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Graph structural properties of non-Yutsis graphs allowing fast recognition

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Abstract

Yutsis graphs are connected simple graphs which can be partitioned into two vertex-induced trees. Cubic Yutsis graphs were introduced by Jaeger as cubic dual Hamiltonian graphs, and these are our main focus.

Cubic Yutsis graphs also appear in the context of the quantum theory of angular momenta, where they are used to generate summation formulae for general recoupling coefficients. Large Yutsis graphs are of interest for benchmarking algorithms which generate these formulae.

In an earlier paper we showed that the decision problem of whether a given cubic graph is Yutsis is NP-complete. We also described a heuristic that was tested on graphs with up to 300,000 vertices and found Yutsis decompositions for all large Yutsis graphs very quickly.

In contrast, no fast technique was known by which a significant fraction of bridgeless non-Yutsis cubic graphs could be shown to be non-Yutsis. One of the contributions of this article is to describe some structural impediments to Yutsisness and to provide experimental evidence that almost all non-Yutsis cubic graphs can be rapidly shown to be non-Yutsis by their application. Combined with the algorithm

described in the earlier paper this gives an algorithm that, according to experimental evidence, runs efficiently on practically every large random cubic graph and can decide on whether the graph is Yutsis or not.

The second contribution of this article is a set of construction techniques for non-Yutsis graphs implying, for example, the existence of 3-connected non-Yutsis cubic graphs of arbitrary girth and with few non-trivial 3-cuts.

Key words: Yutsis graph; dual Hamiltonian graph; decision problem; general recoupling coefficient

1 Introduction

A Yutsis graph is a multigraph in which the vertex set can be partitioned in two parts such that each part induces a tree. Cubic Yutsis graphs appear in the quantum theory of angular momenta as a graphical representation of general recoupling coefficients. They can be manipulated following certain rules in order to generate so-called summation formulae for the general recoupling coefficient. Details can be found in [1–3].

Consider a multigraph $G = (V, E)$ and a subgraph $H \subseteq G$. We will use $V(H)$ to denote the vertex set of H and $E(H)$ to refer to the (multi-)set of edges of H. In case V_1 and V_2 are disjoint subsets of V, $E(V_1, V_2)$ denotes the multiset of edges with one endpoint in V_1 and one in V_2. For disjoint subgraphs G_1, G_2 of G we also write $E(G_1, G_2)$ for $E(V(G_1), V(G_2))$. By $\langle S \rangle$ we denote the subgraph induced by $S \subseteq V$.

A graph is *cyclic* if it contains a cycle. A graph with two vertex-disjoint cycles is *cyclically k-edge-connected* if any edge-cut $E(S, V-S)$ such that $\langle S \rangle$ and $\langle V-S \rangle$ are cyclic contains at least k edges.

Due to the applications in physics and the conjecture by Jaeger [4] that all cyclically 4-edge-connected cubic graphs are Yutsis (which he proves for the planar case), the Yutsis property is especially interesting for simple cubic graphs and results about these are also the main aim of this paper. Nevertheless during some of our constructions, non-cubic graphs and also multigraphs occur and therefore we will define it in a more general context.

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Let Y be a Yutsis graph for which the vertex set can be partitioned into two parts which induce trees B and R. We call such a tree pair (B, R) a *defining tree pair* and each tree of such a pair a *defining tree* for Y. The edge-cut $E(B, R)$ is called a *defining edge-cut* for Y. Note that in general the partitioning of a Yutsis graph into two defining trees is not unique. An example of a Yutsis graph and two defining trees is given in Figure 1. If there are defining trees B, R of the same size, we call the graph *strongly Yutsis*. A simple counting argument shows that regular graphs of degree at least three which are Yutsis are in fact strongly Yutsis.

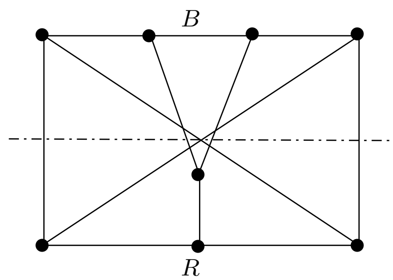

Fig. 1. A Yutsis graph with defining tree pair (B, R) and the corresponding edge-cut $E(B, R)$.

For convenience, we use colours to classify the vertices and edges with respect to a defining tree pair (B, R): we refer to the vertices and edges of B as blue, to those of R as red, and to the edges in the defining edge-cut $E(B, R)$ as white.

In the mathematical literature, simple cubic Yutsis graphs are also known as cubic dual Hamiltonian graphs [4,5].

2 Preliminaries

Let $G = (V, E)$ be a multigraph and $\{V_1, V_2, \ldots, V_k\}$ a partition of V. Then we call $\{\langle V_1\rangle, \langle V_2\rangle, \ldots, \langle V_k\rangle\}$ a *decomposition* of G into k induced subgraphs (induced by the partition $\{V_1, V_2, \ldots, V_k\}$). We call it a *cyclic decomposition* of G if each $\langle V_i\rangle$, $1 \leq i \leq k$, contains a cycle.

By $\overline{V_i}$ we denote the set of vertices $V - V_i$. For a subgraph X of a graph $G = (V, E)$ we define $\overline{X} = \langle V - V(X)\rangle$. An edge in $E(V_i, V_j)$ is called a cross-edge from V_i to V_j, $1 \leq i < j \leq k$.

An edge-cut $E(S, \overline{S})$ is called *cyclic* if $\{\langle S\rangle, \langle\overline{S}\rangle\}$ is a cyclic decomposition of G. Hence, for $k \geq 2$ and $1 \leq i \leq k$, each $E(V_i, \overline{V_i})$ of a cyclic decomposition $\{\langle V_1\rangle, \ldots, \langle V_k\rangle\}$ is a cyclic edge-cut.

Let $G = (V, E)$ be a multigraph and $W = \{W_1, \ldots, W_k\}$ be a decomposition of G. The *contraction graph* $G_W = (V_W, E_W)$ is a multigraph with vertices $W_1, \ldots, W_k$ and $|E(W_i, W_j)|$ edges connecting W_i with W_j, $1 \leq i < j \leq k$. (There are no loops.) We always assume a bijection between the $|E(W_i, W_j)|$ edges $\{W_i, W_j\}$ and the edges in $E(W_i, W_j)$ to be given, so that, by abuse of language, we can speak for example of the colour of an edge in the contraction graph if the edges in the original graph are coloured.

3 Sufficient criteria for being non-Yutsis

The following properties follow directly from the definition of a Yutsis graph and will be used in proofs later on.

Remark 1 *For any defining tree pair (B, R) of a Yutsis graph $G = (V, E)$, it holds that:*

(1) a white edge has endvertices of different colour,
(2) a cycle contains at least two white edges, and thus vertices of both colours,
(3) every cyclic edge-cut $E(S, \overline{S})$ contains at least one blue and one red edge.

We will repeatedly use the fact that every 2-edge-cut in a cubic multigraph is a cyclic edge-cut.

In the following we will present some sufficient criteria for non-Yutsisness.

Lemma 2 *Let $G = (V, E)$ be a multigraph and $W = \{W_1, \ldots, W_k\}$ a cyclic decomposition of G. If G is Yutsis then the contraction graph G_W must contain two connected edge-disjoint spanning subgraphs.*

PROOF. Assume that G is Yutsis with defining tree pair (B, R).

By Remark 1 each cyclic edge-cut $E(W_i, \overline{W_i})$ contains a blue and a red edge. Consequently, every vertex of G_W is incident with at least one blue edge and one red edge in G, so the red and blue subgraphs are spanning and of course edge-disjoint. Moreover, the red and blue subgraphs of G_W are contractions of trees and therefore connected.

The fact that connected spanning subgraphs with k vertices have at least $k-1$ edges gives us a criterion for a graph to be non-Yutsis.

Corollary 3 *Given a multigraph $G = (V, E)$. If there is a cyclic decomposition $W = \{W_1, \ldots, W_k\}$ of G so that the contraction graph G_W contains less than $2k - 2$ edges, then G is not Yutsis.*

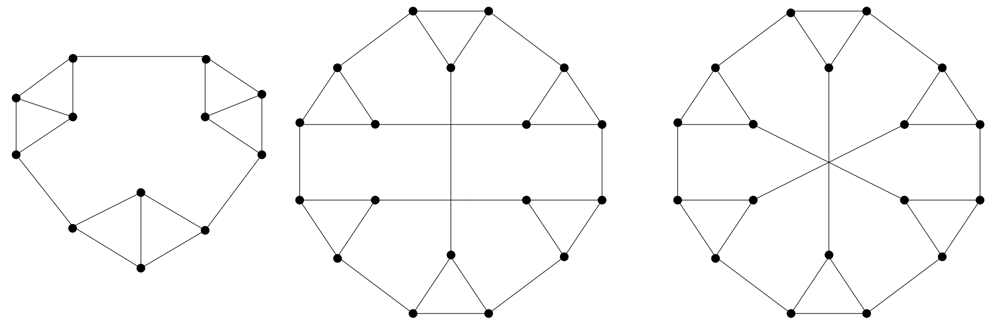

Fig. 2. The smallest simple 2-connected cubic non-Yutsis graph and the smallest simple 3-connected cubic non-Yutsis graphs. The graph on the right is also the smallest simple nonplanar cubic non-Yutsis graph.

Remark 4 *Whenever Lemma 2 is applicable to show non-Yutsisness of a graph G using a decomposition W, Corollary 3 can be applied to show non-Yutsisness using a cyclic decomposition coarser than or equal to W.*

PROOF. This is an easy consequence of a theorem by Tutte and Nash-Williams [6] that a multigraph contains two edge-disjoint spanning trees if and only if every decomposition into k subgraphs contains $2k-2$ cross-edges.

Figure 2 shows the smallest bridgeless simple cubic non-Yutsis graph and the smallest 3-connected cubic non-Yutsis graphs, which are all non-Yutsis by Corollary 3 with $k = 3$ and $k = 6$ respectively. The graph on the right is also the smallest non-planar simple cubic non-Yutsis graph.

The smallest contraction graphs of cubic non-Yutsis graphs described by Corollary 3 are illustrated in Figure 3, for $k = 2, \ldots, 5$. Contraction graphs which can be further contracted to a smaller graph to which the lemma applies are not listed. For example, the 4-cycle is not present in the figure because it can be further contracted to a triangle.

It is possible to relax the upper bound on the number of edges a bit by strengthening the requirement on the elements of the decomposition.

Lemma 5 *Let $G = (V, E)$ be a multigraph, $W = \{W_1, \ldots, W_{k-1}, W_k = X\}$ a cyclic decomposition of G and X non-Yutsis.*

If the contraction graph G_W has no two connected edge-disjoint spanning subgraphs, with at least one of them having a cycle through X, then G is non-Yutsis.

PROOF. Assume that G is Yutsis with defining tree pair (B, R). As in Lemma 2, the edges corresponding to $E(B)$ and $E(R)$ form two connected

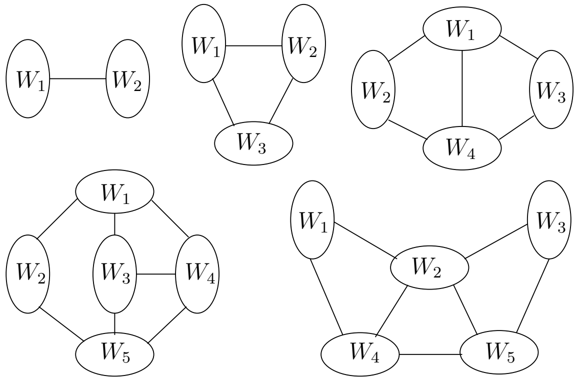

Fig. 3. The possible decomposition structures making graphs non-Yutsis on basis of Corollary 3, for $k = 2, \ldots, 5$. Each ellipse stands for a cyclic subgraph and corresponds to a vertex of the contraction graph.

spanning subgraphs of G_W.

By construction $B' = \langle V(B) \cap V(X) \rangle$ and $R' = \langle V(R) \cap V(X) \rangle$ are induced, cycle-free, and together span X. Since X is non-Yutsis, they cannot be defining trees of a Yutsis decomposition, so at least one of them, say B', is not connected. Let x and y be two vertices from different components of B'. Because B is connected, there is a path P from x to y in B and it must use edges not in B' since x and y are from different components. But these edges must form at least one blue cycle in G_W, in contradiction to the assumption.

Corollary 6 *Let $G = (V, E)$ be a multigraph, $W = \{W_1, \ldots, W_{k-1}, W_k = X\}$ a cyclic decomposition of G and X non-Yutsis.*

If the contraction graph G_W has less than $2k - 1$ edges, then G is non-Yutsis.

Lemma 5 can be used to construct 3-connected cubic non-Yutsis graphs that have a cyclic decomposition as in the lemma such that the role of X is played by a smaller known non-Yutsis graph that need not be 3-connected.

This lemma can also be used to understand the graph structures given in Figure 3: Assuming the first one with a bridge to be non-Yutsis, we get the second one by choosing X as $\langle V(W_1) \cup V(W_2) \rangle$ and analogously for the other structures in that figure. Table 1 in [7] shows that the vast majority of small non-Yutsis graphs are non-Yutsis because they contain a bridge. Lemma 5 implies that a large number of 2-connected and even 3-connected non-Yutsis graphs are non-Yutsis because of a subgraph with a bridge and the rest of the graph not being sufficiently highly connected to repair this deficit.

Remark 7 *A multigraph $G = (V, E)$ can be detected by Lemma 2 as being non-Yutsis if and only if it can be detected by Lemma 5 with an induced non-Yutsis subgraph X that can be divided into two disjoint cyclic subgraphs with a single edge between them.*

PROOF.

First assume a cyclic decomposition $W = \{W_1, \dots, W_{k-1}, W_k\}$ of G is given such that G_W has at most $2k-3$ edges. Because G_W is connected, there is a spanning tree with $k-1$ edges and because there are at most $2k-3$ edges, there are two vertices of G_W, say W_{k-1} and W_k, which are joined by exactly one edge. Now choose $X = \langle V(W_{k-1}) \cup V(W_k)\rangle$. Obviously X is non-Yutsis due to a bridge joining two cyclic parts. Looking at the decomposition $W' = \{W_1, \dots, W_{k-2}, X\}$, the contraction graph $G_{W'}$ has $k' = k-1$ vertices and exactly one edge less than G_W, so at most $2k-4 = 2k'-2$ edges, fulfilling the requirements of Corollary 6 and therefore of Lemma 5.

Now assume that Lemma 5 demonstrates non-Yutsisness with a decomposition $W = \{W_1, \dots, W_{k-1}, W_k = X\}$ such that $V(X) = V(W'_k) \cup V(W'_{k+1})$, where W'_k and W'_{k+1} are disjoint cyclic subgraphs and there is a single cross-edge e between W'_k and W'_{k+1}. Let $W' = \{W_1, \dots, W_{k-1}, W'_k, W'_{k+1}\}$. We will show that Lemma 2 demonstrates non-Yutsisness with decomposition W'. For, if not, consider two connected edge-disjoint spanning subgraphs T_1, T_2 in $G_{W'}$. At least one of them, say T_1, does not use e, so it contains a path from W'_k to W'_{k+1} that avoids e. This means that T_1, T_2, with e contracted, provide connected edge-disjoint spanning subgraphs in G_W, with T_1 having a cycle through X. This contradicts our assumption that Lemma 5 applies.

If Lemma 5 can be applied to show non-Yutsisness, then Corollary 6 can not necessarily be applied with a non-trivial partition. However, such exceptions are covered by Corollary 3:

Remark 8 *If Lemma 5 can be applied to show non-Yutsisness of a multigraph G with X an induced subgraph of G, $X \neq G$, then there is a non-trivial decomposition of G with which Corollary 3 or Corollary 6 can be applied to show non-Yutsisness.*

PROOF.

Assume a decomposition $W = \{W_1, \dots, W_{k-1}, W_k = X\}$ with $k > 1$ given so that G_W has no two connected edge-disjoint spanning subgraphs with at least one of them having a cycle through the non-Yutsis part X, but Corollary 3 cannot be applied.

Remove an arbitrary edge e from G_W incident with X. If $G_W - e$ had two edge-disjoint spanning trees, then, together with e, one of them would form a cycle in G_W. So $G_W - e$ does not have two edge-disjoint spanning trees and therefore there is a coarser nontrivial decomposition $W' = \{W'_1, \dots, W'_{l-1}, W'_l\}$ with less than $2l-2$ crossing edges. Without loss of generality $X \subset W'_l$. Reinserting e

gives rise to a decomposition of G with at most $2l-2$ crossing edges to which Corollary 6 can be applied, unless W'_l is Yutsis.

So assume that W'_l is Yutsis. Because W' is a coarser decomposition than W, for every $S \in W$ we have $S \cap W'_l = \emptyset$ or $S \subseteq W'_l$. Without loss of generality assume that those S with an empty intersection come first, so for some $m \in \mathbb{N}$, $W'' = \{W_1, \ldots, W_m, W'_l\}$ is a cyclic decomposition of G, which by assumption leads to two edge-disjoint spanning trees in $G_{W''}$. Now look at the decomposition $\bar{W} = \{W_{m+1}, \ldots, W_k = X\}$ of W'_l. Because W'_l is Yutsis, this must give rise to two edge disjoint spanning subgraphs of $(W'_l)_{\bar{W}}$ one of which has a cycle through X. But together with the two edge-disjoint spanning trees in $G_{W''}$ these give two edge disjoint spanning subgraphs of G_W one of which has a cycle through X, a contradiction.

Another sufficient criterion for a regular multigraph to be non-Yutsis can be obtained by relaxing the conditions for the subgraph X but strengthening them for the remaining W_i:

Lemma 9

(i) *Let $G = (V, E)$ be an r-regular multigraph, $X \subset G$ an induced subgraph that is not strongly Yutsis. If there exists a decomposition $\{W_1, \ldots, W_k, X\}$ of G such that, for $1 \le i \le k$, W_i is cyclic, and $|E(W_i, \overline{W_i})| = |E(W_i, X)| = 2$, then G is not Yutsis.*

(ii) *If a 2-connected r-regular non-Yutsis graph G has a 2-edge-cut K, then at least one of the components of $G - K$ is not strongly Yutsis. Thus (i) can be applied to demonstrate the non-Yutsisness in this case.*

PROOF.

(i): Assume that G is Yutsis with a defining tree pair (B, R). By Remark 1, there must be two vertices with degree $r-1$ in each of the W_i and they must belong to different trees. By counting outgoing edges it can be easily seen that for each W_i there must be equally many vertices in $W_i \cap B$ and $W_i \cap R$, so there must also be equally many vertices in $X \cap B$ and $X \cap R$.

Due to B, R being defining trees, $X \cap B$ and $X \cap R$ are also induced and they are connected, because there can be no path between two vertices in X through any of the W_i that uses just edges of one colour (because the outgoing edges have different colours).

So $X \cap B$ and $X \cap R$ are defining trees of equal size for X, in contradiction to X not being strongly Yutsis.

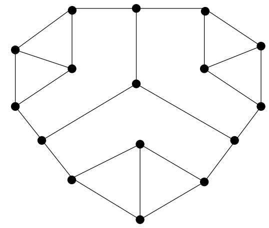

Fig. 4. The smallest non-Yutsis graph for which Lemmas 2 and 5 are not applicable.

(ii): Suppose that a 2-edge-cut $K = \{e_1, e_2\}$ is given and both components $\{C_1, C_2\}$ of $G - K$ are strongly Yutsis. Let B, R be trees that form a strongly Yutsis decomposition of C_1. Counting the number w of white edges leaving B we get (with D_B the number of edges of K incident with vertices of B): $w = r|B| - 2(|B|-1) - D_B = (r-2)|B| + 2 - D_B$ and analogously for the red tree R: $w = (r-2)|R| + 2 - D_R$. Since $|B| = |R|$ and $D_B + D_R = 2$, this implies that $D_B = D_R = 1$. Of course the same can be done for C_2, so in both parts the endpoints of the cut belong to different trees and therefore the two strongly Yutsis decompositions of the parts can be extended to form a Yutsis decomposition of G, a contradiction.

Useful applications of this lemma occur in cases where X is pretty small and has a simple structure, so that it is easy to show that it is not strongly Yutsis. So, in contrast to part (ii) of Lemma 9, in general the decomposition will have much more than just two parts. The smallest simple cubic graph where Lemmas 2 and 5 cannot be applied is given in Figure 4. With X the graph induced by its centre vertex together with its neighbours, Lemma 9 proves that this graph is non-Yutsis.

Lemma 9 can also be used for the construction of cubic non-Yutsis graphs: Define a tree $T = (V, E)$ to be *edge central* if it contains an edge e so that $T - e$ has two components of equal size. A tree with at least two vertices is strongly Yutsis if and only if it is edge-central. Now take a tree that is not edge-central and has maximum degree 3 and an even number n of vertices. Furthermore take $(n+2)/2$ simple cubic graphs, split an edge in each of them and iteratively connect the two half-edges to vertices of the tree that still have a degree less than 3. The result is a simple cubic non-Yutsis graph. The graph in Figure 4 can be constructed this way by taking the graph $K_{1,3}$ as the tree and the cubic graphs as disjoint copies of K_4.

We will use Lemma 2 to define what we call a *blow-up construction*. This construction can be defined in a more general way, but we will describe it for the most important case of cubic graphs only, in order to allow simpler proofs.

Let G be a connected cubic multigraph with vertices $g_1, \ldots, g_n$, and let G_1, $\ldots, G_n$ be cubic graphs, with each G_i having a distinguished vertex v_i. For each i, label the neighbours of v_i as $w_{i,1}, w_{i,2}, w_{i,3}$ in any order. Now take the disjoint union H of the graphs $G_1, \ldots, G_n$ with the vertices $v_1, \ldots, v_n$ removed. For each edge $\{g_i, g_j\}$ of G, add an edge of the form $\{w_{i,k}, w_{j,l}\}$ to H, in such a way that the final graph H is cubic. We denote such a graph by $B(G, G_1, \ldots, G_n)$ and say that $\{g_i, g_j\}$ *corresponds to* $\{w_{i,k}, w_{j,l}\}$. Note that G is the contraction graph of $B(G, G_1, \ldots, G_n)$ associated with the decomposition $W = \{G_1 - v_1, \ldots, G_n - v_n\}$.

Remark 10 *Given 3-edge-connected cubic multigraphs $G, G_1, \ldots, G_n$.*

(i) *The girth of $B(G, G_1, \ldots, G_n)$ is at least as large as the minimum of the girths of $G_1, \ldots, G_n$.*
(ii) *If all the graphs G_i are cyclically 4-edge-connected then $B(G, G_1, \ldots, G_n)$ is 3-connected. Moreover, the non-trivial 3-edge-cuts of $B(G, G_1, \ldots, G_n)$ are precisely the sets of edges which correspond to the 3-edge-cuts of G, trivial or not.*
(iii) *If $|V(G)| \geq 6$ and $G_1 - v_1, \ldots, G_n - v_n$ are cyclic, then $B(G, G_1, \ldots, G_n)$ is non-Yutsis.*

PROOF.

(i): Cycles that do not use newly inserted edges are already present in one of the G_i while cycles C using new edges must use two edges leaving each $G_i - v_i$ that contain vertices of C. But the path between the endpoints of these edges in G_i together with two edges leading to v_i form a cycle in G_i that is at least as short as C.

(ii): Let C be a minimal non-trivial edge-cut of $B(G, G_1, \ldots, G_n)$. Since $B(G, G_1, \ldots, G_n)$ is cubic, the minimality implies that C consists of at most 3 mutually non-adjacent edges. The structure of $B(G, G_1, \ldots, G_n)$ implies either that $G - C$ is disconnected (so C is a cut of G), or that one of the subgraphs $G_i - v_i - C$ is disconnected. In the latter case we must have that $|C| = 3$ and $C \subset E(G_i - v_i)$, since a cyclically 4-edge-connected cubic graph cannot be disconnected by deleting a vertex and two non-adjacent edges. Moreover, each component of $G_i - v_i - C$ contains one of the vertices $w_{i,k}$ since otherwise it would be a component of $G_i - C$ (violating the cyclic 4-edge-connectivity of G_i). This, together with the fact that $G - g_i$ is connected, implies that $B(G, G_1, \ldots, G_n) - C$ is connected, contradicting that C is a cut.

Finally, the edges corresponding to a 3-edge-cut of G form a non-trival 3-edge-cut of $B(G, G_1, \ldots, G_n)$.

(iii): A cubic multigraph G with $n \geq 6$ vertices has $3n/2 < 2n - 2$ edges. Therefore Corollary 3 shows that $B(G, G_1, \ldots, G_n)$ is non-Yutsis due to the cyclic decomposition $\{(G_1 - v_1), \ldots, (G_n - v_n)\}$.

The fact that $K_{3,3}$ does not have nontrivial 3-edge-cuts gives us:

Theorem 11 *For every $g \geq 3$, there exist cubic 3-connected non-Yutsis graphs with girth at least g. In fact the graphs can even be constructed in a way that they contain only 6 nontrivial 3-cuts.*

PROOF.

This follows from Remark 10 applied with $G = K_{3,3}$ and $G_1, \ldots, G_6$ arbitrary cubic graphs with girth and cyclic connectivity g. Such cubic graphs exist due to results by Wormald, Nedela and Skoviera, see [8] and [9].

For cubic polyhedra, that is, planar cubic 3-connected graphs, every possible girth can also be achieved by a non-Yutsis graph.

Theorem 12 *For every $g \in \{3, 4, 5\}$ there exist cubic non-Yutsis polyhedra with girth at least g. In fact the graphs can even be constructed in a way that they contain only 7 nontrivial 3-cuts.*

PROOF.

In this case we can choose the planar cubic graph with 6 vertices (the trigonal prism) as G and the tetrahedron, the cube or the dodecahedron as the G_i. We only have to make sure that the new edges are chosen in such a way that the new graph is still planar.

4 A fast heuristic for bridgeless cubic non-Yutsis graphs

Deciding whether a given cubic graph is Yutsis is an NP-complete problem [7]. In [7] we developed a fast randomized heuristic algorithm able to recognize simple cubic Yutsis graphs by finding a Yutsis decomposition with some probability. By repeated applications (which we will call "runs") of the heuristic, the probability that a Yutsis graph is recognised as Yutsis is improved. Each run requires time $O(n^2)$ for general cubic graphs and $O(n)$ for plane graphs.

This heuristic recognized more than 99.9% of the small Yutsis cubic graphs (up to 30 vertices) in less than 10 runs, and all of the tested large random cubic Yutsis graphs in very few runs on average. For 300,000 vertices the average number of runs required to find a decomposition was 1.16.

Except in trivial cases, the heuristic in [7] cannot certify that a graph is non-Yutsis. This is the deficiency which we aim to correct in the present paper. Of course non-Yutsis graphs with bridges can be recognized in linear time, but no polynomial time method was known to recognize a significant fraction of the cubic 2-connected non-Yutsis graphs.

The concept of a *cyclic* decomposition used in the previous section was just a means of guaranteeing that the subgraph induced by each part of the composition must contain vertices of both trees of any Yutsis decomposition. Since the algorithm we will describe includes other means of making this guarantee, we adopt the following definition.

Definition 13 *During the execution of the algorithm, a subgraph of a cubic graph is called* validated *if it has been determined to contain vertices from both parts of every Yutsis decomposition of the cubic graph (if such exist). Subgraphs which are not validated may or may not have this property.*

We will also say that an edge-cut C of a cubic graph G is validated *if all the components of $G - C$ are validated.*

Initially all cyclic subgraphs are implicitly validated, on account of Remark 1(2). The rationale for validation is that Corollary 3 is true in greater generality, with the same proof:

Lemma 14 *Given a multigraph $G = (V, E)$. If there is a decomposition $W = \{W_1, \ldots, W_k\}$ of G into validated subgraphs so that the contraction graph G_W contains less than $2k - 2$ edges, then G is not Yutsis.*

The principal idea of the algorithm is as described in the following pseudocode for the function **test** which is initially applied to the cubic graph being tested. The algorithm constructs decompositions by dividing the graph into validated parts until Lemma 14 can be applied.

There are two global variables `components` and `edges` that keep track of the number of parts and cross-edges for the current decomposition.

Global variable initialization:
`components` $= 1$
`edges` $= 0$

Function **test**(C : connected graph)

a: If `edges` $<2 \times$ `components` -2, return *non-Yutsis*.
b: If C has no validated 1- 2- or 3-edge cut that splits the graph into two connected components, return *undecided*.
c: Choose a random smallest validated 1- 2- or 3-edge cut K that splits C into two connected components and let C_1, C_2 be the two validated connected components of $C - K$.
d: Increase `components` by one and `edges` by the number of edges in K.
e: Run test(C_1). If the result is *non-Yutsis* return *non-Yutsis*.
f: Run test(C_2). If the result is *non-Yutsis* return *non-Yutsis*.
g: Return *undecided*.

Due to Lemma 14, whenever the heuristic returns *non-Yutsis*, this is in fact the case, because a decomposition has been found that proves it.

A first obvious improvement is to remember the values of the global variables `edges` and `components` before testing C_1 and to restore them if the recursive call increased `edges` $-2 \times$ `components`; this corresponds to considering C_1 as not split. Similarly for C_2.

A much more important improvement in performance of the algorithm can be achieved by accumulating knowledge about colour relations between vertices. In many cases we can compute that some edges or vertices must have the same colour in any possible Yutsis decomposition. Clearly, colour relations between edges can be expressed by colour relations between their endvertices. We can express these local colour relations by using a family a_i, a_{-i}, $i \in \mathbb{N}$ of variables to label the vertices. The possible values for the variables are B which stands for *blue tree* and R for *red tree*. The relation between a_i and a_{-i} is that $\{a_i, a_{-i}\} = \{R, B\}$, so $a_i \neq a_{-i}$, but we never know which variable is R and which is B. All vertices marked a_i must be in the same tree and all vertices marked a_{-i} must be in the other tree. Colour relations which are disjoint from each other are expressed by variables with a different index. On the other hand, colour relations may be found to overlap, in which case they either contradict each other (proving that the graph is not Yutsis) or are compatible and can be merged into a single colour relation.

Colour relations give us an additional criterion for validating a subgraph, namely that it contains vertices labelled both a_i and a_{-i} for some i. The algorithm develops the colour relations using the following observations.

Lemma 15 *Suppose that procedure* **test** *is being applied to a subgraph C, and* *let K be the validated edge cut chosen by* **test**.

(i) *If* `edges` $= 2 \times$ `components` $- 2$, *then*

(a) *if K is a 2-edge-cut, then the two edges in K are in different defining trees for every Yutsis partition;*

(b) *if K is a bridge for which one component of $C - K$ is validated and the other not, then all vertices in the non-validated component of $C - K$ and the endpoints of the bridge belong to the same tree for every Yutsis partition.*

(ii) *If* `edges` $= 2 \times$ `components` $- 1$ *and K is a bridge, then K is in one of the trees for every pair of defining trees.*

PROOF. Edge counting arguments give in (i) that the contraction graph cannot have two edge-disjoint connected spanning subgraphs so that one has a cycle through any of the components (especially not through C). So for every defining tree pair (B, R), $B \cap C$ and $R \cap C$ must be connected. In case (a) each tree must contain vertices from both components of $C - K$, which gives the result. In case (b) there must be a path with vertices from just one tree inside C from the non-validated component to the validated one. So the whole bridge must belong to this tree and also all vertices in the non-validated component due to the connectedness of the other tree.

Note that in case (i,b), the bridge is not used to refine the decomposition, but only to derive colour relations. If a validated bridge is found under the same conditions, the algorithm will stop and return *non-Yutsis*.

In case (ii) one connected spanning subgraph of the contraction graph can have a cycle through C, but not both of them, so the bridge must be contained in one of the trees for every defining tree pair.

We use this result as follows. Whenever (i,a) is applicable, we mark the two endpoints of one of the edges a_i and of the other a_{-i} with a new value of i. In case (i,b) is applicable we mark the endpoints of the bridge and the vertices inside the non-validated component a_i. In case of (ii) we mark both the endpoints of the bridge a_i. As mentioned above, the labelling may cause a contradiction between two colour relations, or may lead to compatible overlapping colour relations being merged together.

In Figure 5 we give an example of how the algorithm works. The example chosen is the graph from Figure 4. The graph in the figure cannot be proved to be non-Yutsis using Lemma 2. In fact it is not difficult to show that all graphs that can be constructed by the method described after Lemma 9, using trees that are not edge central, can be detected.

It should be noted that whether or not a graph is detected can depend on the (random) order in which the cuts are chosen. An example of an order that leads to the detection of non-Yutsisness and another order that does not, is given in Figure 6.

An important observation is that the computed colour relations are intrinsic properties of the graph that hold regardless of the sequence of cuts that led to finding them. This means that we can keep the labels between successive runs of the algorithm on the same graph, so later runs can take advantage of the knowledge developed by earlier inconclusive runs.

Furthermore it happens during the labelling process that the two ends of an edge e are labelled a_j and a_{-j} for the same j. This means that in no possible Yutsis decomposition of G, the edge e can belong to one of the trees. So if $G-e$ has no Yutsis decomposition, G also has none. In fact we can even require strong Yutsis decompositions because the initial graph is 3-regular. So after each run we remove such edges and start the next run on the smaller graph, which is likely to have new small edge-cuts, thus enlarging the probability of finding new colour relations and/or a validated decomposition proving non-Yutsisness. Removing such edges during a run would allow an earlier detection of labelling possibilities but would result in a more difficult and error sensitive implementation, so we did not choose that option.

Having removed these edges, some easy checks can be performed on the graphs. For each label x, let V_x be the set of vertices labelled x and assume without losing generality that x represents B. Then G is non-Yutsis if any one of the following cases occurs:

(1) $\langle V_x \rangle$ is cyclic.
(2) $G - V_x$ has two validated components: this would mean that the red vertices induce a disconnected subgraph.
(3) $G - V_x$ has a validated component C, with $|C| \leq |V|/2$: this would imply that R must be completely contained in C together with at least one vertex of B. This is only possible if the graph is not strongly Yutsis and thus non-Yutsis because we started with a cubic graph.
(4) $G - V_x$ contains a component C, with $|C| < |V|/2$ but $|C \cup V_x| > |V|/2$: because both trees are connected, C must either be completely blue or completely red which always implies that the graph is not strongly Yutsis and thus non-Yutsis.

These tests detected some graphs that were otherwise not detected, but the number of such graphs was very small relative to the number detected by the main routines.

The labelling routine is not only important in increasing the number of recognizable graphs, it also decreases the number of runs necessary to prove a graph non-Yutsis. For 24 vertices we ran the algorithm on all bridgeless non-Yutsis graphs[1] and applied the heuristic up to 800 times. The final version of the algorithm using labels detected more than 99.989% of the non-Yutsis graphs with an average of 1.0016 runs for the graphs that were detected. Without labelling only 97.42% of the graphs were detected and on average 1.36 runs were necessary. For further results on the performance of the algorithm, see Table 1 and Table 2. There are additional techniques that can improve the performance still further, but because the amount of improvement is very small we will not describe them. The numbers in the tables were computed by programs that implement the methods we have described.

Our implementation was intended as a proof of concept and not as realising an asymptotically optimal algorithm. It uses straightforward methods to detect validated cuts and has a theoretical upper bound of $O(n^5)$ for the running time. It nevertheless ran very quickly even on large graphs. The reason is that, for all the large graphs tested (unlike for the complete lists of small graphs), no 3-cuts and only a small number of 2-cuts plus one final 1-cut were required in order to establish that the graphs were not Yutsis. For 150 vertices, 95% of the 2-connected non-Yutsis graphs were detected after removing a single 2-cut (so the contraction graph was a triangle) and no more than four 2-cuts were ever used. For 250 vertices this improved further to 97% being detected after removing a single 2-cut and at most three 2-cuts were required.

For planar graphs combining the duality between hamiltonian triangulations and Yutsis polyhedra observed by Jaeger [4] and results about the asymptotic ratio of hamiltonian triangulations by Richmond, Robinson and Wormald [10,11] proves that asymptotically all cubic polyhedra are non-Yutsis, the computational results suggest that for all cubic graphs the ratio of Yutsis graphs quickly converges towards 1. This made it difficult to find many random non-Yutsis graphs for large vertex numbers. Already for 250 vertices the ratio was only 0.0015% and among them 99.4% had a bridge, so that out of 27,499,965,781 randomly generated connected cubic graphs only 2,381 were 2-connected and non-Yutsis. For more than 250 vertices these statistics become even worse, so it presently appears very difficult to test our heuristic properly for such sizes.

The programs were once implemented in `C` and once in `C++` and all the results given were checked independently by both programs. In fact it was even checked whether the set of non-detectable graphs was the same after a sufficiently large number of iterations. Furthermore we tested the programs by checking whether no Yutsis graphs with up to 26 vertices were erroneously

[1] We determined the non-Yutsis graphs by an exhaustive search method with intensive pruning described in [7], which is much slower on non-Yutsis graphs than the heuristic described here.

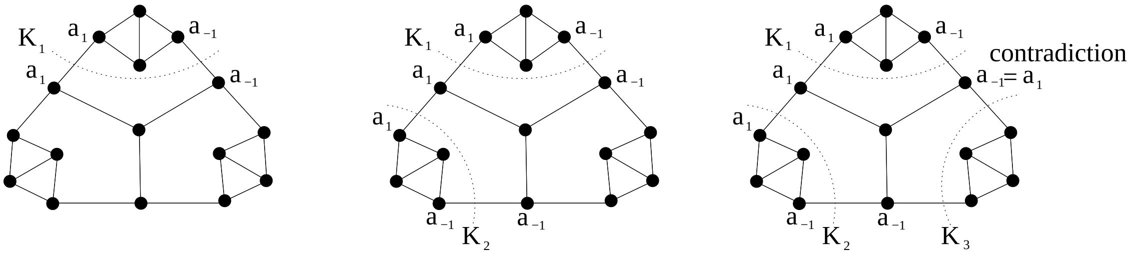

Fig. 5. An illustration of the algorithm

number of vertices	non-Yutsis graphs	with bridge (ratio of non-Yutsis graphs)	ratio of 2-conn. non-Yutsis graphs recognized after 1 application	ratio of 2-conn. n-Y graphs rec. after at most 800 applic. and avg. number of applic.
10	1	1 (100%)		
12	5	4 (80.0%)	100%	100%, 1
14	34	29 (85.3%)	100%	100%, 1
16	224	186 (83.0%)	100%	100%, 1
18	1746	1435 (82.2%)	100%	100%, 1
20	15444	12671 (82.0%)	>99.96%	100%, 1.0004
22	159751	131820 (82.5%)	>99.92%	>99.99%, 1.0012
24	1900079	1590900 (83.7%)	>99.90%	>99.98%, 1.0016
26	25698855	21940512 (85.4%)	>99.85%	>99.98%, 1.0028
28	389715827	339723835 (87.1%)	>99.82%	>99.97%, 1.0037
30	6549111460	5821548438 (88.9%)	>99.79%	>99.97%, 1.0045

Table 1
The ratios of small non-Yutsis graphs recognized by one and by up to 800 applications of the heuristic with a reset of the labels after every 50 applications and a reinsertion of the deleted edges after 250 iterations. The tests were run on complete lists of cubic graphs generated by *minibaum*; see [12].

number of vertices	number of graphs tested	non-Yutsis graphs	with bridge (ratio of non-Yutsis graphs)	ratio of 2-conn non-Yutsis graphs recognized after 1 application	ratio of 2-conn. n-Y graphs rec. after at most 800 applic. and avg. number of applic.
50	27499003875	13877865 (0.05%)	13310579 (96%)	>99.95%	>99.99%, <1,0013
100	27499776389	2862811 (0.01%)	2817068 (98.4%)	>99.99%	100%, <1,0001
150	27499903748	1203722(0.004%)	1191899 (99%)	100%	100%, 1
200	27499946319	659402 (0.0024%)	654532 (99.3%)	100%	100%, 1
250	27499965781	415559 (0.0015%)	413178 (99.4%)	100%	100%, 1

Table 2
The ratios of random large non-Yutsis graphs recognized by one and by up to 800 applications of the heuristic with a reset of the labels after every 50 applications. The tests were run on lists of random cubic graphs generated with equal probability for every labelled graph by *genrang*; see [13].

detected as being non-Yutsis. Because of the large number of graphs to be tested, for up to 20 vertices the heuristic was applied 50 times and from 22 to 26 vertices only 5 times per graph. For 24 and 26 vertices this last test was done for only one of the two implementations.

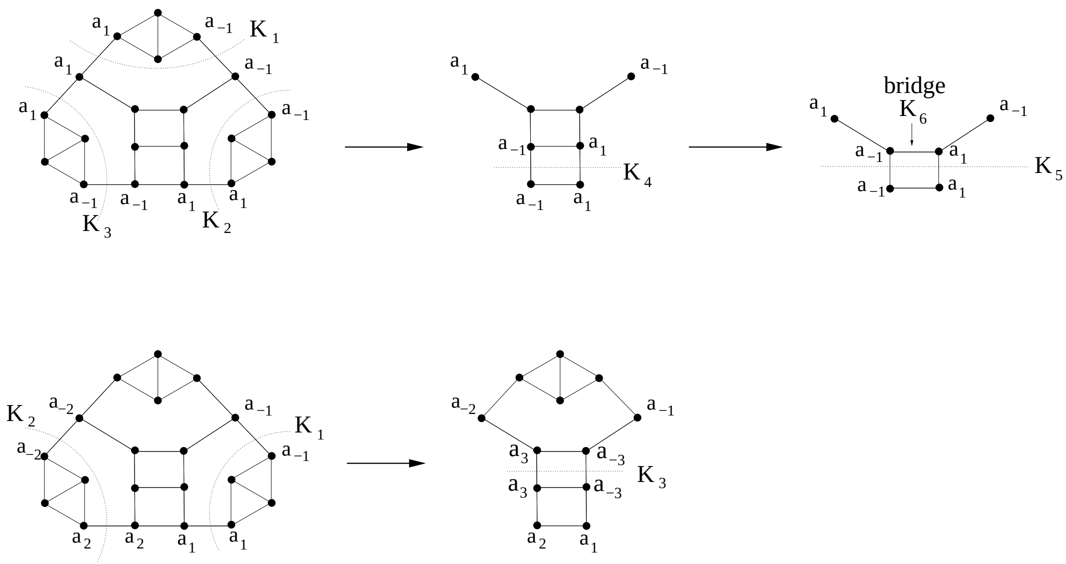

Fig. 6. An example where one sequence of choices of cuts leads to detecting non-Yutsisness and the other does not.

5 Conclusions

Although the problem of deciding whether a cubic graph is Yutsis is NP-complete, fast heuristics exist for recognizing the vast majority of both Yutsis [7] and non-Yutsis graphs. The algorithm described in this article together with the one described in [7] give a heuristic that is practically sure to be able to give a decisive answer if applied to large random cubic graphs. The key to the fast heuristic algorithm developed in this article is the application of the simple Lemma 14 combined with the detection of contradictory label relations between vertices.

Furthermore the lemmas given can be used to construct simple cubic non-Yutsis graphs and cubic non-Yutsis graphs with arbitrarily large girth and to make 3-connected cubic non-Yutsis graphs out of smaller 1- or 2-edge-connected examples.

All our constructions imply the existence of at least a few nontrivial 1- 2- or 3-edge cuts, so they do not provide counterexamples to Jaeger's interesting conjecture. However, we have shown that the conjecture cannot be significantly strengthened.

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