Voorwoord

Differentiaalvergelijkingen kennen hun toepassing in talrijke gebieden. Ze worden gebruikt als modellen om bepaalde verschijnselen te beschrijven. In de literatuur vindt men hier allerhande voorbeelden van terug. Zogenaamde oscillerende elektrische circuits, die gebruikt worden in televisies, radio's, radars en andere elektronische apparaten kunnen bestudeerd worden door middel van differentiaalvergelijkingen. Maar ook de bevolkingsgroei kan gemodelleerd worden via een differentiaalvergelijking.

Deze thesis handelt over polynomiale Liénard vergelijkingen, die we kortweg Liénard vergelijkingen zullen noemen. Liénard vergelijkingen zijn een bijzondere vorm van tweede orde gewone differentiaalvergelijkingen. Tweede orde gewone differentiaalvergelijkingen zijn wiskundige vergelijkingen waaraan een bepaalde functie voldoet die naast de functie zelf ook de eerste en tweede afgeleide van de functie bevatten.

Laat ons, ter verduidelijking, de begrippen functie en de eerste en tweede afgeleide van een functie nader verklaren. Een functie drukt in de wiskunde een afhankelijkheid uit van één element van een ander. Zo is bijvoorbeeld $f(x) = 2x$ een functie die aan elk reël getal x, het dubbele van dit getal $2x$ relateert. Iedere functie f heeft een visuele voorstelling in het vlak, ook wel grafiek genoemd. Deze grafiek kan men verkrijgen door de koppels $(x, f(x))$ uit te zetten in een rechthoekig assenstelsel. Een rechthoekig assenstelsel bestaat uit een horizontale as, de x–as, en een verticale as, de y–as. In ieder punt x bestaat er een raaklijn rakend aan de grafiek van f. De richtingscoëfficient, of helling, van deze raaklijn wordt de afgeleide van f in x genoemd en genoteerd als $f'(x)$. De tweede afgeleide van f is gegeven door de afgeleide van de functie f'. Deze wordt genoteerd als f''.

Laten we, bij wijze van voorbeeld, het systeem van de zogenaamde gedempte oscillator omschrijven d.m.v. een differentiaalvergelijking. Het systeem van de gedempte oscillator ziet er als volgt uit. Beschouw een deeltje met massa m, bevestigd aan het uiteinde van een veer. Het andere uiteinde van de veer is bevestigd aan een vast punt. Heel het systeem wordt opgezet in een bepaalde vloeistof. Wanneer het deeltje uit zijn evenwichtspositie wordt gehaald, oefent het een oscillerende op– en neerwaartse beweging uit. Noteer $f(t)$ als de tijdsafhankelijke uitwijking van de massa t.o.v. zijn evenwichtspositie.

De differentiaalvergelijking waaraan $f(t)$ voldoet, bekomen we door het toepassen van de tweede wet van Newton: $F_{tot} = mf''(t)$, waarbij F_{tot} de totale kracht is die het deeltje ondervindt. Deze totale kracht is hier gegeven door de som van twee krachten werkende op het deeltje. Enerszijds is er de terugroepende kracht uitgeoefend door de veer en gegeven door $-kf(t)$ waarbij k de zogenaamde veerconstante van de veer voorstelt. Anderszijds is er de wrijvingskracht veroorzaakt door de vloeistof. Deze kracht is tegengesteld aan de snelheid $f'(t)$ van het deeltje en gegeven door $-\lambda f'(t)$ waarbij λ een constante is afhankelijk van de aard van de vloeistof. Zodus bekomen we als totale kracht $F_{tot} = -\lambda f'(t) - kf(t)$ zodat uit de tweede wet van Newton volgt dat $-\lambda f'(t) - kf(t) = mf''(t)$ of nog:

$$f''(t) + \frac{\lambda}{m} f'(t) + \frac{k}{m} f(t) = 0.$$

Bovenstaande differentiaalvergelijking is in het bijzonder gegeven door een Liénard vergelijking en is afhankelijk van een zogenaamde parameter λ. Afhankelijk van in welke vloeistof het systeem wordt opgezet, zal λ een andere waarde aannemen. Stellen we het systeem op in het luchtledige, dan ondervindt de massa m geen enkele wrijving en kan λ als nul beschouwd worden.

Een oplossing van een differentiaalvergelijking is niet uniek. Afhankelijk van de begintoestand van het systeem krijgt men steeds een andere oplossing. Deze begintoestand kan beschreven worden met zogenaamde beginvoorwaarden. Indien men in het bovenstaande voorbeeld als begintoestand, de toestand kiest waarin de veer niet is uitgerokken, dan zal het deeltje stil blijven hangen. De bijbehorende oplossing is gegeven door de functie $f = 0$ en wordt ook wel een singulariteit van het systeem genoemd. In het luchtledige echter (dus $\lambda = 0$) zal, afhankelijk van de mate waarin men het deeltje uit zijn evenwichtspositie brengt, het deeltje een periodieke op– en neerwaartse beweging uitoefenen corresponderend met een periodieke oplossing f.

Men noemt een periodieke oplossing geïsoleerd wanneer de meest geringe aanpassing van de begintoestand van het systeem ervoor zorgt dat de bijbehorende oplossing niet meer periodiek is. Een geïsoleerde periodieke oplosssing wordt ook wel een limietcyclus genoemd. Men kan zich nu de vraag stellen hoeveel van deze zogenaamde limietcycli er maximaal mogelijk zijn in een Liénard vergelijking? Deze vraag stelde de beroemde Duitse wiskundige David Hilbert(1862-1943) zich ook, weliswaar op een wat algemenere manier waar we nu niet verder op ingaan. Wiskundigen refereren naar dit probleem als Hilberts 16e probleem en zijn al meer dan 100 jaar op zoek naar de oplossing hiervan.

In een poging tot het zoeken naar een oplossing van dit probleem is een kwalitatieve studie van de vergelijking onontbeerlijk. Hierin gaat men niet op zoek naar een expliciet voorschrift van de functies die voldoen aan een bepaalde differentiaalvergelijking, maar tracht men, met allerhande wiskundige technieken, de meest essentiële aspecten van de oplossingen te achterhalen.

Een belangrijk aspect in het onderzoek i.v.m. Hilberts 16e probleem zijn de limiet periodieke verzamelingen. Differentiaalvergelijkingen die afhangen van een parameter kunnen beschouwd worden als een familie van differentiaalvergelijkingen, waarbij elk lid van de familie correspondeert met een welbepaalde parameterwaarde. Een limiet periodieke verzameling van deze familie is dan een unie van oplossingen die kan voorkomen als een limiet van geïsoleerde periodieke oplossingen binnen deze familie. In deze thesis beschrijven we alle mogelijke limiet periodieke verzamelingen die kunnen voorkomen in een Liénard vergelijking en bestuderen we er enkele van om zodoende een stapje dichter te komen in de richting van de oplossing van Hilberts 16e probleem.

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First of all, I wish to express my gratitude to my supervisor, Prof. dr. Freddy Dumortier, for guiding me through the world of dynamical systems during the writing of this work. I would also like to thank Prof. dr. Leopold Verstraelen for the encouragement when I started my PhD. Also many thanks to the members of the jury for reading everything thoroughly.

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Stijn Luca,
December 2007

Contents

Introduction

In 1928, the French physicist A. Liénard studied differential equations of the following form [22]:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1}$$

These so–called *Liénard equations* (1) appear to be very useful to model oscillating behaviour. As an illustration of such an oscillating behaviour in an electrical circuit, one can consider the famous example of Van der Pol's equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \tag{2}$$

where μ is some constant. It is well known that when $\mu < 0$, the system will be damped; all solutions will tend asymptotically to zero as $t \to \infty$. This corresponds to an electrical circuit that is dead, i.e. after some period of time all the currents and voltages will approach 0. But as μ crosses zero, the circuit becomes alive. For $\mu > 0$, there exists an unique periodic solution to which every nontrivial solution tends as $t \to \infty$. The circuit oscillates.

Physically $f(x)\,\dot{x}$ in (1) should be regarded as a friction or resistance, that opposes the motion induced by a force $g(x)$. Equation (1) can be written in the phase plane as:

$$\left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & -g(x) - f(x)y, \end{array} \right. \tag{3}$$

where $f(x)$ is referred to as the *friction term* and $g(x)$ as the *forcing term*.

Often it can be convenient to study system (3) in the so–called *Liénard plane*. Herefore, one applies the transformation

$$\overline{y} = y + F(x), \tag{4}$$

with $F(x) = -\int_0^x f(u)\,du$ to obtain the system:

$$\left\{\begin{array}{rcl} \dot{x} & = & \overline{y} - F(x), \\ \dot{\overline{y}} & = & -g(x). \end{array}\right. \tag{5}$$

In this thesis, we study *polynomial Liénard systems of type* (m, n); (m, n) arbitrarily but fixed:

$$X : \left\{\begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & P(x) + yQ(x), \end{array}\right. \tag{6}$$

where P and Q are polynomials of respective precise degrees m and n. Shortly, we will refer to these systems as Liénard systems (of a certain type (m, n)). Using linear dilatations in x, y and t we can assume that P and Q are given by

$$P(x) = -(Ax^m + \sum_{i=0}^{m-1} a_i x^i), \qquad Q(x) = -(x^n + \sum_{i=0}^{n-1} b_i x^i), \tag{7}$$

with $A = 1$ or -1, if $m \neq 2n+1$, and $A \in \mathbb{R}\backslash\{0\}$, if $m = 2n+1$.

To study the behaviour near infinity we use a *Poincaré–Lyapunov compactification*. The phase plane will be compactified in the so–called *Poincaré–Lyapunov disc*. For a general introduction to as well the Poincaré compactification as the Poincaré–Lyapunov compactification, we refer to [12] and [14].

Using an appropriate quasi–homogeneous compactification, the singularities at infinity will be of a rather simple nature. We use a Poincaré–Lyapunov compactification of type $(\alpha, \beta) \in \mathbb{N}_1 \times \mathbb{N}_1$ to compactify the phase space in the Poincaré–Lyapunov disc of degree (α, β) [12]. This means that, near infinity, we set:

$$x = \frac{\cos\theta}{v^\alpha}, \quad y = \frac{\sin\theta}{v^\beta},$$

with $\theta \in \mathbb{S}^1$, and multiply the obtained vector field by a factor v^d to extend the obtained vector field to the Poincaré–Lyapunov disc. Depending on (m, n) the degree (α, β) and the power d in the factor v^d is chosen to be:

1. $(\alpha, \beta) = (1, n + 1)$, $d = n$, when $m \leq 2n + 1$,
2. $(\alpha, \beta) = (2, m + 1)$, $d = m - 1$, when $m > 2n + 1$ and m is even,
3. $(\alpha, \beta) = (1, \frac{1}{2}(m + 1))$, $d = (m - 1)/2$, when $m > 2n + 1$ and m is odd.

The compactification, indicated in the previous list, is called the *appropriate compactification* for the respective Liénard system of type (m, n) [10]. The associated disc is called the appropriate Poincaré–Lyapunov disc and is denoted by $D^{(m,n)}$, or shortly D. We endow D with the usual topology of a disc. The vector field on $D^{(m,n)}$ obtained from X, after such an appropriate Poincaré–Lyapunov compactification and multiplication by a factor v^d, is denoted as $\overline{X}$. The study near infinity of $\overline{X}$ can also be done by means of different charts.

Denote by $L^{(m,n)}(D)$, the space of all vector fields $\overline{X}$ on $D^{(m,n)}$ obtained, after an appropriate Poincaré–Lyapunov compactification, from a Liénard system X of type (m, n) with P and Q as in (7). The systems obtained from a system (6) with P or Q zero can be interpreted as lying on the boundary of this space of Liénard systems. These system are of Hamiltonian or singular type [10].

In the first two chapters, we try to get a better idea of what is going on in Liénard systems. In Chapter 1, we study the *singularities* of a Liénard system $\overline{X} \in L^{(m,n)}(D)$. A complete classification of them is made relying on [12] and [14].

For a given degree N, *Hilbert's 16th problem* asks for a maximum number of limit cycles that an Nth degree polynomial vector field in the plane can have. Using the localisation method of Roussarie [27], the proof of the existence of such a finite upperbound can be reduced to the *finite cyclicity* of any limit periodic set occuring in the flow of a polynomial vector field of degree N. Concerning Hilbert's 16th problem for polynomial Liénard systems, one uses herefore an appropriate Poincaré–Lyapunov compactificiation of the phase plane [12] together with a compactification of the chosen space of Liénard systems [10]. An attempt to solve the existence part of Hilbert's 16th problem for polynomial Liénard systems of a given type (m, n) could hence start by listing all limit periodic sets occuring in Liénard systems of type (m, n).

Therefore in Chapter 2, we describe all possible *limit periodic sets* occuring in a Liénard system $\overline{X} \in L^{(m,n)}(D)$. We also give necessary conditions on (m, n) for a limit periodic set to occur in $\overline{X} \in L^{(m,n)}(D)$.

A major advantage of Liénard systems is the simplicity. Proofs or calculations related to general systems often simplify considerably in the context of Liénard systems. This is already used in Chapters 1 and 2, but also in Chapter 3, we illustrate this by proving some results concerning *structural stability* of Liénard systems in the respective spaces $L^{(m,n)}(D)$.

When dealing with cyclicity problems, one often uses *normal forms*, near singularities of a vector field, to simplify calculations. Chapter 4 deals with some normal forms that we will need in the chapters that follow.

Chapter 5 concerns a search for so–called *alien limit cycles* inside an unfolding of a Hamiltonian 2–saddle cycle. The theory behind this search has already been presented in [16]. In view of applying the theory to specific systems, we add in Chapter 5 some interesting formulas enabling to check in practice the conditions given in [16] to track such alien limit cycles. Like for the results on normal forms, also this Chapter is worked out in a way that it can be applied to more general systems than Liénard systems.

Chapter 6 deals with *a cyclicity problem of an unbounded semi–hyperbolic 2–saddle cycle* $\mathcal{L}_0$, i.e. a 2–saddle cycle consisting of two semi–hyperbolic saddles at infinity and including one connection that lies at infinity. In this chapter we generalize results that have been obtained in [5].

In Chapter 7, we describe all possible boundaries of any *period annulus* occuring in a Liénard system, the bounded ones as well as the ones that extend to infinity.

Chapter 1

Singularities in Liénard systems

Let $\overline{X} \in L^{(m,n)}(D)$, as in the introduction obtained after an appropriate Poincaré–Lyapunov compactification from a Liénard system X of type (m,n):

$$X : \left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & P(x) + yQ(x), \end{array} \right. \tag{1.1}$$

where P and Q are given by

$$P(x) = -(Ax^m + \sum_{i=0}^{m-1} a_i x^i), \qquad Q(x) = -(x^n + \sum_{i=0}^{n-1} b_i x^i), \tag{1.2}$$

with $A = 1$ or -1, if $m \neq 2n+1$, and $A \in \mathbb{R}\backslash\{0\}$, if $m = 2n+1$.

In this chapter we want to describe all possible singularities of $\overline{X}$. The study of the singularities at infinity is already done in [12]. For sake of completeness and for later use, we recall the results. Figures 1.1, 1.2, 1.3 and 1.4 show the behaviour near infinity of $\overline{X}$ in the three cases: $m < 2n+1$, $m = 2n+1$ and $m > 2n+1$. Note that the behaviour near infinity only depends on the degree (m,n) and the coefficient A. In these pictures simple arrows on different curves near a singularity indicate that the singularity is hyperbolic, while for semi–hyperbolic singularities we use simple arrows for the centre behaviour and double arrows for the hyperbolic behaviour. Other singularities do not occur in the appropriate Poincaré–Lyapunov compactification.

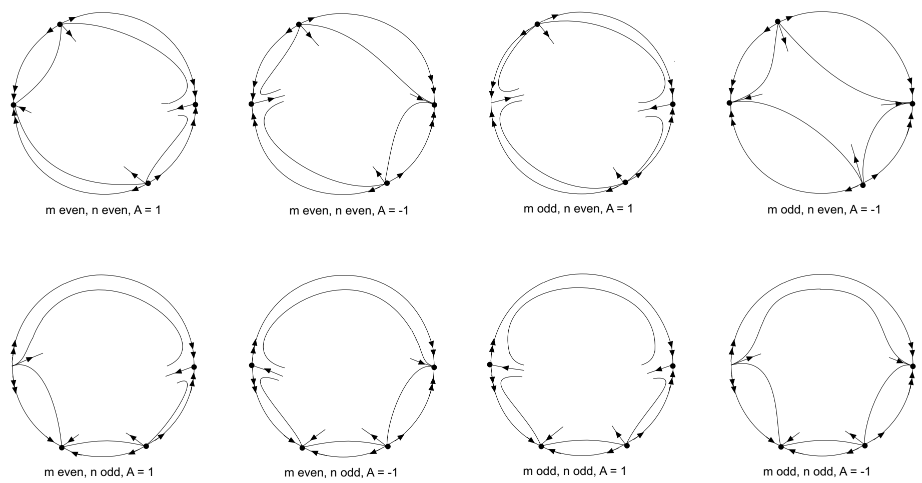

Figure 1.1: Behaviour near infinity for $m < 2n+1$ on the Poincaré–Lyapunov disc of degree $(1, n+1)$.

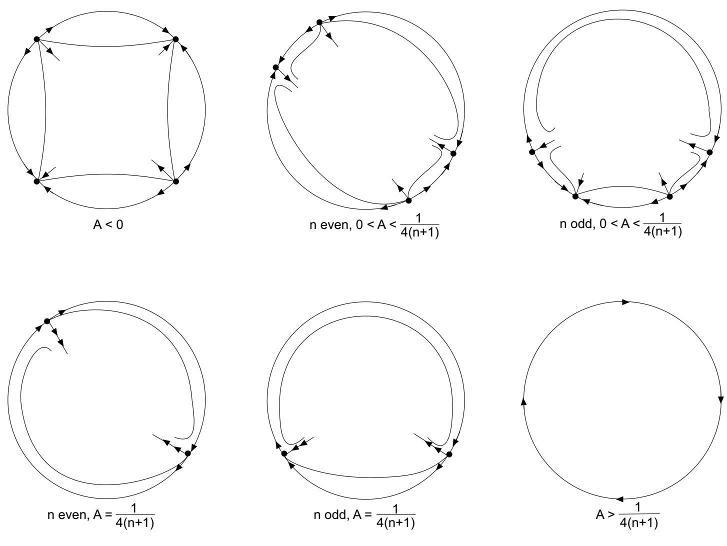

Figure 1.2: Behaviour near infinity for $m = 2n+1$ on the Poincaré–Lyapunov disc of degree $(1, n+1)$.

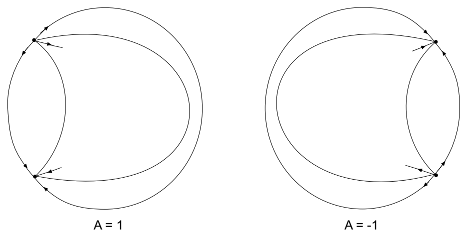

Figure 1.3: Behaviour near infinity for $m > 2n+1$, m even, on the Poincaré–Lyapunov disc of degree $(2, m+1)$.

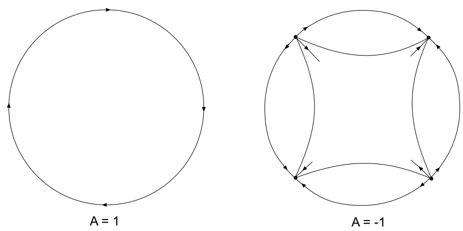

Figure 1.4: Behaviour near infinity for $m > 2n+1$, m odd, on the Poincaré–Lyapunov disc of degree $(1, \frac{1}{2}(m+1))$.

The study of the singularities of $\overline{X}$ situated in the interior of $D^{(m,n)}$ is equivalent to the study of the singularities of the original Liénard system X of type (m, n). Because $\dot{x} = y$ we say that the flow of X points to the right above the x–axis and to the left below the x–axis. Regular orbits of X will intersect the x–axis transversally.

A singularity s of X is given by a point $s = (x_0, 0)$ with $P(x_0) = 0$. The linear part of X at s is equal to

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ P'(x_0) & Q(x_0) \end{pmatrix}, \tag{1.3}$$

with eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(Q(x_0) \pm \sqrt{Q(x_0)^2 + 4P'(x_0)}\right)$ and corresponding eigenspaces V_1 and V_2 spanned by respectively $(1, \lambda_1)$ and $(1, \lambda_2)$.

Moreover $\lambda_1\lambda_2 = -P'(x_0)$. Further we write:

$$\begin{aligned} P(x) &= a(x-x_0)^k + o(x-x_0)^k, \\ Q(x) &= b(x-x_0)^l + o(x-x_0)^l, \end{aligned} \tag{1.4}$$

where $ab \neq 0$ and with $(k,l) \in \mathbb{N}^2$, $1 \leq k \leq m$ and $0 \leq l \leq n$. We call (k,l) *the Liénard degree of the singularity* s.

The propositions below describe the singularities of a Liénard system (1.1) with respect to their Liénard degree and the coefficients (a,b). We will present all possible phase portraits of the singularities of a Liénard system. A singulariy s always lies on the x–axis. In case s is not a focus nor a center, a dotted line indicates the direction in which orbits can approach the singularity. Notice that in presenting the phase portraits, it is not possible to stress out the exact position of the orbits, nor their exact contact with the directions in which they approach s.

Proposition 1.1 *If $k = 1$ and:*

1. *if $a > 0$, s is a hyperbolic saddle as in Figure 1.5 (a).*
2. *if $a < 0$, $l = 0$ and $b^2 + 4a \geq 0$, then s is a stable node when $b < 0$ and an unstable node when $b > 0$. When $b^2 + 4a > 0$, all orbits except two will approach the singularity along the eigenspace that corresponds with the eigenvalue that is biggest in absolute value (see Figure 1.5, (b) and (c)). In case $b^2+4a = 0$ all orbits will approach s along the unique eigenspace (see Figure 1.5, (d) and 1.5 (e)). In particular s cannot be a star node.*
3. *if $a < 0$, $l = 0$ and $b^2 + 4a < 0$, then s is a stable focus when $b < 0$ and an unstable focus when $b > 0$ (see Figure 1.5, (f) and (g)).*
4. *if $a < 0$ and $l > 0$, then s is linearly a center. The singularity is a focus or a center (see Figure 1.5 (h)).*

Proof: Because X is analytic, s is a saddle, node or focus when it is for the linear part $\mathcal{A}$. When s is a node, one easily verifies that $\mathcal{A}$ is diagonalizable if and only if the eigenvalues are different such that a star

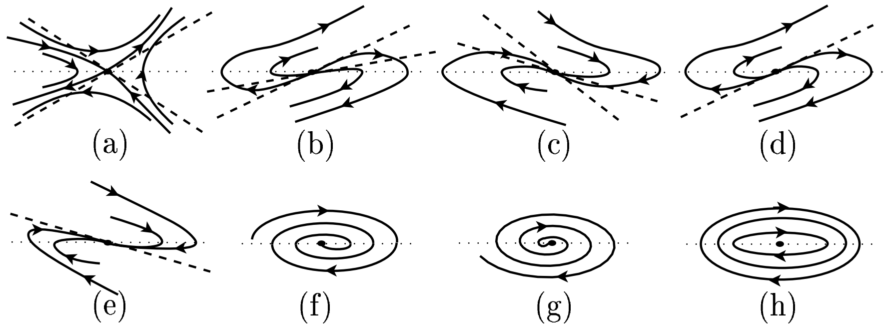

Figure 1.5: Phase portraits of hyperbolic singularities and linear centers of a Liénard system.

node is out of the question. The directions in which the orbits approach the singularity can be found by blowing up, see for instance [25].In case $\mathcal{A}$ has a center in s, it can be a center or focus for X. We are then left with the well–known center–focus problem.

The phase portraits are obtained by using the fact that the flow of X points to the right above the x–axis and to the left below the x–axis. In case of a saddle or node, no eigenspace coincides with the x–axis because $\lambda_1\lambda_2 \neq 0$. Note also that the y–axis can not be an eigenspace. When s is a saddle, the stable manifold theorem ensures the existence of C^ω–invariant manifolds tangent to the eigenspaces $y = \lambda_1(x - x_0)$ and $y = \lambda_2(x - x_0)$. $\square$

Proposition 1.2 *The singularity s of X is semi–hyperbolic if and only if $k \geq 2$ and $l = 0$. Every center manifold has at s a contact of order $k-1$ with the x–axis and on it, the vector field behaves like*

$$\dot{x} = -\frac{a}{b}(x - x_0)^k + o(x - x_0)^k.$$

Depending on whether b is positive or negative there is an unstable respectively stable manifold tangent to $y = b(x - x_0)$. Denoting $Q(x) - b = c(x - x_0)^p + o(x - x_0)^p$ for some $p \geq 1$, the (un)stable manifold has a contact at s of order $\min\{k-1, p\}$ *with $y = b(x - x_0)$. Furthermore:*

1. *if k is odd and $a > 0$ the singularity is of saddle–type (Figure 1.6, (a) and (b)).*

2. *if k is odd and $a < 0$ the singularity is a stable node when $b > 0$ and an unstable node when $b < 0$ (Figure 1.6, (c) and (d)).*

3. *if k is even the singularity is a saddle–node (Figure 1.6, (e), (f), (g) and (h)).*

Moreover orbits not lying on the (un)stable manifold and adherent to s belong to a center manifold, having at s a contact of order $k-1$ with the x–axis.

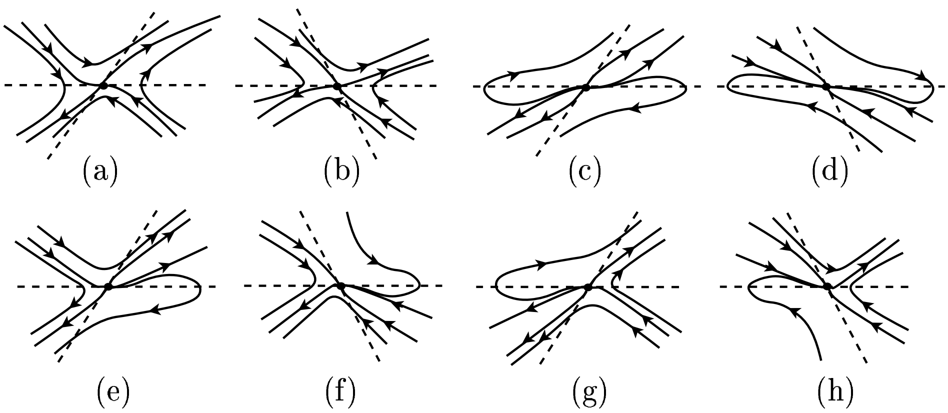

Figure 1.6: Phase portraits of semi–hyperbolic singularities of a Liénard system.

Proof: The singularity is semi–hyperbolic if and only if $\lambda_1\lambda_2 = 0$, $\lambda_1 + \lambda_2 \neq 0$ which is equivalent to $P'(x_0) = 0$, $Q(x_0) = b \neq 0$. The linear part of X at s is given by:

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & b \end{pmatrix}$$

with eigenvalues $0, b$ and respective eigenspaces $y = 0$ and $y = b(x - x_0)$.

The center manifold theorem ensures the existence of at least one C^∞

manifold $y = c(x)$ tangent to the x–axis and invariant under the flow of X (see e.g. [3] or [14]), i.e.

$$c'(x)c(x) = P(x) + c(x)Q(x) \quad \text{or} \quad c(x)(-b + O(x - x_0)) = P(x)$$

such that $P(x) = O(x - x_0)^k$ implies $c(x) = O(x - x_0)^k$. Comparing coefficients one sees that

$$c(x) = -\frac{a}{b}(x - x_0)^k + o(x - x_0)^k.$$

Suppose now $y = s(x)$ is the (unique) invariant manifold tangent to the (un)stable direction $y = b(x - x_0)$. Just as before the invariance implies

$$s(x)(s'(x) - Q(x)) = P(x). \tag{1.5}$$

Substituting $Q(x) = b + O(x - x_0)^p$ and $s(x) = b(x - x_0) + O(x - x_0)^2$ one obtains

$$(b + O(x - x_0))(s'(x) - b) = O(x - x_0)^{k-1} + O(x - x_0)^p,$$

such that $s'(x) - b = O(x - x_0)^{\min\{k-1,p\}}$, meaning that at x_0 the contact between $y = b(x - x_0)$ and $y = s(x)$ is of order $\min\{k - 1, p\}$.

If γ is an orbit not lying on the unique (un)stable manifold and it approaches s, then we know (e.g by [14]) that it lies on a center manifold. Because all center manifolds are mutually infinitely tangent [14], γ has to have at s a contact of order $k - 1$ with the x–axis.

The phase portraits are easily obtained using the fact that $\dot{x} = y$. □

Proposition 1.3 *The singularity s of X is nilpotent if and only if $k \geq 2$ and $l \geq 1$. Moreover:*

1. *if $k = 2p$ is even and $k < 2l + 1$, the singularity is a cusp for which the separatrices have at s a contact of order p (Figure 1.7, (a) and (b)).*

2. *if k is even and $k > 2l + 1$, the singularity is a saddle–node. Orbits can only approach s along the x–axis having at s mutual contact of order l (Figure 1.7, (c), (d), (e) and (f)).*

3. *if $k = 2p + 1$ is odd and $a > 0$, then the singularity is a saddle where the separatrices are tangent to the x–axis having at s mutual contact of order p when $k \leq 2l + 1$ and of order l when $k > 2l + 1$ (Figure 1.7 (g)).*

4. *if k is odd and $a < 0$ and:*

   (a) *if $k < 2l + 1$, or $k = 2l + 1$ and $b^2 + 4a(l + 1) < 0$, then the singularity is a focus or center (Figure 1.7, (h), (i) and (j)),*

   (b) *if l is odd and either $k > 2l+1$, or $k = 2l+1$ and $b^2+4a(l+1) \geq 0$, then s is a singularity with one elliptic and one hyperbolic sector where orbits can only approach s along the x–axis having at s mutual contact of order l (Figure 1.7, (k) and (l)),*

   (c) *if l is even and either $k > 2l+1$, or $k = 2l+1$ and $b^2+4a(l+1) \geq 0$ then s is an attractive node when $b < 0$ and a repelling node when $b > 0$. All orbits will approach s along the x–axis having at s mutual contact of order l (Figure 1.7, (m) and (n)).*

Proof: If $Q(x_0) = 0$ and $P'(x_0) = 0$, we see immediately that the linear part of X is nilpotent.

The study of the singularity relies on quasi–homogeneous blow up. In case $k \leq 2l + 1$ and k is odd, one uses the blow up

$$x = u, \quad y = u^q \overline{y}, \tag{1.6}$$

with $q = \frac{k+1}{2}$ and we divide by u^{q-1}. If k is even, one uses:

$$x = u^2, \quad y = u^{k+1} \overline{y}, \tag{1.7}$$

and we divide by u^{k-1}. In case $k > 2l + 1$, one uses the blow up:

$$x = u, \quad y = u^{l+1} \overline{y}, \tag{1.8}$$

and we divide by u^l.

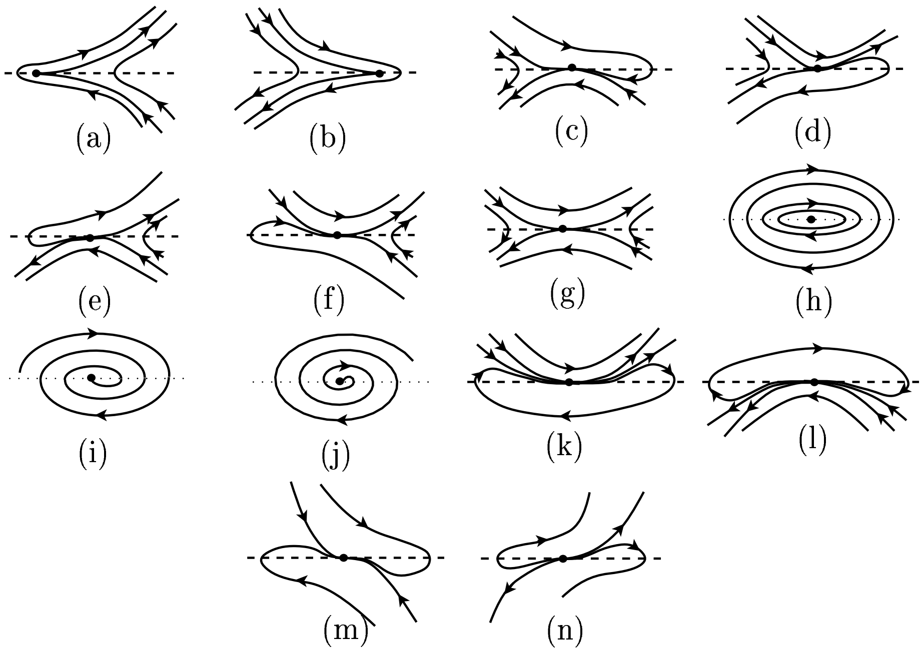

Figure 1.7: Phase portraits of nilpotent singularities of a Liénard system.

In the directional charts $\{\overline{y} = \pm 1\}$, one won't find any singularity in the origin. We can therefore restrict our study in the $\{u = \pm 1\}$–charts. However using the transformations (1.6) and (1.7), the information found in the $\{u = 1\}$–chart also covers the information in the $\{u = -1\}$–chart. We refer to [14] where all calculations are present.

Let us clarify the statements about the contacts. In blow–up coordinates the $\overline{y}$–axis is invariant under the flow and corresponds to the origin in the original coordinates. Suppose $y = y(x)$ is an orbit that approaches the singularity s, then this orbit will correspond to an orbit $\overline{y} = \overline{y}(u)$ not lying on the $\overline{y}$–axis that approaches a singularity on the $\overline{y}$–axis.

If $k \leq 2l + 1$, one finds no singularities at the origin after blow up [14]. Therefore $\overline{y}(u) = \alpha + o(1), u \to 0$ with $\alpha \neq 0$. After blowing down, we get $y(x) = \alpha x^{\frac{k+1}{2}} + o(x^{\frac{k+1}{2}}), x \to 0$.

In case $k > 2l + 1$ one finds, after blowing up, two singularities on the $\overline{y}$–axis [14]. The origin is a semi–hyperbolic singularity with a unique (un)stable manifold lying on the $\overline{y}$–axis. All other orbits are given by $\overline{y}(u) = o(u^{k-2l-2})$, $u \to 0$. Blowing down gives $y(x) = o(x^{k-l-1}), x \to 0$. The other singularity is a saddle having a unique invariant manifold, transverse to the $\overline{y}$–axis, given by $\overline{y}(u) = \alpha + o(1), u \to 0$ with $\alpha \neq 0$. Blowing down yields $y(x) = o(x^l), x \to 0$. Because $k > 2l + 1$, we get certainly $y(x) = o(x^l), x \to 0$ in both cases. □

The previous propositions classify the singularities of Liénard systems (1.1) according to their Liénard degree (k, l) and the coefficients (a, b), defined in (1.4). It is clear that a singularity of Liénard degree (k, l) can only occur in a Liénard system of type (m, n) when $m \geq k$ and $n \geq l$. As a direct consequence of the former propositions, we can state the following theorem that presents all possible singularities of a Liénard system of type $(m, n) \in \mathbb{N}^2$.

Theorem 1.4 *A Liénard system X of type (m, n) as defined in (1.1) has at most m singularities, where we count a singularity $(x_0, 0)$, $k+1$ times when $P(x_0) = P'(x_0) = \cdots = P^{(k)}(x_0) = 0$. Denote by s a singularity of X, then:*

1. *s can be a hyperbolic singularity: a saddle, a node (no star node) or a focus, see Figure 1.5, (a)–(g). Near a node, all orbits approach s along one of the eigenspaces of the linear part of X at s.*

2. *when $n \geq 1$, s can be a linear center, i.e. a center or a focus, see Figure 1.5, (f), (g) or (h).*

3. *if $m \geq 2$, s can be a saddle–node. When $m \geq 3$, s is possibly a semi–hyperbolic node or a semi–hyperbolic saddle. The unique (un)stable manifold intersects the x–axis transversally. The center manifolds will have at the singularity a contact of order at most $m - 1$ with the x–axis. See Figure 1.6.*

4. *if $m \geq 2$, $n \geq 1$, s can be a cusp, when $m \geq 3$, $n \geq 1$, s can be a nilpotent focus, center, saddle or a singularity with an elliptic sector, if $m \geq 4$, $n \geq 1$, s can be a nilpotent saddle–node, when*

$m \geq 5$, $n \geq 2$, s can be a nilpotent node. In case s is not a center nor a focus, orbits can only approach the singularity s along the x–axis. See Figure 1.7.

Let us now approach the former results from a different point of view that will enable us to say something more about possible successions on the x–axis of singularities of X.

A singularity $s = (x_0, 0)$ induces a local behaviour on the vector field X restricted to the x–axis. Because $X(x, 0) = (0, P(x))$ this behaviour is completely determined by the local behaviour of $P(x)$ at the zero x_0. In particular according to the sign of P in a right– and left semi–neighbourhood of x_0 we can distinguish four cases:

I. $P(x) > 0$ for $x < x_0$ and $P(x) > 0$ for $x > x_0$,

II. $P(x) < 0$ for $x < x_0$ and $P(x) < 0$ for $x > x_0$,

III. $P(x) < 0$ for $x < x_0$ and $P(x) > 0$ for $x > x_0$,

IV. $P(x) > 0$ for $x < x_0$ and $P(x) < 0$ for $x > x_0$.

We speak respectively of *singularities of up–up, down–down, down–up and up–down type.* We will now present all singularities of (1.1) according to their type. Of course again, we cannot pay attention to exact positions of orbits nor to exact contacts.

In cases I and II, the smoothness of P implies its tangency to the x–axis. Depending on the sign of the lowest order coefficient, the local behaviour of P looks like in Figure 1.8, (a) or (b). In particular P is of the form

$$P(x) = a(x - x_0)^{2k_0} + o(x - x_0)^{2k_0}, \quad a \neq 0, \quad k_0 \in \mathbb{N}_1$$

such that Propositions 1.2 and 1.3 imply that the singularity under consideration can only be a cusp or a saddle–node. In Figures 1.9 and 1.10, we present all singularities of up–up and down–down type.

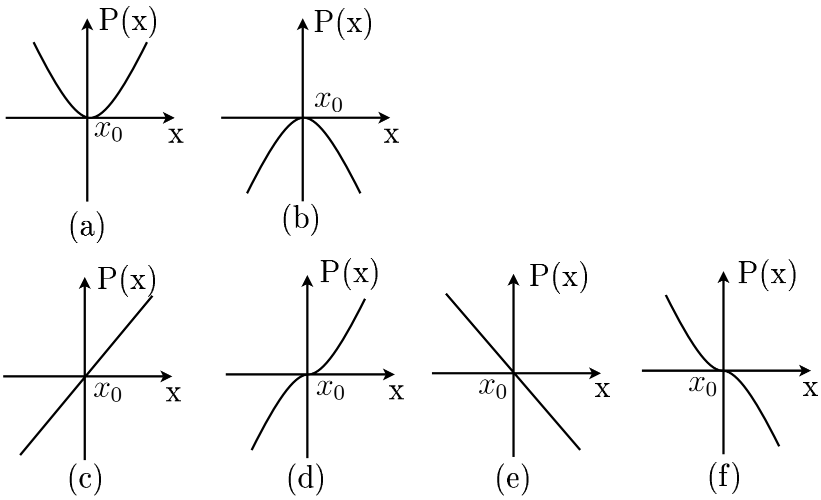

Figure 1.8: Types of local behaviour at x_0 of P.

In cases III and IV and when P cuts the x–axis transversally, the singularity is hyperbolic or a linear center, Figure 1.8, (c) and (e). In case P is not transverse to the x–axis, s is semi–hyperbolic or nilpotent, Figure 1.8, (d) and (f). In any case k has to be odd such that P is of the form

$$P(x) = a(x - x_0)^{2k_0+1} + o(x - x_0)^{2k_0+1}, \quad a \neq 0, \quad k_0 \in \mathbb{N}.$$

When $a > 0$, the singularity is of down–up type and according to Propositions 1.1, 1.2 and 1.3 it has to be a saddle. On the other hand when $a < 0$, the singularity is of up–down type given by a node, focus, center or a singularity with one elliptic sector. Figures 1.11 and 1.12 present all singularities of down–up and up–down type.

Let us now consider possible successions of singularities on the x–axis. Suppose $s_1 = (x_1, 0)$ and $s_2 = (x_2, 0)$ are two successive singularities such that $x_1 < x_2$. Then P cannot change sign in $]x_1, x_2[$ because of the intermediate value theorem. One can easily prove the following proposition (where $*$ denotes "down" or "up").

Proposition 1.5 *Suppose $s_1 = (x_1, 0)$ and $s_2 = (x_2, 0)$ are two singularities of a Liénard system such that $x_1 < x_2$, then:*

1. *if s_1 is of $*$–up type and s_2 is of down–$*$ type, then there is a singularity of up–down type situated on the x–axis between s_1 and s_2.*

2. *if s_1 is of $*$-down type and s_2 of up–$*$ type, then there is a singularity of down–up type situated on the x–axis between s_1 and s_2.*

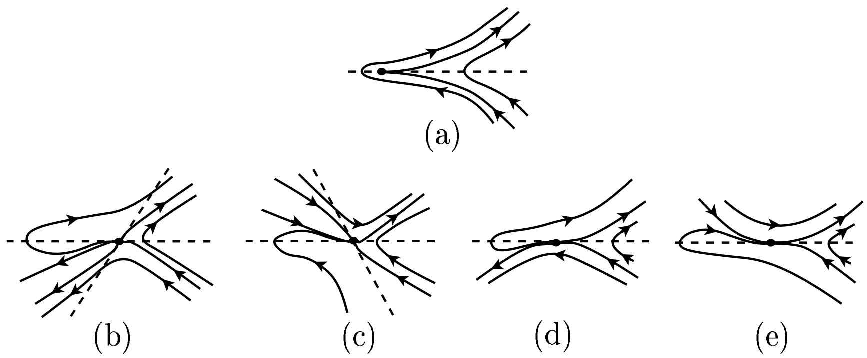

Figure 1.9: Singularities of up–up type: a cusp of up–up type (a) and saddle–nodes of up–up type; (b) and (c) are semi–hyperbolic saddle-nodes while (d) and (e) are nilpotent saddle-nodes.

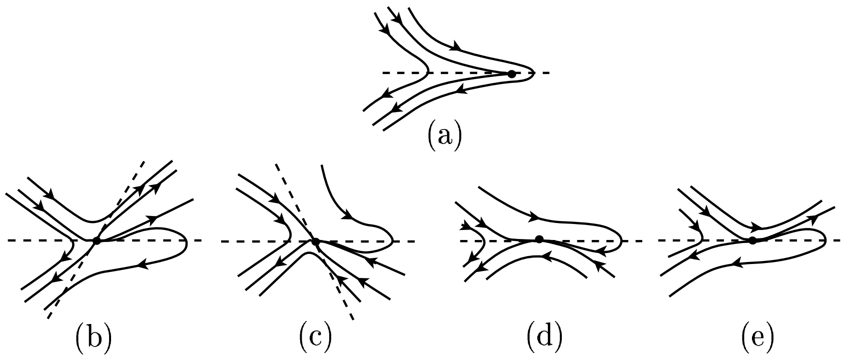

Figure 1.10: Singularities of down–down type: a cusp of down–down type (a) and saddle–nodes of down–down type; (b) and (c) are semi–hyperbolic saddle-nodes while (d) and (e) are nilpotent saddle–nodes.

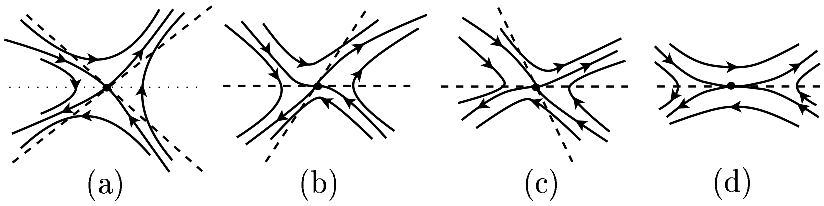

Figure 1.11: Singularities of down–up type are all given by saddles; (a) a linear saddle, (b) and (c) semi–hyperbolic saddles, (d) a nilpotent saddle.

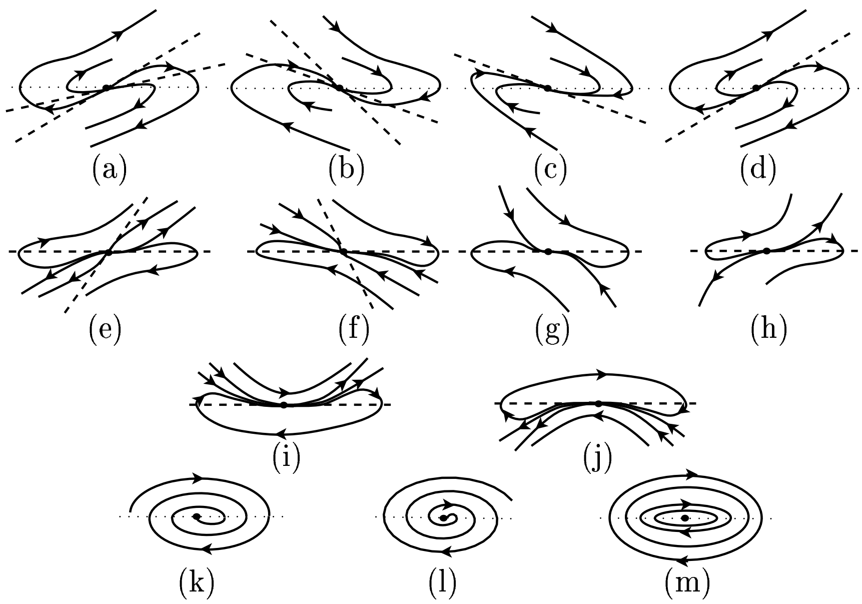

Figure 1.12: Singularities of up–down type. Nodes of up–down type can be hyperbolic (a),(b),(c) or (d) semi–hyperbolic (e),(f) or nilpotent (g),(h). (i) and (j) present the singularities with one elliptic and one hyperbolic sector. Foci, (k) and (l), can be nilpotent or hyperbolic. (m) presents a center.

Chapter 2

Limit periodic sets in Liénard systems

After a study of all possible singularities occuring in a Liénard system $\overline{X} \in L^{(m,n)}(D)$, we are able to describe all possible limit periodic sets occuring in $\overline{X}$. The other way around, we also describe necessary conditions on (m,n) for a certain limit periodic set to occur in $\overline{X}$.

For being able to treat limit periodic sets (and later on structurally stability) we will endow $L^{(m,,n)}(D)$ with a topology. We endow $L^{(m,n)}(D)$ with *the coefficient topology*. This is the topology $\mathcal{T}^{(m,n)}$ such that the following map is a homeomorphism:

$$\begin{array}{rlcl} \pi : & (L^{(m,n)}(D), \mathcal{T}^{(m,n)}) & \mapsto & (\mathcal{B}, \mathcal{U}) \\ & \overline{X} & \mapsto & (A, a_{m-1}, \ldots, a_0, b_{n-1}, \ldots, b_0), \end{array}$$

associating to each $\overline{X} \in L^{(m,n)}(D)$, the coefficients

$$(A, a_{m-1}, \ldots, a_0, b_{n-1}, \ldots, b_0)$$

of the polynomials P and Q of the original system X (1.1) and where $\mathcal{B}$ is given by $\{-1, 1\} \times \mathbb{R}^{n+m}$, when $m \neq 2n+1$, and $\mathbb{R} \backslash \{0\} \times \mathbb{R}^{n+m}$, when $m = 2n+1$, equipped with the induced topology $\mathcal{U}$ from $\mathbb{R}^{m+n+1}$.

Define $\mathcal{H}(D)$ as the set of all non-empty compact subsets of D provided with *the Hausdorff metric*, i.e.

$$d_H(K_1, K_2) = \sup\{\sup_{x\in K_1} d(x, K_2), \sup_{y\in K_2} d(K_1, y)\}; \quad K_1, K_2 \in \mathcal{H}(\mathbb{R}^2),$$

where $d(a, K) = \inf_{x\in K}\{\rho(a, x)\}$, ρ the Euclidean metric on $\mathbb{R}^2$.

A compact subset $\mathcal{L}$ of D is said to be *a limit periodic set of* $\overline{X} \in L^{(m,n)}(D)$ if for each $\varepsilon > 0$ and for each neighbourhood V of $\overline{X}$, there exists an $\overline{Y} \in V$ whose flow contains a limit cycle γ such that $d_H(\gamma, \mathcal{L}) < \varepsilon$. Limit periodic sets were first introduced in [18] and further worked out in [28].

A limit periodic set can only be a singularity situated in $D\backslash\partial D$, a periodic orbit or a graphic of $\overline{X}$ [28]. *A graphic* of $\overline{X}$ consists of a finite number of singularities $s_1, \ldots, s_r$, not necessarily different, and a finite number of regular orbits $\gamma_1, \ldots, \gamma_r$ connecting them in the sense that for $i = 1, \ldots, r$: $\omega(\gamma_i) = s_{i+1}$ and $\alpha(\gamma_{i+1}) = s_i$, where $s_{r+1} = s_1$. We take care of taking r as small as possible avoiding useless coincidences in the graphic. A graphic that is homeomorphic with the unit circle S^1 is called *a cycle*. Depending on whether a limit periodic set, a periodic orbit or a graphic fits in $D\backslash\partial D$ or not, we call it *bounded* resp. *unbounded*.

A pathwise connected subset of a graphic constitutes *a (singular) connection* between two points A and B on $\mathcal{L}$. Depending on the sense of the flow of $\overline{X}$ we speak of a connection from A to B or from B to A.

Let us define a *passage* $\{\mathcal{O}_\omega, s, \mathcal{O}_\alpha\}$ at a singularity s of X_{λ_0} as the union of s together with a regular orbit $\mathcal{O}_\omega$ with s as ω-limit and a regular orbit $\mathcal{O}_\alpha$ with s as α-limit. It is clear that every singularity s of a graphic has to possess at least one passage.

Let $\varphi : A \subset [-1, 1] \mapsto D$ be a C^ω function on an open connected subset $A \subset [-1, 1]$. Here $[-1, 1]$ is endowed with the induced topology from $\mathbb{R}$. Take $\Sigma = \varphi(A)$ with the induced topology. If $\varphi : A \subset [-1, 1] \mapsto \Sigma$ is a homeomorphism such that for every $a \in A$, $\varphi'(a)$ and $\overline{X}(\varphi(a))$ are linearly independent, then Σ is called *a transverse section of* $\overline{X}$.

From the fact that $\dot{x} = y$ in a Liénard system, it is easily seen that for abritrarily x, the sets $\Sigma_x^+ = \{(x,y) \in D \mid y > 0\}$ and $\Sigma_x^- = \{(x,y) \in D \mid y < 0\}$ are transverse sections for any $\overline{X} \in L^{(m,n)}(D)$.

The following proposition is an easy consequence of the proof of the Poincaré–Bendixson theorem. It can be found in [28] but we repeat it here for sake of completeness.

Proposition 2.1 *Suppose* $\mathcal{L}$ *is a limit periodic set of* $\overline{X}$ *and* Σ *is a transverse section of* $\overline{X}$. *Then* Σ *intersects* $\mathcal{L}$ *in at most one point.*

Proof: By contraposition, suppose that a, b are two distinct points of $\mathcal{L} \cap \Sigma$. Consider a neighbourhood V of $\overline{X}$ such that Σ is a transverse section for each $\overline{Y} \in V$. Then for each $\varepsilon > 0$ there exists an $\overline{Y} \in V$ whose flow contains a limit cycle γ such that $d_H(\gamma, \mathcal{L}) < \varepsilon$. In particular, for ε small enough, one finds a γ that intersects Σ at two points α and β. However this will lead to a contradiction.

Consider therefore the segment $[\alpha, \beta]$ lying on Σ together with the arc $A = \{\gamma(t) \mid t \in [t_1, t_2]\}$ part of γ such that $\gamma(t_1) = \alpha$ and $\gamma(t_2) = \beta$. Because $J = A \cup [\alpha, \beta]$ forms a simple closed curve, the Jordan curve theorem insures us that J devides D into two connected components. However points $\gamma(t)$ with $t < t_1$ will be in another connected set as points with $t > t_2$. Suppose for instance that that points $\gamma(t)$ with $t < t_1$ are inside J and points $\gamma(t)$ with $t > t_1$ are outside J. Then the flow of $\overline{X}$ on $[\alpha, \beta]$ will be pointing outward the region enclosed by J, (indeed otherwise the mean value theorem would ensure us a point on $[\alpha, \beta]$ where $\overline{X}$ is tangent to Σ). Hence γ cannot be periodic. When points $\gamma(t)$ with $t < t_1$ are outside J, a similar argument can be used. □

2.1 Bounded limit periodic sets

Let us describe the bounded limit periodic sets $\mathcal{L}$ occuring in a $\overline{X} \in L^{(m,n)}(D)$ for certain $(m, n) \in \mathbb{N}^2$.

In Chapter 1, we have already desribed the non–hyperbolic singularities lying in $D\backslash\partial D$. If $\mathcal{L}$ is a bounded periodic orbit, it will be C^ω diffeomorphic to a circle cutting the x–axis transversally in two distinct points. Indeed this follows immediately from the fact that $\dot{x} = y$ in a Liénard system. Let us now study the bounded graphics.

We already know that all singularities of a bounded graphic have to lie on the x–axis possessing at least one passage $\{\mathcal{O}_\omega, s, \mathcal{O}_\alpha\}$. Locally both $\mathcal{O}_\alpha$ and $\mathcal{O}_\omega$ have the option to lie under or above the x–axis. We come to four different passages:

i. *a left–right–passage (lr–passage)*: when both $\mathcal{O}_\alpha$ and $\mathcal{O}_\omega$ lie above the x–axis,

ii. *a right–left–passage (rl-passage)*: when both $\mathcal{O}_\alpha$ and $\mathcal{O}_\omega$ lie under the x–axis,

iii. *an upgoing–passage*: when $\mathcal{O}_\alpha$ lies above and $\mathcal{O}_\omega$ under the x–axis,

iv. *a downgoing–passage*: when $\mathcal{O}_\alpha$ lies under and $\mathcal{O}_\omega$ above the x–axis.

We call lr–passages and rl–passages also *horizontal passages* and the upgoing- and downgoing passages also *vertical passages*. In Figure 2.1 the four kind of passages are illustrated.

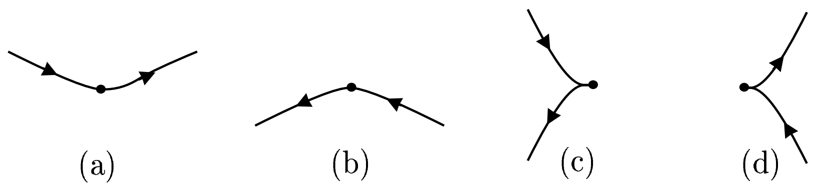

Figure 2.1: Horizontal and vertical passages of a singularity of a Liénard system.

Only the singularities containing passages can be part of a graphic. Let us give an overview, based on the results in Chapter 1, of all singularities possessing passages.

We begin with the singularities containing a horizontal passage. Saddles contain as well a lr–passage as a rl–passage (Figure 2.2). Saddle–nodes and singularities with one elliptic sector contain either a lr–passage (Figure 2.3) or a rl–passage (Figure 2.4).

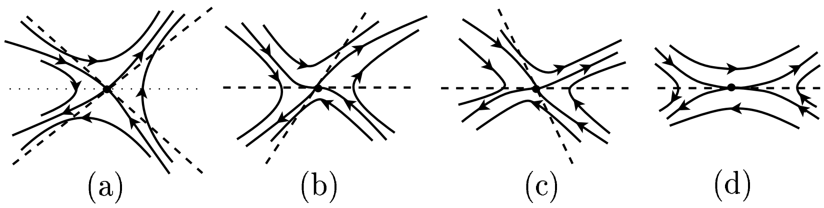

Figure 2.2: Saddles contain both lr– and rl–passages as well as both upgoing and downgoing passages.

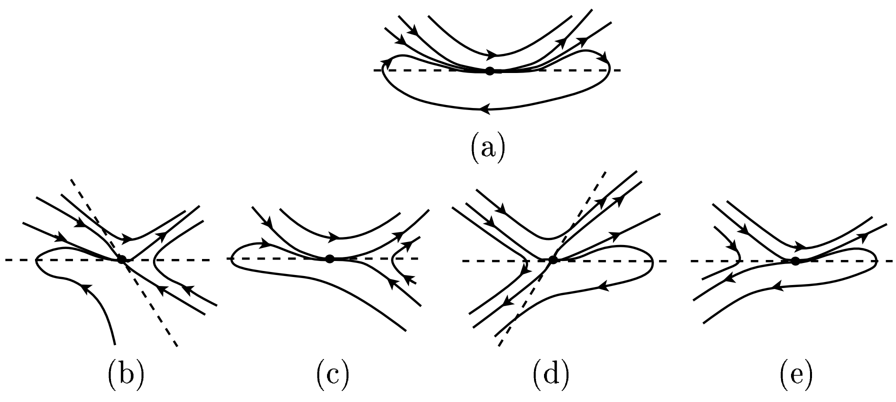

Figure 2.3: Singularities containing lr–passages but no rl-passages.

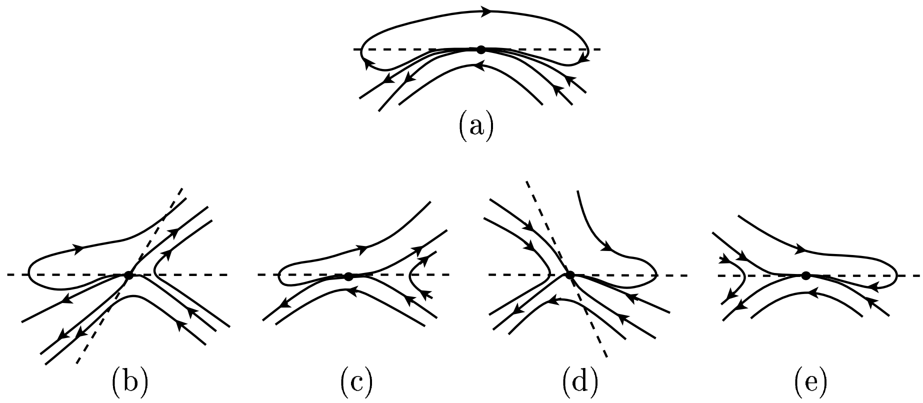

Figure 2.4: Singularities containing rl–passages but no lr–passages.

The singularities with a vertical passage are the saddles containing upgoing as well as donwgoing passages (Figure 2.2), the singularities of up–up type containing upgoing but no downgoing passages (Figure 2.5) and the singularities of down–down type containing downgoing but no upgoing passages (Figure 2.6).

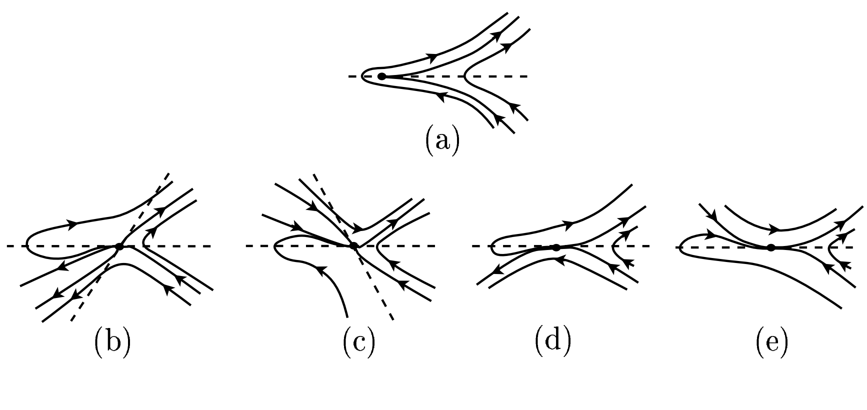

Figure 2.5: Singularities of up–up type all contain upgoing passages.

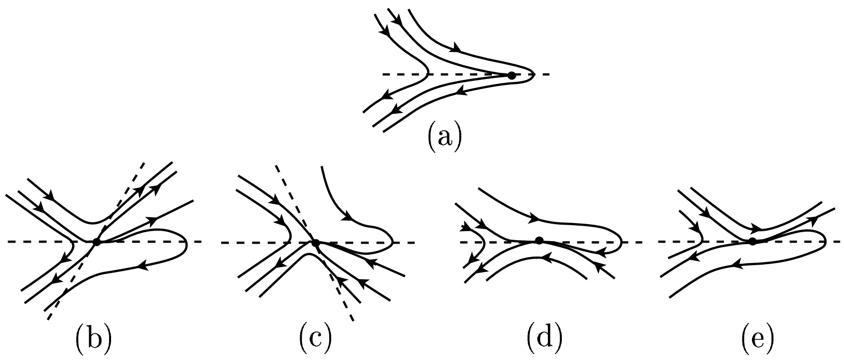

Figure 2.6: Singularities of down–down type all contain downgoing passages.

Remark that the singularities in Figures 2.3 and 2.4, (b) and (c), also contain upgoing passages while the ones in (d) and (e) also contain downgoing passages. The singularities in Figures 2.5 and 2.6, (b) and (c), contain lr–passages while the ones in (d) and (e) contain rl–passages. Saddles contain all four kind of passages. Furthermore horizontal passages nor vertical passages have to be unique. For instance saddle–nodes can contain infinitely many horizontal or vertical passages.

As a matter of example we list all possible bounded limit periodic sets containing at most 2 singularities. Note that an exact position of the orbits or exact contacts cannot be illustrated in these pictures. To shorten the list, we work up to reflections with respect to the x–axis, y–axis or the origin, i.e. up to the transformations

$$(y,t) \mapsto (-y,-t), \qquad (x,t) \mapsto (-x,-t), \qquad (x,y) \mapsto (-x,-y).$$

First of all a bounded limit periodic set $\mathcal{L}$ can be a bounded periodic orbit or a singularity, Figure 2.7, (a) and (b). Let $\mathcal{L}$ be a bounded graphic. If $\mathcal{L}$ contains one singularity, then depending on the kind of passage of the singularity, we find 3 different pictures. The singularity can have a vertical passage, one horizontal passage or two horizontal passages lying on $\mathcal{L}$, respectively illustrated in Figure 2.7, (c), (d) and (e). If $\mathcal{L}$ contains two singularities $\{s_1, s_2\}$, one distincts another 7 different pictures according to the kind of passages at s_1 and s_2 that lie on $\mathcal{L}$.

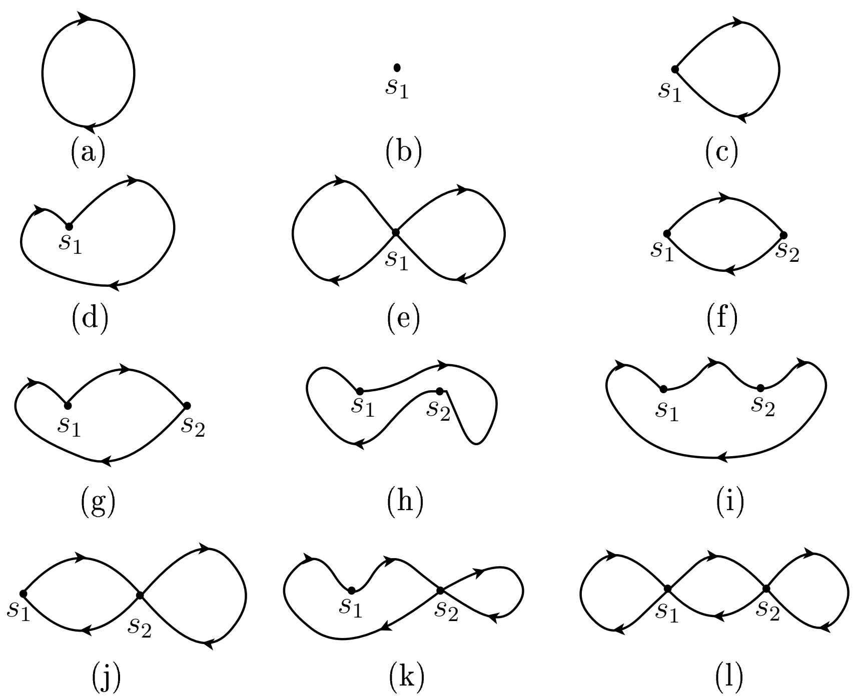

Figure 2.7: Possible bounded limit periodic sets containing at most two singularities.

Concerning the possible topological types of the different singularities in Figure 2.7, we can say the following, making use of the above presentation of all singularities containing passages. In Figure 2.7 (b) every topological type can a priori occur. Of course if the singularity is a saddle, then it cannot be a hyperbolic one, but it can e.g. be a nilpotent saddle of codimension 3, similar for a focus. In Figure 2.7 (e), s_1 has as well a lr–passage as a rl–passage implying that it has to be a saddle (hyperbolic or non–hyperbolic). Singularities with a vertical passage on $\mathcal{L}$ are possibly saddles, saddle–nodes or cusps. Those with a horizontal passage on $\mathcal{L}$ can only be saddles, saddle-nodes or singularities with one elliptic sector.

The following theorem describes any bounded limit periodic set.

Theorem 2.2 *Suppose $\mathcal{L}$ is a bounded limit periodic set of $\overline{X} \in L^{(m,n)}(D)$. Then $\mathcal{L}$ is a non–hyperbolic singularity of $\overline{X}$ situated in $D \backslash \partial D$ or it is the union of two continuous graphs $y = h_1(x)$ and $y = h_2(x)$, $x \in [a_0, b_0]$, $a_0 < b_0$ such that:*

i. $\forall x \in [a_0, b_0] : h_1(x) \geq 0$ *and* $h_2(x) \leq 0$. *Moreover* $h_1(a_0) = h_2(a_0) = 0$ *and* $h_1(b_0) = h_2(b_0) = 0$.

ii. for $x \in]a_0, b_0[$, $h_1(x)h_2(x) = 0$ *if and only if* $(x, 0)$ *is a singularity of* $\overline{X}$ *lying on* $\mathcal{L}$.

Moreover concerning the singularities, we have the following:

a. the zeros of h_1 *(resp.* h_2*) with abscis lying in* $]a_0, b_0[$ *can only be saddles or one of the singularities presented in Figure 2.3 (resp. Figure 2.4),*

b. if $(a_0, 0)$ *(resp.* $(b_0, 0)$*) is a singularity, then it has to be a saddle or a singularity of up–up (resp. down–down) type, Figure 2.5 (resp. Figure 2.6),*

c. if $h_1(x) = h_2(x) = 0$, $x \in]a_0, b_0[$, *then* $(x, 0)$ *can only be a saddle.*

Proof: Denote the singularities and regular orbits that constitute $\mathcal{L}$ as $s_1, \ldots, s_r$ and $\gamma_1, \ldots, \gamma_r$.

We first show that it is justified to assume that $\gamma_i \neq \gamma_j$ for every $i, j \in \{1, \ldots, n\}$. Afterall suppose $\gamma_i = \gamma_j$ for some $i \neq j$, then also $s_i = s_j$ and $s_{i+1} = s_{j+1}$. For $\mathcal{L}$ being closed there has to correspond connections Ω_i from s_{i+1} to s_i and Ω_j from s_{j+1} to s_j. These connections have to coincide because otherwise they will result in two intersections of $\mathcal{L}$ with a transveral Σ_x^+ or Σ_x^- contradicting Proposition 2.1. So $\mathcal{L}_i = \{s_i, s_{i+1}, \gamma_i, \Omega_i\}$ describes the same graphic as $\mathcal{L}_j = \{s_j, s_{j+1}, \gamma_j, \Omega_j\}$ implying that the chosen configuration is not minimal, i.e. r is not as small as possible.

Because $\mathcal{L}$ is compact, it is meaningful to set $a_0 = \min\{x \mid (x,y) \in \mathcal{L}\}$ and $b_0 = \max\{x \mid (x,y) \in \mathcal{L}\}$. Let us denote the connection from a_0 to b_0 as Γ_1 and the one from b_0 to a_0 as Γ_2.

We now prove that Γ_1 is entirely situated in the half plane $\{y \geq 0\}$ implying that it is the graph of a continuous function h_1. Indeed suppose Γ_1 intersects the half plane $\{y < 0\}$, then it also intersects a transversal Σ_z^-, $z \in]a_0, b_0[$. Because $\dot{x} = y$ the connection Γ_2 also intersects Σ_z^-. For not contradicting Proposition 2.1 this intersection has to coincide with that of Γ_1 such that Γ_1 and Γ_2 have at least one regular orbit in common. But we just assumed that all regular orbits constituting $\mathcal{L}$ were different.

Analogous arguments show that also Γ_2 is a continuous graph $y = h_2(x)$ lying in the half plane $\{y \leq 0\}$.

Furthermore Γ_1 cannot cut the x–axis at regular points with abscis lying in $]a_0, b_0[$. Indeed because the vector field is transverse to the x–axis, such an intersection would also imply an intersection with the half plane $\{y < 0\}$. Therefore in $]a_0, b_0[$, h_1 can only be zero at singularities on $\mathcal{L}$. Of course also $h_1(a_0) = h_1(b_0) = 0$. Analogous h_2 can only be zero at singularities on $\mathcal{L}$ with an abscis in $]a_0, b_0[$ and $h_2(a_0) = h_2(b_0) = 0$.

A singularity with an abscis in $]a_0, b_0[$ that lies on Γ_1 clearly has a lr–passage induced by Γ_1, while it has a rl–passage if it lies on Γ_2. In particular in the region $a_0 < x < b_0$, Γ_1 only contains saddles or singularities like in Figure 2.3 while Γ_2 only contains saddles or singularities like in Figure 2.4. Singularities $(x, 0)$ with $x \in]a_0, b_0[$ and lying on both connections have both kind of horizontal passages implying that they are saddles. If $(a_0, 0)$ (resp. $(b_0, 0)$) is a singularity, it has to contain an upgoing passage (resp. a downgoing passage) such that is has to be a singularity of up–up type (resp. down–down type). $\square$

Remark:

1. Property ii) follows immediately from property i). Indeed suppose

$h_1(x)$ intersects the x–axis at $(x_0, 0)$ with $x_0 \in]a_0, b_0[$ such that $(x_0, 0)$ is a regular point of $\overline{X}$. Then using same arguments as in the above proof, one easily sees that $h_1(x)$ cannot completely lie above the x–axis in the region $a_0 \leq x \leq b_0$.

2. As a direct consequence of this theorem, a bounded limit periodic set $\mathcal{L}$ is a non–hyperbolic singularity or it is the union of cycles of which all singularities are situated on the x–axis and that only cross each other in a saddle.

2.2 Unbounded limit periodic sets

Concerning the unbounded limit periodic sets, we have the following theorem.

Theorem 2.3 *Suppose $\overline{X} \in L^{(m,n)}(D)$ has an unbounded limit periodic set $\mathcal{L}$. Then one of the below mentioned conditions has to be satisfied; A denotes the highest order coefficient of P in* (1.1).

1. *When*

   (a) $m = 2n+1$, $A > \frac{1}{4(n+1)}$ *or,*

   (b) $m > 2n+1$, m *odd and* $A = 1$,

   $\mathcal{L}$ is given by a periodic orbit lying at infinity, Figure 2.8 (a).

2. *When*
$$m = 2n+1, n \text{ even}, A = \frac{1}{4(n+1)},$$
   $\mathcal{L}$ looks like in Figure 2.8 (b). The two singularities are saddle–nodes with a center behaviour at infinity and a repelling behaviour on their hyperbolic separatrices.

3. *When*
$$m = 2n+1, n \text{ odd}, A = \frac{1}{4(n+1)}$$

$\mathcal{L}$ looks like in Figure 2.8, (c), (d1) or (d2). The singularities s_1 and s_2 at infinity are both saddle–nodes with a center seperatix at infinity. The behaviour on the hyperbolic separatrix of s_1 is attractive, while that of s_2 is repelling.

4. *When*

$$m = 2n+1, n \text{ odd}, \, 0 < A < \frac{1}{4(n+1)},$$

$\mathcal{L}$ looks like in Figure 2.8, (d1) or (d2). In this case the singularities s_1 and s_2 are both hyperbolic saddles.

5. *When*

$$m < 2n+1, m \text{ odd}, \, n \text{ odd and } A = 1,$$

$\mathcal{L}$ looks like in Figure 2.8, (e1) or (e2) containing at infinity two semi–hyperbolic saddles s_1 and s_2 such that s_1 (resp. s_2) has an unstable (resp. stable) separatrix lying at infinity and an attractive (resp. repelling) center separatrix in $D\backslash\partial D$.

Moreover in all cases the singularities on $\mathcal{L}$ that are situated inside $D\backslash\partial D$ have to be saddles or one of the singularities pictured in Figure 2.4.

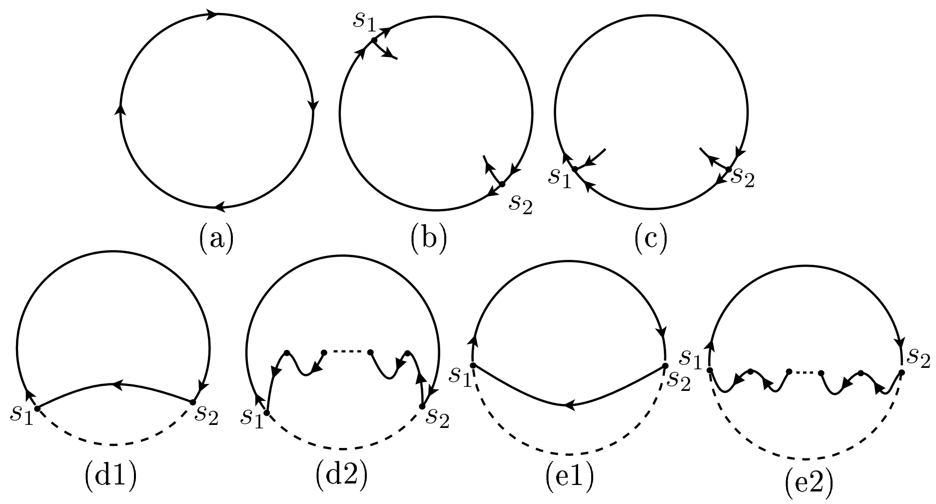

Figure 2.8: All possible unbounded limit periodic sets (we do not pay attention to the exact position of orbits nor to exact contacts).

Proof: The study of the behaviour of $\overline{X}$ near infinity immediately implies that the above mentioned cases are the only potentials in which $\overline{X}$ can have a limit periodic set. Indeed we can distinguish two cases: $\mathcal{L}$ includes whole infinity or partly infinity. In the first case, it can be a regular periodic orbit at infinity or a graphic at infinity that only contains singularities possessing a passage at infinity. It is obvious that this is only possible when:

a. $m = 2n + 1$, $A \geq \frac{1}{4(n+1)}$,

b. $m > 2n + 1$, m odd and $A = 1$.

When a limit periodic set $\mathcal{L}$ is not lying enterily at infinity, it has to contain at least one connection on ∂D bordered by two singularities p_1 and p_2 that both contain a passage. For $\mathcal{L}$ being closed, p_1 and p_2 have to possess a hyperbolic sector such that the flow on the separatrix lying in $D \backslash \partial D$ is attractive in one singularity and repelling in the other. This is only possible in the cases:

a. $m < 2n + 1$, m odd, n odd and $A = 1$,

b. $m = 2n + 1$, n odd, $0 < A \leq \frac{1}{4(n+1)}$.

The topological type of the singularities at infinity immediately follows from the study of the behaviour near infinity of $\overline{X}$ in the above mentioned cases. In Figure 2.8, (d1), (d2), (e1) and (e2), $\mathcal{L}$ contains the upper semicircle at infinity. Therefore for not being in contradiction with proposition 2.1 the remaining part of $\mathcal{L}$ has to be situated in the half plane $\{y \leq 0\}$. In particular all regular orbits of $\mathcal{L}$ lying in $D \backslash \partial D$ are situated below the x–axis implying that all singularities on $\mathcal{L}$ not lying at infinity have to contain a rl–passage. □

2.3 Occurence of limit periodic sets

Let $\overline{X} \in L^{(m,n)}(D)$. We try to find necessary conditions on (m, n) for a limit periodic set $\mathcal{L}$ to occur in the flow of $\overline{X}$.

Suppose first $\mathcal{L}$ is a bounded limit periodic set of $\overline{X}$. If $\mathcal{L}$ is a singularity with Liénard degree (k,l), then obviously $m \geq k$ and $n \geq l$. When $\mathcal{L}$ is a bounded periodic orbit, then $mn \geq 1$. Indeed it is generally known that inside a periodic orbit, there has to exist a singularity together with a point where the divergence of X, i.e. $Q(x)$, disappears.

Suppose now $\mathcal{L}$ is not a singularity nor a periodic orbit. Denote by $\{s_1,\dots,s_t\}$ with $s_i = (p_i, 0)$ the singularities on $\mathcal{L}$. We know from Theorem 2.2, that $\mathcal{L}$ is the union of cycles that can only cross each other in a saddle. Denote n_0 as the number of cycles constituting $\mathcal{L}$. The *Liénard degree* (K,L) *of* $\mathcal{L}$ is defined as

$$(K,L) = \left(\sum_{i=1}^{t} k_i, \sum_{i=1}^{t} l_i\right),$$

where (k_i, l_i) is the Liénard degree of the singularity s_i on $\mathcal{L}$.

We recall from Chapter 1 that a singularity s_i on $\mathcal{L}$ induces a local behaviour on P near p_i that depends on the type of s_i; s_i can be of up–up, down–down, down–up or up–down type. Moreover if the endpoints a_0 and b_0 of $\mathcal{L}$ are not singularities, then it is clear that $P(a_0) > 0$ and $P(b_0) < 0$. From Proposition 1.5, it follows that there exists a minimum number m_0 of singularities, being all of up–down or down–up type, that X is forced to have for P being continuous on $[a_0, b_0]$.

The following proposition now states necessary conditions for $\mathcal{L}$ to occur in the flow of X.

Theorem 2.4 *Suppose* $\mathcal{L}$ *is a bounded limit periodic set of an* $\overline{X} \in L^{(m,n)}(D)$*, then:*

1. *if* $\mathcal{L}$ *is a singularity with Liénard degree* (k,l)*, then* $m \geq k$ *and* $n \geq l$*,*

2. *if* $\mathcal{L}$ *is a periodic orbit, then* $mn \geq 1$*,*

3. *if $\mathcal{L}$ is a graphic, then if m_0 and n_0 is defined as above and (K, L) is the Liénard degree of $\mathcal{L}$, m and n have to satisfy*

$$m \geq K + m_0 \quad \textit{and} \quad n \geq L + n_0.$$

Proof: Suppose $\mathcal{L}$ is a graphic. Then the degree of the polynomial P has to be at least the number of its zeros, counted with their multiplicity. Thus $m \geq K + m_0$. Furthermore in each area enclosed by a cycle, there has to exist a point where the divergence of X, i.e. $Q(x)$, is zero such that $n \geq L + n_0$. $\square$

When $\overline{X}$ contains an unbounded limit periodic set, m has to be odd and $A > 0$ such that $\lim_{x\to-\infty} P(x) < 0$ and $\lim_{x\to+\infty} P(x) > 0$. Moreover if $\mathcal{L}$ contains a singularity $s = (p, 0)$ in $D\backslash\partial D$, then it induces a local behaviour on P near p as desribed in Chapter 1. Again the continuity of P, forces X to have a minimum number m_0 of singularities, of up–down and down–up type.

Further we define the Liénard degree of an unbounded limit periodic set $\mathcal{L}$ as:

$$(K, L) = \left(\sum_{i=1}^{t} k_i, \sum_{i=1}^{t} l_i \right),$$

where $(k_i, l_i)_i$, $i = 1, \ldots, t$ are the Liénard degrees of the singularities on $\mathcal{L}$ situated in $D\backslash\partial D$. As a direct consequence of Theorem 2.3, we can state necessary conditions on (m, n) for an unbounded limit periodic set $\mathcal{L}$ to occur in a Liénard system of type (m, n).

Theorem 2.5 *Suppose $\mathcal{L}$ is an unbounded limit periodic set of $\overline{X} \in L^{(m,n)}(D)$. Then if m_0 is defined as above and (K, L) is the Liénard degree of $\mathcal{L}$, the following applies:*

1. *if $\mathcal{L}$ is a periodic orbit at infinity, then $m \geq 2n + 1$, m odd,*

2. *if $\mathcal{L}$ is like in Figure 2.8 (b), then $m = 2n + 1$, n even,*

3. *if $\mathcal{L}$ is like in Figure 2.8, (c) or (d1), then $m = 2n + 1$ and n is odd,*

4. *if $\mathcal{L}$ is like in Figure 2.8 (d2), then* $m = 2n + 1$, *n is odd,* $m \geq K + m_0$ *and* $n \geq L$,

5. *if $\mathcal{L}$ is like in Figure 2.8 (e1), then* $m < 2n + 1$, *m and n are odd,*

6. *if $\mathcal{L}$ is like in Figure 2.8 (e2), then* $m < 2n + 1$, *m and n are odd,* $m \geq K + m_0$ *and* $n \geq L$.

One can now ask oneself whether a limit periodic set does occur in a system satisfying the minimal conditions given in Theorems 2.4 and 2.5. Let us verify this in some examples.

A singularity with a Liénard degree (k, l) can indeed occur in a Liénard system of type (k, l) as already discussed in Chapter 1. Periodic orbits obviously occur in the Liénard system:

$$X : \begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + xy \end{cases}$$

having, by symmetry reasons, a center at the origin. Furthermore, from Chapter 1 we know that the system:

$$X : \begin{cases} \dot{x} &= y, \\ \dot{y} &= -\frac{1}{2}x + y \end{cases}$$

has a strong stable focus in the origin and a periodic orbit lying at infinity. However a bounded periodic orbit isn't possible here because the divergence of X does not disappear anywhere.

Let us take a view to a loop as in Figure 2.7 (c). If the loop is based at a hyperbolic saddle, we found that m has to be at least 2 and n at least 1. It is well known that such a hyperbolic saddle–loop does indeed occur in a Liénard system of type $(2, 1)$. Let us verify this in the next proposition.

Proposition 2.6 *The family*

$$X_\mu : \begin{cases} \dot{x} &= y, \\ \dot{y} &= -x(x-1) - y(x-\mu) \end{cases}$$

admits a loop as in Figure 2.7 (c), based at a hyperbolic saddle, for some $\mu \in]0, 3/2[$.

Proof: The family (X_μ) admits a hyperbolic saddle at the origin $O = (0,0)$ for all parameter values. At $(1,0)$, X_μ admits a stable focus when $\mu < 1$, a linear center when $\mu = 1$ and an unstable focus when $\mu > 1$.

Denote $\gamma^s(\mu, t)$ and $\gamma^u(\mu, t)$ as the stable and unstable separatrix of O respectively, that lie locally near O, in the half plane $\{x \geq 0\}$. Consider the section $\Sigma_1^- = \{(1, y) \mid y < 0\}$. Because $\dot{x} = y$, it is clear that $\gamma^s(\mu, t)$ intersects the section Σ_1^- at some point $(1, p(\mu))$. We now show that $\gamma^u(\mu, t)$ will intersect the x–axis right of $(1, 0)$ after which it will intersect Σ_1^- at some point $(1, q(\mu))$.

The flow of X_μ is pointing upward on the x–axis for $0 < x < 1$ and downward for $x > 1$. Therefore if $\gamma^u(\mu, t)$ intersect the x–axis, it has to in a point lying right of $(1, 0)$. Entering the half plane $\{y < 0\}$, $\gamma^u(\mu, t)$ clearly has to intersect Σ_1^-. So it suffices to show that $\gamma^u(\mu, t)$ intersects the x–axis. Suppose by contraposition, that $\gamma^u(\mu, t) = (x_\mu(t), y_\mu(t))$ stays in the half plane $\{y > 0\}$. Then it is clear that for some $t_0 > 0$ and $M > 0$:

$$y_\mu(t) = y_\mu(t_0) + \int_{t_0}^{t} \dot{y}_\mu(s)ds < y_\mu(t_0) - M(t - t_0), \quad t \geq t_0$$

contradicting the fact that $y_\mu(t) > 0, \forall t$.

The function $d(\mu) = q(\mu) - p(\mu)$ is clearly smooth. We now show that $d(0) > 0$ and $d(3/2) < 0$ implying the result using the mean value theorem. Consider herefore the algebraic curve:

$$y^2 - x^2(\frac{2}{3}x - 1) = 0$$

which is a loop based at the origin and contained in the flow of the Hamiltonian vector field:

$$X_H : \begin{cases} \dot{x} &= y, \\ \dot{y} &= -x(x - 1). \end{cases}$$

This loop is the union of two graphs $y = f(x)$, lying in $\{y \geq 0\}$, and $y = g(x)$, lying in $\{y \leq 0\}$, both defined on $[0, 3/2]$. Denote the region enclosed by the loop as R.

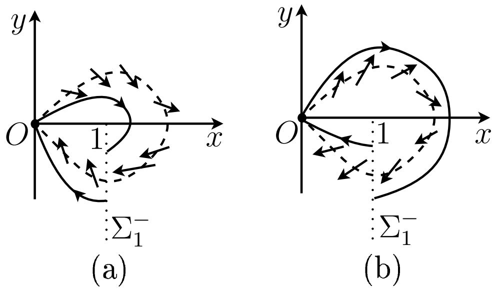

Figure 2.9: Phase portraits of X_μ: (a) when $\mu = 0$, (b) when $\mu = \frac{3}{2}$. The dotted orbit indicates the loop of X_H. The other orbits are the unstable and stable separatrices of O of X_μ.

Locally near the origin $\gamma^u(\mu, t)$ and $\gamma^s(\mu, t)$ can be written as graphs $y = y^s_\mu(x)$ and $y = y^u_\mu(x)$. The vector field X_0 clearly points inwards in the region R. Moreover for x near zero, one sees that:

$$\begin{aligned} y^u_0(x)\frac{dy^u_0}{dx}(x) &= -x(x-1) - xy^u_0(x) \\ &< -x(x-1) \\ &= f(x)\frac{df}{dx}(x). \end{aligned}$$

Integrating both sides, this inequality becomes $(y^u_0(x))^2 < (f(x))^2$ implying $y^u_0(x) < f(x)$ for x near zero. On the other hand, one finds in an analogous way that $y^s_0(x) < g(x)$, x near zero. Since the vector field is pointing inwards in R, it is easily seen that $d(0) > 0$, see Figure 2.9 (a). One can use totally similar arguments on the vector field $X_{3/2}$ to show that $d(3/2) < 0$, see Figure 2.9 (b). $\Box$

We finish with an example containing an unbounded 2–saddle cycle, where the saddles at infinity are both hyperbolic. Consider the Liénard

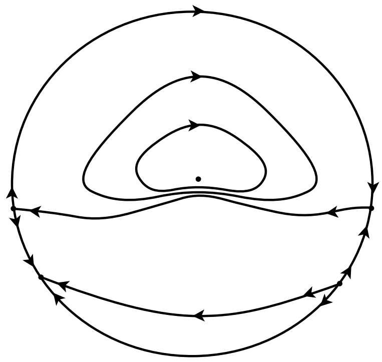

Figure 2.10: Phase portrait of system (2.1)

system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -\frac{1}{10}x(x^2+2) - yx. \end{cases} \tag{2.1}$$

Because the system is symmetric with respect to the x–axis, i.e. it is invariant under the transformation $(x,t) \to (-x,-t)$, the linear center at the origin is a center. Moreover, it is the only singularity of the system inside the Poincaré–Lyapunov disc of degree $(1,2)$. Looking at the behaviour at infinity, Chapter 1, Figure 1.2, one sees that there is only one possibility for the global flow of the above system, Figure 2.10. The system has an unbounded 2–saddle cycle. From Theorem 2.5, we indeed know that m has to be at least 3 and n at least 1 for such an unbounded 2–saddle cycle to occur.

Chapter 3

Structural stability of Liénard systems

In this chapter, we give some results concerning the structural stability of Liénard systems. The results are already known for general vector fields, but in the context of Liénard systems, their proofs can be considerably simplified. We want to treat structural stability of Liénard systems inside the space $L^{(m,n)}(D)$ endowed with the coefficient topology $\mathcal{T}^{(m,n)}$, see Chapter 2.

We first treat the question whether all singularities in $D\backslash\partial D$ of a structurally stable Liénard system have to be hyperbolic. Afterwards we also prove that a structurally stable Liénard system cannot have any saddle–connection lying inside $D\backslash\partial D$.

A Liénard system $\overline{X} \in L^{(m,n)}(D)$ is said to be *structurally stable* if there exists a neighbourhood V of $\overline{X}$ in $(L^{(m,n)}(D), \mathcal{T}^{(m,n)})$ such that all $Y \in V$ are topologically equivalent with $\overline{X}$ on D. Two vector fields $\overline{X}, \overline{Y} \in (L^{(m,n)}(D), \mathcal{T}^{(m,n)})$ are *topologically equivalent* when there exists a homeomorphism $h : D \mapsto D$ which sends orbits of $\overline{X}$ to orbits of $\overline{Y}$ preserving the orientation.

3.1 Structural stability and hyperbolic singularities

It is well known that the hyperbolic singularities can be classified in three topological types. Besides the saddles, we have the class of stable singularities (stable nodes and stable foci) and the class of the unstable singularities (unstable nodes and foci). For proving Theorem 3.2 below, we will need the persistence of hyperbolic singularities after perturbation.

Theorem 3.1 *If s is a hyperbolic singularity of an $\overline{X} \in L^{(m,n)}(D)$, then there exists a neigbourhood V of $\overline{X}$ in $(L^{(m,n)}(D), \mathcal{T}^{(m,n)})$ and a neighbourhood N of s such that every $\overline{Y} \in V$ has a unique hyperbolic singularity in N that has the same topological type as s.*

Proof: Suppose s is a hyperbolic singularity of $\overline{X}$ lying at infinity. Referring to the study near infinity (see Chapter 1), one sees that in the cases where at least one singularity at infinity is hyperbolic, the behaviour near infinity only depends of m, n and sgn A. So, every vector field lying in a sufficiently small neigbourhood of $\overline{X}$ has the same behaviour near infinity.

It remains to treat the case where s lies in $D \backslash \partial D$. Denote by X_η a Liénard system:

$$X_\eta : \left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & P_\eta(x) + yQ_\eta(x), \end{array} \right.$$

with

$$P_\eta(x) = -(Ax^m + \sum_{i=0}^{m-1} a_i x^i), \qquad Q_\eta(x) = -(x^n + \sum_{i=0}^{n-1} b_i x^i),$$

and $\eta := (A, a_0, \ldots, a_{m-1}, b_0, \ldots b_{n-1}) \in (\mathcal{B}, \mathcal{U})$, where $\mathcal{B} = \{-1, 1\} \times \mathbb{R}^{m+n}$, when $m \neq 2n+1$, and $\mathcal{B} = \mathbb{R}\backslash\{0\} \times \mathbb{R}^{m+n}$, when $m = 2n+1$, endowed with the induced topology $\mathcal{U}$ from $\mathbb{R}^{m+n+1}$.

Let $\eta_0 \in \mathcal{B}$ such that X_{η_0} admits a hyperbolic singularity s. By definition of the coefficient topology, it is clear that one has to show the existence of a neighbourhood W of η_0 in $(\mathcal{B},\mathcal{U})$ such that each Liénard system X_η with $\eta \in W$ admits a hyperbolic singularity lying in a neighbourhood of s and having the same topological type as s.

Because s is a hyperbolic singularity of X_{η_0}, we have that $X(\eta_0, s) = 0$ and $\det\big(D_z X(\eta_0, s)\big) \neq 0$, where D_z denotes the differential of X with respect to $z = (x, y)$. The implicit function theorem guarantees the existence of a neighbourhood W_0 of η_0 in $(\mathcal{B},\mathcal{U})$ and a neighbourhood N of s together with a C^ω function $s : W_0 \mapsto N$ such that

$$X(\eta, s(\eta)) = 0, \forall \eta \in W_0 \quad \text{and} \quad s(\eta_0) = s.$$

Further let $s(\eta) = (x(\eta), 0)$ and $a(\eta) := P'_\eta(x(\eta))$, $b(\eta) := Q'_\eta(x(\eta))$. It is clear that if s is a saddle, then $a(\eta_0) > 0$ implying, by continuity, that one can choose the neighbourhood W_0 such that $a(\eta) > 0, \forall \eta \in W_0$. If s is a stable (resp. unstable) singularity, then $a(\eta_0) < 0$ and $b(\eta_0) < 0$ (resp. $b(\eta_0) > 0$) which stays this way for $\eta \in W_0$, W_0 chosen small enough. This ends the proof. □

Theorem 3.2 *If $\overline{X} \in L^{(m,n)}(D)$ is structurally stable, then*

1. *all singularities of $\overline{X}$ in $D\backslash\partial D$ are hyperbolic, and*
2. *if $m \geq 2n+1$, all singularities at infinity are hyperbolic. When $m < 2n+1$, the singularities are either hyperbolic or semi–hyperbolic having a hyperbolic separatrix lying at infinity and a center separatrix in $D\backslash\partial D$.*

Proof: Let $\overline{X} \in L^{(m,n)}(D)$, obtained after an appropriate Poincaré–Lyapunov compactification from a Liénard system X:

$$X : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x). \end{cases}$$

Following Theorem 3.1, the topological behaviour near any hyperbolic singularity will not be affected by any small perturbation of X. We search an appropriate perturbation of X that will change the topological type of non–hyperbolic singularities. From Chapter 1, we know that, for a non–hyperbolic singularity $s = (x, 0)$, either $P(x) = P'(x) = 0$ or s is a linear center.

Suppose first that every non–hyperbolic singularity of X is a linear center. We consider the family $(\overline{X}_\eta)$ obtained, after an appropriate Poincaré–Lyapunov compactification, from the family:

$$(X_\eta) : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x) + \eta y. \end{cases}$$

The linear centers perturb into strong unstable foci for small $\eta > 0$, while for small $\eta < 0$ they perturb into strong stable foci. So, for η near zero, $\overline{X}_\eta$ is not topological equivalent to $\overline{X}_{-\eta}$ implying the structural instability of $\overline{X}$.

Suppose that X has at least one non–hyperbolic singularity that is not a linear center. Denote by $\{s_1, \ldots s_k\}$, with $s_i = (x_i, 0)$, the non–hyperbolic singularities of X satisfying $P(x_i) = P'(x_i) = 0$, $1 \leq i \leq k$. It is clear that $m \geq 2k$. Consider a polynomial $\widetilde{P}(x)$ such that

$$\widetilde{P}(s_i) = 0 \quad \text{and} \quad \widetilde{P}'(s_i) > 0, \quad \forall 1 \leq i \leq k.$$

Such a polynomial has degree at least $2k - 1 < m$. Let $(\overline{X}'_\eta)$ be the family, obtained after an appropriate Poincaré–Lyapunov compactification from the family:

$$(X'_\eta) : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x) + \eta\widetilde{P}(x). \end{cases} \tag{3.1}$$

For $\eta \neq 0$ small, the non–hyperbolic singularities $\{s_1, \ldots, s_k\}$ change into hyperbolic singularities. They will be saddles for $\eta > 0$ and nodes or foci for $\eta < 0$. Using the same techniques as in Theorem 3.1, one verifies that the linear centers persist as linear centers or strong foci for $\eta \neq 0$ small enough. One concludes that for $\eta > 0$ small, X'_η has at least one saddle more than for $\eta < 0$ small. This implies immediately that $\overline{X}$ cannot be structurally stable.

The second statement easily follows from the study of the behaviour near infinity (see Chapter 1). Indeed when $m \neq 2n+1$, the behaviour near infinity is always structurally stable. In case $m > 2n+1$ all singularities are hyperbolic, in case $m < 2n+1$ they are either hyperbolic or semi-hyperbolic having a hyperbolic separatrix at infinity and a center separatrix in $D \backslash \partial D$. If $m = 2n+1$, $\overline{X}$ can only be structurally stable if $A \neq \frac{1}{4(n+1)}$. Then all singularities at infinity are hyperbolic. □

3.2 Structural stability and saddle–connections

A saddle–connection is an orbit Γ such that both $\omega(\Gamma) = S$ and $\alpha(\Gamma) = s$ are saddles. When $s \neq S$ we speak of *a heteroclinic connection* and when $s = S$ we speak of *a homoclinic connection.*

Saddle–connections are called *bounded* when as well s as S are situated in the finite plane. When one of them lies at infinity we speak of an *unbounded* saddle–connection.

It is possible that a saddle–connection of a Liénard system $\overline{X} \in L^{(m,n)}(D)$ lies completely at infinity. From Chapter 1, it is clear that this is only possible in the following cases:

1. when $m < 2n+1$, m odd and n are odd and $A = 1$, then $\overline{X}$ has a saddle–connection at infinity between two semi–hyperbolic saddles of which the hyperbolic seperatrices lie at infinity,
2. when $m = 2n+1$, n is odd and $0 < A < \frac{1}{4(n+1)}$, $\overline{X}$ has a saddle–connection between two hyperbolic saddles at infinity.

Obviously these saddle–connections will persist for any small perturbation of $\overline{X}$ in $L^{(m,n)}(D)$. This section will be devoted to the proof of the following theorem.

Theorem 3.3 *A structurally stable system $\overline{X} \in L^{(m,n)}(D)$ cannot have any saddle–connection lying in $D \backslash \partial D$.*

It is sufficient to prove that any saddle–connection lying in $D \backslash \partial D$ breaks after a certain perturbation. Indeed two topological equivalent vector fields have to possess the same number of saddle–connections. One way to prove this break up of any connection is the analytical way. One can calculate a Melnikov integral to be non–zero. However we will prefer another, more geometric way, to approach the problem making systematical use of the fact that we are dealing with Liénard systems.

Let us start with treating the bounded saddle–connections. By Theorem 3.2, it is enough to restrict to those saddle–connections between two hyperbolic saddles. We will need to study the influence of a certain perturbation on the separatrices of a hyperbolic saddle lying in $D \backslash \partial D$.

Herefore let $(\overline{X}_\eta)$ be a family in $L^{(m,n)}(D)$, with η varying in a neighbourhood $\mathcal{P} \subset \mathbb{R}$ of zero, that is obtained after an appropriate Poincaré–Lyapunov compactification from the family:

$$(X_\eta) : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x) + \eta y f(x), \end{cases} \tag{3.2}$$

where $f(x)$ is a strictly positive polynomial of degree at most n.

Suppose $X = X_0$ admits a hyperbolic saddle $s = (x_0, 0)$. It is clear that the saddle persists at $(x_0, 0)$ inside the family (X_η). Consider the sections

$$\Sigma_z^+ = \{(z, y) \mid y > 0\} \quad \text{and} \quad \Sigma_z^- = \{(z, y) \mid y < 0\}.$$

By the implicit function theorem, one can suppose that any intersection between one of the separatrices of s and the sections Σ_z^+ or Σ_z^- persists for $\eta \in \mathcal{P}$. We now have the following lemma.

Lemma 3.4 *Let (X_η), $\eta \in \mathcal{P}$, be the family (3.2), where each X_η has a hyperbolic saddle $s = (x_0, 0)$. Denote Γ_η^s and Γ_η^u as a stable and unstable separatrix of the saddle s of X_η respectively. Consider the sections $\Sigma_z^\pm$ for $z > x_0$. Then:*

1. *every intersection p between Γ_0^u and Σ_z^+, $z > x_0$ will move upward on Σ_z^+ when η changes from zero into a positive number, and*

2. *every intersection q between Γ_0^s and Σ_z^- will move upward on Σ_z^- when η changes from zero into a positive number.*

Proof: The saddle s has two unstable hyperbolic separatrices for each η. Suppose first that Γ_η^u is the unstable separatrix lying locally above the x-axis and let p_1 be the first intersection of Γ^u with Σ_z^+. By this we mean that before p_1 the separatrix does not intersect the section Σ_z^+.

We now distinguish two cases. If Γ^u does not intersect the x–axis before it reaches p_1, we get a situation displayed like in Figure 3.1 (a). For $\eta > 0$ the separatrix Γ^u will locally move upward. Indeed the eigenspace belonging to the positive eigenvalue $\lambda_1(\eta)$ of the linear part of X_η at s is spanned by $(1, \lambda_1(\eta))$ where $\lambda_1(\eta)$ satisfies:

$$\lambda_1(\eta) = \frac{1}{2}\left((Q(x_0) + \eta) + \sqrt{(Q(x_0) + \eta)^2 + 4P'(x_0)}\right) > \lambda_1(0), \, \forall \eta > 0$$

with $P'(x_0) > 0$. Because X_η points upward on Γ^u for $\eta > 0$, Γ_η^u stays above Γ^u implying that the intersection point p_1 indeed moves upward.

In the other case Γ^u does intersect the x–axis before it reaches p_1. The separatrix Γ^u has to intersect the x–axis an even number of times to get back above the x–axis for reaching Σ_z^+. Denote this even number of intersections as $\{i_1, \ldots i_{2k}\}$, ordered such that it is increasing along Γ^u, i.e. ordered with increasing flight time along Γ^u. Remark that this sequence is not arbitrarily. It has to satisfy $i_{2i+1} > i_{2i-1}$ and $i_{2i} < i_{2(i-2)}$, for all indices i that make sense. The reason is obvious; else the unstable separatrix would obstruct itself to reach Σ_z^+.

The first intersection i_1 will move to the right when η changes from zero into a positive number. Indeed X_η points upward on that part of Γ^u lying between s and i_1 while, locally, the separatrix moves upward. The part of Γ^u lying between i_1 and i_2 lies under the x-axis and X_η will point downard on that part for $\eta > 0$ implying that i_2 will move to the left. Proceeding this argument by induction one concludes that i_{2k} will move to the left. However on the part of Γ^u lying between i_{2k} and p_1, X_η points upward for $\eta > 0$ and that's why p_1 has to move upward. For

$k = 1$, these arguments are illustrated in Figure 3.1 (b).

Eventually all intersections after the first one will move upward. Indeed denote the sequence of intersections as $(p_i)_i$, ordered such that it is increasing along Γ^u. The corresponding sequence of y–coordinates $(y_i)_i$ is strictly monotone. Suppose first $(y_i)_i$ is increasing. Then if the first intersection moves upward, so does the second one p_2 when η changes from zero into a positive number. Indeed, the vector field X_η points outward on that part of Γ^u lying between p_1 and p_2 obstructing Γ^u_η to reach any point on Σ^+_z under p_2, when $\eta > 0$, see Figure 3.1 (c). Proceeding by induction one concludes that all intersections have to move upward. When the sequence $(y_i)_i$ is decreasing, same arguments will also prove that all intersections move upward.

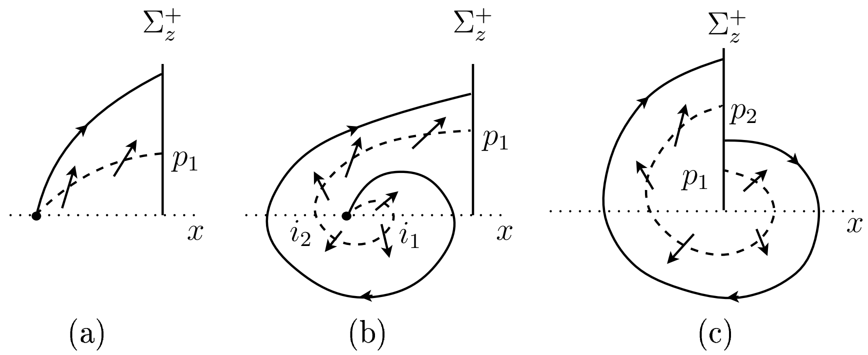

Figure 3.1: Movement of the unstable separatrix of a hyperbolic saddle in the family (3.2).

By totally analogous arguments, one proves that the same result holds for the other unstable separatrix of s lying locally under the x–axis.

Concerning the stable separatrices, one sees that the negative eigenvalue

$\lambda_2(\eta)$ of the linear part of X_η at s satisfies:

$$\begin{aligned}\lambda_2(\eta) &= \frac{1}{2}\bigg((Q(x_0)+\eta) - \sqrt{(Q(x_0)+\eta)^2 + 4P'(x_0)}\bigg) \\ &= \lambda_2(0) + \frac{\eta}{2}\bigg(1 - \frac{Q(x_0)}{\sqrt{Q(x_0)^2 + 4P'(x_0)}}\bigg) + O(\eta^2)\end{aligned}$$

implying that $\lambda_2(\eta) > \lambda_2(0)$ for $0 < \eta << 1$. One proceeds similar as before to prove that any intersection between a stable separatrix Γ_0^s of s and Σ_z^- will move upward on Σ_z^- when η changes from zero into a positive number. □

Lemma 3.5 *Let $(X_\eta) \in L^{(m,n)}(D)$, $\eta \in \mathcal{P}$, be the family (3.2), where each X_η has a hyperbolic saddle $s = (x_0, 0)$. Denote Γ_η^s and Γ_η^u as a stable and unstable separatrix of the saddle s of X_η respectively. Consider the sections $\Sigma_z^\pm$ for $z < x_0$. Then:*

1. *every intersection p between Γ_0^u and Σ_z^- will move downward on Σ_z^- when η changes from zero into a positive number, and*

2. *every intersection q between Γ_0^s and Σ_z^+ will move downward on Σ_z^+ when η changes from zero into a positive number.*

Proof: The proof is totally analogous to that of Lemma 3.4. For instance for the unstable separatrix lying locally under the x-axis, the arguments are illustrated in Figure 3.2. □

These two lemmas will imply that any structurally stable system cannot have any bounded saddle–connections, which is the content of the next theorem.

Theorem 3.6 *If $\overline{X} \in L^{(m,n)}(D)$ is structurally stable, then $\overline{X}$ cannot have any bounded saddle–connection.*

Proof: Suppose $\overline{X}$ has a bounded saddle–connection Γ between two saddles $s = (x_1, 0)$ and $S = (x_2, 0)$ with $s = \alpha(\Gamma)$ and $S = \omega(\Gamma)$. The

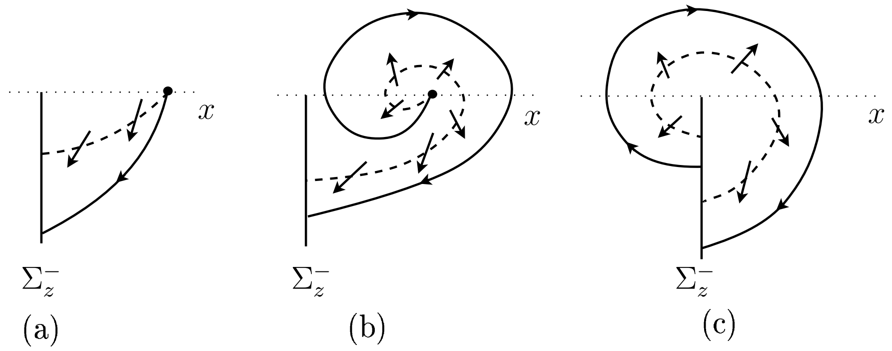

Figure 3.2: Movement of the unstable separatrix of a hyperbolic saddle in the family (3.2).

study of this connection is equivalent with the corresponding saddle–connection of the original Liénard system X:

$$X : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x). \end{cases}$$

By Theorem 3.2, one can suppose that the saddles s and S are hyperbolic. The following perturbation of X:

$$X_\eta : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x) + \eta y, \end{cases} \tag{3.3}$$

will cause Γ to break, for $\eta > 0$ small.

First consider the case where $x_1 = x_2$. Then s together with the homoclinic connection are part of a loop based at s. From our study of the bounded limit periodic sets Chapter 2, Theorem 2.2, we know what such loops look like. Up to transformations

$$(x,t) \mapsto (-x,-t), \qquad (y,t) \mapsto (-y,-t), \qquad (x,y) \mapsto (-x,-y),$$

the loop will look like in Figure 3.3 (a) or (b). It is easily seen that the new vector field points outwards of the region R enclosed by the loop. However choosing sections $\Sigma_z^\pm$ with $z > x_1$ that crosses the homoclinic connection transversally, Lemma 3.4 implies that one separatrix moves to the interior of R and one moves to the exterior of R for $\eta > 0$ small. So the homoclinic connection cannot persist after perturbation, see Figure 3.3 (a) or (b).

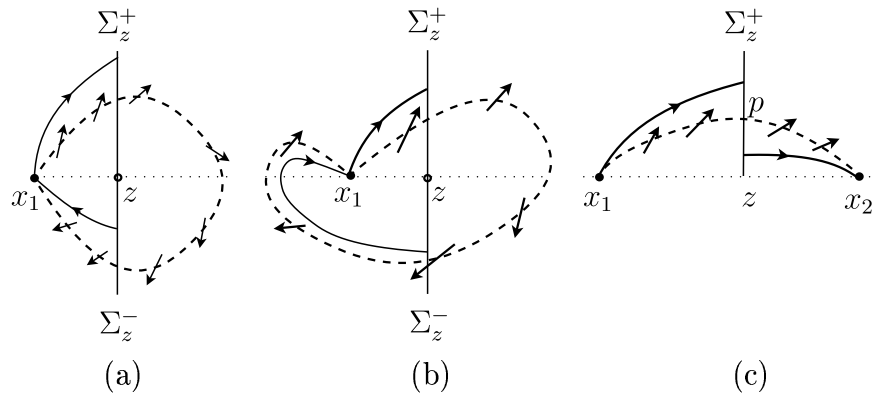

Figure 3.3: Perturbation of a bounded saddle–connection inside the family (3.3).

Suppose $x_1 < x_2$, then choose a section Σ_z^+ with $z \in]x_1, x_2[$. Because the flow points to the right above the x-axis, Γ certainly has to intersect Σ_z^+ at least once, in particular there is an unstable separatrix Γ^u of s and a stable separatrix Γ^s of S both intersecting Σ_z^+ in some point p. For $\eta > 0$ small it follows from Lemma 3.4 that Γ_η^u will intersect Σ_z^+ in a point lying above p, while from Lemma 3.5 it follows that Γ_η^s will intersect Σ_z^+ in a point lying under p, see Figure 3.3 (c). So Γ breaks.

If $x_2 < x_1$, one takes a section Σ_z^- with $z \in]x_2, x_1[$. Lemmas 3.4 and 3.5 imply that Γ breakes for $\eta > 0$ small. $\square$

We are left with the study of the unbounded saddle–connections of an $\overline{X} \in L^{(m,n)}(D)$. By Theorem 3.2 one can suppose that the saddles in $D \backslash \partial D$ are hyperbolic while the ones at infinity have hyperbolic separatrices at infinity. We treat the cases $m = 2n + 1$ and $m < 2n + 1$ seperately. When $m > 2n + 1$ all singularities at infinity are nodes such that an unbounded saddle–connection is out of the question.

Proposition 3.7 *When $m = 2n+1$, a structurally stable vector field $\overline{X} \in L^{(m,n)}(D)$ cannot have any unbounded saddle–connection.*

Proof: Consider $\overline{X} \in L^{(m,n)}(D)$ and denote X as the Liénard system of which $\overline{X}$ is obtained after an appropriate Poincaré–Lyapunov compactification. We write X as:

$$X : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x), \end{cases} \tag{3.4}$$

where P and Q are given as in (1.2) chosen such that $\overline{X}$ has an unbounded saddle–connection Γ with $\alpha(\Gamma) = s$ and $\omega(\Gamma) = S$. Along Γ the unstable separatrix Γ^u of s and the stable separatrix Γ^s of S coincides.

Unbounded saddle–connections are only possible in the cases

$$n \text{ odd}, \; 0 < A < \tfrac{1}{4(n+1)} \quad \text{and} \quad n \text{ even}, \; 0 < A < \tfrac{1}{4(n+1)},$$

where we have two hyperbolic saddles at infinity, see Chapter 1, Figure 1.2.

We will show that, when n is odd, a perturbation of $\overline{X}$ corresponding with a perturbation of X:

$$X_\eta : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x) + \eta y x^{n-1} \end{cases} \tag{3.5}$$

will cause the connection Γ to break for $\eta > 0$ small. When n is even, we use a perturbation determined by

$$X_\eta : \begin{cases} \dot{x} &= y \\ \dot{y} &= P(x) + yQ(x) + \eta y x^{n-2}. \end{cases} \tag{3.6}$$

Let us first treat the case where n is odd. Then the saddle at infinity in the positive x–direction has an unstable separatrix lying in $D \backslash \partial D$ and the one in the negative x–direction has a stable separatrix in $D \backslash \partial D$, see Chapter 1, Figure 1.2. Three possibilities of Γ have to be considered. In the first case Γ is a connection from the saddle at infinity in the positive x–direction to a saddle in $D\backslash\partial D$. In the second case Γ goes from a saddle in $D\backslash\partial D$ to the saddle in the negative x–direction. It is clear that this case can easily be reduced to the first case by means of the transformation $(x,t) \mapsto (-x,-t)$. Finally the case where Γ is a connection between the two saddles lying at infinity has to be considered.

Assume first that Γ has as α–limit, the saddle in the positive x–direction and as ω-limit a saddle lying in $D\backslash\partial D$. Consider the section Σ_M^- with M chosen such that the section lies right of $\omega(\Gamma) = S$ while the part of Γ lying between Σ_M^- and s is situated entirely under the x–axis, see Figure 3.4 (a). One can suppose that S is hyperbolic and using Lemma 3.4 the intersection of Γ^s with Σ_M^- moves upward for $\eta > 0$ small. We will now show that the intersection of Γ^u with Σ_M^- will move downward causing Γ to break, see Figure 3.4 (a).

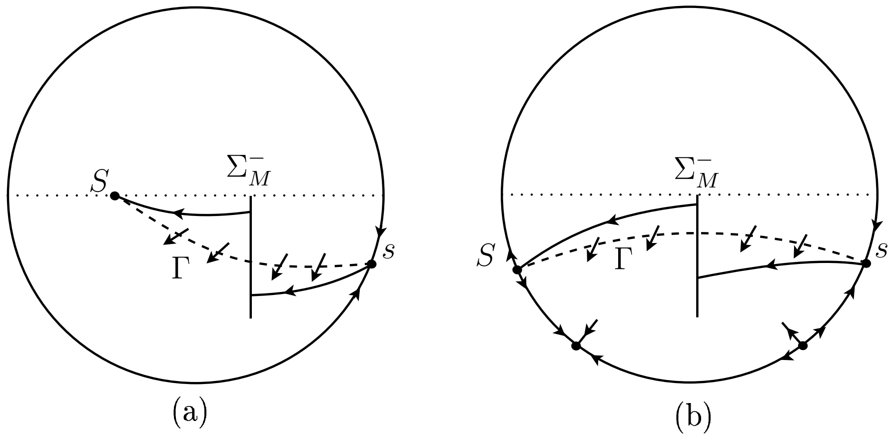

Figure 3.4: Perturbation of an unbounded saddle–connection in case $m = 2n+1$. The dotted orbits are the ones of the unperturbed vector field.

Herefore, we study X_η in the positive x–direction by means of the transformation

$$x = 1/u, \qquad y = v/u^{n+1}.$$

Multiplying the result with a factor u^n, we obtain the vector field:

$$X_\eta^+ : \begin{cases} \dot{u} &= -uv, \\ \dot{v} &= -A - \sum_{k=0}^{2n} a_k u^{2n-k+1} - v(1 + \sum_{k=0}^{n-1} b_k u^{n-k}) \\ & \quad -(n+1)v^2 + \eta v u. \end{cases}$$

Because $0 < A < \frac{1}{4(n+1)}$, we find for $\eta = 0$ a saddle at $s_+ = (0, v_0)$, with $v_0 = (-1 + \sqrt{1 - 4A(n+1)})/(2(n+1)) < 0$ that will persist for $\eta \neq 0$. Locally near s_+, the unstable manifold of X_η^+ can be written as the graph of a smooth function $f_\eta^+(u)$, with $f_\eta^+(0) = v_0$. For $u > 0$ small, $f_\eta^+(u)$ corresponds with the unstable separatrix of the saddle s of $\overline{X}_\eta$ lying at infinity.

We now show that locally for $u > 0$, the unstable manifold will move downward for $\eta > 0$. Indeed the linear part of X_η^+ at $(0, v_0)$ is given by

$$A^+ = \begin{pmatrix} -v_0 & 0 \\ -a_{2n} - v_0(b_{n-1} - \eta) & -\sqrt{1 - 4A(n+1)} \end{pmatrix}$$

such that the slope ω_η^+ of f_η^+ at zero equals

$$\omega_\eta^+ := \frac{a_{2n} + v_0(b_{n-1} - \eta)}{v_0 - \sqrt{1 - 4A(n+1)}}.$$

Because $v_0 < 0$ and $v_0 - \sqrt{1 - 4A(n+1)} < 0$, the slope ω_η^+ will be decreasing in η such that the unstable separatrix of s will locally move downward for $\eta > 0$ small.

For $\eta > 0$ the perturbed vector field $\overline{X}_\eta$ points downward on that part of Γ^u lying between s and Σ_M^- such that it is easily seen that the intersection of Γ^u with Σ_M^- will move downward for $\eta > 0$ small, see Figure 3.4 (a).

Suppose now Γ is a connection between the two saddles at infinity. From the previous calculations we know that the perturbation of $\overline{X}$ defined in (3.5) will result, for $\eta > 0$ small, in a locally downward motion of the unstable separatrix Γ^u of s. Analogous calculations in the negative

x–direction show that the stable separatrix Γ^s of S will locally move upward. Indeed transforming X_η by means of

$$x = -1/u, \qquad y = v/u^{n+1}$$

and multiplying the result with a factor u^n one obtains:

$$X_\eta^- : \begin{cases} \dot{u} &= uv, \\ \dot{v} &= A - \sum_{k=0}^{2n}(-1)^k a_k u^{2n-k+1} - v((-1)^n + \sum_{k=0}^{n-1}(-1)^k b_k u^{n-k}) \\ & \quad +(n+1)v^2 + \eta v u. \end{cases}$$

Again there is a saddle $s_- = (0, v_0)$ that will persists for $\eta \neq 0$, where $v_0 = (-1 + \sqrt{1 - 4A(n+1)})/(2(n+1)) < 0$. The linear part of X_η^- at s_- is given by

$$A^- = \begin{pmatrix} v_0 & 0 \\ -a_{2n} - v_0(b_{n-1} - \eta) & \sqrt{1 - 4A(n+1)} \end{pmatrix}.$$

If $\{v = f_\eta^-(u)\}$ denotes the stable manifold of X_η^- near s_-, then the slope of f_η^- at zero equals:

$$\omega_\eta^- = -\frac{a_{2n} + v_0(b_{n-1} - \eta)}{v_0 - \sqrt{1 - 4A(n+1)}} = -\omega_\eta^+.$$

As before, one easily verifies that, for $\eta > 0$ small, the chosen perturbation will cause the stable separatrix of S lying on Γ to locally move upward.

Because for $\eta > 0$, the perturbed vector field $\overline{X}_\eta$ points downward on the connection Γ one easily sees that Γ will break, see Figure 3.4 (b).

We now treat the case where n is even. Again there are two saddles, one in the positive x–direction and one in the negative x–direction, both having an unstable separatrix in $D \backslash \partial D$. We only have to treat the case where Γ goes from the saddle at infinity lying in the positive x–direction to a saddle in $D \backslash \partial D$. By means of the transformation $(x, y) \mapsto (-x, -y)$, one easily obtains the case where Γ goes from the saddle at infinity lying in the negative x–direction to a saddle in $D\backslash\partial D$. A saddle–connection between the two saddles at infinity is not possible here.

So suppose $\alpha(\Gamma) = s$ lies at infinity in the positive x–direction and $\omega(\Gamma) = S$ lies in $D\backslash\partial D$. As before, we take a section Σ_M^- lying right of S such that the part of the unstable separatrix of s lying between Σ_M^- and s is enterily situated under the x–axis. We perturb $\overline{X}$ by means of (3.6). Notice that $n \neq 0$ because else, X is a linear vector field having a node in the origin implying that a saddle–connection like Γ is impossible. So we may assume that $n \geq 2$.

Because n is even x^{n-2} is positive on whole the plane. Assuming that S is hyperbolic, one knows from Lemma 3.4 that the intersection of the stable separatrix of S with Σ_M^- will move upward for $\eta > 0$. On the other hand the intersection of the unstable separatrix of s with Σ_M^- will move downward. A study of the vector field in the positive x–direction gives:

$$X_\eta^+ : \begin{cases} \dot{u} &= -uv, \\ \dot{v} &= -A - \sum_{k=0}^{2n} a_k u^{2n-k+1} - v(1 + \sum_{k=0}^{n-1} b_k u^{n-k}) \\ & \quad -(n+1)v^2 + \eta v u^2, \end{cases}$$

with a saddle at $s_+ = (0, v_0)$ where $v_0 = (-1 + \sqrt{1 - 4A(n+1)})/(2(n+1)) < 0$. The unstable manifold of the saddle $s_+ = (0, v_0)$ of X_η^+ can locally be written as the graph of a C^∞ function $g_\eta^+(u)$ with $g_\eta^+(0) = v_0$. Let us write

$$g_\eta^+(u) = v_0 + v_1 u + v_2(\eta) u^2 + O(u^3).$$

Using the invariance of the unstable manifold under the flow of X_η^+, we find that v_1 and $v_2(\eta)$ are given by:

$$\begin{aligned} v_1 &= \frac{a_{2n} + v_0 b_{n-1}}{v_0 - \sqrt{1 - 4A(n+1)}}, \\ v_2(\eta) &= -\frac{a_{2n-1} + n\, v_1^2 + v_1\, b_{n-1} + v_0(b_{n-2} - \eta)}{1 + 2n v_0}. \end{aligned}$$

Noticing that $v_0 < 0$ and $1 + 2nv_0 > 0$, one sees that the unstable separatrix of s will locally move downward for $\eta > 0$. For $\eta > 0$, the vector field $\overline{X}_\eta$ points downward on the part of the unstable separatrix of s lying under the x–axis. The connection will break after perturbation, see also Figure 3.4 (a). $\square$

Proposition 3.8 *If* $\overline{X} \in L^{(m,n)}(D)$, $m < 2n+1$, *is structurally stable, then* $\overline{X}$ *cannot have any unbounded saddle–connection.*

Proof: Let $\overline{X} \in L^{(m,n)}(D)$, obtained after an appropriate Poincaré-Lyapunov compactification of a Liénard system X:

$$X : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x), \end{cases}$$

where P and Q are polynomials given as in (1.2), chosen such that $\overline{X}$ has an unbounded saddle–connection Γ with $\alpha(\Gamma) = s$ and $\alpha(\Gamma) = S$.

Referring to the study near infinity in Chapter 1, saddles at infinity are only possible in the following cases:

$$m \text{ odd, } n \text{ even, } A = 1 \quad \text{and} \quad m \text{ odd, } n \text{ odd, } A = 1.$$

The saddles at infinity are semi–hyperbolic. So it is the center separatrix Γ^c of these saddles that possibly lies on Γ.

Same arguments are used as in Proposition 3.7 that can be analogously illustrated as in Figure 3.4. Let us start with studying the center separatrices of the semi–hyperbolic saddles at infinity. Transforming X in the positive x–direction by means of the transformation

$$x = 1/u, \qquad y = v/u^{n+1}$$

and multiplying the obtained vector field with a factor u^n yields:

$$X^+ : \begin{cases} \dot{u} &= -uv, \\ \dot{v} &= -u^{2n-m+1} - \sum_{k=0}^{m-1} a_k u^{2n-k+1}, \\ & \quad -v(1 + \sum_{k=0}^{n-1} b_k u^{n-k}) - (n+1)v^2. \end{cases} \tag{3.7}$$

One finds a semi–hyperbolic saddle in $(0,0)$. It is the center manifold of this saddle that can be part of Γ. Locally near $(0,0)$, the center manifold is a graph of a C^∞ function $v^+(u)$. Using the invariance of the center manifold for the flow of X^+, one finds:

$$v^+(u) = v_0 u^{2n-m+1} + v_1 u^{2n-m+2} + v_2 u^{2n-m+3} + O(u^{2n-m+4}), \qquad (3.8)$$

in which the coefficients v_0, v_1 and v_2 are obtained by substitution of (3.8) into (3.7):

$$v_0 = -1, \qquad v_1 = b_{n-1} + c(\overline{a}), \qquad v_2 = b_{n-2} + d(b_{n-1}, \overline{a}),$$

where c only depends the coefficient vector $\overline{a} = (a_0, \ldots, a_{m-1})$ and d depends on b_{n-1} and $\overline{a}$. The explicit expressions of c and d depend on the difference $(2n+1) - m > 0$.

The saddle in the negative x–direction can be studied by means of the transformation

$$x = -1/u, \qquad y = v/u^{n+1}$$

yielding, after multiplication with a factor u^n:

$$X^- \left\{ \begin{array}{rcl} \dot{u} & = & uv, \\ \dot{v} & = & (-1)^{m+1} u^{2n-m+1} - \sum_{k=0}^{m-1} (-1)^k a_k u^{2n-k+1}, \\ & & -v((-1)^n + \sum_{k=0}^{n-1} (-1)^k b_k u^{n-k}) + (n+1)v^2. \end{array} \right. \qquad (3.9)$$

The origin is a semi–hyperbolic saddle. Using again the invariance of the center manifold for the flow of X^-, one finds that the center manifold is the graph of some C^∞ function $v^-(u)$ with

$$\overline{v}(u) = \overline{v}_0 s^{2n-m+1} + \overline{v}_1 s^{2n-m+2} + O(s^{2n-m+3}), \qquad (3.10)$$

where

$$\overline{v}_0 = (-1)^{n+m+1}, \qquad \overline{v}_1 = \overline{v}_0 b_{n-1} + \overline{c}(\overline{a}),$$

where $\overline{c}$ only depends on the coefficient vector $\overline{a} = (a_0, \ldots, a_{m-1})$.

Suppose first that we are the case where m and n are odd and $A = 1$.

There are two saddles lying at infinity, the one in the positive x–direction with a unstable separatrix in $D\backslash\partial D$ and the one in the negative x–direction with a stable separatrix in $D\backslash\partial D$. Suppose Γ is a connection going from the saddle s at infinity to a hyperbolic saddle S in $D\backslash\partial D$. A transformation $(x,t)\mapsto(-x,-t)$ will treat the case where the ω–limit of Γ lies at infinity in the negative x–direction and the α-limit of Γ is given by a hyperbolic saddle in $D\backslash\partial D$.

Choose the section Σ_M^- to lie right of S such that the part of the center separatrix Γ^c of s lying between Σ_M^- and s is situated enterily under the x–axis. We perturb X as follows:

$$X_\eta : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x) + \eta y x^{n-1}, \end{cases} \tag{3.11}$$

corresponding with a change of the coefficient b_{n-1} of Q into $b_{n-1}-\eta$. Suppose now that $\eta>0$, then looking at the coefficient v_2 in (3.8) and noticing that A is equal to 1, one sees that the change of b_{n-1} corresponds in a locally downward motion of the center separatrix of s. Moreover the perturbed vector field points downward on that part of the center separatrix lying between s and Σ_M^- such that one sees that the intersection of Γ^c with Σ_M^- will move downward. On the other hand Lemma 3.4 implies, that the intersection of the unstable separatrix of S and Σ_M^- will move upward.

When Γ is an unbounded connection such that as well $\alpha(\Gamma)=s$ as $\omega(\Gamma)=S$ lies at infinity, the perturbation in (3.11) will cause, for $\eta>0$, the unstable separatrix of s to locally move downward. On the other hand, looking at the coefficient $\overline{v}_1$ in (3.10), one sees that this perturbation will lead to a locally upward motion of the stable separatrix of S. For $\eta>0$ the perturbed vector field points downward on the connection Γ, such that Γ will certainly break.

Suppose m is odd, n is even and $A=1$. In this case there are two saddles lying at infinity, both with an unstable separatrix in $D\backslash\partial D$. So in any case $\alpha(\Gamma)=s$ has to be one of the saddles at infinity and S can only be a saddle in $D\backslash\partial D$, supposed to be hyperbolic. Suppose s lies in the positive x–direction. The case where s lies in the negative x–direction can be studied by means of the transformation $(x, y) \mapsto (-x, -y)$.

We use the following perturbation:

$$X_\eta : \begin{cases} \dot{x} &= y, \\ \dot{y} &= P(x) + yQ(x) + \eta y x^{n-2}. \end{cases} \tag{3.12}$$

Using total analogous arguments as before, one sees that for $\eta > 0$ small, the intersection of Γ^c with Σ_M^- will move downward while the intersection of the unstable separatrix of S lying on Γ with Σ_M^- will move upward. □

We end this chapter with the following corollary of Theorem 3.3.

Corollary 3.9 *Suppose $\overline{X} \in L^{(m,n)}(D)$ is a Liénard system with a limit periodic set $\mathcal{L}$, containing at least one singularity. Then $\overline{X}$ cannot be structurally stable.*

Proof: Let us first treat the case where $\mathcal{L}$ is bounded. When $\mathcal{L}$ is a non–hyperbolic singularity, the result follows immediately from Theorem 3.2.

Suppose $\mathcal{L}$ contains at least one singularity and one regular orbit. From Theorems 2.2 and 3.2, one concludes that all singularities on $\mathcal{L}$ have to be hyperbolic saddles implying that $\mathcal{L}$ contains at least one bounded saddle–connection. So from Theorem 3.3, $\overline{X}$ cannot be structurally stable.

Consider now the case where $\mathcal{L}$ is unbounded. From Theorem 2.3, we know that this is only possible in 5 cases. We prove that in all these cases $\overline{X}$ will be structurally unstable.

Case 1 is excluded because we have supposed that $\mathcal{L}$ contains at least one singularity. In case 2 and 3, a slight increasement of the coefficient A will remove all singularities at infinity such that $\overline{X}$ cannot be structurally stable. In case 4 and 5, a similar reasoning as in the bounded case shows that $\mathcal{L}$ has to contain at least one unbounded saddle–connection. □

Remark: A structurally stable Liénard system cannot have a closed orbit being part of an *annulus* (an annulus is an open subset of D filled by closed orbits). Indeed a perturbation like in (3.2) with $f > 0$ would imply that, for $\varepsilon > 0$, the perturbed vector field points outward on all closed orbits in the annulus. This means that the annulus will not persist for $\varepsilon > 0$ small.

Chapter 4

Normal forms

In this chapter, we start by refreshing some elementary notions about *normal forms at singularities of vector fields.* For later use, we will focus our attention on the singularities at which the linear part of the vector field is diagonalisable. Afterwards we will try to obtain a further simplification of the obtained normal forms in case of a Hamiltonian vector field admitting a hyperbolic saddle and in case of a family of vector fields of which each member has a non–flat center behaviour of a same order near a semi–hyperbolic singularity.

The obtained normal forms and the necessary techniques to obtain them will be needed in the following chapters. Temporarily we will not limit ourselves anymore to Liénard systems, needing a normal form for general vector fields.

4.1 Singularities at which the linear part is diagonalisable

We repeat how to obtain a formal normal form at singularities at which the linear part of the vector field is diagonalisable. In particular we focus our attention on the specific transformations that one has to perform in order to obtain this normal form. In the appendix we implement these necessary transformations in a Maple–procedure that can be helpful in performing these kind of calculations, see Appendix A.1.

Let us start by repeating the formal normal form theorem for vector fields on $\mathbb{R}^n$ [19]. The original proof of the theorem can be found in [30].

Let X be a C^∞ vector field on $\mathbb{R}^n$ admitting a singularity s. After a translation the singularity can supposed to lie at the origin such that one can write $X = A + f$, where A is linear and f is a C^∞ function such that $f(0) = 0$ and $Df(0) = 0$.

Consider the so–called *adjoint action* or *Lie bracket operator*:

$$\begin{array}{rlcl} L_A: & H^r(\mathbb{R}^n) & \mapsto & H^r(\mathbb{R}^n) \\ & X & \mapsto & [A, X], \end{array}$$

where $H^r(\mathbb{R}^n)$ denotes the set of polynomial vector fields which are homogeneous of degree r and $[A, X]$ is given by

$$[A, X] = A \circ X - X \circ A,$$

where A and X are seen as differential operators. Let $B^r = L_A(H^r(\mathbb{R}^n))$ and G^r some complement, i.e. $B^r \oplus G^r = H^r(\mathbb{R}^n)$.

Theorem 4.1 *Let $k \in \mathbb{N}$ and X a C^k vector field on $\mathbb{R}^n$ with $X(0) = 0$ and $DX_0 = A$. Then for each $l \leq k$, there exists, locally near 0, an analytic coordinate change $w = \phi(z)$ such that the l–jet of $X'(w) = D\phi_z(X(z))$ at 0 reads*

$$j_l(X')(0) = A + g_2 + \ldots + g_l,$$

where $g_i \in G^i, \forall i : 2, \ldots, l$.

Proof: The proof proceeds by induction on $2 \leq r \leq l$. Suppose

$$j_r(X)(0)(z) = Az + g_2(z) + \ldots + g_{r-1}(z) + X_r(z), \tag{4.1}$$

where $g_i \in G^i, \forall i = 2, \ldots, r-1$ and X_r is homogeneous of degree r. The term X_r can be decomposed in $B^r \oplus G^r$ as $b_r + g_r$. In particular, there exists some polynomial $P \in H^r(\mathbb{R}^n)$ such that

$$X_r = L_A(P) + g_r. \tag{4.2}$$

We will show that, near the origin, the coordinate transformation

$$z = h(w) = w + P(w)$$

will remove the component b_r of X_r in (4.1).

Substitution gives

$$(I+DP(w))\,\dot{w} = A(w+P(w))+g_2(w)+\ldots+g_{r-1}(w)+X_r(w)+O(\| y \|^{r+1}).$$

where we have used that

$$P(y) = O(\| y \|^r) \quad \text{and} \quad Q_i(y + O(\| y \|^r) = Q_i(y) + O(\| y \|^{r+1})$$

for each $Q_i \in H^i(\mathbb{R}^n), i \geq 2$. Because $Dh_0 = I$, h is a diffeomorphism near the origin such that:

$$\begin{aligned}
\dot{w} &= (I + DP_w)^{-1}[Aw + AP(w) + \cdots] \\
&= (I - DP_w + O(\| y \|^r))[Aw + AP(w) + \ldots] \\
&= Aw + g_2(w) + \ldots + g_{r-1}(w) + X_r(w) \\
&\qquad +AP(w) - DP(w)Aw + O(\| w \|^{r+1}).
\end{aligned}$$

The $(r-1)$–jet of X remains unchanged, while the terms of order r change into $X_r(w) - L_A(P)(w)$, yielding the desired result. □

Notice that by Borel's theorem [2], the case where $l = k = \infty$ is not excluded in Theorem 4.1. Indeed the above construction guarantees the existence of a sequence of C^∞ coordinate changes $\{w = \psi^r(z)\}_{r\in\mathbb{N}_1}$, defined locally near the origin, such that the r–jet of $X'_r(w) = D\psi^r_x(X(z))$ at 0 reads

$$j_r(X'_r)(0)(w) = D\psi^r_z(X(z)) = Aw + g_2(w) + g_3(w) + \cdots + g_r(w),$$

where each $g_i \in G^i$, $2 \leq i \leq r$. Moreover $j_{r-1}(\psi^{r-1})(0) = j_{r-1}(\psi^r)(0)$. By Borel's theorem, there exists a C^∞ function ψ such that $j_r(\psi)(0) =$

$j_r(\psi^r)(0)$ for each $r \in \mathbb{N}_1$. One easily verifies that for $X'(w) = D\psi_z(X(z))$, with $w = \psi(z)$:

$$j_\infty X'(0) = A + \sum_{i=2}^{\infty} g_i,$$

with $g_i \in G^i$, $i \geq 2$.

As a special case, let us calculate normal forms of planar vector fields for which the linear part A of X is diagonalisable. After a linear coordinate transformation, we can suppose that A is given by

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 denote the eigenvalues of A. The set

$$\mathcal{B} := \{x^m y^{r-m} \frac{\partial}{\partial x} \mid 0 \leq m \leq r\} \cup \{x^m y^{r-m} \frac{\partial}{\partial y} \mid 0 \leq m \leq r\}$$

is a basis for $H^r(\mathbb{R}^2)$ and

$$\begin{aligned} L_A(x^m y^{r-m} \frac{\partial}{\partial x}) &= [\lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}, x^m y^{r-m} \frac{\partial}{\partial x}] \\ &= ((m-1)\lambda_1 + (r-m)\lambda_2)\, x^m y^{r-m} \frac{\partial}{\partial x}, \end{aligned} \tag{4.3}$$

and totally analogously

$$L_A(x^m y^{r-m} \frac{\partial}{\partial y}) = (m\lambda_1 + (r-m-1)\lambda_2)\, x^m y^{r-m} \frac{\partial}{\partial y}, \tag{4.4}$$

such that $\mathcal{B}$ is a basis consisting of eigenvectors of L_A. Therefore $G^r = \ker L_A$ is a well chosen complementary space of B^r with basis:

$$\begin{aligned} \{x^m y^{r-m} \frac{\partial}{\partial x} \mid (m-1)\lambda_1 + (r-m)\lambda_2 = 0, 0 \leq m \leq r\} \quad \cup \\ \{x^m y^{r-m} \frac{\partial}{\partial y} \mid m\lambda_1 + (r-m-1)\lambda_2 = 0, 0 \leq m \leq r\}. \end{aligned} \tag{4.5}$$

From the formal normal form theorem, one concludes that there exist, locally near the origin, C^∞ coordinates $(u,v) = \varphi(x,y)$ in which the ∞–jet of X reads:

$$(\lambda_1 u + \sum{}^1 a^r_m u^m v^{r-m})\frac{\partial}{\partial u} + (\lambda_2 v + \sum{}^2 b^r_m u^m v^{r-m})\frac{\partial}{\partial v}, \qquad (4.6)$$

where Σ^i is the sum taken over those (m,r) satisfying $\lambda_i = m\lambda_1 + (r-m)\lambda_2$, $i = 1,2$.

We emphase the fact that Theorem 4.1 only guarantees a formal normal form; X can be brought to (4.6) up to flat terms. However it can be shown that the above normal form (4.6) is not only formal; there indeed exists a near–identity transformation (a transformation that equals the identity map up to flat terms) removing possible flat terms appearing in the normal form. We will not go into further detail here and refer the reader to [14]. As a special case however, we will show in Section 4.3 how to remove the flat terms in case the singularity is semi–hyperbolic.

Furthermore, let us remark that, from Sternberg [29], a given C^∞ vector field is always C^k conjugate to its $N(k)$–jet in a neighbourhood of a given hyperbolic singularity for some $N(k) \geq k$, $k \in \mathbb{N}$.

Theorem 4.2 (Sternberg) *For each $k \in \mathbb{N}_1$, there exists $N(k) \geq k$ such that a given C^∞ vector field X is always C^k conjugate to its $N(k)$–jet in a neighbourhood of a given hyperbolic singularity.*

We continue by treating families of vector fields. Let (X_μ) be a C^∞ family of planar vector fields, μ varying in some open set $\mathcal{P} \subset \mathbb{R}^p$ such that $\forall \mu \in \mathcal{P}$, X_μ admits a singularity s_μ at which the Jordan form of the linear part of X_μ is given by

$$A_\mu = \begin{pmatrix} \lambda_1(\mu) & 0 \\ 0 & \lambda_2(\mu) \end{pmatrix}.$$

We have the following theorem of which the proof is based on techniques introduced in [26]. The jets are only taken in the (x,y)–direction and the C^∞ conjugacies are conjugacies by means of a diffeomorphism depending C^∞ on (x,y,μ).

Theorem 4.3 *Let $(X_\mu)_{\mu\in\mathcal{P}}$ be a family as above. Let $\mu_0 \in \mathcal{P}$. Then for each $l \in \mathbb{N}$, there exists a neighbourhood $\mathcal{P}_l$ of μ_0 in parameter space such that, for $\mu \in \mathcal{P}_l$, the l–jet of X_μ at s_μ is locally C^∞ conjugate to:*

$$\begin{aligned} j_l(X_\mu)(s_\mu) \sim \left(\lambda_1(\mu)x + \sum_{k\le l}{}^{1} a_m^k(\mu)x^m y^{k-m}\right)\frac{\partial}{\partial x} + \\ \left(\lambda_2(\mu)y + \sum_{k\le l}{}^{2} b_m^k(\mu)x^m y^{k-m}\right)\frac{\partial}{\partial y}, \end{aligned} \tag{4.7}$$

for some coefficients $a_m^k(\mu), b_m^k(\mu)$ smooth in μ and where $\sum_{k\le l}{}^{i}$ is the sum taken over those (m,k) for which $k \le l$ and $\lambda_i(\mu_0) = m\lambda_1(\mu_0) + (k-m)\lambda_2(\mu_0)$, $i = 1, 2$.

Proof: First one can perform a suitable, parameter dependent, linear transformation such that the singularity s_μ lies at the origin, for each parameter value $\mu \in \mathcal{P}$, and such that the linear part of (X_μ) at the origin is given by:

$$A_\mu = \begin{pmatrix} \lambda_1(\mu) & 0 \\ 0 & \lambda_2(\mu) \end{pmatrix}.$$

One proceeds by induction on $2 \le r \le l$. Suppose one has found a neighbourhood $\mathcal{P}_{r-1}$ of μ_0 in parameter space such that $\forall \mu \in \mathcal{P}_{r-1}$, the r–jet of (X_μ) at s_μ is locally C^∞ conjugate to:

$$\begin{aligned} j_r(X_\mu)(s_\mu) \sim \left(\lambda_1(\mu)x + \sum_{k\le r-1}{}^{1} a_m^k(\mu)x^m y^{k-m} + X_\mu^r(x,y)\right)\frac{\partial}{\partial x} + \\ \left(\lambda_2(\mu)y + \sum_{k\le r-1}{}^{2} b_m^k(\mu)x^m y^{k-m} + Y_\mu^r(x,y)\right)\frac{\partial}{\partial y}, \end{aligned} \tag{4.8}$$

where X_μ^r and Y_μ^r are homogeneous polynomials in (x, y) of degree r. Denote $Z_\mu^r = X_\mu^r \frac{\partial}{\partial x} + Y_\mu^r \frac{\partial}{\partial y} \in H^r(\mathbb{R}^2)$. Denote ρ_μ^r as the Lie bracket operator on $H^r(\mathbb{R}^2)$, $r \ge 2$:

$$\begin{aligned} \rho_\mu^r : \; H^r(\mathbb{R}^2) &\mapsto H^r(\mathbb{R}^2) \\ X &\mapsto [A_\mu, X]. \end{aligned}$$

For $\mu = \mu_0$, we already know that $H^r(\mathbb{R}^2)$ can be decomposed in $B_0^r \oplus G_0^r$, with $B_0^r = \text{Im } \rho_{\mu_0}^r$ and $G_0^r = \ker \rho_{\mu_0}^r$. A basis for G_0^r is given like in (4.5), where the appearing eigenvalues of A are replaced by the eigenvalues $\lambda_1(\mu_0)$ and $\lambda_2(\mu_0)$ of A_{μ_0}.

Suppose $G_0^r = \{0\}$, then $\rho_{\mu_0}^r$ is inversible such that ρ_μ^r is inversible for μ in a neighbourhood $\mathcal{P}_r \subset \mathcal{P}_{r-1}$ of μ_0. In particular, one can solve the equation

$$[A_\mu, P_\mu^r] = Z_\mu^r,$$

for some homogeneous polynomial P_μ^r of degree r in (x, y). Analogous as in the proof of Theorem 4.1, one verifies that changing to the C^∞ coordinates (u, v) determined by $(x, y) = (I + P_\mu^r)(u, v)$ will, locally near the origin, remove the homogeneous term Z_μ^r in (4.8).

Suppose now $G_0^r \neq \{0\}$. One easily verifies that ρ_0^r is an isomorphism of B_0^r onto itself. By continuity the space $B_\mu^r = \rho_\mu^r(B_0^r)$ is of codimension 2 for μ near μ_0. Taking perhaps a smaller neighbourhood $\mathcal{P}_r \subset \mathcal{P}_{r-1}$ of μ_0, B_μ^r is transversal to G_0^r, $\forall \mu \in \mathcal{P}_r$. So one concludes that $H^r(\mathbb{R}^2) = B_\mu^r \oplus G_0^r$ such that one can find C^∞ polynomials $P_\mu^r \in B_0^r$ and $g_\mu^r \in G_0^r$ satisfying

$$Z_\mu^r = [A_\mu, P_\mu^r] + g_\mu^r. \tag{4.9}$$

Changing to C^∞ coordinates (u, v) determined by $(x, y) = (I + P_\mu^r)(u, v)$ will, locally near the origin bring (4.8) into the desired normal form. □

The proof of Theorem 4.3 reveals an inductive way for calculating the normal form (4.6) up to any finite order $l \in \mathbb{N}$. Let us specify the transformations that have to be performed in each induction step in the proof of Theorem 4.3.

These transformations are determined by equation (4.9). Denote

$$Z_\mu^r(x, y) = \sum_{m \leq r} \left(\alpha_m^r(\mu) x^m y^{r-m}\right) \frac{\partial}{\partial x} + \sum_{m \leq r} \left(\beta_m^r(\mu) x^m y^{r-m}\right) \frac{\partial}{\partial y}.$$

For obtaining an expression for P^r_μ, we decompose Z^r_μ in $G^r_0 \oplus B^r_\mu$ as $g^r_\mu + b^r_\mu$ with

$$b^r_\mu(x,y) := \sum_{m\leq r}{}^1 \left(\alpha^r_m(\mu)x^m y^{r-m}\right)\frac{\partial}{\partial x} + \sum_{m\leq r}{}^2 \left(\beta^r_m(\mu)x^m y^{r-m}\right)\frac{\partial}{\partial y},$$

where the sums appearing in this equation are taken over those m satisfying $m \leq r$ and $\lambda_i(\mu_0) \neq m\lambda_1(\mu_0)+(r-m)\lambda_2(\mu_0)$, $i = 1,2$ respectively.

Replacing in the equations (4.3) and (4.4), the appearing eigenvalues of A by the eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$ of A_μ, one sees immediately that $P^r_\mu(x,y)$ is given by

$$\begin{aligned} P^r_\mu(x,y) &\\ = &\\ \sum_{m\leq r}{}^1 \left(\frac{\alpha^r_m(\mu)}{(m-1)\lambda_1(\mu)+(r-m)\lambda_2(\mu)}x^m y^{r-m}\right)\frac{\partial}{\partial x} &\\ +\sum_{m\leq r}{}^2 \left(\frac{\beta^r_m(\mu)}{m\lambda_1(\mu)+(r-m-1)\lambda_2(\mu)}x^m y^{r-m}\right)\frac{\partial}{\partial y}, & \end{aligned} \tag{4.10}$$

for μ near μ_0, is an element of B^r_0 satisfying $[A_\mu, P^r_\mu] = b^r_\mu$.

So, for μ near μ_0, we conclude that, after a translation, moving the singularities s_μ to the origin, and a linear transformation, bringing the linear parts in their diagonal form, one can obtain the normal form (4.7) of the l–jet of (X_μ) at s_μ by inductively changing to coordinates (u,v) determined by

$$(x,y) = (I + P^r_\mu)(u,v),$$

where P^r_μ is like in (4.10), $2 \leq r \leq l$.

The above procedure for obtaining the normal form (4.7) can easily be implemented in a Maple program, see Appendix A.1.

4.2 Normal forms at hyperbolic saddles

As an application of the previous section, we treat normal forms at hyperbolic saddles. We will get to a further simplification in case of a Hamiltonian vector field admitting a hyperbolic saddle. We will need this normal form in Chapter 5, where alien limit cycles are searched for near a 2–saddle cycle in an unfolding of a Hamiltonian vector field.

Consider a C^∞ family of planar vector fields (X_μ) with parameter values μ varying in some open set $\mathcal{P}$ of $\mathbb{R}^p$. Suppose that for some $\mu_0 \in \mathcal{P}$, X_{μ_0} admits a hyperbolic saddle s. Denote the jordan form of $DX_{\mu_0}(s)$ as

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

with $\lambda_2 < 0 < \lambda_1$. The *ratio of hyperbolicity of* X_{μ_0} *at* s is given by $-\lambda_2/\lambda_1$.

As an easy consequence of the implicit function theorem, one can suppose that the saddle will persist for all X_μ, $\mu \in \mathcal{P}$. By this we mean that there exists a C^∞ function $s : \mathcal{P} \mapsto \mathbb{R}^2$ such that each X_μ admits a hyperbolic saddle at $s_\mu := s(\mu)$ with $s_{\mu_0} = s$. As a direct application of Theorem 4.3, we have the following theorem.

Theorem 4.4 *Let* $(X_\mu)_{\mu\in\mathcal{P}}$ *be a* C^∞ *family as above such that* X_{μ_0} *admits a hyperbolic saddle* s*. Suppose the ratio of hyperbolicity of* X_{μ_0} *at* s *is rational, given by* p/q *with* $p, q \in \mathbb{N}_1, (p,q) = 1$*. Then for each* $N \in \mathbb{N}$*, there exists a neighbourhood* $\mathcal{P}_N$ *of* μ_0 *in parameter space such that the* M*–jet,* $M = N(p+q)+1$*, at* s_μ *of each* X_μ*,* $\mu \in \mathcal{P}_N$*, is locally* C^∞*–conjugate to:*

$$j_M(X_\mu)(s_\mu) \sim \begin{cases} \dot{x} &= x\Big(\lambda_1 + \sum_{i=0}^N a_i(\mu)(x^p y^q)^i\Big), \\ \dot{y} &= y\Big(\lambda_2 + \sum_{i=0}^N b_i(\mu)(x^p y^q)^i\Big), \end{cases}$$

where the coefficients $a_i(\mu)$ *and* $b_i(\mu)$ *are smooth in* μ*.*

In case the ratio of hyperbolicity of X_{μ_0} *at* s *is irrational, then for every* $N \in \mathbb{N}$*, there exists a neighbourhood* $\mathcal{P}_N$ *of* μ_0 *in parameter space such that the* N*–jet of* X_μ*,* $\mu \in \mathcal{P}_N$*, is, locally near* s_μ*,* C^∞ *linearisable.*

Notice that the above theorem only applies to every finite jet of the family of vector fields, while for an individual vector field we have a normal form for its infinite jet at our disposal. However from Theorem 4.2, which is also available for families of vector fields ([1] or [21]), one can restrict to a finite jet of the family, but then one has to cope with a finite smoothness of the conjugacy.

Theorem 4.5 *Let* $(X_\mu)_{\mu\in\mathcal{P}}$ *be a* C^∞ *family such that* X_{μ_0} *admits a hyperbolic saddle* s*. Then for any* $k \in \mathbb{N}_1$*, there exists some neighbourhood* $\mathcal{P}_k$ *of* μ_0 *such that, when the ratio of hyperbolicity of* X_{μ_0} *at* s *is given by* p/q*,* $p,q \in \mathbb{N}_1$*,* $(p,q)=1$*, the family* $(X_\mu)_{\mu\in\mathcal{P}_k}$ *is, locally near* s_μ*,* C^k*–conjugate to:*

$$\left\{\begin{array}{rcl} \dot{x} & = & x\left(\lambda_1 + \sum_{i=0}^{N(k)} a_i(\mu)(x^p y^q)^i\right), \\ \dot{y} & = & y\left(\lambda_2 + \sum_{i=0}^{N(k)} b_i(\mu)(x^p y^q)^i\right), \end{array}\right.$$

for some $N(k) \geq k$*. When the ratio is irrational, the family* $(X_\mu)_{\mu\in\mathcal{P}_k}$ *is, locally near* s_μ*,* C^k*–linearisable.*

Let us now treat the case of a Hamiltonian vector field admitting a hyperbolic saddle and come to a further simplification. Similar reductions of Morse functions are already obtained [20]. However results in [20] are only valid near critical points that are no saddle points.

The following propositions apply to a more general class of vector fields, the class of *integrable vector fields.* A C^∞ vector field X on an open set $U \subset \mathbb{R}^n$ is called integrable if there exists a smooth function H on U which is constant on all solution curves contained in U. The function H is called *a first integral of* $X = X_1\frac{\partial}{\partial x} + X_2\frac{\partial}{\partial y}$ and obviously it has to satisfy $XH = X_1\frac{\partial H}{\partial x} + X_2\frac{\partial H}{\partial y} = 0$.

A hyperbolic saddle s of X corresponds to *a Morse point of* H, meaning that $DH(s) = 0$ and the Hessian of H at s is invertible. From Morse's lemma [24], there exist C^∞ coordinates (u, v) near s in which H reads uv. In these coordinates X will take a very simple normal form up to C^∞ equivalence.

Proposition 4.6 *Let X be an integrable vector field with integral H and admitting a hyperbolic saddle s. Denote by (u, v) the coordinates near s, given by Morse's lemma, in which H reads uv. Then, near s and expressed in the coordinates (u, v), X reads:*

$$\begin{cases} \dot{u} &= -u, \\ \dot{v} &= v, \end{cases} \tag{4.11}$$

up to C^∞ equivalence and a possible coordinate switch in (u, v).

Proof: Denote by Y, the vector field, defined locally near the origin, that one obtains after expressing X in the coordinates (u, v). It is clear that uv is a first integral of Y such that:

$$vY_1(u, v) + uY_2(u, v) = 0, \tag{4.12}$$

already implying that

$$Y_1 = u\overline{Y}_1 \quad \text{and} \quad Y_2 = v\overline{Y}_2, \tag{4.13}$$

for some C^∞ functions $\overline{Y}_1$ and $\overline{Y}_2$ with $\overline{Y}_1(0)\overline{Y}_2(0) \neq 0$. Substituting (4.13) into (4.12), one obtains $\overline{Y}_1 = -\overline{Y}_2$ or:

$$\begin{cases} \dot{u} &= -u\overline{Y}_2(u, v), \\ \dot{v} &= v\overline{Y}_2(u, v). \end{cases}$$

Clearly $\overline{Y}_1(0)\overline{Y}_2(0) < 0$ such that after a coordinate switch in the coordinates (u, v) one can always suppose that $\overline{Y}_2(0) > 0$ implying the desired result. □

One can ask oneself whether the opposite of Proposition 4.6 is true. Let

X be an integrable vector field with integral H and admitting a hyperbolic saddle s. Denote by (u, v) the C^∞ coordinates in which X reads like in (4.11), up to C^∞ equivalence. Will H read uv when expressed in the coordinates (u, v)? The answer is no, but we can prove a weaker result, stating that H restricted to each of the half planes $\{u \geq 0\}$, $\{u \leq 0\}$, $\{v \geq 0\}$ or $\{v \leq 0\}$ can be written as uv. Let us first prove the following lemma.

Lemma 4.7 *Suppose $F : V \subset \mathbb{R}^2 \mapsto \mathbb{R}$ is a C^∞ function, defined on a neighbourhood V of the origin, satisfying*

$$x\frac{\partial F}{\partial x}(x, y) - y\frac{\partial F}{\partial y}(x, y) = 0. \tag{4.14}$$

Let $\mathcal{H}$ be one of the half planes: $\{x \geq 0\}$, $\{x \leq 0\}$, $\{y \geq 0\}$ or $\{y \leq 0\}$. Then on $\mathcal{H}$ and locally near the origin, F can be written as a C^∞ function in $z = xy$. By this we mean that there exists a C^∞ function $f : I \subset \mathbb{R} \mapsto \mathbb{R}$, defined on a neighbourhood I of zero, such that $F(x, y) = f(xy)$, $\forall (x, y) \in V \cap \mathcal{H}$, with $xy \in I$.

Proof: Let us suppose that $\mathcal{H}$ is given by $\{x \geq 0\}$, the other cases being totally analogous. Consider the system:

$$\begin{cases} \dot{x} &= -x, \\ \dot{y} &= y. \end{cases} \tag{4.15}$$

The orbits of this system lying in $\mathcal{H}$ are given by the curves $xy = c$, $c \neq 0$ together with the origin as a singularity and the positive x–axis as well as the positive and negative y–axis. Equation (4.14) clearly implies that on V, F is a first integral of system (4.15). In particular F stays constant on the orbits of (4.15) lying in $\mathcal{H} \cup V$. Let us define, for c near zero, $f(c) = F(c_0, \frac{c}{c_0})$, c_0 chosen such that $(c_0, 0) \in \mathcal{H} \cup V$. It is easily seen that f is C^∞.

We now check that $f(xy) = F(x, y)$ on $\mathcal{H}$ and locally near the origin. When $x > 0$, this follows immediately by the definition of f. Suppose now $x = 0$, then because $f(0) = F(x, 0), \forall x > 0$, the continuity of F implies

$F(0,0) = f(0)$. Because F stays constant on the positive and negative y–axis, it follows that $F(0,y) = F(0,0), \forall y$ implying the required result. □

Proposition 4.8 *Let X be an integrable vector field with first integral H that admits a hyperbolic saddle s. Suppose that there exist C^∞ coordinates (u,v), near s, in which X reads:*

$$\begin{cases} \dot{u} &= -u, \\ \dot{v} &= v, \end{cases} \tag{4.16}$$

up to C^∞ equivalence. Then H, expressed in the coordinates (u,v), can be written as a C^∞ function in uv, locally near the origin, on each of the half planes $\{u \geq 0\}$, $\{u \leq 0\}$, $\{v \geq 0\}$ or $\{v \leq 0\}$.

Proof: Denote φ as the coordinate transformation expressing the old coordinates (x,y) in function of the new ones (u,v). Clearly $H \circ \varphi$ is an integral of the system (4.16) such that, near the origin,

$$u\frac{H \circ \varphi}{\partial u}(u,v) - v\frac{H \circ \varphi}{\partial v}(u,v) = 0.$$

Applying Lemma 4.7, the result immediately follows. □

We now apply the above propositions on *a Hamiltonian vector field X_H* given by:

$$\begin{cases} \dot{x} &= -\dfrac{\partial H}{\partial y}, \\ \dot{y} &= \dfrac{\partial H}{\partial x}, \end{cases} \tag{4.17}$$

for some C^∞ function H that is called the *Hamiltonian* of X_H. Suppose X_H admits a hyperbolic saddle s.

Using Proposition 4.6, one sees that X_H can be brought to a very simple normal form up to C^∞ equivalence. We now want to describe another way to obtain this normal form which is not based on Morse's lemma but on Proposition 4.8. As starting point, we use the formal normal form obtained in (4.6) which can be an advantage when performing calculations in practice. The formal normal form (4.6) can easily be obtained using the Maple program in Appendix A.1.

The divergence of X_H is everywhere zero. In particular the eigenvalues of the linear part of X_H at s are opposite, $\pm\lambda$, $\lambda > 0$ implying that the ratio of hyperbolicity of X_H at s equals 1. So, from (4.6), we know that, locally near s, there exist C^∞ coordinates (u, v) in which the ∞–jet of X_H reads:

$$\left\{\begin{array}{rcl} \dot{u} &=& -u\Big(\lambda + \sum_{i\geq 1} a_i(uv)^i\Big), \\ \dot{v} &=& v\Big(\lambda + \sum_{i\geq 1} b_i(uv)^i\Big), \end{array}\right. \tag{4.18}$$

for some $a_i, b_i \in \mathbb{R}$, $i \in \mathbb{N}_1$. In the next proposition, we prove that $a_i = b_i$ in the above normal form.

Proposition 4.9 *For a Hamiltonian vector field X_H (4.17) that admits a saddle s, there exist C^∞ coordinates (u, v), near s, in which the ∞–jet of X_H reads:*

$$\left\{\begin{array}{rcl} \dot{u} &=& -u\Big(\lambda + \sum_{i\geq 1} a_i(uv)^i\Big), \\ \dot{v} &=& v\Big(\lambda + \sum_{i\geq 1} a_i(uv)^i\Big), \end{array}\right. \tag{4.19}$$

for some $a_i \in \mathbb{R}, i \in \mathbb{N}_1$.

Proof: We already know that there exists a C^∞ coordinate transformation $(x, y) = \varphi_1(u, v)$ bringing X_H in (4.18). We prove that the coefficients a_i and b_i in (4.18) are identical.

It is easily verified that the coordinate transformation $(x, y) = \varphi_1(u, v)$ transforms the Hamiltonian vector field into:

$$\left\{\begin{array}{rcl} \dot{u} &=& -\dfrac{1}{\det D\varphi_1(u,v)}\dfrac{\partial H \circ \varphi_1}{\partial v}(u, v), \\ \dot{v} &=& \dfrac{1}{\det D\varphi_1(u,v)}\dfrac{\partial H \circ \varphi_1}{\partial u}(u, v). \end{array}\right. \tag{4.20}$$

On the other hand using Borel's theorem on the realisation of formal power series, one finds smooth functions f and g such that

$$j_\infty f(0)(z) = \sum_{i\geq 1} a_i z^i, \qquad j_\infty g(0)(z) = \sum_{i\geq 1} b_i z^i.$$

In particular φ_1 brings X_H into

$$\left\{ \begin{array}{rcl} \dot{u} &=& -u\Big(\lambda + f(uv)\Big) + R(u,v), \\ \dot{v} &=& v\Big(\lambda + g(uv)\Big) + S(u,v), \end{array} \right. \tag{4.21}$$

with $j_\infty R(0) = j_\infty S(0) = 0$. After a suitable near–identity transformation, one can suppose that $R = S = 0$ [14]. Comparing (4.20) with (4.21), one sees that:

$$\left\{ \begin{array}{rcl} \dfrac{\partial H \circ \varphi_1}{\partial v} &=& u \det D\varphi_1(u,v)(\lambda + f(uv)), \\ \dfrac{\partial H \circ \varphi_1}{\partial u} &=& v \det D\varphi_1(u,v)(\lambda + g(uv)). \end{array} \right. \tag{4.22}$$

Abbreviating $\det D\varphi_1(u,v)$ as $D(u,v)$, one obtains

$$\lambda\left(u\frac{\partial D}{\partial u}(u,v) - v\frac{\partial D}{\partial v}(u,v)\right) + u\frac{\partial D}{\partial u}(u,v)f(uv) - v\frac{\partial D}{\partial v}(u,v)g(uv)$$
$$+D(u,v)\big(f(uv) - g(uv) + uv(f'(uv) - g'(uv))\big) = 0. \tag{4.23}$$

By comparing terms in $(uv)^k$ in the $2k$–jet at $u = v = 0$ of (4.23), we prove by induction on $k \geq 1$ that $a_k = b_k, \forall k \in \mathbb{N}_1$.

It is easily seen that in every $2k$–jet, $k \in \mathbb{N}_1$, at zero of the expression

$$u\frac{\partial D}{\partial u}(u,v) - v\frac{\partial D}{\partial v}(u,v) = 0,$$

no terms in $(uv)^i, i \leq k$ appear. Therefore comparing terms in uv of the 2–jet at zero of (4.23), one sees immediately that $a_1 = b_1$. Suppose now by induction that $a_i = b_i, i \leq k$, then no terms in $(uv)^i, i \leq k+1$ appear in the $(2k+2)$–jet at zero of the expression

$$u\frac{\partial D}{\partial u}(u,v)f(uv) - v\frac{\partial D}{\partial v}(u,v)g(uv)$$

yielding, by comparing terms in $(uv)^{k+1}$ in the $(2k+2)$–jet at zero of (4.23), $a_{k+1} = b_{k+1}$. □

Theorem 4.10 *Let X_H be a Hamiltonian vector field that admits a hyperbolic saddle s at which the eigenvalues of $DX_H(s)$ are given by $\pm\lambda, \lambda > 0$. Then there exist C^∞ coordinates (n, m) near s in which X_H reads:*

$$\left\{ \begin{array}{rcl} \dot{n} & = & -n, \\ \dot{m} & = & m, \end{array} \right. \tag{4.24}$$

up to a C^∞–equivalence factor $\lambda + d(nm)$, for some smooth function d with $d(0) = 0$.

Denote by φ the coordinate transformation expressing the old coordinates (x, y) in function of the new ones (n, m) and let $\mathcal{H}$ be one of the half planes $\{n \geq 0\}$, $\{n \leq 0\}$, $\{m \geq 0\}$ or $\{m \leq 0\}$. Then φ can be chosen such that the Hamiltonian expressed in the coordinates (n, m) reads nm on $\mathcal{H}$ and the equivalence factor equals $1/\det D\varphi(n, m) = \lambda + d(nm)$ on $\mathcal{H}$.

Proof: From Proposition 4.9, one already knows that there exists a C^∞ coordinate change $(x, y) = \varphi_1(u, v)$ bringing the ∞–jet of X_H near s into:

$$\left\{ \begin{array}{rcl} \dot{u} & = & -u\Big(\lambda + \sum_{i\geq 1} a_i(uv)^i\Big), \\ \dot{v} & = & v\Big(\lambda + \sum_{i\geq 1} a_i(uv)^i\Big), \end{array} \right. \tag{4.25}$$

for some $a_i \in \mathbb{R}$, $i \in \mathbb{N}_1$. Using Borel's theorem on the realisation of formal power series, (4.25) can be written as:

$$\left\{ \begin{array}{rcl} \dot{u} & = & -u(\lambda + f(uv)) + R(u, v), \\ \dot{v} & = & v(\lambda + f(uv)) + S(u, v), \end{array} \right. \tag{4.26}$$

for some C^∞ functions f, with $j_\infty f(0)(z) = \sum_{i\geq 1} a_i z^i$. Applying a suitable near–identity transformation, one can suppose that the flat terms R and S are zero [14], such that the normal form of X_H for C^∞ equivalence reads:

$$\left\{\begin{array}{rcl} \dot{u} & = & -u, \\ \dot{v} & = & v. \end{array}\right. \tag{4.27}$$

From Proposition 4.8, one knows that H, expressed in the new coordinates (u, v), is a function in uv locally near the origin on each of the half planes $\{u \geq 0\}$, $\{u \leq 0\}$, $\{v \geq 0\}$ or $\{v \leq 0\}$. Suppose $H(u, v) = uvH_0(uv)$ on $\{u \geq 0\}$ with $H_0(0) \neq 0$. After a reflection with respect to the u–axis, one can suppose that $H_0(0) > 0$. One performs the local transformation $(n, m) = \varphi_2(u, v)$, with

$$n = uG_0(uv), \qquad m = vG_0(uv), \tag{4.28}$$

with $G_0(uv) = \sqrt{H_0(uv)}$. This transformation will leave (4.27), up to C^∞ equivalence, invariant but will bring the Hamiltonian into nm on the half plane $\{n \geq 0\}$. In the new coordinates (n, m), X_H reads:

$$\left\{\begin{array}{rcl} \dot{n} & = & -\dfrac{1}{\det D\varphi(n,m)} \dfrac{\partial H \circ \varphi}{\partial m}(n, m), \\ \dot{m} & = & \dfrac{1}{\det D\varphi(n,m)} \dfrac{\partial H \circ \varphi}{\partial n}(n, m), \end{array}\right.$$

where $\varphi = \varphi_1 \circ \varphi_2^{-1}$. On the other hand it is a straightforward calculation to verify, using Lemma 4.7, that $f(uv)$ can be written as function d in nm locally near the origin on the half plane $\{n \geq 0\}$. Therefore applying the transformation (4.28) on (4.26) (with $R = S = 0$), one finds:

$$\left\{\begin{array}{rcl} \dot{n} & = & -n(\lambda + d(nm)), \\ \dot{m} & = & m(\lambda + d(nm)), \end{array}\right.$$

for some C^∞ function d with $d(0) = 0$ implying the result on $\{n \geq 0\}$. The same arguments can be used for obtaining the result on the half planes $\{n \leq 0\}$, $\{m \geq 0\}$ or $\{m \leq 0\}$. □

Remark that after Section 4.1, one is able to calculate any finite jet of the transformation φ_1 used in the proof of the above theorem. Also φ_2 is determined up to any finite order such that one can calculate any finite jet of $\varphi = \varphi_1 \circ \varphi_2^{-1}$. We will illustrate these calculations in a specific example in Chapter 5.

4.3 Normal forms at semi–hyperbolic singularities

Suppose $(X_\mu)_{\mu \in K}$ is a C^∞ family of vector fields, μ varying in some compact subset $K \subset \mathbb{R}^p$, such that each X_μ admits a semi–hyperbolic singularity that can supposed to lie at the origin after a suitable translation. By the Jordan normal form theorem, we can write the family as:

$$(X_\mu) : \begin{cases} \dot{x} &= \lambda(\mu)x + F(x, y, \mu), \\ \dot{y} &= G(x, y, \mu), \end{cases} \tag{4.29}$$

where λ, F and G are C^∞ functions satisfying $\lambda(\mu) \neq 0$, $F(0, 0, \mu) = G(0, 0, \mu) = 0$ and $D_z F(0, 0, \mu) = D_z G(0, 0, \mu) = 0, \forall \mu \in K$ where D_z represents the differential with respect to $z = (x, y)$.

In Theorem 4.3, we have already found a normal form of a family like in (4.29). In this section however, we try to continue this simplification to come to a better normal form. We will need this normal form in Chapter 6, where we treat a cyclicity problem near an unbounded 2–saddle cycle, containing 2 semi–hyperbolic saddles at infinity.

Let us assume that all $X_\mu, \mu \in K$ have a *non–flat center behaviour of a same order*. By this we mean the following. Take $\mu \in K$ arbitrarily but fixed, then the singularities of X_μ in a neighbourhood of the origin are lying on

$$\lambda(\mu)x + F(x, y, \mu) = 0,$$

which by the implicit function theorem is locally given by a graph $x = f_\mu(y)$ for some C^∞ function f_μ. f_μ is analytic in case F is. So the ordinates of the singularities of X_μ are solutions of

$$G(f_\mu(y), y, \mu) = 0.$$

Saying that all X_μ have a same non-flat center behaviour of a same order $k \in \mathbb{N}_2$ is saying that $\forall \mu \in K$:

$$G(f_\mu(y), y, \mu) = y^k h_\mu(y),$$

for some h_μ, C^∞ in y with $h_\mu(0) \neq 0$. Without loss of generality, one can suppose that $h_\mu(0) > 0, \forall \mu \in K$. After a linear change in y, we can take $h_\mu(y) = 1 + y\overline{h}(y, \mu)$ for some C^∞ function $\overline{h}$.

The rest of this section will be devoted to proving the existence of a coordinate transformation putting (4.31), up to C^∞–equivalence, into the form:

$$\left\{ \begin{array}{rcl} \dot{x} & = & x, \\ \dot{y} & = & \dfrac{y^k}{1 - a(\mu) y^{k-1}}, \end{array} \right. \tag{4.30}$$

where $a(\mu)$ is C^∞ and where we have supposed that $\lambda(\mu) > 0, \forall \mu \in K$. When $\lambda_\mu(0) < 0, \forall \mu \in K$, there will appear a minus sign in the first component.

We will heavily rely on the calculations made in [10], where one already obtains a normal form like in (4.30) for an individual vector field admitting a semi–hyperbolic singularity.

In all these calculations we will frequently use the following notation. If $f(u, \mu)$, with $u \in \mathbb{R}$ or $u \in \mathbb{R}^2$, denotes an arbitrarily C^∞ function, then for each $\mu \in K$ the map $u \mapsto f(u, \mu)$ is denoted as f_μ.

The first thing we note is that the normal form in (4.7) also applies to the ∞–jet of (X_μ). Indeed referring to the proof of Theorem 4.3, one sees that, because $\lambda(\mu) \neq 0, \forall \mu \in K$, each B^r_μ, $r \in \mathbb{N}_1$ stays transversal to G^r_0, $\forall \mu \in K$. In particular, one can perform the transformations $(x, y) = (I + P^r_\mu)(u, v)$, $\forall r \in \mathbb{N}_1$, where P^r_μ is defined as in (4.10), $\forall \mu \in K$.

This yields a sequence of C^∞ coordinate changes $\{\varphi_\mu^r(x,y)\}_{r\in\mathbb{N}_1}$, $\mu \in K$ defined locally near the origin bringing for each r, the r–jet of (X_μ) into:

$$\begin{cases} \dot{x} &= \lambda(\mu)x + \sum_{i=2}^r a_i(\mu)xy^{i-1}, \\ \dot{y} &= \sum_{i=2}^r b_i(\mu)y^i, \end{cases}$$

for some coefficients $a_i(\mu)$ and $b_i(\mu)$, $i = 2,\dots,r$, C^∞ dependent on μ. By Borel's theorem, we find a C^∞ function $\varphi : (x,y,\mu) \mapsto (\varphi_\mu(x,y),\mu)$ such that $j_\infty\varphi_\mu(0,0) = j_r(\varphi_\mu^r)(0,0)$, $\forall r \in \mathbb{N}$. In particular φ will bring (X_μ) into:

$$\begin{cases} \dot{x} &= x(\lambda(\mu) + yf(y,\mu)) + p(x,y,\mu), \\ \dot{y} &= y^k(1 + yg(y,\mu)) + q(x,y,\mu), \end{cases} \tag{4.31}$$

for some C^∞ functions f, g, p and q with $j_\infty p_\mu(0,0) = 0$, $j_\infty q_\mu(0,0) = 0$ for each $\mu \in K$. Notice that we have also applied a linear change in y in order to change the coefficient in front of y^k, in the second component of (4.31), into 1.

We continue by applying the stable manifold theorem, inducing the existence of an invariant manifold for (4.31) tangent to the $x\mu$–plane that can be straightened to $\{y = 0\}$ by a near–identity transformation. A near–identity transformation $\varphi : \mathbb{R}^2 \times K \mapsto \mathbb{R}^2$ satisfies $j_\infty(\varphi_\mu - I)(0,0) = 0, \forall\mu \in K$ and will not affect the ∞–jet of (4.31).

Straightening this invariant manifold, will change expression (4.31) into:

$$\begin{cases} \dot{x} &= x(\lambda(\mu) + yf(y,\mu)) + \tilde{p}(x,y,\mu), \\ \dot{y} &= y^k(1 + yg(y,\mu)) + y\tilde{q}(x,y,\mu), \end{cases} \tag{4.32}$$

where λ, f, g are as in (4.31) and $\tilde{p}$, $\tilde{q}$ are C^∞ functions with $j_\infty\tilde{p}_\mu(0,0) = 0$, $j_\infty\tilde{q}_\mu(0,0) = 0$ for each $\mu \in K$.

We can even do better, by linearizing $X_\mu|_{\{y=0\}}$ by a near–identity transformation.

Proposition 4.11 *Let*

$$\dot{u} = \lambda(\mu)u(1 + h(u,\mu)) \tag{4.33}$$

be a differential equation on $\mathbb{R}$*, where* λ, h *are* C^∞*,* $\lambda(\mu) \neq 0$*,* $j_\infty h_\mu(0) = 0, \forall \mu \in K$*. Then there exists a unique change of coordinates*

$$u = x(1 + \alpha(x, \mu)), \tag{4.34}$$

with α *a* C^∞ *function and* $j_\infty \alpha_\mu(0) = 0, \forall \mu \in K$*, changing (4.33) into* $\dot{x} = \lambda(\mu)x$*.*

Proof: The coordinate change (4.34) changes (4.33) into

$$(1 + \alpha(x, \mu) + x\frac{d\alpha}{dx}(x, \mu))\dot{x} = \lambda(\mu)x(1 + \alpha(x, \mu))(1 + h(x(1 + \alpha(x, \mu)))),$$

such that α has to satisfy

$$1 + \alpha(x, \mu) + x\frac{d\alpha}{dx}(x, \mu) = (1 + \alpha(x, \mu))(1 + h(x(1 + \alpha(x, \mu)))).$$

A straightforward calculation shows that $\alpha(x, \mu)$ needs to be solution of the differential equation

$$\frac{dy}{dx} = (1 + y)\overline{h}(x, y, \mu),$$

with $\overline{h}(x, y, \mu) = h(x(1+y), \mu)/x$. This last equation clearly has a unique C^∞ solution α with $j_\infty \alpha_\mu(0) = 0, \forall \mu \in K$. $\square$

One finds a near–identity transformation linearising the behaviour of $X_\mu \mid \{y = 0\}$ or changing (4.32) into:

$$\begin{cases} \dot{x} &= x(\lambda(\mu) + yf(y, \mu)) + y\alpha(x, y, \mu), \\ \dot{y} &= y^k(1 + yg(y, \mu)) + y\beta(x, y, \mu), \end{cases} \tag{4.35}$$

where α and β are C^∞ functions with $j_\infty \alpha_\mu(0, 0) = 0$, $j_\infty \beta_\mu(0, 0) = 0$ and where f, g are as in (4.31).

We continue by looking for a near–identity transformation changing (4.35) into a similar expression with $\beta(x, y, \mu) = y^k\tilde{\beta}(x, y, \mu)$ where $\tilde{\beta}$ is C^∞. We write

$$\beta(x, y, \mu) = x\beta_0(x, \mu) + y\overline{\beta}_0(x, y, \mu),$$

and let $\gamma_\mu(y) := \gamma(y,\mu)$ be a C^∞ solution of:

$$\begin{cases} \lambda(\mu)\dfrac{d\gamma}{dx}(x) & = -\beta_0(x,\mu)\gamma(x), \\ \gamma(0) & = 1, \end{cases}$$

then $j_\infty(\gamma_\mu(x)-1)(0) = 0$, $\forall \mu \in K$ and $(x,y) = (x, Y\gamma(x,\mu))$ is a C^∞ coordinate change transforming (4.31) into a similar expression with

$$\beta(x,y,\mu) = y\beta_1(x,y,\mu),$$

for some C^∞ function $\beta_1(x,y,\mu)$. Next we proceed by induction. If we start with an expression:

$$\begin{cases} \dot{x} & = x(\lambda(\mu) + yf(y,\mu)) + y\alpha(x,y,\mu), \\ \dot{y} & = y^k(1+yg(y,\mu)) + y^l(x\beta_l(x,\mu) + y\overline{\beta}_l(x,y,\mu)), \end{cases}$$

for some $2 \le l \le k$, then by using the C^∞ flat function

$$\gamma(x,\mu) = -\frac{1}{\lambda(\mu)}\int_0^x \beta_l(u,\mu)du,$$

we find that $(x, y + y^l\gamma(x,\mu))$ are new C^∞ variables for which (X_μ) gets a similar expression as (4.35) with

$$y\beta(x,y,\mu) = y^{l+1}\beta_{l+1}(x,y,\mu),$$

for some C^∞ function β_{l+1}.

As such, the expression that we get after the previous steps is:

$$\begin{cases} \dot{x} & = x(\lambda(\mu) + yf(y,\mu)) + y\tilde{\alpha}(x,y,\mu), \\ \dot{y} & = y^k(1+yg(y,\mu) + y\tilde{\beta}(x,y,\mu)), \end{cases} \tag{4.36}$$

where f, g are as in (4.35), $\tilde{\alpha}$ and $\tilde{\beta}$ are C^∞ functions with $j_\infty\tilde{\alpha}_\mu(0,0) = j_\infty\tilde{\beta}_\mu(0,0)$, $\forall \mu \in K$, $k \ge 2$.

We are now ready to prove the existence of a (not necessarily unique) C^∞ manifold $x = x(y,\mu) = x_\mu(y)$ such that for each $\mu \in K$, $x = x_\mu(y)$

is an invariant manifold for (4.36) tangent to the y-axis. The function $x(y,\mu)$ is a solution of the scalar differential equation

$$y^k \frac{dx}{dy} = x(\lambda(\mu) + yh(y,\mu) + yC(x,y,\mu)) + yD(x,y,\mu),$$

for some C^∞ functions h, C, D with $j_\infty C_\mu(0,0) = j_\infty D_\mu(0,0) = 0$, $\forall \mu \in K$ and $k \geq 2$. The graph $(x_\mu(y), y)$ is also called a center manifold of (4.36).

Lemma 4.12 *Consider a scalar differential equation*

$$y^k \frac{dx}{dy} = x(\lambda(\mu) + yF(x,y,\mu)) + G(y,\mu) \tag{4.37}$$

where F and G are C^∞ functions, $j_\infty G_\mu(0) = 0$ and $\lambda(\mu) \neq 0, \forall \mu \in K$. Then there is at least one C^∞ solution $x = \alpha(y,\mu)$ with $\alpha : (-\varepsilon, \varepsilon) \times K \to \mathbb{R}$ for some $\varepsilon > 0$ sufficiently small, such that $j_\infty \alpha_\mu(0) = 0, \forall \mu \in K$.

Moreover, whenever $\beta : (0,\varepsilon) \times K \to \mathbb{R}$ (respectively $\beta : (-\varepsilon, 0) \times K \to \mathbb{R}$) is a solution, for some $\varepsilon > 0$, with $\lim_{y\to 0} \beta(y,\mu) = 0, \forall \mu \in K$, then $\overline{\beta} : [0,\varepsilon) \times K \to \mathbb{R}$ (respectively $\overline{\beta} : (-\varepsilon, 0] \times K \to \mathbb{R}$) defined by $\overline{\beta}(y,\mu) = \beta(y,\mu)$ for $x \neq 0$ and $\overline{\beta}(0,\mu) = 0$, is everywhere C^∞ on its domain of definition and $j_\infty \overline{\beta}_\mu(0) = 0$.

Proof: We attach to the scalar differential equation (4.37) the planar system:

$$\begin{cases} \dot{x} &= x(\lambda(\mu) + yF(x,y,\mu)) + G(y,\mu), \\ \dot{y} &= y^k. \end{cases}$$

Let us first take $\lambda(\mu) > 0, \forall \mu \in K$ and restrict to $\{y \geq 0\}$. If we choose $l \in \mathbb{N}$ with $l \geq 1$ and $c \in \mathbb{R} \setminus \{0\}$, we see that at any point of the curves (cy^l, y), and for $y > 0$ sufficiently small,

$$\left| \frac{dx}{dy}(cy^l, y) \right| > |c| l y^{l-1}, \tag{4.38}$$

see figure 4.1(a). As such all solutions $x(y,\mu)$, for $y > 0$, have 0 as their

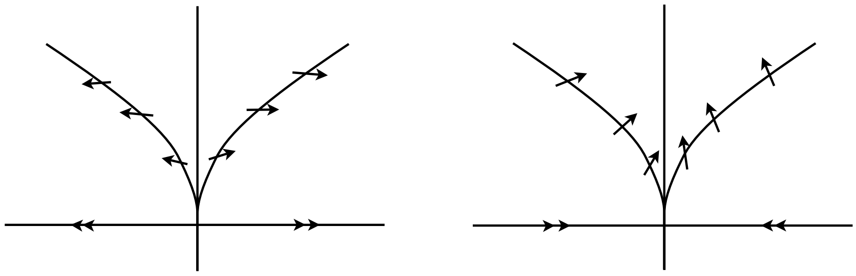

Figure 4.1: Flow near a center manifold when (a) $\lambda(\mu) > 0$ and when (b) $\lambda(\mu) < 0$.

limit as $y \to 0$ (see Figure 4.1(a)). They are clearly C^∞ for $y > 0$ and satisfy the flatness property:

$$x(y,\mu) = y^l u_l(y,\mu), \quad \forall l \in \mathbb{N}_1, \tag{4.39}$$

for some $u_l(y,\mu)$ smooth for $y > 0$ and bounded near $y = 0$.

Since they are solutions of equation (4.37), the flatness property (4.39) also holds for the derivative $x'_\mu(y)$, and by differentiating (4.37), we can prove inductively that it holds for all derivatives $x_\mu^{(n)}(y)$. Define $\overline{x}(y,\mu)$, as:

$$\overline{x}(y,\mu) = \begin{cases} x(y,\mu), & y > 0, \\ 0, & y = 0, \end{cases}$$

then $\overline{x}$ is clearly C^∞ for $y > 0$. One can easily verify that the flatness property (4.39) holds on all mixed derivatives:

$$\frac{\partial^{i+j} x}{\partial y^i \partial \mu^j}(y,\mu), \quad \forall i,j \in \mathbb{N},$$

such that $\overline{x}(y,\mu)$ is C^∞, with $j_\infty \overline{x}_\mu(0) = 0, \forall \mu \in K$.

Second we consider the case $\lambda(\mu) < 0$, $\forall \mu \in K$, still keeping $y \geq 0$. Property (7.6) still holds along the curves (cy^l, y) for any $l \in \mathbb{N}$ with $l \geq 1$ and any $c \in \mathbb{R} \setminus \{0\}$. It induces a picture as in Figure 4.1(b). It easily follows that there exists at least one solution $x(y,\mu)$, for $y \geq 0$ sufficiently small, such that $\lim_{y\to 0} x(y,\mu) = 0, \forall \mu \in K$. Based on the same reasoning as in the previous case it follows that $\overline{x}$, with $\overline{x}(y,\mu) = x(y,\mu)$ for $y > 0$ and $\overline{x}(0,\mu) = 0$ is also C^∞ at points $(0,\mu)$, $\mu \in K$ and $j_\infty \overline{x}_\mu(0) = 0$.

The side $\{y \leq 0\}$ can be reduced to $\{y \geq 0\}$ by a reflection in y. $\square$

Choose now such a C^∞ manifold $\{x = \psi(y,\mu)\}$ satisfying $j_\infty \psi_\mu(0) = 0$, $\forall \mu \in K$. The change of variables $(x,y) \longmapsto (x - \psi_\mu(y), y)$ changes expression (4.36) into:

$$\begin{cases} \dot{x} &= x(\lambda(\mu) + yf(y,\mu) + y\overline{\alpha}(x,y,\mu)), \\ \dot{y} &= y^k(1 + yg(y,\mu) + y\overline{\beta}(x,y,\mu)), \end{cases} \tag{4.40}$$

for some $\overline{\alpha}$ and $\overline{\beta}$ that are C^∞ functions with $j_\infty \overline{\alpha}_\mu(0,0) = j_\infty \overline{\beta}_\mu(0,0) = 0$.

We postpone the proof of the elimination of the flat terms $\overline{\alpha}$ and $\overline{\beta}$ in expression (4.40), by a C^∞ coordinate change φ, to the end of this section. Since the proof of this fact is independent of what follows, we will take this for granted for the moment. We hence suppose, the semi–hyperbolic vector field X to have the following simple normal form at the origin:

$$\begin{cases} \dot{x} &= x(\lambda(\mu) + yf(y,\mu)), \\ \dot{y} &= y^k(1 + yg(y,\mu)). \end{cases} \tag{4.41}$$

This is the general C^∞ normal form for semi–hyperbolic singularities with a non–flat center behavior. We now describe a simplification of the center behavior.

Proposition 4.13 *Consider the differential equation*

$$\dot{u} = u^k(1 + uh(u,\mu)), \tag{4.42}$$

for some C^∞ function $h(u,\mu)$ and $k \in \mathbb{N}$ with $k \geq 2$. Then there exists a change of coordinates

$$u = y(1 + y\alpha(y,\mu)),$$

with $\alpha(y,\mu)$ a C^∞ function, changing the above differential equation into

$$\dot{y} = \frac{y^k}{1 - a(\mu)y^{k-1}}, \tag{4.43}$$

for some C^∞ function $a(\mu)$. If h is analytic then α can be chosen to be analytic.

Proof: We will treat both the C^∞ and the C^ω results at once, each time writing C^∞ (respectively C^ω). So we start with an equation (4.42) where h is C^∞ (respectively C^ω). We write

$$h(u,\mu) = \sum_{i=0}^{N-2} a_i(\mu)u^i + O(u^{N-1}),$$

for some arbitrarily chosen $N \geq k$. By inductively using a coordinate change

$$u = y(1 + \overline{\alpha}(\mu)y^l),$$

for a well chosen $\overline{\alpha}(\mu)$, and $1 \leq l \leq k-2$, we can change $h(u,\mu)$ into

$$h(u,\mu) = a(\mu)u^{k-2} + O(u^{k-1}).$$

The coefficient in front of u^{k-2} cannot be changed. However the obtained equation can be written as

$$\dot{u} = \frac{u^k}{1 - a(\mu)u^{k-1}} \left(1 + \tilde{h}(u,\mu)\right),$$

for some C^∞ (resp. C^ω) function $\tilde{h}(u,\mu) = O(u^k)$. Writing

$$\tilde{h}(u,\mu) = \sum_{i=k}^{N-1} a_i(\mu)u^i + O(u^N),$$

one verifies that the induction on $l \geq k$ can be continued to obtain:

$$\dot{u} = \frac{u^k}{1 - a(\mu)u^{k-1}}(1 + \overline{h}(u,\mu)), \tag{4.44}$$

where $\overline{h}$ is C^∞ (respectively C^ω) and $\overline{h}(u,\mu) = O(u^N)$, for some $N > k$.

We do not yet need to take a precise value for N, but we will see that $N \geq 2k-2$ will permit us to prove the proposition. We now try a coordinate change

$$u = y(1 + x(y,\mu)), \tag{4.45}$$

where $x(y,\mu)$ is C^∞ (respectively C^ω) and check the necessary conditions. Substituting (4.45) into (7.7), we immediately see that $x(y,\mu)$ needs to be a solution of

$$(1 + x + y\frac{dx}{dy}) = \frac{1 - a(\mu)y^{k-1}}{1 - a(\mu)y^{k-1}(1+x)^{k-1}}(1+x)^k(1 + O(y^N)),$$

with $O(y^N)$ some function in (x,y,μ) which is C^∞ (respectively C^ω).

A straightforward calculation reduces this to:

$$y\frac{dx}{dy} = x((k-1) + A(x,y,\mu)) + y^N B(y,\mu), \tag{4.46}$$

where A and B are C^∞ (respectively C^ω functions), $A_\mu(x,y) = O(\| (x,y) \|)$.

If we write $x = y^{N-1}X$, then (4.46) changes into

$$y\frac{dX}{dy} = X((k-N) + A(y^{N-k}X, y, \mu)) + yC(y,\mu),$$

for a function $C(y,\mu)$ which is C^∞ (respectively C^ω).

This last differential equation can be represented by the system:

$$\begin{cases} \dot{X} &= X((k-N) + A(y^{N-1}X, y, \mu)) + yC(y,\mu) \\ \dot{y} &= y. \end{cases}$$

Now if $N = 2k-2$ then $k - N \leq 0$ such that the system has a hyperbolic saddle at $(0,0)$ when $k \geq 3$ and a semi–hyperbolic singularity for

$k = 2$. In any case there exists a C^∞ (respectively C^ω) invariant manifold $(X(y,\mu), y)$ such that for each $\mu \in K$, $X = X_\mu(y)$ represents an unstable manifold at the origin. The function $x(y,\mu) = y^{N-1}X(y,\mu)$ provides a solution of (4.46). □

Applying this proposition to expression (4.51), we can put a semi–hyperbolic singularity with non–flat center behavior in a very simple normal form for C^∞–equivalence. If $\lambda > 0$, we first multiply (4.51) by $(\lambda(\mu) + f(y,\mu))^{-1}$, leading to

$$\begin{cases} \dot{x} &= x, \\ \dot{y} &= y^k \overline{g}(y,\mu). \end{cases}$$

After a linear coordinate change, we can require that $\overline{g}(0,\mu) = 1 + y\overline{g}(y,\mu)$, for some C^∞ function $\overline{g}$. Applying Proposition 4.13 to $y^k(1 + y\overline{g}(y,\mu))$ we can change the differential equation into:

$$\begin{cases} \dot{x} &= x, \\ \dot{y} &= \dfrac{y^k}{1 - a(\mu)y^{k-1}}, \end{cases}$$

for some $a(\mu)$, C^∞ dependent on μ. This is the final normal form for C^∞ equivalence, in the sense that the coefficient $a(\mu)$ is an invariant if we want to keep the expression as it is. If $\lambda < 0$ we first multiply (4.51) by $-(\lambda(\mu) + f(y,\mu))^{-1}$ leading to the normal fom:

$$\begin{cases} \dot{x} &= -x, \\ \dot{y} &= \dfrac{y^k}{1 - a(\mu)y^{k-1}}, \end{cases}$$

for some $a(\mu)$, C^∞ dependent on μ.

Removal of the flat terms

In the former elaboration, we have left the problem of the removal of flat terms in (4.40) in order to get the required C^∞ normal form. From now on we prefer to swith the x and y coordinate such that the x–axis is a center manifold:

$$\begin{cases} \dot{x} = x^k(1 + \overline{f}(x, \mu) + A(x, y, \mu)), \\ \dot{y} = y(\lambda(\mu) + \overline{g}(x, \mu) + B(x, y, \mu)), \end{cases} \tag{4.47}$$

where $\overline{f}(x, \mu) = x\overline{f}_0(x, \mu), \overline{g}(x, \mu) = x\overline{g}_0(x, \mu)$, all functions are C^∞ and $\lambda(\mu) \neq 0$, $j_\infty A_\mu(0, 0) = j_\infty B_\mu(0, 0) = 0$, $\forall \mu \in K$.

First we claim that there exists a C^∞ coordinate change $\psi : \mathbb{R}^2 \times K \mapsto \mathbb{R}^2$ such that $j_\infty(\psi_\mu - I)(0, 0) = 0, \forall \mu \in K$, where I is the identity isomorphism, that adapts (4.47) in a way that $j_\infty A_\mu(x, 0) = j_\infty B_\mu(x, 0) = 0$, $\forall \mu \in K$. To prove the claim we work with Taylor expansions in y, having as coefficients C^∞ functions in x. The proof relies first on an induction procedure on the powers of y^n followed by an application of Borel's theorem for representing such semi–formal developments by genuine C^∞ functions in two variables (see for example [2]).

We write (4.47) as:

$$\begin{cases} \dot{x} = x^k(1 + \overline{f}(x, \mu) + a(x, \mu) + y\overline{A}(x, y, \mu)), \\ \dot{y} = y(\lambda(\mu) + \overline{g}(x, \mu) + b(x, \mu) + y\overline{B}(x, y, \mu)), \end{cases} \tag{4.48}$$

with $j_\infty \overline{A}_\mu(0, 0) = j_\infty \overline{B}_\mu(0, 0) = 0$, $j_\infty a_\mu(0) = j_\infty b_\mu(0) = 0$, $\forall \mu \in K$.

To obtain a normal form as in (4.53) we could of course immediately incorporate $a(x, \mu)$ in $\overline{f}(x, \mu)$. We will however prove that in fact also $a(x, \mu)$ can be removed, in the sense that in (4.51) only the ∞–jet of $\overline{f}$ matters and not its precise C^∞ realisation. The same holds for $\overline{g}(x, \mu)$.

We first try a near–identity coordinate change

$$(x, y) = (X(1 + \alpha_\mu(X), y), \tag{4.49}$$

proving the existence of a C^∞ function $\alpha_\mu(X) = \alpha(X, \mu)$ that is flat in X and that permits us to remove $a(x, \mu)$ in (4.48). Substituting (4.49) into (4.48) and writing x instead of X gives

$$\begin{aligned}
&\dot{x}(1+\alpha_\mu(x)) + x\alpha'_\mu(x)) \\
&= \\
&x^k(1+\alpha_\mu(x))^k\Big(1+\overline{f}_\mu(x(1+\alpha_\mu(x))) + a_\mu(x(1+\alpha_\mu(x)))\Big) \\
&+x^k y(1+\alpha_\mu(x))^k(\overline{A}_\mu(x(1+\alpha_\mu(x)),y).
\end{aligned}$$

We now choose $\alpha(x,\mu)$ such that

$$\begin{aligned}
&(1+\alpha_\mu(x)+x\alpha'_\mu(x))(1+\overline{f}_\mu(x)) \\
&= \\
&(1+\overline{f}_\mu(x(1+\alpha_\mu(x))) + a_\mu(x(1+\alpha_\mu(x))))(1+\alpha_\mu(x))^k,
\end{aligned} \tag{4.50}$$

leading to

$$\dot{x} = x^k(1+\overline{f}_\mu(x) + y\tilde{A}_\mu(x,y)),$$

with $j_\infty \tilde{A}_\mu(0,0) = 0$. The equation (4.50) has a solution α such that $j_\infty \alpha_\mu(0) = 0, \forall \mu \in K$. Writing (4.50) as

$$(1+\overline{f}_\mu(x))x\alpha'_\mu(x) = [(k-1)+O(x)+O(\alpha_\mu(x))]\alpha_\mu(x) + a_\mu(x),$$

it is clear that such a solution is guaranteed by the unstable manifold theorem.

The coordinate change does not change the form of the second equation of (4.48) such that (4.49) reduces (4.48) to:

$$\left\{\begin{array}{l}
\dot{x} = x^k(1+\overline{f}(x,\mu) + y\tilde{A}(x,y,\mu)), \\
\dot{y} = y(\lambda(\mu) + \overline{g}(x,\mu) + \tilde{b}(x,\mu) + y\tilde{B}(x,y,\mu)),
\end{array}\right. \tag{4.51}$$

with $\overline{f}, \overline{g}$ as in (4.48), $j_\infty \tilde{A}_\mu(0,0) = j_\infty \tilde{B}_\mu(0,0) = 0$ and $j_\infty \tilde{b}_\mu(0) = 0$, $\forall \mu \in K$.

We continue similar by trying a transformation

$$(x,y) = (x, Y(1+\beta(x,\mu))), \tag{4.52}$$

for some C^∞ function $\beta_\mu(x) = \beta(x,\mu)$ that is flat in x and of which we prove that it can be chosen such that the transformation (4.52) will remove $\tilde{b}(x,\mu)$ in (4.51). Writing y instead of Y, one gets

$$\dot{y}(1+\beta_\mu(x)) + y\beta'_\mu(x)x^k\Big(1+\overline{f}_\mu(x)+y(1+\beta_\mu(x))\tilde{A}_\mu(x,y(1+\beta_\mu(x)))\Big)$$
$$=$$
$$y(1+\beta_\mu(x))\Big(\lambda(\mu)+\overline{g}_\mu(x)+\tilde{b}_\mu(x)+y(1+\beta_\mu(x))\tilde{B}_\mu(x,y(1+\beta_\mu(x)))\Big).$$

We choose $\beta(x,\mu)$ such that

$$\tilde{b}_\mu(x)(1+\beta_\mu(x)) = \beta'_\mu(x)\,x^k(1+\overline{f}_\mu(x))$$

which has clearly a C^∞ solution such that $j_\infty\beta_\mu(0)=0$. This yields into

$$\dot{y}(1+\beta_\mu(x)) = y(1+\beta_\mu(x))(\lambda(\mu)+y\overline{g}_\mu(x)) + y^2H_\mu(x,y)$$

with $j_\infty H_\mu(0,0)=0, \forall\mu\in K$. In particular

$$\dot{y} = y(\lambda(\mu)+\overline{g}_\mu(x)+yw_\mu(x,y))$$

where $j_\infty w_\mu(0,0)=0, \forall\mu\in K$.

The form of the first equation of (4.51) stays unchanged after the transformation (4.52) leading to:

$$\begin{cases} \dot{x} = x^k(1+\overline{f}(x,\mu)+yu(x,y,\mu)), \\ \dot{y} = y(\lambda(\mu)+\overline{g}(x,\mu)+yw(x,y,\mu)), \end{cases} \tag{4.53}$$

with $j_\infty u_\mu(0,0) = j_\infty w_\mu(0,0) = 0, \forall\mu\in K$ and $\overline{f},\overline{g}$ as in (4.51).

We now proceed by induction. Suppose that we can write (4.53) as:

$$\begin{cases} \dot{x} = x^k(1+\overline{f}(x,\mu)+y^n\overline{u}(x,y,\mu)), \\ \dot{y} = y(\lambda(\mu)+\overline{g}(x,\mu)+y^n\overline{w}(x,y,\mu)), \end{cases} \tag{4.54}$$

for some $n\geq 1$. We search for transformations that change (4.54) into a similar expression with n replaced by $n+1$.

We first try

$$(x,y) = (X(1+X^{k-1}y^n\alpha(X,\mu)),y), \tag{4.55}$$

proving the existence of a C^∞ function $\alpha_\mu(X) = \alpha(X,\mu)$ that is flat in X and that permits us to change $y^n\overline{u}(x,y,\mu)$ into $y^{n+1}\tilde{u}(X,y,\mu)$ for some C^∞ function $\tilde{u}$ such that $j_\infty\tilde{u}_\mu(0,0) = 0, \forall\mu \in K$. We write $h(x,\mu) = \overline{u}_\mu(x,0)$.

Substituting (4.55) into the first equation of (4.54), and writing x instead of X, gives

$$\begin{aligned}\dot{x}(1+kx^{k-1}y^n\alpha_\mu(x)+x^ky^n\alpha'_\mu(x)) + nx^ky^{n-1}\alpha_\mu(x)\dot{y} = \\ x^k(1+x^{k-1}y^n\alpha_\mu(x))^k(1+\overline{f}_\mu(x(1+x^{k-1}y^n\alpha_\mu(x)))+ \\ y^nh_\mu(x(1+x^{k-1}y^n\alpha_\mu(x)))+O(y^{n+1})).\end{aligned}$$

Working modulo $O(y^{n+1})$, this gives the following equation:

$$\begin{aligned}&\dot{x}(1+kx^{k-1}y^n\alpha_\mu(x)+x^ky^n\alpha'_\mu(x)) = \\ &\quad x^k\left[(1+kx^{k-1}y^n\alpha_\mu(x))(1+\overline{f}_\mu(x)+nx^ky^n\alpha_\mu(x)\overline{f}'_\mu(x)+y^nh_\mu(x))+\right. \\ &\quad\left. ny^n\alpha_\mu(x)(\lambda(\mu)+\overline{g}_\mu(x))\right].\end{aligned}$$

A straightforward calculation of the terms in y^n shows that we have to solve the following equation:

$$(1+\overline{f}_\mu(x))x^k\alpha'_\mu(x) = (n(\lambda(\mu)+\overline{g}_\mu(x))+x^k\overline{f}'_\mu(x))\alpha_\mu(x)+h_\mu(x).$$

Because $j_\infty h_\mu(0) = 0, \forall\mu \in K$, lemma 6.3 guarantees the existence of a C^∞ solution $\alpha(x,\mu)$ with $j_\infty\alpha_\mu(0) = 0, \forall\mu \in K$.

Substituting (4.55) into the second equation of (4.54) does not change the form of it. Coordinate change (4.55) hence reduces (4.54) to some expression

$$\left\{\begin{array}{l}\dot{x} = x^k(1+\overline{f}(x,\mu)+y^{n+1}\tilde{u}(x,y,\mu)), \\ \dot{y} = y(\lambda(\mu)+\overline{g}(x,\mu)+y^n\tilde{w}(x,y,\mu)),\end{array}\right. \tag{4.56}$$

of which the ∞–jet is the same as in (4.54).

We continue similar and try a near–identity transformation

$$(x, y) = (x, Y(1 + Y^n \beta(x, \mu)), \tag{4.57}$$

proving the existence of a flat C^∞ function $\beta_\mu(x) = \beta(x, \mu)$ that permits us to change $y^n \overline{w}(x, y, \mu)$ into $Y^{n+1}\tilde{w}(x, Y, \mu)$ for some C^∞ function $\tilde{w}$ such that $j_\infty \tilde{w}_\mu(0, 0) = 0, \forall \mu \in K$. We write $h(x, \mu) = \overline{w}_\mu(x, 0)$.

Substituting (4.57) into the second equation of (4.56), and writing y instead of Y, gives:

$$\dot{y}(1 + (n+1)y^n \beta_\mu(x)) + y^{n+1}\beta'_\mu(x)\dot{x} =$$
$$y(1 + y^n \beta_\mu(x))(\lambda(\mu) + \overline{g}_\mu(x) + y^n h_\mu(x) + O(y^{n+1})).$$

Working modulo $O(y^{n+2})$ this gives the following equation:

$$\dot{y}(1 + (n+1)y^n \beta_\mu(x)) =$$
$$y(\lambda(\mu) + \overline{g}_\mu(x) + y^n h_\mu(x) + (\lambda(\mu) + \overline{g}_\mu(x))y^n \beta_\mu(x) -$$
$$y^n \beta'_\mu(x) x^k (1 + \overline{f}_\mu(x))).$$

A straightforward calculation of the terms in y^{n+1} shows that we have to solve the following equation:

$$\beta'_\mu(x) x^k (1 + \overline{f}_\mu(x)) = -n(\lambda(\mu) + \overline{g}_\mu(x))\beta_\mu(x) + h_\mu(x).$$

Because $j_\infty h_\mu(0) = 0$, Lemma 6.3 guarantees the existence of a C^∞ solution $\beta_\mu(x)$ with $j_\infty \beta_\mu(0) = 0, \forall \mu \in K$.

Substituting (4.57) into the first equation of (4.56) does not change the form of it. The successive coordinate changes (4.55) and (4.57) prove that the induction works. We hence know that near the origin (X_μ) is C^∞–conjugate to a family:

$$\begin{cases} \dot{x} = x^k(1 + \overline{f}(x, \mu) + y\hat{A}(x, y, \mu)), \\ \dot{y} = y(\lambda(\mu) + \overline{g}(x, \mu) + y\hat{B}(x, y, \mu)), \end{cases} \tag{4.58}$$

where the ∞–jet is as in (4.47) and $j_\infty \hat{A}_\mu(x, 0) = j_\infty \hat{B}_\mu(x, 0) = 0$, $\forall \mu \in K$.

For eliminating the term $\hat{A}(x,y,\mu)$ and $\hat{B}(x,y,\mu)$ in the above expression, we proceed in a more general context. Let us denote the family (4.58) as $(X_\mu + Y_\mu)$ where (X_μ) and (Y_μ) are both C^∞ families and where each member is defined in a neigbourhood of the origin, and $j_\infty Y_\mu(x,0) = 0, \forall \mu \in K$. We now want to find locally near the origin a C^∞ diffeomorphism $\varphi : \mathbb{R}^2 \times K \to \mathbb{R}^2$ with $\varphi_\mu(0) = 0$ such that $(\varphi_\mu)_*(X_\mu) = X_\mu + Y_\mu$ and $j_\infty(\varphi_\mu - I)(x,0) = 0, \forall \mu \in K$.

To that end, we will use the so called *homotopic method* (see, e.g. [21]). We introduce a parameter $\tau \in [0,1]$, consider $(X_\mu^\tau) = (X_\mu + \tau Y_\mu)$, and look for a family (φ_μ^τ) of transformations with the property that

$$(\varphi_\mu^\tau)_* X_\mu = X_\mu + \tau Y_\mu, \forall \mu \in K, \tag{4.59}$$

and $j_\infty(\varphi_\mu^\tau - I)(x,0) = 0, \forall \mu \in K, \forall \tau \in [0,1]$.

It now suffices to look for a τ–dependent vector field Z_μ^τ, such that $j_\infty(Z_\mu^\tau)(x,0) = 0$, satisfying

$$[X_\mu + \tau Y_\mu, Z_\mu^\tau] = Y_\mu, \forall \mu \in K, \tag{4.60}$$

where $[\cdot,\cdot]$ denotes the usual Lie–bracket operation. We prove that this is sufficient in the next lemma, in which we use $z = (x,y)$, $Z(z,\mu,\tau) = Z_\mu^\tau(z)$, $X_\mu(z) = X(z,\mu)$ and $\varphi(z,\mu,\tau) = \varphi_\mu^\tau(z)$.

Lemma 4.14 *Let Z_μ^τ be a solution of equation (4.60). Then φ_μ^τ defined by*

$$\frac{\partial \varphi}{\partial \tau}(z,\mu,\tau) = Z(\varphi(z,\mu,\tau),\mu,\tau) \tag{4.61}$$

is a solution of equation (4.59).

Proof: It clearly follows from $j_\infty(Z_\mu^\tau)(x,0) = 0$ that $j_\infty(\varphi_\mu^\tau - I)(x,0) = 0, \forall \mu \in K$. We furthermore have to check condition (4.59), which we can write as

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(z,\mu,\tau) & \frac{\partial \varphi_1}{\partial y}(z,\mu,\tau) \\ \frac{\partial \varphi_2}{\partial x}(z,\mu,\tau) & \frac{\partial \varphi_2}{\partial y}(z,\mu,\tau) \end{pmatrix} \begin{pmatrix} X_1(z,\mu) \\ X_2(z,\mu) \end{pmatrix} = \begin{pmatrix} (X_1 + \tau Y_1)(\varphi(z,\mu,\tau),\mu) \\ (X_2 + \tau Y_2)(\varphi(z,\mu,\tau),\mu) \end{pmatrix}. \tag{4.62}$$

To find the relation with equation (4.60), we differentiate (4.62) with respect to τ:

$$\begin{pmatrix} Y_1(\varphi(z,\mu,\tau),\mu) \\ Y_2(\varphi(z,\mu,\tau),\mu) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(\frac{\partial \varphi_1}{\partial \tau})(z,\mu,\tau) & \frac{\partial}{\partial y}(\frac{\partial \varphi_1}{\partial \tau})(z,\mu,\tau) \\ \frac{\partial}{\partial x}(\frac{\partial \varphi_2}{\partial \tau})(z,\mu,\tau) & \frac{\partial}{\partial y}(\frac{\partial \varphi_2}{\partial \tau})(z,\mu,\tau) \end{pmatrix} \begin{pmatrix} X_1(z,\mu) \\ X_2(z,\mu) \end{pmatrix} -$$

$$\begin{pmatrix} \frac{\partial (X_1+\tau Y_1)}{\partial x}(\varphi(z,\mu,\tau),\mu) & \frac{\partial (X_1+\tau Y_1)}{\partial y}(\varphi(z,\mu,\tau),\mu) \\ \frac{\partial (X_2+\tau Y_2)}{\partial x}(\varphi(z,\mu,\tau),\mu) & \frac{\partial (X_2+\tau Y_2)}{\partial y}(\varphi(z,\mu,\tau),\mu) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial \tau}(z,\mu,\tau) \\ \frac{\partial \varphi_2}{\partial \tau}(z,\mu,\tau) \end{pmatrix}. \tag{4.63}$$

In this expression, we can write

$$\frac{\partial \varphi_i}{\partial \tau}(z,\mu,\tau) = Z_i(\varphi(z,\mu,\tau),\mu,\tau) \tag{4.64}$$

and by differentiating (4.64) we get

$$\begin{pmatrix} \frac{\partial}{\partial x}(\frac{\partial \varphi_1}{\partial \tau})(z,\mu,\tau) & \frac{\partial}{\partial y}(\frac{\partial \varphi_1}{\partial \tau})(z,\mu,\tau) \\ \frac{\partial}{\partial x}(\frac{\partial \varphi_2}{\partial \tau})(z,\mu,\tau) & \frac{\partial}{\partial y}(\frac{\partial \varphi_2}{\partial \tau})(z,\mu,\tau) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{\partial Z_1}{\partial x}(\varphi(z,\mu,\tau),\mu,\tau) & \frac{\partial Z_1}{\partial y}(\varphi(z,\mu,\tau),\mu,\tau) \\ \frac{\partial Z_2}{\partial x}(\varphi(z,\mu,\tau),\mu,\tau) & \frac{\partial Z_2}{\partial y}(\varphi(z,\mu,\tau),\mu,\tau) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(z,\mu,\tau) & \frac{\partial \varphi_1}{\partial y}(z,\mu,\tau) \\ \frac{\partial \varphi_2}{\partial x}(z,\mu,\tau) & \frac{\partial \varphi_2}{\partial y}(z,\mu,\tau) \end{pmatrix}$$

This relation, together with (4.62) and (4.64), shows that (4.63) is the same as (4.60). As such (4.62) follows from (4.60) by integration, i.e. solving (4.61). $\square$

We can suppose that $\lambda(\mu) < 0, \forall \mu \in K$. If not we just change orientation of the orbits by multiplying $(X_\mu + Y_\mu)$ by -1. For simplifying notation, we denote the x–axis as M and the distance of some $z \in \mathbb{R}^2$ to M as $\| z \|_M$. From now on take $\tau \in [0,1]$ arbitrarily but fixed. An orbit of $(X_\mu + \tau Y_\mu)$ will be denoted by $\gamma(z,\mu,\tau,t) = \gamma_\mu(z,\tau,t)$.

We will see that equation (4.60) can locally be solved by defining:

$$\begin{cases} Z_\mu^\tau(z) &= -\displaystyle\int_0^\infty (F_\mu(\gamma_\mu(z,\tau,t)))^{-1}(Y_\mu(\gamma_\mu(z,\tau,t)))dt, \quad z \notin M, \\ Z_\mu^\tau(z) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z \in M, \end{cases} \tag{4.65}$$

where $F_\mu(\gamma_\mu(z,\tau,t))$ represents, along an orbit $\gamma_\mu(z,\tau,t)$, the matrix solution of the related variational equation:

$$\begin{cases} \dfrac{d}{dt}F_\mu(\gamma_\mu(z,\tau,t)) = [D_z(X_\mu+\tau Y_\mu)(\gamma_\mu(z,\tau,t))](F_\mu(\gamma_\mu(z,\tau,t))), \\ F_\mu(\gamma_\mu(z,\tau,0)) = I. \end{cases} \tag{4.66}$$

We will prove that $Z^\tau(z,\mu)$ defined as in (4.65) is, locally near the origin, C^∞ and ∞–flat along M. It is then easy to check that Z_μ^τ satisfies the Lie–bracket relation required in (4.60), since equation (4.60) can clearly be written as

$$\frac{d}{dt}(Z_\mu^\tau(\gamma_\mu(z,\tau,t))) =$$

$$(D_z(X_\mu+\tau Y_\mu)(\gamma_\mu(z,\tau,t)))(Z_\mu^\tau(\gamma_\mu(z,\tau,t))) + Y_\mu(\gamma_\mu(z,\tau,t)),$$

from which it is clear that Z_μ^τ, as defined above is a solution.

The hyperbolic contraction towards M, locally around the origin and close to M, will appear to play a crucial role in proving the flatness of Z_μ^τ along M. The hyperbolic contraction towards M implies the existence of some $\nu > 0$, $C > 0$ such that uniformly in $\tau \in [0,1]$ and $\forall \mu \in K$:

$$\| \gamma_\mu(z,\tau,t) \|_M < Ce^{-\nu t} \| z \|_M, \tag{4.67}$$

for initial values z in some ε–neigbourhood of M.

The definition in (4.65) requires that the orbits $\gamma_\mu(z,\tau,t)$ exist for all $t \in [0,\infty[$, which might not be the case for the equation under consideration. However because one is only interested in the local behaviour of (4.58), we can replace this vector field by one that stays unchanged near the origin but of which the x–component is zero away from the origin and such that $\lambda(\mu)+\overline{g}(x,\mu)+y\hat{B}(x,y,\mu)$ stays uniformly negative for y near zero and x in the support of $X_\mu + \tau Y_\mu \mid_M$.

By compactness of K there exists some $\overline{\lambda} < 0$ such that $\lambda(\mu) < \overline{\lambda} < 0$, $\forall \mu \in K$. Choose a neighbourhood $V_\varepsilon =]-\varepsilon, \varepsilon[\times \overline{V}$ of the origin such that on V_ε

$$\lambda(\mu)+\overline{g}(x,\mu)+y\hat{B}(x,y,\mu) < \frac{\overline{\lambda}}{2}.$$

Consider a bump function $\chi(x)$ that is zero when $|x| \geq \varepsilon$ and 1 when $|x| \leq \eta$, for an $0 < \eta < \varepsilon$. Replace the vector field (4.58) by the vector field:

$$\begin{cases} \dot{x} = \chi(x)\Big(x^k(1+\overline{f}(x,\mu)+y\hat{A}(x,y,\mu))\Big), \\ \dot{y} = y\Big(\lambda(\mu) + (\overline{g}(x,\mu)+y\hat{B}(x,y,\mu))\chi(x)\Big), \end{cases} \tag{4.68}$$

that equals (4.58) for $|x| \leq \eta$ and of which the x–component is zero for $|x| \geq \varepsilon$. Because the hyperbolic character in the y–component is preserved in a uniform way, one sees that on V_ε, (4.67) stays valid and all orbits exist for $t \to \infty$.

We now search appropriate upperbounds on

$$\| (F_\mu(\gamma_\mu(z,\tau,t)))^{-1} \| = \frac{\| F_\mu(\gamma_\mu(z,\tau,t)) \|}{|\det(F_\mu(\gamma_\mu(z,\tau,t)))|}, \quad t \geq 0, \tag{4.69}$$

and on

$$\| Y_\mu(\gamma_\mu(z,\tau,t)) \|, \quad t \geq 0 \tag{4.70}$$

appearing in the integral (4.65), implying the flatness of it along M.

The divergence at the origin of (4.68) is a strictly negative number. Thus on V_ε the divergence has a lower bound that is strictly negative. Using Liouville's formula,

$$|\det(F_\mu(\gamma_\mu(z,\tau,t)))| = \exp\left(\int_0^t \operatorname{div}(X_\mu+\tau Y_\mu)(\gamma_\mu(z,\tau,s))ds\right), \tag{4.71}$$

one shows the existence of some $N_1 > 0$ such that

$$|\det(F_\mu(\gamma_\mu(z,\tau,t)))| \geq e^{-N_1 t}, \quad \forall z \in V_\varepsilon, \quad \forall t \geq 0, \quad \forall \mu \in K. \tag{4.72}$$

Further, using (4.66), one sees that there exists some $N_2 > 0$ such that:

$$\| F_\mu(\gamma_\mu(z,\tau,t) \| \;\leq\; 1 + N_2 \int_0^t \| F_\mu(\gamma_\mu(z,\tau,s)) \| \, ds,$$

$\forall z \in V_\varepsilon, \forall t \geq 0, \forall \mu \in K$, implying, using Gronwall's inequality,

$$\| F_\mu(\gamma_\mu(z,\tau,t)) \| \leq e^{N_2 t}, \quad \forall z \in V_\varepsilon, \forall t \geq 0, \forall \mu \in K. \tag{4.73}$$

Now (4.73) together with (4.72) and (4.69) results in

$$\| (F_\mu(\gamma_\mu(z,\tau,t)))^{-1} \| \leq e^{Nt}, \quad \forall z \in V_\varepsilon, \forall t \geq 0, \forall \mu \in K, \tag{4.74}$$

with $N = N_1 + N_2 > 0$ leading to an upperbound on (4.69).

Further, using the flatness of Y_μ along M, one finds that for all $k \in \mathbb{N}_1$, there exists some $L_0(k)$ such that

$$\| Y_\mu(\gamma_\mu(z,\tau,t) \| < L_0(k) \| \gamma_\mu(z,\tau,t) \|_M^k, \quad \forall z \in V_\varepsilon, \forall t \geq 0, \forall \mu \in K,$$

as $\| z \|_M \to 0$. Together with (4.67) this implies for each $k \in \mathbb{N}_1$, the existence of some $L(k) > 0$ such that uniformly in $\tau \in [0,1]$:

$$\| Y_\mu(\gamma_\mu(z,\tau,t) \| < L(k) \| z \|_M^k \, e^{-k\nu t}, \quad \forall t \geq 0, \forall \mu \in K, \tag{4.75}$$

as $\| z \|_M \to 0$, leading to an upperbound of (4.70).

If we now choose $k \in \mathbb{N}$ such that $N - k\nu < 0$, then, using (4.74) and (4.75) together with (4.65), one comes, for each $j \geq k$, to the existence of constants $\rho < 0$ and $A(j) > 0$ such that:

$$\begin{aligned} \| Z_\mu^\tau(z) \| &\leq A(j) \| z \|_M^j \int_0^{+\infty} e^{\rho t} dt \\ &\leq -\frac{A(j)}{\rho} \| z \|_M^j, \end{aligned}$$

as $\| z \|_M \to 0$ implying that the integral in (4.65) converges and that Z_μ^τ is flat along M.

Let us proceed by proving that the partial derivatives of $Z_\mu^\tau(z)$ with respect to (x, y, μ, τ) are flat along M implying the smoothness of $Z_\mu^\tau(z)$. Define the function $H_t(z, \mu, \tau)$, $z = (x, y)$ as:

$$H_t(z, \mu, \tau) = F_\mu(\gamma_\mu(z, \tau, t))^{-1} Y_\mu(\gamma_\mu(z, \tau, t)). \tag{4.76}$$

We will prove that the partial derivatives of $Z_\mu^\tau(z)$ with respect to x and y are given by:

$$\frac{\partial^{n+m} Z_\mu^\tau}{\partial x^n \partial y^m}(z) = -\int_0^{+\infty} \frac{\partial^{n+m} H_t}{\partial x^n \partial y^m}(z, \mu, \tau) dt, \quad (n, m) \in \mathbb{N} \times \mathbb{N}. \tag{4.77}$$

Herefore we search for appropriate upperbounds on the partial derivatives of H with respect to x and y.

An expression for the partial derivatives $\dfrac{\partial^{n+m} H_t}{\partial x^n \partial y^m}$ can be obtained by induction on n and m respectively. To simplify notation let us write $G_\mu(z, \tau, t) := F_\mu(\gamma_\mu(z, \tau, t))$. Differentiating H_t with respect to x gives:

$$\begin{aligned} \frac{\partial H_t}{\partial x}(z, \mu, \tau) = &\left(\frac{\partial}{\partial x}(G_\mu(z, \tau, t)^{-1})\right) Y_\mu(\gamma_\mu(z, \tau, t)) + \\ &G_\mu(z, \tau, t)^{-1} \left(\frac{\partial}{\partial x}(Y_\mu(\gamma_\mu(z, \tau, t)))\right). \end{aligned}$$

and by induction on n, one proves:

$$\frac{\partial^n H_t}{\partial x^n}(z, \mu, \tau) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\partial^k}{\partial x^k}(G_\mu(z, \tau, t)^{-1})\right) \frac{\partial^{n-k}}{\partial x^{n-k}}(Y_\mu(\gamma_\mu(z, \tau, t))).$$

Using a similar induction on m, one gets:

$$\begin{aligned} &\frac{\partial^{n+m} H_t}{\partial x^n \partial y^m}(z, \mu, \tau) = \\ &\sum_{k=0}^{n} \sum_{l=0}^{m} c_{k,l} \left(\frac{\partial^{k+l}}{\partial x^k \partial y^l}(G_\mu(z, \tau, t)^{-1})\right) \frac{\partial^{n+m-(k+l)}}{\partial x^{n-k} \partial y^{m-l}}(Y_\mu(\gamma_\mu(z, \tau, t))), \end{aligned} \tag{4.78}$$

with $c_{k,l} = \binom{n}{k}\binom{m}{l}$.

We now search for appropriate upperbounds on the partial derivatives of $Y_\mu(\gamma_\mu(z,\tau,t))$ and $G_\mu(z,\tau,t)^{-1}$. Because $Y_\mu(z)$ is ∞–flat along M, property (4.75) stays valid on all its derivatives. So for each $(i,j) \in \mathbb{N}\times\mathbb{N}$ and $q \in \mathbb{N}_1$, one finds a constant $L_{i,j}(q)$ such that:

$$\parallel \frac{\partial^{i+j}}{\partial x^i \partial y^j}(Y_\mu(\gamma_\mu(z,\tau,t)) \parallel < L_{i,j}(q) \parallel z \parallel_M^q e^{-q\nu t}, \quad \forall t \geq 0, \tag{4.79}$$

as $\parallel z \parallel_M \rightarrow 0$. Furthermore, we prove by induction on $i+j$, $(i,j) \in \mathbb{N}\times\mathbb{N}$, the existence of positive constants $C_{i,j}$ and $N_{i,j}$ such that

$$\parallel \frac{\partial^{i+j}}{\partial x^i \partial y^j}(G_\mu(z,\tau,t)^{-1}) \parallel \leq C_{i,j} e^{N_{i,j}t}, \quad \forall t \geq 0. \tag{4.80}$$

For $i+j=0$, this is clear from (4.74). Suppose (4.80) is true for every partial derivative of order less than or equal to Q. We then prove that (4.80) is also valid for every partial derivative of order $Q+1$. Deriving $G_\mu(z,\tau,t)^{-1}$ with respect to x, one gets:

$$\frac{\partial}{\partial x}(G_\mu(z,\tau,t)^{-1}) = -G_\mu(z,\tau,t)^{-1}\frac{\partial}{\partial x}(G_\mu(z,\tau,t))G_\mu(z,\tau,t)^{-1},$$

and analogously:

$$\frac{\partial}{\partial y}(G_\mu(z,\tau,t)^{-1}) = -G_\mu(z,\tau,t)^{-1}\frac{\partial}{\partial y}(G_\mu(z,\tau,t))G_\mu(z,\tau,t)^{-1}.$$

By induction on $i+j$, one sees that for $i+j>0$:

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j}(G_\mu(z,\tau,t)^{-1}) =$$
$$-G_\mu(z,\tau,t)^{-1}\frac{\partial^{i+j}}{\partial x^i \partial y^j}(G_\mu(z,\tau,t))G_\mu(z,\tau,t)^{-1} -$$
$$\sum_b c_b \frac{\partial^{k_1+l_1}}{\partial x^{l_1}\partial y^{l_1}}(G_\mu(z,\tau,t)^{-1})\frac{\partial^{k_2+l_2}}{\partial x^{k_2}\partial y^{l_2}}(G_\mu(z,\tau,t))\frac{\partial^{k_3+l_3}}{\partial x^{k_3}\partial y^{l_3}}(G_\mu(z,\tau,t)^{-1}), \tag{4.81}$$

for some constants $c_b \in \mathbb{R}$ and where the sum is taken over the multiple index $b = (k_1, l_1, \ldots, k_3, l_3)$, with $k_p + l_p < Q$ for each $p \in \{1, 2, 3\}$.

By using Gronwall's inequality, we now search appropriate upperbounds on $\frac{\partial^{i+j}}{\partial x^i \partial y^j}(G_\mu(z, \tau, t))$, $\forall (i, j) \in \mathbb{N} \times \mathbb{N}$. From (4.73), we already have an upperbound on $G_\mu(z, \tau, t))$. Define:

$$\tilde{H}_t(z, \mu, \tau) = [D_z(X_\mu + \tau Y_\mu)(\gamma_\mu(z, \tau, t))](G_\mu(z, \tau, t)).$$

Derivation of $\tilde{H}_t$ with respect to x gives:

$$\frac{\partial \tilde{H}_t}{\partial x}(z, \mu, \tau) = D_z^2(X_\mu + \tau Y_\mu)(\gamma_\mu(z, \tau, t))(\frac{\partial}{\partial x}(\gamma_\mu(z, \tau, t)), G_\mu(z, \tau, t))+$$
$$D_z(X_\mu + \tau Y_\mu)(\gamma_\mu(z, \tau, t))(\frac{\partial}{\partial x}(G_\mu(z, \tau, t)), \quad \forall t \geq 0.$$

From (4.66), one now sees:

$$\| \frac{\partial}{\partial x}(G_\mu(z, \tau, t)) \| \leq D_1 \int_0^t \| \frac{\partial}{\partial x}(\gamma_\mu(z, \tau, s)) \| \| G_\mu(z, \tau, s) \| \, ds+$$
$$D_2 \int_0^t \| \frac{\partial}{\partial x}(G_\mu(z, \tau, s)) \| \, ds, \quad \forall t \geq 0,$$

with D_1 and D_2 positive constants. Because the column vectors of $G_\mu(z, \tau, t)$ are given by the partial derivatives of $\gamma_\mu(z, \tau, t)$ with respect to x and y, one finds:

$$\| \frac{\partial}{\partial x}(G_\mu(z, \tau, t)) \| \leq \overline{D}_1 e^{2N_2 t} + D_2 \int_0^t \| \frac{\partial}{\partial x}(G_\mu(z, \tau, s)) \| \, ds, \quad \forall t \geq 0,$$

with N_2 the positive constant given in (4.73). Using Gronwall's inequality leads to:

$$\| \frac{\partial}{\partial x}(G_\mu(z, \tau, t)) \| \leq \overline{C}_{1,0} e^{\overline{N}_{1,0} t}, \quad \forall t \geq 0,$$

for some positive constants $\overline{C}_{1,0}$ and $\overline{N}_{1,0}$. Totally analogous, it can be shown that:

$$\| \frac{\partial}{\partial y}(G_\mu(z, \tau, t)) \| \leq \overline{C}_{0,1} e^{\overline{N}_{0,1} t}, \quad \forall t \geq 0,$$

for some positive constants $\overline{C}_{0,1}$ and $\overline{N}_{0,1}$. One now proceeds by induction on the order $i+j$ of the partial derivatives of $G_\mu(z,\tau,t)$ with respect to x and y. Higher order derivatives of $\tilde{H}$ with respect to x and y are given by:

$$\frac{\partial^{i+j}\tilde{H}_t}{\partial x^i \partial y^j}(z,\mu,\tau) = D_z(X_\mu + \tau Y_\mu)(\gamma_\mu(z,\tau,t))(\frac{\partial^{i+j}}{\partial x^i \partial y^j}(G_\mu(z,\tau,t))+$$
$$\sum_{p=1}^{i+j}\sum_{l=1}^{b(p)} d_l^p\, D_z^{p+1}(X_\mu + \tau Y_\mu)(\gamma_\mu(z,\tau,t))(w_l^{p+1}), \tag{4.82}$$

for some constants $d_p \in \mathbb{R}$ and where each $b(p)$ is some natural number and each w_l^{p+1} is a vector of length $p+1$ with components partial derivatives of $G_\mu(z,\tau,t)$ with respect to x and y of order at most $i+j-1$. Using Gronwall's inequality, on a similar way as before, one can show, for each $i+j$, $(i,j) \in \mathbb{N}\times\mathbb{N}$, the existence of positive constants $\overline{C}_{i,j}$ and $\overline{N}_{i,j}$ such that:

$$\parallel \frac{\partial^{i+j}}{\partial x^i \partial y^j}(G_\mu(z,\tau,t)) \parallel \leq \overline{C}_{i,j} e^{\overline{N}_{i,j}t}, \quad \forall t \geq 0. \tag{4.83}$$

Substituting (4.83) in (4.81), the upperbounds in (4.80) indeed follow by induction.

By substituting (4.79) and (4.80) in (4.78), one finds, as $\parallel z \parallel_M \to 0$:

$$\parallel \frac{\partial^{n+m}H_t}{\partial x^n \partial y^m}(z,\mu,\tau) \parallel \leq \sum_{k=}^{n}\sum_{l=0}^{m} \overline{c}_{k,l} e^{\rho_{k,l}t} \parallel z \parallel_M^{q_{k,l}}, \quad \forall t \geq 0, \tag{4.84}$$

for some constants $\overline{c}_{k,l} > 0$ and where for each (k,l), $q_{k,l}$ is chosen such that $\rho_{k,l} := N_{k,l} - q_{k,l}\nu < 0$. So for each $(n,m) \in \mathbb{N}\times\mathbb{N}$ and $k \in \mathbb{N}$, there exists a constant $A_{n,m}^k$ together with a constant $\rho_{n,m}^k < 0$ such that:

$$\parallel \frac{\partial^{n+m}H_t}{\partial x^n \partial y^m}(z,\mu,\tau) \parallel \leq A_{n,m}^k e^{\rho_{n,m}^k t} \parallel z \parallel_M^k, \quad \forall t \geq 0. \tag{4.85}$$

In particular the integrals in (4.77) indeed converges and all partial derivatives of Z_μ^τ with respect to (x,y) are flat along M.

Let us now look to the case where we also derive with respect to one of the parameters $(\mu_1, \dots, \mu_p, \tau)$. Let $\overline{\mu} \in \{\mu_1, \dots, \mu_p, \tau\}$, we search for appropriate upperbounds on the partial derivatives of H with respect to $(x, y, \overline{\mu})$ to prove:

$$\frac{\partial^{n+m+r} Z}{\partial x^n \partial y^m \overline{\mu}^r}(z, \mu, \tau) = -\int_0^{+\infty} \frac{\partial^{n+m+r} H_t}{\partial x^n \partial y^m \overline{\mu}^r}(z, \mu, \tau) dt.$$

The derivatives of H with respect to $(x, y, \overline{\mu})$ are given by:

$$\frac{\partial^{n+m+r} H_t}{\partial x^n \partial y^m \partial \overline{\mu}^r}(z, \mu, \tau) =$$

$$\sum_i c_i \left(\frac{\partial^{i_1+i_2+i_3}}{\partial x^{i_1} \partial y^{i_2} \partial \overline{\mu}^{i_3}} (G_\mu(z, \tau, t)^{-1}) \right) \frac{\partial^{n+m+r-(i_1+i_2+i_3)}}{\partial x^{n-i_1} \partial y^{m-i_3} \partial \overline{\mu}^{r-i_3}} (Y_\mu(\gamma_\mu(z, \tau, t))), \tag{4.86}$$

where the sum is taken over the mulitple index $i = (i_1, i_2, i_3)$, with i_1 varying from 0 to n, i_2 from 0 to m and i_3 from 0 to r.

It will appear to be necessary to have upperbounds on the partial derivatives of $\gamma_\mu(z, \tau, t)$ with respect to $\overline{\mu}$. We prove, for each $k \in \mathbb{N}_1$, the existence of positive constants $\tilde{C}_k$ and $\tilde{N}_k$ such that:

$$\| \frac{\partial^k}{\partial \overline{\mu}^k} (\gamma_\mu(z, \tau, t)) \| \leq \tilde{C}_k e^{\tilde{N}_k t}, \quad \forall t \geq 0. \tag{4.87}$$

Define herefore

$$V(z, \mu, \tau) = (X_\mu + \tau Y_\mu)(z). \tag{4.88}$$

Derivation of $V(\gamma_\mu(z, \tau, t), \mu, \tau)$ with respect to $\overline{\mu}$, leads to:

$$\frac{\partial}{\partial t} \left(\frac{\partial}{\partial \mu} (\gamma_\mu(z, \tau, t)) \right) = \frac{\partial}{\partial \overline{\mu}} (V(\gamma_\mu(z, \tau, t), \mu, \tau)) =$$

$$D_z V(\gamma_\mu(z, \tau, t), \mu, \tau) (\frac{\partial}{\partial \overline{\mu}} (\gamma_\mu(z, \tau, t))) + D_{\overline{\mu}} V(\gamma_\mu(z, \tau, t), \mu, \tau)(1),$$

where $D_z V$ denotes the differential of V with respect to $z = (x, y)$ and $D_{\overline{\mu}} V$ denotes the differential with respect to $\overline{\mu}$. Therefore:

$$\frac{\partial}{\partial \overline{\mu}}(\gamma_\mu(z, \tau, t)) =$$

$$\int_0^t \Big(D_z V(\gamma_\mu(z, \tau, s), \mu, \tau)(\frac{\partial}{\partial \overline{\mu}}(\gamma_\mu(z, \tau, s))) + D_{\overline{\mu}} V(\gamma_\mu(z, \tau, s), \mu, \tau)(1)\Big) ds,$$

yielding:

$$\| \frac{\partial}{\partial \overline{\mu}}(\gamma_\mu(z, \tau, t)) \| \leq \Omega_0 t + \Omega_1 \int_0^t \| \frac{\partial}{\partial \overline{\mu}}(\gamma_\mu(z, \tau, s)) \| ds,$$

for some positive constants Ω_0 and Ω_1. Using Gronwall's inequality, we find:

$$\| \frac{\partial}{\partial \overline{\mu}}(\gamma_\mu(z, \tau, t)) \| \leq \tilde{C}_1 e^{\tilde{N}_1 t}, \quad \forall t \geq 0,$$

for some positive constants $\tilde{C}_1$ and $\tilde{N}_1$. On a totally analogous way as before, one can proceed by induction to come to the upperbounds in (4.87).

Using (4.79) and (4.87) and the infinitely flatness of $Y_\mu(z)$ along M, one can verify by induction, for each $q \in \mathbb{N}_1$ and $(i, j, k) \in \mathbb{N}^3$, the existence of constants $L_{i,j,k}(q)$ such that:

$$\| \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial \overline{\mu}^k}(Y_\mu(\gamma_\mu(z, \tau, t)) \| < L_{i,j,k}(q) \| z \|_M^q \, e^{-q\nu t}, \quad \forall t \geq 0, \quad (4.89)$$

as $\| z \|_M \to 0$. Formula (4.81) stays valid when we take all appearing partial derivatives of $G_\mu(z, \tau, t)$ and $G_\mu(z, \tau, t)^{-1}$ with respect to $(x, y, \overline{\mu})$. As before, we assume by induction that for $t \geq 0$:

$$\| \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial \overline{\mu}^k}(G_\mu(z, \tau, t)^{-1}) \| \leq C_{i,j,k} e^{N_{i,j,k} t}, \quad (4.90)$$

for some positive constants $C_{i,j,k}$ and $N_{i,j,k}$ with $i + j + k \leq Q$ and prove that (4.90) also holds for $i + j + k = Q + 1$. Using the upperbounds in (4.89) and (4.90), one finds an upperbound on all partial derivatives of H

with respect to $(x, y, \overline{\mu})$ on a totally similar way as before (see equations (4.84) and (4.85)).

Appropriate upperbounds on $\frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial \overline{\mu}^k}(G_\mu(z,\tau,t))$, $\forall (i,j,k) \in \mathbb{N}^3$ can be found by using Gronwall's inequality. Herefore one needs a formula for the partial derivatives of $\tilde{H}_t$ with respect to $(x, y, \overline{\mu})$. This formula can be deduced by induction on the order of the partial derivatives yielding:

$$\frac{\partial^{i+j+k}\tilde{H}_t}{\partial x^i \partial y^j \partial \overline{\mu}^k}(z) = D_z V(\gamma_\mu(z,\tau,t),\mu,\tau)(\frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial \overline{\mu}^k}(G_\mu(z,\tau,t))+$$
$$\sum_{p=1}^{i+j+k} \sum_{l=1}^{\overline{b}(p)} \overline{d}_l^p \, D_{z\overline{\mu}}^{p+1} V(\gamma_\mu(z,\tau,t),\mu,\tau)(w_l^{p+1}), \tag{4.91}$$

where $d_l^p \in \mathbb{R}$, each $\overline{b}(p)$ is some natural number, V is the function defined in (4.88), $D_{z\overline{\mu}}^{p+1} V$ is a differential of V of order $p+1$ with respect to the directions $z = (x, y)$ and $\overline{\mu}$ and each w_l^{p+1} is a vector of length $p+1$ that contains as components, the scalar 1, partial derivatives of $G_\mu(z,\tau,t)$ with respect to $(x, y, \overline{\mu})$ of order at most $i+j+k-1$ and partial derivatives of $\gamma_\mu(z,\tau,t)$ with respect to $(x, y, \overline{\mu})$ of order at most $i+j+k$.

Because the column vectors of $G_\mu(z,\tau,t)$ are given by the partial derivatives of order 1 of $\gamma_\mu(z,\tau,t)$ with respect to x and y, every partial derivative, with respect to $(x, y, \overline{\mu})$, of $\gamma_\mu(z,\tau,t)$ of order $i+j+k$ with $i+j>0$ corresponds to a partial derivatives of $G_\mu(z,\tau,t)$ of order $(i+j-1)+k$. Making use of formula (4.91) and using the upperbounds in (4.87), one is able to prove, by induction, on the order of the partial derivatives, and using Gronwall's inequality, for each $(i,j,k) \in \mathbb{N}_3$, the existence of positive constants $\overline{C}_{i,j,k}$ and $\overline{N}_{i,j,k}$ such that:

$$\| \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial \overline{\mu}^k}(G_\mu(z,\tau,t)) \| \leq \overline{C}_{i,j,k} e^{\overline{N}_{i,j,k} t}. \tag{4.92}$$

Totally similar arguments can be used if, instead of succesively deriving with respect to a same $\overline{\mu} \in \{\mu_1, \ldots, \mu_p, \tau\}$, one, at each step, makes an abitrary choice from $(\mu_1, \ldots, \mu_p, \tau)$.

Chapter 5

2–saddle cycles producing alien limit cycles

In [16] it is proven that a Hamiltonian 2–saddle cycle can produce limit cycles not covered by zeros of the related Abelian integral. Herefore, one can consider an unfolding, leaving one connection of the 2–saddle cycle unbroken, that is generic and of codimension 4 and in addition satifies an extra generic condition concerning the derivatives of the transition maps along the saddle–connections, which we will specify later on. The limit cycles, produced after perturbation, that are not covered by zeros of the Abelian integral are also called *alien limit cycles*, see also [4] or [6].

To check all these necessary conditions on an unfolding can be quite involved. This chapter will present necessary techniques in order to check the required generic conditions on specific examples. Let us first repeat some elementary notions and specify the necessary conditions.

5.1 Settings and necessary conditions

We deal with perturbations of Hamiltonian systems:

$$(X_{(\overline{\mu},\varepsilon)}) : \begin{cases} \dot{x} &= -\frac{\partial H}{\partial y} + \varepsilon f, \\ \dot{y} &= \frac{\partial H}{\partial x} + \varepsilon g, \end{cases} \tag{5.1}$$

where $H(x, y), f(x, y, \overline{\mu}, \varepsilon), g(x, y, \overline{\mu}, \varepsilon)$ are C^∞ functions, ε is considered to take small positive values and $\overline{\mu}$ varies in some compact subset $K \subset \mathbb{R}^p$. Further we abbreviate $\mu = (\overline{\mu}, \varepsilon)$.

We suppose that the flow of $X_{(\overline{\mu},0)} = X_H$ contains *a period annulus* bounded by a hyperbolic 2–saddle cycle $\mathcal{L}$ as in Figure 5.1. A period annulus is a subset of the plane filled by closed orbits of X_H. The hyperbolic 2–saddle cycle consists of two saddle–connections Γ_1 and Γ_2 and two hyperbolic saddles s_1 and s_2 such that $s_1 := \alpha(\Gamma_1) = \omega(\Gamma_2)$ and $s_2 := \alpha(\Gamma_2) = \omega(\Gamma_1)$. We choose H to be zero on the 2–saddle cycle and strictly positive on the nearby closed orbits.

A difficult problem is to study the number of limit cycles of $X_{(\overline{\mu},\varepsilon)}$ that can be found for $\overline{\mu} \in K$, ε near zero and (x, y) in a neighbourhood of $\mathcal{L}$.

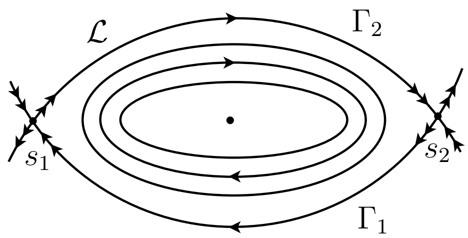

Figure 5.1: A 2–saddle cyle lying on the boundary of a period annulus.

In studying these limit cycles, one considers the difference map near the period annulus. In first order approximation with respect to ε, it is given by the *Abelian integral*:

$$I_{\overline{\mu}}(h) = I(h, \overline{\mu}) = \int_{\gamma_h} f dy - g dx, \tag{5.2}$$

where γ_h is the regular orbit in the annulus on which H takes the constant value h. Let us make this more precise.

Choose sections Σ_1 and Σ_3, near s_1 and s_2 respectively, transverse to

Γ_2. Denote by u and z, the regular parameters used to parameterise Σ_1 and Σ_3, with Γ_2 represented by $u = 0$ and $z = 0$. Similar choose sections Σ_2 and Σ_4, near s_1 and s_2 respectively, transverse to Γ_1 and parameterised by regular parameters v and w respectively, with Γ_1 represented by $v = 0$ and $w = 0$, see Figure 5.2.

Consider now the regular transition maps R^2_μ from Σ_1 to Σ_3 along Γ_2 and R^1_μ from Σ_2 to Σ_4 along Γ_1. Let D^1_μ and D^2_μ be the corner passages near the saddle s_1, s_2 respectively, see Figure 5.2. All these transition maps are expressed in function of the chosen regular parameter on the sections Σ_i, $i = 1, \ldots, 4$. They are only locally defined: as well ε as the chosen regular parameter take small positive values.

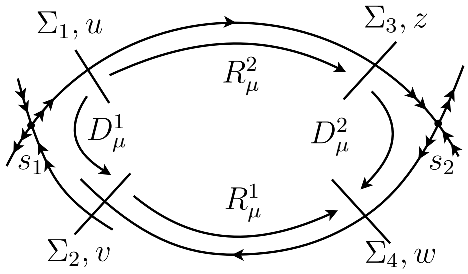

Figure 5.2: Transition maps near a 2–saddle cycle.

The difference map Δ is locally defined as:

$$\Delta(u, \mu) = \Delta_\mu(u) = \Delta^2(u, \mu) - \Delta^1(u, \mu),$$

for $u > 0$, with

$$\Delta^1(u, \mu) = \Delta^1_\mu(u) = R^1_\mu(u) \circ D^1_\mu, \quad \Delta^2(u, \mu) = \Delta^2_\mu(u) = D^2_\mu \circ R^2_\mu(u).$$

Now the limit cycles of X_μ that can be found in a neighbourhood of $\mathcal{L}$, for μ near $(\overline{\mu}_0, 0), \overline{\mu}_0 \in K$, correspond to the positive zeros, close to $u = 0$, of $\Delta_\mu(u)$ with μ varying close to $(\overline{\mu}_0, 0)$.

We show that Δ is, in first order approximation with respect to ε and for $h > 0$, given by the Abelian integral (5.2) [16]. This follows from the fact that Δ is related to the return map $P(w, \mu) = P_\mu(w)$ on Σ_4 in the following way:

$$P_\mu \circ \Delta^1_\mu = \Delta^2_\mu$$

such that, if one defines $\delta(w, \mu) = \delta_\mu(w) = P_\mu(w) - w$, then

$$\Delta(u, \mu) = P_\mu \circ \Delta^1_\mu - \Delta^1_\mu = \delta_\mu \circ \Delta^1_\mu. \tag{5.3}$$

Replacing w by h in the parametrisation of Σ_4, where h denotes the value taken by H, δ can be written as a function of h. It is generally known that for $h > 0$:

$$\delta_\mu(h) = \varepsilon I_{\overline{\mu}}(h) + O(\varepsilon^2),$$

(see for instance [17]). On the other hand for $\varepsilon = 0$, X_μ has a period annulus near $\mathcal{L}$ implying that

$$\Delta = \varepsilon \overline{\Delta}. \tag{5.4}$$

So comparing terms of first order in ε in equation (5.3), one gets:

$$\overline{\Delta}(u, \overline{\mu}, 0) = I_{\overline{\mu}}(h), \tag{5.5}$$

where $I_{\overline{\mu}}(h)$ is the Abelian integral (5.2) taken over the level curve $H = h$ passing through the point $w = \Delta^1_{(\overline{\mu},0)}(u)$ on Σ_4.

One can ask oneself whether this first order approximation of the difference map, given in (5.5), gives enough information to track limit cycles near $\mathcal{L}$. In the case where $\mathcal{L}$ is a saddle–loop this reveals to be true [23]. However for the 2–saddle cycle case a direct transfer from results on the zero–set of $I_{\overline{\mu}}(h)$ to the results on the set of limit cycles is not always possible as is showed in [16].

More precisely, let (X_μ) be an unfolding of a Hamiltonian vector field X_H like in (5.1) such that X_H admits a period annulus bounded by a 2–saddle cycle like in Figure 5.1. Suppose μ varies in some neighbourhood of $(\overline{\mu}_0, 0)$, $\overline{\mu}_0 \in K$, hence the unfolding is a perturbation of the Hamiltonian $X_{(\overline{\mu}_0,0)} = X_H$. After a translation in parameter space, one can always suppose that $\overline{\mu}_0 = 0$. Suppose that the unfolding leaves one connection (say Γ_2) unbroken. Then in [16], it is proven that if the unfolding is generic and of codimension 4, satisfying an extra generic condition concerning the second derivatives at zero of the transition maps R^1_μ and R^2_μ, $\mathcal{L}$ can produce four limit cycles. However because the codimension of the unfolding is 4, the related Abelian integral can have at most 3 zeros that bifurcate from $\{h = 0\}$ and $\overline{\mu}$ near zero.

Let us explain what is meant by (X_μ) being a generic unfolding of codimension 4 and specify the extra generic condition that (X_μ) has to satisfy. Consider the related Abelian integral $I_{\overline{\mu}}(h)$ defined in (5.2). The family $I_{\overline{\mu}}(h)$, $\overline{\mu}$ varying in a neighbourhood of 0, has a Dulac series linear in $\log u$, see for example [16]. This means that if we consider $\{f_i\}_{i\in\mathbb{N}}$ with:

$$f_0 = 1, \quad f_1 = h\log h, \quad f_2 = h, \quad f_3 = h^2\log h, \quad f_4 = h^2, \ldots$$

then there exists a formal development of $I_{\overline{\mu}}$ at $h = 0$ with respect to this scale:

$$\hat{I}_{\overline{\mu}} = \sum_{i=0}^{\infty} \alpha_i(\overline{\mu}) f_i,$$

where all $\alpha_i(\overline{\mu})$ are smooth. The index of the first non–zero coefficient $\alpha_i(0)$ is called the *codimension of* $I_{\overline{\mu}}$. One can show that the number of zeros of $I_{\overline{\mu}}$ that bifurcate from $\{h = 0\}$ and $\overline{\mu}$ near zero is at most the codimension of $I_{\overline{\mu}}$. This is a simple application of the so–called derivation–division algorithm as it appeared in [26], see also [16].

If the Abelian integral $I_{\overline{\mu}}$ is of codimension k, we say that it is *generic* if the map

$$\overline{\mu} \mapsto (\alpha_0(\overline{\mu}), \ldots, \alpha_{k-1}(\overline{\mu}))$$

is a local submersion at zero of $\mathbb{R}^p$ into $\mathbb{R}^k$ with $k \leq p$.

The unfolding (X_μ) is said to be *generic of codimension* 4 if the related Abelian integral $I_{\overline{\mu}}$ is generic of codimension 3. For a general definition of the codimension of the unfolding (X_μ), we refer to [16]. In particular if the unfolding (X_μ) is generic of codimension 4, then the related Abelian integral reads:

$$I_{\overline{\mu}}(h) = p(\overline{\mu}) + q(\overline{\mu})h\log h + r(\overline{\mu})h + s(\overline{\mu})h^2\log h + O(h^2), \qquad (5.6)$$

for some p, q, r, s smooth in $\overline{\mu}$, with

$$p(0) = q(0) = r(0) = 0, \qquad s(0) \neq 0. \qquad (5.7)$$

Moreover the map

$$\overline{\mu} \mapsto (p(\overline{\mu}), q(\overline{\mu}), r(\overline{\mu})), \qquad (5.8)$$

has to be a local submersion at zero of $\mathbb{R}^p$ into $\mathbb{R}^3$, with $p \geq 3$. In particular this means that at least four parameters (ε taken into account) are needed for (X_μ) being a generic unfolding of X_H of codimension 4.

However, referring to [16], it seems also natural to consider generic unfoldings of codimension 3 with 5 parameters ($\overline{\mu} \in \mathbb{R}^4$ and ε) such that

$$\overline{\mu} \mapsto (p(\overline{\mu}), q(\overline{\mu}), r(\overline{\mu}), \alpha_1(\overline{\mu})) \qquad (5.9)$$

is a local submersion at zero. Here the coefficient $\alpha_1(\overline{\mu}) := \alpha_1(\overline{\mu}, 0)$ is defined such that the ratio $r_1(\mu)$ of eigenvalues of the saddle of X_μ lying near s_1 equals

$$r_1(\mu) = 1 + \varepsilon\alpha_1(\mu).$$

So there are two different types of genericity here, (5.8) or (5.9). In the example that we will treat in this chapter, we will use condition (5.9).

Let us now specify the extra generic condition concerning the transition maps R^1_μ and R^2_μ. In Section 5.4, we show that both transition maps R^1_μ and R^2_μ, when expressed in appropriate normalizing coordinates near the saddles, are the identity map for $\varepsilon = 0$. In particular

$$R^1_\mu(v) = v + \varepsilon(-\beta_1(\mu) + \gamma_1(\mu)v + \eta_1(\mu)v^2 + O(v^3)), \qquad (5.10)$$

where $\beta_1, \gamma_1, \eta_1$ are C^∞ in the parameter $\mu = (\overline{\mu}, \varepsilon)$ and

$$R^2_\mu(u) = u + \varepsilon(-\beta_2(\mu) + \gamma_2(\mu)u + \eta_2(\mu)u^2 + O(u^3)),$$

for some $\beta_2, \gamma_2, \eta_2$ C^∞ dependent on the parameter $\mu = (\overline{\mu}, \varepsilon)$. However because Γ_2 stays unbroken, $\beta_2(\mu) = 0, \forall \mu$, and after performing a parameter dependent coordinate change in u, one can suppose that $\gamma_2(\mu) = 0$ yielding:

$$R^2_\mu(u) = u + \varepsilon(\eta_2(\mu)u^2 + O(u^3)), \tag{5.11}$$

with η_2, C^∞ in μ. The last generic condition now reads

$$\eta_2(0) \neq 2\eta_1(0). \tag{5.12}$$

To start off, let us sketch how one could verify the required conditions (5.7), (5.8) or (5.9) and (5.12) in practice.

The conditions (5.7), (5.8) or (5.9) all concern the Abelian integral $I_{\overline{\mu}}(x)$ defined in (5.2) belonging to the system (5.1). In the calculations concerning the Abelian integral, one can for instance rely on the Picard–Fuchs equations in order to calculate the desired coefficients.

The coefficient $p(\overline{\mu})$, defined in (5.6), can easily be calculated once parametrisations for Γ_1 and Γ_2 are known:

$$p(\overline{\mu}) = I_{\overline{\mu}}(0) = \int_{\Gamma_1} f dy - g dx + \int_{\Gamma_2} f dy - g dx, \tag{5.13}$$

where the saddle–connections Γ_1 and Γ_2 are lying on the curve $H = 0$, which is algebraic if H is a polynomial. Because Γ_2 stays unbroken in the unfolding, the integral in (5.13) taken over Γ_2 is zero (see for instance [31]) such that

$$p(\overline{\mu}) = \int_{\Gamma_1} f dy - g dx.$$

Consider now the case where there exists some curve $\overline{\mu} = \gamma(\varepsilon)$ in parameter space with $\gamma(0) = 0$ such that the connection Γ_1 stays unbroken inside the subfamily

$$(Z_\varepsilon) = (X_{(\gamma(\varepsilon),\varepsilon)}).$$

This is the case when there exists some $i_0 \in \{1, \ldots, p\}$ such that

$$\overline{p}(0) = 0 \quad \text{and} \quad \frac{\partial \overline{p}}{\partial \overline{\mu}_{i_0}}(0) \neq 0, \tag{5.14}$$

see for instance [19].

The family (Z_ε) has a 2–saddle cycle for each $\varepsilon > 0$ sufficiently small. The second derivatives at zero of the transition maps along both saddle–connections Γ_1 and Γ_2 will only depend on ε. In particular, looking at (5.10) and (5.11), they are given by

$$\frac{d^2\tilde{R}^1_\varepsilon}{dy^2}(0) = \varepsilon\tilde{\eta}_1(\varepsilon), \tag{5.15}$$

and

$$\frac{d^2\tilde{R}^2_\varepsilon}{dx^2}(0) = \varepsilon\tilde{\eta}_2(\varepsilon). \tag{5.16}$$

where $\tilde{R}^1_\varepsilon, \tilde{R}^2_\varepsilon$ and $\tilde{\eta}_1, \tilde{\eta}_2$ are the respective restrictions of R^1_μ, R^2_μ and η_1, η_2 on the curve $\overline{\mu} = \gamma(\varepsilon)$ in parameter space. In particular because $\gamma(0) = 0$:

$$\tilde{\eta}_1(0) = \eta_1(0) \quad \text{and} \quad \tilde{\eta}_2(0) = \eta_2(0).$$

Thus when (5.14) is satisfied, one is able to calculate the coefficients $\eta_1(0)$ and $\eta_2(0)$ ones the first and second derivative at zero of $\tilde{R}^1_\varepsilon$ and $\tilde{R}^2_\varepsilon$ are known up to order $O(\varepsilon^2)$. We now continue by treating the transition maps along a saddle–connection within a family that leaves the saddle–connection unbroken. Let us start by refreshing some of the main ideas concerning transition maps.

5.2 Transition maps

This section is meant to recall some elementary notions concerning the transition maps and to calculate their first and second derivative. Let us start with recalling the definition of a transition map of a planar flow.

Denote by X a C^∞ planar vector field with flow $\phi_t(v), v \in \mathbb{R}^2$ meaning that

$$\begin{cases} \dfrac{d}{dt}(\phi_t(v)) = X(\phi_t(v)), \\ \phi_0(v) = v. \end{cases}$$

Take two sections Σ_1 and Σ_2 transverse to the flow of X. Suppose that $\psi_i = (f_i, g_i) : I_i \subset \mathbb{R} \mapsto \Sigma_i$ is a regular parametrisation of Σ_i, for $i = 1, 2$. Let $p = (f_1(s_1), g_1(s_1)) \in \Sigma_1$ and $q = (f_2(s_2), g_2(s_2)) \in \Sigma_2$ and assume that the orbit $\phi_t(p)$ reaches q in finite time τ_0. Then the C^∞ function

$$\theta(s, s', t) = \phi_t(\psi_1(s)) - \psi_2(s') \tag{5.17}$$

has a zero in $(s, s', t) = (s_1, s_2, \tau_0)$. Because Σ_2 is transverse to X and the ψ_i are regular,

$$\begin{vmatrix} f_2'(s_2) & X_1(q) \\ g_2'(s_2) & X_2(q) \end{vmatrix}$$

is not equal to zero. It follows from the implicit function theorem that there exist C^∞ functions T, τ defined on a neighborhood of s_1 such that

$$\theta(s, T(s), \tau(s)) = 0, \tag{5.18}$$

and $T(s_1) = s_2$, $\tau(s_1) = \tau_0$.

The function $T(s)$ is called *the transition map of X from Σ_1 to Σ_2 expressed in the chosen parameters s and s'*. In particular the orbit $\phi_t(\psi_1(s))$ crosses the section Σ_2 at the point $\psi_2(T(s))$. Note that $T(s)$ is a regular map (see Theorem 5.2). The time needed to go from $\psi_1(s)$ to $\psi_2(T(s))$ is given by $\tau(s)$. The function $\tau(s)$ is referred to as *the transition time function of X from Σ_1 to Σ_2 expressed in the chosen parameter s*.

Derivatives of the transition map and the transition time function can be found by means of implicit differentiation of the expression (5.18). Denoting for fixed t, $D_2\phi(t, v)$ as the differential of the function

$$\begin{array}{rccc} \phi_t : & \mathbb{R}^2 & \mapsto & \mathbb{R}^2 \\ & v & \mapsto & \phi_t(v), \end{array}$$

we find

$$D_2\phi(\tau(s), \psi_1(s))\, \psi_1'(s) + X(\phi_{\tau(s)}(\psi_1(s)))\tau'(s) - \psi_2'(T(s))T'(s) = 0. \tag{5.19}$$

To find the desired derivatives, we use Diliberto's theorem [9] to decompose the vectorial equation (5.19) with respect to an appropriate orthogonal basis. See also [7] where formulas for $T'(s)$ and $\tau'(s)$ are obtained using Diliberto's theorem.

The scalar and wedge product between a vector field X with Euclidean coordinates (P,Q) and a vector field $\overline{X}$ with Euclidean coordinates $(\overline{P},\overline{Q})$ are denoted as

$$X \cdot \overline{X} = P\overline{P} + Q\overline{Q}, \quad \text{and} \quad X \wedge \overline{X} := P\overline{Q} - Q\overline{P}.$$

Define the vector field

$$N := \frac{1}{\| X \|^2} X^{\perp},$$

multiple of the orthogonal vector field:

$$X^{\perp} = -Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y},$$

such that $X^{\perp} \cdot N = 1$. The following C^{∞} functions are referred to as the curl, the divergence and the curvature of X at p respectively:

$$\text{curl } X(p) = \frac{\partial Q}{\partial x}(p) - \frac{\partial P}{\partial y}(p), \qquad \text{div } X(p) = \frac{\partial P}{\partial x}(p) + \frac{\partial Q}{\partial y}(p),$$

and:

$$\kappa(p) = \frac{1}{\| X(p) \|}\left(N(p) \cdot \frac{d}{dt} X(\phi_t(p)) \mid_{t=0}\right). \tag{5.20}$$

Theorem 5.1 *(Diliberto, [9]). Let X be a C^{∞} planar vector field with flow $\phi_t(v), v \in \mathbb{R}^2$. Let $p \in \mathbb{R}^2$ with $X(p) \neq 0$. For*

$$w = \alpha X(p) + \beta N(p)$$

the system:

$$\begin{cases} \dot{v} &= DX(\phi_t(p))v, \\ v(0) &= w \end{cases} \tag{5.21}$$

has solution

$$D_2\phi(t,p)w = A(t)X(\phi_t(p)) + B(t)N(\phi_t(p)),$$

where $A(t) := A(t, X, p, w)$ *and* $B(t) := B(t, X, p, w)$ *are given by:*

$$A(t) = \alpha + \int_0^t \left\{ \frac{1}{\| X \|^2} [2\kappa \| X \| - \text{curl } X] \right\} (\phi_r(p))\, B(r)\, dr, \tag{5.22}$$

$$B(t) = \beta \exp \left(\int_0^t \text{div } X(\phi_r(p))\, dr \right). \tag{5.23}$$

Proof: To simplify the notation let us denote $X(\phi_t(p)) = X_t(p)$ and analoguous $N(\phi_t(p)) = N_t(p)$.

It is easily verified that $v(t) = D_2\phi(t, p)w$ satisifies (5.21). The decomposition of this solution with respect to the orthogonal basis $\{X_t(p), N_t(p)\}$ is given by

$$v(t) = A(t)X_t(p) + B(t)N_t(p), \tag{5.24}$$

with

$$A(t) = v(t) \cdot \frac{X_t(p)}{\| X_t(p) \|^2} \quad \text{and} \quad B(t) = v(t) \wedge X_t(p). \tag{5.25}$$

Because $v(0) = w$, we have

$$A(0) = \alpha \qquad \text{and} \qquad B(0) = \beta. \tag{5.26}$$

We now search for differential equations which $A(t)$ and $B(t)$ must satisfy enabling us to find the formulas in (5.22) and (5.23).

Let us start by calculating $B(t)$. Deriving the expression of $B(t)$ in (5.25), one finds:

$$\begin{aligned} B'(t) &= v'(t) \wedge X_t(p) + v(t) \wedge DX(\phi_t(p))X_t(p) \\ &= DX(\phi_t(p))v(t) \wedge X_t(p) + v(t) \wedge DX(\phi_t(p))X_t(p) \\ &= \text{tr } DX(\phi_t(p))(v(t) \wedge X_t(p)) \\ &= \text{div } X(\phi_t(p))B(t), \end{aligned}$$

concluding that $B(t)$ is the unique solution of the differential equation:

$$\begin{cases} B'(t) &= \text{div } X(\phi_t(p))B(t), \\ B(0) &= \beta, \end{cases}$$

which obviously result in formula (5.23).

The formula for $A(t)$ can be obtained analogously. Denote

$$I(t) = \frac{1}{2}\frac{d}{dt}(\| X_t(p) \|^2) = \frac{d}{dt}X_t(p) \cdot X_t(p).$$

Differentiating $v(t) \cdot X_t(p) = \| X_t(p) \|^2 A(t)$ one finds

$$v'(t) \cdot X_t(p) + v(t) \cdot \frac{d}{dt}X_t(p) = 2I(t)A(t) + \| X_t(p) \|^2 A'(t). \quad (5.27)$$

Using (5.24) one computes:

$$\begin{aligned} v'(t) \cdot X_t(p) &= DX(\phi_t(p))v(t) \cdot X_t(p) \\ &= A(t)I(t) + B(t)DX(\phi_t(p))N_t(p) \cdot X_t(p), \quad (5.28) \end{aligned}$$

and

$$v(t) \cdot \frac{d}{dt}X_t(p) = A(t)I(t) + B(t)N_t(p) \cdot \frac{d}{dt}X_t(p). \quad (5.29)$$

Substituting (5.28) and (5.29) into (5.27), one finds

$$A'(t) = \frac{B(t)}{\| X_t(p) \|^2}\Big(DX(\phi_t(p))N_t(p) \cdot X_t(p) + N_t(p) \cdot DX(\phi_t(p))X_t(p)\Big).$$

It is a straightforward calculation to show:

$$DX(\phi_t(p))N_t(p) \cdot X_t(p) - N_t(p) \cdot DX(\phi_t(p))X_t(p) = -\text{ curl } X(\phi_t(p)$$

such that using

$$\kappa(\phi_t(p)) = \frac{1}{\| X_t(p) \|}\Big(N_t(p) \cdot DX(\phi_t(p))X_t(p)\Big),$$

one finds

$$A'(t) = \left\{ \frac{1}{\| X \|^2} (2\kappa \| X \| - \text{curl } X) \right\} (\phi_t(p)) B(t)$$

which together with the initial condition $A(0) = \alpha$ implies the formula for $A(t)$ in (5.95). □

Theorem 5.2 *Let X be a C^∞ vector field. Consider the transition map $T(s)$ between two sections Σ_1 and Σ_2 transverse to the flow of X. Suppose ψ_1 and ψ_2 are regular parametrisations of these sections and Γ_s is the orbit starting at $\psi_1(s)$ and ending in $\psi_2(T(s))$. Defining the quantities*

$$\Delta_i(s) := \Delta(s, X, \psi_i) = X(\psi_i(s)) \wedge \psi_i'(s),$$

and

$$\sigma_i(s) := \sigma(s, X, \psi_i) = \frac{\Delta_i'(s)}{\Delta_i(s)} - \frac{X(\psi_i(s)) \cdot \psi_i'(s)}{\| X(\psi_i(s)) \|^2} \text{div } X(\psi_i(s)),$$

with $i = 1$ or $i = 2$, the derivatives of first and second order of T are given by:

$$T'(s) = \frac{\Delta_1(s)}{\Delta_2(T(s))} \exp \int_{\Gamma_s} \frac{\text{div } X}{\| X \|} d\overline{s}, \tag{5.30}$$

$$T''(s) = T'(s) \left(\sigma_1(s) - T'(s)\sigma_2(T(s)) + \Delta_1(s) \int_{\Gamma_s} \frac{\mathcal{A}\mathcal{B}}{\| X \|^3} d\overline{s} \right) \tag{5.31}$$

where $d\overline{s}$ represents the arc length element of Γ_s and where $\mathcal{A}(z) := \mathcal{A}(z, X)$ and $\mathcal{B}(z) := \mathcal{B}(z, X)$, $z = (x, y) \in \mathbb{R}^2$, are given by:

$$\begin{aligned} \mathcal{A}(z) &= D(\text{div } X)_z(X^\perp(z)) - \left\{ (2\kappa \| X \| - \text{curl } X) \text{ div } X \right\}(z), \\ \mathcal{B}(z) &= \exp \int_{\Gamma_s(z)} \frac{\text{div } X}{\| X \|} d\overline{s}, \end{aligned} \tag{5.32}$$

with $\Gamma_s(z)$ the orbit starting at $\psi_1(s)$ and ending in z.

Proof: To shorten notation during the proof let us denote $\tilde{s} = T(s)$. The transition map T and the transition time function $\tau(s)$ are defined by the equation (5.18):

$$\theta(s, \tilde{s}, \tau(s)) = \phi_{\tau(s)}(\psi_1(s)) - \psi_2(\tilde{s}) = 0. \tag{5.33}$$

Derivation of (5.33) gives

$$D_2\phi(\tau(s), \psi_1(s))\psi_1'(s) + X(\psi_2(\tilde{s}))\tau'(s) - \psi_2'(\tilde{s})T'(s) = 0. \tag{5.34}$$

Formulas for $T'(s)$ and $\tau'(s)$ will follow from a decomposition of (5.34) with respect to the orthogonal basis $\{X, N\}$ introduced in Theorem 5.1.

Decomposing $\psi_i'(s)$ as

$$\psi_i'(s) = \alpha_i(s)X(\psi_i(s)) + \beta_i(s)N(\psi_i(s)) \tag{5.35}$$

with

$$\alpha_i(s) = \frac{X(\psi_i(s)) \cdot \psi_i'(s)}{\| X(\psi_i(s)) \|^2}, \qquad \beta_i(s) = X(\psi_i(s)) \wedge \psi_i'(s),$$

it follows from Theorem 5.1 that

$$D_2\phi(t, \psi_1(s))(\psi_1'(s)) = A(t)X(\phi_t(\psi_1(s)) + B(t)N(\phi_t(\psi_1(s))), \tag{5.36}$$

where $A(t) = A(t, X, \psi_1(s), \psi_1'(s)$ and $B(t) = B(t, X, \psi_2(\tilde{s}), \psi_1'(s))$ are defined as in Theorem 5.1. Substitution of (5.35) and (5.36) into (5.34) yields

$$\begin{aligned}\Big(\alpha_2(\tilde{s})T'(s) - \tau'(s) - A(\tau(s))\Big)X(\psi_2(\tilde{s}))\\ +\Big(\beta_2(\tilde{s})T'(s) - B(\tau(s))\Big)N(\psi_2(\tilde{s})) = 0.\end{aligned} \tag{5.37}$$

In particular $\beta_2(\tilde{s})T'(s) - B(\tau(s)) = 0$ such that

$$T'(s) = \frac{\beta_1(s)}{\beta_2(\tilde{s})} \exp \int_0^{\tau(s)} \text{div } X(\gamma_s(t))\, dt \tag{5.38}$$

with $\gamma_s(t) = \phi(t, \psi_1(s))$. Formula (5.30) follows.

Derivation of (5.38) gives

$$\begin{aligned} T''(s) = T'(s)\Bigg(&\frac{\beta_2(\tilde{s})}{\beta_1(s)}\frac{d}{ds}\left(\frac{\beta_1(s)}{\beta_2(\tilde{s})}\right) + \tau'(s)\,\mathrm{div}\ X(\psi_2(\tilde{s})) \\ &+ \int_0^{\tau(s)} \frac{d}{ds}\Big(\mathrm{div}\ X(\gamma_s(t))\Big)\,dt\Bigg). \end{aligned} \tag{5.39}$$

The formula can be simplified by applying the technique of partial integration on the integral $\int_0^{\tau(s)} \frac{d}{ds}\left(\mathrm{div}\ X(\gamma_s(t))\right) dt$ and using the formula

$$\tau'(s) = \alpha_2(\tilde{s})T'(s) - A(\tau(s)) \tag{5.40}$$

that follows from (5.37). Because:

$$\begin{aligned} \frac{d}{ds}\Big(\mathrm{div}\ X(\gamma_s(t))\Big) &= D(\mathrm{div}\ X)_{\gamma_s(t)}\left(\frac{d}{ds}\gamma_s(t)\right) \\ &= D(\mathrm{div}\ X)_{\gamma_s(t)}(D_2\phi(t, \psi_1(s))\psi_1'(s)), \end{aligned}$$

one finds, after substituting formula (5.36),

$$\begin{aligned} \frac{d}{ds}\Big(\mathrm{div}\ X(\gamma_s(t))\Big) &= A(t)D(\mathrm{div}\ X)_{\gamma_s(t)}(X(\gamma_s(t))) \\ &\quad + B(t)D(\mathrm{div}\ X)_{\gamma_s(t)}(N(\gamma_s(t))). \end{aligned} \tag{5.41}$$

Since $D(\mathrm{div}\ X)_{\gamma_s(t)}(X(\gamma_s(t))) = \frac{d}{dt}\,\mathrm{div}\ X(\gamma_s(t))$ one can use the technique of partial integration on the integral

$$\int_0^{\tau(s)} A(t)D(\mathrm{div}\ X)_{\gamma_s(t)}(X(\gamma_s(t)))\,dt.$$

Using (5.41) this yields

$$\int_0^{\tau(s)} \frac{d}{ds}\Big(\mathrm{div}\ X(\gamma_s(t))\Big)dt = \Big[A(t)\,\mathrm{div}\ X(\gamma_s(t))\Big]_0^{\tau(s)} + I, \tag{5.42}$$

where I is given by

$$\int_0^{\tau(s)} B(t)\frac{\mathcal{A}(\gamma_s(t))}{\parallel X(\gamma_s(t)) \parallel^2}\, dt = \beta_1(s)\int_{\Gamma_s} \frac{\mathcal{A}\mathcal{B}}{\parallel X \parallel^3}\, d\overline{s},$$

with $\mathcal{A}$ and $\mathcal{B}$ defined as in (5.32). Substituting (5.42) and (5.40) into (5.39) and using:

$$\frac{\beta_2(\tilde{s})}{\beta_1(s)}\frac{d}{ds}\left(\frac{\beta_1(s)}{\beta_2(\tilde{s})}\right) = \frac{\beta_1'(s)}{\beta_1(s)} - T'(s)\frac{\beta_2'(\tilde{s})}{\beta_2(\tilde{s})},$$

the formula of $T''(s)$ follows. □

Let us formulate a special case of Theorem 5.2 which will often be used in the next sections. See also [11], where the same formulas can be found.

Corollary 5.3 *Let $X = P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}$ be a C^∞ vector field. Suppose Γ is an orbit lying on the x-axis and $\Sigma_i = \{x = x_i\}$ are sections locally transverse to the flow of X at $(x_i, 0)$, $i = 1, 2$ and parametrised by $y \mapsto (x_i, y)$. Then the first two derivatives of the transition map T along Γ from Σ_1 to Σ_2 read:*

$$\begin{aligned} T'(0) &= \exp\left(\int_{x_1}^{x_2} \frac{Q_y}{P}(x,0)dx\right), \\ T''(0) &= T'(0)\int_{x_1}^{x_2} \frac{PQ_{yy}-2P_yQ_y}{P^2}(x,0)\exp\left(\int_{x_1}^{x} \frac{Q_y}{P}(u,0)du\right)dx. \end{aligned}$$

Similarly if Γ' is an orbit lying on the y-axis and $\Sigma_i' = \{y = y_i\}$ are sections locally transverse to the flow of X at $(0, y_i)$, $i = 1, 2$ and parametrised by $x \mapsto (x, y_i)$. Then the first two derivatives of the transition map T along Γ from Σ_1' to Σ_2' read:

$$\begin{aligned} T'(0) &= \exp\left(\int_{y_1}^{y_2} \frac{P_x}{Q}(0,y)dy\right), \\ T''(0) &= T'(0)\int_{y_1}^{y_2} \frac{QP_{xx}-2Q_xP_x}{Q^2}(0,y)\exp\left(\int_{y_1}^{y} \frac{P_x}{Q}(0,u)du\right)dy. \end{aligned}$$

Proof: We only prove the first part of the theorem. The second part then easily follows by a switch of the x and y coordinate.

Using the x–coordinate as parameter on Γ, one easily computes that the formulas of Theorem 5.2 simplify to:

$$T'(0) = \frac{P(x_1,0)}{P(x_2,0)} \exp\left(\int_{x_1}^{x_2} \frac{\text{div } X}{P}(x,0)dx\right), \tag{5.43}$$

and

$$\begin{aligned} T''(0) &= T'(0)\left[\frac{P_y}{P}(x_1,0) - T'(0)\frac{P_y}{P}(x_2,0)\right. \\ &\left. + P(x_1,0)\int_{x_1}^{x_2} \overline{P}(x,0)\exp\left(\int_{x_1}^{x} \frac{\text{div } X}{P}(u,0)du\right)dx\right], \end{aligned} \tag{5.44}$$

with

$$\overline{P}(x,0) = \frac{P(P_{xy}+Q_{yy}) - P_y(P_x+Q_y)}{P^3}(x,0).$$

These formulas can be further simplified. Let $x \in [x_1, x_2]$. Setting $g(r) = P(r,0)$ and integrating $\dfrac{g'(r)}{g(r)} = \dfrac{P_x}{P}(r,0)$ on $[x_1, x]$ gives

$$\ln P(x,0) - \ln P(x_1,0) = \int_{x_1}^{x} \frac{P_x}{P}(u,0)du,$$

such that

$$\frac{P(x,0)}{P(x_1,0)} = \exp\left(\int_{x_1}^{x} \frac{P_x}{P}(x,0)dx\right). \tag{5.45}$$

Setting $x = x_2$ and substituting (5.45) into (5.43) yields the formula of $T'(0)$.

Setting

$$F(x) = \frac{P_y}{P^2}(x,0)\exp\left(\int_{x_1}^{x} \frac{\text{div } X}{P}(u,0)du\right),$$

the fundamental theorem of calculus reads

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} F'(x)dx,$$

which using (5.43) results in

$$T'(0)\frac{P_y}{P}(x_2,0) - \frac{P_y}{P}(x_1,0) = \tag{5.46}$$

$$P(x_1,0)\int_{x_1}^{x_2} \frac{P_{xy}P - P_xP_y + P_yQ_y}{P^3}(x,0)\exp\left(\int_{x_1}^{x} \frac{\text{div } X}{P}(u,0)du\right)dx.$$

Substituting (5.45) and (5.46) into (5.44) yields the formula of $T''(0)$. □

5.3 Transition along a saddle–connection

We continue by considering the particular transition maps between sections transverse to a saddle–connection, expressed in normalizing coordinates near the saddles. Moreover we don't restrict to individual vector fields but treat the transition in a family that leaves the saddle–connection unbroken.

Consider a C^∞ family of vector fields $(X_\mu)_{\mu\in\mathcal{P}}$ with parameter values μ varying in some subset $\mathcal{P} \subset \mathbb{R}^p$. Suppose that for $\mu_0 \in \mathcal{P}$, the vector field X_{μ_0} admits a saddle–connection Γ, with $\alpha(\Gamma) = s_1$ and $\omega(\Gamma) = s_2$, s_1 and s_2 hyperbolic saddles of X_{μ_0}. The coordinates in which (X_μ) is expressed are denoted as (x, y).

We demand that for μ near μ_0, the connection stays unbroken. In particular we suppose that for μ near μ_0, X_μ has two hyperbolic saddles $s_1(\mu)$ and $s_2(\mu)$ lying in a neigbourhood of s_1 respectively s_2 such that there exists a saddle–connection Γ_μ between them that coincides with Γ for $\mu = \mu_0$.

Normalizing coordinates near the saddles

Let $i = 1$ or $i = 2$. Denote the eigenvalues of the linear part $DX_{\mu_0}(s_i)$ at the saddle s_i as λ_i and ν_i with $\nu_i < 0 < \lambda_i$. We already know from Chapter 4 that for μ near μ_0, locally near the saddle s_i, (X_μ) can be brought into a normal form depending on the ratio of hyperbolicity

$$r_i = -\frac{\nu_i}{\lambda_i}.$$

For μ near μ_0 the hyperbolic saddles s_1 and s_2 persist as $s_1(\mu)$ and $s_2(\mu)$ with $s_1(\mu_0) = s_1$ and $s_2(\mu_0) = s_2$. Denote the eigenvalues of $DX_\mu(s_i(\mu))$ as $\lambda_i(\mu)$ and $\nu_i(\mu)$ with $\lambda_i(\mu_0) = \lambda_i$ and $\nu_i(\mu_0) = \nu_i$. The corresponding ratio of hyperbolicity of $s_i(\mu)$ is denoted as $r_i(\mu)$ and

$$r_i(\mu) = -\frac{\nu_i(\mu)}{\lambda_i(\mu)} = r_i + \tilde{r}_i(\mu),$$

for some C^∞ function $\tilde{r}_i(\mu)$ with $\tilde{r}_i(\mu_0) = 0$.

In what follows, we suppose that μ varies in some subset $\mathcal{P}_0 \subset \mathcal{P}$ such that s_1 and s_2 persist as $s_1(\mu)$ and $s_2(\mu)$ respectively, $\forall \mu \in \mathcal{P}_0$, and $(X_\mu)_{\mu\in\mathcal{P}_0}$ can be brought into a normal form at both saddles s_1 and s_2. The normal form at s_i depends on the ratio of hyperbolicity r_i. From now on, (n, m) will denote the normalizing coordinates near s_1 or s_2 depending on near which saddle, s_1 or s_2, we are working.

In case r_i is given by $p_i/q_i, p_i, q_i \in \mathbb{N}_1$, $(p_i, q_i) = 1$, there exist some C^k $(k \geq 2)$ coordinates, near the saddle s_i, in which the family $(X_\mu)_{\mu\in\mathcal{P}_0}$ reads:

$$\begin{cases} \dot{n} &= n(\lambda_i(\mu) + a_i(\mu)n^{p_i}m^{q_i} + P_i(n^{p_i}m^{q_i}, \mu)), \\ \dot{m} &= m(\nu_i(\mu) + b_i(\mu)n^{p_i}m^{q_i} + Q_i(n^{p_i}m^{q_i}, \mu)), \end{cases} \tag{5.47}$$

where $P_i(z, \mu)$ and $Q_i(z, \mu)$ are polynomials in $z = n^{p_i}m^{q_i}$ of degree $N(k) \geq k$ and of order $O(z^2)$. The dependences on the parameter μ in (5.47) are all C^∞.

On the other hand if the ratio of hyperbolicity r_i of $DX_{\mu_0}(s_i)$ is irrational, then $(X_\mu)_{\mu\in\mathcal{P}_0}$ is C^k linearisable near the saddle s_i. In particular there exists some C^k coordinates near the saddle s_i in which $(X_\mu)_{\mu\in\mathcal{P}_0}$ reads:

$$\begin{cases} \dot{n} &= \lambda_i(\mu)\, n, \\ \dot{m} &= \nu_i(\mu)\, m. \end{cases} \tag{5.48}$$

The coordinate transformations expressing the coordinates (x, y) in function of (n, m) are denoted as φ^1_μ and φ^2_μ near s_1 and s_2 respectively. We choose normalizing coordinates near s_1 (resp. s_2) such that points on the positive n–axis correspond to points on the unstable (resp. stable) separatrix of s_1 (resp. s_2), lying on Γ for $\mu = \mu_0$. This can always be achieved by performing a suitable linear transformation in (n, m).

Let us denote the determinants of the corresponding jacobians of these transformations as:

$$A^i_\mu(n, m) := \det D\varphi^i_\mu(n, m), \quad i = 1, 2. \tag{5.49}$$

Further we also define

$$\theta^i_\mu(n, m) \;=\; \frac{\frac{\partial\varphi^i_\mu}{\partial n}(n, m).\frac{\partial\varphi^i_\mu}{\partial m}(n, m)}{\parallel \frac{\partial\varphi^i_\mu}{\partial n}(n, m) \parallel^2}. \tag{5.50}$$

Remark that geometrically $A^i_\mu(n, m)$ represents the area of the paralellogram spanned by the vectors $\frac{\partial\varphi^i_\mu}{\partial n}(n, m)$ and $\frac{\partial\varphi^i_\mu}{\partial m}(n, m)$. The angle between these two vectors is strongly related to the function $\theta^i_\mu(n, m)$.

The above normal form, together with the defined quantities can be calculated using the techniques elaborated in Section 4.1 of Chapter 4. In performing these calculations, the program in Appendix A.1 can be a helpful tool.

Let us state the following lemma, which will be of use later on. It can be applied near both saddles s_1 and s_2 inside the family $(X_\mu)_{\mu\in\mathcal{P}_0}$.

Lemma 5.4 *Suppose* (X_μ) *is a family of vector fields such that* X_{μ_0} *admits a hyperbolic saddle* $\overline{s}$ *persisting as* $\overline{s}(\mu)$ *for* μ *near* μ_0. *Denote by* (n, m) *the normalizing coordinates in which the family is, near* $\overline{s}$, *expressed as the normal form* $\overline{N}_\mu = \overline{N}^1_\mu \frac{\partial}{\partial n} + \overline{N}^2_\mu \frac{\partial}{\partial m}$ *in (5.47) or (5.48). Let* $(x, y) = \overline{\varphi}_\mu(n, m)$ *be the corresponding* C^k *coordinate change. Denote by* $\overline{\lambda}(\mu)$ *and* $\overline{\nu}(\mu)$ *the eigenvalues of* $DX_\mu(\overline{s}(\mu))$ *with* $\overline{\nu}(\mu) < 0 < \overline{\lambda}(\mu)$. *Then*

$$X_\mu(\overline{\varphi}_\mu(n,m)) \wedge \frac{\partial \overline{\varphi}_\mu}{\partial m}(n,m) = \det D\overline{\varphi}_\mu(n,m)\overline{N}^1_\mu(n,m), \tag{5.51}$$

and

$$\overline{\lambda}(\mu)\, n \,\frac{X_\mu(\overline{\varphi}_\mu(n,0)).\frac{\partial \overline{\varphi}_\mu}{\partial m}(n,0)}{\| X_\mu(\overline{\varphi}_\mu(n,0)) \|^2} = \frac{\frac{\partial \overline{\varphi}_\mu}{\partial u}(u,0).\frac{\partial \overline{\varphi}_\mu}{\partial m}(n,0)}{\| \frac{\partial \overline{\varphi}_\mu}{\partial n}(n,0) \|^2}. \tag{5.52}$$

Proof: The identity

$$X_\mu(\overline{\varphi}_\mu(n,m)) = D\overline{\varphi}_\mu(n,m)\overline{N}_\mu(n,m) \tag{5.53}$$

together with

$$\frac{\partial \overline{\varphi}_\mu}{\partial m}(n,m) = D\overline{\varphi}_\mu(n,m)e_2,$$

where $e_2 = (0,1)$, implies

$$X_\mu(\overline{\varphi}_\mu(n,m)) \wedge \frac{\partial \overline{\varphi}_\mu}{\partial m}(n,m) = \det D\overline{\varphi}_\mu(n,m)\big(\overline{N}_\mu(n,m) \wedge e_2\big),$$

resulting in the identity (5.51).

Further (5.53) and the expression of the normal form, (5.47) or (5.48), learns us:

$$\begin{aligned} X_\mu(\overline{\varphi}_\mu(n,0)) &= D\overline{\varphi}_\mu(n,0)\big(\overline{\lambda}(\mu)ne_1\big) \\ &= \overline{\lambda}(\mu)nD\overline{\varphi}_\mu(n,0)e_1 \\ &= \overline{\lambda}(\mu)n\frac{\partial \overline{\varphi}_\mu}{\partial n}(n,0). \end{aligned}$$

which easily implies (5.52). □

Expressing the transition using normalizing coordinates

We choose sections Σ_μ^1, Σ_μ^2 transverse to the flow of $(X_\mu)_{\mu\in\mathcal{P}_0}$ that correspond to $\{n = 1\}$ in the normalizing coordinates near s_1 and s_2 respectively in the following sense.

Let $i = 1$ or $i = 2$. Consider the C^k coordinate transformation $(x, y) = \varphi_\mu^i(n, m)$, with (n, m) the normalizing coordinates in which the family (X_μ) is expressed as the normal form (5.47) or (5.48). Denote this normal form as (N_μ^i). Let C_{r_i} be a circle centered at the origin with radius $r_i > 0$ small enough such that C_{r_i} lies in the domain of φ_μ^i.

The circle C_{r_i} will cut the n–axis in a point c_i in a transverse way. Locally near c_i, C_{r_i} is transverse to the flow of (N_μ^i). Thus both C_{r_i} and $\{n = 1\}$ are sections locally transverse to the flow of (N_μ^1). Choose m as regular parameter on $\{n = 1\}$, there is no need to parametrise C_{r_i}. Consider the transition map $T_\mu^i(m)$ from $\{n = 1\}$ to C_{r_i} and the corresponding transition time function $\tau_\mu^i(m)$, both locally defined: m lying near zero and μ varying near μ_0. Denote the flow of the family $(X_\mu)_{\mu\in\mathcal{P}_0}$ as $\Phi_\mu(t, p)$. The section Σ_μ^i is then defined as:

$$\Sigma_\mu^i = \{\Phi_\mu(-\tau_\mu^i(m), \varphi_\mu^i(T_\mu^i(m))) \mid m \text{ near zero}\}. \tag{5.54}$$

The normalizing coordinate m can be used in order to parametrise the section. Indeed denote $\psi_\mu^i(m) = \Phi_\mu(-\tau_\mu^i(m), \varphi_\mu^i(T_\mu^i(m)))$, then one easily verifies that ψ_μ^i is a regular parametrisation of Σ_μ^i.

The transition map from Σ_μ^1 to Σ_μ^2 expressed in the normalizing coordinate m is denoted as $R_\mu(m)$. Remark that $R_\mu(m)$ is only defined for m near zero and μ near μ_0.

Calculating the derivatives of R_μ directly using Theorem 5.2 is not possible. Indeed only a finite jet of φ_μ^1 and φ_μ^2 at $(0, 0)$ can be calculated implying that one isn't able to calculate the derivatives of the parametrisations of the sections Σ_μ^i, $i = 1, 2$. However this can be dealt with as we will show in a moment (see also [15]).

Take $K_0 > 0$ and $\varepsilon_0 > 0$ such that $\{(x, y) \mid 0 \leq n < K_0, -\varepsilon_0 < m < \varepsilon_0\}$ lies in the domains of φ^1_μ and φ^2_μ. For some $0 < K < K_0$ fixed, consider the sections

$$\varphi^i_\mu(\{(K, m) \mid -\varepsilon_0 < m < \varepsilon_0\}) = C^i_{\mu,K}, \qquad i = 1, 2,$$

parametrised respectively by

$$\varphi^i_\mu \mid_{\{n=K\}} : m \mapsto \varphi^i_\mu(K, m), \qquad i = 1, 2.$$

Consider the part of Γ_μ lying between the sections $C^1_{\mu,K}$ and $C^2_{\mu,K}$, denoted as $\Gamma_{\mu,K}$, and write $Z = \overline{T}_{\mu,K}(Y)$ as the transition map along $\Gamma_{\mu,K}$ from $C^1_{\mu,K}$ to $C^2_{\mu,K}$ expressed in the parameter m. Further let $F_{\mu,K}$ and $G_{\mu,K}$ be the transition maps from $\{n = 1\}$ to $\{n = K\}$ near s_1 and s_2 respectively, expressed using as parameter the normalizing coordinate m. Then the transition map R_μ can be seen as the composition (see Figure 5.3)

$$R_\mu = G_{\mu,K} \circ \overline{T}_{\mu,K} \circ F_{\mu,K}. \tag{5.55}$$

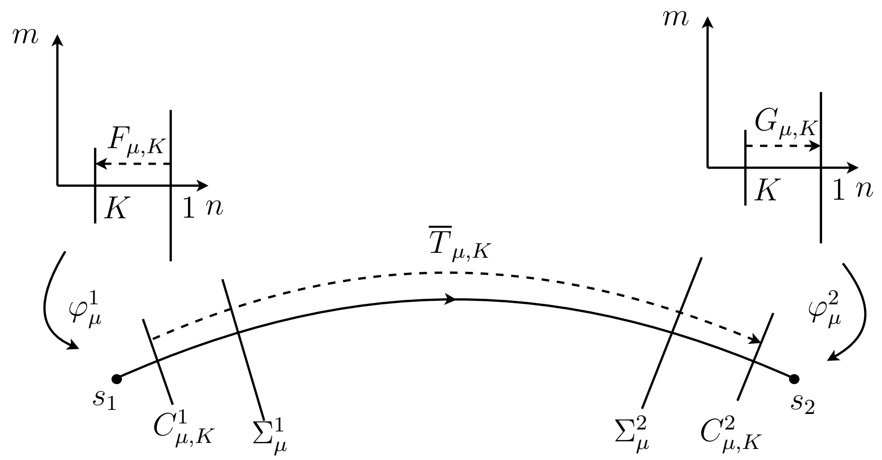

Figure 5.3: The transition map R_μ as the composition $G_{\mu,K} \circ \overline{T}_{\mu,K} \circ F_{\mu,K}$.

The first two derivatives of R_μ at zero are now given by

$$R'_\mu(0) = G'_{\mu,K}(0)\overline{T}'_{\mu,K}(0)F'_{\mu,K}(0),$$

and

$$R''_\mu(0) = G''_{\mu,K}(0)\,[\,\overline{T}'_{\mu,K}(0)\,F'_{\mu,K}(0)\,]^2 + G'_{\mu,K}(0)\,\overline{T}''_{\mu,K}(0)\,[\,F'_{\mu,K}(0)\,]^2 + G'_{\mu,K}(0)\,\overline{T}'_{\mu,K}(0)\,F''_{\mu,K}(0).$$

Because this equalities apply for every $0 < K < K_0$, one can switch over to the limit for $K \to 0$ causing the chosen sections $C^1_{\mu,K}$ and $C^2_{\mu,K}$ to tend arbitrarily close to the saddles. This proces will enable us to calculate the derivatives as stated in the following theorem.

Theorem 5.5 *Let (X_μ) be a C^∞ family admitting for each parameter two hyperbolic saddles $s_1(\mu)$ and $s_2(\mu)$ with a saddle–connection Γ_μ between them. Let R_μ be the transition map from Σ^1_μ to Σ^2_μ along Γ_μ expressed using normalizing coordinates. Consider the normal form at s_i, (5.47) or (5.48), and the corresponding coordinate transformation φ^i_μ, $i = 1, 2$. Let $\Gamma_{\mu,K}$ be the part of Γ_μ starting at $\varphi^1_\mu(K, 0)$ and ending in $\varphi^2_\mu(K, 0)$.*

Then one has

$$R'_\mu(0) = \frac{A^1_\mu(0,0)\,\lambda_1(\mu)}{A^2_\mu(0,0)\,\lambda_2(\mu)} \lim_{K\to 0}\left[K^{r_2(\mu)-r_1(\mu)}\,\exp\left(\int_{\Gamma_{\mu,K}} \frac{\text{div } X_\mu}{\parallel X_\mu \parallel}\, d\overline{s}\right)\right], \tag{5.56}$$

where $d\overline{s}$ represents the arc length element of Γ_μ. Suppose that $R'_\mu(0) = 1$, then

$$R''_\mu(0) = \lim_{K\to 0}\left[U_\mu(K) + I_\mu(K) + \lambda_1(\mu)K^{1-r_1(\mu)}A^1_\mu(K,0)\int_{\Gamma_{\mu,K}} \frac{\mathcal{A}_\mu\,\mathcal{B}_\mu}{\parallel X_\mu \parallel^3}\, d\overline{s}\right], \tag{5.57}$$

where $\mathcal{A}_\mu(p) := \mathcal{A}(p, X_\mu)$, $\mathcal{B}_\mu(p) := \mathcal{B}(p, X_\mu)$ are defined as in Theorem 5.2 and where $U_\mu(K)$ is the difference $U^1_\mu(K) - U^2_\mu(K)$ with

$$U^i_\mu(K) := K^{-r_i(\mu)}\left(\frac{\frac{\partial A^i_\mu}{\partial y}(K,0)}{A^i_\mu(K,0)} - \frac{\theta^i_\mu(K,0)}{\lambda_i(\mu)K}\,\text{div } X_\mu(\varphi^i_\mu(K,0))\right), \quad i = 1, 2.$$

The function $I_\mu(K)$ disappears for $r_i \notin \mathbb{N}$. When $r_i \in \mathbb{N}$ it is given by the difference $I^1_\mu(K) - I^2_\mu(K)$ with

$$I^i_\mu(K) := \frac{a_i(\mu)}{\lambda_i(\mu)}K^{-\tilde{r}_i(\mu)} + 2\frac{b_i(\mu) + a_i(\mu)r_i(\mu)}{\lambda_i(\mu)}\int_1^K u^{-(1+\tilde{r}_i(\mu))}du, \quad i = 1, 2.$$

Proof: We have already noticed that the derivative $R'_\mu(0)$ equals

$$R'_\mu(0) = G'_{\mu,K}(0)\overline{T}'_{\mu,K}(0)F'_{\mu,K}(0). \tag{5.58}$$

We will successively calculate all derivatives appearing in the right part of the above formula (5.58). Because the identity applies for all $0 < K < K_0$, one can take the limit for $K \to 0$ to find the desired derivative $R'_\mu(0)$.

For calculating the derivatives $F'_{\mu,K}(0)$ and $G'_{\mu,K}(0)$, one can use the formulas of Corollary 5.3. One computes:

$$\begin{aligned} F'_{\mu,K}(0) &= \exp\left(\int_1^K -\frac{r_1(\mu)}{u}\,du\right) \\ &= \exp\Big(-r_1(\mu)\ln K\Big) \\ &= K^{-r_1(\mu)} \end{aligned} \tag{5.59}$$

and:

$$\begin{aligned} G'_{\mu,K}(0) &= \exp\left(\int_K^1 -\frac{r_2(\mu)}{z}\,dz\right) \\ &= \exp\Big(r_2(\mu)\ln K\Big) \\ &= K^{r_2(\mu)}, \end{aligned} \tag{5.60}$$

such that

$$R'_\mu(0) = K^{(r_2(\mu)-r_1(\mu)}\,\overline{T}'_{\mu,K}(0). \tag{5.61}$$

The derivative $\overline{T}'_{\mu,K}(0)$ can be calculated using formula (5.30) in Theorem 5.2. From Lemma 5.4 it easily follows that

$$\overline{T}'_{\mu,K}(0) = \frac{A^1_\mu(K,0)\,\lambda_1(\mu)}{A^2_\mu(K,0)\,\lambda_2(\mu)}\exp\left(\int_{\Gamma_{\mu,K}} \frac{\text{div }X_\mu}{\|X_\mu\|}d\overline{s}\right). \tag{5.62}$$

One now substitutes (5.62) into (5.61) and takes the limit for $K \to 0$. Because the coordinate transformations φ_μ^1 and φ_μ^2 are locally diffeomorphisms, A_μ^i, $i = 1, 2$ stays away from zero for K near 0 implying (5.56).

The second order derivative $R''_\mu(0)$ equals

$$R''_\mu(0) = G''_{\mu,K}(0)\,[\,\overline{T}'_{\mu,K}(0)\,F'_{\mu,K}(0)\,]^2 + G'_{\mu,K}(0)\,\overline{T}''_{\mu,K}(0)\,[\,F'_{\mu,K}(0)\,]^2 + \\ G'_{\mu,K}(0)\,\overline{T}'_{\mu,K}(0)\,F''_{\mu,K}(0). \tag{5.63}$$

Assuming that $R'_\mu(0) = 1$, i.e. $G'_{\mu,K}(0)\,\overline{T}'_{\mu,K}(0)\,F'_{\mu,K}(0) = 1$, the above equation simplifies to

$$R''_\mu(0) = \frac{G''_{\mu,K}(0)}{G'_{\mu,K}(0)^2} + \frac{\overline{T}''_{\mu,K}(0)}{\overline{T}'_{\mu,K}(0)}\,F'_{\mu,K}(0) + \frac{F''_{\mu,K}(0)}{F'_{\mu,K}(0)}. \tag{5.64}$$

Again we calculate all ingredients of the right part in this identity after which we let K tend to zero.

The quotient $\overline{T}''_{\mu,K}(0)/\overline{T}'_{\mu,K}(0)$ can be computed by use of Theorem 5.2. We define

$$\sigma_\mu^1(K) := \sigma_1(0, X_\mu, \varphi_\mu\mid_{v=K}), \qquad \sigma_\mu^2(K) := \sigma_2(0, X_\mu, \psi_\mu\mid_{w=K}),$$

where σ_1 and σ_2 are defined as in Theorem 5.2 and find that

$$\frac{\overline{T}''_{\mu,K}(0)}{\overline{T}'_{\mu,K}(0)} = \Bigg(\sigma_\mu^1(K) - \overline{T}'_{\mu,K}(0)\sigma_\mu^2(K) + \\ \left(X_\mu(\varphi_\mu^1(K,0)) \wedge \frac{\partial \varphi_\mu^1}{\partial y}(K,0)\right) \int_{\Gamma_{\mu,K}} \frac{\mathcal{A}_\mu\,\mathcal{B}_\mu}{\|\,X_\mu\,\|^3} d\overline{s}\Bigg). \tag{5.65}$$

From (5.61) and the assumptation that $R'_\mu(0) = 1$, it follows that

$$\overline{T}'_{\mu,K}(0) = K^{r_1(\mu) - r_2(\mu)}. \tag{5.66}$$

Substituting (5.59) and (5.66) into (5.65) and using Lemma 5.4, one finds

$$F'_{\mu,K}(0)\frac{\overline{T}''_{\mu,K}(0)}{\overline{T}'_{\mu,K}(0)} = \Big(K^{-r_1(\mu)}\sigma^1_\mu(K) - K^{-r_2(\mu))}\sigma^2_\mu(K) + +\lambda_1(\mu)K^{1-r_1(\mu)}A^1_\mu(K,0)\int_{\Gamma_{\mu,K}}\frac{\mathcal{A}_\mu\mathcal{B}_\mu}{\parallel X_\mu \parallel^3}d\overline{s}\Big).$$

The expressions for the functions $\sigma^i_\mu(K)$ can be simplified by applying Lemma 5.4. When $r_1 \notin \mathbb{N}$, in particular when $q_1 > 1$, we find

$$\sigma^1_\mu(K) = \frac{\frac{\partial A^1_\mu}{\partial y}(K,0)}{A^1_\mu(K,0)} - \frac{\theta^1_\mu(K,0)}{\lambda_1(\mu)K}\,\mathrm{div}\ X_\mu(\varphi^1_\mu(K,0),$$

while in the case where $r_1 \in \mathbb{N}$, we find

$$\sigma^1_\mu(K) = \frac{\frac{\partial A^1_\mu}{\partial y}(K,0)}{A^1_\mu(K,0)} - \frac{\theta^1_\mu(K,0)}{\lambda_1(\mu)K}\,\mathrm{div}\ X_\mu(\varphi^1_\mu(0,K) + \frac{a_1(\mu)}{\lambda_1(\mu)}K^{r_1}.$$

Totally similar expressions are obtained for $\sigma^2_\mu(K)$.

For the expression $F''_{\mu,K}(0)/F'_{\mu,K}(0)$, we use Corollary 5.3. One calculates that for $r_1 \notin \mathbb{N}$ this quantity vanishes and that for $r_1 \in \mathbb{N}$:

$$\begin{aligned}\frac{F''_{\mu,K}(0)}{F'_{\mu,K}(0)} &= 2\frac{b_1(\mu)+a_1(\mu)r_1(\mu)}{\lambda_1(\mu)}\int_1^K x^{r_1-1}\exp\left(\int_1^x -\frac{r_1(\mu)}{z}dz\right)dx \\ &= 2\frac{b_1(\mu)+a_1(\mu)r_1(\mu)}{\lambda_1(\mu)}\int_1^K x^{-(1+\tilde{r}_1(\mu))}dx.\end{aligned}$$

Computations of the same sort also reveal an expression for $G''_{\mu,K}(0)/G'_{\mu,K}(0)$ when $r_2 \notin \mathbb{N}$:

$$\begin{aligned}\frac{G''_{\mu,K}(0)}{G'_{\mu,K}(0)^2} &= 2\frac{b_2(\mu)+a_2(\mu)r_2(\mu)}{\lambda_2(\mu)}K^{-r_2(\mu)}\int_K^1 x^{r_2-1}\exp\left(\int_K^y -\frac{r_2(\mu)}{z}dz\right)dx \\ &= -2\frac{b_2(\mu)+a_2(\mu)r_2(\mu)}{\lambda_2(\mu)}\int_1^K x^{-(1+\tilde{r}_2(\mu)}dx\end{aligned}$$

Doing the obvious substitutions into (5.64) and taking the limit for $K \to 0$ yields the formula for $R''_\mu(0)$ in (5.57). $\square$

Notice that in practice Theorem 5.5 can only be used when $R'_\mu(0) = 1$, which is true after a coordinate transformation $\tilde{m} = R'_\mu(0)m$ in normalizing coordinates near s_1.

5.4 Transition along a Hamiltonian saddle–connection

In this section, we apply the formulas obtained in the previous section on an unfolding (X_μ) of a Hamiltonian vector field.

Consider:

$$(X_\mu) : \begin{cases} \dot{x} &= -\dfrac{\partial H}{\partial y}(x,y) + \varepsilon f(x,y,\mu), \\ \dot{y} &= \dfrac{\partial H}{\partial x}(x,y) + \varepsilon g(x,y,\mu), \end{cases} \tag{5.67}$$

where H, f, g are all C^∞ and $\mu = (\overline{\mu}, \varepsilon)$ is varying near $\mu_0 = (\overline{\mu}_0, 0)$ with $\overline{\mu}_0 \in \mathbb{R}^p$. Suppose that for $\varepsilon = 0$, $X_{(\overline{\mu}_0,0)} = X_H$ has a saddle–connection Γ on which the Hamiltonian takes constant value 0. Denote s_1 and s_2 as the hyperbolic saddles that are respectively given by the α–limit and ω–limit of Γ. Assume that Γ persists in the family (X_μ).

Appropriate normalizing coordinates near the saddles

Relying on Chapter 4, we describe how to choose *appropriate normalizing coordinates*, simplifying the formulas obtained in the previous section.

Consider first the Hamiltonian vector field X_H near the saddles s_1 and s_2. Let $i = 1$ or $i = 2$. Denote the eigenvalues of $DX_H(s_i)$ as $\pm\lambda_i, \lambda_i > 0$. Let $(\overline{n}, \overline{m})$ denote coordinates near s_i in which X_H reads

$$\begin{cases} \dot{\overline{n}} &= \overline{n}, \\ \dot{\overline{m}} &= -\overline{m}, \end{cases} \tag{5.68}$$

up to a non–zero factor $E_i^0(\overline{n},\overline{m})$ that equals $-\lambda_i$ for $\overline{n}\,\overline{m} = 0$. Denote by ψ_0^i the coordinate transformation expressing the old coordinates (x,y) in function of the new ones $(\overline{n},\overline{m})$. Let $\mathcal{H}_i$ denote the half plane that contains, in its interior, the separatrix corresponding to the separatrix of s_i, lying on Γ for $\varepsilon = 0$. One can choose ψ_0^i such that H expressed in the new coordinates reads $\overline{n}\,\overline{m}$ on $\mathcal{H}_i$ and such that $E_i^0(\overline{n},\overline{m}) = -1/\det D\psi_0^i(\overline{n},\overline{m})$ on $\mathcal{H}_i$.

From Chapter 4, Section 4.2, we know that such coordinate transformations ψ_0^1 and ψ_0^2 can be obtained by applying Morse's lemma on the Hamiltonian H near s_1 and s_2 respectively. On the other hand, one can also start from the formal normal form and use the techniques elaborated in the proof of Theorem 4.10. In this chapter, we prefer to use the last technique, applying the program in Appendix A.1.

If necessary, one can perform a coordinate switch $(\overline{n},\overline{m}) \mapsto (\overline{m},\overline{n})$ and/or a reflection with respect to the origin $(\overline{n},\overline{m}) \mapsto (-\overline{n},-\overline{m})$ near the saddles s_1 and s_2 such that points on the positive $\overline{n}$–axis correspond to points on the unstable and stable separatrix of s_1 and s_2 respectively. These transformations leave the expression of H and of the normal form (5.68), up to a non–zero factor, invariant such that one can suppose that $H \circ \psi_0^i = \overline{n}\,\overline{m}$ on $\{\overline{n} \geq 0\}$. However the sign of $\det D\psi_0^i(0,0)$ can alter. Let us write $d_i = 1/\det D\psi_0^i(0,0) = \pm\lambda_i$.

We continue by peforming the transformation $(x,y) = \psi_0^i(\overline{n},\overline{m})$ on the family (X_μ) yielding:

$$\left\{\begin{array}{rcl} \dot{\overline{n}} &=& \overline{n} + \varepsilon \tilde{f}_i(\overline{n},\overline{m},\mu), \\ \dot{\overline{m}} &=& -\overline{m} + \varepsilon \tilde{g}_i(\overline{n},\overline{m},\mu), \end{array}\right. \tag{5.69}$$

up to the factor $E_i^0(\overline{n},\overline{m})$. One can now apply the techniques of Theorem 4.3 to simplify the expression of the functions $\tilde{f}_i(\overline{n},\overline{m},\mu)$ and $\tilde{g}_i(\overline{n},\overline{m},\mu)$. This will yield C^k transformations

$$(\overline{n},\overline{m}) = (I + \varepsilon\phi_\mu^i)(n,m), \quad i = 1,2, \tag{5.70}$$

near s_i and for μ near μ_0, transforming (5.69) into the normal form:

$$\begin{cases} \dot{n} &= n\big(1+\varepsilon\big(\tilde{\lambda}_i(\mu)+\tilde{a}_i(\mu)nm+P_i(nm,\mu)\big)\big), \\ \dot{m} &= -m\big(1+\varepsilon\big(\tilde{\nu}_i(\mu)-\tilde{b}_i(\mu)nm-Q_i(nm,\mu)\big)\big), \end{cases} \tag{5.71}$$

where $P_i(z,\mu)$ and $Q_i(z,\mu)$ are polynomials in $z = nm$ of finite degree $N(k) \geq k$ and of order $O(z^2)$.

Composing all the above performed transformations, one obtains C^k transformations $(x,y) = \varphi_\mu^1(n,m)$ and $(x,y) = \varphi_\mu^2(n,m)$ near the saddles s_1 and s_2 respectively with μ varying near μ_0. In the normalizing coordinates (n,m), near s_i, the family (X_μ) reads as in (5.71) up to the factor $E_i(n,m,\mu) = \big(E_i^0 \circ (I+\varepsilon\phi_\mu^i)\big)(n,m)$, $i = 1,2$. For $\varepsilon = 0$, this equivalence factor is given by $-1/\det D\psi_0^i(n,m)$ on $\{n \geq 0\}$. Points on the positive n–axis correspond to points on the unstable and stable separatrix of s_1 and s_2 respectively. For $\varepsilon = 0$ the transformations φ_μ^1 and φ_μ^2 coincide with ψ_0^1 and ψ_0^2 respectively. Moreover the Hamiltonian H expressed in normalizing coordinates is, for $\varepsilon = 0$, given by nm, near both saddles in the half plane $\{n \geq 0\}$.

Expressing the transition along a Hamiltonian saddle–connection using appropriate normalizing coordinates

Choose appropriate normalizing coordinates near s_1 and s_2 as in the previous paragraph. As in (5.54), choose transverse sections Σ_μ^1 and Σ_μ^2 corresponding to $\{n = 1\}$ in normalizing coordinates near s_1 and s_2 respectively. Consider the transition map $R_\mu(m)$ between these sections expressed in the second normalizing coordinate m.

Because we have chosen the normalizing coordinates in such way that for $\varepsilon = 0$ the Hamiltonian H reads nm in the normalizing coordinates near the saddles on $\{n \geq 0\}$, it is easily verified that

$$R_\mu = I + O(\varepsilon).$$

On the half plane $\{n \geq 0\}$, one can define for $i = 1, 2$:

$$\begin{aligned} -E_i(n,m,\mu)\det\left(D\varphi^i_\mu(n,m)\right) &= 1 + \varepsilon \overline{A}^i_{\overline{\mu}}(n,m) + O(\varepsilon^2), \\ \overline{\theta}^i_0(n,m) &= \frac{\frac{\partial \psi^i_0}{\partial n}(n,m) \cdot \frac{\partial \psi^i_0}{\partial m}(n,m)}{\| \frac{\partial \psi^i_0}{\partial n}(n,m) \|^2}. \end{aligned} \tag{5.72}$$

Further, we let $\tilde{a}^0_i(\overline{\mu}) = \tilde{a}_i(\mu)\mid_{\varepsilon=0}$ and $\tilde{b}^0_i(\overline{\mu}) = \tilde{b}_i(\mu)\mid_{\varepsilon=0}$. We can now state the following proposition.

Proposition 5.6 *Suppose (X_μ) is a perturbation of a Hamiltonian vector field as in (5.67), with μ varying in a neighbourhood of some $(\overline{\mu}_0, 0)$, with $\overline{\mu}_0 \in \mathbb{R}^p$, such that X_H admits a saddle–connection $\Gamma : H = 0$, between two hyperbolic saddles s_1 and s_2, that persists in the family (X_μ). Choose appropriate normalizing coordinates (n, m) near the saddles in which (X_μ) reads as in (5.71) and consider the functions $\overline{\theta}^i_0$ and $\overline{A}^i_\mu$ defined in (5.72) together with the coefficients $\tilde{a}^0_i(\overline{\mu})$ and $\tilde{b}^0_i(\overline{\mu})$.*

Consider the transition map from Σ^1_μ to Σ^2_μ expressed in the appropriate normalizing coordinates. Then we have

$$R'_\mu(0) = 1 + O(\varepsilon). \tag{5.73}$$

Denoting Γ_K as the part of Γ lying between $\psi^1_0(K, 0)$ and $\psi^2_0(K, 0)$ and $f_{\overline{\mu}}$, $g_{\overline{\mu}}$ as the restrictions of f and g to $\varepsilon = 0$, we have

$$R''_\mu(0) = \varepsilon \lim_{K\to 0}\left[\alpha(\overline{\mu}) + \delta(\overline{\mu})\ln K + \overline{U}_{\overline{\mu}}(K) - \int_{\Gamma_K} \frac{\overline{\mathcal{A}}}{\| X_H \|^3}\, ds\right] + O(\varepsilon^2), \tag{5.74}$$

where $\overline{\mathcal{A}}(z)$ is defined as

$$\overline{\mathcal{A}}(z) := D(\operatorname{div}(f_{\overline{\mu}}, g_{\overline{\mu}}))_z(X^\perp_H(z)) - \left\{(2\kappa_0 \| X_H \| - \operatorname{curl} X_H)\operatorname{div}(f,g)\right\}(z),$$

with $\kappa_0(z)$ the curvature of X_H at z, as defined in (5.20), and where $\overline{U}_{\overline{\mu}}(K)$ is given by the difference $\overline{U}^1_{\overline{\mu}}(K) - \overline{U}^2_{\overline{\mu}}(K)$ with

$$\overline{U}^i_{\overline{\mu}}(K) := \frac{1}{K}\left(\frac{\partial \overline{A}^i_{\overline{\mu}}}{\partial m}(K,0) + \frac{\overline{\theta}^i_0(K,0)}{d_i\, K}\operatorname{div}(f,g)(\psi^i_0(K,0))\right), \quad i = 1, 2.$$

The coefficient $\alpha(\overline{\mu})$ *is given by* $\tilde{a}_1^0(\overline{\mu}) - \tilde{a}_2^0(\overline{\mu})$ *and* $\delta(\overline{\mu})$ *is given by* $\delta_1(\overline{\mu}) - \delta_2(\overline{\mu})$ *with* $\delta_i^0(\overline{\mu}) = 2(\tilde{b}_i^0(\overline{\mu}) + \tilde{a}_i^0(\overline{\mu}))$.

Proof: We choose appropriate normalizing coordinates (n, m) near the saddles as before and apply Theorem 5.5. In the normalizing coordinates, all transitions occur in the half plane $\{n \geq 0\}$, even in $\{n \geq 0\} \cap \{m \geq 0\}$. So all calculations in normalizing coordinates can be restricted to the half plane $\{n \geq 0\}$, such that, for $\varepsilon = 0$, one can assume that the Hamiltonian reads nm when expressed in normalizing coordinates and the equivalence factor of the normal form is given by $E_i(n, m, \mu)$ which, for $\varepsilon = 0$, coincides with $-1/\det D\psi_0^i(n, m)$.

Formula (5.73) is just the consequence of the fact that $R_\mu = I + O(\varepsilon)$. However it can also be seen by applying the formula (5.56). Let us explain how formula (5.74) follows from Theorem 5.5.

In the formulas of Lemma 5.4, we have to take the equivalence factors $E_i(n, m, \mu)$ into account, $i = 1, 2$. Equation (5.51) stays valid up to this equivalence factor:

$$X_\mu(\varphi_\mu^i(n, m)) \wedge \frac{\partial \varphi_\mu^i}{\partial m}(n, m) = E_i(n, m, \mu) \det D\varphi_\mu^i(n, m) N_\mu^1(n, m), \tag{5.75}$$

for $i = 1, 2$ and where $N_\mu = N_\mu^1 \frac{\partial}{\partial n} + N_\mu^2 \frac{\partial}{\partial m}$ denotes the normal form (5.71) at s_1 or s_2 depending near which saddle we apply the identity. Equation (5.52) is translated into:

$$(1 + \varepsilon \tilde{\lambda}_i(\mu))\, n E_i(n, 0, \mu) \frac{X_\mu(\varphi_\mu^i(n, 0)) . \frac{\partial \varphi_\mu^i}{\partial m}(n, 0)}{\| X_\mu(\varphi_\mu^i(n, 0)) \|^2} = \frac{\frac{\partial \varphi_\mu^i}{\partial n}(n, 0) . \frac{\partial \varphi_\mu^i}{\partial m}(n, 0)}{\| \frac{\partial \varphi_\mu^i}{\partial u}(n, 0) \|^2}. \tag{5.76}$$

Formula (5.75) implies that the area A_μ^i, in the proof of Theorem 5.5 is now replaced by:

$$\begin{aligned} A_\mu^i &= E_i(n, m, \mu) \det \left\{ D\varphi_\mu^i(n, m) \right\} \\ &= -\left(1 + \varepsilon \overline{A}_\mu^i(n, m) + O(\varepsilon^2)\right), \quad i = 1, 2. \end{aligned}$$

Further, for $\varepsilon = 0$, formula (5.76) leads to

$$-d_i\, n\, \frac{X_H(\psi_0^i(n,0)).\frac{\partial\psi_0}{\partial m}(n,0)}{\| X_H(\psi_0^i(n,0)) \|^2} = \frac{\frac{\partial\psi_0^i}{\partial n}(n,0).\frac{\partial\psi_0^i}{\partial m}(n,0)}{\| \frac{\partial\psi_0^i}{\partial n}(n,0) \|^2} = \overline{\theta}_0^i(n,0), \quad (5.77)$$

such that $\theta_\mu^i(n,0)$ appearing in the formulas of Theorem 5.5 now equals $\overline{\theta}_0^i(n,0) + O(\varepsilon), \varepsilon \to 0$.

Noticing that the divergence of X_μ reads div $X_\mu = \varepsilon\, \mathrm{div}\,(f,g) + O(\varepsilon^2)$ and referring to the normal form in (5.71), it should be clear for the reader that formula (5.56) in Theorem 5.5 reduces to

$$R''_\mu(0) = \varepsilon r_1(K,\overline{\mu}) + \varepsilon^2 r_2(K,\mu), \quad (5.78)$$

where $r_1(K,\overline{\mu})$ is the function given by

$$r_1(K,\overline{\mu}) = \alpha(\overline{\mu}) + \delta(\overline{\mu}) \ln K + U_\mu(K) - \int_{\Gamma_K} \frac{\overline{\mathcal{A}}}{\| X_H \|^3}\, ds,$$

with all appearing functions defined as above.

Notice that the transformation $\tilde{m} = R'_\mu(0)m$ in normalizing coordinates leaves the equality (5.78) invariant up to order $O(\varepsilon^2)$. Therefore, one can always assume that the condition $R'_\mu(0) = 1$ is satisfied such that it is justified to apply formula (5.57) for obtaining a formula for $R''_\mu(0)$ up to order $O(\varepsilon^2)$.

Because $R_\mu = I + O(\varepsilon)$, we can write:

$$R''_\mu(0) = \varepsilon\eta(\overline{\mu}) + O(\varepsilon^2), \quad (5.79)$$

for some function $\eta(\overline{\mu})$, C^∞ dependent on $\overline{\mu}$. In particular comparing (5.78) with (5.79), one sees:

$$r_1(K,\overline{\mu}) = \eta(\overline{\mu}),$$

for all $0 < K < K_0$, K_0 near zero. This implies:

$$\eta(\overline{\mu}) = \lim_{K\to 0} r_1(K,\overline{\mu}),$$

resulting in formula (5.74). □

5.5 Calculation of η_1 and η_2

We are mainly interested to apply the techniques of the previous sections within the framework of Section 5.1. Consider:

$$(X_\mu) : \begin{cases} \dot{x} &= -\dfrac{\partial H}{\partial y}(x,y) + \varepsilon f(x,y,\mu), \\ \dot{y} &= \dfrac{\partial H}{\partial x}(x,y) + \varepsilon g(x,y,\mu), \end{cases} \tag{5.80}$$

where H, f, g are all C^∞ and $\mu = (\overline{\mu}, \varepsilon)$ is varying near $(\overline{\mu}_0, 0)$ with $\overline{\mu}_0 \in \mathbb{R}^p$. Suppose that for $\varepsilon = 0$, $X_{(\overline{\mu}_0,0)} = X_H$ admits a period annulus bounded by a hyperbolic 2–saddle cycle $\mathcal{L}$. We choose H to be zero on the 2–saddle cycle and strictly positive on the nearby closed orbits. Denote s_1 and s_2 as the hyperbolic saddles and Γ_1 and Γ_2 as the saddle–connections constituting $\mathcal{L}$ as in Figure 5.1.

In this section, we obtain formulas for $\eta_1(\overline{\mu}, 0)$ (resp. $\eta_2(\overline{\mu}, 0)$) in the case where one can find a curve in parameter space passing through $(\overline{\mu}, 0)$ along which Γ_1 (resp. Γ_2) persists.

Appropriate normalizing coordinates near the saddles of a Hamiltonian 2–saddle cycle

One proceeds similar as in Section 5.4 to obtain a C^k transformation $(x,y) = \varphi^i_\mu(n,m)$, near s_i, such that (X_μ) is transformed into (5.71) up to a non–zero factor $E_i(n,m,\mu)$. All notations are kept the same as in the previous section.

The normalizing coordinates (n,m) can be chosen such that for $\varepsilon = 0$: points on the positive n–axes correspond to points on the stable and unstable separatrix (coinciding along Γ_1 for $\varepsilon = 0$) of s_1 and s_2 respectively; similar, points on the positive m–axes correspond to points on the unstable and stable separatrix (coinciding along Γ_2 for $\varepsilon = 0$) of s_1 and s_2 respectively; moreover, near s_i, for $\varepsilon = 0$ and on $\{n \geq 0\}$, the Hamiltonian expressed in normalizing coordinates reads nm and the equivalence factor reads $-1/\det D\psi^i_0(n,m)$, $i = 1, 2$.

The above choice of coordinates implies an orientation on the normalizing coordinate axes. In particular ψ_0^1 and ψ_0^2 have to be chosen such that $\det D\psi_0^1(0,0) > 0$ and $\det D\psi_0^2(0,0) < 0$. This can be achieved by performing, if necessary, coordinate switches near the saddles such that $\det D\psi_0^1(0,0)$ equals $1/\lambda_1$ and $\det D\psi_0^2(0,0)$ equals $-1/\lambda_2$. Afterwards, if necessary, a reflection with respect to the origin will be sufficient in order that the positive $\overline{n}$–axes and $\overline{m}$–axes correspond to the appropriate separatrices.

Formulas for calculating η_1 and η_2

Choose appropriate normalizing coordinates near the saddles s_1 and s_2. As in (5.54), choose transverse sections Σ_μ^1 and Σ_μ^2 corresponding to $\{n = 1\}$ in normalizing coordinates near s_1 and s_2 respectively. Similar choose transverse sections Σ_μ^2 and Σ_μ^4 corresponding to $\{m = 1\}$ in normalizing coordinates near s_1 and s_2 respectively. The sections Σ_μ^1 and Σ_μ^3 are parametrised using the normalizing coordinate m while the normalizing coordinate n is used in order to parametrise the sections Σ_μ^2 and Σ_μ^4.

Consider the transition maps $R_\mu^1(m)$ and $R_\mu^2(n)$ from Σ_1 to Σ_3 and Σ_2 to Σ_4 respectively, see Figure 5.2. The coefficients $\eta_1(\mu)$ and $\eta_2(\mu)$ are defined as in Section 5.1:

$$R_\mu^1(m) = m + \varepsilon(-\beta(\mu) + u(\mu)m + \eta_1(\mu)m^2 + O(m^3)), \tag{5.81}$$

and

$$R_\mu^2(n) = n + \varepsilon(\eta_2(\mu)n^2 + O(n^3)). \tag{5.82}$$

We assume that Γ_1 as well as Γ_2 stay unbroken along a curve

$$\overline{\mu} = \gamma(\varepsilon),$$

with $\gamma(0) = \overline{\mu}_0$.

We have the following corollary of Proposition 5.6.

Corollary 5.7 *Suppose (X_μ) is a perturbation of a Hamiltonian vector field $X_H = X_{(\overline{\mu}_0,0)}$ like in (5.80). Suppose X_H contains a 2–saddle cycle in its flows, constituted by two saddles s_1, s_2 and two saddle–connection Γ_1, Γ_2 which stay unbroken along a curve $\overline{\mu} = \gamma(\varepsilon)$ in parameter space passing through $(\overline{\mu}_0, 0)$. Choose appropriate normalizing coordinates near the saddles in which (X_μ) reads as in* (5.71) *and consider the functions $\overline{\theta}_0^i$ and $\overline{A}_\mu^i$ defined in* (5.72) *together with the coefficients $\tilde{a}_i^0(\overline{\mu})$ and $\tilde{b}_i^0(\overline{\mu})$.*

Let Γ_1^K be the part of Γ^1 lying between $\psi_0^1(K,0)$ and $\psi_0^2(K,0)$. Analogously let Γ_2^K be the part of Γ_2 lying between $\psi_0^1(0,K)$ and $\psi_0^2(0,K)$. Denote f_{μ_0} and g_{μ_0} as the restrictions of f and g to $\mu = (\mu_0, 0)$ respectively. Then the coefficient $\eta_1(\overline{\mu}_0, 0)$ as defined in (5.81) reads

$$\eta_1(\overline{\mu}_0, 0) = \lim_{K\to 0}\left[\alpha(\overline{\mu}_0) + \delta(\overline{\mu}_0)\ln K + \overline{V}_{\overline{\mu}_0}(K) - \int_{\Gamma_K^1} \frac{\overline{\mathcal{A}}}{\| X_H \|^3}\, ds\right], \tag{5.83}$$

where $\overline{\mathcal{A}}(z)$ is defined as

$$\overline{\mathcal{A}}(z) := D(\mathrm{div}\,(f_{\mu_0}, g_{\mu_0}))_z(X_H^\perp(z)) - \Big\{(2\kappa_0\|X_H\| - \mathrm{curl}\, X_H)\mathrm{div}\,(f_{\mu_0}, g_{\mu_0})\Big\}(z)$$

with $\kappa_0(z)$ the curvature of X_H at z, defined as in (5.20)*, and $\overline{V}_{\overline{\mu}_0}(K)$ is given by the difference $\overline{V}_{\overline{\mu}_0}^1(K) - \overline{V}_{\overline{\mu}_0}^2(K)$ with:*

$$\begin{aligned}
\overline{V}_{\overline{\mu}_0}^1(K) &:= \frac{1}{K}\left(\frac{\partial \overline{A}_{\overline{\mu}_0}^2}{\partial m}(K,0) - \frac{\partial \overline{A}_{\overline{\mu}_0}^1}{\partial m}(K,0)\right),\\
\overline{V}_{\overline{\mu}_0}^2(K) &:= \frac{1}{K}\left(\frac{\overline{\theta}_0^1(K,0)}{\lambda_1 K}\,\mathrm{div}_1 + \frac{\overline{\theta}_0^2(K,0)}{\lambda_2 K}\,\mathrm{div}_2\right),
\end{aligned}$$

where $\mathrm{div}_i := \mathrm{div}\,(f_{\mu_0}, g_{\mu_0})(\psi_0^i(K,0))$. The coefficient $\alpha(\overline{\mu}_0)$ is given by $\tilde{a}_2^0(\overline{\mu}_0) - \tilde{a}_1^0(\overline{\mu}_0)$ and $\delta(\overline{\mu}_0)$ is given by $\delta_2(\overline{\mu}_0) - \delta_1(\overline{\mu}_0)$ with $\delta_i^0(\overline{\mu}_0) = 2(\tilde{b}_i^0(\overline{\mu}_0) + \tilde{a}_i^0(\overline{\mu}_0))$.

Proof: Applying Proposition 5.6 on the family $(Z_\varepsilon) = (X_{(\gamma(\varepsilon),\varepsilon)})$, one easily obtains from formula (5.78):

$$\frac{d^2 R^1_{(\gamma(\varepsilon),\varepsilon)}}{dm^2}(0) = \varepsilon r_1(K,\varepsilon) + O(\varepsilon^2),$$

for each $0 < K < K_0$, K_0 near zero, with

$$r_1(K,\varepsilon) =$$

$$\left[\alpha(\gamma(\varepsilon)) + \delta(\gamma(\varepsilon))\ln K + \overline{U}_{\gamma(\varepsilon)}(K) - \int_{\Gamma_K} \frac{\overline{\mathcal{A}}}{\| X_H \|^3}\, ds\right] + O(\varepsilon).$$

Remark here that Γ_1 runs from s_2 to s_1. So, compared with the previous section, the roles of s_1 and s_2 are interchanged. On the other hand

$$\frac{d^2 R^1_{(\gamma(\varepsilon),\varepsilon)}}{dm^2}(0) = \varepsilon\eta_1(\gamma(\varepsilon),\varepsilon) + O(\varepsilon^2)$$

such that

$$\eta_1(\gamma(\varepsilon),\varepsilon) + O(\varepsilon) = r_1(K,\varepsilon) + O(\varepsilon).$$

Substituting $\varepsilon = 0$, one obtains:

$$\eta_1(\overline{\mu}_0, 0) = r_1(K, 0),$$

for each $0 < K < K_0$, resulting in the stated formula. □

Corollary 5.8 *With the same notations and considerations as in Corollary 5.7, the coefficient $\eta_2(\overline{\mu}_0, 0)$ as defined in (5.82) reads:*

$$\eta_2(\overline{\mu}_0, 0) = \lim_{K\to 0}\left[\beta(\overline{\mu}_0) + \delta(\overline{\mu}_0)\ln K + \tilde{V}_\mu(K) - \int_{\Gamma^2_K} \frac{\overline{\mathcal{A}}}{\| X_H \|^3}\, ds\right], \tag{5.84}$$

where $\overline{\mathcal{A}}(z)$ is defined as in Corollary 5.7 and $\tilde{V}_{\overline{\mu}_0}(K)$ is given by the difference $\tilde{V}^2_{\overline{\mu}_0}(K) - \tilde{V}^1_{\overline{\mu}_0}(K)$, with:

$$\begin{aligned}
\tilde{V}^1_{\overline{\mu}_0}(K) &:= \frac{1}{K}\left(\frac{\partial \overline{A}^2_{\overline{\mu}_0}}{\partial n}(0,K) - \frac{\partial \overline{A}^1_{\overline{\mu}_0}}{\partial n}(0,K)\right),\\
\tilde{V}^2_{\overline{\mu}_0}(K) &:= \frac{1}{K}\left(\frac{\overline{\theta}^1_0(0,K)}{\lambda_1 K}\,\mathrm{div}_1 + \frac{\overline{\theta}^2_0(0,K)}{\lambda_2 K}\,\mathrm{div}_2\right),
\end{aligned}$$

where $\mathrm{div}_i := \mathrm{div}\,(f_{\mu_0}, g_{\mu_0})(\psi^i_0(0,K))$. The coefficient $\beta(\overline{\mu}_0)$ is given by $\tilde{b}^0_2(\overline{\mu}) - \tilde{b}^0_1(\overline{\mu})$ and $\delta(\overline{\mu}_0)$ is given by $\delta_2(\overline{\mu}) - \delta_1(\overline{\mu})$ with $\delta^0_i(\overline{\mu}_0) = 2(\tilde{b}^0_i(\overline{\mu}_0) + \tilde{a}^0_i(\overline{\mu}_0))$.

Proof: Locally near the saddles, the unbroken connection Γ_2 corresponds to the m–axis in the normal forms. After a coordinate switch $(n,m) \mapsto (m,n)$, we can apply Corrollary 5.7. The coefficients in the normal form (5.71) switch roles and change sign, if we want to keep the expression of (5.71) as it is. Because Γ_2 runs from s_1 to s_2, the roles of the saddles are interchanged compared with Corollary 5.7. $\square$

5.6 Unfolding a Hamiltonian 2–saddle cycle

In this section, we apply the formulas obtained in the last section in a specific example. Consider a Hamiltonian vector field X_H with Hamiltonian

$$H(x,y) = y(x^2 + \frac{1}{12}y^2 - 1),$$

which we unfold as $(X_{(\overline{\mu},\varepsilon)})$ given by:

$$\begin{cases} \dot{x} &= 1 - \frac{1}{4}y^2 - x^2 + \varepsilon[\overline{\mu}_3 xy + \overline{\mu}_4 y^2 x + y(x^2 + \frac{1}{12}y^2 - 1)(x - \frac{\sqrt{3}\pi}{8}xy)], \\ \dot{y} &= 2xy + \varepsilon y(\overline{\mu}_1 + \overline{\mu}_2 x). \end{cases} \tag{5.85}$$

The phase portrait of X_H looks like in Figure 5.4. It contains four singularities: two centers at $(0,\pm 2)$ and two saddles given by $s_1 = (-1,0)$ and $s_2 = (1,0)$ where both saddles have eigenvalues ± 2. The singularities stay fixed after perturbation, as well as the saddle–connection between them lying on the x–axis. The saddles and the saddle–connection on the x–axis are part of two 2–saddle cycles, one lying in the half plane $\{y \geq 0\}$ and one lying in the half plane $\{y \leq 0\}$. On the 2–saddle cycles, H takes the value zero, inside the annulus lying in the half plane $\{y \leq 0\}$, H is strictly positive.

We suppose that μ varies in an neighbourhood of $(0,0)$ and verify conditions (5.7), (5.9) and (5.12) near the 2–saddle cycle $\mathcal{L}$ lying in the half plane $\{y \leq 0\}$. From [16], one concludes that $\mathcal{L}$ produces at least one alien limit cycle inside the family (5.85), μ varying near $(0,0)$.

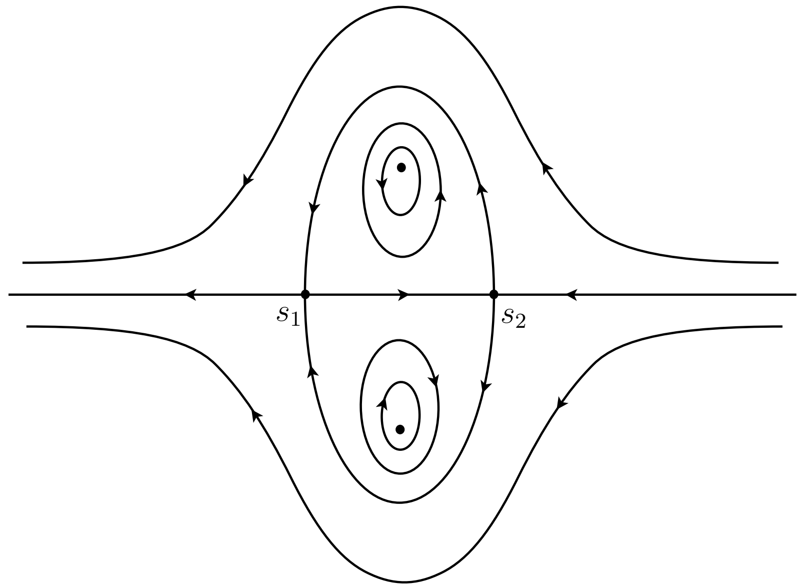

Figure 5.4: Phase portrait of $X_H = X_{(\overline{\mu},0)}$.

The conditions concerning the transition maps

Let us start by verifying the condition concerning the transition maps along the 2–saddle cycle $\mathcal{L}$. We suppose that $\mu = (\overline{\mu}, \varepsilon)$ varies in a neighbourhood of $(0,0)$ and verify condition (5.12). Along $\{\overline{\mu} = 0\}$ the perturbation is zero on both connections of $\mathcal{L}$, implying that they persist in the subfamily $X_{(0,\varepsilon)}$. Hence it is justified to use the formulas (5.81) and (5.82) in order to calculate $\eta_1(0)$ and $\eta_2(0)$. In what follows, notations are kept the same as in Corollaries 5.7 and 5.8.

Calculation of appropriate normalizing coordinates

We calculate the appropriate normalizing coordinates near the saddles of the subfamily $(X_\varepsilon) = (X_{(0,\varepsilon)})$. The unfolding (X_ε) reads:

$$(X_\varepsilon) : \begin{cases} \dot{x} &= 1 - \frac{1}{4}y^2 - x^2 + \varepsilon y(x^2 + \frac{1}{12}y^2 - 1)(x - \frac{\sqrt{3}\pi}{8}xy), \\ \dot{y} &= 2xy \end{cases}$$

with $X_0 = X_H$. It will appear to be sufficient to perform normal form calculations up to order 4.

The Hamiltonian vector field X_H has two hyperbolic saddles, one at $(-1, 0)$ and one at $(1, 0)$. Near $(1, 0)$, we proceed as follows. We start with calculating the 3–jet of $(x, y) = \psi_0^2(\overline{n}, \overline{m})$, the coordinate change transforming X_H into the normal form (5.68) up to C^∞ equivalence. First we translate the singularity to the origin, yielding:

$$\left\{ \begin{array}{rcl} \dot{x} & = & -2x - \frac{1}{4}y^2 - x^2, \\ \dot{y} & = & 2y + 2xy. \end{array} \right.$$

The linear part at the origin is already in its jordan form. All terms of order 2 are non–resonant such that the transformation

$$(x, y) = (\overline{x} + \frac{1}{2}\overline{x}^2 - \frac{1}{24}\overline{y}^2, \overline{y} - \overline{x}\,\overline{y}), \tag{5.86}$$

will remove all of them. The 3–jet of the obtained vector field reads, writing (x, y) instead of $(\overline{x}, \overline{y})$:

$$\left\{ \begin{array}{rcl} \dot{x} & = & -2x - x^3 + \frac{7}{12}xy^2, \\ \dot{y} & = & 2y - x^2y - \frac{1}{12}y^3. \end{array} \right.$$

Continuing this procedure, one obtains the transformation

$$(x, y) = (\overline{x} + \frac{1}{4}\overline{x}^3 + \frac{7}{48}\overline{xy}^2, \overline{y} + \frac{1}{4}\overline{x}^2\overline{y} - \frac{1}{48}\overline{y}^3), \tag{5.87}$$

that will remove all terms of order 3. One concludes that the 3–jet of X is C^ω linearisable by the transformation given by the composition of the translation $(x, y) = (\overline{x} + 1, \overline{y})$ with (5.86) and (5.87).

The Hamiltonian expressed in the new coordinates already reads $2xy$ up to order 5. Therefore the transformation that brings the Hamiltonian in xy is up to order 4 given by a dilatation that one can choose to be $(x, y) = (\overline{x}, \frac{\overline{y}}{2})$. After a switch of the normalizing coordinates and a reflection $(x, y) \mapsto (-x, -y)$, the positive x and y–axis in normalizing coordinates correspond respectively to the unstable and the stable separatrix of s_2 lying on $\mathcal{L}$. One obtains the following 3–jet of the transformation $(x,y) = \psi_0^2(\overline{n},\overline{m})$:

$$(x,y) = \Big(1 - \overline{m} + \tfrac{1}{2}\overline{m}^2 - \tfrac{1}{96}\overline{n}^2 - \tfrac{1}{4}\overline{m}^3 - \tfrac{7}{192}\overline{n}^2\overline{m}, \\ -\tfrac{1}{2}\overline{n} - \tfrac{1}{2}\overline{n}\,\overline{m} - \tfrac{1}{8}\overline{n}\,\overline{m}^2 + \tfrac{1}{384}\overline{n}^3\Big). \tag{5.88}$$

The 3–jet of (X_ε) is transformed into:

$$\begin{cases} \dot{\overline{n}} &= \overline{n} + \frac{1}{2}\varepsilon\overline{n}^2\overline{m}, \\ \dot{\overline{m}} &= -\overline{m} - \frac{1}{2}\varepsilon\overline{n}\,\overline{m} - \varepsilon\frac{\sqrt{3}}{32}\pi\overline{n}^2\overline{m}, \end{cases} \tag{5.89}$$

up to a factor $2 + O(|\overline{n}\,\overline{m}|^2)$.

Near $s_1 = (-1,0)$, one can proceed totally analogous as above. However, one can also make use of the symmetry of X_H with respect to the y–axis, leading to the same results.

The Hamiltonian vector field is invariant under the transformation

$$(x,y,t) \mapsto (-x,y,-t),$$

such that the behaviour of X_H in the region $\{(x,y) \mid -1 < x < -1+\varepsilon_0\}$ is exactly given by the behaviour of $-X_H$ in the region $\{(x,y) \mid 1-\varepsilon_0 < x < 1\}$. Choosing $\psi_0^1 := S \circ \psi_0^2$, where $S(x,y) = (-x,y)$, the 3–jet of (X_ε) is near s_1 transformed into:

$$\begin{cases} \dot{\overline{n}} &= \overline{n} - \frac{1}{2}\varepsilon\overline{n}^2\overline{m}, \\ \dot{\overline{m}} &= -\overline{m} + \frac{1}{2}\varepsilon\overline{n}\,\overline{m} + \varepsilon\frac{\sqrt{3}}{32}\pi\overline{n}^2\overline{m}, \end{cases} \tag{5.90}$$

up to a factor $-2 + O(|\overline{n}\,\overline{m}|^2)$. Moreover the Hamiltonian expressed in new coordinates reads $\overline{n}\,\overline{m}$, up to order 5.

One continues by performing a transformation of the form $I + \varepsilon\varphi_\varepsilon^i$, $i = 1,2$ near s_1 and s_2 respectively, keeping the unperturbed vector field unchanged but removing non–resonant terms of order less then 4, appearing after the parameter ε in expressions (5.89) and (5.90).

Writing at each step again (x, y) as the old coordinates and $(\overline{x}, \overline{y})$ as the new coordinates, the term $\frac{\varepsilon}{2}xy\frac{\partial}{\partial y}$ in (5.89) can be removed by means of a transformation

$$(x, y) = (\overline{x}, \overline{y} - \frac{1}{2}\varepsilon\overline{x}\,\overline{y}), \tag{5.91}$$

yielding in the vector field:

$$\begin{cases} \dot{x} &= x + \frac{1}{2}\varepsilon x^2 y, \\ \dot{y} &= -y + \varepsilon\left(\frac{1}{4}\varepsilon - \frac{\sqrt{3}\pi}{32}\right)x^2 y. \end{cases}$$

The non–resonant term

$$\varepsilon\left(\frac{1}{4}\varepsilon - \frac{\sqrt{3}\pi}{32}\right)x^2 y\frac{\partial}{\partial y},$$

can be removed in the 3–jet by a coordinate change

$$(x, y) = (\overline{x}, \overline{y} - \frac{1}{2}\varepsilon\left(\frac{1}{4}\varepsilon - \frac{\sqrt{3}\pi}{32}\right)\overline{x}^2\overline{y}) \tag{5.92}$$

yielding in the following normal form at s_2 for the 3–jet of (X_ε):

$$\begin{cases} \dot{n} &= n + \frac{1}{2}\varepsilon n^2 m, \\ \dot{m} &= -m, \end{cases} \tag{5.93}$$

up to a factor $2 + O(|\overline{n}\,\overline{m}|^2)$. Analogously, performing the transformation

$$(\overline{n}, \overline{m}) = \left(n, m + \frac{1}{2}\varepsilon(nm + \frac{1}{2}\left(\frac{1}{4}\varepsilon + \frac{\sqrt{3}\pi}{32}\right)n^2 m)\right),$$

the 3–jet of (5.90) will, locally near s_1, be transformed into:

$$\begin{cases} \dot{n} &= n - \frac{1}{2}\varepsilon n^2 m, \\ \dot{m} &= -m, \end{cases}$$

up to a factor $-2 + O(|\overline{n}\,\overline{m}|^2)$.

Remark that one can use the program in Appendix A.1 for calculating the above normal forms.

Calculation of $\eta_1(0)$ and $\eta_2(0)$

We use formulas (5.83) and (5.84) to calculate $\eta_1(0)$ and $\eta_2(0)$.

Using the above normal form calculations, one computes:

$$\overline{\theta}_0^1(n,m) = \overline{\theta}_0^2(n,m) = \frac{13}{12}x - \frac{11}{24}xy + O(\| (x,y) \|^3), \tag{5.94}$$

and:

$$\begin{cases} \overline{A}_0^1(n,m) &=& \frac{1}{2}n + \frac{1}{64}\sqrt{3}\pi n^2 + O(\| (n,m) \|^3), \\ \overline{A}_0^2(n,m) &=& -\frac{1}{2}n - \frac{1}{64}\sqrt{3}\pi n^2 + O(\| (n,m) \|^3), \end{cases} \tag{5.95}$$

together with:

$$\operatorname{div}(f_0, g_0)(x,y) = 2xy(x - \frac{\sqrt{3}\pi}{8}xy) + H(x,y)(1 - \frac{\sqrt{3}\pi}{8}y), \tag{5.96}$$

where f_0 and g_0 are the functions appearing after the parameter ε in the expression of (X_ε). In particular, one gets:

$$\operatorname{div}(f_0, g_0)(\psi_0^i(0,K)) = 0, \quad i = 1,2, \quad \forall 0 < K < K_0,$$

such that $\tilde{V}_0^2(K) = 0$, for each K near zero, in formula (5.84). On the other hand, using (5.95), one sees that the asymptotic behaviour of $\tilde{V}_0^1(K)$ as $K \to 0$ is given by:

$$\tilde{V}_0^1(K) = -\frac{1}{K} + O(K), \quad K \to 0.$$

One concludes that the function $\tilde{V}_0(K)$ in (5.84) has the following asymptotic behaviour as $K \to 0$:

$$\tilde{V}_0(K) = \frac{1}{K} + O(K), \quad K \to 0.$$

Further, from (5.94), we have:

$$\overline{\theta}_0^1(K,0) = \overline{\theta}_0^2(K,0) = \frac{13}{12}K + O(K^3), \quad K \to 0$$

and, from (5.96),

$$\text{div}\,(f_0, g_0)(\psi_0^1(K,0)) = \text{div}\,(f_0, g_0)(\psi_0^2(K,0)) = -K + O(K^2), \quad K \to 0$$

such that the function $\overline{V}_0^2(K)$ in formula (5.83) is given by:

$$\overline{V}_0^2(K) = -\frac{13}{24} + O(K), K \to 0.$$

Using (5.95), one easily gets

$$\overline{V}_0^1(K) = O(K), K \to 0$$

implying that $\overline{V}_0(K)$ in formula (5.83) is given by:

$$\overline{V}_0(K) = \frac{13}{24} + O(K), K \to 0.$$

We are left with the calculations of the integrals appearing in the formulas (5.83) and (5.84) of $\eta_1(0)$ and $\eta_2(0)$ respectively.

Consider the integral in formula (5.84), along the orbit Γ_2 lying on the x–axis. Parametrising the orbit using the x–coordinate leads to an integral over $x \in [r_1^1(0,K), r_1^2(0,K)]$ with $\psi_0^i = (r_1^i, r_2^i)$ and K varying in $]0, K_0[$, K_0 near zero. A direct calculation yields the following primitive of the integrandum:

$$F(x) := \ln\left(\frac{1-x}{1+x}\right) - \frac{x}{x^2-1}.$$

The integral then equals $F(r_1^2(0,K)) - F(r_1^1(0,K))$ or because $r_1^1(0,K) = -r_1^2(0,K)$:

$$I := -2\frac{r_1^2(0,K)}{r_1^2(0,K)^2 - 1} + 2\ln\left(\frac{1 - r_1^2(0,K)}{1 + r_1^2(0,K)}\right).$$

Using (5.88), one easily finds that

$$r_1^2(0,K) = 1 - K + \frac{1}{2}K^2 + O(K^3), \quad K \to 0$$

yielding in

$$I = \frac{1}{K} - 2\ln 2 + 2\ln K + O(K), \quad K \to 0.$$

Consider now the integral along the connection Γ_1 in formula (5.83),

$$\Gamma_1 : x^2 + \frac{1}{12}y^2 - 1 = 0,$$

which can be parametrised by the x–coordinate yielding an integral over $x \in [r_1^2(K,0), r_1^1(K,0)]$. A direct calculation shows that

$$G(x) := g(x) + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

is a primitive of the integrandum, with g a function of a rather long expression satisfying

$$\lim_{x\downarrow -1} g(x) = -\lim_{x\uparrow 1} g(x) = \frac{1}{24} - \frac{3}{8}\pi^2.$$

Thus, the integral equals $G(-r_1^2(K,0)) - G(r_1^2(K,0))$. Using (5.88), one sees

$$r_1^2(K,0) = 1 - \frac{K^2}{96} + O(K^3), \quad K \to 0$$

such that the integral equals

$$I := 2c_0 + 2\ln K - \ln 192 + o(1), \quad K \to 0,$$

with $c_0 = \lim_{x\downarrow -1} g(x)$.

Substituting all the obtained data in the formulas (5.83) and (5.84), one gets:

$$\eta_1(0) = \frac{35}{24} + \ln 192 + \frac{3}{4}\pi^2, \qquad \eta_2(0) = 2\ln 2.$$

The conditions concerning the Abelian integral

Suppose $h \geq 0$ and denote γ_h as one of the closed curves inside the annulus of which $\mathcal{L}$ is the boundary. The Abelian integral is defined as:

$$\int_{\gamma_h} f dy - g dx, \tag{5.97}$$

with $f(x, y, \mu)$ and $g(x, y, \mu)$ the functions that appear after the parameter ε in the expression of (X_μ), (5.85). We now check the conditions (5.7) and (5.9).

In what follows, we show how to calculate the coefficients in the expansion

$$I(h, \overline{\mu}) = p(\overline{\mu}) + q(\overline{\mu}) h \log h + r(\overline{\mu}) h + s(\overline{\mu}) h^2 \log h + O(h^2). \tag{5.98}$$

The Abelian integral is given by:

$$\begin{aligned} I(h, \overline{\mu}) &= \int_{\gamma_h} \overline{\mu}_3 xy dy - (\overline{\mu}_1 + \overline{\mu}_2 x) y dx \\ &\quad + \int_{\gamma_h} \overline{\mu}_4 y^2 x dy + \underbrace{y(x^2 + \frac{1}{12} y^2 - 1)}_{H(x,y)} (x - \tfrac{\sqrt{3}\pi}{8} yx) dy \\ &= \overline{\mu}_4 \int_{\gamma_h} y^2 x dy + \left(\overline{\mu}_3 - \tfrac{\sqrt{3}\pi}{8} h\right) \int_{\gamma_h} xy dy \\ &\quad + h \int_{\gamma_h} x dy - \overline{\mu}_1 \int_{\gamma_h} y dx - \overline{\mu}_2 \int_{\gamma_h} xy dx \end{aligned}$$

Using

$$\begin{aligned} \int_{\gamma_h} d(xy) = 0 \quad &\Rightarrow \quad \int_{\gamma_h} y dx = -\int_{\gamma_h} x dy, \\ \int_{\gamma_h} d(\tfrac{x^2}{2} y) = 0 \quad &\Rightarrow \quad \int_{\gamma_h} xy dx = -\tfrac{1}{2} \int_{\gamma_h} x^2 dy, \\ \int_{\gamma_h} x^2 dy \quad &= \quad \int_{D_h} 2x \, dx \, dy = 0, \end{aligned}$$

implies

$$I(h, \overline{\mu}) = \overline{\mu}_4 I_2(h) + (\overline{\mu}_3 - \frac{\sqrt{3}\pi}{8} h) I_1(h) + (\overline{\mu}_1 + h) I_0(h), \tag{5.99}$$

with $I_k(h) = \int_{\gamma_h} y^k x dy$. Now by direct computation, one easily verifies that:

$$\begin{aligned}
\lim_{h\to 0} I_0(h) &= -\sqrt{3}\pi, \\
\lim_{h\to 0} I_1(h) &= 8, \\
\lim_{h\to 0} I_2(h) &= -3\sqrt{3}\pi,
\end{aligned}$$

In particular the condition to have a 2–saddle cycle is given by:

$$I(0, \overline{\mu}) = -3\sqrt{3}\pi\overline{\mu}_4 + 8\overline{\mu}_3 - \sqrt{3}\pi\overline{\mu}_1 = 0.$$

Referring to [13], the Picard–Fuchs equation are given by:

$$D(h)\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4}h^2 - 2 & -\frac{3}{4}h & \frac{2}{3} \\ h & \frac{9}{8}h^2 & -h \\ -\frac{3}{2}h^2 & -3h & \frac{3}{2}h^2 \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix}, \tag{5.100}$$

with $D(h) = \frac{9}{8}h(h^2 - \left(\frac{4}{3}\right)^2)$. Writing:

$$\begin{aligned}
I_0(h) &= -\sqrt{3}\pi + a_1 h + a_2 h\log h + a_3 h^2 \log h + O(h^2), \\
I_1(h) &= 8 + b_1 h + b_2 h\log h + b_3 h^2 \log h + O(h^2), \\
I_2(h) &= -3\sqrt{3}\pi + c_1 h + c_2 h\log h + c_3 h^2 \log h + O(h^2),
\end{aligned}$$

and substituting this in (5.100) gives:

$$\left(\tfrac{9}{8}h^3 - 2h\right)\left[\, a_1 + a_2(1 + \log h) + a_3(2h\log h + h) + O(h)\,\right]$$
$$= (\tfrac{3}{4}h^2 - 2)I_0(h) - \tfrac{3}{4}hI_1(h) + \tfrac{2}{3}I_2(h)$$
$$\left(\tfrac{9}{8}h^3 - 2h\right)\left[\, b_1 + b_2(1 + \log h) + b_3(2h\log h + h) + O(h)\,\right]$$
$$= hI_0(h) + \tfrac{9}{8}h^2 I_1(h) - hI_2(h)$$
$$\left(\tfrac{9}{8}h^3 - 2h\right)\left[\, c_1 + c_2(1 + \log h) + c_3(2h\log h + h) + O(h)\,\right]$$
$$= -\tfrac{3}{2}h^2 I_0(h) - 3hI_1(h) + \tfrac{3}{2}h^2 I_2(h)$$

Comparing the coefficient with h in every equation leads respectively to:

$$-2a_1 - 2a_2 = -2a_1 - 6 + \frac{2}{3}c_1, \quad -2b_1 - 2b_2 = 2\sqrt{3}\pi, \quad -2c_1 - 2c_2 = -24,$$

Those with $h \log h$ are given by:

$$-2a_2 = -2a_2 + \frac{2}{3}c_2, \quad -2b_2 = 0, \quad -2c_2 = 0.$$

So

$$a_2 = -1, \qquad b_1 = -\sqrt{3}\pi, \qquad b_2 = 0, \qquad c_1 = 12, \qquad c_2 = 0.$$

We conclude that:

$$\begin{aligned} I_0(h) &= -\sqrt{3}\pi + a_1 h - h\log h + a_3 h^2 + a_4 h^2 \log h + O(h^2), \\ I_1(h) &= 8 - \sqrt{3}\pi h + b_3 h^2 + b_4 h^2 \log h + O(h^2), \\ I_2(h) &= -3\sqrt{3}\pi + 12h + c_3 h^2 + c_4 h^2 \log h + O(h^2). \end{aligned}$$

Using (5.99), the coefficients in (5.98) are given by:

$$\begin{aligned} p(\overline{\mu}) &= -3\sqrt{3}\pi\overline{\mu}_4 + 8\overline{\mu}_3 - \sqrt{3}\pi\overline{\mu}_1, \\ q(\overline{\mu}) &= -\overline{\mu}_1, \\ r(\overline{\mu}) &= 12\overline{\mu}_4 - \sqrt{3}\pi\overline{\mu}_3 + a_1\overline{\mu}_1, \\ s(\overline{\mu}) &= c_4\overline{\mu}_4 + b_4\overline{\mu}_3 + a_4\overline{\mu}_1 - 1, \end{aligned}$$

So $p(0) = q(0) = r(0) = 0$, but $s(0) \neq 0$. Moreover, it is easily seen that the map

$$\overline{\mu} \mapsto (p(\overline{\mu}), q(\overline{\mu}), r(\overline{\mu}), \alpha_1(\overline{\mu})),$$

with $\alpha_1(\overline{\mu}) = \frac{1}{2}(\overline{\mu}_1 - \overline{\mu}_2)$ is a local diffeomorphism at zero.

Chapter 6

Cyclicity of an unbounded semi–hyperbolic 2–saddle cycle

We will study the cyclicity of an unbounded limit periodic set inside a family of Liénard systems of type (m, n) with $m < 2n+1$, m and n odd. We have already seen in Chapter 2 that it is possible for such Liénard systems to contain a heteroclinic connection between two semi–hyperbolic saddles at infinity, giving rise to an *unbounded semi–hyperbolic 2–saddle cycle*, see Figure 6.1.

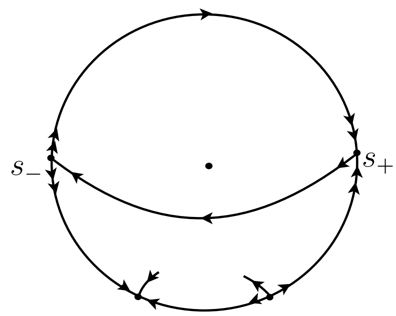

Figure 6.1: An unbounded semi–hyperbolic 2–saddle cycle.

In this chapter we are aiming to find an upperbound on the number of limit cycles that can perturb from such an unbounded semi–hyperbolic 2–saddle cycle $\mathcal{L}_0$, generalising the techniques used in [5].

Let (m,n) be natural numbers chosen fixed such that $m < 2n+1$, m and n are odd. Consider a family of general Liénard systems:

$$(X^0_{(a,b)}) : \left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & -(x^m + \sum_{i=0}^{m-1} a_i x^i) - y(x^n + \sum_{i=0}^{n-1} b_i x^i), \end{array} \right. \quad (6.1)$$

with paramater values $(a,b) = (a_0, a_1, \ldots, a_{m-1}, b_0, \ldots, b_{n-1})$ lying in a neighbourhood of a parameter value $(\overline{a}, \overline{b}) = (\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_{m-1}, \overline{b}_0, \ldots, \overline{b}_{n-1})$ for which $X^0_{(\overline{a},\overline{b})}$ has an unbounded semi–hyperbolic 2–saddle cycle $\mathcal{L}_0$.

Introducing the *difference map* $\Delta^{(a,b)}(w)$ *near* $\mathcal{L}_0$ with $\Delta^{(a,b)}(0) = 0$, we will search an upperbound on *the cyclicity of* $\mathcal{L}_0$ *at* $(\overline{a}, \overline{b})$ inside the family $(X^0_{(a,b)})$. This cyclicity is defined as:

$$cycl(X^0_{(a,b)}, \mathcal{L}_0, (\overline{a}, \overline{b})) = \limsup_{(a,b)\to(\overline{a},\overline{b}), w\downarrow 0} \{\text{number of isolated zeros } w \text{ of } \Delta^{(a,b)}\}.$$

For $m = 1$, the above family (6.1) is a family of classical Liénard systems. As is shown in [5], $\mathcal{L}_0$ has finite cyclicity inside a family of classical Liénard systems. However the techniques used in that article can also be used in a more general context which we want to describe in this chapter. Unfortunally a complete generalisation for $m \geq 3$ will be difficult as we will see below. For the moment, the techniques introduced in [5] only permit us to prove finite cyclicity of $\mathcal{L}_0$ in certain subfamilies of (6.1). We will obtain a quite good estimate for the cyclicity.

6.1 The difference map

In this section, we introduce the difference map near the unbounded semi–hyperbolic 2–saddle cycle. This map will enable us to study the limit cycles that can perturb from $\mathcal{L}_0$ inside the family $(X^0_{(a,b)})$. Such limit cycles are also called *large amplitude limit cycles.*

It will appear to be useful to switch to the Liénard plane by means of the transformation

$$\overline{y} = y + F(x), \quad (6.2)$$

with $F'(x) = (x^n + \sum_{i=0}^{n-1} b_i x^i)$ and $F(0) = 0$. In order to study large amplitude limit cycles an appropriate compactification of the phase plane has to be performed. Notice however that the behaviour obtained at infinity is not essentialy different from the behaviour at infinity of $X^0_{(a,b)}$ described in Chapter 1. Indeed the transformation (6.2) is extendable to infinity [10].

Switching to the Liénard plane and compactification

As is well known, the study at infinity of $(X^0_{(a,b)})$ can be done by means of charts. Both semi–hyperbolic saddles s_1 and s_2 in Figure 6.1 cannot be studied at once in one chart.

Therefore, we switch to the so–called Liénard plane bringing (6.1) into:

$$\begin{cases} \dot{x} &= y - \left(\frac{1}{n+1}x^{n+1} + \sum_{i=0}^{n-1} \frac{a_i}{i+1} x^{i+1}\right), \\ \dot{y} &= -\left(x^m + \sum_{i=0}^{m-1} b_i x^i\right), \end{cases} \tag{6.3}$$

where we have switched the roles of a_i and b_i to agree with the notation used in [5]. We can simplify (6.3) by removing the coefficient in front of x^{n+1} by means of a coordinate change

$$(x, y, t) \to (\mu x, \beta y, \gamma t),$$

with

$$\mu = (n+1)^{\frac{2}{m-(2n+1)}}, \qquad \beta = (n+1)^{\frac{m+1}{m-(2n+1)}} \quad \text{and} \quad \gamma = \left(\frac{1}{n+1}\right)^{\frac{m-1}{m-(2n+1)}}.$$

Because m is odd, (6.3) always has a singularity. After a translation a singularity can always be supposed to lie at the origin, implying that $b_0 = 0$. Adapting again the exact value of (a, b), one sees that it is sufficient to study a family $(X_{(a,b)})$ of the form:

$$(X_{(a,b)}) : \begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^{n} a_i x^{n+1-i}\right), \\ \dot{y} &= -\left(x^m + \sum_{i=1}^{m-1} b_i x^{m-i}\right), \end{cases} \tag{6.4}$$

with $(m,n)\in\mathbb{N}^2$ fixed such that $m<2n+1$, m and n are odd, and with parameter values

$$(a,b)=(a_1,\dots,a_n,b_1,\dots,b_{m-1})\in\mathcal{W},$$

varying in a neighbourhood $\mathcal{W}$ of a parameter value

$$(\overline{a},\overline{b})=(\overline{a}_1,\dots,\overline{a}_n,\overline{b}_1,\dots,\overline{b}_{m-1})$$

for which $X_{(\overline{a},\overline{b})}$ has an unbounded semi–hyperbolic 2–saddle cycle. Remark that we have also rearranged the coefficients which is more natural realising that in the calculations at infinity, the coefficients of the higher order terms will play a more prominent role.

We show that for a system $X_{(a,b)}$ like in (6.4), both semi–hyperbolic saddles at infinity lie in the chart in the positive y–direction. A study in the positive y–direction can be done by means of the tranformation

$$x=u/s,\qquad y=1/s^{n+1}$$

and by multiplying the result with s^n, leading to the family

$$(\hat{X}_{(a,b)}):\left\{\begin{array}{rcl}\dot{u}&=&1-u^{n+1}-\displaystyle\sum_{i=1}^{n}a_iu^{n+1-i}s^i\\ &&+\dfrac{u^{m+1}}{n+1}s^{2n+1-m}+\dfrac{1}{n+1}u\displaystyle\sum_{i=1}^{m-1}b_iu^{m-i}s^{2n+1-m+i},\\ \dot{s}&=&\dfrac{1}{n+1}s^{2n+2-m}\left(u^m+\displaystyle\sum_{i=1}^{m-1}b_iu^{m-i}s^i\right).\end{array}\right. \tag{6.5}$$

The blown up vector field has two singularities $s_\pm=(\pm1,0)$ that are both semi–hyperbolic saddles with linear part

$$\pm\begin{pmatrix}-(n+1)&a_1\\0&0\end{pmatrix}.$$

To understand the behaviour near $s_+=(1,0)$, we study the behaviour on the center manifold that locally can be written as a graphic

$$\{(U(s,a,b),s)\mid s\geq0\},\tag{6.6}$$

for some smooth function U with $U(0,a,b) = 1$. From (6.5), one sees immediately that the behaviour on the center manifold is given by

$$\dot{s} = \frac{1}{n+1} s^{2n+2-m} + O(s^{2n+3-m}), \quad s \to 0. \tag{6.7}$$

The behaviour near s_- follows easily from the behaviour near s_+ using symmetry arguments. The flow of $\hat{X}_{(a,b)}$ is invariant under the transformation

$$(t,u,s) \mapsto (-t,-u,-s).$$

Therefore the behaviour of the flow of $-\hat{X}_{(a,b)}$ near s_- in the region $\{(u,s) \mid s > 0, -1 < u < -\varepsilon\}$ is found precisly in the behaviour of the flow of $\hat{X}_{(a,b)}$ near s_+ in the region $\{(u,s) \mid s < 0, \varepsilon < u < 1\}$, where $\varepsilon > 0$.

Furthermore by means of the transformation

$$x = -u/s, \qquad y = 1/s^{n+1},$$

one can easily verify that in the negative y–direction there are no singularities. The transformation

$$x = \pm 1/s, \qquad y = u/s^{n+1}$$

permits to study respectively the behaviour of $X_{(a,b)}$ in the positive and negative x–direction. One finds a repelling respectively attracting node at the origin yielding the behaviour at infinity that is shown in Figure 6.2 (a).

As before we denote $(\overline{X}_{(a,b)})$ as the family of vector fields on the Poincaré–Lyapunov disc of degree $(1, n+1)$ obtained from the family $(X_{(a,b)})$ after compactification.

Definition of the difference map

Denote Γ_1 as the connection at infinity between the saddles s_- and s_+

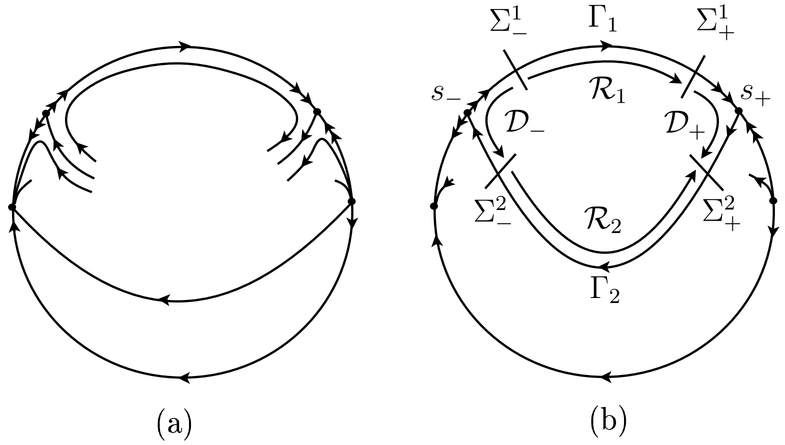

Figure 6.2: (a) Behaviour near infinity of system (6.4), (b) A semi–hyperbolic 2–saddle cycle at infinity.

of $\overline{X}_{(a,b)}$, i.e. the part of the u–axis lying between $(-1,0)$ and $(1,0)$ in $\hat{X}_{(a,b)}$. This connection stays fixed for all parameter values $(a,b) \in \mathcal{W}$. Denote $\Gamma^-_{2,a,b}$ and $\Gamma^+_{2,a,b}$ as the local center manifolds of s_- and s_+ respectively. For $(a,b) = (\overline{a}, \overline{b})$, these two local manifolds coincide along a heteroclinic connection, being part of $\mathcal{L}_0$. Denote this connection as Γ_2 (see Figure 6.2 (b)).

Choose sections $\Sigma^i_\pm$ transverse to Γ_i near the saddles and parametrised by a regular parameter. Furthermore, if the regular parameter on Σ^1_- is denoted by w, $w > 0$, then we suppose that the intersection $\Gamma_1 \cap \Sigma^1_-$ corresponds to $w = 0$. One can define (see also Figure 6.2 (b)):

1. *the Dulac maps* $\mathcal{D}^{(a,b)}_\pm$ describing the corner passage near $s_\pm$ from $\Sigma^1_\pm$ to $\Sigma^2_\pm$ defined by the flow of $\pm\overline{X}_{(a,b)}$,

2. *the regular transition maps* $\mathcal{R}^{(a,b)}_i$ near Γ_i from Σ^i_- to Σ^i_+, $i = 1,2$, defined by the flow of $\pm\overline{X}_{(a,b)}$ respectively.

The *difference map* $\Delta^{(a,b)} : \Sigma^1_- \mapsto \Sigma^2_+$, expressed in the chosen parameters on the sections Σ^1_- and Σ^2_+, is defined as:

$$\Delta^{(a,b)}(w) = \Delta(w,(a,b)) = (\mathcal{R}^{(a,b)}_2 \circ \mathcal{D}^{(a,b)}_- - \mathcal{D}^{(a,b)}_+ \circ \mathcal{R}^{(a,b)}_1)(w),$$

with w lying near zero and (a, b) varying close to $(\overline{a}, \overline{b})$. The dependence on (a, b) in all above defined maps is smooth.

The large amplitude limit cycles correspond to small positive zeros of $\Delta^{(a,b)}$. The cyclicity $Cycl(\overline{X}_{(a,b)}, (\mathcal{L}_0, (\overline{a}, \overline{b})))$ is equal to the least upper bound of the number of isolated zeros of $\Delta^{(a,b)}$, for $w \downarrow 0, (a, b) \mapsto (\overline{a}, \overline{b})$. An upperbound on this cyclicity will be found by applying a *division–derivation algorithm* to $\Delta^{(a,b)}$, based on Rolle's theorem.

A suitable choice of the sections $\Sigma^i_\pm$ together with their regular paramatrisations will enable us to calculate the difference map. This choice is determined by the normal form coordinates near s_+ in which $\hat{X}_{(a,b)}$, up to C^∞ equivalence, reads:

$$N_\alpha : \begin{cases} \dot{z} &= -z \\ \dot{w} &= w^{2n+2-m}(1 + \alpha(a, b)w^{2n+1-m})^{-1}, \end{cases} \tag{6.8}$$

with $\alpha = \alpha(a, b)$, C^∞ dependent of (a, b). Denote $\varphi = \varphi^{(a,b)} : \mathcal{U} \mapsto \mathcal{U}'$ as the C^∞ coordinate change expressing locally the normalizing coordinates (z, w) in function of the coordinates (u, s). In Chapter 4, Section 4.3, we proved that such normalizing coordinates exist. Remark that after some dilatation we can suppose that $[0, 1] \times [0, 1] \subset \mathcal{U}'$. In the future we will often not mention explicitly the dependence of φ on (a, b) for not overloading the notation. We now choose sections as follows.

Let

$$\sigma^1_\pm = \{(1, \pm w) \mid w \geq 0\} \cap \mathcal{U}' \text{ and } \sigma^2_\pm = \{(z, \pm 1) \mid z \geq 0\} \cap \mathcal{U}'$$

and then choose:

$$\Sigma^1_+ = \varphi^{-1}(\sigma^1_+), \quad \Sigma^2_+ = \varphi^{-1}(\sigma^2_+),$$

that are clearly transverse to $\Gamma_1, \Gamma^+_{2,a,b}$ respectively (see Figure 6.3). By symmetry of the flow of $\hat{X}_{(a,b)}$, one can choose:

$$\Sigma^1_- = \{(u, s) \mid \varphi(-u, -s) \in \sigma^1_-\}, \quad \Sigma^2_- = \{(u, s) \mid \varphi(-u, -s) \in \sigma^2_-\},$$

transverse to $\Gamma_1, \Gamma^-_{2,a,b}$ respectively. Notice that by this definition the sections $\Sigma^i_\pm$, $i = 1, 2$ depend on (a, b). In a natural way, we can parametrise $\Sigma^\pm_1$ by the first normalizing coordinate w and $\Sigma^\pm_2$ by the second normalizing coordinate z.

To obtain a cyclicity result, it will be sufficient to study the Dulac maps and the transition map $\mathcal{R}^{(a,b)}_1$. Herefore we will express $\mathcal{R}^{(a,b)}_1$ and the Dulac maps $\mathcal{D}^{(a,b)}_\pm$ in function of the normalizing coordinates near s_+. Let us start with the calculation of the Dulac maps.

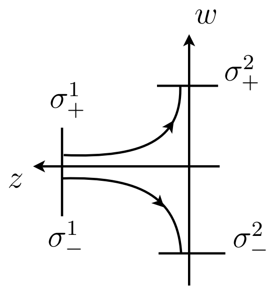

Figure 6.3: Corner passage in normal form.

The Dulac maps

In normalizing coordinates (z, w) the corner passage near s_+ in the half plane $\{w > 0\}$ reads:

$$D_+ = D^\alpha_+ : \sigma^1_+ \mapsto \sigma^2_+ : (1, w) \mapsto (D(w), 1), \tag{6.9}$$

for some C^∞ function $D = D^\alpha$. The Dulac map $\mathcal{D}_+$ is then completely described by D:

$$\mathcal{D}^{(a,b)}_+ : \Sigma^1_+ \mapsto \Sigma^2_+ : (u, s) \mapsto \varphi^{-1}(D_+(\varphi(u, s))).$$

It is the map D that is the expression of $\mathcal{D}_+$ in normalizing coordinates. Using symmetry arguments also $\mathcal{D}_-$ can be completely described by D. Indeed by symmetry of the flow of $\hat{X}_{(a,b)}$, the Dulac map $\mathcal{D}_-$ reads:

$$\mathcal{D}^{(a,b)}_- : \Sigma^1_- \mapsto \Sigma^2_- : (u, s) \mapsto \varphi^{-1}(D_-(\varphi(u, s))),$$

where $D_- = D_-^\alpha : \sigma_1^- \mapsto \sigma_2^-$ is the corner passage near s_+ in the half plane $\{w < 0\}$ in normalizing coordinates. This corner passage equals:

$$D_- = D_-^\alpha : \sigma_-^1 \mapsto \sigma_-^2 : (1, w) \mapsto (D(w), -1).$$

where we have used the invariance of the normal form N_α under the reflection $(t, z, w) \mapsto (t, z, -w)$.

So we are mainly interested in the map D that is calculated in the following proposition.

Proposition 6.1 *Let $D = D^\alpha$ be the C^∞ map defined in (6.9), completely determining the corner passages near the saddles of $\hat{X}_{(a,b)}$. Then*

$$D(w) = w^\alpha \exp\left(\frac{1 - w^{m-(2n+1)}}{2n+1-m}\right).$$

Proof: Indeed D is determined by the integral equation:

$$\int_w^1 \frac{1 + \alpha v^{2n+1-m}}{v^{2n+2-m}} dv = -\int_1^{D(w)} \frac{1}{z} dz$$

meaning that

$$-\frac{1}{2n+1-m} + \frac{w^{m-(2n+1)}}{2n+1-m} - \alpha \ln w = -\ln D(w)$$

which immediately implies the result. □

The transition map along Γ_1

Also the transition map $\mathcal{R}_1^{(a,b)}$ can be expressed in normalizing coordinates. We define the C^∞ map $R_1 = R_1^{(a,b)}$ as:

$$\varphi(\mathcal{R}_1^{(a,b)}(-\varphi^{-1}(1, -w))) = (1, R_1^{(a,b)}(w)), \quad (1, -w) \in \sigma_-^1. \tag{6.10}$$

Analogously let $R_2 = R_2^{(a,b)}$ be the map describing the transition map $\mathcal{R}_2^{(a,b)}$ from Σ_-^2 to Σ_+^2 using normalizing coordinates.

The rest of this paragraph will be devoted to the calculation of the derivatives of $R_1^{(a,b)}$ at zero. Herefore, we use a similar technique as in Chapter 5 in which we pass to the limit of a regular transition as the sections come arbitrarily close to the semi–hyperbolic saddles.

The derivatives of $R_1^{(a,b)}$ will be strongly related to an intermediate normal form. From Chapter 4, Section 4.3, we know that there exists a C^∞ family of transformations, defined on a neighbourhood of s_+, that brings $(\hat{X}_{(a,b)})$ into an intermediate normal form $(N_{(a,b)}^{int})$:

$$(N_{(a,b)}^{int}) : \begin{cases} \dot{V} &= -\overline{h}^{(a,b)}(S)V, \\ \dot{S} &= \overline{g}^{(a,b)}(S)S^{2n+2-m}, \end{cases} \tag{6.11}$$

where $\overline{h} = \overline{h}^{(a,b)}$ and $\overline{g} = \overline{g}^{(a,b)}$ are strictly positive smooth functions. Again by symmetry arguments, this normal form can also be used in the neighbourhood of s_-.

Denote (v, s) as the C^∞ coordinates in which this center manifold in (6.6) has been straightened:

$$(v, s) = (-u + U(s, a, b), s).$$

From (6.7), we know that the behaviour of $\hat{X}_{(a,b)}$ on the center manifold is of the order $O(s^{2n+2-m})$. Normal form calculations imply that $\overline{h}^{(a,b)}$ and $\overline{g}^{(a,b)}$ in (6.11) are given by:

$$\begin{aligned} \overline{g}^{(a,b)}(S) &= \frac{1}{n+1}\left(U(S)^m + \sum_{i=1}^{m-1} b_i U(S)^{m-i}S^i\right) + O(S^{2n+2-m}), \\ \overline{h}^{(a,b)}(S) &= (n+1)U(S)^n + \sum_{i=1}^{n}(n+1-i)a_i U(S)^{n-i}S^i + O(S^{2n+1-m}), \end{aligned} \tag{6.12}$$

as $S \to 0$, where we have omitted the dependence of U on (a, b). Writing

$$(n+1)^2\frac{\overline{g}^{(a,b)}(S)}{\overline{h}^{(a,b)}(S)} = 1 + \sum_{i=1}^{2n-m} g_i(a,b)S^i + O(S^{2n+1-m}), S \to 0, \tag{6.13}$$

where $g_i = g_i(a,b)$ are polynomials in the parameters (a,b), we can state the following theorem.

Theorem 6.2 *Consider the map $R_1 = R_1^{(a,b)}$, as introduced in (6.10), describing the transition of the flow of $\overline{X}_{a,b}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1. Let g_i be the coefficients defined in* (6.13)*. Then we have:*

$$R_1(0) = 0 \text{ and } R_1'(0) = 1$$

If $0 \leq k \leq \frac{2n+1-m}{2}$ *such that* $g_1 = g_3 = \cdots g_{2k-3} = 0$ *in (6.13), then*

$$R_1^{(j)} = 0, \qquad \forall 2 \leq j \leq 2k-1,$$

and

$$R_1^{(2k)}(0) = r_k g_{2k-1}, \quad R_1^{(2k+1)}(0) = 0,$$

with

$$r_k = 2(2k)!\,(n+1)^{\frac{2(2k-1)}{2n+1-m}} \frac{1}{2n+2-m-2k}.$$

Notice that in the above theorem, we have omitted the dependence on (a,b) in the notation of R_1 and the coefficients g_i for not overloading the notations. We will now give a sketch of the proof of the above theorem. The proof proceeds in a totally similar way as in [5]. For further details we refer the reader to [5].

Let $u_0 > 0$, $s_0 < 1$ be fixed and (u_0, s_0) sufficiently close to s_+ such that $\mathcal{P} = \{(u,s) \mid u_0 < u < 1, |s| < s_0\}$ is contained in $\mathcal{U}$. Let

$$\Pi_u^{\pm} = \{(\pm u, s) \mid 0 \leq s < s_0\}$$

and consider the transition from Π_u^- to Π_u^+ defined by the flow of $\hat{X}_{(a,b)}$:

$$\Pi_u^- \mapsto \Pi_u^+ : (-u, s) \mapsto (u, H_u^{(a,b)}(s)). \tag{6.14}$$

Choose the w–coordinate as a regular parameter on $\sigma = \sigma_+^1 \cup \sigma_-^1$. Put $\pi_u^{(a,b)} = \varphi^{(a,b)}(\Pi_u^+)$, there is no need to choose a parametrisation on $\pi_u^{(a,b)}$. Denote by

$$\Psi_u = \Psi_u^{(a,b)} : \sigma \mapsto \pi_u^{(a,b)},$$

the regular transition map from σ to $\pi_u^{(a,b)}$ defined by the flow of N_α (6.8). Let the inverse map of $\varphi^{(a,b)}$ be denoted by $\psi = \psi^{(a,b)} = (\psi_1, \psi_2)$ defined on $\mathcal{U}' = \varphi^{(a,b)}(\mathcal{U})$. For $(1, w) \in \sigma$, one can write:

$$\psi(\Psi_u^{(a,b)}(w)) = (u, \psi_2(\Psi_u^{(a,b)}(w))) \in \Pi_u^+.$$

By construction, we now obtain the following commutative diagram:

$$\begin{array}{ccccc}
\sigma_-^1 & \xrightarrow{R_1^{(a,b)} \circ (-Id)} & \sigma_+^1 & \xrightarrow{\Psi_u^{(a,b)}} & \pi_{u,(a,b)} \\
\Psi_u^{(a,b)} \downarrow & & & & \uparrow \varphi^{(a,b)} \\
\pi_{u,(a,b)} & \underset{\psi_2^{(a,b)}}{\longrightarrow} & \Pi_u^+ \cap \{s < 0\} & \underset{H_u^{(a,b)} \circ (-Id)}{\longrightarrow} & \Pi_u^+ \cap \{s > 0\}
\end{array} \tag{6.15}$$

Equivalently, we can write, where the parameter $w < 0$ corresponds with the point $(1, w) \in \sigma_-^1$:

$$\Psi_u^{(a,b)}(R_1^{(a,b)}(-w)) = \varphi^{(a,b)}(u, H_u^{(a,b)}(-\psi_2(\Psi_u^{(a,b)}(w)))). \tag{6.16}$$

We have already noticed that $\hat{X}_{(a,b)}$ can be put in an intermediate normal form (6.11) but also in the normal form N_α, (6.8). Denote $\tilde{\varphi} = \tilde{\varphi}^{(a,b)} : \mathcal{U} \mapsto \mathcal{U}''$ as the transformations bringing the vector field $\hat{X}_{(a,b)}$ into the intermediate normal form $N_{(a,b)}^{int}$. Furthermore denote by:

$$\begin{array}{rlcl}
\tilde{\varphi} = \tilde{\varphi}^{(a,b)} : & \mathcal{U}' & \mapsto & \mathcal{U}'' \\
& (V, S) & \mapsto & (V, T^{(a,b)}(S)),
\end{array} \tag{6.17}$$

the transformation bringing the vector field $\frac{1}{\bar{h}(S)} N_\alpha^{int}$ into the vector field N_α. Let $L_1 = L_1^{(a,b)}$ be the transition map $R_1^{(a,b)}$ expressed in the intermediate normalising coordinate $S > 0$. Then $L_1^{(a,b)}$ satisfies a similar scheme as $R_1^{(a,b)}$, with φ replaced by $\tilde{\varphi}$ defined locally around s_+ and with the transition map $\Psi_u^{(a,b)}$ together with the sections $\sigma_\pm^1, \pi_u^{(a,b)}$ defined as above but in the coordinates (V, S) (see Figure 6.4).

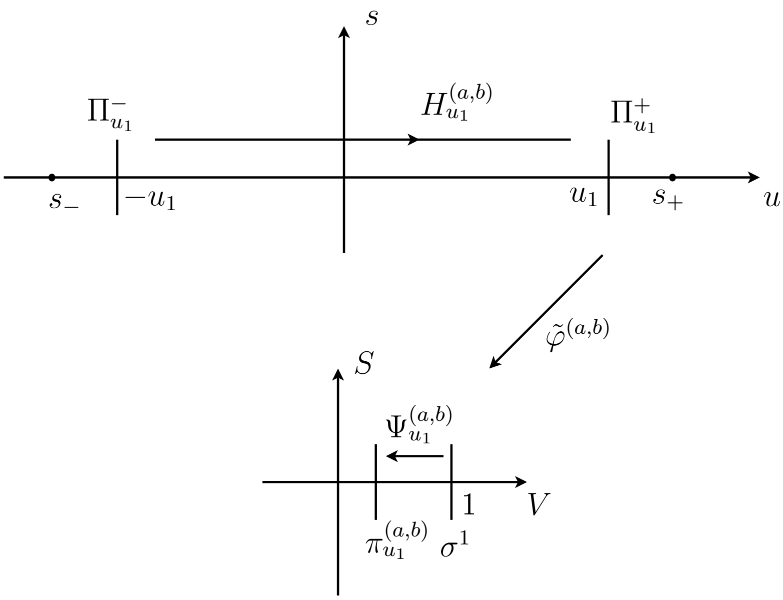

Figure 6.4: Illustration of the commutative diagram (6.15) for an arbitrarily but fixed $u_0 < u_1 < 1$, using intermediate normalizing coordinates near the saddles.

Totally similar as in [5], one proves that $H_u^{(a,b)}(s) = s + O(s^{2n+3-m}), s \to 0$. Performing the scheme (6.15) for $L_1^{(a,b)}$, one finds that the first $2n + 2 - m$ derivatives of L_1 are strongly related to the derivatives:

$$\frac{\partial^{2i}\tilde{\varphi}_2}{\partial s^{2i}}(1,0), \quad \forall 1 \le i \le \frac{2n+1-m}{2}.$$

In particular, if $1 \le k \le \frac{2n+1-m}{2}$ such that

$$\frac{\partial^{2i}\tilde{\varphi}_2}{\partial s^{2i}}(1,0) = 0, \quad \forall 1 \le i \le k-1,$$

then

$$\begin{cases} \dfrac{\partial^n L}{\partial S^n}(0) & = & 0, \qquad \forall 2 \leq n \leq 2k-1, \\ \dfrac{\partial^{2k} L}{\partial S^{2k}}(0) & = & 2\left(\dfrac{\partial \tilde{\varphi}_2}{\partial s}(1,0)\right)^{-2k} \dfrac{\partial^{2k} \tilde{\varphi}_2}{\partial s^{2k}}(1,0), \\ \dfrac{\partial^{2k+1} L}{\partial S^{2k+1}}(0) & = & 0. \end{cases}$$

This relationship can be proved totally similar as in [5].

One proceeds by proving that

$$\tilde{\varphi}_2(V,S) = S + O(s^{2n+2-m}), \quad s \to 0.$$

The proof given in [5] can be generalised in a rather straightforward way. One only has to keep in mind that the order of the center behaviour is $2n+2-m$ implying the necessary adaptions. Notice that the coefficients b_i come to play a rol in the calculations, but at the end cancel to give the same result. One concludes that

$$L_1(S) = S + O(S^{2n+3-m}), \quad S \to 0.$$

Finally, using the derivatives of L_1, one calculates that the derivatives of $R_1^{(a,b)}$ will be related to those of the transformation $T^{(a,b)}$ defined in (6.17) which are related to the coefficient $g_i(a,b)$ defined in (6.13), yielding Theorem 6.2.

From Theorem 6.2, one sees that the knowledge of the first $2n+2-m$ derivatives of $R_1^{(a,b)}$ at zero relies on the knowledge of the coefficients g_{2i+1}, $0 \leq i \leq \frac{2n-m-1}{2}$. We remind the reader that these coefficients are defined as

$$(n+1)^2 \frac{\overline{g}^{(a,b)}(S)}{\overline{h}^{(a,b)}(S)} = 1 + \sum_{i=1}^{2n-m} g_i(a,b) S^i + O(S^{2n+1-m}), S \to 0,$$

where $\overline{g}$ and $\overline{h}$ are the C^∞ functions appearing in the intermediate normal form (6.11).

We are interested in the expression of a $g_{2k+1}(a,b)$ with $k \leq \frac{2n-1-m}{2}$, assumming that all previous ones with odd index are zero. For $m = 1$, this expression is just a multiple of a_{2k+1}. In the general case, the expression of g_{2k+1} will be an algebraic expression in (a,b) of which the complexity increases considerably as k is getting bigger. However, when all the coefficients a_{2i+1} and b_{2i+1} are supposed to be zero for $i < k$, the expression of g_{2k+1} is rather simple as we will show in Theorem 6.4. This result will not be strong enough to obtain a finite cyclicity result of $\mathcal{L}_0$ inside a general family $(X_{(a,b)})$ as in (6.4), but only for certain subfamilies of $(X_{(a,b)})$.

For calculating the coefficient g_{2i+1}, $0 \leq i \leq \frac{2n-1-m}{2}$ we first have to calculate $U(s,a,b)$, expressing the center manifold.

Lemma 6.3 *Let $\{u = U(s,a,b)\}$ be the center manifold at s_+. Then*

$$U(0,a,b) = 1 \text{ and } \frac{\partial U}{\partial s}(0,a,b) = -\frac{1}{n+1}a_1.$$

Moreover if $1 \leq k \leq \frac{2n-1-m}{2}$ and $k \leq \frac{n-1}{2}$ such that $a_1 = a_3 = \cdots = a_{2k-1} = 0$, then:

$$\begin{cases} \dfrac{\partial^{2i+1}U}{\partial s^{2i+1}}(0,a,b) & = \quad 0, \quad \forall 0 \leq i \leq k-1, \\ \dfrac{1}{(2k+1)!}\dfrac{\partial^{2k+1}U}{\partial s^{2k+1}}(0,a,b) & = \quad -\dfrac{1}{n+1}a_{2k+1}. \end{cases}$$

Proof: Using the invariance of $u = U(s) = U(s,a,b)$ under the flow of $\hat{X}_{(a,b)}$, one finds:

$$\frac{dU}{ds}(s)\frac{1}{n+1}s^{2n+2-m}\left(U(s)^m + \sum_{i=1}^{m-1} b_i U(s)^{m-i}s^i\right)$$
$$=$$
$$1 - U(s)^{n+1} - \sum_{i=1}^{n} a_i U(s)^{n+1-i}s^i + \frac{1}{n+1}U(s)^{m+1}s^{2n+1-m}$$
$$+\frac{1}{n+1}U(s)\sum_{i=1}^{m} b_i U(s)^{m-i}s^{2n+1-m+i}.$$

Up to order $O(s^{2n+1-m})$, $s \to 0$ the above equation is given by:

$$1 - U(s)^{n+1} - \sum_{i=1}^{n} a_i U(s)^{n+1-i} s^i = 0. \tag{6.18}$$

Write

$$U(s) = 1 + \gamma_1 s + \gamma_2 s^2 \cdots \gamma_{2n-m} s^{2n-m} + O(s^{2n+1-m}), s \to 0,$$

with $\gamma_i = \frac{1}{i!}\frac{\partial^i U}{\partial s^i}(0)$. Comparing coefficients with s in (6.18), we find:

$$\gamma_1 = -\frac{a_1}{n+1}.$$

If we assume that $a_1 = \cdots = a_{2k-1} = 0$ and so by induction also $\gamma_1 = \cdots = \gamma_{2k-1} = 0$, then equation (6.18) reduces to

$$1 - \left(1 + \sum_{i=1}^{k} \gamma_{2i} s^{2i} + \gamma_{2k+1} s^{2k+1}\right)^{n+1} - \sum_{i=1}^{k} a_{2i} U(s)^{n+1-2i} s^{2i}$$
$$-a_{2k+1} U(s)^{n+1-2k-1} s^{2k+1} = O(s^{2k+2}), s \to 0,$$

implying, comparing coefficients with s^{2k+1}:

$$\gamma_{2k+1} = -\frac{a_{2k+1}}{n+1}.$$

□

This lemma permits to prove the following proposition.

Theorem 6.4 *Let $\overline{g}^{(a,b)}$ and $\overline{h}^{(a,b)}$ be the C^∞ functions appearing in the intermediate normal form (6.11). Write $G^{(a,b)}(S) = \overline{g}^{(a,b)}(S)(\overline{h}^{(a,b)}(S))^{-1}$ and let $g_i(a,b)$ be the coefficients defined in (6.13). Then*

$$G(S) = \frac{1}{(n+1)^2}(1 + (b_1 - \frac{m}{n+1} a_1)S + O(S^2)), S \to 0. \tag{6.19}$$

Further let $k \in \mathbb{N}$ such that $2k+1 \leq 2n-m$, then:

1. *If k is chosen such that $2k-1 < m-2$ and $2k-1 < n$ with*

$$a_1 = a_3 = \cdots a_{2k-1} = 0 \text{ and } b_1 = b_3 = \cdots b_{2k-1} = 0,$$

then

$$g_{2i-1} = 0, \forall 1 \le i \le k \quad \text{and} \quad g_{2k+1} = b_{2k+1} - \frac{m-2k}{n+1} a_{2k+1}.$$

2. *If k is chosen such that $2k-1 < n$ and $2k-1 \ge m-2$ with*

$$a_1 = a_3 = \cdots a_{2k-1} = 0 \text{ and } b_1 = b_3 = \cdots b_{m-2} = 0,$$

then

$$g_{2i-1} = 0, \forall 1 \le i \le k \quad \text{and} \quad g_{2k+1} = -\frac{m-2k}{n+1} a_{2k+1}.$$

3. *If k is chosen such that $2k-1 < m-2$ and $2k-1 \ge n$ with*

$$a_1 = a_3 = \cdots a_n = 0 \text{ and } b_1 = b_3 = \cdots b_{2k-1} = 0,$$

then

$$g_{2i-1} = 0, \forall 1 \le i \le k \quad \text{and} \quad g_{2k+1} = b_{2k+1}.$$

Proof: Recall that:

$$\begin{aligned} \overline{g}^{(a,b)}(S) &= \frac{1}{n+1}\left(U(S)^m + \sum_{i=1}^{m-1} b_i U(S)^{m-i} S^i\right) + O(S^{2n+2-m}), \\ \overline{h}^{(a,b)}(S) &= (n+1)U(S)^n + \sum_{i=1}^{n} (n+1-i) a_i U(S)^{n-i} S^i + O(S^{2n+1-m}). \end{aligned} \tag{6.20}$$

From Lemma 6.3, one knows that $U(S) = 1 - \frac{a_1}{n+1} S + O(S^2)$ implying:

$$\begin{aligned} \overline{g}^{(a,b)}(S) &= \frac{1}{n+1}\left(1 - m\frac{a_1}{n+1} S + b_1 S\right) + O(S^2), \\ \overline{h}^{(a,b)}(S) &= (n+1)(1 - n\frac{a_1}{n+1} S) + n a_1 S + O(S^2), \end{aligned}$$

such that (6.19) immediately follows.

We will only prove case 1, the other two being totally analogous. Suppose that

$$a_1 = a_3 = \cdots a_{2k-1} = 0 \text{ and } b_1 = b_3 = \cdots b_{2k-1} = 0,$$

with k chosen such that $2k-1 < m-2$ and $2k-1 < n$ together with $2k+1 \leq 2n-m$. From Lemma 6.3, one sees that

$$U(S) = 1 + \sum_{i=1}^{k} \gamma_{2i} S^{2i} - \frac{a_{2k+1}}{n+1} S^{2k+1} + O(S^{2k+2}), S \to 0,$$

for some $\gamma_{2i}, 1 \leq i \leq k$. One calculates, by substitution in (6.20):

$$\overline{g}^{(a,b)}(S)$$

$$=$$

$$\frac{1}{n+1}\left(1 + \sum_{j=1}^{k} \chi_j^1(a,b) S^{2j} + (b_{2k+1} - \frac{m}{n+1} a_{2k+1}) S^{2k+1} + O(S^{2k+2})\right),$$

and

$$\overline{h}^{(a,b)}(S) = (n+1)\left(1 + \sum_{j=1}^{k} \chi_j^2(a,b) S^{2j} - \frac{2k}{n+1} a_{2k+1} S^{2k+1} + O(S^{2k+2})\right),$$

for some polynomials χ_j^1 and χ_j^2 in (a,b), $1 \leq j \leq k$. Therefore

$$\frac{1}{\overline{h}^{(a,b)}(S)} = \frac{1}{n+1}\left(1 + \sum_{j=1}^{k} \tilde{\chi}_j(a,b) S^{2j} + \frac{2k}{n+1} a_{2k+1} S^{2k+1} + O(S^{2k+2})\right),$$

for some polynomials $\tilde{\chi}_j$ in (a,b), $1 \leq j \leq k$ which implies the result. □

6.2 Cyclicity result

The result in Theorem 6.4 will not be strong enough to obtain the finite cyclicity of any possible unbounded semi–hyperbolic 2–saddle cycle $\mathcal{L}_0$

inside a general family $(X_{(a,b)})$ of Liénard systems as in (6.4). We will have to impose extra conditions on $(X_{(a,b)})$, considering appropriate subfamilies (Y_μ) of $(X_{(a,b)})$. Let us describe these appropriate subfamilies.

Appropriate subfamilies

We consider families of general Liénard systems like in (6.4)

$$(X_{(a,b)}) : \begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^{n} a_i x^{n+1-i} \right), \\ \dot{y} &= -\left(x^m + \sum_{i=1}^{m-1} b_i x^{m-i} \right), \end{cases} \tag{6.21}$$

where each a_{2i+1} or b_{2i+1} is considered to be zero inside the family and this for each index $0 \leq i < r$ with $2r - 1 = \min\{m-2, n\}$. Such an *appropriate subfamily* will be denoted as (Y_μ) with parameter values μ, given by the non–zero coefficients among $(a, b) \in \mathbb{R}^{m+n-1}$. The parameter μ takes values in $\mathbb{R}^p$, with $p = m + n - r - 1$. We prefer to order $(\mu_1, \ldots, \mu_p)$ in such way that all non–zero coefficients among (a, b) with odd index are placed in front, ordered with increasing index.

For instance, the family of Liénard systems with odd friction term is an appropriate subfamily:

$$(Y_{(\alpha,b)}) : \begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^{(n-1)/2} a_{2i} x^{n+1-2i} \right), \\ \dot{y} &= -\left(x^m + \sum_{i=1}^{m-1} b_i x^{m-i} \right), \end{cases}$$

with parameter values $\mu = (b_1, b_3, \ldots, b_{m-2}, \ldots)$. Another example is the family of Liénard systems with odd forcing term:

$$(Y_{(a,\beta)}) : \begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^{n} a_i x^{n+1-i} \right), \\ \dot{y} &= - \left(x^m + \sum_{i=1}^{(m-1)/2} b_{2i} x^{m-2i} \right), \end{cases}$$

with parameter values $\mu = (a_1, a_3, \ldots, a_n, \ldots)$.

Consider an appropriate subfamily (Y_μ) as described above with μ varying in a neighbourhood $\mathcal{V}$ of some parameter value μ^0 for which Y_{μ^0} has an unbounded semi–hyperbolic 2–saddle cycle $\mathcal{L}_0$.

Theorem 6.2 only permits to calculate the first $2n + 2 - m$ derivatives. Therefore in order to be able to calculate enough derivatives at zero of $R_1^{(a,b)}$, we restrict to families of Liénard systems of type (m, n) for which $2n - m \geq d = \max\{m - 2, n\}$. In the future we will also write $d = 2l - 1$, for an $l \in \mathbb{N}$.

Because the subfamilies $(Y_\mu)_{\mu \in \mathcal{V}}$ are in particular families of general Liénard systems all the previous calculations apply to them, where we just have to replace the proper coefficients by zero. In particular compactifying the phase plane results in a family $(\overline{Y}_\mu)$ which in the positive y–direction takes the form $(\hat{Y}_\mu)$ given by (6.5) with the proper coefficients replaced by zero.

Because we suppose that Y_{μ^0} admits an unbounded semi–hyperbolic 2–saddle cycle, the vector field $\overline{Y}_{\mu^0}$ has a connection Γ_1 between the saddles s_- and s_+ that stays fixed for all parameter values $\mu \in \mathcal{V}$. On the other hand the connection Γ_2 between s_- and s_+ of $\overline{Y}_{\mu^0}$ crossing the finite plane can break after perturbation. Moreover locally around the saddle s_+ the normal form of $\hat{Y}_\mu$ reads:

$$N_\alpha : \begin{cases} \dot{z} &= -z, \\ \dot{w} &= w^{2n+2-m}(1 + \alpha(\mu) w^{2n+1-m})^{-1}, \end{cases}$$

where $\alpha = \alpha(\mu)$ is given like in (6.8), with the proper coefficients considered to be zero. Like before symmetry reasons imply that this normal form can also be used locally around s_-.

Dulac maps describing the corner passages at the saddles s_- and s_+ at infinity and transition maps along the connections Γ_1 and Γ_2 are respectively denoted as: $\mathcal{D}^\mu_\pm$ and $\mathcal{R}^\mu_{1,2}$. In normalizing coordinates the Dulac maps are described by $D = D^{\alpha(\mu)}$ and the transition maps along Γ_1 and Γ_2 by R^μ_1 and R^μ_2 respectively. The difference map along $\mathcal{L}_0$ reads Δ^μ which in normalizing coordinates is expressed as:

$$\Delta^\mu(w) = \Delta(w,\mu) = (R^\mu_2 \circ D^\mu - D^\mu \circ R^\mu_1)(w), \tag{6.22}$$

where (w, μ) varies in $[0, W_0[\times\mathcal{V}$ for $W_0 > 0$ small.

Finite cyclicity

This paragraph will be devoted to the proof of the following theorem.

Theorem 6.5 *Consider an appropriate subfamily* (Y_μ) *as in* (6.21) *and let* μ^0 *be a parameter value for which* Y_{μ^0} *has an unbounded semi–hyperbolic 2–saddle cycle. Let* $d = \max\{m-2, n\} = 2l-1$*, then*

$$Cycl(\overline{Y}_\mu, (\mathcal{L}_0, \mu^0)) \le l.$$

Let $(Y_\mu)_{\mu\in\mathcal{V}}$ denote an appropriate subfamily as in (6.21) admitting an unbounded semi–hyperbolic 2–saddle cycle $\mathcal{L}_0$ for some parameter value μ^0. To simplify the notations, we will omit the dependence on μ of the maps $R^\mu_{1,2}, D^\mu, \Delta^\mu$.

Consider the difference map Δ, as defined in (6.22). By Rolle's theorem Δ has at most $N+1$ zeros in a neighbourhood of zero if $\frac{\partial}{\partial w}\Delta$ has at most N zeros for w near zero, multiplicity taken into account. Therefore an upperbound on $cycl(\overline{Y}_\mu, (\mathcal{L}_0, \mu^0))$ is found by searching the number of solutions of the equation:

$$R_2'(D(w))D'(w) = D'(R_1(w))R_1'(w) \tag{6.23}$$

in which, as an immediate consequence of Proposition 6.1,

$$D'(w)=w^{\alpha+m-(2n+2)}\exp\left(\frac{1}{2n+1-m}\right)\exp\left(-\frac{w^{m-(2n+1)}}{2n+1-m}\right)(1+\alpha w^{2n+1-m}),$$

One sees that both sides in (6.23) are exponentially flat. For removing this exponentally flatness, we introduce a reduced difference map $\overline{\Delta}$, in such a way that its zeroes represent the roots of (6.23) and hence the zeroes of $\frac{\partial \Delta}{\partial w}$.

The reduced difference map $\overline{\Delta}$ is defined as follows. One takes logarithms on both sides of (6.23) and suppresses the factor $\exp\left(\frac{1}{2n+1-m}\right)$ on both sides of the equation. In this way, the left hand side is reduced to

$$\log(R_2'(D(w)))+(\alpha+m-(2n+2))\log w-\frac{w^{m-(2n+1)}}{2n+1-m}+\log\left(1+\alpha w^{2n+1-m}\right), \tag{6.24}$$

and the right hand side is reduced to

$$\begin{aligned}(\alpha+m-(2n+2))\log\left(R_1(w)\right)-\tfrac{R_1(w)^{m-(2n+1)}}{m-(2n+1)}+\\ \log\left(1+\alpha R_1(w)^{2n+1-m})\right)+\log\left(R_1'(w)\right).\end{aligned} \tag{6.25}$$

We introduce a smooth function $\overline{R}_1$ by the relation

$$R_1(w)=w\overline{R}_1(w).$$

Noticing that $R_2'(0)>0$, we define the C^∞ map $\overline{\Delta}^\mu(w)=\overline{\Delta}(w,\mu)$ by the difference of the equations (6.24) and (6.25) multiplied by w^{2n-m}:

$$\begin{aligned}\overline{\Delta}^\mu(w) &= \frac{1}{m-(2n+1)}\left(\frac{1-\overline{R}_1(w)^{m-(2n+1)}}{w}\right)\\ &- (\alpha+m-(2n+2))w^{2n-m}\log\overline{R}_1(w)\\ &+ w^{2n-m}\log\left(\frac{1+\alpha w^{2n+1-m}}{1+\alpha R_1(w)^{2n+1-m}}\right)\\ &+ w^{2n-m}\log R_2'(D(w))-w^{2n-m}\log R_1'(w).\end{aligned} \tag{6.26}$$

The map $\overline{\Delta}(w,\mu)$ is well defined on $[0,W_0[\times\mathcal{V}$ for an $W_0>0$. Indeed because $R_2'(0)>0$, $\log R_2'(D(w))$ makes sense and all other logarithms are taken near 1 for w near zero. Moreover, we know from Theorem 6.2 that

$$\overline{R}_1(w)=1+O(w),\ w\to 0$$

and hence

$$\frac{1}{m-(2n+1)}\left(1-\overline{R}_1(w)^{m-(2n+1)}\right)=O(w),\ w\to 0. \tag{6.27}$$

We now have the tool at our disposal for searching an upperbound on the number of limit cycles bifurcating from the cycle $\mathcal{L}_0$ at infinity. Indeed by construction zeros of $\overline{\Delta}^\mu$ correspond to zeros of $\frac{\partial\Delta}{\partial w}$. By Rolle's theorem, it follows that the maximum number of zeros of Δ is at most one unit more than this number for $\overline{\Delta}$. In order to obtain a good upperbound on the number of limit cycles near $\mathcal{L}_0$, for μ near μ^0, it hence suffice to count small positive zeros of $\overline{\Delta}$. As in [5], we call $\overline{\Delta}$ *a reduced difference map for the family* $(\overline{Y}_\mu)$.

For calculating this reduced difference map, we will need to know $R_1(w)$. Combining Theorem 6.2 and 6.4 leads to the following proposition.

Proposition 6.6 *Consider the map* $R_1=R_1^\mu$*, describing the transition of the flow of* $\overline{Y}_\mu$ *from* σ_-^1 *to* σ_+^1 *along the unbroken connection* Γ_1*, expressed in terms of the normalizing coordinate* $w>0$*. If* $0\le k\le l-1$ *such that*

$$\mu_i=0,\qquad \forall 1\le i\le k,$$

then $R_1(0)=0$*,* $R_1'(0)=1$*, all derivatives*

$$\frac{\partial^n R_1}{\partial w^n}(0),\quad 2\le n\le 2k+1,$$

are zero and

$$\frac{\partial^{2k+2}R_1}{\partial w^{2k+2}}(0)=c_{k+1}\mu_{k+1},\qquad \frac{\partial^{2k+3}R_1}{\partial w^{2k+3}}(0)=0,$$

for some constant $c_{k+1}\neq 0$.

This proposition permits to prove the following result.

Proposition 6.7 *Let* $\overline{\Delta} : [0, W_0[\times \mathcal{V} \mapsto \mathbb{R}$ *be the reduced difference map as defined above. If* $\exists 0 \le k \le l-1$ *such that*

$$\mu_i = 0, \qquad \forall 1 \le i \le k,$$

then

$$\overline{\Delta}(w, \mu) = C_{k+1}\mu_{k+1}w^{2k} + O(w^{2k+1}), \quad w \to 0, \tag{6.28}$$

for some constant C_{k+1}.

Proof: From Proposition 6.6, one knows that

$$\overline{R}_1(w) = 1 + c_{k+1}\mu_{k+1}w^{2k+1} + O(w^{2k+2}), \quad w \to 0,$$

for some constant c_{k+1}. From (6.26) and $d \le 2n - m$, it easily follows that:

$$\begin{aligned} \overline{\Delta}(w, \mu) &= \frac{1}{m-(2n+1)} \left(\frac{1 - \overline{R}_1(w)^{m-(2n+1)}}{w} \right) + O(w^{2n-m}) \\ &= C_k \mu_{k+1} w^{2k} + O(w^{2k+1}), \end{aligned}$$

where $C_k = \dfrac{c_{k+1}}{m-(2n+1)}$. $\square$

As a consequence we can state the following proposition.

Proposition 6.8 *Consider a subfamily* (Y_μ) *as in* (6.21) *and let* μ^0 *be a parameter value for which* Y_{μ^0} *has an unbounded semi–hyperbolic 2–saddle cycle. Let* $\overline{\Delta}$ *be the reduced difference map as defined in (6.26). Then the following conditions are equivalent:*

1. $\exists 0 < W_1 < W_0$ *such that* $\overline{\Delta}(w, \mu^0) = 0, \forall w \in [0, W_1[$;
2. $\mu_1^0 = \mu_2^0 = \ldots \mu_l^0 = 0$.

Proof: Condition 1 immediately implies condition 2 by Proposition 6.7. Condition 2 implies that Y_{μ^0} is invariant under the reflection

$$(x, y, t) \mapsto (-x, y, -t)$$

implying that the flow of Y_{μ^0} has to contain an unbounded annulus filled by periodic orbits and bounded by $\mathcal{L}_0$ at infinity. $\square$

For the proof of Theorem 6.5, we have to distinguish two cases. First the finite cyclicity is proved in the *regular case*, this is the case where $\overline{\Delta}(w, \mu^0) \neq 0$ or equivalently

$$\exists k : \mu_j^0 = 0, \forall j \leq k \text{ and } \mu_{k+1}^0 \neq 0.$$

Then the so-called *center case* is treated. This is when $\overline{\Delta}^{\mu^0} = 0$ or $\mu_1^0 = \cdots = \mu_l^0 = 0$.

We first need the following lemma where we denote by $\kappa_i(\mu)$ the natural projection:

$$\kappa_i : \mathbb{R}^p \mapsto \mathbb{R}^{p-i+1} : (\mu_1, \ldots \mu_p) \mapsto (\mu_i, \mu_{i+1}, \ldots \mu_p).$$

Lemma 6.9 *Concerning the difference map $\overline{\Delta}$ defined in (6.26), there exist $0 < W_1 < W_0$ and C^∞ functions $\Phi_i : [0, W_1[\times \mathcal{V}_i \mapsto \mathbb{R}$, with $\kappa_i(\mathcal{V}) \subset \mathcal{V}_i$, $1 \leq i \leq l$, such that $\forall (w, \mu) \in [0, W_1[\times \mathcal{V}$:*

$$\overline{\Delta}(w, \mu) = \sum_{i=1}^{l} \mu_i \Phi_i(w, \kappa_i(\mu)), \tag{6.29}$$

with $\forall 1 \leq i \leq l$:

$$\Phi_i(w, \kappa_i(\mu)) = C_i w^{2i-2} + O(w^{2i-1}), \quad w \to 0,$$

where C_i is given as (6.28).

Proof: Taylor's theorem applied on $\overline{\Delta}(w, \mu)$ with respect to $\mu_1 = 0$ guarantees us the existence of some C^∞ function $\Phi_1 : [0, W_0[\times \mathcal{V} \mapsto \mathbb{R}$ such that $\forall (w, \mu) \in [0, W_0[\times \mathcal{V}$:

$$\overline{\Delta}(w, \mu) = \overline{\Delta}(w, \mu) \mid_{\mu_1 = 0} + \mu_1 \Phi_1(w, \mu).$$

Applying the same principle on $\overline{\Delta}(w, \mu) \mid_{\mu_1=0}$, we find a C^∞ function $\Phi_2 : [0, W_0[\times \mathcal{V}_2 \mapsto \mathbb{R}$ with $\kappa_2(\mathcal{V}) \subset \mathcal{V}_2$ such that $\forall (w, \mu) \in [0, W_0[\times \mathcal{V}$:

$$\overline{\Delta}(w, \mu) = \overline{\Delta}(w, \mu) \mid_{\mu_1 = \mu_2 = 0} + \mu_2 \Phi_2(w, \kappa_2(\mu)) + \mu_1 \Phi_1(w, \mu).$$

Proceeding by induction we find C^∞ functions $\Phi_i : [0, W_0[\times \mathcal{V}_i \mapsto \mathbb{R}$ with $\kappa_i(\mathcal{V}) \subset \mathcal{V}_i$ such that $\forall (w, \mu) \in [0, W_0[\times \mathcal{V}$:

$$\overline{\Delta}(w,\mu) = \overline{\Delta}(w,\mu) \mid_{\mu_1=\mu_2=\cdots\mu_l=0} + \sum_{i=1}^{l} \mu_i \Phi_i(w, \kappa_i(\mu)).$$

From Proposition 6.8 we know that

$$\overline{\Delta}(w,\mu) \mid_{\mu_1=\mu_2=\cdots\mu_l=0} = 0, \quad \forall (w,\mu) \in [0, W_1[\times \mathcal{V}$$

for an $0 < W_1 < W_0$ implying (6.29).

From (6.29) it now follows that $\forall 1 \leq k < l$:

$$\overline{\Delta}(w,\mu) \mid_{\mu_1=\cdots=\mu_{k-1}=0} = \mu_k \Phi_k(w, \kappa_k(\mu)) + \sum_{i=k+1}^{l} \mu_i \Phi_i(w, \kappa_i(\mu)). \quad (6.30)$$

On the other hand from Proposition 6.7, one sees that $\forall 1 \leq k < l$:

$$\overline{\Delta}(w,\mu) \mid_{\mu_1=\cdots=\mu_{k-1}=0} = C_k \mu_k w^{2k-2} + O(w^{2k-1}), \quad w \to 0. \quad (6.31)$$

Equation (6.30) together with (6.31) implies that

$$\mu_k(\Phi_k(w,\mu) - C_k w^{2k-2}) + \sum_{i=k+1}^{l} \mu_i \Phi_i(w, \kappa_i(\mu)) = O(w^{2k-1}), w \to 0.$$

The terms behind the summation sign are independent of $\mu_1, \ldots, \mu_k$ such that the above equation immediately implies:

$$\Phi_k(w,\mu) = C_k w^{2k-2} + O(w^{2k-1}), \quad w \to 0.$$

□

Proposition 6.10 *Consider a subfamily* (Y_μ) *as in* (6.21) *and let* μ^0 *be a parameter value for which* Y_{μ^0} *has an unbounded semi–hyperbolic 2–saddle cycle. Moreover suppose that*

$$\exists k < l : \mu_j^0 = 0, \forall j \leq k \text{ and } \mu_{k+1}^0 \neq 0,$$

then

$$Cycl(\overline{Y}_\mu, (\mathcal{L}_0, \mu^0)) \leq k+1.$$

Proof: By Lemma 6.9, one can write

$$\overline{\Delta}(w,\mu) = \sum_{i=1}^{l} \mu_i \Phi_i(w, \kappa_i(\mu)), \quad \forall (w,\mu) \in [0, W_1[\times \mathcal{V}, \tag{6.32}$$

for some $W_1 > 0$ and smooth functions $\Phi_i : [0, W_1[\times \mathcal{V}_i \mapsto \mathbb{R}$, with $\kappa_i(\mathcal{V}) \subset \mathcal{V}_i$, $1 \leq i \leq l$. Moreover $\forall 1 \leq i \leq l$:

$$\Phi_i(w, \kappa_i(\mu)) = C_i w^{2i-2} + O(w^{2i-1}), \quad w \to 0.$$

for a constant $C_i \neq 0$ given in (6.28).

We now apply a devision–derivation procedure on $\overline{\Delta}(w, \mu^0)$. Divide $\overline{\Delta}$ by the non–zero function Φ_1, for $w > 0$ close to zero, and then derive with respect to w to obtain

$$\overline{\Delta}^1(w,\mu) = \sum_{i=1}^{l-1} \mu_{i+1} \Phi_i^1(w, \kappa_i(\mu)),$$

for w near zero and with

$$\overline{\Delta}^1 := \frac{\partial}{\partial w}\left(\frac{\overline{\Delta}}{\Phi_1}\right) \quad \text{and} \quad \Phi_i^1 := \frac{\partial}{\partial w}\left(\frac{\Phi_{i+1}}{\Phi_1}\right), \quad \forall 1 \leq i \leq l-1.$$

Then $\forall 1 \leq i \leq l-1$

$$\Phi_i^1 = C_i^1 w^{2i-1} + O(w^{2i}), \quad w \to 0,$$

where $C_i^1 := 2iC_{i+1}/C_0 \neq 0$. Continuing this devision–derivation procedure, after $k < l$ derivations and divisions by a non–zero function, for $w > 0$ close to zero, we find C^∞ functions $\overline{\Delta}^k, \Phi_1^k, \ldots \Phi_1^{l-k}$ and non–zero constants $C_1^k, \ldots, C_{l-k}^k$ such that:

$$\overline{\Delta}^k(w,\mu) = \sum_{i=1}^{l-k} \mu_{i+k} \Phi_i^k(w, \kappa_i(\mu)),$$

for w near zero and with $\forall 1 \leq i \leq l-k$

$$\Phi_i^k = C_i^k w^{2i-1} + O(w^{2i}), \quad w \to 0.$$

As a consequence since $\mu^0_{k+1} \neq 0$:

$$\overline{\Delta}^k(w, \mu^0) = \mu^0_{k+1} C^k_1 w(1 + o(1)), \quad w \to 0.$$

So by continuity, there exists a constant $0 < \overline{W}_1 < W_0$ and a neighbourhood $\mathcal{V}_0 \subset \mathcal{V}$ of μ^0 in $\mathbb{R}^p$ such that $\forall \mu \in \mathcal{V}_0$, the map $\overline{\Delta}(., \mu)$ has at most k zeros in $[0, \overline{W}_1[$ implying the result. $\square$

This proposition immediately implies that in the regular case the cyclicity $Cycl(\overline{Y}_\mu, (\mathcal{L}_0, \mu^0))$ is always bounded by l. We are left with treating the center case.

Proposition 6.11 *Consider a subfamily* (Y_μ) *as in* (6.21) *and let* μ^0 *be a value for which* Y_{μ^0} *has an unbounded semi–hyperbolic 2–saddle cycle. Moreover suppose that*

$$\mu^0_1 = \mu^0_2 = \cdots = \mu^0_l = 0,$$

then

$$Cycl(\overline{Y}_\mu, (\mathcal{L}_0, \mu^0)) \leq l.$$

Proof: From Proposition 6.8, we know that Y_{μ^0} has no limit cycles near $\mathcal{L}_0$. Moreover all parameter values $\mu \in \mathcal{V}$ for which $\mu_1 = \mu_2 = \cdots = \mu_l = 0$ correspond with a vector field Y_μ not having any limit cycle near $\mathcal{L}_0$. Let us denote:

$$\mathcal{C} = \{\mu \in \mathbb{R}^p : \mu_j = 0, \forall 1 \leq j \leq l\}.$$

Because $\mathcal{V}$ is a neigbourhood of μ^0 in $\mathbb{R}^p$, there exists $\rho_0 > 0$ with

$$\mathcal{V}_0 = \{\mu \in \mathbb{R}^p : \sum_{j=1}^{l} \mu_j^2 < \rho_0^2 \text{ and } \left|\mu_j - \mu_j^0\right| < \rho_0, \forall j > l\} \subset \mathcal{V}.$$

For all parameter values $\mu \in \mathcal{V}_0 \backslash \mathcal{C}$, we have:

$$\exists ! 0 < \rho < \rho_0 \ : \ \mu_j = \rho \overline{\mu}_j, \ \forall 1 \leq j \leq l \text{ and } \sum_{j=1}^{l} \overline{\mu}_j^2 = 1. \qquad (6.33)$$

Lemma 6.9 guarantees the existence of an $0 < W_1 < W_0$ and functions $\Phi_i : [0, W_1[\times \mathcal{V}_i \mapsto \mathbb{R}$ with $\kappa_i(\mathcal{V}) \subset \mathcal{V}_i$ such that on $[0, W_1[\times \mathcal{V}$:

$$\overline{\Delta}(w, \mu) = \sum_{i=1}^{l} \mu_i \Phi_i(w, \kappa_i(\mu)),$$

where each $\Phi_i(w, \kappa_i(\mu))$ can be written as:

$$\Phi_i(w, \kappa_i(\mu)) = C_i w^{2i-2} + O(w^{2i-1}), \quad w \to 0,$$

for some constant $C_i > 0$. Now on $[0, W_1[\times \mathcal{V}_0 \backslash \mathcal{C}$, one can express $\overline{\Delta}$ in terms of $(\rho, \overline{\mu})$ as:

$$\overline{\Delta}(w, \chi(\rho, \overline{\mu})) = \rho \sum_{j=1}^{l} \overline{\mu}_j \Phi_j(w, \kappa_i(\chi(\rho, \overline{\mu}))),$$

where $\mu = \chi(\rho, \overline{\mu})$ is defined by (6.33) and $\mu_j = \overline{\mu}_j$, $\forall j > l$. Isolated zeros of $\overline{\Delta}(., \mu)$ for $\mu \in \mathcal{V}_0 \backslash \mathcal{C}$ correspond with isolated zeros of the map

$$\Psi(w, (\rho, \overline{\mu}) = \sum_{j=1}^{l} \overline{\mu}_j \Phi_j(w, \kappa_i(\chi(\rho, \overline{\mu}))),$$

with $0 < \rho < \rho_0$ and $\overline{\mu}$ varying in the set

$$\mathcal{B} = \{\overline{\mu} \in \mathbb{R}^p : \sum_{j=1}^{l} \overline{\mu}_j^2 = 1 \text{ and } \left|\overline{\mu}_j - \mu_j^0\right| < \rho_0, \forall j > l\}.$$

For any $b^0 = (\overline{\mu}_1^0, \ldots \overline{\mu}_l^0)$ lying on the unit sphere $\mathbb{S}^{l-1} \subset \mathbb{R}^l$, we find an $k = k(b^0) \leq l - 1$ such that $\overline{\mu}_j^0 = 0, \forall 1 \leq j \leq k$ but $\overline{\mu}_{k+1}^0 \neq 0$. By a totally analogous derivation–division argument as in Proposition 6.10, we find a neighbourhood $\mathcal{V}_{b^0}$ of $\overline{\mu}^0$ in $\mathbb{S}^{l-1}$ and a constant $0 < W_{b^0} < W_1$ such that the map $\Psi(., (\rho, \overline{\mu})$ has at most k zeros in $]0, W_{b^0}[$, for all $(\rho, \overline{\mu})$ with $\chi(\rho, \overline{\mu}) \in \mathcal{V}_0$ and $(\overline{\mu}_1, \ldots \overline{\mu}_l) \in \mathcal{V}_{b^0}$. By compactness of the sphere $\mathbb{S}^{l-1}$, we can now take a constant $0 < W < W_1$ independent of b^0, such that $\overline{\Delta}(w, \mu)$ has at most $l - 1$ zeros in $[0, W[$, $\forall \mu \in \mathcal{V}_0$. This ends the proof. $\square$

Chapter 7

Boundaries of period annuli in Liénard systems

In the previous chapter, we met an unbounded semi–hyperbolic 2–saddle cycle as *exterior boundary* of an *unbounded period annulus* occuring in a Liénard system of type (m, n), with $m < 2n + 1$, m and n odd, reading:

$$X : \begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^{(n-1)/2} a_{2i} x^{n+1-2i} \right), \\ \dot{y} &= -\left(x^m + \sum_{i=1}^{(m-1)/2} b_{2i} x^{m-2i} \right). \end{cases}$$

We also say that $\overline{X}$ has a centrum at infinity. One way for $\overline{X}$ having a centrum at infinity is $\overline{X}$ having a *Hopf centrum* in $D \backslash \partial D$ that extends to infinity, also called an *unbounded Hopf centrum*.

In this chapter, we describe all possible exterior and interior boundaries of as well bounded as unbounded period annuli. This permits to describe the boundary of all possible Hopf centra, the ones that extend to infinity as well as the ones that do not extend to infinity. Further, we find some sufficient conditions for a Hopf centrum to be bounded or to be unbounded.

7.1 Boundaries of period annuli

Let $\overline{X} \in L^{(m,n)}(D)$, obtained after a suitable Poincaré–Lyapunov compactification from a Liénard system X of type (m, n):

$$X : \left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & P(x) + yQ(x), \end{array} \right. \tag{7.1}$$

where P and Q are polynomials of respective precise degrees m and n. In this section, we will describe the possible boundaries of period annuli occuring in $\overline{X}$. We start with some definitions.

An open connected subset $\mathcal{A}$ of the plane filled by closed orbits of $\overline{X}$ is called *a period annulus of* X; we will shortly call it an annulus. We say that the annulus is *bounded* when $\mathcal{A}$ fits in some compact $K \subset D \backslash \partial D$, if not, $\mathcal{A}$ is said to be *unbounded*.

Let $\mathcal{A}$ be an annulus of $\overline{X}$. We can provide $\mathcal{A}$ with an *order relation* $\preceq$. Herefore let $p_1, p_2 \in \mathcal{A}$. Then, one can find two closed orbits of $\overline{X}$, $\gamma_1 \subset \mathcal{A}$ and $\gamma_2 \subset \mathcal{A}$ such that $p_1 \in \gamma_1$ and $p_2 \in \gamma_2$. Denote A_1 and A_2 as the regions enclosed by γ_1 and γ_2 respectively. We say that $p_1 \preceq p_2$ if and only if $A_1 \subseteq A_2$. If both $p_1 \preceq p_2$ and $p_2 \preceq p_1$, then we say that p_1 is equivalent to p_2, denoted as $p_1 \sim p_2$; this happens when p_1 and p_2 both belong to the same closed orbit.

We say that a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ is *monotonically increasing in* $\mathcal{A}$ if $p_i \preceq p_{i+1}$, $\forall i \in \mathbb{N}$. If $p_{i+1} \preceq p_i$, $\forall i \in \mathbb{N}$, we say that the sequence is *monotonically decreasing in* $\mathcal{A}$.

We define *the exterior boundary of* $\mathcal{A}$ as

$$\partial^e\mathcal{A} = \{q \in \partial\mathcal{A} \mid \exists\, (p_n)_n \text{ monotonically increasing in } \mathcal{A} : p_n \to q\},$$

and *the interior boundary of* $\mathcal{A}$ as

$$\partial^i\mathcal{A} = \{q \in \partial\mathcal{A} \mid \exists\, (p_n)_n \text{ monotonically decreasing in } \mathcal{A} : p_n \to q\}.$$

Clearly $\partial^e\mathcal{A} \cap \partial^i\mathcal{A} = \phi$.

Any periodic orbit of $\overline{X}$ intersects the x–axis in exactly two points. In particular the intersection of $\mathcal{A}$ with the x–axis is the union of two open intervals $]a, b[\cup]c, d[$ lying on the x–axis, see Figure 7.1. Notice, when a or d lie on the circle at infinity, the annulus $\mathcal{A}$ is unbounded.

For each $x \in]a, d[$, the sections $\Sigma_x^+ = \{(x, y) \in D \mid y > 0\}$ and $\Sigma_x^- = \{(x, y) \in D \mid y < 0\}$ both intersect $\mathcal{A}$ in an unique open interval:

$$\begin{aligned} \mathcal{A} \cap \Sigma_x^+ &= \{x\} \times]p_x^+, q_x^+[, \\ \mathcal{A} \cap \Sigma_x^- &= \{x\} \times]q_x^-, p_x^-[, \end{aligned} \tag{7.2}$$

where for each x, $p_x^\pm$ and $q_x^\pm$ are uniquely determined, with $(x, p_x^\pm)$ and $(x, q_x^\pm)$ lying either in the interior of the disc D or on the circle at infinity.

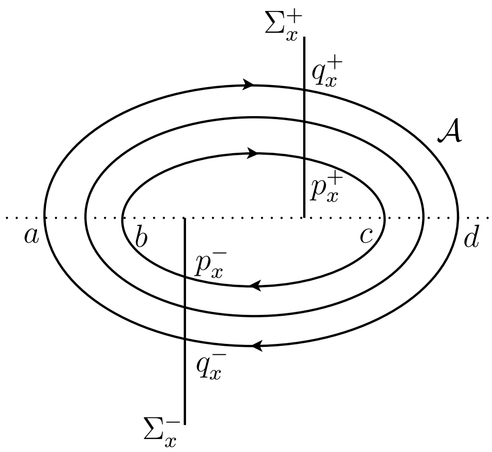

Figure 7.1: An annulus $\mathcal{A}$ of $\overline{X}$, the dotted line represents the x–axis.

Proposition 7.1 *Suppose $\mathcal{A}$ is a period annulus of $\overline{X} \in L^{(m,n)}(D)$ that intersects the x–axis in $]a, b[\cup]c, d[$. Consider*

$$\begin{aligned} \Gamma_e^+ = \{(x, q_x^+) \mid x \in]a, d[\}, \quad \Gamma_i^+ = \{(x, p_x^+) \mid x \in]b, c[\}, \\ \Gamma_e^- = \{(x, q_x^-) \mid x \in]a, d[\}, \quad \Gamma_i^- = \{(x, p_x^-) \mid x \in]b, c[\}, \end{aligned} \tag{7.3}$$

with $p_x^{\pm}, q_x^{\pm}$ defined as in (7.2). Then the exterior and interior boundaries of $\mathcal{A}$ are invariant subsets for the flow of $\overline{X}$, given by

$$\begin{aligned} \partial^e \mathcal{A} &= \{(a,0)\} \cup \{(d,0)\} \cup \Gamma_e^+ \cup \Gamma_e^-, \\ \partial^i \mathcal{A} &= \{(b,0)\} \cup \{(c,0)\} \cup \Gamma_i^+ \cup \Gamma_i^-. \end{aligned} \tag{7.4}$$

Further $\{(a,0)\} \cup \{(d,0)\} \cup \Gamma_e^{\pm}$ and $\{(b,0)\} \cup \{(c,0)\} \cup \Gamma_i^{\pm}$ are all continuous graphs.

Proof: Let us first prove the invariance, for the flow $\phi_t : D \mapsto \mathbb{R}^2$ of $\overline{X}$, of the exterior and interior boundary of $\mathcal{A}$. Let $p \in \partial^e \mathcal{A}$ (resp. $p \in \partial^i \mathcal{A}$), then there exists a sequence $(p_n)_n$ in $\mathcal{A}$ that is monotonically increasing (resp. decreasing) in $\mathcal{A}$ and that converges to p. Because of the continuity of the flow ϕ_t of $\overline{X}$, one immediately concludes that $(\phi_t(p_n))_n$ is a monotonically increasing (resp. decreasing) sequence in $\mathcal{A}$ converging to $\phi_t(p)$.

We proceed by proving the equalities in (7.4). Clearly the boundary of $\mathcal{A}$ is given by:

$$\mathcal{F} := \{(a,0),(b,0),(c,0),(d,0)\} \cup \Gamma_e^{\pm} \cup \Gamma_i^{\pm}.$$

Indeed, one easily verifies that $\forall p \in \mathcal{F}, \forall \varepsilon > 0 : B(p,\varepsilon) \cap \mathcal{A} \neq \phi$ and $B(p,\varepsilon) \cap \mathcal{A}^c \neq \phi$. Since $\partial^e \mathcal{A} \cap \partial^i \mathcal{A} = \phi$, it clearly suffices to prove that

$$\{(a,0)\} \cup \{(d,0)\} \cup \Gamma_e^+ \cup \Gamma_e^- \subset \partial^e \mathcal{A}$$

and

$$\{(b,0)\} \cup \{(c,0)\} \cup \Gamma_i^+ \cup \Gamma_i^- \subset \partial^i \mathcal{A},$$

for showing (7.4). We prove the inclusion concerning the exterior boundary, the other one can be proved completely analogously.

Let us first remark that if $p_1 = (x, y_1)$ and $p_2 = (x, y_2)$ with $x \in]a, d[$ and $y_1, y_2 > 0$, then one has

$$y_1 < y_2 \text{ if and only if } p_1 \prec p_2.$$

This is evident and follows immediately from the fact that the closed orbits trough p_1 and p_2 do not intersect.

Suppose $(x, q_x^+) \in \Gamma_e^+$, then the sequence $(x, q_x^+ - \frac{1}{n})_{n \geq N}$, with N big enough, lies in $\mathcal{A}$ and converges to (x, q_x^+). Moreover from the above remark, one easily sees that $(x, q_x^+ - \frac{1}{n})_n$ is a monotonically increasing sequence in $\mathcal{A}$. It follows that $\Gamma_e^+ \subset \partial^e \mathcal{A}$. Similar $\Gamma_e^- \subset \partial^e \mathcal{A}$. One also sees that $(a, 0)$ and $(d, 0)$ belong to $\partial^e \mathcal{A}$ by choosing appropriate sequences in $\mathcal{A}$ that are lying on the x–axis and that converge to $(a, 0)$ and $(d, 0)$.

Concerning the statement of the continuity, we will only treat the interior boundary. The exterior boundary can be treated in the same way. In fact in the unbounded case, the result follows from the classification that we will give in Theorem 7.2 and which does not rely on the previous knowledge of the continuity in the unbounded case.

Let $\mathcal{G} = \{(a, 0)\} \cup \{(d, 0)\} \cup \Gamma_i^+$. Then $\mathcal{G}$ is the graph of the function

$$P^+ : x \mapsto p_x^+,$$

where we set $p_x^+ = 0$ when $x = a$ or $x = d$. We prove that P^+ is continuous in any $x_0 \in [a, d]$. Suppose first $(x_0, P^+(x_0))$ is a regular point of the flow of $\overline{X}$. Because the interior boundary is invariant for the flow of $\overline{X}$, it is clear that for $x \in [a, d]$ and near x_0, the graph of P^+ is given by a connected piece of the regular orbit of X passing through $(x_0, P^+(x_0))$ such that P^+ is continuous at x_0. If $(x_0, P^+(x_0))$ is a singularity, then it has to contain a hyperbolic sector induced by the annulus $\mathcal{A}$. From the list of singularities in Chapter 1, Figures 1.9, 1.10, 1.11 and 1.12, it is clear that the boundary of such a hyperbolic sector is always continuous. Analogously, one shows that $\{(a, 0)\} \cup \{(d, 0)\} \cup \Gamma_i^-$ is a continuous graph. □

We can now completely describe all possible boundaries of annuli occuring in a Liénard system $\overline{X} \in L^{(m,n)}(D)$. Let us first treat the exterior boundaries.

Theorem 7.2 *Suppose $\mathcal{A}$ is a period annulus of $\overline{X} \in L^{(m,n)}(D)$, then*

1. *if $\mathcal{A}$ is bounded, then $\partial^e \mathcal{A}$ is a loop, as in Figure 7.2 (a) or (b), or a cycle with 2 singularities, as in Figure 7.2 (c). The singularities on the exterior boundary can only be saddles (as well hyperbolic*

as non–hyperbolic), saddle–nodes (as well semi–hyperbolic as nilpotent) or cusps as described in Chapter 1,

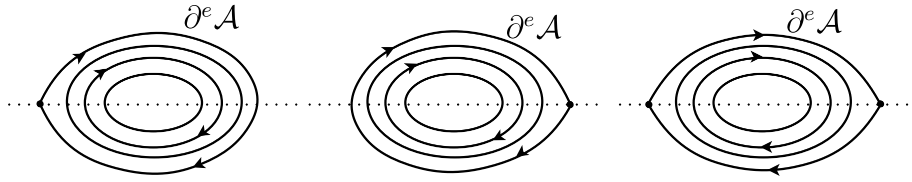

Figure 7.2: Possible exterior boundaries of a bounded annulus $\mathcal{A}$ of $\overline{X}$.

2. *if $\mathcal{A}$ is not bounded, then one of the below mentioned conditions have to be satisfied (as in Chapter 1, A denotes the highest order coefficient of the polynomial P in* (1.2)*):*

   (a) *When*

   $$m = 2n+1, A > \frac{1}{4(n+1)} \text{ or } m > 2n+1, m \text{ odd and } A = 1,$$

   then $\partial^e\mathcal{A}$ is given by a periodic orbit lying at infinity, see Figure 7.3 (a).

   (b) *When*

   $$m < 2n+1, m \text{ and } n \text{ odd }, \text{ and } A = 1,$$

   the exterior boundary looks like in Figure 7.3 (b) containing at infinity two semi–hyperbolic saddles s_1 and s_2 such that s_1 (resp. s_2) has an unstable (resp. stable) separatrix lying at infinity and an attractive (resp. repelling) center separatrix in $D\backslash\partial D$.

   (c) *When*

   $$m = 2n+1, n \text{ odd }, 0 < A \leq \frac{1}{4(n+1)},$$

   $\partial^e\mathcal{A}$, looks like in Figure 7.3 (c). When $A = 1/(4(n+1))$, the singularities at infinity are saddle-nodes of which the center separatrix lies at infinity. The behaviour on the hyperbolic

seperatric is attractive for s_1, while it is repelling for s_2. When $A < 1/(4(n+1))$, s_1 and s_2 are both hyperbolic saddles.

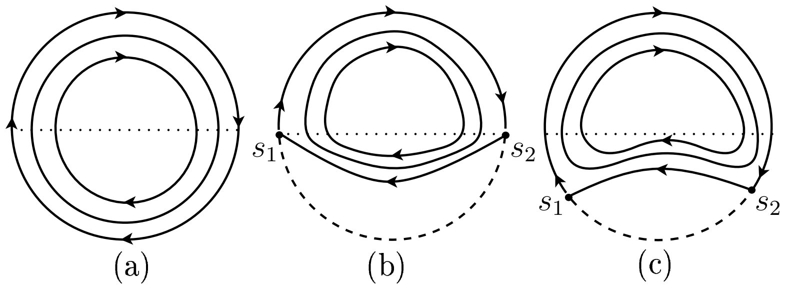

Figure 7.3: Possible exterior boundaries of an unbounded annulus $\mathcal{A}$ of $\overline{X}$.

Proof: From Proposition 7.1, one concludes that the exterior boundary of $\mathcal{A}$ is an invariant set for the flow of $\overline{X}$, being (at least in the bounded case) the union of two continuous graphs, one lying in the half plane $\{y \geq 0\}$ and one lying in $\{y \leq 0\}$. Moreover the graphs are given by

$$\{(a,0)\} \cup \{(d,0)\} \cup \Gamma_e^+ \quad \text{and} \quad \{(a,0)\} \cup \{(d,0)\} \cup \Gamma_e^-,$$

with $\Gamma_e^+ \subset \{y > 0\}$, $\Gamma_e^- \subset \{y < 0\}$, defined in (7.3), lying enterily in the region $a < x < d$.

In particular if $\mathcal{A}$ is bounded, then $\mathcal{A}$ lies enterily in some compact $K \subset D \backslash \partial D$ such that the only possible singularities of $\overline{X}$ lying on $\partial^e \mathcal{A}$ are $(a,0)$ and $(d,0)$. This completely determines the structure of $\partial^e \mathcal{A}$. Furthermore, because of the analyticity of the Liénard system X, $\partial^e \mathcal{A}$, has to contain at least one singularity.

If only $(a,0)$ (resp. $(d,0)$) is a singularity, then $\partial^e \mathcal{A}$ is a loop consisting of the singulariy $(a,0)$ (resp. $(d,0)$) and a regular orbit of $\overline{X}$ given by $\Gamma_e^+ \cup \Gamma_e^- \cup \{(d,0)\}$ (resp. $\Gamma_e^+ \cup \Gamma_e^- \cup \{(a,0)\}$. If both $(a,0)$ and $(d,0)$ are singularities, the exterior boundary of $\mathcal{A}$ consists of the two singularities $(a,0)$ and $(d,0)$ together with regular orbits Γ_e^+ and Γ_e^- connecting them.

Obvious the annulus $\mathcal{A}$ induces a hyperbolic sector on the singularities of $\overline{X}$ lying on its exterior boundary. So referring to Chapter 1, the only possibilities for the singularities are saddles, saddle–nodes and cusps.

Suppose now $\mathcal{A}$ is unbounded. Then $\partial^e\mathcal{A}$ includes the whole circle at infinity or part of it. When it includes the whole circle at infinity, the boundary is given by a regular periodic orbit at infinity. When it includes partly the circle at infinity, $\partial^e\mathcal{A}$ has to contain two singularities, s_1 and s_2, at infinity, with a hyperbolic sector induced by the annulus $\mathcal{A}$. Referring to the study at infinity of $\overline{X} \in L^{(m,n)}(D)$ in Chapter 1, it is clear that only the graphics in Figure 7.3 can occur as exterior boundaries of unbounded annuli.

In case (a), $\partial^e\mathcal{A}$ contains no singularities and is given by a periodic orbit at infinity. In case (b), $(a,0)$ and $(d,0)$ are given by singularities of $\overline{X}$ at infinity, Γ_e^+ is a regular orbit lying at infinity and Γ_e^- is a regular orbit that lies completely under the x–axis. In case (c), the singularities are lying on Γ_e^-, $(a,0)$ and $(d,0)$ are part of a regular orbit lying at infinity between the singularities s_1 and s_2. $\square$

Concerning the interior boundaries, we have the following theorem.

Theorem 7.3 *The interior boundary of some period annulus $\mathcal{A}$ of $\overline{X} \in L^{(m,n)}(D)$, is given by*

1. *a center, or*

2. *the union of the two bounded (i.e. not intersecting ∂D) continuous graphs*

$$\{(a,0)\} \cup \{(d,0)\} \cup \Gamma_i^+ \quad \textit{and} \quad \{(a,0)\} \cup \{(d,0)\} \cup \Gamma_i^-,$$

lying respectively in the half planes $\{y \geq 0\}$ and $\{y \leq 0\}$ and containing at least one singularity of $\overline{X}$. Concerning the type of the singularities, we have the following:

(a) *the zeros of the continuous graph* Γ_i^+ *(resp.* Γ_i^-*) are singularities that can only be saddles or one of the singularities presented in Chapter 2, Figure 2.2 and Figure 2.3 (resp. Figure 2.4),*

(b) *the intersections between* Γ_i^+ *and* Γ_i^- *are saddles,*

(c) *if* $(a, 0)$ *(resp.* $(d, 0)$*) is a singularity, then it has to be a cusp of up–up (resp. down–down) type.*

Proof: From Proposition 7.1, one concludes that the interior boundary of $\mathcal{A}$ is an invariant set for the flow of $\overline{X}$, being the union of two continuous graphs, one lying in the half plane $\{y \geq 0\}$ and one lying in $\{y \leq 0\}$. The graphs are given by

$$\{(b, 0)\} \cup \{(c, 0)\} \cup \Gamma_i^+ \quad \text{and} \quad \{(b, 0)\} \cup \{(c, 0)\} \cup \Gamma_i^-, \tag{7.5}$$

with $\Gamma_i^+ \subset \{y \geq 0\}$, $\Gamma_i^- \subset \{y \leq 0\}$, defined in (7.3), lying enterily in the region $b < x < c$. The two graphs intersect the x–axis in $(b, 0)$ and $(c, 0)$ and possibly other points, with abscis lying in $]b, c[$.

Whether the annulus is bounded or unbounded, $\partial^i \mathcal{A}$ is clearly situated in $D \backslash \partial D$. Because of the analyticity of the Liénard system X, $\partial^i \mathcal{A}$, has to contain at least one singularity. Furthermore, the singularities lying on $\partial^i \mathcal{A}$ are clearly given by its intersections with the x–axis.

If $b = c$, then obviously $\partial^i \mathcal{A}$ has to be a center. If $b \neq c$, then $\partial^i \mathcal{A}$ is a graphic being the union of the two continuous graphs in (7.5). If $(b, 0)$ (resp. $(c, 0)$) is a singularity it clearly has to contain an upgoing (resp. downgoing passage) and a hyperbolic sector lying left (resp. right) of this vertical passage. Referring to the study of the singularities in Chapter 1, one sees that this is only possible when $(b, 0)$ (resp. $(c, 0)$) is a cusp of up–up type (resp. down–down type). The singularities lying on Γ_i^+ (resp. Γ_i^-), and with abscis in $]b, c[$, clearly contain a lr–passage (resp. rl–passage) determining the type of the singularities (see Chapter 2, Figures 2.2, 2.3 and 2.4). □

From Theorem 7.2, it is clear that the exterior boundaries of bounded (resp. unbounded) annuli form a subset of the list of bounded (resp. unbounded) limit periodic sets that are *inner polycycles*. An inner polycycle is a cycle that can be approached by limit cycles lying in the open region enclosed by the cycle. Similarly, from Theorem 7.3, the interior boundaries of the annuli form a subset of the list of bounded limit periodic set that are *outer polycycles*. An outer polycycle is a cycle that can be approached by limit cycles lying in the exterior of the region enclosed by the cycle.

Not every limit periodic set is the boundary of some annulus. There are some restrictions on the structure of a limit periodic set making it the boundary of some annulus. The question remains whether every boundary of some annulus is indeed a limit periodic set. Herefore, one has to search an appropriate unfolding in $L^{(m,n)}(D)$, of a Liénard system with an annulus $\mathcal{A}$, producing a limit cycle that can be put arbitrarily close to the exterior (resp. interior) boundary of $\mathcal{A}$ for the Hausdorff metric. At the moment, we won't focus our attention to this question. Let us instead, as a consequence of the former results, give a complete description of the boundary of any Hopf centrum occuring in a Liénard system.

7.2 Boundaries of Hopf centra

We say that $\overline{X}$ has *a Hopf centrum* at s if s is a linear center of $\overline{X}$ of which an open neighbourhood is filled by closed orbits of $\overline{X}$. In particular $\overline{X}$ contains an annulus $\mathcal{A}$ with $\partial^i\mathcal{A} = \{s\}$. We say that s is *a bounded Hopf centrum* if $\mathcal{A}$ is bounded. When $\mathcal{A}$ is unbounded, s is *an unbounded Hopf centrum.* Evidently a Hopf centrum s can never lie at infinity.

In this section we will describe all possible boundaries of, as well bounded as unbounded, Hopf centra. We find some sufficient conditions for a Hopf centrum to be bounded or unbounded.

From Theorem 7.2, we can describe all possible exterior boundaries of the annulus $\mathcal{A}$ of $\overline{X} \in L^{(m,n)}(D)$ occuring when $\overline{X}$ has a Hopf centrum

s. We come to three possibilities, shown in Figure 7.4. The singularities on $\partial^e\mathcal{A}$ can only be saddles, saddle–nodes or cusps.

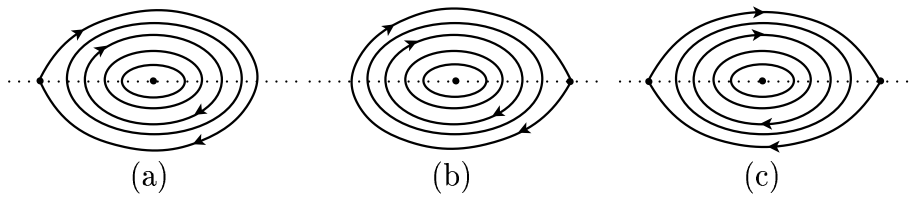

Figure 7.4: Possible bounded Hopf centra of $\overline{X}$. The Hopf centra are bounded by a loop in (a) and (b) and a cycle with two singularities in (c).

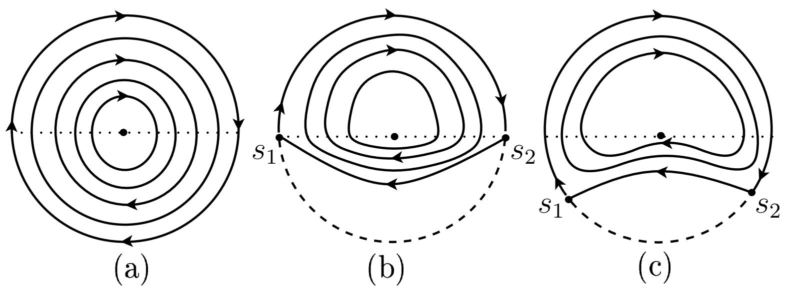

Figure 7.5: Possible unbounded Hopf centra of $\overline{X}$.

For an unbounded Hopf centrum, we have the options illustrated in Figure 7.5. As Theorem 7.2 states, case (a) can only occur when $m = 2n+1$, $A > 1/(4(n+1))$; or when $m > 2n+1$, m odd, $A = 1$. Case (b) can only occur when $m < 2n+1$, m and n odd, $A = 1$. The singularities s_1 and s_2 at infinity are semi–hyperbolic saddles where s_1 (resp. s_2) has an unstable (resp. stable) separatrix lying at infinity and an attractive (resp. repelling) center separatrix in $D\backslash\partial D$. Case (c) can only occur when $m = 2n+1$, n odd and $0 < A \leq 1/(4(n+1))$. When $A = 1/(4(n+1))$ the singularities at infinity are given by saddle–nodes of which the center separatrix lies at infinity. The behaviour on the hyperbolic separatrix is attractive for s_1, while it is repelling for s_2. When $A < 1/(4(n+1))$, the singularities at infinity are hyperbolic saddles.

Let us now concern about the question whether a Hopf centrum is bounded or unbounded. Using the above classification of unbounded Hopf centra, one easily proves the following theorem.

Theorem 7.4 *Suppose* $\overline{X} \in L^{(m,n)}(D)$ *has a Hopf centrum* s *in* $D\backslash\partial D$, *then it is unbounded if and only if* X *has an unique singularity. Such an unbounded Hopf centrum can only occur in the cases:*

1. $m = 2n+1$, $A > 1/(4(n+1))$,
2. $m = 2n+1$, n *odd,* $0 < A \leq 1/(4(n+1))$,
3. $m > 2n+1$, m *odd,* $A = 1$,
4. $m < 2n+1$, m *odd,* n *odd,* $A = 1$.

For verifying whether a given singularity s of $\overline{X} \in L^{(m,n)}(D)$ is a Hopf centrum, one can use the following theorem. Notice that after a suitable translation, s can supposed to lie at the origin.

Theorem 7.5 (C. Christopher [8]) *Let* $\overline{X} \in L^{(m,n)}(D)$ *obtained after an appropriate Poincaré–Lyapunov compactification from a Liénard system* X (7.1), *with* $P'(0) < 0$, *then* $\overline{X}$ *has a centrum at the origin if and only if there exists a polynomial,* $M(x) = x^2 + O(x^3)$ *and polynomials* $k(z), l(z)$ *with* $k(0) = l(0) = 0$ *such that* X *is given by:*

$$\begin{cases} \dot{x} &= y - k(M(x)), \\ \dot{y} &= -\frac{d}{dx}(l(M(x))). \end{cases} \tag{7.6}$$

The above theorem completely characterises the Hopf centra in Liénard systems. Another tool for characterising Hopf centra are Lyapunov coefficients. For a general introduction to the Lyapunov coefficients as well as an algorithm to compute them, we refer to [14], see also appendix A.2. In particular X reads like (7.6) if and only if all Lyapunov coefficient at s are zero.

Theorem 7.6 *Let $\overline{X} \in L^{(m,n)}(D)$, obtained after a suitable Poincaré–Lyapunov compactification from a Liénard system X that is given by:*

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - k(M(x)), \\ \dot{y} & = & -\frac{d}{dx}(l(M(x))), \end{array} \right. \tag{7.7}$$

for some polynomials $M(x), k(z), l(z)$ with $M(x) = x^2 + O(x^3), k(0) = 0, l(0) = 0, l'(0) > 0$ and the degree of M odd. Then the origin is a bounded Hopf centrum.

Proof: From Theorem 7.5, we already know that the origin is a Hopf centrum. Using the previous theorem, we only have to show that the origin is not the only singularity of $\overline{X}$ in $D \backslash \partial D$.

The degree of M is odd, thus

$$M(x) = x^2 + m_1 x^3 + \cdots + m_{2k-1} x^{2k+1},$$

for some $k \geq 1$. Therefore:

$$M'(x) = x(2 + 3m_1 x + \cdots (2k+1) m_{2k-1} x^{2k-1}),$$

implying that $M'(x) = x M_0(x)$ with $M_0(0) \neq 0$ and the degree of M_0 odd. So M_0 has to possess a zero $x_0 \neq 0$ implying that X has a singularity not lying at the origin. $\square$

Appendix A

Some Maple programs

In this appendix, we present a Maple–library containing programs that can be very helpful when performing calculations related to vector fields. The library is stored as *vfields*. In particular executing the single command `with(vfields)` all programs, contained in the library, are at the users disposal.

A.1 Calculating normal forms with Ndiag

We present a procedure *Ndiag* that calculates a normal form for a local family of vector field (X_μ) of which every member admits a singularity s_μ at which the linear part of (X_μ) is diagonalisable. After a parameter dependent linear coordinate transformation, one can suppose that s_μ is fixed, lying at the origin for all parameter values and that the linear part of (X_μ) at s_μ is in its diagonal form:

$$\begin{pmatrix} \lambda_1(\mu) & 0 \\ 0 & \lambda_2(\mu) \end{pmatrix}.$$

The procedure has also the ability to calculate the normal form of an individual vector field.

The program contains four, possible five parameters. The first four are respectively given by:

vf: the family (X_μ) of planar vector fields as a list containing the first and second component of (X_μ),

coords: a list containing the name of the two coordinates in which (X_μ) is expressed,

para: a list containing the name of the parameters; when calculating the normal form of an individual vector field, one has to enter the parameter list as empty,

r: the order up to one wishes to know the normal form.

Notice that the family (X_μ) is supposed to have a singularity at the origin at which the linear part is already in diagonal form. So, if necessary, one merely has to perform a (parameter dependent) linear transformation before applying the procedure *Ndiag*.

The fifth parameter is optional. If present, Maple will store in it the corresponding coordinate transformation bringing (X_μ) in the normal form up to order r.

Concerning coordinate transformations, we make the following agreement in this section. All transformations express the old coordinates in function of the new. In particular if X_0 is a vector field and X_1 is the transformed vector field by means of a coordinate transformation T, then locally

$$X_1(T^{-1}(x,y)) = DT^{-1}_{(x,y)} X_0(x,y).$$

Let us illustrate the procedure by an example. The following command

```
> Ndiag([2*x+x^2+y^2,y+x*y],[x,y],[],3,T);
```

returns the normal form of $(2x+x^2+y^2)\frac{\partial}{\partial x} + (y+xy)\frac{\partial}{\partial y}$ up to order 3:

$$[2\,x + y^2,\; y].$$

The corresponding transformation is stored in T, such that the command `T;` returns:

$$[x + \frac{1}{2}\,x^2,\; y + \frac{1}{2}\,x\,y].$$

The procedure *Ndiag*

Before stating the main procedure, we will give some subprocedures that *Ndiag* will need and that are also useful as independent procedures. Remark that before being able to use the following procedures, one has to load the library *linalg* in Maple. This is possible by the command `with(linalg);`

The following procedure *Transformation* performs any transformation on a vector field *vf*. As before *vf* is given as a list with entries the components of *vf*. The second paramater is given by *cds*, a list with entries the names of the coordinates in which *vf* is expressed. The parameter *Tr* is a list with entries the components of the transformation that one would like to perform on *vf*.

```
> vfields[Transformation]:=proc(vf::list,cds::list,Tr::list)
  local J,V;

  J:=jacobian(Tr,cds);
  V:=inverse(J)&*subs(cds[1]=Tr[1],cds[2]=Tr[2],vf);
  RETURN(convert(evalm(V),list));
  end:
```

The following subprocedure *Tay* returns the MacLaurin polynomial of the vector field up to the given order r minus 1. The paramaters are *vf* and *cds* and the order up to one wishes to know the MacLaurin polynomial.

```
> vfields[Tay]:=proc(vf::list,cds::list,r::posint)
  local V1,V2;

  V1:=convert(mtaylor(vf[1],cds,r),polynom);
  V2:=convert(mtaylor(vf[2],cds,r),polynom);
  RETURN([V1,V2]);
  end:
```

The procedure Ndiag reads as follows.

```
> vfields[Ndiag]:=proc(vf::list,cds::list,para::list,
  r::posint,T::evaln) local e,V,A,gr,P,Q,i,j,c,m,d,L,M;

  p:=nops(para);
  V:=Tay(vf,cds,r);
  A:=subs({cds[1]=0,cds[2]=0},jacobian(V,cds));
  e:=[seq(A[i,i],i=1..2)];

  if nargs>=4 then L:=[cds[1],cds[2]]; fi;

  for gr from 2 to r-1 do
    P:=0;Q:=0;

    for i from 0 to gr do
      c:=V[1];
      c:=coeff(coeff(c,cds[1],i),cds[2],gr-i);
      m:=(i-1)*e[1] + (gr-i)*e[2];
      if c<>0 and subs({seq(para[i]=0,i=1,...,p)},m)<>0 then
        P:=(c/m)*cds[1]^i*cds[2]^(gr-i)+P;
      fi;
    od;

    for j from 0 to gr do
      d:=V[2];
      d:=coeff(coeff(d,cds[1],j),cds[2],gr-j);
      m:=j*e[1] + (gr-j-1)*e[2];
      if d<>0 and subs({seq(para[i]=0,i=1,...,p)},m)<>0 then
        Q:=(d/m)*cds[1]^j*cds[2]^(gr-j)+Q;
      fi;
    od;

    V:=Transformation(V,cds,[cds[1]+P,cds[2]+Q]);
    V:=simplify(Tay(V,cds,r));

    if nargs>=4 then
      L:=subs({cds[1]=cds[1]+P,cds[2]=cds[2]+Q},L);
```

```
    L:=Tay(L,cds,r);
    T:=L;
  fi;
od;

RETURN(V);
end:
```

A.2 Calculating Lyapunov coefficients

The following program calculates Lyapunov coefficients of an individual vector field X having a linear center at the origin. The paramaters are given by: *vf*, a list of which the entries are given by the components of X; *cds*, a list with entries the name of the coordinates in which X is expressed and l, a natural number such that the number of Lyapunov coefficients that are calculated does not exceed $(l-1)/2$.

For a general introduction to Lyapunov coefficients as well as the algorithm implementated in this procedure, we refer the reader to [14].

```
> vfields[Lyapunov]:=proc(vf::list,cds::list,l)
  local cplex,G,F,H,Z,R,V,k,i,n,h;

  cplex:=proc(vf::list,cds::list) local Z,rcds;
   rcds:=[(cds[1]+cds[2])/2,(cds[1]-cds[2])/(2*I)];
   Z:=transformation(vf,cds,rcds)[1];
   RETURN(Z);
  end:

  G:=proc(R,cds::list) local c,d,GR,i,j;
   if R=0 then RETURN(0) fi;
   d:=degree(R,op(cds));
   GR:=0;

   for i from 0 to d do
```

```
    for j from 0 to d do
      if i<>j then
        c:=2*coeff(coeff(R,cds[1],i),cds[2],j)/(i-j);
        GR:=GR+c*x^i*y^j;
      fi;
    od:
  od:

  RETURN(GR);
end:

F:=proc(R,Rk,cds::list) local f,Imf,cds2;
  f:=unapply(G(diff(R*Rk,cds[1]),cds),(op(cds)));
  cds2:=[cds[1]+I*cds[2],cds[1]-I*cds[2]];
  Imf:=-coeff(simplify(f(op(cds2))),I,1);
  Imf:=unapply(Imf,op(cds));
  RETURN(Imf((cds[1]+cds[2])/2,(cds[1]-cds[2])/(2*I)));
end:

H:=proc(R,Rk,cds::list) local n,k,RS,A,B,DAB,ret,i,j,c;
  if (R=0) or (Rk=0) then RETURN(0) fi;

  n:=degree(R,x,y);
  k:=degree(Rk,x,y);
  RS:=unapply(R*Rk,cds);
  A:=-1/(2*I)*subs(I=-I,RS(cds[2],cds[1]));
  B:=1/(2*I)*RS(op(cds));
  for i from 0 to n+k do
   DAB:=-diff(A,cds[2])+diff(B,cds[1]);
   c[i]:=coeff(coeff(DAB,cds[1],i),cds[2],i);
  od:

  ret:=-sum(c[j]*(2*rho)^(j+1)/(j+1),j=0..n+k);
  RETURN(((2*Pi*I)/((2*rho)^((n+1+k)/(2))))*ret);
end:
```

```
#MAIN PROGRAM

Z:=cplex(vf,cds);

for k from 2 to l do
  R[k]:=sum(coeff(coeff(Z,cds[1],i),cds[2],k-i)
  *cds[1]^i*cds[2]^(k-i),i=0..k);
od:

k:='k';
h[1]:=F(1,R[2],cds);

for k from 2 to l-2 do
  h[k]:=F(1,sum(R[i]*h[k+1-i],i=2..k)+R[k+1],cds);
od:

k:='k'; i:='i';

n:=trunc((l-1)/2);
k:=n;

V:=simplify(H(1,R[2*k+1],cds));
for i from 2 to 2*k do
  V:=simplify(V + H(R[i],h[2*k+1-i],cds));
od:

RETURN(simplify(V/(2*Pi)));
end:
```

Finally all programs presented in this appendix can be stored in a library, called *vfields*, by means of the single command:

```
> save(vfields,"c:/.../maple/lib/vfields.m");
```

where "..." represents the directory in which Maple is stored (mostly *program files*).

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