab}{6} \tag{3.50}$$

$$\mu_3' = \lambda \frac{4a^3 + b^3 + 3a^2 b + 2ab^2}{10} \tag{3.51}$$

$$\mu_4' = \lambda \frac{5a^4 + b^4 + 4a^3 b^+ 3a^2 b^2 + 2ab^3}{15} + 3\lambda^2 \left[\frac{3a^2 + b^2 + 2ab}{6} \right]^2. \tag{3.52}$$

When the mean and variance of the aggregated demand are given as M and V, then a and b can be expressed in their values as:

$$a = \frac{M - \sqrt{2\lambda V - 2M^2}}{\lambda} \tag{3.53}$$

$$b = \frac{M + 2\sqrt{2\lambda V - 2M^2}}{\lambda}. \tag{3.54}$$

For similar reasons as above, a valid lower and upper bound for the value of λ can be determined as:

$$\frac{M^2}{V} \leq \lambda \leq \frac{3M^2}{2V}. \tag{3.55}$$

3.3.4 Symmetric triangular demand size

In case the demand size follows a symmetric triangular distribution with density function

$$f(x) = \begin{cases} \frac{x-a}{l^2} & \text{if } a \leq x \leq a + l; \\ \frac{2}{l} - \frac{x-a}{l^2} & \text{if } a + l \leq x \leq a + 2l; \\ 0 & \text{else,} \end{cases} \tag{3.56}$$

this leads to:

$$\mu_2' = \lambda\left[\frac{7}{6}l^2 + 2al + a^2\right] \tag{3.57}$$

$$\mu_3' = \lambda\left[a^3 + 3a^2 l + \frac{7}{2}al^2 + \frac{3}{2}l^3\right] \tag{3.58}$$

$$\mu_4' = \lambda\left[a^4 + 4a^3 l + 7a^2 l^2 + 6al^3 + \frac{31}{15}l^4\right] + 3\lambda\left[\frac{7}{6}l^2 + 2al + a^2\right]^2. \tag{3.59}$$

When the mean and variance of the aggregated demand are given as M and V, then the lower bound a and the distance from the bounds till the mode l of the symmetric triangular distribution can be expressed in their values as:

$$a = \frac{M - \sqrt{6(\lambda V - M^2)}}{\lambda} \tag{3.60}$$

$$l = \sqrt{\frac{6}{\lambda^2}(\lambda V - M^2)}. \tag{3.61}$$

For similar reasons as above, bounds on the value of λ can be determined as:

$$\frac{M^2}{V} \leq \lambda \leq \frac{7M^2}{6V} \tag{3.62}$$

3.4 3-parameter distributions

3.4.1 Asymmetric triangular demand size

In case the demand follows an asymmetric triangular distribution with density function

$$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & \text{if } a \leq x \leq c; \\ \frac{2}{l} - \frac{2(b-x)}{(b-a)(b-c)} & \text{if } c \leq x \leq b; \\ 0 & \text{else,} \end{cases} \tag{3.63}$$

this leads to:

$$\mu_2' = \lambda\left[\frac{a^2 + b^2 + c^2 + ab + ac + bc}{6}\right] \tag{3.64}$$

$$\mu_3' = \lambda\left[\frac{a^3 + b^3 + c^3 + a^2 b + a^2 c + ab^2 + ac^2 + abc + b^2 c + bc^2}{10}\right] \tag{3.65}$$

$$\mu_4' = \lambda \left[\frac{a^4 + a^3b + a^3c + a^2b^2 + a^2bc + a^2c^2 + ab^3 + ab^2c}{15} \right.$$
$$\left. + \frac{abc^2 + ac^3 + b^4 + b^3c + b^2c^2 + bc^3 + c^4}{15} \right]$$
$$+3\lambda^2 \left[\frac{a^2 + b^2 + c^2 + ab + ac + bc}{6} \right]^2 . \tag{3.66}$$

When the mean and variance of the aggregated demand are given as M and V, then the mode c and the upper bound b of the symmetric triangular distribution can be expressed in their values, and the additional parameter a, as:

$$b = \frac{3M - a\lambda + \sqrt{D_0}}{2\lambda} \tag{3.67}$$
$$c = \frac{3M - a\lambda - \sqrt{D_0}}{2\lambda} \tag{3.68}$$

with

$$D_0 = 24\lambda V - 27M^2 - 3a^2\lambda^2 + 6a\lambda M. \tag{3.69}$$

The parameter a (≥ 0) can be chosen with some freedom. Looking at $D_0 = D(a)$, the positivity requirement of the discriminant

$$D_1 = 288(\lambda^3 V - \lambda^2 M^2) \tag{3.70}$$

implies the bounds for λ:

$$\frac{M^2}{V} \leq \lambda \leq \frac{3M^2}{2V}. \tag{3.71}$$

Real positive values for a are obtained if

$$\max\left\{0, \frac{6\lambda M - \sqrt{288\lambda^3 V - 288\lambda^2 M^2}}{6\lambda^2}\right\} \leq a \leq$$
$$\min\left\{\frac{6\lambda M + \sqrt{288\lambda^3 V - 288\lambda^2 M^2}}{6\lambda^2},\right.$$
$$\left.\frac{24\lambda M - \sqrt{1152\lambda^3 V - 1152\lambda^2 M^2}}{24\lambda^2}\right\}. \tag{3.72}$$

3.5 Results

In the previous sections, for several demand size distributions, the second, third and fourth central moment of the distribution for demand during lead time are calculated when a compound Poisson is assumed for the demand process. Using these moments and the given mean and variance of the distribution of demand during lead time, the parameters of the demand size distribution can be expressed in terms of the given mean and variance of the distribution of demand during lead time and a valid range for λ can be calculated in terms of this mean and variance.

For several combinations of mean and variance, the parameters β_1 and β_2 are calculated, which, for every combination, leads to a single point on the Pearson chart and it is possible to determine characteristics as asymmetry and kurtosis of the lead time demand distribution. Figures A.1, A.2, A.3, A.4, A.5, A.6, A.7 and A.8 in Appendix A show the Pearson charts with results for the Poisson distributions compounded with deterministic, uniform in [0,b], exponential, uniform, right triangular, left triangular, symmetric triangular and asymmetric triangular distributions. The charts show all the lines, which form the boundaries of the various types of Pearson-distributions. The area filled with the compound distributions under study is shown between a line marked with the symbol + and a line marked with the symbol *o*. For the 1-parameter distributions (deterministic, exponential and uniform in [0,b]) there is not an area, but a line marked with the symbol + that indicates the compound distributions under study.

These figures show that when the mean is high with respect to the variance, the values of β_1 (resp. β_2) are close to 0 (resp. 3), which means the demand distribution is close to normal. The smaller the mean and/or the greater the variance, the more β_1 and β_2 move away from their normal values. The demand follows now a unimodal Beta-distribution. If the mean decreases further relative to the variance, β_1 and β_2 further increase and the distributions is now a J-shaped Beta-distribution.

This holds for all the individual demand size distributions under study. The asymmetry and kurtosis region vary a bit when a different distribution for the individual demand size is used. This results in a small difference in size of the possible (β_1, β_2)-region but the type of distribution that is best chosen is the same. Only the point where the transition from a unimodal Beta-distribution to a J-shaped Beta-distribution takes place is slightly different. In general, when μ^2/σ^2 exceeds 15, the demand is close to normal. When μ^2/σ^2 is below 0.65, the demand follows a J-shaped Beta-distribution. When μ^2/σ^2 lies between 0.65 and 15, the unimodal Beta-distribution may be used to describe the demand.

3.6 Validation

The elaboration of this chapter is based on the assumption that the shape characteristics of a Compound Poisson distribution might be similar to the shape of the various distributions of the Pearson family. Probably the most interesting characteristic might be the aspect of unimodality. As it is shown in the graphs in this chapter that under certain conditions - based on the analogy with the Pearson distributions - a Compound Poisson distribution might be bi-modal, it is important to validate this aspect, as it is of relevance to the performance of inventory policies.

The validation procedure might be described as follows: based on a sample of drawings from a Compound Poisson distribution, a specific type of distribution is proposed based on the graphs produced in this chapter, e.g. a Normal or a Beta-distribution. The hypothesis whether or not the sample fits its indicated distribution is tested both on the Kolmogorov-Smirnov and the χ^2-tests. The confirmation by the test of the distribution indicated, offers information on the shape of the Compound Poisson distribution and more specifically on the unimodality.

Furthermore a tool is used in which a ranking of fitness of the sample data towards a finite subset of distributions is made. The candidate distributions are: Beta, Erlang, Exponential, Gamma, Lognormal, Normal, Triangular, Uniform and Weibull. The ranking is based on the mean square error. As a software tool to serve this purpose the Arena (Rockwell Software) input analyzer has been used.

The sample is generated by the following procedure:

Let

- N : the number of sample data required
- λ : the parameter of the Poisson distribution
- Sample : vector containing the sample data

- Depending on the used distribution of the order size, one or more parameters have to be specified to generate the order sizes.

```
For Sample_counter := 1 to N do
    Begin
         Running_Order_Size := 0;
         K := Poisson_variate (parameter value Lambda);
         For  Order_counter := 1 to K do
             Begin
                  O := Order_size_distribution_variate (parameter values);
                   Running_Order_Size := Running_Order_Size + O;
              End;
          Sample_value := Running_Counter;
          Sample[Sample_Counter] := Sample_value;
     End;
```

The experimental data that are used for the validation regarding a Poisson distribution compounded with a deterministic distribution are shown in Table 3.1. Tables A.1, A.2, A.3, A.4, A.5 and A.6 in Appendix A contain the experimental data for the validation of the Poisson distribution compounded with an exponential distribution, a uniform distribution, a triangular distribution with mode=b, a triangular distribution with mode=a, a symmetric triangular distribution and an asymmetric triangular distribution. Except for the deterministic and the exponential distribution, all distributions for the order sizes are defined on the interval $[a, b]$.

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
First moment	1000	1000	100
Second moment	13333	266666	133333
$\boldsymbol{\lambda}$	75	3.75	0.075
a	13.3	266.67	1333.33

Table 3.1: Experimental data for the validation of the Poisson distribution compounded with a deterministic distribution

The results of the validation process for each of the order size distributions are summarized in Tables 3.2, 3.3, 3.4, 3.5, 3.6, 3.7 and 3.8. As stated before, two tests are used to validate the results: the χ^2 and Kolmogorov-Smirnov (KS) goodness-of-fit hypothesis tests. These are standard hypothesis tests that can be used to assess whether a fitted theoretical distribution is a good fit to the data. Corresponding p-values less than about 0.05 indicate that the distribution is not a very good fit. A high p-value indicates a lack of evidence against the fit. Furthermore a ranking of fitness of the sample data towards a finite subset of distributions is made. This ranking is based on the mean square error. The larger this square error value, the further away the fitted distribution is from the actual data. In the results tables, the three best fitted distributions are given with their square error value. If these square error values lie close to each other, there is probably no difference in accuracy between the fitted distributions.

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
Fitted distribution	Normal(1000;115)	Beta(2.02;3.33)	Beta(0.391;7.44)
χ^2-value	78.6	2000	866
p-value χ^2	<0.005	<0.005	<0.005
Conclusion χ^2	rejected	rejected	rejected
KS-value	0.0261	0.111	6.55
p-value KS	>0.15	<0.01	<0.01
Conclusion KS	not rejected	rejected	rejected
Ranking (MSE)	Normal (0.00413)	Normal (0.1)	Weibull (0.00934)
	Beta (0.00435)	Beta (0.101)	Beta (0.15)
	Weibull (0.0044)	Triangular (0.102)	Exponential (0.189)

Table 3.2: Results for the validation of the Poisson distribution compounded with a deterministic distribution

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
Fitted distribution	Normal(1000;115)	Beta(2.34;5.29)	Beta(0.398;7.79)
χ^2-value	26.4	22.1	395
p-value χ^2	0.0451	0.234	<0.005
Conclusion χ^2	not rejected	not rejected	rejected
KS-value	0.0307	0.0356	0.602
p-value KS	>0.15	>0.15	<0.01
Conclusion KS	not rejected	not rejected	rejected
Ranking (MSE)	Normal (0.00107)	Weibull (0.000731)	Weibull (0.0067)
	Beta (0.00125)	Beta (0.000814)	Beta (0.0867)
	Beta (0.00152)	Gamma (0.00116)	Exponential (0.107)

Table 3.3: Results for the validation of the Poisson distribution compounded with an exponential distribution

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
Fitted distribution	Normal(1000;115)	Beta(2.21;4.76)	Beta(0.425;9.56)
χ^2-value	30.6	27.2	484
p-value χ^2	0.0455	0.0782	<0.005
Conclusion χ^2	not rejected	not rejected	rejected
KS-value	0.0253	0.0324	0.231
p-value KS	>0.15	>0.15	<0.01
Conclusion KS	not rejected	not rejected	rejected
Ranking (MSE)	Normal (0.000929)	Weibull (0.00114)	Weibull (0.00558)
	Beta (0.0013)	Beta (0.00131)	Beta (0.108)
	Weibull (0.00164)	Normal (0.00214)	Exponential (0.12)

Table 3.4: Results for the validation of the Poisson distribution compounded with a uniform distribution

Hypothesis	Normal	Unimodal Beta	J-shaped Beta
Fitted distribution	Normal(1000;115)	Beta(2.12;3.81)	Beta(0.384;7.12)
χ^2-value	23.2	19	766
p-value χ^2	0.287	0.522	<0.005
Conclusion χ^2	not rejected	not rejected	rejected
KS-value	0.0199	0.0339	2.4
p-value KS	>0.15	>0.15	<0.01
Conclusion KS	not rejected	not rejected	rejected
Ranking (MSE)	Normal (0.00091)	Beta (0.0011)	Weibull (0.00364)
	Weibull (0.000923)	Normal (0.00132)	Beta (0.137)
	Beta (0.00108)	Weibull (0.00277)	Exponential (0.193)

Table 3.5: Results for the validation of the Poisson distribution compounded with a triangular distribution with mode=b

Hypothesis	Normal	Unimodal Beta	J-shaped Beta
Fitted distribution	Normal(1000;115)	Beta(2.24;3.89)	Beta(0.384;7.09)
χ^2-value	18.2	44.5	688
p-value χ^2	0.449	<0.005	<0.005
Conclusion χ^2	not rejected	rejected	rejected
KS-value	0.0196	0.0394	2.39
p-value KS	>0.15	0.0905	<0.01
Conclusion KS	not rejected	not rejected	rejected
Ranking (MSE)	Normal (0.000789)	Normal (0.00207)	Weibull (0.00629)
	Beta (0.00108)	Beta (0.00229)	Beta (0.129)
	Weibull (0.00108)	Triangular (0.0048)	Exponential (0.192)

Table 3.6: Results for the validation of the Poisson distribution compounded with a triangular distribution with mode=a

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
Fitted distribution	Normal(1000;115)	Beta(2.31;4.11)	Beta(0.399;7.84)
χ^2**-value**	15.9	23.9	882
p-value χ^2	0.599	0.249	<0.005
Conclusion χ^2	not rejected	not rejected	rejected
KS-value	0.0199	0.0374	1.17
p-value KS	>0.15	0.122	<0.01
Conclusion KS	not rejected	not rejected	rejected
Ranking (MSE)	Normal (0.000715)	Beta (0.00105)	Weibull (0.00499)
	Weibull (0.0011)	Normal (0.0012)	Beta (0.122)
	Beta (0.00123)	Triangular (0.00414)	Exponential (0.166)

Table 3.7: Results for the validation of the Poisson distribution compounded with a symmetric triangular distribution

Hypothesis	Normal	Unimodal Beta	J-shaped Beta
Fitted distribution	Normal(1000;115)	Beta(2.21;4.59)	Beta(0.368;6.32)
χ^2-value	22.3	31.9	672
p-value χ^2	0.392	0.0341	<0.005
Conclusion χ^2	not rejected	not rejected	rejected
KS-value	0.0281	0.0405	0.446
p-value KS	>0.15	0.0767	<0.01
Conclusion KS	not rejected	not rejected	rejected
Ranking (MSE)	Weibull (0.000918)	Weibull (0.00124)	Weibull (0.00738)
	Normal (0.000982)	Beta (0.00137)	Beta (0.139)
	Beta (0.0011)	Normal (0.00156)	Exponential (0.192)

Table 3.8: Results for the validation of the Poisson distribution compounded with an asymmetric triangular distribution

These results indicate that for $\mu^2/\sigma^2 > 15$ the demand process can be approximated by the normal distribution. Both the χ^2- and the KS-test have p-values that indicate that the normal distribution is a good fit for the experimental data. Furthermore, the normal distribution is, except for the asymmetric triangular distribution, always first in the ranking of fitness. For the asymmetric triangular distribution, the Weibull distribution provides a better fit. However, the mean square error of the normal distribution is so close the mean square error of the Weibull distribution that the normal distribution is probably just as accurate as the Weibull.

When $0.65 < \mu^2/\sigma^2 < 15$, the goodness of fit of the unimodal β-distribution is tested. Except for the deterministic distribution, the χ^2- and KS-test indicate that the hypothesis of a unimodal Beta-distribution for demand cannot be rejected. In the ranking of fitness, the unimodal Beta-distribution is either at the first or at the second place. However, when the Beta-distribution is at the second place, the mean square error of the distribution at the first place is close to the mean square error of the Beta-distribution, indicating that both fitted distributions have the same accuracy towards the experimental data.

Both tests indicate that for $\mu^2/\sigma^2 < 0.65$, the hypothesis of a J-shaped Beta-distribution for demand has to be rejected. When the ranking of fitness is considered, the J-shaped Beta-distribution takes a second place, after the Weibull distribution, which has a significantly better fit than the Beta-distribution.

3.7 Concluding remarks

In this chapter we assumed that the same mean and standard deviation of demand can be obtained by various patterns regarding demand frequency and size. The frequency of demand is modelled by a Poisson process and for the demand size, various distributions were examined.

Each experiment leads to a single point on the Pearson chart, a two-dimensional chart indicating a range of distributions in terms of an asymmetry characteristic β_1 and a kurtosis characteristic β_2.

Based on the results, one can say that the normal distribution is a good choice when the mean is high with respect to the variance. Otherwise, one should better opt to use a Beta-distribution in inventory management systems, if a Poisson distribution for the order frequency is used and the individual demand size is one of the distributions examined in this chapter.

Chapter 4

Bounds on performance measures in inventory decision-making

4.1 Introduction

In the previous chapter, the demand process under the condition of limited information was described. It was assumed that the same mean and standard deviation can be obtained by various patterns regarding demand frequency and size. In this chapter, bounds on performance measures are calculated under the condition of limited information (Figure 4.1). Two specific cases of limited information are considered: the case of a known range, expected value and variance and the case of a known range, expected value, variance and unique mode.

Performance measures are an important managerial tool in inventory decision-making (Van Landeghem and Persoons 2001). Almost every inventory system contains uncertainty. Some of the uncertainty (such as lead time, quantity and quality) depends on the suppliers. If the suppliers introduce too much uncertainty, corrective action should be taken. Some uncertainty, however, is attributable to customers, especially demand. If insufficient inventory is held, a stock-out may occur leading to shortage costs. Shortage costs are usually high in relation to holding costs, i.e. the cost of keeping the goods during some time period in the warehouse. Companies are

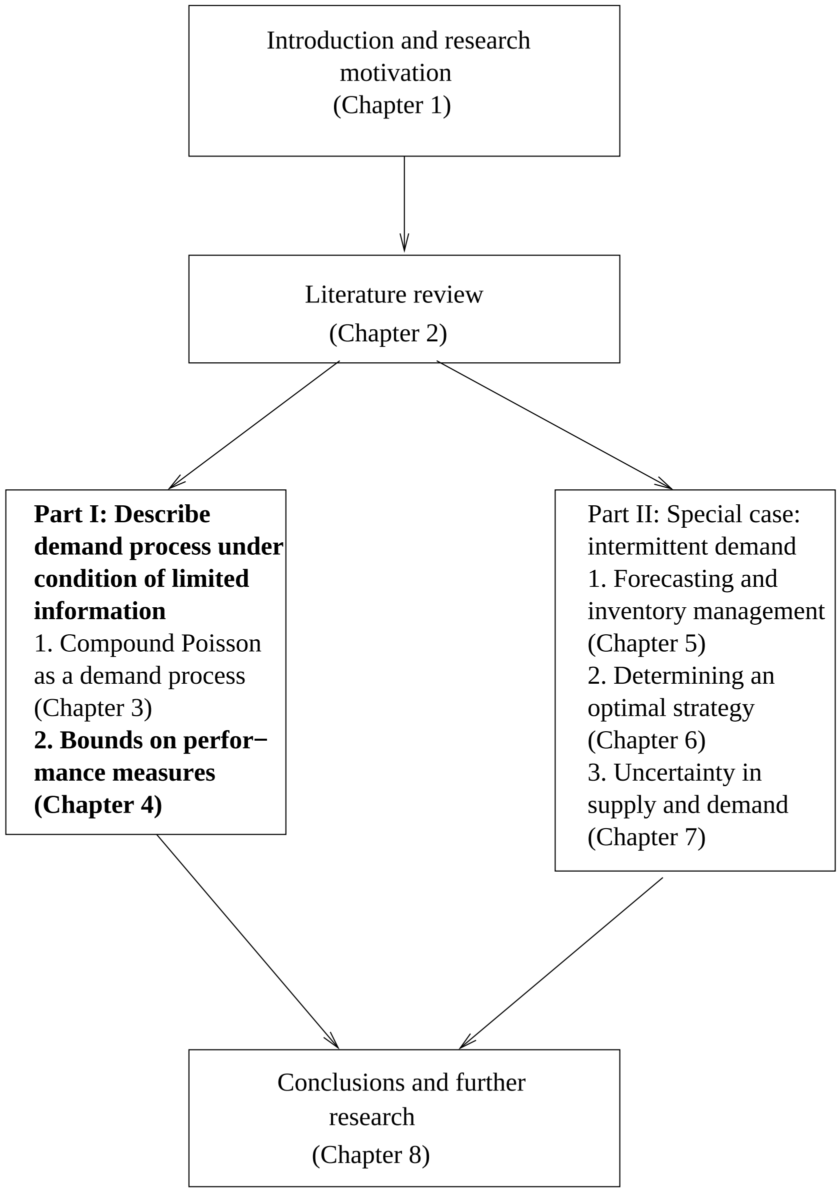

Figure 4.1: Outline of the thesis - Chapter 4

willing to hold additional inventory, above their forecasted needs, to add a margin of safety.

Determination of an inventory replenishment policy, of the quantities to order, of the review period are typical decisions to be taken by logistics managers. Decisions are made making use of optimisation models taking a performance characteristic into consideration which might be cost-oriented or service-oriented. Performance characteristics of the service-oriented type may be expressed relatively as a proportion of customer demand met from inventory, or may be expressed absolutely in terms of number of stock-out units, which is a direct indication for lost sales.

In insurance mathematics, a lot of results have been obtained for deriving bounds on the stop-loss premium $E((X-d)_+)$ where X is allowed to vary under some constraints such as given first order moments, unimodality etc (Heijnen and Goovaerts 1989; De Vylder and Goovaerts 1982; Heijnen 1988). A stop-loss premium limits the risk X of an insurance company to a certain amount d. Furthermore, several authors deduced bounds on tail probabilities (De Schepper and Heijnen 1995).

The same formulas may be useful in the performance evaluation of inventory management in case of uncertain demand during lead time. When a company holds d units of a specific product in inventory starting a period between order and delivery, any demand less than d is satisfied while any demand X greater than d results in a shortage of $X-d$ units. A lesser number of stock-out units results in a better service to the customer. In this way bounds on $E((X-d)_+)$ are a measure for customer service in inventory management. Bounds on tail probabilities can be seen as bounds on the stock-out probability in inventory management.

Let W be the number of stock-out units and d the inventory level. The relationship between W and X is:

$$W = \begin{cases} 0 & \text{if } x \leq d; \\ x-d & \text{if } x > d. \end{cases} \tag{4.1}$$

If U is defined as the stock-out probability, the relationship between U and X is:

$$U = \begin{cases} 0 & \text{if } x \leq d; \\ 1 & \text{if } x > d. \end{cases} \tag{4.2}$$

Before we move towards the remainder of this chapter, it should be stated that the bounds and their use in applications can be translated from any distribution defined on $[a, b]$ into the bounds with a distribution defined on $[0, b_0]$. If $a \neq 0$ and a, b, μ_1 and μ_2 are known, the parameters for the distribution defined on $[0, b_0]$ can be calculated using the following formulas:

$$a_0 = 0, \tag{4.3}$$

$$b_0 = b - a, \tag{4.4}$$

$$\mu_{1,0} = \mu_1 - a, \tag{4.5}$$

$$\mu_{2,0} = \mu_2 - 2a\mu_1 + a^2. \tag{4.6}$$

In the following paragraphs we will work, without loss of generalisation, with distributions defined on $[0, b_0]$.

An example is used throughout the chapter to demonstrate the use of bounds on performance measures. In this example, limited information on demand is known: the mean μ_1 equals 20, the second moment μ_2 equals 600 and the range of demand is [0,b] with $b = 50$. If the unique mode m exists and is known, it equals 15. By calculating bounds on performance measures, it is possible to calculate bounds on the inventory level given a desired level of performance measure. Depending on the degree of optimism of the company, the upper or lower bound on the inventory level can be used to determine the inventory level that has to be held at the beginning of a period. If it is known that a unique mode exists and the value of it is known, the extra information can be used to make the bounds more tight.

The organization of the chapter is as follows: section 4.2 describes the method used to calculate the bounds; section 4.3 presents the lemmas needed to generate two-point and three-point distributions; in section 4.4 the number of stock-out units is discussed; in section 4.5 the stock-out probability is dealt with and in section 4.6 conclusions are drawn.

4.2 Method

4.2.1 E(X) and $E(X^2)$ are known

This section describes the method (Heijnen and Goovaerts 1989; De Schepper and Heijnen 1995) to calculate upper and lower bounds on the number of stock-out units and the stock-out probability, when only the first and second moment of the demand distribution are known.

We define $\mu_1 = E(X)$ and $\mu_2 = E(X^2)$. The demand X is always positive and it has an upper bound b. From a mathematical point of view, when calculating bounds on number of stock-out units, the problem is to find

$$\sup_{F \in \Phi} \int_0^b f(x) dF(x) \tag{4.7}$$

and

$$\inf_{F \in \Phi} \int_0^b f(x) dF(x) \tag{4.8}$$

where Φ is the class of all distribution functions with range [0,b] and moments μ_1 and μ_2 and where $f(x) = (x - d)_+$.

For any polynomial P(x) of degree 2 or less, the integral $\int_0^b P(x)dF(x)$ only depends on μ_1 and μ_2, so it takes the same value for all distributions in Φ. We will look for such polynomials P such that

- $P \geq f$ on $[0, b]$ (in case of upper bound) or $P \leq f$ on $[0, b]$ (in case of lower bound)
- there is some distribution G in Φ for which equality holds:

$$\int_0^b P(x) dG(x) = \int_0^b f(x) dG(x) \tag{4.9}$$

The left hand side only depends on known parameters and determines the best upper bound. As distribution G we will use a two- or three-point distribution in Φ. In section 4.3 one can find how to generate such distributions. For such distributions the equality mentioned above is attained when P(x) and f(x) are equal in the masspoints of G.

To apply this method the formula for a unique parabola g(x) taking values f(u) and f(v) in u and v with derivative f'(v) in v (u and v any points in $[0, b]$) is needed:

$$\begin{aligned} g(x) = \frac{1}{(v-u)^2} & [f(v)(v-u)(x-u) + f(u)(u-v)(x-v) \\ & + [f'(v)(v-u) - f(v) + f(u)](x-u)(x-v)]. \end{aligned} \tag{4.10}$$

When calculating bounds on tail probabilities, the problem is to find

$$\sup_{F\in\Phi}\int_0^b f(x)dF(x) \tag{4.11}$$

and

$$\inf_{F\in\Phi}\int_0^b f(x)dF(x) \tag{4.12}$$

where Φ is the class of all distribution functions with range $[0, b]$ and with moments μ_1 and μ_2 known and where

$$f(x) = \begin{cases} 0 & \text{if } x \leq d; \\ 1 & \text{if } x > d. \end{cases} \tag{4.13}$$

The method for calculating these bounds on tail probabilities is similar to the one explained above for bounds on the number of stock-out units.

4.2.2 E(X), $E(X^2)$ and the unique mode m are known

This section describes the method (Heijnen and Goovaerts 1989; De Schepper and Heijnen 1995) to calculate upper and lower bounds on the number of stock-out units and the stock-out probability, when only the first and second moment and the mode of the demand distribution are known.

This problem will be transformed to the previous one, using a technique analogous to the method of Brockett and Cox (1985). We need a characterisation theorem of unimodal distributions due to Khinchine. A proof can be found in Feller (1984).

Theorem *A stochastic variable Z is unimodal with mode 0 and range I if and only if it has the same distribution as the product U*V of two independent variables U and V such that U is distributed uniformly on [0,1] and V has range I*

For X with mode m and range $[0, b]$ we can use the theorem with $Z = X - m$, so V will have range $[-m, b - m]$. In that case we know for any function f that

$$E[f(Z)] = E[f^*(V)] \tag{4.14}$$

with

$$f^*(x) = E[f(UV)|V = x] = \frac{1}{x}\int_0^x f(t)dt. \tag{4.15}$$

We can define $Y = V + m$, so Y has range $[0, b]$. Now is

$$E[f(X)] = E[\widetilde{f}(Y)] \tag{4.16}$$

with

$$\widetilde{f}(x) = \frac{1}{x-m}\int_0^{x-m} f(t+m)dt. \tag{4.17}$$

We call $\widetilde{f}(x)$ the "Khinchine" transform of f(x).
Using 4.17 we can calculate the moments $\nu_1 = E(Y)$ and $\nu_2 = E(Y^2)$.

$$\nu_1 = 2\mu_1 - m \tag{4.18}$$

$$\nu_2 = 3\mu_2 - 2m\mu_1. \tag{4.19}$$

We will use the same method as in section 4.2.1 to determine

$$\sup_{F\in\Phi'}\int_0^b \widetilde{f}(x)dF(x) \tag{4.20}$$

and

$$\inf_{F\in\Phi'}\int_0^b \widetilde{f}(x)dF(x) \tag{4.21}$$

with Φ' the class of the distribution functions on $[0, b]$ with moments ν_1 and ν_2.

4.3 Generating two-point and three-point distributions

The following lemma is needed to find useful two-point distributions (Jansen, Haezendonck, and Goovaerts 1986).

Lemma 1 Suppose X is a (not constant) demand with range $[0, b]$ and with moments $E(X) = \mu_1$ and $E(X^2) = \mu_2$. Let

$$r' = \frac{\mu_2 - \mu_1 r}{\mu_1 - r} = \mu_1 + \frac{\mu_2 - (\mu_1)^2}{\mu_1 - r} \tag{4.22}$$

for every $r \in [0, b]$ and $r \neq \mu_1$. Then there exists for every $r \in [0, b']$ a unique two point distribution on $[0', b]$ with given moments μ_1 and μ_2, i.e. the distribution in (r, r') with masses:

$$q_r = \frac{\mu_1 - r'}{r - r'}, q_{r'} = \frac{\mu_1 - r}{r' - r} \tag{4.23}$$

In addition the inequalities

$$0 < b' < \mu_1 < 0' < b \tag{4.24}$$

hold and r' is a strictly increasing function of r from $[0, b']$ upon $[0', b]$.
An immediate consequence of this lemma is that $r'' = r$.

To find useful three-point distributions we will use the next lemma (Jansen, Haezendonck, and Goovaerts 1986).

Lemma 2 Suppose X is a (not constant) demand with range $[0, b]$ and with moments $E(X) = \mu_1$ and $E(X^2) = \mu_2$. For every $\alpha \in [0, \mu_1[$, for every $\gamma \in [0', b[$ such that $\alpha' < \gamma$ (or $\alpha < \gamma'$), for every β such that $\gamma' < \beta < \alpha'$, there exists a unique three-point distribution in (α, β, γ) with moments μ_1 and μ_2 and masses

$$q_\alpha = \frac{\sigma^2 + (\mu_1 - \beta)(\mu_1 - \gamma)}{(\beta - \alpha)(\gamma - \alpha)} > 0 \tag{4.25}$$

$$q_\beta = \frac{-\sigma^2 - (\mu_1 - \alpha)(\mu_1 - \gamma)}{(\beta - \alpha)(\gamma - \beta)} > 0 \tag{4.26}$$

$$q_\gamma = \frac{\sigma^2 + (\mu_1 - \alpha)(\mu_1 - \beta)}{(\gamma - \alpha)(\gamma - \beta)} > 0 \tag{4.27}$$

From the conditions it follows immediately that $\alpha < \gamma' < \beta < \alpha' < \gamma$.

4.4 Number of stock-out units

Using the results which have been obtained in insurance mathematics, upper and lower bounds on the number of stock-out units can be obtained in inventory management, given the safety inventory and various levels of information about the demand distribution. However, from a production or trading company's point of view, it is more interesting to know, given an expected number of stock-out units the company wants to face, what the safety inventory should be at least or at most.

Two levels of information about the demand distribution are discussed in this section: the case of a known range, expected value and variance and the case of a known range, expected value, variance and unique mode. For each case, upper and lower bounds on the number of stock-out units are determined using the results of insurance mathematics (Heijnen and Goovaerts 1989; De Vylder and Goovaerts 1982; Heijnen 1988). Next, the optimal inventory level is calculated given the desired maximum number of stock-out units. We consider the special case of a compound Poisson distribution and the section is concluded with a numerical example.

4.4.1 E(X) and $E(X^2)$ are known

UPPER BOUNDS

As already stated before, the problem is to find:

$$\sup_{F\in\Phi}\int_0^b f(x)dF(x) \tag{4.28}$$

where Φ is the class of all distribution functions with range $[0,b]$ and moments μ_1 and μ_2 and where $f(x)=(x-d)_+$.

The two-point distribution that will be used depends on the position of the parameter d in the interval $[0,b]$. Three situations can be distinguished (Heijnen and Goovaerts 1989):

- The parameter d is "rather small" (see Figure 4.2). In this situation, the parabola P is equal and tangent to $(x-d)_+$ at $0'$ and equal to $(x-d)_+$ at 0. These three conditions determine P uniquely. A fourth condition needs to be imposed to avoid P to become negative in $[0,b]$. This last condition forces d to be "rather small".

- The parameter d is "not too big and not too small" (see Figure 4.3). Now, the parabola P is equal and tangent to $(x-d)_+$ at r and r', so P has to fulfill four conditions. Because P is determined uniquely by three conditions, the fourth condition will say in a mathematical way that d should be "not too big and not to small".

- The parameter d is "rather big" (see Figure 4.4). In this situation, the parabola P is equal and tangent to $(x-d)_+$ at b' and equal to $(x-d)_+$ at b. To make

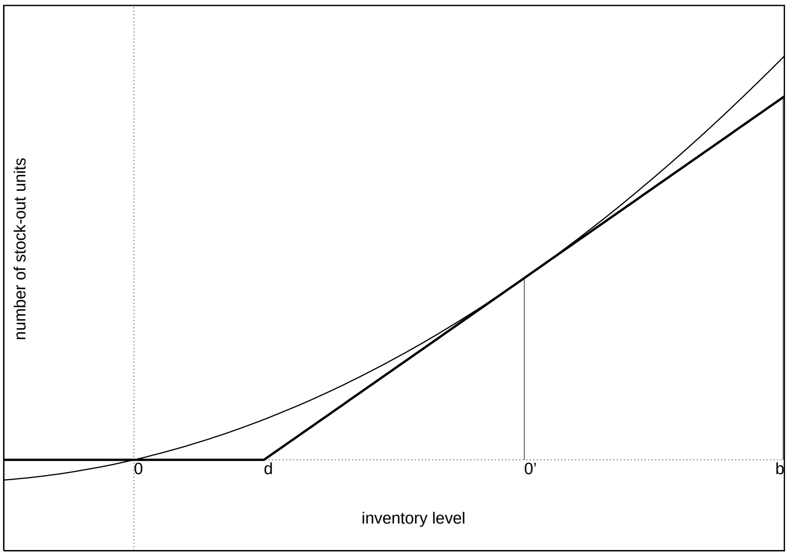

Figure 4.2: Upper bound on $(x-d)_+$ if parameter d is small

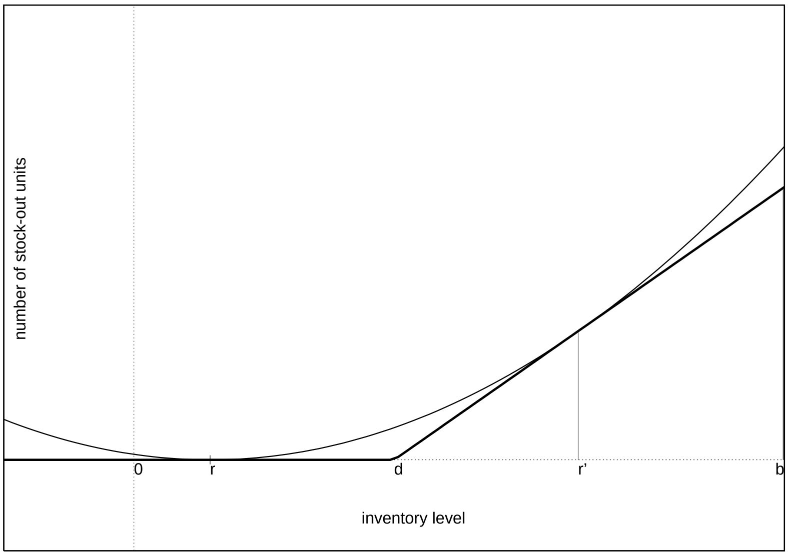

Figure 4.3: Upper bound on $(x-d)_+$ if parameter d is not too big and not too small

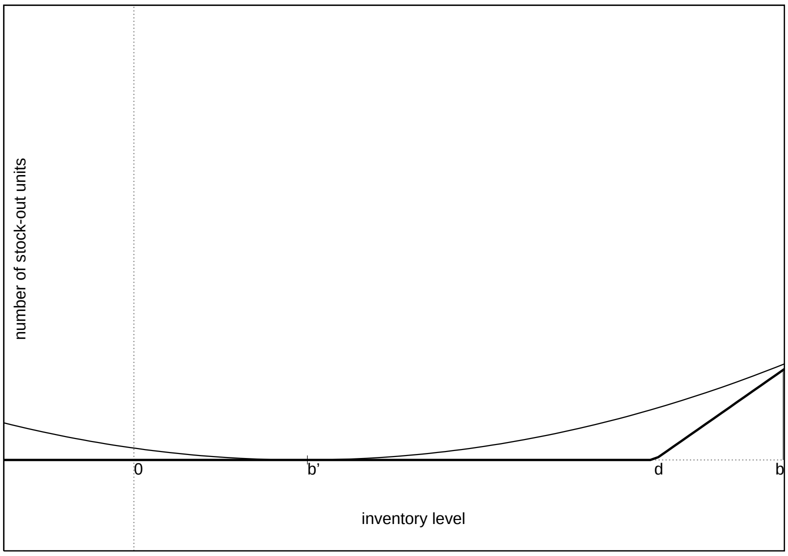

Figure 4.4: Upper bound on $(x-d)_+$ if parameter d is big

sure P is an upper bound, a fourth condition has to be imposed and will say in a mathematical way that d is "rather big".

Parabola through (0,0) and (0',f(0'))

According to Lemma 1 there exists a two point distribution with moments μ_1 and μ_2 in (0,0'). Formula 4.10 can be used with $u = 0$, $v = 0'$ and $f(0) = 0$.

$$g(x) = \frac{1}{0'^2}[f(0')0'x + (f'(0')0' - f(0'))x(x - 0')]. \tag{4.29}$$

To assure that $g \geq 0$ on $[0, d]$, we impose $g'(0) \geq 0$ which means that

$$f'(0') \leq \frac{2f(0')}{0'} \tag{4.30}$$

or

$$d \leq \frac{0'}{2}. \tag{4.31}$$

The best upper bound is $q_{0'}f(0')$ or

$$\frac{\mu_1}{\mu_2}(\mu_2 - \mu_1 d). \tag{4.32}$$

Parabola through (r,0) and (r',f(r'))

Formula 4.10 is used with $v = r$, $u = r'$, $f(v) = 0$ and $f'(v) = 0$. This gives us:

$$g(x) = \frac{f(r')(x-r)^2}{(r'-r)^2}. \tag{4.33}$$

The condition $g'(u) = f'(r')$ leads to

$$2f(r') = (r'-r)f'(r') \tag{4.34}$$

or

$$d = \frac{r+r'}{2}. \tag{4.35}$$

Because of Lemma 1 a unique solution (r,r') can be assured by imposing the condition

$$\frac{0'}{2} \leq d \leq \frac{b+b'}{2}. \tag{4.36}$$

Under this condition the best upper bound is $q_{r'}f(r')$ or

$$\frac{\mu_1 - d + \sqrt{(\mu_2 - \mu_1^2) + (d-\mu_1)^2}}{2}. \tag{4.37}$$

Parabola through (b',0) and (b,f(b))

In this case we take $u = b$, $v = b'$, $f(v) = 0$ and $f'(v) = 0$ and obtain

$$g(x) = \frac{f(b)(x-b')^2}{(b-b')^2}. \tag{4.38}$$

To assure $g \geq f$ we impose $g'(b) \leq f'(b)$ or

$$2f(b) \leq (b-b')f'(b) \tag{4.39}$$

or

$$d \geq \frac{b+b'}{2}. \tag{4.40}$$

In that case the upper bound is $q_b f(b)$ or

$$\frac{(\mu_2 - \mu_1^2)(b-d)}{(\mu_2 - \mu_1^2) + (b-\mu_1)^2}. \tag{4.41}$$

The results for the best upper bounds on number of units short when only the mean and variance of demand are known, are summarized in Table B.1 in Appendix B.

LOWER BOUNDS

Now, the problem is to find:

$$\inf_{F \in \Phi} \int_0^b f(x) dF(x) \tag{4.42}$$

where Φ is the class of all distribution functions with range $[0, b]$ and moments μ_1 and μ_2 and where $f(x) = (x - d)_+$.

Here, also three situations can be distinguished, depending on the position of d in the interval $[0, b]$ (De Vylder and Goovaerts 1982).

$0 \leq$ d $\leq$ b'

A solution is found when P is the straight line through $(d, 0)$, $(\mu_1, f(\mu_1))$ and $(b, f(b))$. The three-point distribution will have masses:

$$q_d = \frac{\mu_2 - \mu_1^2}{(d - \mu_1)(d - b)}; q_{\mu_1} = \frac{\mu_2 - \mu_1^2 + (\mu_1 - d)(\mu_1 - b)}{(\mu_1 - d)(\mu_1 - b)}; q_b = \frac{\mu_2 - \mu_1^2}{(b - d)(b - \mu_1)} \tag{4.43}$$

The lower bound equals $q_{\mu_1} f(\mu_1) + q_b f(b)$ or

$$\mu_1 - d. \tag{4.44}$$

b' $<$ d $<$ 0'

In this case, P is the parabola through $(0, 0)$, $(d, 0)$ and $(b, f(b))$. The best lower bound is $q_b f(b)$ or

$$\frac{\mu_2 - \mu_1 d}{b}. \tag{4.45}$$

0' $\leq$ d $\leq$ b

Here, a solution is found when P is the straight line through $(0, 0)$, $(\mu, 0)$ and $(d, 0)$. The best lower bound is equal to 0.

The results for the best lower bounds on number of units short when only the mean and variance of demand are known, are summarized in Table B.2 in Appendix B.

OPTIMAL INVENTORY LEVEL

To determine the optimal inventory level given the desired maximum number of stock-out units, using the upper and lower bounds derived in the previous sections, numerical analysis can be used.

However, in the case where only the range and first two moments of the demand distribution are known, it is possible to calculate the optimal inventory level given the desired maximum number of stock-out units. In Table 4.1 the optimal inventory level is expressed in terms of the desired number of stock-out units, using the upper bounds on number of stock-out units obtained in the previous section. In Table 4.2 the lower bounds on number of stock-out units are used to calculate the optimal inventory level in terms of the desired number of stock-out units.

Conditions	**Inventory level**
$W \leq \frac{\mu_2-\mu_1^2}{2(b-\mu_1)}$	$b - \frac{W[(\mu_2-\mu_1^2)+(b-\mu_1)^2]}{\mu_2-\mu_1^2}$
$\frac{\mu_2-\mu_1^2}{2(b-\mu_1)} \leq W \leq \frac{\mu_1}{2}$	$\frac{(\mu_2-\mu_1^2)-4W^2+4W\mu_1}{4W}$
$W \geq \frac{\mu_1}{2}$	$\frac{(\mu_1-W)\mu_2}{\mu_1^2}$

Table 4.1: Optimal inventory level using the upper bounds of number of stock-out units when E(X) and $E(X^2)$ are known

COMPOUND POISSON

In the previous chapter, the demand process is described under the condition of limited information using a compound Poisson process. Demand frequency is modelled by a Poisson process and various types of distributions are investigated for the demand size. If we want to apply the formulas of this chapter to the compound Poisson distribution with known mean and standard deviation, b has to be set to infinity. In Table 4.3 and 4.4 the upper and lower bounds on the number of stock-out units are given when $b = \infty$.

Conditions	**Inventory level**
$W \leq \frac{\mu_2-\mu_1^2}{b-\mu_1}$	$\frac{\mu_2-bW}{\mu_1}$
$W \geq \frac{\mu_2-\mu_1^2}{b-\mu_1}$	$\mu_1 - W$

Table 4.2: Optimal inventory level using the lower bounds of number of stock-out units when E(X) and $E(X^2)$ are known

Conditions	**Upper bound**
$d \leq \frac{0'}{2}$	$\frac{\mu_1}{\mu_2}(\mu_2 - \mu_1 d)$
$d \geq \frac{0'}{2}$	$\frac{\mu_1-d+\sqrt{(\mu_2-\mu_1^2)+(d-\mu_1)^2}}{2}$

Table 4.3: Upper bounds on number of stock-out units when E(X) and $E(X^2)$ are known and $b = \infty$

The formulas for the optimal inventory level can also be calculated when $b = \infty$ and are shown in Table 4.5 and 4.6.

4.4.2 E(X), $E(X^2)$ and the unique mode m are known

As mentioned in Section 4.2.2 for unimodal X with mode m

$$\int_d^b (x-d)dF_X(x) = \int_d^b f(x)dF_Y(x) \tag{4.46}$$

where Y has range $[0, b]$. If X has moments μ_1 and μ_2, then Y has moments ν_1 and ν_2 as defined in 4.18 and 4.19. The function f(x) in 4.46 is the Khinchine transform of $(x-d)_+$ and is calculated using 4.17. When we impose the condition $d > m$ the function in 4.46 is

$$f(x) = \frac{(x-d)^2}{2(x-m)}. \tag{4.47}$$

Conditions	Lower bound
$0 \leq d \leq b'$	$\mu_1 - d$
$b' \leq d \leq b$	0

Table 4.4: Lower bounds on number of stock-out units when E(X) and $E(X^2)$ are known and $b = \infty$

Conditions	Inventory level
$W \leq \frac{\mu_1}{2}$	$\frac{(\mu_2-\mu_1^2)-4W^2+4W\mu_1}{4W}$
$W \geq \frac{\mu_1}{2}$	$\frac{(\mu_1-W)\mu_2}{\mu_1^2}$

Table 4.5: Optimal inventory level using the upper bounds of number of stock-out units when E(X) and $E(X^2)$ are known and $b = \infty$

By imposing $d > m$ some generality is lost, but if $d \leq m$ there is enough data to use the classical estimator for E(W) so there is no need for bounds.

For technical reasons one more condition is imposed on the parameters of the problem:

$$\frac{4}{3}m < 0' = \frac{\nu_2}{\nu_1} = \frac{3\mu_2 - 2m\mu_1}{2\mu_1 - m}. \tag{4.48}$$

This condition imposes some skewness on the distribution of X, which is mostly satisfied in practice.

The same formulas as in section 4.4.1 can now be applied (Heijnen and Goovaerts 1989; Heijnen 1988).

Conditions	**Inventory level**
$W \geq 0$	$\mu_1 - W$

Table 4.6: Optimal inventory level using the lower bounds of number of stock-out units when E(X) and $E(X^2)$ are known and $b = \infty$

UPPER BOUNDS

Parabola through (0,0) and (0',f(0'))

Formula 4.29 and 4.30 are valid here too and lead to the following restriction on d:

$$d \leq \frac{0'^2}{30' - 2m}. \tag{4.49}$$

The best upper bound is $q_{0'} f(0')$ or

$$\frac{\nu_1(\nu_2 - d\nu_1)^2}{2\nu_2(\nu_2 - m\nu_1)}. \tag{4.50}$$

Parabola through (r,0) and (r',f(r'))

Condition 4.34 gives here

$$d = \frac{r'^2 - 2mr + rr'}{3r' - 2m - r}. \tag{4.51}$$

d is an increasing function of r and r'. Because of Lemma 1 we get as condition on d:

$$\frac{0'^2}{30' - 2m} \leq d \leq \frac{b^2 - 2mb' + bb'}{3b - 2m - b'}. \tag{4.52}$$

The best upper bound is now $q_{r'} f(r')$. The upper bound can be written as a function of r':

$$\frac{(\nu_2 - \nu_1^2)(r' - d)^2}{2(\nu_2 - 2r'\nu_1 + r'^2)(r' - m)} \tag{4.53}$$

where r' is the root of $r'^3 + Ar'^2 + Br' + C = 0$ with $A = -3d$, $B = 4\nu_1 d + 2md - 2m\nu_1 - \nu_2$ and $C = 2m\nu_2 - 2m\nu_1 d - d\nu_2$.

Parabola through (b',0) and (b,f(b))

Condition 4.39 here implies

$$\frac{b^2 - 2mb' + bb'}{3b - 2m - b'} \leq d \tag{4.54}$$

The upper bound is $q_b f(b)$ or

$$\frac{(\nu_2 - \nu_1^2)(b-d)^2}{2(\nu_2 - 2b\nu_1 + b^2)(b-m)}. \tag{4.55}$$

The results for the best upper bounds on number of units short when only the mean, variance and unique mode of demand are known, are summarized in Table B.3.

LOWER BOUNDS

If $0' \leq d$, the solution is the straight line through $(0,0)$ and $(0',0)$. The best lower bound is 0.

Parabola through (0,0) and (0',f(0'))

Formula 4.10 can be used with u=0, v=0' and f(0)=0. To assure that the best lower bound is found, the condition g(b) ≤ f(b) needs to be imposed. This leads to:

$$\frac{b0'^2 + 0'(0'-m)\sqrt{b(b-m)}}{b0' + (0'-m)(0'+b-m)} \leq d < 0'. \tag{4.56}$$

The best lower bound is $q_{0'} f(0')$ or

$$\frac{\nu_1(\nu_2 - d\nu_1)^2}{2\nu_2(\nu_2 - m\nu_1)}. \tag{4.57}$$

Parabola through (0,0), (r,f(r)) and (b,f(b))

Here, the condition $g(b) = f(b)$ needs to be imposed and leads to

$$\frac{bb'^2 + b'(b'-m)\sqrt{b(b-m)}}{bb' + (b'-m)(b'+b-m)} \leq d \leq \frac{b0'^2 + 0'(0'-m)\sqrt{b(b-m)}}{b0' + (0'-m)(0'+b-m)} \tag{4.58}$$

if $b' > d$. Else, the first inequality drops. The best lower bound equals $q_r f(r) + q_b f(b)$ or

$$\frac{1}{2(b-r)}\left[\frac{(b\nu_1-\nu_2)(r-d)^2}{r(r-m)}+\frac{(\nu_2-\nu_1 r)(b-d)^2}{b(b-m)}\right] \tag{4.59}$$

with

$$r=\frac{d^2(b-m)+d(d-m)\sqrt{b(b-m)}}{b(b-m)+(b-d)^2}. \tag{4.60}$$

Parabola through (b',f(b')) and (b,f(b))

This situation is only possible when $b' > d$. Formula 4.10 can be used with $u = b$ en $v = b'$. To assure that the best lower bound is found, we impose $g(0) \leq 0$ which leads to

$$d \leq \frac{bb'^2+b'(b'-m)\sqrt{b(b-m)}}{bb'+(b'-m)(b'+b-m)}. \tag{4.61}$$

Under this condition the best lower bound is $q_{b'}f(b') + q_b f(b)$ or

$$\frac{1}{2(b-b')}\left[\frac{(b-\nu_1)(b'-d)^2}{(b'-m)}+\frac{(\nu_1-b')(b-d)^2}{(b-m)}\right]. \tag{4.62}$$

The results for the best upper bounds on number of units short when only the mean, variance and unique mode of demand are known, are summarized in Table B.4.

OPTIMAL INVENTORY LEVEL

To determine the optimal inventory level given the desired maximum number of stock-out units or the desired maximum stock-out probability, using the upper and lower bounds derived in the previous sections, numerical analysis can be used. In the case that the range, first two moments and mode of the demand distribution are known, it is not possible to calculate analytical formulas for the optimal inventory level.

COMPOUND POISSON

In Table 4.7 and 4.8 the upper and lower bounds on the number of stock-out units are given when $b = \infty$. These formulas can be applied to a compound Poisson distribution with known mean, standard deviation and mode.

Conditions	**Upper bound**
$d \leq \frac{0'^2}{30'-2m}$	$\frac{\nu_1(\nu_2-d\nu_1)^2}{2\nu_2(\nu_2-m\nu_1)}$
$d \geq \frac{0'^2}{30'-2m}$	$\frac{(\nu_2-\nu_1^2)(r'-d)^2}{2(\nu_2-2r'\nu_1+r'^2)(r'-m)}$ where r' root of $r'^3 + Ar'^2 + Br' + C = 0$ with $A = -3d$, $B = 4\nu_1 d + 2md - 2m\nu_1 - \nu_2$ and $C = 2m\nu_2 - 2m\nu_1 d - d\nu_2$

Table 4.7: Upper bounds on number of stock-out units when E(X), $E(X^2)$ and m are known and $b = \infty$

Conditions	**Lower bound**
$b' \leq d$	0
$d < b'$	$\frac{b'm+b'\nu_1-2b'd-m\nu_1+d^2}{2(b'-m)}$

Table 4.8: Lower bounds on number of stock-out units when E(X), $E(X^2)$ and m are known and $b = \infty$

4.4.3 Numerical example

As stated in the introduction, in the numerical example, $\mu_1 = 20$, $\mu_2 = 600$ and $b = 50$. The upper and lower bounds on the number of stock-out units are presented in Tables B.5 and B.6 in Appendix B. Figure 4.5 shows the upper and lower bounds on the number of stock-out units for a given inventory level.

When the mode m is not known, it is possible to calculate the optimal inventory level, given the desired level of maximum number of stock-out units. The results can be found in Tables B.7, B.8 in Appendix B. Figure 4.6 shows the upper and lower bounds on the inventory level for a given number of stock-out units.

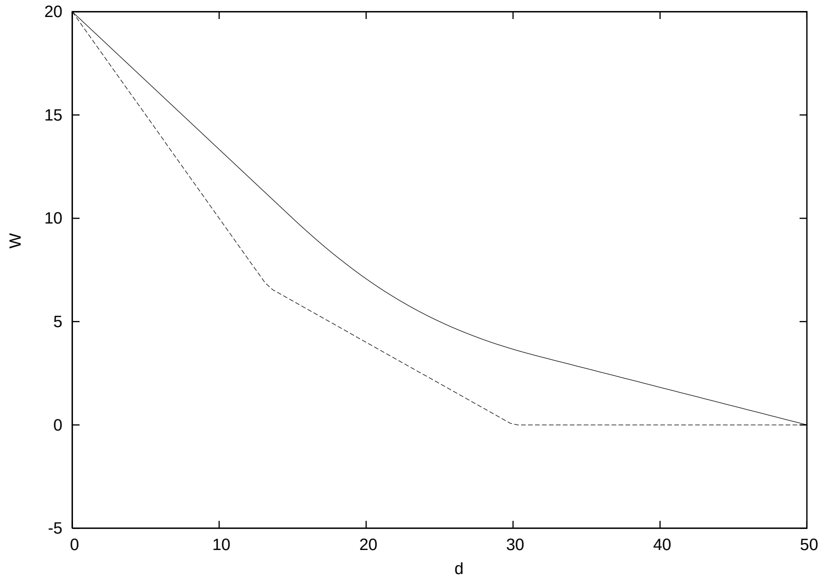

Figure 4.5: Upper and lower bounds on number of stock-out units given the inventory level d

If we assume that the unique mode exists and equals 15, the upper and lower bounds on the number of stock-out units are presented in Tables B.9 and B.10 in Appendix B. To calculate the optimal inventory level analytically, given the desired level of maximum number of stock-out units, is not possible when the unique mode is known but numerical analysis can be used to determine the optimal inventory level stock, given the desired level of maximum number of stock-out units.

In the following paragraphs, these results are used in some specific cases. If, for example, the company wants to face a maximum of 5 stock-out units in a period, the upper bound on the inventory level equals 25 and the lower bound on the inventory level equals 17.5. This means that if the company is very risk averse, an inventory level of 25 units is held, if the company is more risk seeking, an inventory level of 17.5 units is held. If the company only wants to face 2 stock-out units per period, an inventory level of 39 units should be held if the company is risk averse. If the company is more risk seeking, only 25 units of inventory should be held.

If the unique mode of demand is known in the example, tighter bounds can be cal-

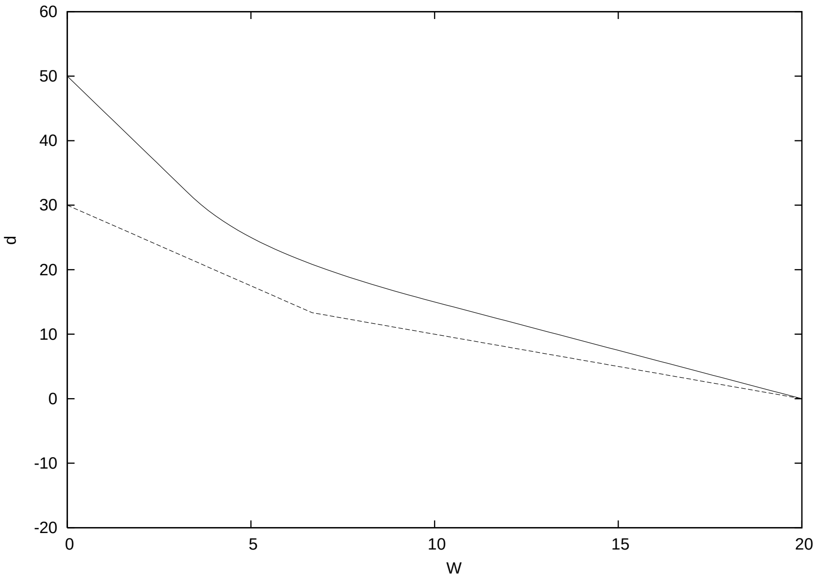

Figure 4.6: Optimal inventory level using upper and lower bounds on the number of stock-out units

culated. However, no closed-form formulas exist for the optimal safety stock. Using numerical analysis, an upper bound on the inventory level of 23 and a lower bound of 22.5 are found when the company wants to face a maximum of 5 stock-out units per period. When only 2 stock-out units per period are allowed, a risk averse company holds an inventory level of 33 units and a risk seeking company holds an inventory level of 32 units.

4.5 Stock-out probability

The bounds on tail probabilities that have been obtained in insurance mathematics can be used to determine upper and lower bounds on the stock-out probability in inventory management, given the inventory level and various levels of information about the demand distribution. These results can be found in Appendix B (De Schepper and Heijnen 1995). However, in inventory management, it is also more interesting to know, given an expected stock-out probability the company wants to face, what the inventory level should be at least or at most.

The outline of this section is as follows: two levels of information about the demand distribution are discussed: the case of a known range, expected value and variance and the case of a known range, expected value, variance and unique mode. For each case, the optimal inventory level is calculated given the desired maximum stock-out probability. Next, the special case of a compound Poisson demand distribution is considered. For each case a numerical example is worked out.

4.5.1 E(X) and $E(X^2)$ are known

OPTIMAL INVENTORY LEVEL

To determine the optimal inventory level given the desired maximum stock-out probability, using the upper and lower bounds derived in Appendix B, numerical analysis can be used.

However, if the mode m is not known, it is possible to calculate the optimal inventory level given the desired maximum number of stock-out units or the desired maximum stock-out probability. Table 4.9 presents the results for the optimal inventory level, given the desired stock-out probability, using the upper bounds on the stock-out probability that were derived in the previous section. Table 4.10 shows the optimal inventory level in terms of the desired stock-out probability, using the lower bound on the stock-out probability.

Conditions	**Inventory level**
$U \leq \frac{\mu_1^2}{\mu_2}$	$\frac{\mu_1 U + \sqrt{(U^2-U)(\mu_1^2-\mu_2)}}{U}$
$U \geq \frac{\mu_1^2}{\mu_2}$	$\frac{b\mu_1-\mu_2}{bU-\mu_1}$

Table 4.9: Optimal inventory level using the upper bounds of the stock-out probability when E(X) and $E(X^2)$ are known

Conditions	Inventory level
$U \geq \frac{\mu_2-\mu_1^2}{b^2-2b\mu_1+\mu_2}$	$\frac{\mu_1(U-1)+\sqrt{\mu_1^2(U-1)^2-(U-1)(\mu_2 U-\mu_1^2)}}{U-1}$
$U \leq \frac{\mu_2-\mu_1^2}{b^2-2b\mu_1+\mu_2}$	$\frac{\mu_2-Ub^2}{\mu_1-Ub}$

Table 4.10: Optimal inventory level using the lower bounds of the stock-out probability when E(X) and $E(X^2)$ are known

COMPOUND POISSON

If the demand process is a compound Poisson process with known mean and standard deviation, in Table 4.11 and 4.12 the upper and lower bounds on the stock-out probability are given when $b = \infty$.

Conditions	Upper bound
$0 \leq d \leq b'$	1
$b' < d \leq 0'$	$\frac{\mu_1}{d}$
$0' < d \leq b$	$\frac{(\mu_2-\mu_1^2)}{(\mu_2-\mu_1^2)+(\mu_1-d)^2}$

Table 4.11: Upper bounds on stock-out probability when E(X) and $E(X^2)$ are known and $b = \infty$

The formulas for the optimal inventory level can also be calculated when $b = \infty$ and are shown in Table 4.13 and 4.14.

Conditions	**Lower bound**
$0 \leq d \leq b'$	$\frac{(\mu_1-d)^2}{\mu_2-\mu_1^2+(\mu_1-d)^2}$
$b' < d \leq b$	0

Table 4.12: Lower bounds on stock-out probability when E(X) and $E(X^2)$ are known and $b = \infty$

Conditions	**Inventory level**
$U \leq \frac{\mu_1^2}{\mu_2}$	$\frac{\mu_1 U+\sqrt{(U^2-U)(\mu_1^2-\mu_2)}}{U}$
$U \geq \frac{\mu_1^2}{\mu_2}$	$\frac{\mu_1}{b}$

Table 4.13: Optimal inventory level using the upper bounds of the stock-out probability when E(X) and $E(X^2)$ are known and $b = \infty$

Conditions	**Inventory level**
$U \geq 0$	$\frac{\mu_1(U-1)+\sqrt{\mu_1^2(U-1)^2-(U-1)(\mu_2 U-\mu_1^2)}}{U-1}$

Table 4.14: Optimal inventory level using the lower bounds of the stock-out probability when E(X) and $E(X^2)$ are known and $b = \infty$

4.5.2 E(X), $E(X^2)$ and the unique mode m are known

OPTIMAL INVENTORY LEVEL

To determine the optimal inventory level given the desired maximum number of stock-out units or the desired maximum stock-out probability, using the upper and lower bounds derived in the previous sections, numerical analysis can be used.

COMPOUND POISSON

In Table 4.15, 4.16, 4.17 and 4.18 the upper and lower bounds on the stock-out probability are given when $b = \infty$. These formulas can be applied to a compound Poisson distribution with known mean, standard deviation and mode.

Conditions	**Upper bound**
$m < d \leq \frac{b'^2}{2b'-m}$ and $b' > c_1$	$\frac{b'-d}{b'-m}$
$m < d < \frac{0'^2}{20'-m}$ and $b' \leq c_1$ or $\frac{b'^2}{2b'-m} < d < \frac{0'^2}{20'-m}$ and $b' > c_1$	$\frac{\mu_1(r-d)}{r(r-m)}$ with $r = d + \sqrt{d(d-m)}$
$\frac{0'^2}{20'-m} \leq d \leq \frac{20'^2-m0'}{30'-2m}$	$\frac{\nu_1^2(\nu_2-d\nu_1)}{\nu_2(\nu_2-m\nu_1)}$
$\frac{20'^2-m0'}{30'-2m} \leq d$	0

Table 4.15: Upper bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d > m$ and $b = \infty$

Conditions	**Lower bound**
$0 \leq d \leq b'$	$\frac{(\nu_1-d)^2}{(\nu_1-m)(\nu_1-d)+\nu_2-\nu_1^2}$
$b' < d$	0

Table 4.16: Lower bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d > m$ and $b = \infty$

Conditions	**Upper bound**
$0 \leq d \leq b'$	1
$b' < d \leq 0'$	$\frac{\nu_1+m-d}{m}$
$0' < d \leq b$	$\frac{1}{d-d'}\left[(\nu_1 - d') + \frac{(d-\nu_1)(m-d)}{m-d'}\right]$

Table 4.17: Upper bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d \leq m$ and $b = \infty$

Conditions	**Lower bound**
$d \leq \frac{0'm}{2m+0'}$	$\frac{\nu_1^2 d+\nu_2(m-d)}{m\nu_2}$
$\frac{0'm}{2m+0'} \leq d$	$\frac{m-d}{m-r} \cdot \frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1 r+r^2} + \frac{(\nu_1-r)^2}{\nu_2-2\nu_1 r+r^2}$ with r root of $r^3 + Ar^2 + Br + C$ with $A = -\frac{1}{2}(2\nu_1 + m + 3d)$, $B = 2d\nu_1 + dm$ and $C = \frac{1}{2}(\nu_2 m - \nu_2 d - 2\nu_1 dm)$.

Table 4.18: Lower bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d \leq m$ and $b = \infty$

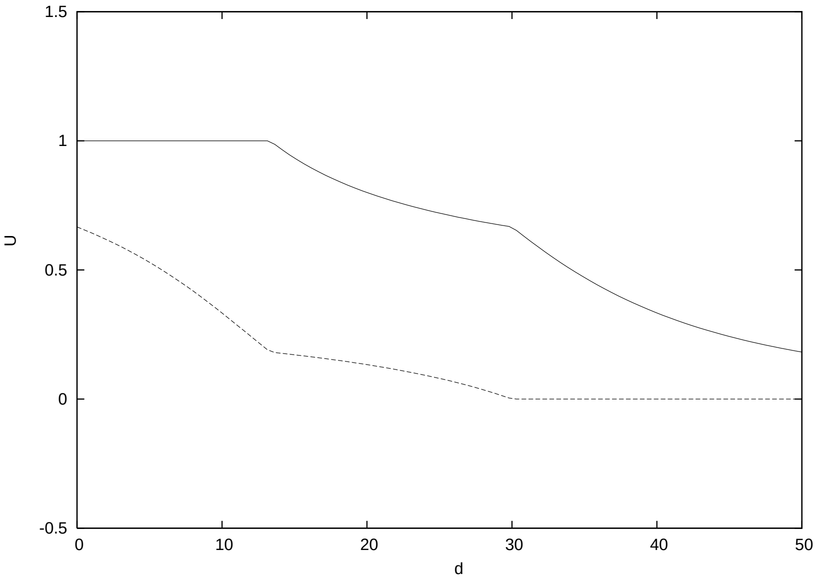

Figure 4.7: Upper and lower bounds on stock-out probability given the inventory level d

4.5.3 Numerical example

In the numerical example, $\mu_1 = 20$, $\mu_2 = 600$ and $b = 50$. The upper and lower bounds on the stock-out probability are presented in Tables B.17 and B.18 in Appendix B. Figure 4.7 shows the upper and lower bounds on the stock-out probability for a given inventory level.

When the mode m is not known, it is possible to calculate the optimal inventory level, given the desired level of maximum number of stock-out units or the desired maximum stock-out probability. The results can be found in Tables B.19 and B.20 in Appendix B. Figure 4.8 shows the upper and lower bounds on the optimal inventory level given a given stock-out probability.

If we assume that the unique mode exists and equals 15, the upper and lower bounds on the stock-out probability are presented in Tables B.21 and B.22 in Appendix B. To calculate the optimal inventory level analytically, given the desired level of maximum number of stock-out units, is not possible when the unique mode is known but numerical analysis can be used to determine the optimal inventory level

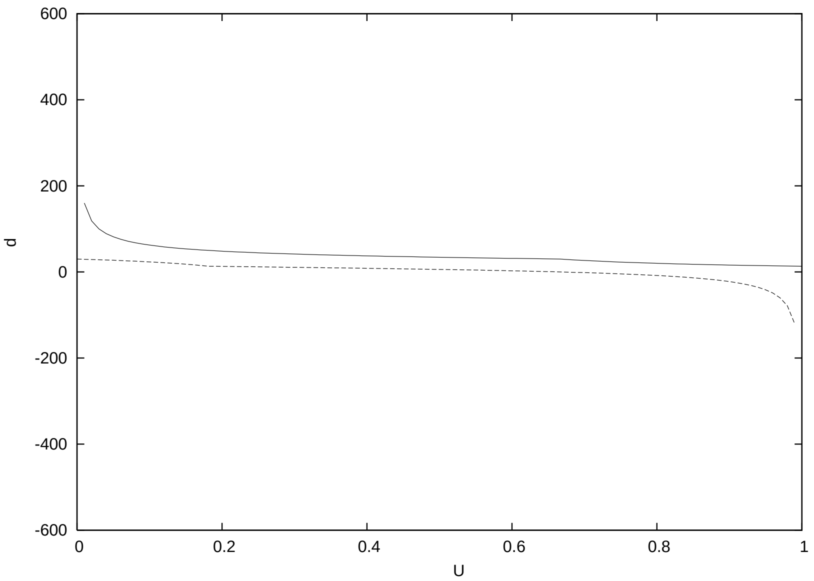

Figure 4.8: Optimal inventory level using the upper and lower bounds on the stock-out probability

stock, given the desired stock-out probability.

In the following paragraphs, these results are used in some specific cases. For example, if the company wants to face a maximum stock-out probability of 10% in a period, the upper bound on the inventory level equals 62 and the lower bound on the inventory level stock equals 23. This means that if the company is very risk averse, an inventory level 62 units is held, if the company is more risk seeking, an inventory level of 23 units is held. If the company only wants to have 5% stock-out probability per period, an inventory level of 96 units should be held if the company is risk averse. If the company is more risk seeking, only 27 units of inventory level should be held.

If the unique mode of demand is known in the example, tighter bounds can be calculated. However, no closed-form formulas exist for the optimal safety stock. Using numerical analysis, an upper bound on the inventory level of 43 and a lower bound of 41 are found when the company wants to have a stock-out probability of 10% per period. When only 5% stock-out probability per period is allowed, a risk averse company holds an inventory level of 47 units and a risk seeking company holds an inventory level of 44 units.

4.6 Concluding remarks

When little information is available on the demand distribution during lead time, which is relevant in inventory decision making, interesting results can be used from the actuarial problem where limited information is known on the claim size distribution. In this chapter, upper and lower bounds are determined for the number of units short and the stock-out probability, given various levels of information about the demand distribution.

In inventory decision making, the opposite question, what should be the inventory level at least or at most given an expected number of units short or an expected stock-out probability the company wants to face, is more interesting. Therefore, results for the optimal inventory level given the desired maximum number of stock-out units or the desired maximum stock-out probability are calculated for the case where $E(X)$ and $E(X^2)$ are known. If the mode m is known, numerical analysis has to be used to determine the optimal inventory level given the desired maximum number of stock-out units or the desired maximum stock-out probability.

Chapter 5

Forecasting and inventory management for intermittent demand

5.1 Introduction

In the previous chapters, the demand process is described under the condition of limited information. A special type of demand, where information on the demand process is often limited, is intermittent demand. This special type of demand is treated in Part II of this thesis (Figure 5.1). Intermittent demand is the type of demand that does not occur in every period and, if it appears, it shows high variability. Items with intermittent demand include service spare parts and high-priced capital goods, such as heavy machinery.

Demand forecasting is one of the most crucial aspects of inventory management (Willemain, Smart, and Schwarz 2004). However, for intermittent demand, forecasting is difficult, and errors in prediction may be costly in terms of obsolescent stock or unmet demand (Syntetos and Boylan 2005). The standard forecasting method for intermittent demand items is considered to be Croston's method (Croston 1972). This method builds estimates taking into account both demand size and the interval between demand occurrences.

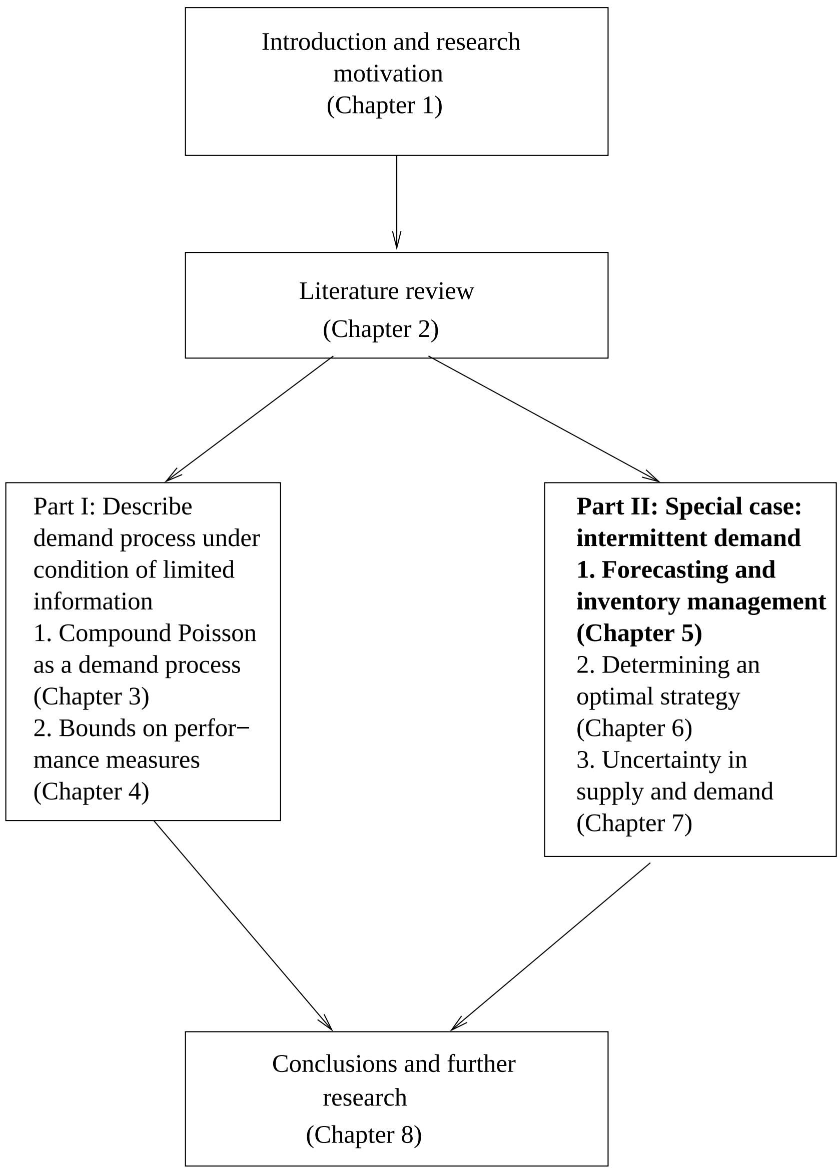

Figure 5.1: Outline of the thesis - Chapter 5

Despite the theoretical superiority of such an estimation procedure, empirical evidence suggests modest gains in performance when compared with simpler forecasting techniques (Syntetos and Boylan 2001). In this chapter[1], we study a single-product inventory system of a wholesaler facing demand of the intermittent type. Different forecasting methods are used in order to compare the performance of those methods. In addition, the results are analysed to see if there is an interaction between inventory decision making and demand forecasting.

Most of the research on inventory control is focused on the replenishment of single items. However, in many situations, considerable savings may be achieved by the coordination of replenishment of orders for groups of items. Therefore, at the end of this chapter, the simulation model is expanded to include joint replenishment to see if joint replenishment can be beneficial when dealing with intermittent demand.

The organization of the chapter is as follows: section 5.2 introduces the research approach; section 5.3 describes the inventory system and the forecasting methods; section 5.4 presents the experimental set-up; section 5.5 discusses the results; section 5.6 adds joint replenishment to the model and in section 5.7 conclusions are drawn.

5.2 Simulation approach

Because of the uncertainty present in the inventory system, often mathematical models cannot accurately describe the system. Therefore, simulation models will be used. The main advantage of simulation is that most complex, real-world systems which cannot be accurately described by a mathematical model can be evaluated analytically (Law and Kelton 1991). However, simulation results can be difficult to interpret. Each simulation run leads to just an estimation of the model's characteristic and, as a result, these estimations can differ greatly from the corresponding true characteristics of the model. Thus, appropriate statistical techniques must be used to analyze and interpret the simulation experiments. Two main issues will be discussed for that matter: output data analysis and common random numbers. At the end of this section, generating intermittent demand will be briefly considered.

[1]This chapter is based on Ramaekers and Janssens (2004).

5.2.1 Output data analysis

As mentioned before, a single run of a simulation model can lead to serious errors and poor decisions. In order to obtain a point estimate and confidence interval for a simulation output, several runs of the simulation model using different random numbers are needed. Several methods exist to estimate the outputs confidence intervals. The most popular method is the replication/deletion approach because it is a simple approach that gives good statistical performance (Law and Kelton 1991). It can easily be used to estimate several different parameters for the same simulation model and to compare different system configurations. Other methods that estimate confidence intervals for simulation output are the batch means method, the autoregressive method the spectral method, the regenerative method and the standardized time series method. In this chapter, we use the replication/deletion method to analyse simulation output.

Suppose we make n replications of the simulation. The independence of replications is accomplished by using different random numbers for each replication. Let Y_j be the jth replication (for j=1,2,...,n) of the measure of performance of interest. An unbiased point estimator for the mean $\mu = E(Y)$ is given by

$$\overline{Y}(n) = \frac{\sum_{j=1}^{n} Y_j}{n}. \tag{5.1}$$

and an approximate $100(1-\alpha)$ percent confidence interval for μ is given by

$$\overline{Y}(n) \pm t_{n-1,1-\alpha/2}\sqrt{\frac{S^2(n)}{n}} \tag{5.2}$$

where $t_{n-1,1-\alpha/2}$ is the upper $1-\frac{\alpha}{2}$ critical point for the t distribution with n-1 degrees of freedom and $S^2(n)$ is the sample variance given by

$$S^2(n) = \frac{\sum_{j=1}^{n} (Y_j - \overline{Y}(n))^2}{n-1}. \tag{5.3}$$

Comparing the expected responses of two alternatives

When comparing alternative systems, Law and Kelton (1991) point out that decisions based on the output of a single simulation run can be unreliable. When comparing only two alternative systems, a confidence interval can be constructed for the difference in the two expectations. This does not only results in a "reject" or "fail-to-reject" conclusion but also quantifies how much the measures differ. When the number of replications for each alternative is the same ($n_1 = n_2 = n$), a paired-t confidence interval can be built. Let Y_{j1} and Y_{j2} be the corresponding outputs of two alternatives and define $Z_j = Y_{j1} - Y_{j2}$, for j=1,2,...,n. Then $E(Z_j) = \nu$ is the quantity for which we want to construct a confidence interval. Assuming that Z_j's are IID random variables, let

$$\overline{Z}(n) = \frac{\sum_{j=1}^{n} Z_j}{n} \tag{5.4}$$

and

$$S^2(n) = \frac{\sum_{j=1}^{n} (Z_j - \overline{Z}(n))^2}{n-1}, \tag{5.5}$$

the approximate $100(1-\alpha)$ percent confidence interval is

$$\overline{Z}(n) \pm t_{n-1,1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}. \tag{5.6}$$

This is called the paired-t confidence interval. If this interval does not contain zero, it can be concluded that the two responses are different.

Comparing the expected responses of more than two alternatives

To compare more than two alternative systems, it is still possible to use a confidence-interval approach. One approach is to do all pairwise comparisons of responses (Law and Kelton 1991). In this case, the individual confidence levels have to be adjusted upward so that the overall confidence level of all intervals covering their respective target is at the desired level $(1-\alpha)$. The Bonferroni inequality is used to ensure that the overall confidence level is at least $(1-\alpha)$. The all pairwise comparisons for k responses requires $\frac{k(k-1)}{2}$ evaluations. The individual confidence level for each interval should therefore be $1-\alpha\frac{k(k-1)}{2}$ in order to have a confidence level of at least $(1-\alpha)$ for all the intervals together. Stoline (1981) compares several methods for all pairwise comparisons and shows that the Tukey test is one of the best methods to perform such comparisons. For the balanced cases, the $100(1-\alpha)$ percent simultaneous Tukey confidence intervals for k pairwise comparisons are

$$\overline{Y}_i - \overline{Y}_j \pm (q_{\alpha,k,\nu})\sqrt{\frac{S^2}{n}} \tag{5.7}$$

where $q_{\alpha,k,\nu}$ is the upper α point of the Studentized range distribution with parameter k and $\nu = k(n-1)$ and

$$S^2 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n} (Y_{ij} - \overline{Y}_i)^2}{\nu}. \tag{5.8}$$

5.2.2 Common random numbers

Variance-reduction techniques can improve the statistical efficiency, measured by the variances of the output random variables, of a simulation study. If it is possible to reduce the variance of an output random variable of interest without disturbing its expectation, a greater precision, e.g. smaller confidence intervals, for the same amount of simulating can be obtained. A commonly used variance-reduction technique is common random numbers (Heikes, Montgomory, and Rardin 1976; Wright and Ramsay 1979). Unlike the other methods for variance reduction, this method applies when comparing two or more alternative system configurations. Using common random numbers also increases the confidence that observed differences in performance are due to differences in the system configurations rather than to fluctuations in the generated random variates. In this simulation study, we use multiple runs for each alternative using the same random numbers for each corresponding pair of alternatives.

5.2.3 Generating intermittent demand

To generate intermittent demand, demand occurrence and demand size are separately generated. To decide on how to generate the intermittent demand, data of Alcon Couvreur N.V. is used.

Alcon Couvreur N.V. is an establishment in Puurs of Alcon Laboratories, an American company. In the establishment in Belgium, a new assembly unit for Custom-Paks was opened in 1998. Custom-Pak are surgical packs that contain a customized single-use surgical procedure tray and consumable products used by eye surgeons for specific ophthalmic procedures. Each individual Custom-Pak tray is manufactured to the surgeon's unique specifications and contains virtually every item needed for a single surgery ranging from surgical devices, drugs and solutions. Additionally, the materials are packed in the exact sequence requested and used by the surgeon. Demand for Custom-Paks is intermittent because each Custom-Pak is made for a specific eye surgeon or team of eye surgeons. Customers alternately order small and large quantities and the frequency of orders can differ greatly.

Each product item has a unique identification number or core. Core 06406, one of the best sold Custom-Paks of Alcon, is used for the analysis. In Table 5.1, the observed frequencies of the interarrival times between orders (in weeks) are given. These observed frequencies have a mean equal to 2.927. This mean is used to calculate the expected frequencies when independent Poisson arrivals are assumed. The observed frequencies are compared to the expected frequencies in Table 5.1.

	1	**2**	**3**	**4**	**5**	**6**	**>7**
Observed	4	15	12	2	6	2	0
Expected	9.95	7.07	5.03	3.57	2.54	1.8	11.03

Table 5.1: Comparison of observed and expected frequencies of the interarrival times

To compare the observed results to the expected results, the χ^2-test is used as a goodness-of-fit test. The χ^2 statistic equals

$$\sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed frequency for class i, E_i is the expected frequency for class i and k is the number of classes. The test statistic equals 38.59, which is above the critical value of the χ^2-distribution with 5 degrees of freedom on the 95% confidence level, 11.07. Therefore, the assumption of Poisson arrivals is rejected and dependency between arrivals can be assumed. In Table 5.2 the interarrival times are grouped based on the present (n) and the previous ($n-1$) interarrival time. Between brackets, the expected frequencies are given when the assumption is made that interarrival times are independent, i.e. the present interarrival times do not depend on the previous interarrival times.

				n			
		1	**2**	**3**	**4**	**5**	**6**
	1	0 (2/30)	1 (2.5)	1 (13/6)	1 (1/3)	0 (1)	1 (1/3)
	2	2 (2/30)	4 (2.5)	5 (13/6)	0 (1/3)	3 (1)	1 (1/3)
n-1	**3**	2 (2/30)	5 (2.5)	2 (13/6)	1 (1/3)	2 (1)	0 (1/3)
	4	0 (2/30)	1 (2.5)	1 (13/6)	0 (1/3)	0 (1)	0 (1/3)
	5	0 (2/30)	3 (2.5)	2 (13/6)	0 (1/3)	1 (1)	0 (1/3)
	6	0 (2/30)	1 (2.5)	1 (13/6)	0 (1/3)	0 (1)	0 (1/3)

Table 5.2: Observed and expected frequencies of the interarrival times

In the matrix of the observed frequencies, there are too many classes with no observations which makes it impossible to perform a χ^2-test. Therefore, Table 5.2 is replaced by Table 5.3, in which classes are put together to be able to perform a χ^2-test.

		n	
		1-3	**4-6**
n-1	**1-3**	22 (15.5)	9 (5)
	4-6	9 (15.5)	1 (5)

Table 5.3: Observed and expected frequencies of the interarrival times in 2 classes

The test statistic to compare the observed frequencies to the expected frequencies equals 11.85. The critical value of the χ^2-distribution with 2 degrees of freedom on the 95% confidence level is 5.99. This means we can reject the null hypothesis that demand arrivals are independent.

Because of the dependency of the arrivals, the demand occurrence is generated according to a first-order Markov process with transition matrix

$$\mathbf{T} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix},$$

where p_{00} is the probability of no order in the next period when there has been no order in this period and p_{10} is the probability of no order in the next period when there has been an order in the current period.

Individual order sizes are generated using a Gamma distribution with shape parameter γ and scale parameter β.

If we define I as the probability of having demand in a certain period and D as the size of an individual demand, the mean and variance of the aggregated demand can be calculated as:

$$E(X) = E(I) * E(D) \tag{5.9}$$

and

$$Var(X) = E(I) * Var(D) + E^2(D) * Var(I). \tag{5.10}$$

5.3 Experimental framework

This section describes the inventory system that is used to compare the performance of different forecasting methods for intermittent demand and to obtain some initial results on the interaction between inventory decision making and demand forecasting. The study focuses on a single-product inventory system of a wholesaler facing demand of the intermittent type. The simulation model is developed in Microsoft Excel spreadsheets and uses VBA. The simulation model starts by generating intermittent demand as described in the previous section. Next, the inventory system is simulated for 52 periods. At each review-time, a demand forecast and an order decision are made. The total costs and the performance (the number of stock-out periods and the number of stock-out units) of the inventory system are determined.

5.3.1 Inventory system

An inventory system provides the organizational structure and the operating policies for maintaining and controlling goods to be stocked. Silver and Peterson (1985) describe the most common classification of inventory systems. There are two general types of inventory systems: continuous review models and periodic review models. In continuous review models, the stock status is always known whereas in periodic review models, the stock status is determined only every R time units. There are a number of possible inventory control systems. The four most common ones are described below.

The order-point, order quantity (s, Q) system involves continuous review. A fixed quantity Q is ordered whenever the inventory position drops to the reorder point s or lower. The order-point, order-up-to-level (s, S) system again involves continuous review an a replenishment is made whenever the inventory position drops to the order point s or lower. The periodic review, order-up-to-level (R, S) system is a commonly used system in practice. The control procedure is that every R units of time enough is ordered to raise the inventory position to the level S. The (R, s, S) system is a combination of a (s, S) and a (R, S) system. Every R units of time, the inventory position is checked. If it is at or below the reorder point s, enough is ordered to raise it to S.

In this research, two periodic review models are used. The first one is the (R, s, S) system just described. The second system is similar to the (R, s, S) system but uses a fixed order quantity Q instead of an order-up-to-level S.

A deterministic lead time L is assumed. Three possible review periods are considered: review period equal to lead time (R=L), review period equal to twice the lead time (R=2L) and review period equal to half the lead time (L=2R).

When making decisions about the size of the safety stock, the order-point, the order quantity and the order-up-to-level, the following costs must be considered: unit holding cost per period C_h, ordering cost C_o and unit shortage cost per period C_s. The simulation starts with an initial inventory level I_0.

For periodic review systems, reorders are placed at the time of review T and the safety stock SS that must be reordered is

$$SS = z\sigma_{R+L} \tag{5.11}$$

where σ_{R+L} is the standard deviation of demand over the review period and the lead time. The value z can be obtained by solving the following equation for E(z) and using a table provided by Robert Brown ((Chase, Aquilano, and Jacobs 1998), to determine the corresponding z value:

$$E(z) = \frac{\overline{X}R(1-P)}{\sigma_{R+L}} \tag{5.12}$$

where $\overline{X}$ is the average demand per period and P is the service level desired.

The order point s is set equal to the safety stock plus the average demand during one time period:

$$s = SS + \overline{X}. \tag{5.13}$$

The fixed order quantity Q is determined using the formula of the Economic Order Quantity EOQ:

$$Q = \sqrt{\frac{2\overline{X}C_o}{C_h}}. \tag{5.14}$$

The order-up-to-level S is the sum of the safety stock and the average demand over the vulnerable period:

$$S = SS + \overline{X}(R + L). \tag{5.15}$$

To compare the inventory systems and forecasting methods, the total costs and the performance (number of stock-out periods and number of stock-out units) of the inventory system are determined.

5.3.2 Forecasting methods

The standard forecasting method for intermittent demand items is considered to be Croston's method. However, in practice, single exponential smoothing and simple moving averages are often used to deal with intermittent demand. In this section, these three forecasting methods are described. Let X_t be the observed demand in period t ($t = 1, 2, ...T$). This integer demand is often zero, and when it is nonzero it tends to be highly variable.

Single Exponential Smoothing

Exponential smoothing (ES) is probably the most used of all forecasting techniques. The single exponential smoothing (SES) method is easy to apply because only three pieces of data are needed to forecast the future: the most recent forecast, the most recent actual demand and a smoothing constant α (DeLurgio 1998). The smoothing constant determines the weight given to the most recent past observations and therefore controls the rate of smoothing or averaging. It is commonly constrained to be in the range of zero to one. The equation for SES is:

$$F_t = \alpha X_{t-1} + (1 - \alpha)F_{t-1} \tag{5.16}$$

where F_t is the exponentially smoothed forecast for period t and F_{t-1} the exponentially smoothed forecast of the prior period.

Simple Moving Averages

The assumption of the moving average (MA) forecasting method is that a future value will equal an average of past values (DeLurgio 1998). The number of past values used to calculate the forecast can vary. The simple four-period moving average forecast is calculated as:

$$F_t = (X_{t-4} + X_{t-3} + X_{t-2} + X_{t-1})/4. \tag{5.17}$$

Croston's method

Croston's method (Croston 1972) was developed to provide a more accurate forecast of the mean demand per period. Croston's method (CR) applies exponential smoothing separately to the intervals between nonzero demands and their sizes. Let I_t be the smoothed estimate of the mean interval between nonzero demands, and let D_t be the smoothed estimate of the mean size of a nonzero demand. Let q be the time interval since the last nonzero demand. Croston's method works as follows: if $X_t = 0$ then

$$D_t = D_{t-1}; I_t = I_{t-1}; q = q + 1 \tag{5.18}$$

else

$$D_t = \alpha X_t + (1-\alpha)D_{t-1}; I_t = \alpha q + (1-\alpha)I_{t-1}; q = 1. \tag{5.19}$$

where α is the smoothing parameter. Combining the estimates of size and interval provides the forecast:

$$F_t = D_t / I_t. \tag{5.20}$$

These estimates are only updated when demand occurs. When demand occurs every period, Croston's method is identical to single exponential smoothing.

5.4 Experimental set-up

In this section, the experimental set-up of the simulation model is discussed.

Demand occurrence is generated using a first-order Markov process with transition matrix

$$\mathbf{T} = \begin{pmatrix} 0.7875 & 0.2125 \\ 0.85 & 0.15 \end{pmatrix}.$$

The transition matrix of a Markov process can be used to calculate the steady-state probabilities of the Markov process using the following steady-state equations:

$$p_0 = p_0 p_{00} + p_1 p_{10} \tag{5.21}$$

and

$$p_1 = p_0 p_{01} + p_1 p_{11} \tag{5.22}$$

Using these formulas, transition matrix T corresponds with a probability of 20% to have demand in a certain period. The size of demand is generated using a Gamma distribution with $\gamma = 6$ and $\beta = 1$. This corresponds to a mean and variance of the aggregated demand of 1.2 and 6.96.

Two inventory management policies are used: the (R, s, S) system and the (R, s, Q) system. The costs of the inventory system are: $C_o = 100$, $C_h = 2$ and $C_s = 5$. The initial inventory level I_0 equals 5. Three possibilities for the review period R exist: R=L; L=2R and R=2L. For each of these possibilities, the safety stock SS, fixed order quantity Q and order-up-to-level S are calculated and shown in Table 5.4. A desired service level of 90% is used.

	R=L	**L=2R**	**R=2L**
Safety stock SS	6	8	6
Fixed order quantity Q	11	11	11
Order-up-to-level S	9	12	10

Table 5.4: Parameters of the inventory system

Three forecasting methods are compared: single exponential smoothing with smoothing parameter $\alpha = 0.5$, simple moving averages using 4 equally weighted past values to calculate the forecast and Croston's method with smoothing parameter $\alpha = 0.5$.

The simulation run length is set to 52 periods and 10 replications are made for each simulation run.

5.5 Results

For the experimental set-up described in the previous section, the results are summarized in Table 5.5, 5.6 and 5.7. These tables show the average value and variance of the ten replications for each experimental point. 95% confidence intervals are calculated to determine the impact of the forecasting method, the impact of the review period and the impact of the inventory management policy. The paired-t test is used

			Mean	Variance
L=R	**Order-up-to**	total costs	1899	94769
	Level S	stock-out periods	1.4	0.27
		stock-out units	3.3	5.12
	Fixed order	total costs	2347	78370
	Quantity Q	stock-out periods	0.4	0.27
		stock-out units	0.7	1.12
L=2R	**Order-up-to**	total costs	2842	111053
	Level S	stock-out periods	0.8	0.84
		stock-out units	3.3	20.01
	Fixed order	total costs	2675	43835
	Quantity Q	stock-out periods	0.4	0.71
		stock-out units	1.6	14.93
R=2L	**Order-up-to**	total costs	1871	68095
	Level S	stock-out periods	1.4	0.71
		stock-out units	3.9	26.77
	Fixed order	total costs	2390	73388
	Quantity Q	stock-out periods	0.5	0.5
		stock-out units	1.2	3.96

Table 5.5: Results of the 10 simulation runs for exponential smoothing

to compare the two inventory management policies used in the simulation. To compare the three forecasting methods and the three review periods, the Tukey test is performed. The confidence intervals can be found in Table 5.8, 5.9 and 5.10.

For this experimental set-up, there is no significant difference in costs and performance between the forecasting methods when the order-up-to-level inventory management policy is used. When the fixed order quantity policy is used, Croston's method is significantly more expensive than single exponential smoothing and simple moving averages when the review period is half the lead time. When the review period is equal to the lead time or equal to twice the lead time, Croston's method is also more ex-

			Mean	**Variance**
L=R	**Order-up-to**	total costs	1899	94769
	Level S	stock-out periods	1.4	0.27
		stock-out units	3.3	5.12
	Fixed order	total costs	2263	57646
	Quantity Q	stock-out periods	0.4	0.27
		stock-out units	0.7	1.12
L=2R	**Order-up-to**	total costs	2842	111052
	Level S	stock-out periods	0.8	0.84
		stock-out units	3.3	20.01
	Fixed order	total costs	2555	91053
	Quantity Q	stock-out periods	0.4	0.71
		stock-out units	1.6	14.93
R=2L	**Order-up-to**	total costs	1871	68095
	Level S	stock-out periods	1.4	0.71
		stock-out units	3.9	26.77
	Fixed order	total costs	2359	54967
	Quantity Q	stock-out periods	0.4	0.49
		stock-out units	1.1	4.1

Table 5.6: Results of the 10 simulation runs for moving average

			Mean	Variance
L=R	**Order-up-to**	total costs	1899	94769
	Level S	stock-out periods	1.4	0.27
		stock-out units	3.3	5.12
	Fixed order	total costs	2508	112963
	Quantity Q	stock-out periods	0.4	0.27
		stock-out units	0.7	1.12
L=2R	**Order-up-to**	total costs	2834	104907
	Level S	stock-out periods	0.8	0.84
		stock-out units	3.3	20.01
	Fixed order	total costs	3011	124530
	Quantity Q	stock-out periods	0.4	0.71
		stock-out units	1.6	14.93
R=2L	**Order-up-to**	total costs	1871	68094
	Level S	stock-out periods	1.4	0.71
		stock-out units	3.9	26.77
	Fixed order	total costs	2639	196648
	Quantity Q	stock-out periods	0.5	0.5
		stock-out units	1.2	3.96

Table 5.7: Results of the 10 simulation runs for Croston's method

			ES-MA	ES-CR	MA-CR
L=R	**S**	total costs	-341.31;341.31	-341.31;341.31	-341.31;341.31
		stock-out periods	-0.57;0.57	-0.57;0.57	-0.57;0.57
		stock-out units	-2.51;2.51	-2.51;2.51	-2.51;2.51
	Q	total costs	-235.7;403.1	-480.1;158.7	-563.8;75
		stock-out periods	-0.57;0.57	-0.57;0.57	-0.57;0.57
		stock-out units	-1.17;1.17	-1.17;1.17	-1.17;1.17
L=2R	**S**	total costs	-366.04;366.04	-358.84;373.24	-358.84;373.24
		stock-out periods	-1.02;1.02	-1.02;1.02	-1.02;1.02
		stock-out units	-4.96;4.96	-4.96;4.96	-4.96;4.96
	Q	total costs	-205.92;446.12	-661.82;-9.78	-781.92;-129.88
		stock-out periods	-0.93;0.93	-0.93;0.93	-0.93;0.93
		stock-out units	-4.28;4.28	-4.28;4.28	-4.28;4.28
R=2L	**S**	total costs	-289.31;289.31	-289.31;289.31	-289.31;289.31
		stock-out periods	-0.93;0.93	-0.93;0.93	-0.93;0.93
		stock-out units	-5.74;5.74	-5.74;5.74	-5.74;5.74
	Q	total costs	-333.42;396.42	-613.82;116.02	-645.32;84.52
		stock-out periods	-0.68;0.88	-0.78;0.78	-0.88;0.68
		stock-out units	-2.12;2.32	-2.22;2.22	-2.32;2.12

Table 5.8: Confidence intervals for comparing the forecasting methods

			R=L-L=2R	R=L-R=2L	L=2R-R=2L
ES	**S**	total costs	-1277.31;-607.29	-306.81;363.21	635.49;1305.51
		stock-out periods	-0.26;1.46	-0.86;0.86	-1.46;0.26
		stock-out units	-4.61;4.61	-5.21;4.01	-5.21;4.01
	Q	total costs	-611.09;-44.91	-326.29;239.89	1.07;567.89
		stock-out periods	-0.78;0.78	-0.88;0.68	-0.88;0.68
		stock-out units	-3.76;1.96	-3.36;2.36	-2.46;3.26
MA	**S**	total costs	-1277.31;-607.29	-306.81;363.21	635.49;1305.51
		stock-out periods	-0.26;1.46	-0.86;0.86	-1.46;0.26
		stock-out units	-4.61;4.61	-5.21;4.01	-5.21;4.01
	Q	total costs	-580.47;-2.73	-384.27;193.47	-92.67;485.07
		stock-out periods	-0.78;0.78	-0.78;0.78	-0.78;0.78
		stock-out units	-3.77;1.97	-3.27;2.47	-2.37;3.37
CR	**S**	total costs	-1266.33;-603.87	-303.03;359.43	632.07;1294.53
		stock-out periods	-0.26;1.46	-0.86;10.86	-1.46;0.26
		stock-out units	-4.61;4.61	-5.21;4.01	-5.21;4.01
	Q	total costs	-924.86;-81.34	-553.16;290.36	-50.06;793.46
		stock-out periods	-0.78;0.78	-0.88;0.68	-0.88;0.68
		stock-out units	-3.76;1.96	-3.36;2.36	-2.46;3.26

Table 5.9: Confidence intervals for comparing the review periods

			Q-S
ES	**R=L**	total costs	-571.67;-323.93
		stock-out periods	0.52;1.48
		stock-out units	0.94;4.26
	L=2R	total costs	-16.69;349.69
		stock-out periods	-0.1;0.9
		stock-out units	-0.69;4.09
	R=2L	total costs	-656.03;-382.38
		stock-out periods	0.27;1.53
		stock-out units	0.15;5.25
MA	**R=L**	total costs	-517.98;-210.22
		stock-out periods	0.52;1.48
		stock-out units	0.94;4.26
	L=2R	total costs	112.61;460.59
		stock-out periods	-0.1;0.9
		stock-out units	-0.69;4.09
	R=2L	total costs	-594.44;-380.96
		stock-out periods	0.42;1.58
		stock-out units	-0.28;5.32
CR	**R=L**	total costs	-733.42;-483.58
		stock-out periods	0.52;1.48
		stock-out units	0.94;4.26
	L=2R	total costs	-415.94;62.94
		stock-out periods	-0.1;0.9
		stock-out units	-0.68;4.09
	R=2L	total costs	-969.03;-567.17
		stock-out periods	0.27;1.53
		stock-out units	0.15;5.25

Table 5.10: Confidence intervals for comparing the inventory management systems

pensive but the difference is not significant. No significant differences in performance can be found. Since Croston's method is the most accurate of the three forecasting methods, it could be expected that this forecasting method is the most expensive one and has a better performance. However, the results do not show a significantly better performance when Croston's method is used.

The impact of the review period on costs is significant. A review period equal to half the lead time is significantly more expensive than the other two choices of the review period. A possible explanation is the fact that for this choice of the review period orders can be placed more frequently, leading to higher order costs. Difference in costs are more distinct if the order-up-to-level inventory management policy is used. The impact of the review period on performance is not significant.

There are significant differences in costs and performance between the inventory management policies. If the review period is equal to the lead time or equal to twice the lead time, the fixed order quantity policy is significantly cheaper but the order-up-to level policy has a significantly better performance. When the review period is equal to half the lead time, no significant differences can be detected between the two inventory management policies. Only when moving averages is used as forecasting method, the order-up-to-level policy is significantly cheaper.

5.6 Joint replenishment

Joint replenishment is used whenever a number of items are involved in a replenishment and it is possible to share the fixed cost associated with it. In inventory management, this fixed cost is that part of the ordering cost which is independent of the number of items on order. Ordering items jointly may also enable the utilization of the same transportation facility and/or may lead to a group quantity discount.

Both deterministic and stochastic models exist for inventory systems with joint replenishment (Goyal and Satir 1989). An often used stochastic model is the can-order system. Balintfy (1964) was the first to propose the use of this system. In such a system, whenever item i's inventory position drops to its must-order point s_i or lower, it triggers a replenishment action that raises the item's level to its order-up-to-level S_i. At the same time any other item j (within the associated family) with its inventory position at or below its can-order point c_j is included in the replenishment. If item j is included, a quantity is ordered sufficient to raise its level to S_j. This system does not necessarily minimize the total costs of replenishment, inventory carrying and shortage but it does achieve a solution that is close to the best attainable.

When dealing with intermittent demand, there are often periods of zero demand. Therefore, a replenishment order is not frequently placed and it might not be useful to jointly replenish several items. In order to investigate if joint replenishment is beneficial for products with intermittent demand, the simulation model is extended. An inventory system with two products is considered. The situation of joint replenishment is compared to the situation where both products are replenished separately.

For product 1, the same experimental set-up as for the single-item inventory system is used. For product 2, demand occurrence is generated using a first-order Markov process with transition matrix

$$\mathbf{T} = \begin{pmatrix} 0.7875 & 0.2125 \\ 0.85 & 0.15 \end{pmatrix}.$$

This corresponds with a probability of 20% to have demand in a certain period. The size of demand is generated using a Gamma distribution with $\gamma = 12$ and $\beta = 1$. This corresponds to a mean and variance of the aggregated demand of 2.4 and 25.44.

The two inventory management policies and the costs of the inventory system remain the same. The initial inventory level I_0 equals 5 for product 1 and 10 for product 2. Again, three possibilities for the review period R are considered. The desired service level remains 90%. For each of these possibilities, the safety stock SS, fixed order quantity Q and order-up-to-level S are calculated and shown in Table 5.4 for product 1 and in table 5.11 for product 2.

	R=L	**L=2R**	**R=2L**
Safety stock SS	11	13	11
Fixed order quantity Q	16	16	16
Order-up-to-level S	16	21	19

Table 5.11: Parameters of the inventory system for product 2

When using joint replenishment, can order points need to be determined. On the one hand, joint replenishment makes it possible to share the order cost C_o. But on the other hand, more units have to be kept in stock which leads to a higher inventory cost. The can order point is determined as the quantity for which the extra inventory cost equals the order cost that is saved. For product 1, the extra inventory cost is calculated in Table 5.12. The average demand during a period is 1.2 units. So the first 1.2 units that are ordered are on average one period in inventory which costs 2.4, the second 1.2 units that are ordered are on average two periods in inventory which costs 4.8,... For 10.8 units, the extra inventory cost is 108 which is more than the saved order cost of 100. This means the can order point lies between 9.6 and 10.8 units. For 9.6 units, the extra inventory cost equals 86.4. The number of units for which the extra inventory cost equals exactly 100 is:

$$9.6 + \frac{100 - 86.4}{9 * 2} = 10.36,$$

which is rounded to a can order point of 11. If the can order point is set higher than this value, the extra inventory cost is higher than the order cost that is saved by joint replenishment.

Demand	**Number of periods**	**Extra inventory cost**	**Total extra inventory cost**
1.2	1	2.4	2.4
2.4	2	4.8	7.2
3.6	3	7.2	14.4
4.8	4	9.6	24
6	5	12	36
7.2	6	14.4	50.4
8.4	7	16.8	67.2
9.6	8	19.2	86.4
10.8	9	21.6	108

Table 5.12: Calculation of the can order point for product 1

The can order point of product 2 can be calculated using exactly the same procedure. The extra inventory cost is calculated in Table 5.13. The number of units for which the extra inventory cost equals exactly 100 is:

$$12 + \frac{100 - 72}{6 * 2} = 14.33,$$

which is rounded to a can order point of 15.

Demand	**Number of periods**	**Extra inventory cost**	**Total extra inventory cost**
2.4	1	4.8	4.8
4.8	2	9.6	14.4
7.2	3	14.4	28.8
9.6	4	19.2	48
12	5	24	72
14.4	6	28.8	100.8

Table 5.13: Calculation of the can order point for product 2

Again, three forecasting methods are compared: single exponential smoothing with smoothing parameter $\alpha = 0,5$, simple moving averages using 4 equally weighted past values to calculate the forecast and Croston's method with smoothing parameter $\alpha = 0,5$.

The simulation run length is set to 52 periods and 10 replications are made for each simulation run.

The results of the inventory system with joint replenishment are compared to the sum of the results of the two individual inventory systems. These results are summarized in Table 5.14 and 5.15 and 5.16. These tables show the average value and variance of the ten replications for each experimental point. To compare the performance of the inventory systems, only the number of stock-out units is used here since it is impossible to sum the number of stock-out periods. 95% paired-t confidence intervals are calculated to determine the effect of joint replenishment on total costs and performance. The confidence intervals can be found in Table 5.17.

The confidence intervals in Table 5.17 indicate that when using the order-up-to-level inventory management policy, joint replenishment is significantly cheaper than replenishing the two products individually. When the fixed order inventory manage-

			Individual		Joint	
			Mean	**Variance**	**Mean**	**Variance**
L=R	**S**	total costs	4413	79402	4143	58944
		stock-out units	5	18.44	5	18.44
	Q	total costs	5736	108356	5748	87339
		stock-out units	2.2	8.4	1.9	7.43
L=2R	**S**	total costs	6563	146883	6161	95384
		stock-out units	10.5	115.17	9.7	119.34
	Q	total costs	6445	101666	6468	64888
		stock-out units	7.4	107.82	7.4	107.82
R=2L	**S**	total costs	4518	35387	4213	100007
		stock-out units	9.3	20.01	8.2	27.96
	Q	total costs	5877	52491	5931	264039
		stock-out units	3.3	18.23	3.2	18.84

Table 5.14: Results of the 10 simulation runs for individual and joint replenishment using single exponential smoothing

			Individual		Joint	
			Mean	**Variance**	**Mean**	**Variance**
L=R	**S**	total costs	4413	79402	4143	58944
		stock-out units	5	18.44	5	18.44
	Q	total costs	5550	77536	5498	118731
		stock-out units	2.2	8.4	1.9	7.43
L=2R	**S**	total costs	6474	154929	6120	70178
		stock-out units	10.6	114.27	9.8	118.62
	Q	total costs	6149	167852	6037	121606
		stock-out units	9	106.44	8.8	109.51
R=2L	**S**	total costs	4518	35387	4118	5809
		stock-out units	9.3	20.01	9.3	20.01
	Q	total costs	5671	66146	5573	70112
		stock-out units	4	18	4	18

Table 5.15: Results of the 10 simulation runs for individual and joint replenishment using simple moving averages

			Individual		Joint	
			Mean	**Variance**	**Mean**	**Variance**
L=R	**S**	total costs	4413	79402	4143	58944
		stock-out units	5	18.44	5	18.44
	Q	total costs	6579	160682	6346	180769
		stock-out units	1.9	7.43	1.9	7.43
L=2R	**S**	total costs	6620	116997	6245	151311
		stock-out units	10.5	115.17	9.7	119.34
	Q	total costs	7607	163876	7412	98602
		stock-out units	6.8	107.96	6.8	107.96
R=2L	**S**	total costs	4518	35387	4118	5809
		stock-out units	9.3	20.01	9.3	20.01
	Q	total costs	6805	172583	6528	166657
		stock-out units	2.9	19.21	2.9	19.21

Table 5.16: Results of the 10 simulation runs for individual and joint replenishment using Croston's method

			ES	MA	CR
L=R	**S**	total costs	202.14;337.86	202.14;337.86	202.14;337.86
		stock-out units	0;0	0;0	0;0
	Q	total costs	-119.25;95.25	-53.79;157.39	139.12;326.48
		stock-out units	-0.38;0.98	-0.38;0.98	0;0
L=2R	**S**	total costs	200.84;602.36	195.67;511.53	177.63;571.97
		stock-out units	-0.58;2.18	-0.58;2.18	-0.58;2.18
	Q	total costs	-290.05;243.65	-12.05;236.85	100.42;289.98
		stock-out units	0;0	-0.25;0.65	0;0
R=2L	**S**	total costs	15.69;594.31	273.83;526.17	273.83;526.17
		stock-out units	-1.39;3.59	0;0	0;0
	Q	total costs	-401.93;292.92	-81.14;276.74	205.10;348.5
		stock-out units	-0.13;0.33	0;0	0;0

Table 5.17: Confidence intervals for comparing individual and joint replenishment

ment policy is used, joint replenishment is only significantly cheaper when Croston's method is used to make the forecasts. No differences in performance can be found.

Overall, it can be concluded that joint replenishment leads to significantly lower total costs without worsening the performance of the inventory system, except when exponential smoothing is used in combination with a fixed order quantity. The highest cost savings are achieved when the order-up-to-level inventory management policy is used.

5.7 Concluding remarks

In this chapter, the performance of several forecasting methods and inventory systems for intermittent demand is compared. A simulation model is built to obtain some initial results on the interaction between inventory decision making and demand forecasting.

The results of this chapter indicate a rather small impact of the forecasting method. There is a significant difference in costs and performance between the inventory management policies. The impact of the review period is significant on total costs, not on performance.

When joint replenishment is added to the models, total costs decrease without changing the performance of the system. When the fixed order quantity inventory management policy is used, this decrease is only significant when using Croston's forecasting method. Although there is only a demand in some periods and it might seem not so useful to coordinate the replenishment of several products, these results demonstrate that joint replenishment can be beneficial, when dealing with intermittent demand.

Since we can conclude, based on the results of this chapter, that there is an interaction between the forecasting methods and inventory management policies for intermittent demand, it is useful to study this interaction in more detail. Therefore, in the next chapter, the impact of several parameters of the inventory management policies and forecasting method is investigated and the simulation model is optimized to obtain the best strategy in combining inventory decision making and demand forecasting.

Chapter 6

Determining a best strategy in combining forecasting and inventory management for intermittent demand

6.1 Introduction

In the previous chapter, the presence of an interaction between demand forecasting and inventory decision making for intermittent demand is demonstrated using a simulation model to study a single-product inventory system facing demand of the intermittent type. Therefore, in this chapter[1], this interaction is studied in more detail (Figure 6.1). Again, the two inventory systems and three forecasting methods described in the previous chapter are used to study the interaction.

The impact of several parameters of the inventory system and forecasting method is investigated and the simulation model is optimized to obtain the best strategy in combining inventory decision making and demand forecasting. It is important to determine this best strategy since considerable savings can be done when using this strategy as will be demonstrated further on in this chapter.

[1]This chapter is based on Ramaekers and Janssens (2006).

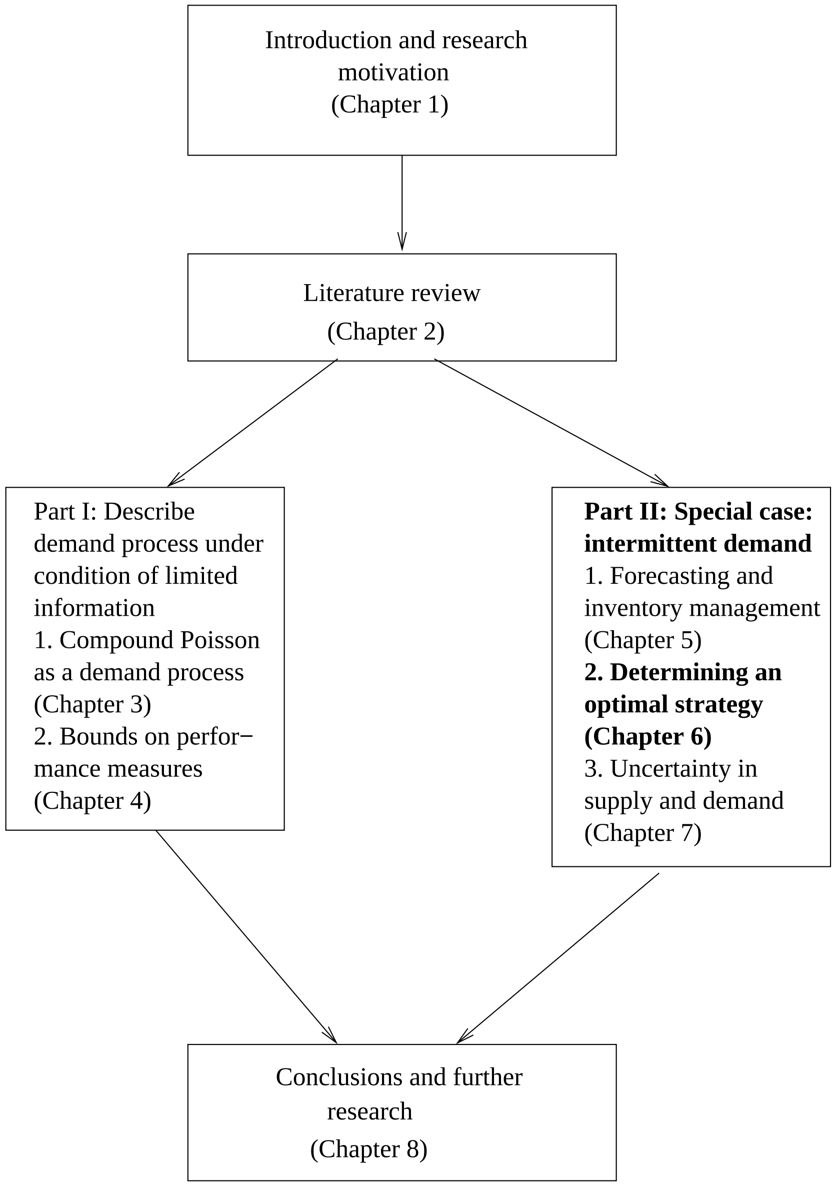

Figure 6.1: Outline of the thesis - Chapter 6

The chapter is organized as follows: section 6.2 presents the experimental design; in section 6.3 a research approach is described to optimise the simulation model in order to obtain the best strategy in combining inventory decision making and demand forecasting; the experimental environment is described in section 6.4; section 6.5 discusses the results and in section 6.6 conclusions are formulated.

6.2 Experimental design

The experimental design includes three qualitative factors: the forecasting method, the inventory management policy and the review period. In addition, depending on the choice of the qualitative factors, a set of quantitative factors are part of the experimental design. If the (R, s, Q) inventory management policy is used, the safety stock SS and order quantity Q are the parameters to optimise. If the (R, s, S) inventory management policy is used, the safety stock SS and order-up-to-level S are the optimising parameters. For single exponential smoothing and Croston's method, the smoothing parameter α is optimised and for moving averages, the weights of the past values are optimised.

In this chapter, we aim to decide on the optimal[2] combination of forecasting method, inventory management policy and review period. Furthermore, the optimal settings for the safety stock, the fixed order quantity or order-up-to-level and the parameter(s) of the forecasting method are determined.

6.3 Research approach

Because of the dependence of the quantitative factors on the choice of the qualitative factors, we use the research approach described in this section.

For every combination of forecasting method, inventory management policy and review period, the optimal values of the quantitative factors are determined. Because of the random outcome of a simulation, there is an additional problem of uncertainty in replication. The combination of simulation-optimisation however offers some answers to this type of problems. Section 6.3.1 gives a general overview of simulation-optimisation methods.

[2]In chapter 6 and chapter 7, by optimal solution, the best solution found by the simulation optimisation method is meant.

In this thesis, three simulation-optimisation methods are compared to determine the optimal values of the quantitative factors: design of experiments, tabu search and response surface methodology. These three methods are chosen because they each belong to a different class of simulation optimisation methods. The methods are described in more detail in sections 6.3.2, 6.3.3 and 6.3.4. The results of the three methods are compared to determine which method works best to optimise the simulation model. Next, the best strategy will be used throughout the remainder of this chapter.

Once the optimal values of the quantitative factors are determined for each combination of forecasting method, inventory management policy and review period, the optimal combination of forecasting method, inventory management policy and review period can be determined.

6.3.1 Simulation optimisation

Simulation and optimisation were often seen as two separate disciplines in the area of operations research. However, in recent years, the combination of simulation and optimisation has developed steadily and has increased in popularity. Simulation optimisation is the practice of linking an optimisation method with a simulation model to determine appropriate settings of certain input parameters so as to maximize the performance of the simulated system (Carson and Maria 1997).

Simulation models have specific features that make the application of classical optimisation methods difficult, or even impossible. Paul and Chanev (1998) list the following features:

- Model behaviour is very complex - a result of the highly non-linear interaction of the model parameters.
- Noisy model output - simulation models are stochastic in nature and their output is not deterministic with respect to the model parameters.
- Inverse problems are often incorrect or ill posed in the sense that small changes in the parameter space can lead to dramatic changes in the behaviour of the model. In this case it is better to attack the problem with a method, which is as robust as possible.

- The parameter space is not continuous - often there is a need for discrete parameters such as integer, logical or linguistic.
- The search space is not compact - there are zones of parameter values that are forbidden or impossible for the model.
- Performance measures can have several extrema. Local extrema may have values close to the global one, or multiple global solutions with the same value may exist.

Bowden and Hall (1998) identify six domains that are common to any simulation optimisation tool. These domains, which they call the cornerstones for a unified strategy for simulation optimisation, are Problem Formulation, Methods, Classification, Strategy and Tactics, Intelligence, and Interfaces.

The *Problem Formulation Domain* addresses the construction of the objective function(s) and constraints to guide the optimiser. This domain considers tools to assist the user in designing appropriate objective functions and constraints.

The *Classification Domain* addresses the analysis and classification of a given optimisation problem. Accurate classification is important for the optimisation tool to select the appropriate optimisation method and strategy. Classification can depend on the types of decision variables, number of decision variables, the variance of the simulation model's output, and number of available runs of the simulation model.

The *Strategy and Tactics Domain* addresses the employment of simulation optimisation in order to make the most efficient use of computing resources and increase the accuracy of the observed optimal solution. Strategic issues may consider the optimisation method or methods selected for a class of problems. Tactical issues consider the use of metamodeling techniques, variance reduction techniques, multiple comparison test, etc. to enhance the efficiency or accuracy of the search.

The *Intelligence Domain* considers the intelligence embedded in the solver to select the strategic approach that will be used for an optimisation study.

The *Interfaces Domain* addresses both the interface between the optimiser and the user and the interface between the optimiser and the simulation model.

The *Methods Domain* addresses those optimisation methods used to optimise simulated systems. Techniques for simulation optimisation vary greatly depending on the exact problem setting.

We present a review of methods that can be used to determine the values of the system parameters that will yield optimal performance of the system. We consider both the case in which the parameters of the system can take a continuous range of values and the case in which the parameter values belong to a discrete set. Four major classes of simulation-optimisation methods can be distinguished: design of experiments, guided search methods, indirect optimisation and statistical methods.

A *first* class of simulation optimisation methods are Design of Experiments techniques. These techniques provide a way to set up the complete experimental design before the experimentation process begins. Design of experiment methods can in general only be applied to discrete variables. Several schemes for setting up experimental designs are known from the literature. Some examples are one factor at a time, full factorial experimental design and the Taguchi method (Ross 1988).

Guided search methods are a *second* class of simulation optimisation methods. In guided search methods, the result of the previous experiment is used to decide on the factor values that will be changed to run the following experiment. The general idea behind this principle is that by using information from previous runs, we will be able to set up the experiments in a more intelligent way so that parts of the search space which are not interesting in terms of optimum seeking are not used for running experiments. Three classes of guided search methods are distinguished. Numerical methods like the Hooke and Jeeves method (Hooke and Jeeves 1961) are based on the idea that if a direction has produced a favourable change in the optimal value, then one should continue to move in this direction (Jacobson and Schruben 1989). Gradient based methods are based on the calculation of gradients in order to move through the search space. Several techniques can be used to estimate the response gradient: finite differences, likelihood ratios, perturbation analysis and frequency domain experimentation (Andradottir 1995; Andradottir 1996). Random search methods include metaheuristics such as tabu search, genetic algorithms and simulated annealing. Although these methods are generally designed for combinatorial optimisation in the deterministic context, they have been quite successful when applied to simulation optimisation (Olafsson and Kim 2002). Paul and Chanev (1998) demonstrate the capability of genetic algorithms to solve problems in the area of complex simulation model optimisation. Haddock and Mittenthal (1992) investigate the feasibility of using a simulated annealing algorithm in conjunction with a simulation model to find the optimal parameter levels at which to operate a system. Dengiz and Alabas (2000) use a tabu search algorithm in conjunction with a simulation model to find the optimal parameter levels.

A *third* class are the statistical methods (Goldsman, Kim, Marshall, and Nelson 2002; Ho, Cassandras, Chen, and Dai 2000; Pichitlamken and Nelson 2001). These methods are mostly used when the optimisation process involves selecting the best of a finite number of alternatives and the parameters are discrete. Some examples of statistical methods are ranking and selection, selection with memory and multiple comparison procedures.

The *fourth* class of simulation optimisation methods, indirect optimisation or response surface methodology (RSM), is useful when input factors are quantitative and continuous. A response surface is a meta-model, i.e. it is a regression model which models the output results of a simulation model (Myers, Khuri, and Carter 1989).

6.3.2 Design of experiments: Taguchi's method

Design of Experiment (DOE) Techniques provide a way to set up the complete experimental design before the experimentation process begins. Van Landeghem and De Backer (1996) demonstrate the use of a Taguchi design of experiments methodology in a simulation based optimisation.

The experimental points are chosen in order to cover the search space as completely as possible. Design of experiment methods can in general only be applied to discrete variables, so the first step before applying a DOE-method consists of choosing a limited number of discrete values in the domain of each continuous variable.

Several schemes for setting up experimental designs are known from literature. The first step is to rank the n relevant values of each decision variable and give them a level number from 1 to n.

Three discrete values are chosen in the domain of each of the three quantitative, continuous factors. These values are shown in Table 6.1, where SS, S and Q are calculated using the following formulas:

$$SS = z\sigma_{R+L} \tag{6.1}$$

where σ_{R+L} is the standard deviation of demand over the review period and the lead time. The value z depends on the desired service level.

The fixed order quantity Q is determined using the formula of the Economic Order Quantity EOQ:

$$Q = \sqrt{\frac{2\overline{X}C_o}{C_h}}. \tag{6.2}$$

The order-up-to-level S is the sum of the safety stock and the average demand over the vulnerable period:

$$S = SS + \overline{X}(R + L). \tag{6.3}$$

Variable	**1**	**2**	**3**
$\boldsymbol{\alpha}$	0.2	0.5	0.8
Safety Stock	SS-2	SS	SS+2
Weights	0.1;0.2;0.3;0.4	0.25;0.25;0.25;0.25	0.4;0.3;0.2;0.1
Order-up-to-level	S-5	S	S+5
Fixed order quantity	Q-5	Q	Q+5

Table 6.1: Discrete values of the variables in the Taguchi method

The next stage is to set up the experiments. This is usually done using specially constructed orthogonal arrays containing a number of rows. Orthogonal arrays are factorial designs, that are usually highly fractional. In every pair of columns of such an array every combination of levels appears the same number of times. This guarantees that the averaged effect of each factor can be determined while the levels of all other factors are varied. If every level-combination for all the factors occurs, then the design is called a full factorial design. If at least one level-combination does not appear, then we have a fractional factorial design (Logothetis and Wynn 1989). Fractional factorial designs provide a way to get good estimates of only the main effects and some interactions at a fraction of the computational effort required by a full factorial design (Law and Kelton 1991). The only drawback of these designs is

Experiment	**Factor 1**	**Factor 2**	**Factor 3**
1	1	1	1
2	1	2	2
3	1	3	3
4	2	1	2
5	2	2	3
6	2	3	1
7	3	1	3
8	3	2	1
9	3	3	2

Table 6.2: Taguchi design: L9 Array

the need to assume that certain interactions can be ignored. However, in many situations, higher interactions are not so important and the fractional designs can be used.

Taguchi has developed a family of fractional factorial design matrices. The selection of which of the orthogonal arrays to use depends on the number of levels for the factors of interest and the number of factors and interactions of interest (Ross 1988).

In this experiment, three factors with each three levels are considered. Six degrees of freedom are needed for the main effects of the factors. An L9 orthogonal array, developed by Taguchi and shown in Table 6.2, can be used because this design has 8 degrees of freedom, which is more than the 6 degrees of freedom that are required. When the two-way interaction effects are also important, another 12 degrees of freedom are needed, which gives a total of 18 degrees of freedom. In this case an L27 orthogonal array, developed by Taguchi and shown in Table 6.3, is needed. Each row defines one experiment to be carried out with the corresponding levels for the variables.

6.3.3 Response surfaces

The goal of response surface methodology (RSM) is to obtain an approximate functional relationship between the continuous input variables and the output objective function. When this is done on the entire domain of interest, the result is often called

Experiment	**Factor 1**	**Factor 2**	**Factor 3**
1	1	1	1
2	1	1	2
3	1	1	3
4	1	2	1
5	1	2	2
6	1	2	3
7	1	3	1
8	1	3	2
9	1	3	3
10	2	1	1
11	2	1	2
12	2	1	3
13	2	2	1
14	2	2	2
15	2	2	3
16	2	3	1
17	2	3	2
18	2	3	3
19	3	1	1
20	3	1	2
21	3	1	3
22	3	2	1
23	3	2	2
24	3	2	3
25	3	3	1
26	3	3	2
27	3	3	3

Table 6.3: Taguchi design: L27 Array

a metamodel. Once a metamodel is obtained, appropriate deterministic optimisation procedures can be applied to obtain an estimate of the optimum (Fu 2002; Myers, Khuri, and Carter 1989; Safizadeh and Thornton 1984). However, in the context of optimisation, RSM usually takes the form of a sequential procedure whereby, through successive experimental stages, one attempts to zoom in on the optimal region where a final polynomial is fitted and the optimum determined through the usual deterministic means. Instead of exploring the entire feasible region, which may be impractical or computationally prohibitive, small subregions are explored in succession, where successive subregions are selected for their potential improvement (Fu 1994).

The basic algorithm for the sequential procedure can be described as follows (Fu 1994):

- In the first phase, first-order experimental designs are used to obtain a linear least-squares fit. Then, a steepest ascent/descent direction is estimated from the model in order to define a new subregion to explore. This process is repeated until the linear fit is deemed inadequate, at which juncture additional points are simulated. Inadequacy is indicated when the slope is approximately zero, by which the interaction effects become larger than the main effects.
- In the second phase, a quadratic response surface is fitted using a more detailed second-order experimental design, and then the optimum is determined analytically from this fit.

RSM has the advantage of having an arsenal of well-known and well-studied statistical tools such as regression analysis and the analysis of variance at its disposal. The method is founded on a statistical theory that is easy to understand and is easy to implement (Jacobson and Schruben 1989).

6.3.4 Tabu search

Tabu search uses a local or neighbourhood search procedure to iteratively move from one solution to the next in the neighbourhood of the first, until some stopping criterion has been satisfied. To explore regions in the search space that would be left unexplored by the local search procedure and escape local optimality, tabu search modifies the neighbourhood structure of each solution as the search progresses. The solutions admitted to the new neighbourhood are determined through the use of special memory structures. Tabu search uses both long-term and short-term memory, and each type of memory has its own special strategies (Glover 1989; Dengiz and Alabas 2000).

One type of short-term memory is the tabu list. This list contains solutions that have been visited in the recent past. Other tabu list structures prohibit solutions that have certain attributes or prevent certain moves. This last type of tabu list contains the moves that are not allowed at the present iteration in order to exclude backtracking moves. Subsequent to each move, the opposite move is appended to the list and the oldest move in the list is removed. To prevent this short-term memory from preventing excellent solutions from being found, aspirations levels are commonly introduced. The tabu status of a solution can be overruled if its solution quality exceeds a certain aspiration level. Long-term memory is used for both diversification and intensification of the search process. Diversification strategies are used to force the search into previously unexplored regions of the solution space. Intensification strategies are used to encourage move combinations that have worked well in the past, or to return the search to attractive regions that have been insufficiently explored (Glover 1989).

Tabu search is a heuristic optimisation technique developed specifically for combinatorial problems. Very few works deal with the application to the global minimization of functions depending on continuous variables. Hu (1992) is the first to adapt tabu search to continuous optimisation. However, the algorithm of Hu is rather far from original tabu search. Siarry and Berthiau (1997) propose an adaptation of tabu search to the optimisation of continuous functions where the purpose is to keep as close as possible to original simple tabu search. As neighbourhood of the current solution, they perform a partition of the space around the current solution using a set of concentric balls. Inside each crown, a random neighbour is selected. The tabu list contains m balls, corresponding to the immediate neighbourhoods of the m last retained solutions. Chelouah and Siarry (2000) improve the algorithm of Siarry and Berthiau (1997) and propose an Enhanced Continuous Tabu Search for the global optimisation of continuous functions. They replace the balls by hyperrectangles for the partition of the current solution neighbourhood and add diversification and intensification concepts to the algorithm. The method we propose here is based on (Siarry and Berthiau 1997) and (Chelouah and Siarry 2000). In Figure 6.2, a general flowchart of the TS algorithm is shown. Two issues must be examined: the generation of current solution neighbours and the elaboration of the tabu list.

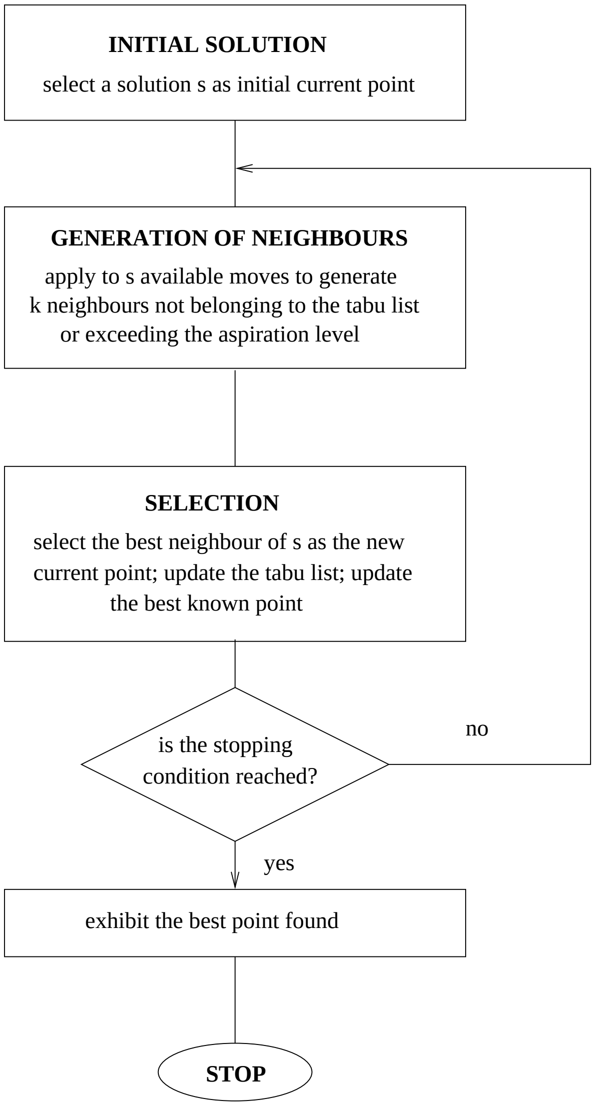

Figure 6.2: General flow chart of tabu search

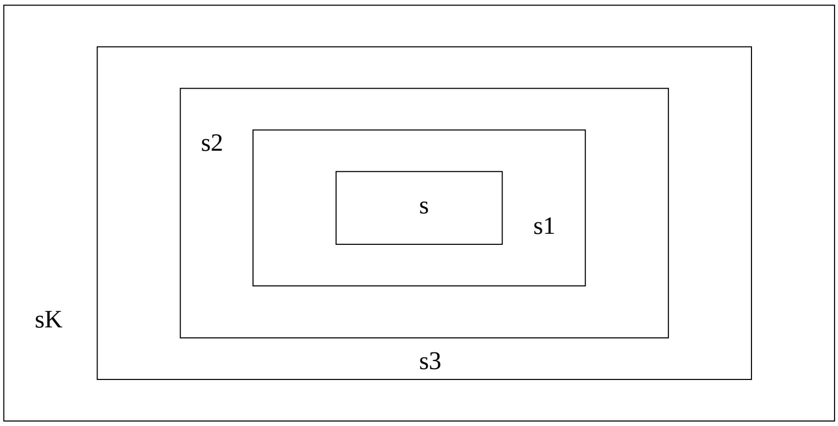

Figure 6.3: Partition of current solution neighbourhood

To define a neighbourhood of the current solution, a set of hyperrectangles is used for the partition of the current solution neighbourhood. The k neighbours of the current solution are obtained by selecting one point at random inside each hyperrectangular zone. In Figure 6.3, a two-dimensional example of such a partition for $k = 4$ neighbours of the current solution is given.

Once a new current solution is determined, the immediate neighbourhood of the previous solution is added to the tabu list. This immediate neighbourhood is also a hyperrectangle. The tabu list containts m hyperrectangles corresponding to the m last retained solutions. A solution belonging to the tabu list can loose its tabu status if its aspiration level is high enough. A solution becomes non-tabu if its objective value is better that the best value obtained at that moment.

As a starting point, the safety stock SS, fixed order quantity Q and order-up-to-level S are calculated using the formulas of the design of experiments (section 6.3.2). A neighbourhood consists of 5 neighbours and the tabu list contains 5 tabu areas. 10 simulation runs are made for each experimental choice. The tabu search is stopped after 200 iterations. The robustness of the results of the tabu search was tested by repeating the algorithm several times. Each replication led to the same optimal values.

6.4 Experimental environment

The experimental environment contains the costs of the inventory system and the parameters for generating intermittent demand. The research approach described above, is executed using a single combination of the costs of the inventory system and demand. However, these factors can have an effect on the results that are obtained. An experimental design is set up for these factors and the optimisation phase is repeated for each experimental point.

Demand occurrence is generated using a first-order Markov process with transition matrices:

$$\mathbf{T_1} = \begin{pmatrix} 0.7875 & 0.2125 \\ 0.85 & 0.15 \end{pmatrix}$$

or

$$\mathbf{T_2} = \begin{pmatrix} 0.5667 & 0.4333 \\ 0.65 & 0.35 \end{pmatrix}.$$

They correspond with a probability of 20% to have demand in a certain period for the first matrix and a probability of 40% to have demand in a period for the second matrix. The *size of demand* is generated using a Gamma distribution with 4 different combinations of the scale parameter γ and the shape parameter β. These values are summarized in Table 6.4. In this table, the mean and variance of the demand size that correspond to each of the parameter combinations are also given.

Combination	γ	$\boldsymbol{\beta}$	**Mean**	**Variance**
1	6	1	6	6
2	12	1	12	12
3	3	2	6	12
4	24	0.5	12	6

Table 6.4: Parameters of the Gamma distribution

The levels of the costs of the inventory system are given in Table 6.5. The initial inventory level I_0 equals 5.

Level	C_o	C_h	C_s
1	100	2	5
2	200	4	10

Table 6.5: Levels for the costs of the inventory system

A fractional factorial design is set up for these six environmental factors. The experimental design is shown in Table 6.6. This design makes it possible to determine the impact of uncontrollable factors as the cost structure and the demand on the optimal strategy in inventory decision making and demand forecasting for intermittent demand. Although only a limited number of experimental points are investigated, the results can be generalised to draw conclusions with respect to an optimal strategy in combining inventory decision making and demand forecasting for intermittent demand since the levels of costs can be seen in relation to each other instead of as absolute values.

6.5 Results

The basic configuration of the factors of the experimental environment is set as follows: demand occurrence is generated using a first-order Markov process with transition matrix

$$\mathbf{T_1} = \begin{pmatrix} 0.7875 & 0.2125 \\ 0.85 & 0.15 \end{pmatrix}.$$

For the demand size, a gamma distribution with scale parameter 6 and shape parameter 1 is used. The ordering cost equals € 100 per order, the unit shortage cost € 5 per period and the unit holding cost € 2 per period.

When Taguchi's method is used as optimisation method, the optimal solution for this experimental environment is an order-up-to-level inventory management system with a review period equal to the lead time. Moving Averages is best used as forecasting method. The weights of the past values do not have a significant impact on the optimal solution. The safety stock SS and the order-up-to-level S are both equal to 4 units in the optimal solution. This means an order is placed every time the inventory

Experiment	C_o	C_h	C_s	**Markov**	γ	β
1	200	4	10	0.4	12	1
2	100	4	5	0.4	12	1
3	200	2	5	0.4	24	0.5
4	100	2	10	0.4	24	0.5
5	200	2	5	0.4	3	2
6	100	2	10	0.4	3	2
7	200	4	10	0.4	6	1
8	100	4	5	0.4	6	1
9	200	2	10	0.2	12	1
10	100	2	5	0.2	12	1
11	200	4	5	0.2	24	0.5
12	100	4	10	0.2	24	0.5
13	200	4	5	0.2	3	2
14	100	4	10	0.2	3	2
15	200	2	10	0.2	6	1
16	100	2	5	0.2	6	1

Table 6.6: Experimental design for uncontrollable factors

level drops below the order-up-to-level S and enough is ordered to raise it again to the order-up-to-level S.

When Response Surface Methodology is used as optimisation method, no optimal solution can be found. The surface of the objective function is not suitable for applying the iterative procedure explained above. Therefore, this method is not used anymore in the remainder of this chapter.

When Tabu Search is used as optimisation method for the quantitative factors, the optimal strategy for the basic configuration of the factors of the experimental environment is an order-up-to-level inventory management policy with a review period equal to the lead time. Exponential smoothing is the forecasting method that leads to the lowest costs. The smoothing parameter α does not have significant impact on the results. The order-up-to-level S is equal to 1, the safety stock SS is negative and the reorder point is 0.

In Table 6.7, the results for Taguchi's method and Tabu Search are compared for the 16 experimental points of the environment. When there is more than one forecasting method shown in the table, it means that both forecasting methods lead to the same result. The results for both methods are quite similar. When using Tabu Search, the safety stock SS is negative, leading to a reorder point of 0. When Taguchi's method is used, the safety stock SS is mostly equal to its lowest value. When the order-up-to-level S is equal to 1 for Tabu Search, the order-up-to-level S or fixed order quantity Q and the safety stock SS are also close together for Taguchi's method, leading to the same inventory management policy i.e. refill the inventory every time a demand occurs. Only for experiments 8 en 9, a completely different result is found for the two optimisation methods. If we examine the results of the two optimisation methods in more detail, the result of the Tabu Search has always lower costs than the Taguchi result. For 11 of the 16 experimental points, this difference in total costs is significant on the 95% confidence level. Confidence intervals for the difference in total costs between the two optimisation methods are shown in Table 6.8. Because Tabu Search leads to better and more accurate results since continuous values are used, in the remainder of the discussion of the results, the results of Tabu Search are used.

Based on these results, it can be concluded that the parameters of the forecasting method have no impact on the results. The impact of the review period is also rather

Experiment	Taguchi	Tabu
1	ES-MA/OUL/R=2; L=1	MA/FOQ/R=L=1
	SS=9; S=21	ROP=0; Q=25
2	ES-MA-CR/OUL/R=L=1	ES/OUL/R=L=1
	SS=13; S=16	ROP=0; S=1
3	ES/OUL/R=2; L=1	MA/OUL/R=L=1
	SS=9; S=21	ROP=0; S=30
4	ES-MA-CR/OUL/R=L=1	MA/OUL/R=L=1
	SS=9; S=16	ROP=0; S=25
5	MA/FOQ/R=L=1	MA/FOQ/R=L=1
	SS=5; Q=22	ROP=0; Q=20
6	CR/OUL/R=2; L=1	MA/OUL/R=L=1
	SS=5; S=10	ROP=0; S=15
7	ES-MA-CR/OUL/R=2; L=1	MA/FOQ/R=L=1
	SS=4; S=9	ROP=0; Q=15
8	ES-MA/OUL/R=2; L=1	ES/OUL/R=2; L=1
	SS=4; S=9	ROP=0; S=1
9	ES-MA-CR/OUL/R=2; L=1	MA/FOQ/R=L=1
	SS=13; S=14	ROP=0; Q=20
10	ES-MA-CR/OUL/R=L=1	ES/OUL/R=L=1
	SS=11; S=11	ROP=0; S=1
11	ES-MA-CR/OUL/R=L=1	CR/OUL/R=L=1
	SS=13; S=11	ROP=0; S=1
12	MA/OUL/R=L=1	ES/OUL/R=L=1
	SS=9; S=11	ROP=0; S=1
13	ES-MA/OUL/R=L=1	MA/OUL/R=L=1
	SS=5; S=5	ROP=0; S=1
14	ES/OUL/R=L=1	MA/OUL/R=L=1
	SS=5; S=5	ROP=0; S=1
15	MA/FOQ/R=L=1	MA/OUL/R=L=1
	SS=4; Q=16	ROP=0; S=15
16	MA/OUL/R=L=1	ES/OUL/R=L=1
	SS=4; S=4	ROP=0; S=1

Table 6.7: Comparison of the results of Taguchi's method and Tabu search

Experiment	Taguchi-Tabu
1	814.99;2091.01
2	382.46;1495.54
3	943.74;1688.26
4	394.77;1009.23
5	621.28;1170.72
6	253.45;1166.55
7	576.90;1717.1
8	83.85;760.15
9	-288.81;1248.81
10	-268.47;580.47
11	190.98;1809.02
12	456.13;1495.87
13	383.42;1702.58
14	-146.34;916.34
15	-612.23;522.23
16	-360.28;398.28

Table 6.8: Confidence intervals for comparing Taguchi's method and Tabu search

small. For 8 experimental points, the best strategy is an order-up-to-level inventory management policy with $S = 1$. For the other 8 experimental points, the best strategy is an order-up-to-level inventory management policy with $S \geq 15$ or a fixed order quantity model with $Q \geq 15$. When the fixed order quantity inventory management policy is best, the best forecasting method is always moving averages. When the order-up-to-level inventory management policy with $S \geq 15$ is best, the best forecasting method is also always moving averages. When the order-up-to-level equal to 1 is the best inventory management policy, no preference for a specific forecasting method can be found. In the next paragraphs, the influence of the uncontrollable factors on the results is examined.

In Table 6.9, results are compared for the two levels of the demand frequency. When the demand frequency is generated using matrix P_1, corresponding to a probability of 20% of having demand in a certain period, an order-up-to-level S of 1 unit is optimal. When the demand frequency is generated using matrix P_2, which corresponds to a probability of 40% of having demand in a certain period, the order-up-to-level S or fixed order quantity Q is a value between 15 and 30. This can be explained because the intermittent character of demand is more distinct when the probability of demand is equal to 20%, leading to an optimal order-up-to-level S of 1 unit. When the intermittent character of demand is less distinct (40%), it is better to order a quantity of at least 15 units. The only exception to this order-up-to-level S of 1 unit for a demand probability of 20% can be found when both the ordering cost and the unit shortage cost are high and the unit holding cost is low. In these circumstances it is better to order a bigger quantity because it is less costly to hold inventory than to have a stock-out or to order a small quantity every time. Inversely, when a demand probability of 40% is used, it is better to use an order-up-to-level S of 1 unit when both the ordering cost and the unit shortage cost are low and the unit holding cost is high. The same reasoning as above can be made here. It can also be noted that when the demand frequency is doubled, Croston's method becomes less useful as forecasting method.

In Table 6.10 and Table 6.11, results are compared for changing the parameters of the demand size. No significant impact of these changes on the results can be detected. This means the only impact of demand on the results is due to the demand frequency, in other words, only the intermittent character of demand has an influence on the strategy that is best chosen, the size of demand has no influence.

Changes in the cost structure of the inventory system have a significant impact on the results. Table 6.12 compares the results of the two possible levels of the ordering cost. When the ordering cost is equal to 100, an order-up-to-level inventory management policy is used with the order-up-to-level S equal to 1, except when the unit holding cost is low, the unit shortage cost is high and the demand probability of a certain period is 40%. The level of these three factors all favour holding more units in inventory. The combination of these three levels therefore changes the best policy to a policy with an order-up-to-level or fixed order quantity between 15 and 30, although the order cost is low. When the ordering cost is equal to 200, the order-up-to-level S or fixed order quantity Q is between 15 and 30, except when the unit holding cost is high, the unit shortage cost is low and the demand probability of a period equals 20%. Here, as an explanation, the opposite reasoning of above can be used.

In Table 6.13, the results for changes in the unit holding cost are compared. When the unit holding cost is equal to 2, an order-up-to-level S or fixed order quantity Q between 15 and 30 is used, unless both the ordering cost and the unit shortage cost are also low and the demand probability of a period equals 20%. When this combination of factor levels occurs, an inventory policy with an order-up-to-level S equal to 1 is better used because all these factor levels give preference to a lower inventory level. When the unit holding cost equals 4, an order-up-to-level S of 1 is the best choice, unless the ordering cost and unit shortage cost are also high and the demand probability of a period is 40%. This combination of factor levels favours a higher inventory level and thus an order-up-to-level or fixed order quantity between 15 and 30 is better used.

Table 6.14 summarizes the results for the two levels of the unit shortage cost. A unit shortage cost of 5 implies an order-up-to-level S of 1 unit, except when the unit holding cost is also low and the probability of demand for a certain period equals 40%. When the shortage cost is low, it is not necessary to keep a lot of units in inventory. Therefore, an order-up-to-level equal to 1 is the best policy. However, if the holding cost is also low and the intermittent character of demand is not so distinct, it is better to have more units in inventory even though the shortage cost is low. Doubling the unit shortage cost leads to an order-up-to-level S or fixed order quantity Q between 15 and 30, except when the unit holding cost is high and the demand frequency is equal to 20%. The same reasoning as before can be used to explain this exception.

Overall, it can be concluded that the uncontrollable factors have an impact on the best strategy for combining inventory decision-making and demand forecasting

Factor 1		**0.4**		**0.2**
	1	MA/FOQ/R=L=1	9	MA/FOQ/R=L=1
		ROP=0; Q=25		ROP=0; Q=20
	2	ES/OUL/R=L=1	10	ES/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1
	3	MA/OUL/R=L=1	11	CR/OUL/R=L=1
		ROP=0; S=30		ROP=0; S=1
	4	MA/OUL/R=L=1	12	ES/OUL/R=L=1
		ROP=0; S=25		ROP=0; S=1
	5	MA/FOQ/R=L=1	13	MA/OUL/R=L=1
		ROP=0; Q=20		ROP=0; S=1
	6	MA/OUL/R=L=1	14	MA/OUL/R=L=1
		ROP=0; S=15		ROP=0; S=1
	7	MA/FOQ/R=L=1	15	MA/OUL/R=L=1
		ROP=0; Q=15		ROP=0; S=15
	8	ES/OUL/R=2; L=1	16	ES/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1

Table 6.9: Comparison of results for the two levels of factor 1, the Markov matrix

for intermittent demand. Furthermore, there is interaction between these factors.

To study this interaction in more detail, a classification tree is constructed using the C4.5 algorithm, a well-known algorithm in data mining ((Quinlan 1993)). The classification three can be found in Figure 6.4. Using this tree, it can be decided which of the two strategies is best: an order-up-to-level inventory management policy with $S = 1$ or an order-up-to-level inventory management policy with $S \geq 15$ or a fixed order quantity model with $Q \geq 15$. Three factors are needed to determine the best strategy in combining inventory decision making and demand forecasting: the frequency of demand, the order cost and the inventory cost. If one of these three factors is not known, the knowledge of the stock-out cost is also sufficient to make a classification. This leads to three other classification trees which can be found in Appendix C. Summarizing, it can be said that if three factors of the four just men-

Factor 2		6		12
	5	MA/FOQ/R=L=1	1	MA/FOQ/R=L=1
		ROP=0; Q=20		ROP=0; Q=25
	6	MA/OUL/R=L=1	2	ES/OUL/R=L=1
		ROP=0; S=15		ROP=0; S=1
	7	MA/FOQ/R=L=1	3	MA/OUL/R=L=1
		ROP=0; Q=15		ROP=0; S=30
	8	ES/OUL/R=2, L=1	4	MA/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=25
	13	MA/OUL/R=L=1	9	MA/FOQ/R=L=1
		ROP=0; S=1		ROP=0; Q=20
	14	MA/OUL/R=L=1	10	ES/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1
	15	MA/OUL/R=L=1	11	CR/OUL/R=L=1
		ROP=0; S=15		ROP=0; S=1
	16	ES/OUL/R=L=1	12	ES/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1

Table 6.10: Comparison of results for the two levels of factor 2, the mean of the Gamma distribution

Factor 3	**6**		**12**	
	3	MA/OUL/R=L=1	1	MA/FOQ/R=L=1
		ROP=0; S=30		ROP=0; Q=25
	4	MA/OUL/R=L=1	2	ES/OUL/R=L=1
		ROP=0; S=25		ROP=0; S=1
	7	MA/FOQ/R=L=1	5	MA/FOQ/R=L=1
		ROP=0; Q=15		ROP=0; Q=20
	8	ES/OUL/R=2; L=1	6	MA/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=15
	11	CR/OUL/R=L=1	9	MA/FOQ/R=L=1
		ROP=0; S=1		ROP=0; Q=20
	12	ES/OUL/R=L=1	10	ES/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1
	15	MA/OUL/R=L=1	13	MA/OUL/R=L=1
		ROP=0; S=15		ROP=0; S=1
	16	ES/OUL/R=L=1	14	MA/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1

Table 6.11: Comparison of results for the two levels of factor 3, the variance of the Gamma distribution

Factor 4		**100**		**200**
	2	ES/OUL/R=L=1	1	MA/FOQ/R=L=1
		ROP=0; S=1		ROP=0; Q=25
	4	MA/OUL/R=L=1	3	MA/OUL/R=L=1
		ROP=0; S=25		ROP=0; S=30
	6	MA/OUL/R=L=1	5	MA/FOQ/R=L=1
		ROP=0; S=15		ROP=0; Q=20
	8	ES/OUL/R=2; L=1	7	MA/FOQ/R=L=1
		ROP=0; S=1		ROP=0; Q=15
	10	ES/OUL/R=L=1	13	MA/FOQ/R=L=1
		ROP=0; S=1		ROP=0; Q=20
	12	ES/OUL/R=L=1	11	CR/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1
	14	MA/OUL/R=L=1	13	MA/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1
	16	ES/OUL/R=L=1	15	MA/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=15

Table 6.12: Comparison of results for the two levels of factor 4, the ordering cost

Factor 5		**2**		**4**
	3	MA/OUL/R=L=1	1	MA/FOQ/R=L=1
		ROP=0; S=30		ROP=0; Q=25
	4	MA/OUL/R=L=1	2	ES/OUL/R=L=1
		ROP=0; S=25		ROP=0; S=1
	5	MA/FOQ/R=L=1	7	MA/FOQ/R=L=1
		ROP=0; Q=20		ROP=0; Q=15
	6	MA/OUL/R=L=1	8	ES/OUL/R=2; L=1
		ROP=0; S=15		ROP=0; S=1
	9	MA/FOQ/R=L=1	11	CR/OUL/R=L=1
		ROP=0; Q=20		ROP=0; S=1
	10	ES/OUL/R=L=1	12	ES/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1
	15	MA/OUL/R=L=1	13	MA/OUL/R=L=1
		ROP=0; S=15		ROP=0; S=1
	16	ES/OUL/R=L=1	14	MA/OUL/R=L=1
		ROP=0; S=1		ROP=0; S=1

Table 6.13: Comparison of results for the two levels of factor 5, the unit holding cost per period

Factor 6	**5**		**10**
2	ES/OUL/R=L=1 ROP=0; S=1	1	MA/FOQ/R=L=1 ROP=0; Q=25
3	MA/OUL/R=L=1 ROP=0; S=30	4	MA/OUL/R=L=1 ROP=0; S=25
5	MA/FOQ/R=L=1 ROP=0; Q=20	6	MA/OUL/R=L=1 ROP=0; S=15
8	ES/OUL/R=2; L=1 ROP=0; S=1	7	MA/FOQ/R=L=1 ROP=0; Q=15
10	ES/OUL/R=L=1 ROP=0; S=1	9	MA/FOQ/R=L=1 ROP=0; Q=20
11	CR/OUL/R=L=1 ROP=0; S=1	12	ES/OUL/R=L=1 ROP=0; S=1
13	MA/OUL/R=L=1 ROP=0; S=1	14	MA/OUL/R=L=1 ROP=0; S=1
16	ES/OUL/R=L=1 ROP=0; S=1	15	MA/OUL/R=L=1 ROP=0; S=15

Table 6.14: Comparison of results for the two levels of factor 6, the unit shortage cost per period

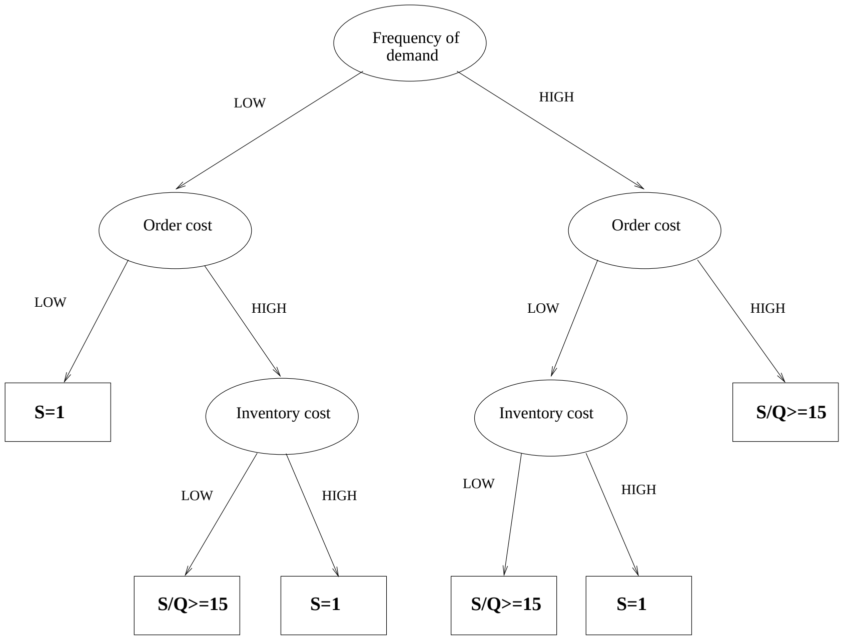

Figure 6.4: Classification tree

tioned (frequency of demand, order cost, inventory cost and stock-out cost) are fixed, the best strategy can be determined.

A good classification is necessary because there is a considerable increase in the costs of the inventory system when using the other strategy. When a fixed order quantity inventory management policy with $Q = 15$ is used instead of an order-up-to-level inventory management policy with $S = 1$, total costs are on average 20% higher. In the opposite case, when an order-up-to-level inventory management policy with $S = 1$ is used instead of an order-up-to-level inventory management policy with $S \geq 15$ or a fixed order quantity model with $Q \geq 15$, total costs increase with more than 40% on average.

6.6 Concluding remarks

In this chapter, the simulation model of the previous chapter is optimised to obtain the best strategy in combining inventory decision making and demand forecasting for intermittent demand. Three different optimisation methods were used to optimise the simulation model: Taguchi's method, Response Surface Methodology and Tabu Search. Only Taguchi's method and Tabu Search were suitable to optimise the quantitative factors.

Based on the results obtained in this chapter, it can be concluded that both optimisation methods lead to roughly similar results but when tabu search is applied, continuous values are used which leads to more accurate results.

The factors of the experimental environment have an impact on the best strategy for combining inventory decision-making and demand forecasting and there is also interaction between these uncontrollable factors.

In general, for intermittent demand, two best policies are found: an order-up-to-level inventory management policy with $S = 1$ or an order-up-to-level inventory management policy with $S \geq 15$ or a fixed order quantity model with $Q \geq 15$. When the fixed order quantity inventory management policy is best, the best forecasting method is always moving averages. When the order-up-to-level inventory management policy with $S \geq 15$ is best, the best forecasting method is also always moving averages. When the order-up-to-level equal to 1 is the best inventory management policy, no preference for a specific forecasting method can be found. The choice between these two policies depends on the uncontrollable factors.

Chapter 7

Forecasting and inventory management for intermittent demand: Uncertainty in supply and demand

7.1 Introduction

In chapter 5 and 6, forecasting and inventory management for intermittent demand is examined using a simulation model. In chapter 5, it is concluded that there is an interaction between the forecasting methods and inventory management systems for intermittent demand. In chapter 6, the simulation model is optimised and a best strategy in combining inventory decision making and demand forecasting for intermittent demand is obtained.

Until now, it is assumed that there is only uncertainty in demand. This means among other things that the lead time is deterministic and that there are no disruptions in the supply. However, uncertainty is also present at the supply side. This type of uncertainty occurs in delivery time, in interruption of delivery during a certain period, or in mismatches in order and delivery in terms of quality or quantity.

In this chapter, the same simulation model is used but the assumption of no uncertainty in supply is relaxed (Figure 7.1). First, the results of the optimal policies found in chapter 6 for a reliable supplier are compared to results when applying the same policies for an unreliable supplier. Next, a new best strategy in combining inventory decision making and demand forecasting is determined for the situation of an unreliable supplier.

Chapter 6 indicates that there is almost no impact of the choice of the review period on the optimal strategy. Furthermore, the parameters of the forecasting method also have no impact on the optimal strategy. Therefore, in the remainder of this chapter, these factors are left out of consideration. This results in an experimental design which includes three forecasting methods, two inventory management policies and two quantitative factors, the safety stock and the order quantity or order-up-to-level. The experimental design remains the same: 6 uncontrollable factors are studied using a fractional factorial design.

The organization of the chapter is as follows: section 7.2 discusses uncertainty in supply in the simulation model, in section 7.3 results for a reliable and an unreliable supplier are compared, in section 7.4 an optimal policy is determined when dealing with uncertainty in supply and section 7.5 formulates some conclusions.

7.2 Uncertainty in supply

As already indicated in the literature review, a lot of literature on uncertainty in supply deals with uncertainty in the lead time. However, next to uncertainty in lead time, supplier reliability also comprises uncertainty in transportation times, in information delays, in quality and in availability of resources.

The focus of this chapter is on uncertainty in availability. The supplier alternates randomly between an available and an unavailable state. When the supplier is available, the order is delivered after the usual lead time. When the supplier is unavailable, the order is executed when the supplier turns available again.

In the simulation model, uncertainty in supply is randomly generated. In every period, there is 20% chance that the supplier is unavailable. If the supplier is unavailable, the order is delivered one lead time after the supplier becomes available again.

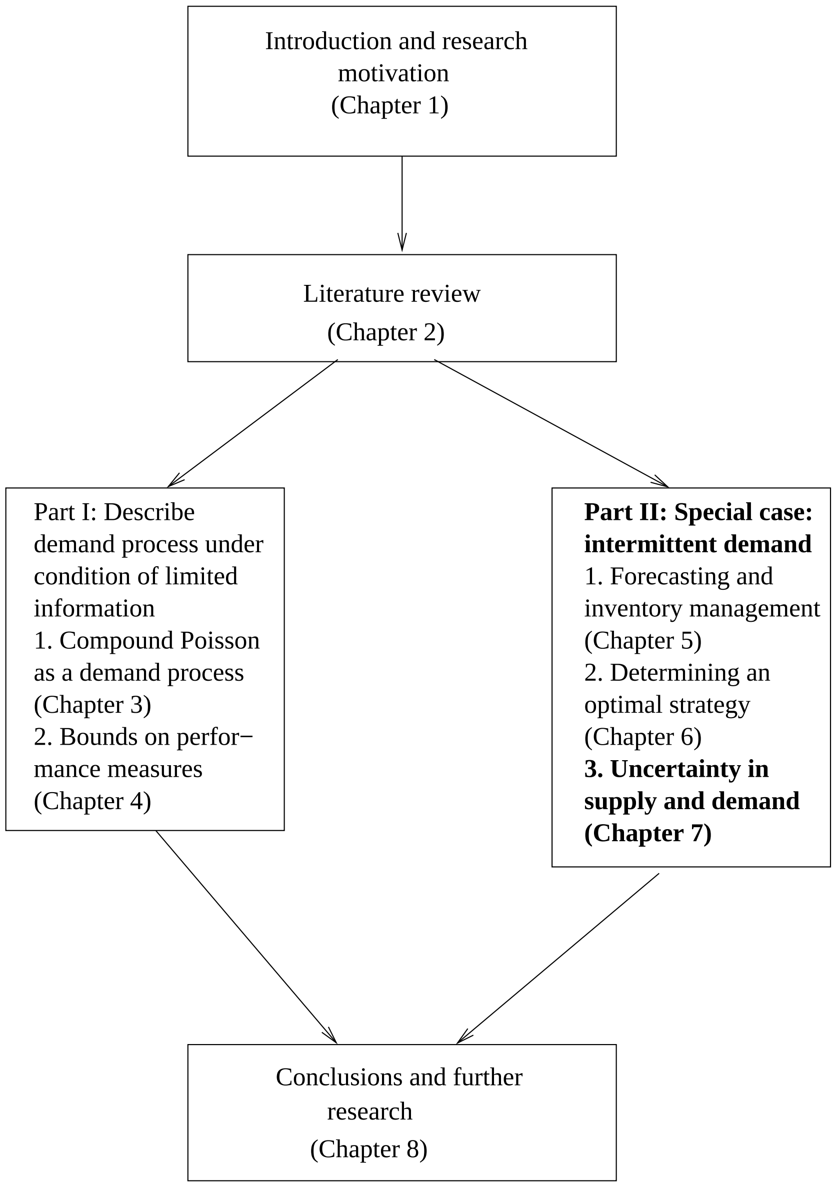

Figure 7.1: Outline of the thesis - Chapter 7

7.3 Comparison of results with and without uncertainty in supply

In this section, results are compared for a reliable and an unreliable supplier. For 16 combinations of uncontrollable factors, the best strategy in combining inventory decision making and demand forecasting for a reliable supplier is determined in chapter 6. Here, this optimal strategy will also be used to determine output measures for the inventory system with an unreliable supplier.

Table 7.1 shows the results for the reliable case, table 7.2 the results for the unreliable case. For each of the output measures (costs, number of stock-out periods and number of stock-out units) the mean and variance is given. Tables 7.3; 7.4 and 7.5 contain 95% confidence intervals for the difference in costs, number of stock-out periods and number of stock-out units respectively.

Experiment	**Costs**		**Periods**		**Units**	
	Mean	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**
1	5369	472.86	4.38	1.37	22.44	9.86
2	3326	469.71	20.42	3.08	218.14	37.57
3	3657	317.57	3.83	1.09	18.82	5.93
4	2746	209.61	4.28	1.45	18.34	6.15
5	2582	357.48	2.61	1.25	9.3	6.46
6	1913	215.77	3.28	1.37	12.56	7.55
7	3768	390.78	2.93	1.42	8.7	5.72
8	2679	419.11	19.68	3.47	98.57	20.12
9	2854	418.13	3.61	1.25	18.91	8.6
10	1709	364.67	10.35	2.45	109.7	26.54
11	2940	636.06	10.67	2.53	112.76	28.93
12	2391	528.32	10.47	2.63	110.67	28.86
13	2507	633.19	9.53	2.97	49.46	18.89
14	1702	407.39	9.33	2.75	48.86	17.31
15	1919	323.32	2.02	1.05	5.8	4.65
16	1385	315.11	9.8	2.64	48.29	15.14

Table 7.1: Results for a reliable supplier

Experiment	Costs		Periods		Units	
	Mean	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**
1	5370	556.41	6.21	2.02	42.29	27.76
2	3709	542.57	21.06	3.36	226.98	44.34
3	3646	362.07	5.06	1.83	31.92	18.81
4	2817	259.67	5.71	2.22	33.74	18.01
5	2522	311	3.25	1.59	13.51	10.68
6	1909	263.15	4.34	1.84	18.47	12.48
7	3750	463.48	3.89	2.18	13.36	10.86
8	2937	381.39	20.25	3.47	106.67	24.27
9	2847	429.51	4.7	1.83	25.77	13.07
10	1951	498.46	11.12	2.8	118.69	36.76
11	3420	857.57	11.38	3.27	117.09	33.02
12	2835	667.06	11.62	2.85	120.89	32.81
13	2791	691.47	9.74	3.31	50.8	21.01
14	2087	532.55	10.56	3.54	54.04	20.95
15	1923	315.09	2.63	1.63	8.19	7.66
16	1518	331.52	10.37	3	51.4	16.47

Table 7.2: Results for an unreliable supplier

Experiment	**Reliable-Unreliable**
1	-150.68;149.88
2	-516.98;-248.38
3	-74.65;97.89
4	-135.08;-5.64
5	-25.81;144.886
6	-65.95;74.03
7	-103.07;139.35
8	-369.92;-144.44
9	-117.4;131.08
10	-359.4;-124.36
11	-679.87;-279.44
12	-597.74;-290.24
13	-459.96;-108.36
14	-507.49;-263.19
15	-100.16;91.3
16	-232.57;-33.19

Table 7.3: Confidence intervals for comparing costs of a reliable and an unreliable supplier

Experiment	Reliable-Unreliable
1	-2.33;-1.33
2	-1.53;0.25
3	-1.67;-0.79
4	-1.96;-0.9
5	-1.06;-0.22
6	-1.51;-0.61
7	-1.49;-0.43
8	-1.53;0.39
9	-1.54;-0.64
10	-1.5;-0.04
11	-1.48;0.06
12	-1.87;-0.42
13	-1.08;0.66
14	-2.04;-0.41
15	-0.99;-0.22
16	-1.44;0.3

Table 7.4: Confidence intervals for comparing stock-out periods of a reliable and an unreliable supplier

Experiment	**Reliable-Unreliable**
1	-25.96;-13.74
2	-19.8;2.12
3	-17.21;-8.99
4	-19.03;-11.77
5	-6.58;-1.84
6	-8.79;-3.03
7	-7.08;-2.24
8	-14.09;-2.11
9	-10.02;-3.7
10	-17.58;-0.4
11	-12.52;3.86
12	-18.35;-2.09
13	-6.67;3.99
14	-10.15;-0.21
15	-4.19;-0.59
16	-7.85;1.63

Table 7.5: Confidence intervals for comparing stock-out units of a reliable and an unreliable supplier

Experiment	C_o	C_h	C_s	**Markov**	γ	β
2	100	4	5	0.4	12	1
8	100	4	5	0.4	6	1
10	100	2	5	0.2	12	1
11	200	4	5	0.2	24	0.5
12	100	4	10	0.2	24	0.5
13	200	4	5	0.2	3	2
14	100	4	10	0.2	3	2
16	100	2	5	0.2	6	1

Table 7.6: Experimental points with significant difference in costs

For eight of the experimental points, the difference in costs between the reliable and the unreliable case is significant. The alternative with the unreliable supplier has higher total costs than the one with the reliable supplier. For these experimental points, there is no significant difference in the number of stock-out periods and the number of stock-out units.

The other eight experimental points indicate no significant difference in costs when comparing a reliable supplier to an unreliable one. When comparing the performance measures, the reliable alternative has significantly better performance measures than the unreliable alternative.

The experimental points with a significant difference in costs are those experimental points for which the best strategy obtained in chapter 6 has an order-up-to-level of 1. For the other experimental points, the best strategy of chapter 6 has an order-up-to-level or fixed order quantity of 15 or more.

Because, as in chapter 6, the optimisation in the next section uses the output measure costs, only the experimental points with a significant difference in costs between the reliable and the unreliable case are considered. These experimental points are listed in Table 7.6.

Experiment	Best strategy
2	ES/OUL; ROP=0; S=1
8	ES/OUL; ROP=0; S=1
10	ES/OUL; ROP=0; S=1
11	CR/OUL; ROP=0; S=1
12	ES/OUL; ROP=0; S=1
13	MA/OUL; ROP=0; S=1
14	MA/OUL; ROP=0; S=1
16	ES/OUL; ROP=0; S=1

Table 7.7: Results of Tabu search for a reliable supplier

7.4 Optimal policy with uncertainty in supply

In this section, a new optimal combination of forecasting method, inventory management policy is determined for the inventory management system with intermittent demand and an unreliable supplier. Furthermore, the optimal settings for the safety stock and the fixed order quantity or order-up-to-level are determined.

The same research approach as in chapter 6 is used. For every combination of forecasting method and inventory management policy, the optimal values of the quantitative factors are determined. This is done using tabu search. Once the optimal values of the quantitative factors are determined for each combination of forecasting method and inventory management policy, the optimal combination of forecasting method and inventory management policy can be determined.

In Table 7.7, the results of the optimisation for a reliable supplier are given for the eight experimental points of interest. Table 7.8 shows the best strategy for the inventory system with an unreliable supplier.

The best strategy for the inventory system with intermittent demand and no uncertainty in supply is an order-up-to-level inventory management policy with $S = 1$. When the results in Table 7.8 are compared to the results in Table 7.7, the experimental points can be divided in two categories. For the experimental points 2, 8, 13 and 16, the best strategy is an inventory management policy with a fixed order

Experiment	Best strategy
2	MA/FOQ; ROP=0; Q=18
8	MA/FOQ; ROP=0; Q=12
10	ES/OUL; ROP=0; S=1
11	CR/OUL; ROP=0; S=1
12	MA/OUL; ROP=0; S=1
13	MA/FOQ; ROP=0; Q=10
14	ES/OUL; ROP=0; S=1
16	MA/FOQ; ROP=0; Q=10

Table 7.8: Results of Tabu search for an unreliable supplier

quantity Q equal to 10 or more. As also noticed in chapter 6, when a fixed order quantity inventory management policy is used, moving averages is always the best forecasting method. For the experimental points 10, 11, 12 and 14, the best strategy is an order-up-to-level inventory management policy with S=1, which is the same strategy as found for the inventory system with a reliable supplier. When the order-up-to-level equal to 1 is the best inventory management policy, no preference for a specific forecasting method can be found.

In Table 7.9 the levels of the uncontrollable factors are given for the eight experimental points. Based on the information in this table, it is difficult to draw some conclusions on the impact of the uncontrollable factors or to construct a classification tree. When the demand frequency is generated using the matrix, corresponded to a probability of 40% of having demand in a certain period, it is always better to use a fixed order quantity equal to 10 or more. In the previous chapter, the results of the reliable case already indicated a strong preference for an order-up-to-level of fixed order quantity equal to 15 or more when a probability of 40% of having demand in a certain period was used. Only for one specific combination of costs, an order-up-to-level of 1 was better used. When the supplier becomes unreliable, it is for this frequency of demand, always better to use an inventory policy with an order-up-to-level or fixed order quantity bigger than 10.

FOQ with $Q \geq 10$							OUL with $S = 1$						
Exp	C_o	C_h	C_s	**Markov**	γ	β	**Exp**	C_o	C_h	C_s	**Markov**	γ	β
2	100	4	5	0.4	12	1	10	100	2	5	0.2	12	1
8	100	4	5	0.4	6	1	11	200	4	5	0.2	24	0.5
13	200	4	5	0.2	3	2	12	100	4	10	0.2	24	0.5
16	100	2	5	0.2	6	1	14	100	4	10	0.2	3	2

Table 7.9: Comparison of the uncontrollable factors for the two categories of results

7.5 Concluding remarks

In this chapter, the simulation model developed in Chapter 5 is extended to include uncertainty in the supply side. Uncertainty in supply consists of many aspects, the focus of this chapter is on uncertainty in availability. The supplier alternates randomly between an available and an unavailable state. When the supplier is available, the order is delivered after the usual lead time. When the supplier is unavailable, the order is executed when the supplier turns available again.

When the best strategy for the inventory system with a reliable supplier is used for the system with an unreliable supplier, 8 of 16 experimental points show a significant difference in total costs. For these 8 experimental points, a new optimal strategy in combining demand forecasting and inventory management decision making is determined using Tabu Search. For four of these points, no better strategy could be determined. For the other four experimental points, it is better to use a fixed order quantity Q of 10 or more instead of an order-up-to-level S equal to 1.

Chapter 8

Conclusions and further research

In this thesis, we described the demand process under the condition of limited information in demand and developed a simulation-optimisation framework for inventory decision support when dealing with intermittent demand. In this chapter, we summarise the main findings and contributions and give directions for future work (Figure 8.1).

8.1 Conclusions

As stated in the introduction, this thesis has four key contributions: (1) to identify characteristics as demand shape and unimodality under the condition of limited information on demand, (2) to determine the optimal safety inventory given a desired performance level under the condition of limited information on demand, (3) to propose a best strategy in combining inventory decision making and demand forecasting for intermittent demand and (4) to describe the impact of uncertainty in the supply side on the best strategy for intermittent demand. Each of these contributions is discussed in more detail in the next paragraphs.

In inventory management, situations exist in which it is realistic to assume that the demand distribution is not completely known (e.g. slow moving products or products recently introduced to the market) because of a lack of sufficient data to decide on the functional form of the demand distribution function. However, in literature

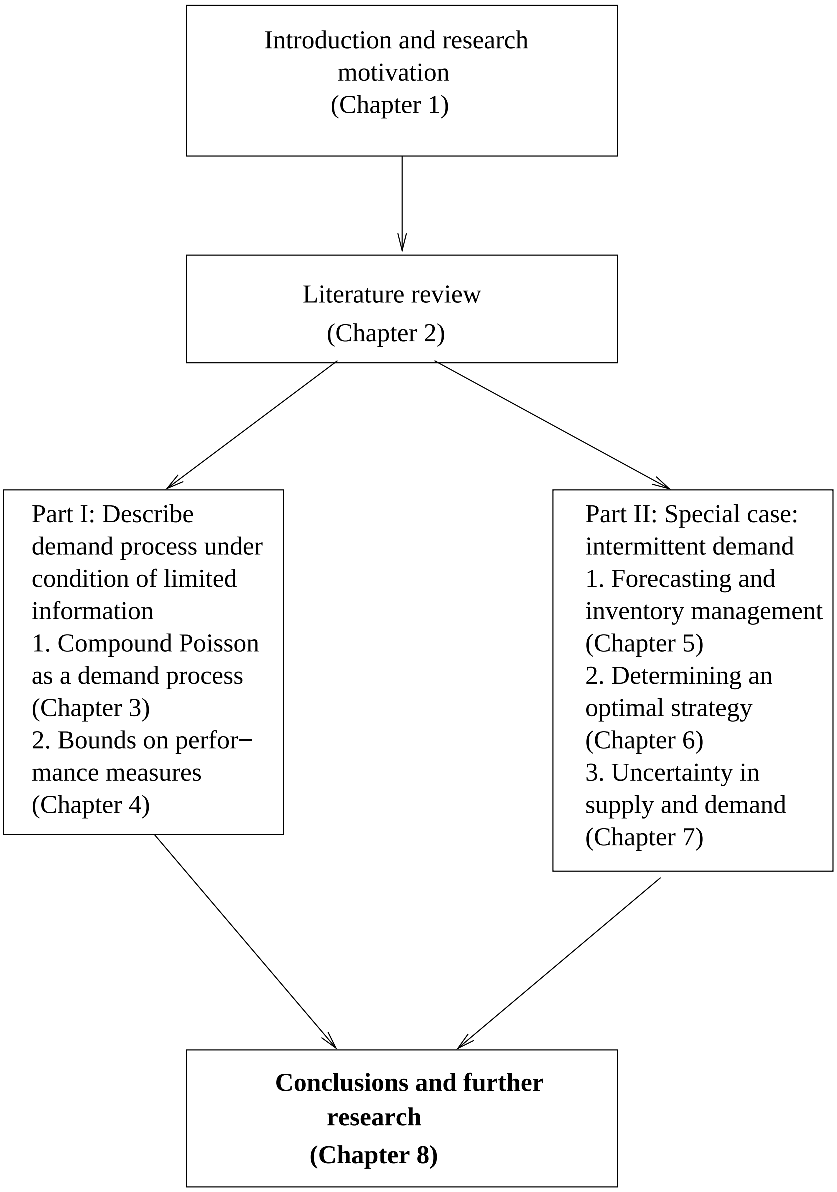

Figure 8.1: Outline of the thesis - Chapter 8

a significant impact of the demand shape on inventory management performance is shown so demand shape is not a secondary factor in the determination of inventories. A different demand shape can increase inventories with more than 100%, given the coefficient of variation. Therefore, it is important to identify characteristics as demand shape and unimodality under the condition of limited information on demand, i.e. only the first two moments are known. A procedure is developed to determine shape characteristics when only the first two moments of the distribution of demand during lead time are known, using a compound Poisson distribution and the Pearson chart. If a compound Poisson distribution is used for modelling demand during lead time, any experiment, choosing a specific frequency of demand, a type of distribution for the demand size, and the first two moments of the distribution, leads to a single point on the Pearson chart, a two-dimensional chart representing an asymmetry measure and a kurtosis measure. Like this, shape and unimodality of the distribution may be recognised. If the mean of the demand distribution during lead time is high with regard to the variance, the normal distribution is a good approximation. If the proportion between the mean of the demand distribution during lead time and the variance decreases, the distribution can be approximated by a unimodal Beta-distribution. When the proportion decreases even further, the demand is similar to a J-shaped Beta-distribution. In literature, the assumption that demand in a certain period of time is continuous and follows a Normal distribution is often made but our results indicate that the Normal distribution is only valid in special cases.

Inventory management decisions make use of optimisation models taking a performance characteristic into consideration. When only limited information is known on the demand distribution, bounds on performance measures, given the inventory level, can be calculated using bounds derived in insurance mathematics. However, in inventory management, it is more interesting to determine which inventory level should be kept at least or at most given a desired level of performance measure. This optimal inventory level is calculated using two performance measures: the number of stock-out units and the stock-out probability in a lead time period. Two cases of limited information are considered: the case of a known range, expected value and variance and the case of a known range, expected value, variance and unique mode. Furthermore, the special case of a compound Poisson demand distribution is discussed. These bounds can be used by an inventory decision maker to calculate the inventory level that has to be held at the beginning of a period, given the desired performance level and the degree of risk aversion of the company. If it is known that the unique mode exists and the value of it is known, the extra information leads to tighter bounds on the inventory level.

A special type of demand, where information on the demand process is limited, is intermittent demand. When demand is of the intermittent type, a best strategy in combining inventory decision making and demand forecasting is proposed, using a simulation model. An experimental design is set up to determine the impact of the cost structure and the demand. Depending on the experimental environment, two options for optimal strategies can be distinguished: an order-up-to level inventory management policy with an order-up-to level equal to 1 and a reorder point equal to 0 or an inventory management policy with a fixed order quantity $Q > 1$ or an order-up-to level $S > 1$ and a reorder point equal to 0. Four factors of the experimental environment have an influence on which of the two strategies is best chosen: the frequency of demand, the inventory holding cost, the order cost and the stock-out cost. When the level of three factors out of these four are fixed, it is possible to determine the optimal strategy. It is important to know which of both strategies is best because there is a significant increase in total costs of the inventory system if the wrong strategy is chosen. Although only a limited number of experimental points are investigated in this thesis, the results can be generalised to draw conclusions with respect to an optimal strategy in combining inventory decision making and demand forecasting for intermittent demand since the levels of costs can be seen in relation to each other instead of as absolute values.

In the last part of the thesis, the impact of uncertainty in the supply side is investigated. One specific type of uncertainty in supply is considered: uncertainty in availability. This uncertainty in availability causes a significant difference in performance measures compared to the reliable situation. For intermittent demand, uncertainty in availability leads to significantly higher total costs for those experimental points that led to the optimal policy of an order-up-to level equal to 1 without uncertainty in availability. A new best strategy in combining inventory decision making and demand forecasting is determined for these points: a fixed order quantity inventory management policy with a fixed order quantity $Q > 1$ and a reorder point of 0.

8.2 Further research

Like any research, this thesis has its limitations in scope. These limitations offer some suggestions for further research.

A best strategy in combining inventory decision making and demand forecasting for intermittent demand is determined using the objective of minimising total costs. However, there are many situations in which total costs are not the most important decision criterion. For example, in Defence or in non-profit organisations, availability is more important than total costs. In future work, the optimisation can be done using service-oriented performance characteristics. In addition, multi-objective optimisation, including total costs and service-oriented performance characteristics, can be used to determine the optimal solution. But since total costs take into account the cost of a stock-out, minor changes in the best strategy are expected.

Three optimisation methods are compared to determine a best strategy in combining inventory decision making and demand forecasting for intermittent demand, using a simulation model. Since a simple Tabu Search gives the best results for this simulation optimisation problem, it may be interesting to enhance this Tabu search or to use other metaheuristics to perform the optimisation of the simulation model.

Depending on the experimental environment, two options for optimal strategies can be distinguished. It is important to know which of both strategies is best because there is a significant increase in total costs of the inventory system if the wrong strategy is chosen. Four factors of the experimental environment have an influence on which of the two strategies is best chosen and a decision tree is constructed to decide which of the two strategies is best chosen, given the levels of the experimental factors. Although the results can be generalised by looking at the proportion of the costs. However, more experiments are required to fine-tune the strategy for a specific situation and to determine a more accurate distinction between the two strategies by means of a cut-off point between the two strategies.

In this thesis, uncertainty in availability is used as uncertainty in the supply side. However, other types of uncertainty in supply, such as uncertainty in quantity or quality, may influence the results differently and therefore deserve further investigation. Next to uncertainty in supply, other types of uncertainty may also have an impact on the best strategy in inventory decision-making for intermittent demand. These types of uncertainty also need to be investigated to test the robustness of the best strategy.

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Appendix A

Compound Poisson as a demand process

A.1 Results: graphs

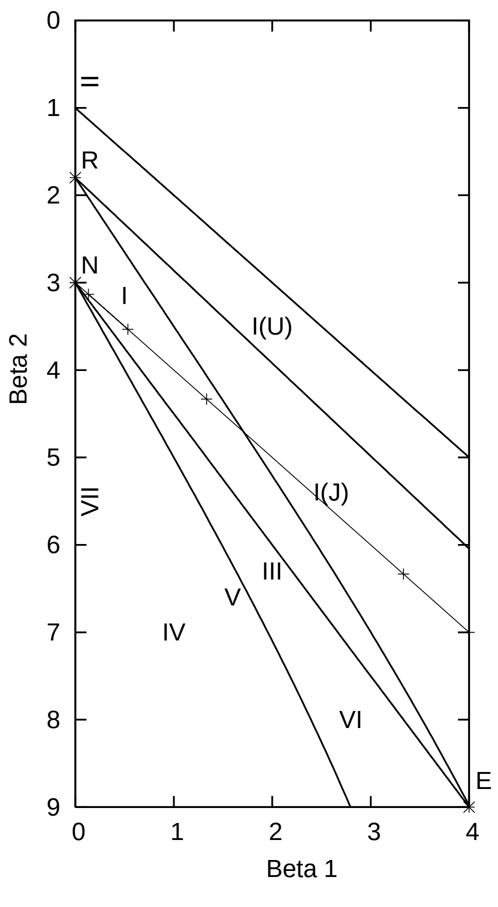

Figure A.1: Pearson chart with values of compound Poisson with deterministic distribution

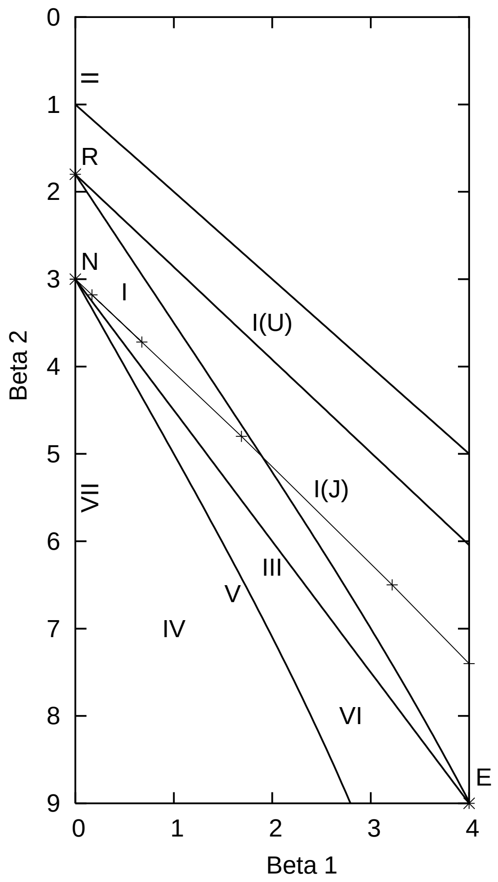

Figure A.2: Pearson chart with values of compound Poisson with uniform distribution in [0,b]

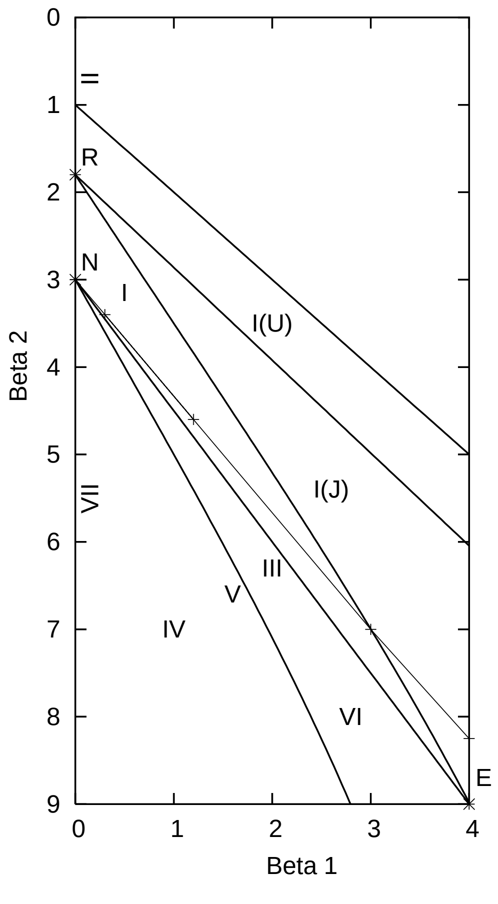

Figure A.3: Pearson chart with values of compound Poisson with exponential distribution

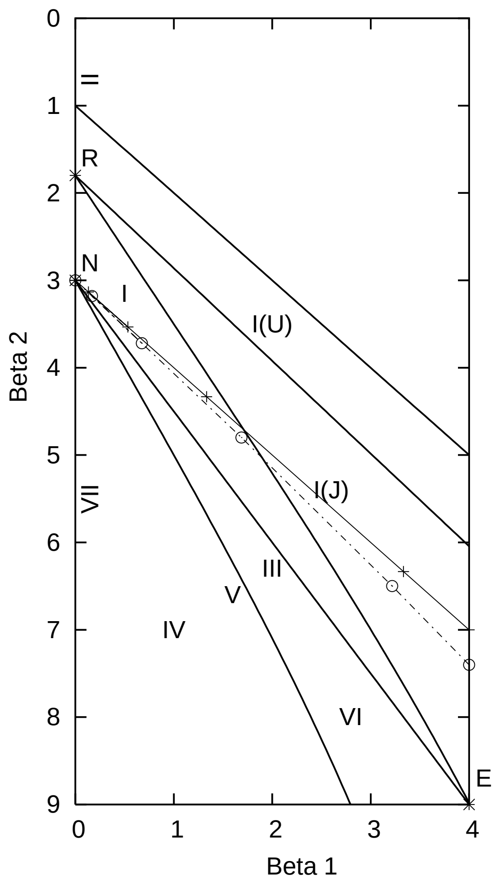

Figure A.4: Pearson chart with area of compound Poisson with uniform distribution

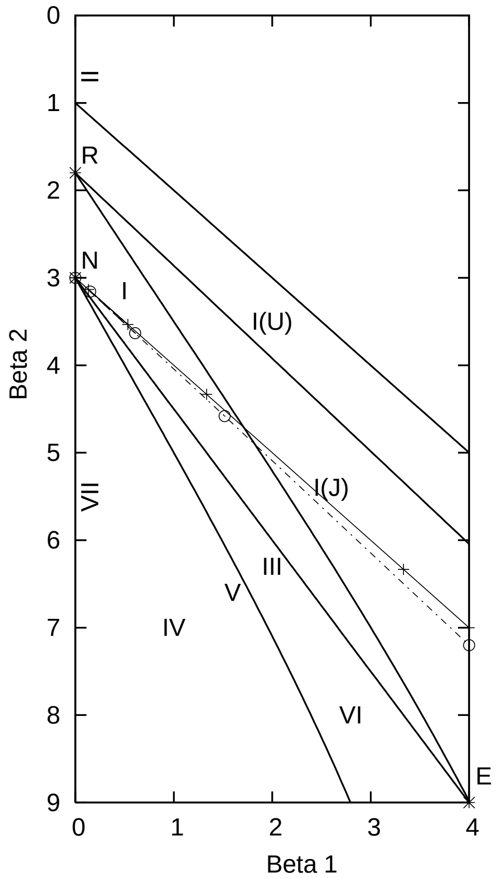

Figure A.5: Pearson chart with area of compound Poisson with triangular distribution with mode=b

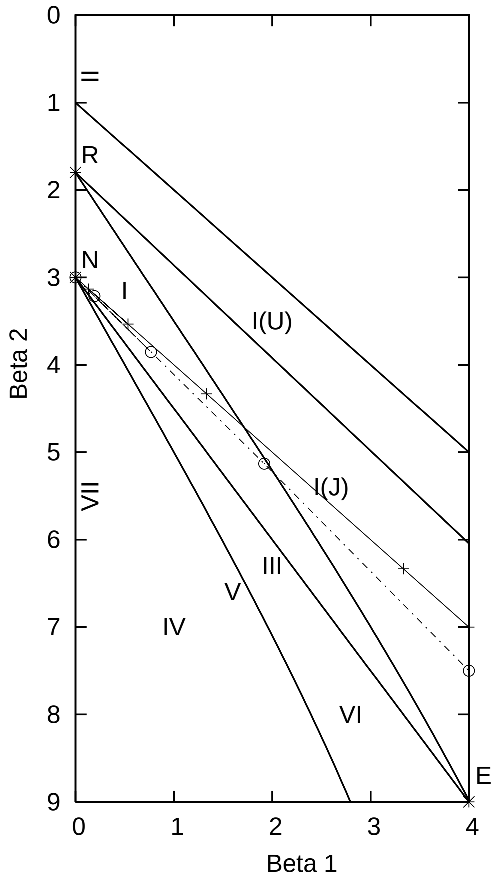

Figure A.6: Pearson chart with area of compound Poisson with triangular distribution with mode=a

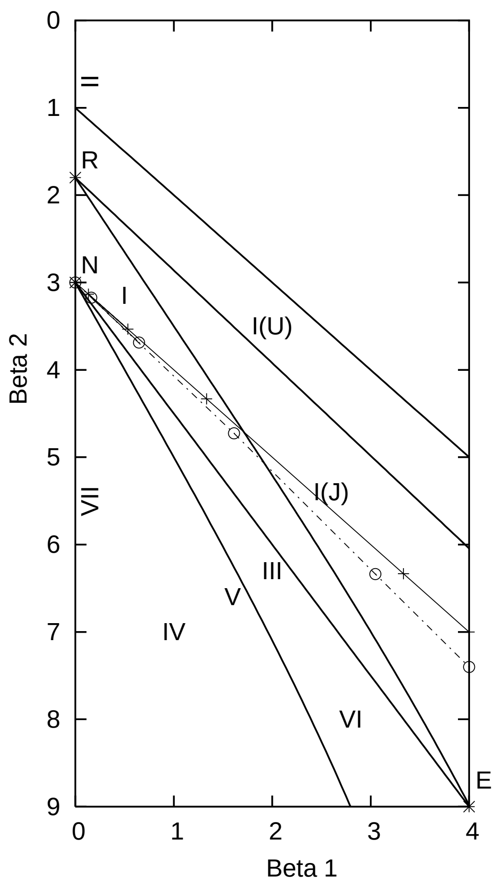

Figure A.7: Pearson chart with area of compound Poisson with symmetric triangular distribution

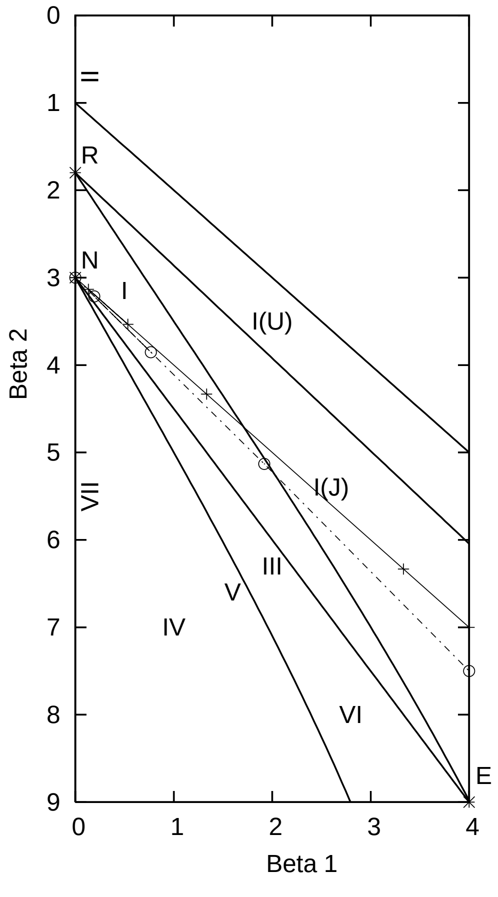

Figure A.8: Pearson chart with area of compound Poisson with asymmetric triangular distribution

A.2 Validation

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
First moment	1000	1000	100
Second moment	13333	266666	133333
$\boldsymbol{\lambda}$	150	7.5	0.15
a	0.15	0.0075	0.0015

Table A.1: Experimental data for the validation of the Poisson distribution compounded with an exponential distribution

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
First moment	1000	1000	100
Second moment	13333	266666	133333
$\boldsymbol{\lambda}$	100	5	0.1
a	0	0	111
b	20	400	1994

Table A.2: Experimental data for the validation of the Poisson distribution compounded with a uniform distribution

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
First moment	1000	1000	100
Second moment	13333	266666	133333
$\boldsymbol{\lambda}$	80	4	0.08
a	3.37	67.43	337.13
b	17.06	341.29	1706.44

Table A.3: Experimental data for the validation of the Poisson distribution compounded with a triangular distribution with mode=b

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
First moment	1000	1000	100
Second moment	13333	266666	133333
$\boldsymbol{\lambda}$	80	4	0.09
a	7.94	158.71	568.94
b	21.63	432.57	2391.52

Table A.4: Experimental data for the validation of the Poisson distribution compounded with a triangular distribution with mode=a

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
First moment	1000	1000	100
Second moment	13333	266666	133333
$\boldsymbol{\lambda}$	80	4	0.09
a	4.59	91.89	124.2
b	20.41	408.11	2228.72

Table A.5: Experimental data for the validation of the Poisson distribution compounded with a symmetric triangular distribution

Hypothesis	**Normal**	**Unimodal Beta**	**J-shaped Beta**
First moment	1000	1000	100
Second moment	13333	266666	133333
$\boldsymbol{\lambda}$	100	5	0.1
a	1	0	0
b	26.3	523.61	2618.03
mode c	2.7	76.39	381.97

Table A.6: Experimental data for the validation of the Poisson distribution compounded with an asymmetric triangular distribution

Appendix B

Bounds on performance measures

B.1 Stock-out units: Tables

Conditions	**Upper bound**
$d \leq \frac{0'}{2}$	$\frac{\mu_1}{\mu_2}(\mu_2 - \mu_1 d)$
$\frac{0'}{2} \leq d \leq \frac{b+b'}{2}$	$\frac{\mu_1 - d + \sqrt{(\mu_2 - \mu_1^2) + (d - \mu_1)^2}}{2}$
$d \geq \frac{b+b'}{2}$	$\frac{(\mu_2 - \mu_1^2)(b-d)}{(\mu_2 - \mu_1^2) + (b - \mu_1)^2}$

Table B.1: Upper bounds on number of stock-out units when E(X) and $E(X^2)$ are known

Conditions	**Lower bound**
$0 \leq d \leq b'$	$\mu_1 - d$
$b' < d < 0'$	$\frac{\mu_2 - \mu_1 d}{b}$
$0' \leq d \leq b$	0

Table B.2: Lower bounds on number of stock-out units when E(X) and $E(X^2)$ are known

Conditions	**Upper bound**
$d \leq \frac{0'^2}{30' - 2m}$	$\frac{\nu_1(\nu_2 - d\nu_1)^2}{2\nu_2(\nu_2 - m\nu_1)}$
$\frac{0'^2}{30' - 2m} \leq d \leq \frac{b^2 - 2mb' + bb'}{3b - 2m - b'}$	$\frac{(\nu_2 - \nu_1^2)(r' - d)^2}{2(\nu_2 - 2r'\nu_1 + r'^2)(r' - m)}$ where r' root of $r'^3 + Ar'^2 + Br' + C = 0$ with $A = -3d$, $B = 4\nu_1 d + 2md - 2m\nu_1 - \nu_2$ and $C = 2m\nu_2 - 2m\nu_1 d - d\nu_2$
$d \geq \frac{b^2 - 2mb' + bb'}{3b - 2m - b'}$	$\frac{(\nu_2 - \nu_1^2)(b - d)^2}{2(\nu_2 - 2b\nu_1 + b^2)(b - m)}$

Table B.3: Upper bounds on number of stock-out units when E(X), $E(X^2)$ and m are known

Conditions	**Lower bound**
$0' \leq d$	0
$\frac{b0'^2+0'(0'-m)\sqrt{b(b-m)}}{b0'+(0'-m)(0'+b-m)} \leq d < 0'$	$\frac{\nu_1(\nu_2-d\nu_1)^2}{2\nu_2(\nu_2-m\nu_1)}$
$b' > d$ and $\frac{bb'^2+b'(b'-m)\sqrt{b(b-m)}}{bb'+(b'-m)(b'+b-m)}$ $\leq d \leq \frac{b0'^2+0'(0'-m)\sqrt{b(b-m)}}{b0'+(0'-m)(0'+b-m)}$ or $b' \leq d$ and $d \leq \frac{b0'^2+0'(0'-m)\sqrt{b(b-m)}}{b0'+(0'-m)(0'+b-m)}$	$\frac{1}{2(b-r)}\left[\frac{(b\nu_1-\nu_2)(r-d)^2}{r(r-m)}+\frac{(\nu_2-\nu_1 r)(b-d)^2}{b(b-m)}\right]$ where $r=\frac{d^2(b-m)+d(d-m)\sqrt{b(b-m)}}{b(b-m)-(b-d)^2}$
$b' > d$ and $d < \frac{bb'^2+b'(b'-m)\sqrt{b(b-m)}}{bb'+(b'-m)(b'+b-m)}$	$\frac{1}{2(b-b')}\left[\frac{(b-\nu_1)(b'-d)^2}{(b'-m)}+\frac{(\nu_1-b')(b-d)^2}{(b-m)}\right]$

Table B.4: Lower bounds on number of stock-out units when E(X), $E(X^2)$ and m are known

Conditions	**Upper bound**
$0 \leq d \leq 15$	$20 - \frac{2}{3}d$
$15 \leq d \leq 31.667$	$10 - \frac{1}{2}d + \frac{1}{2}\sqrt{200+(d-20)^2}$
$31.667 \leq d \leq 50$	$\frac{100-2d}{11}$

Table B.5: Numerical example of upper bounds on the number of stock-out units when E(X) and $E(X^2)$ are known

Conditions	**Lower bound**
$0 \leq d \leq 13.333$	$20 - d$
$13.333 \leq d \leq 30$	$12 - \frac{2}{5}d$
$30 \leq d \leq 50$	0

Table B.6: Numerical example of lower bounds on the number of stock-out units when E(X) and $E(X^2)$ are known

Conditions	**Inventory level**
$W \leq 3.333$	$50 - \frac{11}{2}W$
$3.333 \leq W \leq 10$	$\frac{50-W^2+20W}{W}$
$W \geq 10$	$\frac{60-3W}{2}$

Table B.7: Numerical example of the optimal inventory level using the upper bounds on number of stock-out units when E(X) and $E(X^2)$ are known

Conditions	**Inventory level**
$W \leq 6.667$	$30 - \frac{5}{2}W$
$W \geq 6.667$	$20 - W$

Table B.8: Numerical example of the optimal inventory level using the lower bounds on number of stock-out units when E(X) and $E(X^2)$ are known

Conditions	**Upper bound**
$15 \leq d \leq 20.2$	$\frac{(1200-25d)^2}{79200}$
$20.2 \leq d \leq 21.5$	$\frac{575(r'-d)^2}{2(1200-50r'+r'^2)(r'-15)}$ where r' root of $r'^3 - 3dr'^2 + (130d - 450)r' + 36000 - 1950d = 0$
$21.5 \leq d \leq 50$	$\frac{575(50-d)^2}{84000}$

Table B.9: Numerical example of upper bounds on the number of stock-out units when E(X), $E(X^2)$ and m are known

Conditions	**Lower bound**
$15 \leq d \leq 35.3$	$\frac{1}{2(50-r)}\left[\frac{50(r-d)^2}{r(r-15)} + \frac{(1200-25r)(50-d)^2}{1750}\right]$ where $r = \frac{35d^2+41.83d(d-15)}{1750-(50-d)^2}$
$35.3 \leq d \leq 48$	$\frac{(1200-25d)^2}{79200}$
$48 \leq d \leq 50$	0

Table B.10: Numerical example of lower bounds on the number of stock-out units when E(X), $E(X^2)$ and m are known

B.2 Bounds on tail probabilities

B.2.1 E(X) and $E(X^2)$ are known

When calculating bounds on tail probabilities (De Schepper and Heijnen 1995), the problem is to find:

$$\sup_{F \in \Phi} \int_0^b f(x) dF(x) \tag{B.1}$$

and

$$\inf_{F \in \Phi} \int_0^b f(x) dF(x) \tag{B.2}$$

where Φ is the class of all distribution functions with range $[0, b]$ and with moments μ_1 and μ_2 known and where

$$f(x) = \begin{cases} 0 & \text{if } x \leq d; \\ 1 & \text{if } x > d. \end{cases} \tag{B.3}$$

UPPER BOUNDS

0 ≤ d ≤ b'

A solution is found when P is the straight line through $(b', 1)$ and $(b, 1)$. The upper bound is equal to $q_{b'} f(b') + q_b f(b) = 1$.

b' < d ≤ 0'

In this case, P is the parabola through $(0, 0)$, $(d, 1)$ and $(b, 1)$. According to Lemma 2, the three-point distribution in $(0, d, b)$ will have masses:

$$q_d = \frac{b\mu_1 - \mu_2}{d(b-d)}, q_b = \frac{\mu_2 - \mu_1 d}{b(b-d)}, q_0 = 1 - q_d - q_b. \tag{B.4}$$

The upper bound is $q_d f(d) + q_b f(b)$ or

$$\frac{(b+d)\mu_1 - \mu_2}{bd}. \tag{B.5}$$

0' < d ≤ b

Here, the solution is the parabola through $(d', 0)$ and $(d, 1)$ and tangent to f(x) in d'. The best upper bound is $q_d f'(d) + q_{d'} f(d')$ or

$$\frac{\mu_2 - \mu_1^2}{\mu_2 - \mu_1^2 + (\mu_1 - d)^2}. \tag{B.6}$$

The results for the best upper bounds on the stock-out probability when only the mean and variance of demand are known, are summarized in Table B.11.

Conditions	**Upper bound**
$0 \leq d \leq b'$	1
$b' < d \leq 0'$	$\frac{(b+d)\mu_1 - \mu_2}{bd}$
$0' < d \leq b$	$\frac{(\mu_2 - \mu_1^2)}{(\mu_2 - \mu_1^2) + (\mu_1 - d)^2}$

Table B.11: Upper bounds on stock-out probability when E(X) and $E(X^2)$ are known

LOWER BOUNDS

$0 \leq$ d $\leq$ b'

In this case, P is the parabola through $(d, 0)$ and $d', 1)$ and tangent to f(x) at d'. The lower bound equals $q_d f(d) + q_{d'} f(d')$ or

$$\frac{(\mu_1 - d)^2}{\mu_2 - \mu_1^2 + (\mu_1 - d)^2}. \tag{B.7}$$

b' $<$ d $\leq$ 0'

A solution is found when P is the parabola through $(0, 0)$, $(d, 0)$ and $(b, 1)$. Lemma 2 provides the masses of the three-point distribution which gives a lower bound of

$$\frac{\mu_2 - \mu_1 d}{b(b - d)}. \tag{B.8}$$

0' $<$ d $\leq$ b

Here, P is the line through $(0, 0)$ and $(0', 0)$. The lower bound is 0.

The results for the best lower bounds on the stock-out probability when only the mean and variance of demand are known, are summarized in Table B.12.

Conditions	**Lower bound**
$0 \leq d \leq b'$	$\frac{(\mu_1-d)^2}{\mu_2-\mu_1^2+(\mu_1-d)^2}$
$b' < d \leq 0'$	$\frac{\mu_2-\mu_1 d}{b(b-d)}$
$0' < d \leq b$	0

Table B.12: Lower bounds on stock-out probability when E(X) and $E(X^2)$ are known

B.2.2 E(X), $E(X^2)$ and the unique mode m are known

To solve B.1 and B.2 in case the first two moments and the unique mode is known, the Khinchine transform of the function in B.3 is needed. Therefore, the procedure described in 4.2.2 is used. Since the position of d with respect to m is important to determine the shape of the Khinchine transform, the computations will be split up in two cases:

- $d > m$:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq d; \\ \frac{x-d}{x-m} & \text{if } d < x \leq b. \end{cases} \tag{B.9}$$

- $d \leq m$:

$$f(x) = \begin{cases} \frac{d-m}{x-m} & \text{if } 0 \leq x \leq d; \\ 1 & \text{if } d < x \leq b. \end{cases} \tag{B.10}$$

For each of the two cases, upper and lower bounds can be determined (De Schepper and Heijnen 1995).

I d > m

UPPER BOUNDS

In this section, the best upper bounds for distributions which fulfill the skewness condition

$$\mu_1 > m \tag{B.11}$$

will be derived. Results in case $\mu_1 \leq m$ can be derived using some elementary transformations. The details of this transformation will be explained in section B.2.3.

To distinguish concave and convex parabolas the unique point c_1 in $[d, +\infty[$ such that the tangent to f at c_1 contains the origin, plays an important role. This point c_1 can be calculated as

$$c_1 = d + \sqrt{d(d-m)}. \tag{B.12}$$

If $0' < c_1$ convex parabolas are used, if $0' > c_1$ concave ones are used. If $0' = c_1$ one can use the tangent at c_1 and the two-point distribution in $(0, 0')$. The best upper bound is $q_{0'} f(0')$ or

$$\frac{\nu_1^2(\nu_2 - d\nu_1)}{\nu_2(\nu_2 - m\nu_1)}. \tag{B.13}$$

<u>$0' < c_1$</u>

Parabola through (0,0) and (0',f(0')

Formula 4.10 is used with $u = 0$, $v = 0'$. To assure that $g \geq 0$ on $[0, d]$, the condition $g'(0) \geq 0$ is imposed, which leads to

$$d \leq \frac{20'^2 - m0'}{30' - 2m}. \tag{B.14}$$

The best upper bound equals $q_{0'} f(0')$ or

$$\frac{\nu_1^2(\nu_2 - d\nu_1)}{\nu_2(\nu_2 - m\nu_1)}. \tag{B.15}$$

Parabola through (r,0) and (r',f(r'))

Here we use formula 4.10 with $v = r$, $u = r'$, $f(v) = 0$ and $f'(v) = 0$. This gives us

$$g(x) = \frac{f(r')(x-r)^2}{(r'-r)^2}. \tag{B.16}$$

The condition $g'(r') = f'(r')$ leads to

$$d = \frac{2r'^2 - r'm - rm}{3r' - r - 2m}. \tag{B.17}$$

This function is increasing in r and r'. Because of Lemma 1 a unique solution (r, r') can be assured by imposing the condition

$$\frac{20'^2 - 0'm}{30' - 2m} \leq d \leq \frac{2b^2 - bm - b'm}{3b - b' - 2m}. \tag{B.18}$$

Under this condition the best upper bound is $q_{r'} f(r')$ or

$$\frac{\nu_2 - \nu_1^2}{r'^2 - 2\nu_1 r' + \nu_2} \cdot \frac{r' - d}{r' - m}, \tag{B.19}$$

where r' is the unique root of the polynomial $r'^3 + Ar'^2 + Br' + C$ with $A = -\frac{1}{2}(2\nu_1 + m + 3d)$, $B = 2d\nu_1 + dm$ and $C = \frac{1}{2}(\nu_2 m - \nu_2 d - 2\nu_1 dm)$.

Parabola through (b',0) and (b,f(b))

Formula 4.10 with $u = b$, $v = b'$, $f(v) = 0$ and $f'(v) = 0$) provides us with the parabola

$$g(x) = \frac{f(b)(x - b')^2}{(b - b')^2}. \tag{B.20}$$

To assure $g \geq f$ on $[d, b]$ we impose $g'(b) \leq f'(b)$ which leads to the condition

$$\frac{2b^2 - bm - b'm}{3b - 2m - b'} \leq d. \tag{B.21}$$

The best upper bound is $q_b f(b)$ or

$$\frac{\nu_2 - \nu_1^2}{b^2 - 2\nu_1 b + \nu_2} \cdot \frac{b - d}{b - m}. \tag{B.22}$$

<u>$0' > c_1$</u>

Parabola through (0,0) and (0',f(0'))

Formula 4.10 is used with $u = 0$ and $v = 0'$. To assure that $g \geq f$ on $[d, b]$, we impose $g(b) \geq f(b)$, which gives us

$$\frac{b0'^2}{(0' - m)^2 + b(20' - m)} \leq d. \tag{B.23}$$

The best upper bound is $q_{0'} f(0')$ or

$$\frac{\nu_1^2(\nu_2 - d\nu_1)}{\nu_2(\nu_2 - m\nu_1)}. \tag{B.24}$$

Parabola through (0,0), (r,f(r)) and (b,f(b))

According to Lemma 2, we need r such that $b' < r < 0'$. Formula 4.10 is used with $u = 0$ and $v = r$. The condition $g(b) = f(b)$ will determine r. This leads to

$$d = \frac{br^2}{b(2r - m) + (r - m)^2}. \tag{B.25}$$

d is a strictly increasing function of r on $]m, +\infty[$. To get a solution r in $]b', 0'[$ it is required that

$$\frac{b'^2 b}{b(2b' - m) + (b' - m)^2} < d < \frac{0'^2 b}{b(20' - m) + (0' - m)^2} \tag{B.26}$$

under the condition that $b' > c_1$. Else, the first inequality drops. The solution is then one of the roots of

$$r^2(b - d) - 2rd(b - m) + md(b - m) = 0 \tag{B.27}$$

or

$$r = \frac{d(b - m) + \sqrt{d(b - m)b(d - m)}}{b - d}. \tag{B.28}$$

The best upper equals $q_r f(r) + q_b f(b)$ or

$$\frac{1}{b - r}\left[\frac{(b\nu_1 - \nu_2)(r - d)}{r(r - m)} + \frac{(\nu_2 - \nu_1 r)(b - d)}{b(b - m)}\right]. \tag{B.29}$$

Parabola through (b',f(b')) and (b,f(b))

This case is only valid when $b' > c_1$. The parabola goes through (b',f(b')) and (b,f(b)), touching f in b'. So formula 4.10 is used with $u = b$ and $v = b'$. To assure that $g(x) \geq f(x)$ on $[0, d]$ we impose $g(0) \geq 0$ which can be computed as

$$d \leq \frac{b'^2 b}{b(2b' - m) + (b' - m)^2}. \tag{B.30}$$

The best upper bound is $q_{b'} f(b') + q_b f(b)$ or

$$\frac{1}{b - b'}\left[\frac{(\nu_1 - b')(b - d)}{b - m} + \frac{(b - \nu_1)(b' - d)}{b' - m}\right]. \tag{B.31}$$

The results for the best upper bounds on the stock-out probability when only the mean, variance and unique mode of demand are known for $d > m$, are summarized in Table B.13.

LOWER BOUNDS

0 $\leq$ d $\leq$ b'

A solution is found when the parabola goes through (d,0) and (d',f(d')), touching f in d'. The best lower bound equals $q_{d'}f(d')$ or

$$\frac{(\nu_1 - d)^2}{(\nu_2 - \nu_1^2) + (\nu_1 - m)(\nu_1 - d)}. \tag{B.32}$$

b' < d $\leq$ 0'

The parabola through $(0,0)$, $(d,0)$ and $(b, f(b))$ is smaller than f(x) on $[0, b]$. So the best lower bound is $q_b f(b)$ or

$$\frac{\nu_2 - \nu_1 d}{b(b - m)} \tag{B.33}$$

0' < d $\leq$ b

Now the straight line through $(0,0)$ and $(0',0)$ gives us the solution, which is 0.

The results for the best lower bounds on the stock-out probability when only the mean, variance and unique mode of demand are known for $d > m$, are summarized in Table B.14.

II d $\leq$ m

UPPER BOUNDS

0 $\leq$ d $\leq$ b'

The straight line through $(b', 1)$ and $(b, 1)$ immediately gives the solution. The best upper bound is 1.

Conditions	**Upper bound**
$m < d \leq \frac{bb'^2}{(b'-m)^2+b(2b'-m)}$ and $b' > c_1$	$\frac{1}{b-b'}\left[\frac{(\nu_1-b')(b-d)}{b-m} + \frac{(b-\nu_1)(b'-d)}{b'-m}\right]$
$m < d < \frac{b0'^2}{(0'-m)^2+b(20'-m)}$ and $b' \leq c_1$ or $\frac{bb'^2}{(b'-m)^2+b(2b'-m)} < d$ $< \frac{b0'^2}{(0'-m)^2+b(20'-m)}$ and $b' > c_1$	$\frac{1}{b-r}\left[\frac{(b\nu_1-\nu_2)(r-d)}{r(r-m)} + \frac{(\nu_2-\nu_1 r)(b-d)}{b(b-m)}\right]$ with $r = \frac{d(b-m)+\sqrt{bd(b-m)(d-m)}}{b-d}$
$\frac{b0'^2}{(0'-m)^2+b(20'-m)} \leq d \leq \frac{20'^2-m0'}{30'-2m}$	$\frac{\nu_1^2(\nu_2-d\nu_1)}{\nu_2(\nu_2-m\nu_1)}$
$\frac{20'^2-m0'}{30'-2m} \leq d \leq \frac{2b^2-bm-b'm}{3b-b'-2m}$	$\frac{\nu_2-\nu_1^2}{r'^2-2\nu_1 r'+\nu_2} \cdot \frac{r'-d}{r'-m}$ where r' root of $r'^3 + Ar'^2 + Br' + C$ with $A = -\frac{1}{2}(2\nu_1+m+3d)$, $B = 2d\nu_1+dm$ and $C = \frac{1}{2}(\nu_2 m - \nu_2 d - 2\nu_1 dm)$.
$\frac{2b^2-bm-b'm}{3b-b'-2m} \leq d \leq b$	$\frac{\nu_2-\nu_1^2}{b^2-2\nu_1 b+\nu_2} \cdot \frac{b-d}{b-m}$

Table B.13: Upper bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d > m$

Conditions	**Lower bound**
$0 \leq d \leq b'$	$\frac{(\nu_1-d)^2}{(\nu_1-m)(\nu_1-d)+\nu_2-\nu_1^2}$
$b' < d \leq 0'$	$\frac{\nu_2-\nu_1 d}{b(b-m)}$
$0' < d \leq b$	0

Table B.14: Lower bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d > m$

b' < d ≤ 0'

The parabola through $(0, f(0))$, $(d, 1)$ and $(b, 1)$ is always bigger than f on $[0, b]$. The best upper bound equals $q_0 f(0) + q_d f(d) + q_b f(b)$ or

$$\frac{b(m-d)+(b+d)\nu_1-\nu_2}{bm}. \tag{B.34}$$

0' < d ≤ b

In this case, the parabola goes through $(d, 1)$ and $(d', f(d'))$, touching f in d'. The best upper bound equals $q_d f(d) + q_{d'} f(d')$ or

$$\frac{\nu_1-d'}{d-d'}+\frac{\nu_1-d}{d'-d}\cdot\frac{m-d}{m-d'}. \tag{B.35}$$

The results for the best upper bounds on the stock-out probability when only the mean, variance and unique mode of demand are known for $d \leq m$, are summarized in Table B.15.

LOWER BOUNDS

In this section, the skewness condition $\mu_1 > m$ is imposed. Results in case the condition is not fulfilled can be derived using some elementary transformations. This

Conditions	Upper bound
$0 \leq d \leq b'$	1
$b' < d \leq 0'$	$\frac{b(m-d)+(b+d)\nu_1-\nu_2}{bm}$
$0' < d \leq b$	$\frac{1}{d-d'}\left[(\nu_1 - d') + \frac{(d-\nu_1)(m-d)}{m-d'}\right]$

Table B.15: Upper bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d \leq m$

will be explained in detail in section B.2.3.

To distinguish concave and convex parabolas the point c_2 where the tangent to f contains the point $(b, 1)$ is important. This unique point can be calculated as

$$c_2 = d - \sqrt{(m-d)(b-d)}. \tag{B.36}$$

If $b' < c_2$ convex parabolas are used, if $b' > c_2$ concave ones are used. If $b' = c_2$ one can use the tangent line in c_2 and the two-point distribution in (b, b'). The best lower bound is $q_{b'}f(b') + q_b f(b)$ or

$$\frac{(d-c_2)\nu_1 + (b-c_2)m - bd + c_2^2}{(b-c_2)(m-c_2)}. \tag{B.37}$$

$b' < c_2$

Parabola through (b',f(b')) and (b,1)

Formula 4.10 is used with $u = b$ and $v = b'$. To assure that $g(x) \leq f(x)$ on $[0, d]$ the condition $g(0) \leq f(0)$ is imposed. This leads to

$$d \leq \frac{bm(m - 2b' + b)}{(m-b')^2 + b(m - 2b' + b)} \tag{B.38}$$

under the condition that $b' < m$. The best lower bound is equal to $q_{b'}f(b') + q_b f(b)$ or

$$\frac{1}{b-b'}\left[(\nu_1 - b') + \frac{(b-\nu_1)(m-d)}{m-b'}\right]. \tag{B.39}$$

Parabola through (0,f(0), (r,f(r)) and (b,1)

According to Lemma 2, we need r such that $b' < r < 0'$. Formula 4.10 is used with $u = b$ and $v = r$. The condition $g(0) = f(0)$ will determine r. This leads to

$$d = \frac{bm(m-2r+b)}{(m-r)^2 + b(m-2r+b)}. \tag{B.40}$$

d is a strictly increasing function of r on $]-\infty, m[$. To get a solution r in $]b', 0'[$ it is required that $b' < m$ and

$$\frac{bm(m-2b'+b)}{(m-b')^2 + b(m-2b'+b)} < d \leq m. \tag{B.41}$$

The solution is then one of the roots of

$$r^2 + 2(bm - md - bd)r + (m^2d + bmd + b^2d - bm^2 - bm^2) = 0 \tag{B.42}$$

or

$$r = \frac{bd + md - bm - \sqrt{mb(b-d)(m-d)}}{d}. \tag{B.43}$$

The best lower bound is $q_0 f(0) + q_r f(r) + q_b f(b)$ or

$$\frac{m-d}{m} \cdot \frac{\nu_2 - \nu_1(b+r) + br}{br} + \frac{m-d}{m-r} \cdot \frac{b\nu_1 - \nu_2}{r(b-r)} + \frac{\nu_2 - r\nu_1}{b(b-r)}. \tag{B.44}$$

$b' > c_2$

Parabola through (b',f(b')) and (b,f(b))

Formula 4.10 is used with $u = b$ and $v = b$. To assure that $g(x) \leq f(x)$ on [0,d], the condition $g'(b) \geq 0$ is imposed. This gives us

$$\frac{mb + mb' - 2b'^2}{2m - 3b' + b} \leq d < \frac{mb - b'^2}{m + b - 2b'} \tag{B.45}$$

on the condition that $b' < m$. The best lower bound is equal to $q_{b'} f(b') + q_b f(b)$ or

$$\frac{\nu_1 - b'}{b - b'} + \frac{\nu_1 - b}{b' - b} \cdot \frac{m-d}{m-b'}. \tag{B.46}$$

Parabola through (r,f(r)) and (r',1)

Here the solution is a parabola touching f in r and r'. Formula 4.10 is used with $u = r$ and $v = r'$. The condition $g'(r) = 0$ will determine r. This leads to

$$d = \frac{mr' + mr - 2r^2}{2m - 3r + r'}. \tag{B.47}$$

d is a strictly increasing function of r and r'. To get a solution (r, r') it is required that $b' < m$ and

$$\frac{m0'}{2m + 0'} \leq d < \frac{mb + mb' - 2b'^2}{2m - 3b' + b}. \tag{B.48}$$

or $b' \geq m$ and

$$\frac{m0'}{2m + 0'} \leq d \leq m \tag{B.49}$$

The best lower bound is $q_r f(r) + q_{r'} f(r')$ or

$$\frac{\nu_2 - \nu_1^2}{\nu_2 - 2\nu_1 r + r^2} \cdot \frac{m - d}{m - r} + \frac{(\nu_1 - r)^2}{\nu_2 - 2\nu_1 r + r^2} \tag{B.50}$$

where r is the root of the polynomial $r^3 + Ar^2 + Br + C$ with $A = -\frac{1}{2}(2\nu_1 + m + 3d)$, $B = 2d\nu_1 + dm$ and $C = \frac{1}{2}(\nu_2 m - \nu_2 d - 2\nu_1 dm)$.

Parabola through (0,f(0)) and (0',f(0'))

Again, formula 4.10 is used. To assure that g(x) is always smaller than f, the condition $g'(0) \leq f'(0)$ is imposed. This leads to

$$d \leq \frac{0'm}{2m + 0'}. \tag{B.51}$$

The best lower bound is $q_0 f(0) + q_{0'} f(0')$ or

$$\frac{\nu_1^2 d + \nu_2(m - d)}{m\nu_2}. \tag{B.52}$$

The results for the best lower bounds on the stock-out probability when only the mean, variance and unique mode of demand are known for $d \leq m$, are summarized in Table B.16.

Conditions	Lower bound
$d \leq \frac{0'm}{2m+0'}$	$\frac{\nu_1^2 d+\nu_2(m-d)}{m\nu_2}$
$\frac{0'm}{2m+0'} \leq d \leq m$ and $b' \geq m$ or $\frac{0'm}{2m+0'} \leq d < \frac{mb+mb'-2b'^2}{2m-3b'+b}$ and $b' < m$	$\frac{m-d}{m-r} \cdot \frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1 r+r^2} + \frac{(\nu_1-r)^2}{\nu_2-2\nu_1 r+r^2}$ with r root of $r^3 + Ar^2 + Br + C$ with $A = -\frac{1}{2}(2\nu_1+m+3d)$, $B = 2d\nu_1+dm$ and $C = \frac{1}{2}(\nu_2 m - \nu_2 d - 2\nu_1 dm)$.
$\frac{mb+mb'-2b'^2}{2m-3b'+b} \leq d \leq \frac{bm(m-2b'+b)}{(m-b')^2+b(m-2b'+b)}$ and $b' < m$	$\frac{1}{b-b'}\left[(\nu_1 - b') + \frac{(b-\nu_1)(m-d)}{m-b'}\right]$
$\frac{bm(m-2b'+b)}{(m-b')^2+b(m-2b'+b)} < d \leq m$ and $b' < m$	$\frac{m-d}{m} \cdot \frac{\nu_2-\nu_1(b+r)+br}{br} + \frac{m-d}{m-r} \cdot \frac{b\nu_1-\nu_2}{r(b-r)} + \frac{\nu_2-r\nu_1}{b(b-r)}$ where $r = \frac{bd+md-bm-\sqrt{mb(b-d)(m-d)}}{d}$

Table B.16: Lower bounds on stock-out probability when E(X), $E(X^2)$ and m are known for $d \leq m$

B.2.3 Transformation

The transformation described in this section can be used when $\mu_1 \leq m$ and the best upper bound for the tail probability if $d > m$ has to be derived or the best lower bound if $d \leq m$ (De Schepper and Heijnen 1995).
Denoting this distribution as F, a new distribution function H will be defined as

$$H(x) = 1 - F(b - x). \tag{B.53}$$

Using the notations m^H, μ_1^H and μ_2^H for the mode and the first two moments of this new distribution, the following relations are valid:

$$m^H = b - m \tag{B.54}$$

$$\mu_1^H = b - \mu_1 \tag{B.55}$$

$$\mu_2^H = b^2 - 2b\mu_1 + \mu_2 \tag{B.56}$$

Since $\mu_1 < m$, we have $\mu_1^H > m$. This means that the results of section B.2.2 can be used for H. The only question remaining is how to transform the results. Therefore we use the following identities. The first one is

$$\sup_F \int_0^b f(x)dF(x) = 1 - \inf_H \int_0^b f^*(x)dH(x) \tag{B.57}$$

where

$$f^*(x) = \begin{cases} \frac{d^H - m^H}{x - m^H} & \text{if } 0 \leq x \leq d^H; \\ 1 & \text{if } d^H < x \leq b. \end{cases} \tag{B.58}$$

if we define d^H as $b - d$.

The second identity is

$$\inf_F \int_0^b f(x)dF(x) = 1 - \sup_H \int_0^b f^*(x)dH(x) \tag{B.59}$$

where

$$f^*(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq d^H; \\ \frac{x - d^H}{x - m^H} & \text{if } d^H < x \leq b. \end{cases} \tag{B.60}$$

B.3 Stock-out probability: Numerical example: Tables

Conditions	**Upper bound**
$0 \leq d \leq 13.333$	1
$13.333 \leq d \leq 30$	$\frac{40+2d}{5d}$
$30 \leq d \leq 50$	$\frac{200}{200+(20-d)^2}$

Table B.17: Numerical example of upper bounds on the stock-out probability when E(X) and $E(X^2)$ are known

Conditions	**Lower bound**
$0 \leq d \leq 13.333$	$\frac{(20-d)^2}{200+(20-d)^2}$
$13.333 \leq d \leq 30$	$\frac{600-20d}{50(50-d)}$
$30 \leq d \leq 50$	0

Table B.18: Numerical example of lower bounds on the stock-out probability when E(X) and $E(X^2)$ are known

Conditions	**Inventory level**
$U \leq 0.667$	$\frac{20U+\sqrt{200(U-U^2)}}{U}$
$U \geq 0.667$	$\frac{400}{50U-20}$

Table B.19: Numerical example of the optimal inventory level using the upper bounds on the stock-out probability when E(X) and $E(X^2)$ are known

Conditions	**Inventory level**
$U \leq 0.182$	$\frac{60-250U}{2-5U}$
$U \geq 0.182$	$\frac{20(U-1)+\sqrt{400(U-1)^2-(U-1)(600U-400)}}{U-1}$

Table B.20: Numerical example of the optimal inventory level using the lower bounds on the stock-out probability when E(X) and $E(X^2)$ are known

Conditions	Upper bound
$0 \leq d \leq 2$	1
$2 < d \leq 15$	$\frac{32-d}{30}$
$15 \leq d < 22.4$	$\frac{1}{50-r}\left[\frac{50(r-d)}{r(r-15)} + \frac{(1200-25r)(50-d)}{1750}\right]$ where $r = \frac{35d+\sqrt{1750d(d-15)}}{50-d}$
$22.4 \leq d \leq 34.1$	$\frac{1200-25d}{1584}$
$34.1 \leq d \leq 35.8$	$\frac{575(r'-d)}{(r'^2-50r'+1200)(r'-15)}$ where r' root of $r'^3 + \frac{2415+3d}{2}r'^2 + 65dr' + 9000 - \frac{1575d}{2} = 0$
$35.8 \leq d \leq 50$	$\frac{23(50-d)}{1680}$

Table B.21: Numerical example of upper bounds on the stock-out probability when E(X), $E(X^2)$ and m are known

Conditions	**Lower bound**
$0 \leq d \leq 9.2$	$\frac{720-23d}{720}$
$9.2 \leq d < 10.4$	$\frac{575(15-d)}{(r^2-50r+1200)(15-r)} + \frac{(25-r)^2}{r^2-50r+1200}$ where r root of $r^3 - \frac{1}{2}(65+3d)r^2 + 65dr + 9000 - 975d = 0$
$10.4 \leq d \leq 14.2$	$\frac{674-25d}{624}$
$14.2 < d \leq 15$	$\frac{15-d}{15}\frac{1200-25(50+r)+50r}{50r} + \frac{15-d}{15-r}\frac{50}{r(50-r)} + \frac{1200-25r}{50(50-r)}$ where $r = \frac{61d-750-\sqrt{750(50-d)(15-d)}}{d}$
$15 < d \leq 48$	$\frac{48-d}{70}$
$48 < d \leq 50$	0

Table B.22: Numerical example of lower bounds on the stock-out probability when E(X), $E(X^2)$ and m are known

Appendix C

Classification trees

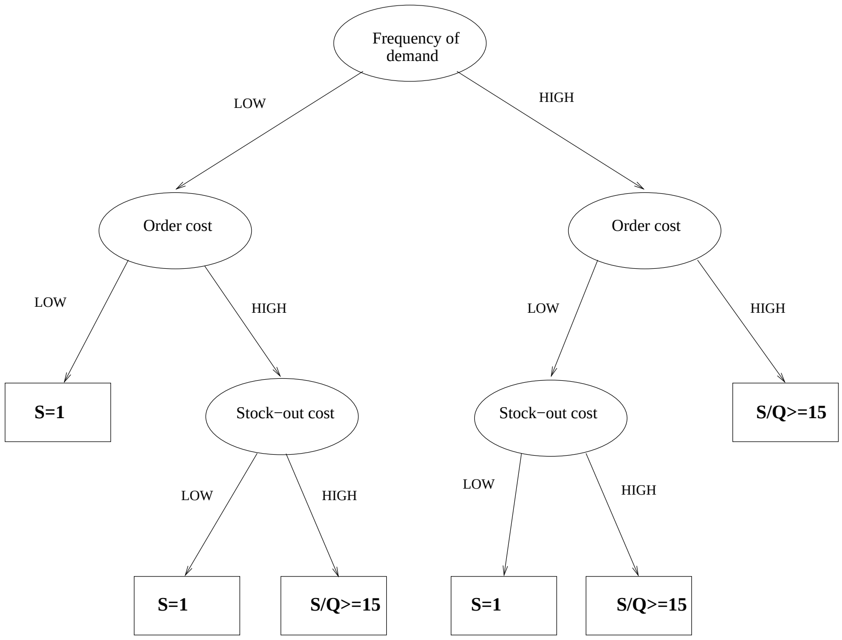

Figure C.1: Classification tree - frequency of demand, order cost and stock-out cost

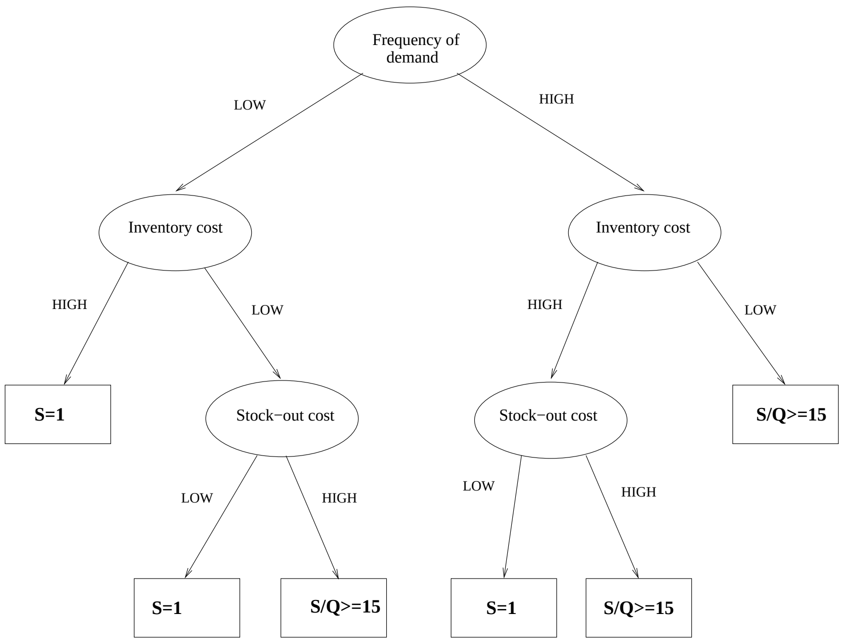

Figure C.2: Classification tree - frequency of demand, inventory cost and stock-out cost

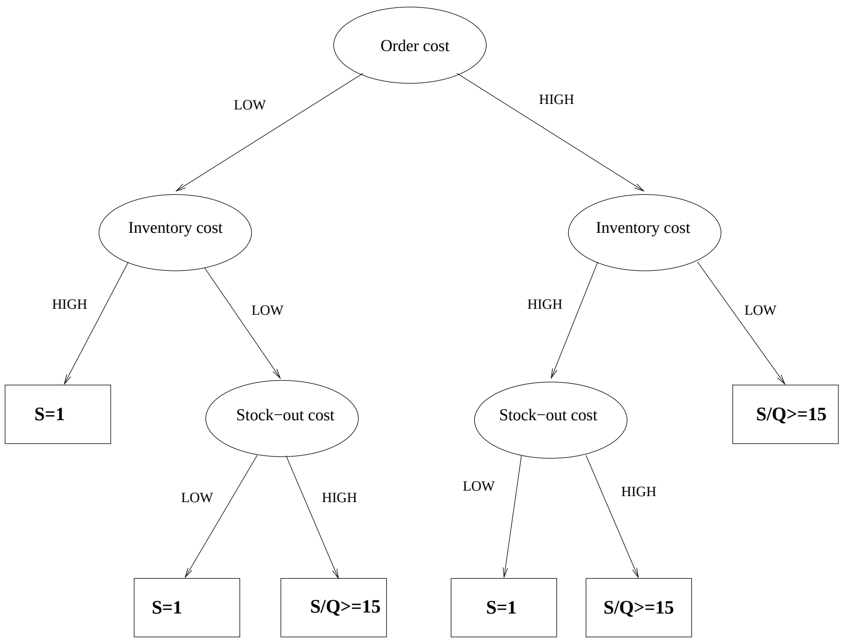

Figure C.3: Classification tree - order cost, inventory cost and stock-out cost

Samenvatting

Logistieke systemen bevatten onzekerheden in vraag, in levertermijn, in transporttijden, in beschikbaarheid van middelen en in kwaliteit. In het modelleren van distributieketens en het analyseren van het gedrag en de prestatie van distributieketens is het dan ook belangrijk met deze onzekerheid rekening te houden. Omdat het bestuderen van de volledige distributieketen te uitgebreid zou zijn voor dit doctoraatsonderzoek, ligt de focus van dit onderzoek op het bestuderen van onzekerheid in voorraadsystemen. In de literatuur gaat men er meestal vanuit dat onzekerheden in parameters kunnen beschreven worden met behulp van een kansverdeling. In de praktijk is de informatie waarover men beschikt echter dikwijls beperkt. In een eerste deel van deze thesis zal het vraagproces beschreven worden wanneer men over beperkte informatie omtrent de vraag beschikt. Na het beschrijven van het vraagproces met beperkte informatie, wordt er aandacht besteed aan een speciaal geval van beperkte informatie: intermitterende vraag. Weinig studies behandelen onregelmatige vraag waarbij er niet elke periode een bestelling geplaatst wordt en, indien er besteld wordt, er een grote variabiliteit is in de hoeveelheid. Daarom zal in het tweede deel van deze thesis de prestatie van verschillende voorspellingsmethoden en de impact van deze methoden op verschillende politieken voor voorraadbeheer onderzocht worden met behulp van een simulatiemodel. Voor de optimalisatie van de parameters van dit simulatiemodel zullen verschillende simulatie-optimalisatie technieken gebruikt en vergeleken worden.

In voorraadbeheer bestaan er heel wat situaties waarvoor de kansverdeling van de vraag tijdens de levertermijn niet volledig gekend is, zoals producten die weinig verkopen of producten die recent op de markt gebracht zijn. Voor deze producten is dikwijls enkel het gemiddelde en de variantie gekend en is er niet voldoende informatie beschikbaar om de vorm van de verdeling te bepalen Nochtans wordt in de literatuur aangetoond dat ook de vorm van de verdeling belangrijk is bij het bepalen van de grootte van een voorraad. Een verschillende vorm van de kansverdeling van de vraag

kan leiden tot een verhoging van de voorraad met 100%. Daarom wordt in hoofdstuk 3 een procedure uitgewerkt voor het bepalen van karakteristieken van de kansverdeling van de vraag die de vorm van de verdeling bepalen zoals de scheefheid en de unimodaliteit, wanneer enkel het gemiddelde en de variantie van de vraag tijdens de levertermijn gekend zijn. De procedure veronderstelt dat de vraag tijdens de levertermijn kan beschouwd worden als een compound Poisson proces, waarbij de frequentie van de vraag en de grootte van de vraag apart gemodelleerd worden. De frequentie van de vraag volgt een Poisson proces, terwijl voor de grootte van de vraag verschillende verdelingen onderzocht worden. Er wordt gebruik gemaakt van de grafiek van Pearson om op basis van de resultaten de vormkarakteristieken van de verdeling te bepalen. Uit de resultaten van dit hoofdstuk blijkt dat, voor de onderzochte verdelingen voor de grootte van de vraag, best gekozen wordt voor een Betaverdeling, tenzij de verhouding tussen gemiddelde en variantie erg groot is.

In voorraadbeheer wordt dikwijls ook gebruik gemaakt van prestatiemaatstaven, zoals het verwachte aantal eenheden tekort in een periode of de kans op een tekort in een periode, om beslissingen te nemen. In hoofdstuk 4 worden grenzen voor prestatiemaatstaven berekend voor een gegeven voorraad, uitgaande van slechts beperkte beschikbare informatie omtrent de vraag tijdens de levertermijn. In voorraadbeheer is het echter interessanter om, gegeven een gewenst niveau voor een prestatiemaatstaf, grenzen te bepalen voor het optimale voorraadniveau. De grenzen worden bepaald voor twee verschillende situaties van beperkte informatie: een gekend bereik, gemiddelde en variantie of een gekend bereik, gemiddelde, variantie en modus. De grenzen kunnen dan door de beleidsvormer gebruikt worden om het niveau van voorraad aan het begin van een periode te bepalen, gegeven het serviceniveau dat men wenst te behalen.

In het tweede deel van het proefschrift wordt dieper ingegaan op een speciaal geval van beperkte informatie, namelijk intermitterende vraag. Bij intermitterende vraag wordt er slechts sporadisch een bestelling geplaatst en bovendien kan de grootte van de bestelling sterk verschillen. Voorbeelden van producten met een intermitterende vraag zijn wisselstukken van vliegtuigonderdelen en kapitaalgoederen. Vooral het voorspellen van intermitterende vraag is moeilijk. Veelgebruikt is de methode van Croston, die afzonderlijk het voorkomen van vraag en de grootte van de vraag gaat voorspellen. Toch wordt deze methode in de literatuur in vraag gesteld omdat eenvoudige voorspellingmethodes zoals Simple Exponential Smoothing en Moving Averages, dezelfde resultaten zouden opleveren. In hoofdstuk 5 wordt een simulatiemodel gebouwd om drie verschillende voorspellingsmethodes en twee verschillende voorraadpolitieken met elkaar te vergelijken. Zowel het effect op totale kosten als op enkele prestatiemaatstaven wordt bekeken.

In hoofdstuk 6 wordt een optimale strategie bepaald voor het voorraadbeheer van producten met een intermitterende vraag. Hierbij wordt gekeken naar de combinatie van voorspellingsmethode en voorraadpolitiek en worden de kwantitatieve parameters van zowel de voospellingsmethode als de voorraadpolitiek geoptimaliseerd. Drie verschillende optimalisatiemethodes worden hiervoor gebruikt en vergeleken: Taguchi's methode, responsie-oppervlakken en Tabu search. Deze laatste methode blijkt de meest accurate resultaten te leveren. Afhankelijk van de omgevingsfactoren (kosten van het voorraadsysteem en de parameters van de vraag) kunnen twee verschillende optimale strategieë onderscheiden worden. De eerste optimale strategie is een voorraadsysteem waarbij periodiek de vooraadpositie herzien wordt. Wanneer het aantal eenheden in voorraad lager is dan het bestelpunt, wordt een order geplaatst om de voorraad aan te vullen tot het maximale niveau $S = 1$. Dit komt er eigenlijk op neer dat telkens er zich een vraag voordoet, het voorraadniveau opnieuw wordt aangevuld tot 1. De tweede optimale strategie is een voorraadsysteem waarbij een vaste bestelhoeveelheid of een maximaal voorraadniveau gebruikt wordt dat groter is dan 1. Er is met andere woorden een grotere voorraad aanwezig en er moet niet elke keer dat er zich een vraag voordoet, bijbesteld worden. Wanneer gebruik gemaakt wordt van een vaste bestelhoeveelheid, wordt best Moving Averages gebruikt als voorspellingsmethode. Welke van de twee strategieën moet gekozen worden, hangt af van de omgevingsfactoren. Vier factoren spelen hierin een beslissende rol: de frequentie van de vraag, de voorraadkost, de bestelkost en de tekortkost. Op basis van deze vier factoren kan er bepaald worden welke van beide strategieën best gebruikt wordt.

In hoofdstuk 7 wordt het simulatiemodel van hoofdstuk 5 en 6 verder uitgebreid met onzekerheid in de aanbodzijde. Het effect van deze onzekerheid op de optimale strategie wordt bekeken. Onzekerheid in de aanbodzijde kan variëren van onzekerheid in levertermijn of transporttijd over onzekerheid in kwaliteit van de geleverde goederen tot onzekerheid in beschikbaarheid. In dit hoofdstuk wordt gekozen voor deze laatste vorm van onzekerheid in de aanbodzijde. Dit betekent dat de leverancier sommige periodes niet beschikbaar is. Indien er een bestelling wordt geplaatst in zo'n periode, wordt de bestelling pas geleverd een levertermijn nadat de leverancier opnieuw beschikbaar is. Wanneer deze onzekerheid aan het simulatiemodel wordt toegevoegd, leidt dit tot een significante stijging van de totale kosten wanneer de optimale strategie met maximaal voorraadniveau gelijk aan 1 dient gebruikt te worden. Als een nieuwe optimale strategie voor deze situatie bepaald wordt met behulp van Tabu search, blijkt dat het nu beter is gebruik te maken van een vaste bestelhoeveelheid die groter is dan 1.

Het proefschrift sluit af met een kijk naar de toekomst. De huidige beperkingen van het onderzoek zijn weergegeven en aanbevelingen voor verder onderzoek zijn geformuleerd.