"Ce que nous évitons de reconnaître en nous-mêmes, nous
le rencontrons plus tard sous la forme du destin".

Carl Gustav Jung

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Chapter 1

Introduction

Stochastic processes serve as mathematical models of random phenomena in time evolution. Lévy processes are stochastic processes with stationary independent increments. They constitute a fundamental class of stochastic processes which possess nice properties and, thus, proved to find a wide range of applications in different areas such as risk theory and finance, physics, biosciences and telecommunications, queueing theory, fragmentation theory etc. For instance, Lévy processes are an excellent tool for modeling price processes in mathematical finance. This fact emphasizes the importance of studying Lévy processes, their path properties and various characteristics. Also the asymptotic behavior of the Lévy process and its important functionals is an important topic. On the other hand, the Lévy processes are important from a theoretical point of view, since other classes of stochastic processes, such as semi-Markov processes, semi-martingales are obtained as generalizations of Lévy processes.

Theory of Lévy processes goes back to the 1920s, and mainly originated from Lévy and Khintchine, when the key stones of modern probability were laid. Their general structure has been gradually discovered by de Finetti, Kolmogorov, Lévy, Khintchine. Lévy processes were named after a famous French mathematician Paul Lévy who himself called them a sub-class of *processus ad-*

ditifs (additive processes). After P. Lévy's characterization in the 1930s of all processes in this class, many researchers have studied properties of the distributions and behavior of sample functions. Later, the fluctuation theory for random walks was developed mainly by Spitzer (1964), Wendel (1960), Feller (1966), and then later for random processes (Borovkov (1972), Rogozin (1966), Fristedt and Pruitt (1971), Fristedt (1974)). More recent work on this topic is Alili and Chaumont (2001). The main weakness of fluctuation identities for general Lévy processes is that they involve the one-dimensional distributions of the process which are seldom explicitly known. In the case of absence of negative (positive) jumps (a so-called spectrally one-sided Lévy process) these identities can be expressed directly in terms of the root of the characteristic equation, involving the Laplace exponent of the process and the scale functions. Excursion theory, use of local times were further developed by Greenwood and Pitman (1980), Doney (2004), Bertoin (1996a), Kesten and Maller (1998). Excellent monographs on Lévy processes are due to Gihman and Skorokhod (1973), Bertoin (1996a), Kyprianou (2006), Sato (1999).

One of the most popular and most applicable problems in the study of Lévy processes is the so called boundary problem (or one- and two-sided exit problem). It is a problem of determining the law of the first passage of a level (first exit time from a fixed interval) by the process and other characteristics such as position of the process at this instant, the value of the overshoot through the level, sojourn time inside the interval, number of intersections of the interval, etc. Motivated by needs of applied sciences, first passage problems and exit problems became a topic of many researchers. These problems were considered starting from Zolotarev (1964), Emery (1973), Suprun (1976) and others. For evaluation of the first passage times for spectrally one-sided Lévy processes see Rogers (2000). The first passage problem in the context of ruin theory, insurance and dam theory has found a lot of applications (see for instance Chiu and Yin (2005), Klueppelberg *et al.* (2004), Klueppelberg and Kyprianou (2006), Avram *et al.* (2009), Huzak *et al.* (2004), Yang and Zhang (2001), Patie (2004), Avram and Usabel (2003)), options pricing (Madan and Schoutens (2007), Mordecki (2002), Chesney and Jeanblanc (2004), Boyarchenko and Levendorskii (2002), Avram *et al.* (2002), Kou *et al.* (2003), Chesney and Jeanblanc (2004)). Asmussen *et al.* (2004) considered first passage problems for phase-type Lévy processes in the context of pricing American options, and also in context of equity default swaps for CGMY Lévy process, see Asmussen *et al.* (2008). For an overview of the applications of Lévy processes in finance we refer to an excellent monograph Schoutens (2003), see also Schoutens and Symens (2003), Cont and Tankov (2004), Schoutens (2006). Later on, one-boundary characteristics were generalized to a more complicated problem, i.e. a two-boundary problem. A lot of significant contributions were made by Doney, Bertoin, Kyprianou and many others. We particulary refer to a very comprehensive book by Kyprianou (2006) on Lévy processes. Two-boundary problems for spectrally one-sided Lévy processes were considered in Bertoin (1996b), (1997), Lambert (2000), see also Doney (2005), Avram *et al.* (2004), Perry *et al.* (2005). Many of these authors proposed new methods for the two-sided exit problem based on martingale approach and the Ito excursions. Analytical properties of the scale function were further studied in Doney (2005), Chan *et al.* (2007), Biffis and Kyprianou (2008). Applications of scale functions in potential analysis of subordinators were presented in Kyprianou and Rivero (2008) and Hubalek and Kyprianou (2007), while Kyprianou *et al.* (2008b) deduced possible applications in control theory. Rogers Rogers (2000) and Surya (2008) provided robust methods for numerical computation of scale functions. Zhou (2005) made an important contribution from the general point of view of Lévy insurance risk processes and introduced the use of so-called scale functions in his analysis of the Gerber-Shiu function. Following Zhou (2005), Biffis and Morales (2008) provided an explicit characterization of a generalized version of the Gerber-Shiu function in terms of scale functions. Generalization of the scale functions for Markov additive processes was considered in Gerber *et al.* (2006).

Scale functions have also found applications in risk insurance, more specifically, in optimal barrier strategies (see: Zhou (2005), Renaud and Zhou (2007), Kyprianou and Palmowski (2007) Albrecher *et al.* (2008), Loeffen

(2008) Kyprianou *et al.* (2008b) Kyprianou and Loeffen (2008). In the context of last ruin times Chiu and Yin (2005) and Baurdoux (2008) also make extensive use of the theory of scale functions. Moreover, generalized analogues of scale functions appear in the first passage problem of a Markov additive process and positive self-similar Markov processes. (see Breuer (2008), Kyprianou *et al.* (2008a), Chaumont *et al.* (2008). Motivated by classical considerations from risk theory, Kyprianou and Loeffen (2008) investigated boundary crossing problems for refracted Lévy processes. The latter is a Lévy process whose dynamics change by subtracting off a fixed linear drift whenever the aggregate process is above a pre-specified level.

For applications of Lévy processes in biosciences, chemistry and theory of branching processes we refer to Le Gall and Le Jan (1998) and Pakes (1996). Another interesting part of the theory of Lévy processes is a study of reflected processes. Stochastic processes with two absorbing or reflecting barriers occur in sequential analysis, queueing theory, insurance risk, mathematical finance and other applications areas (see Bertoin (1997)). For example, the optimal time to exercise a Russian option is the first time that the reflected process crosses a fixed level (see Pistorius (2003), Pistorius (2004)). In the context of dams, for fluid queues with on-off inputs, the reflected Lévy processes were studied in Dube *et al.* (2004), and in the context of queueing models using martingale techniques in Kella *et al.* (2006). Hansen (2006) studied the maximum of the reflected random walk and considered application in structural biology. For applications in sequential analysis we refer to Karlin and Dembo (1992).

Solving optimal stopping problems driven by Lévy processes has been a challenging task and has found many applications in modern theory of mathematical finance.

Asymptotic analysis of the distributions of the boundary functionals of the process constitutes one of the most difficult parts of the theory of Lévy processes. The asymptotic expansions of the distributions of the two-boundary functionals of random walks satisfying the Cramer's condition were derived in Lotov (1979a), Lotov (1979b). Analysis of the asymptotic properties of the two-boundary characteristics of Lévy processes and random walks was made in Lotov and Khodzhibaev (1984), Lotov and Khodzhibayev (1993), Lotov and Khodzhibaev (1998a), Lotov and Khodzhibaev (1998b). Recently Lotov and Orlova (2004), Lotov and Orlova (2005) studied asymptotic behaviour of the two-boundary functionals for random walks defined on a Markov chain.

In the present work, we address both two-sided exit problems and the asymptotic analysis for various two-boundary characteristics of general Lévy processes and the difference of two compound renewal processes. The methodology we use is mainly based on a probabilistic approach, use of one-boundary characteristics of the process and theory of Fredholm equations of the second kind. This approach appears to be quite universal: it works for general Lévy processes, general random walks, and even for certain semi-Markov processes. The solution of most problems is given in the form of a Neumann series. For the special case of Lévy processes, results are given in closed form, namely in terms of the scale function of the process. More details on our results are given in the next section.

This thesis consists of five self-contained parts, and it is structured as follows. Chapter 2, consisting of 7 sections, is devoted to the study of various two-boundary characteristics of Lévy processes. For this class of stochastic processes we determine the Laplace transforms of several functionals connected with the exit from a fixed interval. The results presented in Chapter 2 can be found in the following published articles.

> Kadankova (2003a). On the distribution of the number of the intersections of a fixed interval by the semi-continuous process with independent increments. *Theor. of Stoch. Proc.,* 1-2, 73-81.
>
> Kadankova (2003b). Two-boundary problems for random walks with negative jumps which have geometrical distribution. *Theor. Prob. and Math. Statist.,* 68, 60-67.
>
> Kadankova (2004). On the joint distribution of supremum, infimum and the magnitude of a process with independent increments *Theor. Prob.*

and Math. Statist., 70, 61-70.

Kadankov and Kadankova (2004). On the distribution of duration of stay in an interval of the semi-continuous process with independent increments. Random Operators and Stochastic Equations, 2004 , 12(4) , 365-388

Kadankov and Kadankova (2005a). Intersections of an interval by a process with independent increments *Theor. of Stoch. Proc.,* 11(1-2), 54-68.

Kadankova and Veraverbeke (2005). Several two-boundary problems for Lévy processes. *Proceedings of the 4-th Actuarial and Financial Mathematics Day.* Ed. A. De Schepper et al, KVAB, Brussels, 97–106.

Kadankov and Kadankova (2005b). On the distribution of the first exit time from an interval and the value of overshoot through the borders for processes with independent increments and random walks. *Ukr. Math. J.,* 10(57), 1359-1384.

Kadankova and Veraverbeke (2007). On several two-boundary problems for a particular class of Lévy processes. *J. Theor. Probab.,* 20(4), 1073-1085.

Anderluch and Kadankova (2008). Double-sided knock-in calls in an exponential compound Poisson framework. *Technical report TR08007.*

It appears that the method introduced for Lévy processes can be applied for another class of stochastic processes, namely for a difference of the compound renewal processes and semi-Markov random walks with drift. In Chapter 3 we study several two-boundary characteristics for the difference of a compound Poisson process and a compound renewal process. The results presented rely on the following papers.

Kadankov and Kadankova (2008c). A two-sided exit problem for a difference of a compound Poisson process and a compound renewal process with a discrete phase space. *Stoch. Models*, **24(1)**, 152-172.

Kadankov, Kadankova and Veraverbeke (2009). Intersections of an interval by a difference of a compound Poisson process and a compound renewal process *Stochastic Models*, **25(2)**, 270-300.

Kadankov and Kadankova (2007). Two-boundary problems for a semi-Markov walk with a linear drift. *Random Oper. and Stoch. Equ.* **15**, 223-251.

Chapter 4 is concerned with possible applications in queueing theory. Queueing systems with batch arrivals and finite buffer have wide applications in the performance evaluation, telecommunications, and manufacturing systems. One of the crucial performance issues of the single-server queue with finite buffer room) is losses, namely, customers (packets, cells, jobs) that were not allowed to enter the system due to the buffer overflow. This issue is especially important in the analysis of telecommunication networks. Motivated by this fact, we derived the most important performance measurements of several queueing systems of this type. More precisely, we consider the $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ and $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ queueing systems with finite waiting room (see a rigorous description of such systems in Chapter 4) and their modifications. For these systems we study their main characteristics such as the busy period, the time of the first loss of the customer, virtual waiting time and the number of customers in the system at arbitrary time. The results presented can be found on the following papers.

Kadankov and Kadankova (2008a). Busy period, time of the first loss of a customer and the number of the customers in $M^{\varkappa}|G^{\delta}|1|B$ system *Queueing Systems*, (submitted).

Kadankov and Kadankova (2008b). Busy period, virtual waiting time and number of the customers in $G^{\delta}|M^{\varkappa}|1|B$ system. *Queueing Systems*, (submitted).

Kadankova and Veraverbeke (2008). Exit problems for an oscillating compound Poisson process. *J.Theor. Probab.* (submitted).

Finally, we discuss the results and open problems in Chapter 5.

Chapter 2

Two-boundary problems for Lévy processes

2.1 Overview

In this chapter we will solve several two-boundary problems for a Lévy process whose Laplace exponent is of the general form (2.2.1). We first determine the joint distribution of the first exit time and the value of the overshoot through the boundary. Employing this distribution, we derive the Laplace transform of the joint distribution of the supremum, infimum and the position of the process. Next we find the joint distribution of the number of the upward and downward intersections of the interval. We then apply the results obtained to particular classes of Lévy processes, namely for the spectrally one-sided Lévy process (2.2.7), the compound Poisson process with jumps of both signs (2.2.13) and for the Wiener process.

We will also study the distribution of the total stay time inside and outside the interval, which in our opinion is one of the most difficult two-boundary characteristic of the process. This functional plays an important role in finance applications, more specific in pricing corridor and hurdle options. Other applications are, as suggested in Taleb (1997), to the management of a portfolio for the computation of the expected amount of time a trader will spend in the red. A similar problem for the standard Brownian excursion can be found in chemistry as well, and in particular, in the theory of ring polymers, as explained in Jansons (1997). Fusai (2000) found Laplace transforms of the occupation time inside the interval for the case of a Wiener process with drift. In Section 7 we consider the occupation time for several classes of Lévy processes, such as the compound Poisson process with positive jumps and negative drift, the spectrally one-sided Lévy process and the compound Poisson process with arbitrary positive jumps and exponential negative jumps.

The integral transforms of the two-boundary characteristics are obtained in terms of the integral transforms of the one-boundary functionals of the process. We employ probabilistic methods, the strong Markov property of the process and its spatial and time homogeneity property. The majority of the equations which appear, are linear integral equations. They are solved by means of the method of the successive iterations. The solutions of the equations are given in terms of the Neumann series, which for particular cases become geometric progressions.

As a particular case, we also consider the Wiener process. It appears, that the distributions of its boundary functionals are the limit distributions for the corresponding distributions of the characteristics of the general Lévy processes (after an appropriate scaling of time and space). To illustrate this, we state and prove some limit theorems for the spectrally one-sided Lévy process and for the compound Poisson process (2.2.13).

2.2 One-boundary characteristics of the process

In this section we will give the definition of the general Lévy process and its special subclasses and consider their one-boundary characteristics. It is well known that distributions of the increments of Lévy processes are infinitely divisible (Kyprianou (2006)).

Definition 2.2.1. *A distribution F is said to be infinitely divisible if for every n there exists a distribution F_n such that F is n-fold convolution of F_n : $F = F_n^{n*}$. That is to say, a random variable X has an infinitely divisible distribution F if for every n X can be represented as the sum $X = X_{1,n} + X_{2,n} + \cdots + X_{n,n}$ of n independent random variables with a common distribution F_n.*

It can be shown that a random variable X has an infinitely divisible distribution if its Laplace transform admits the Lévy-Khintchine representation:

$$k(p) = \log \mathbf{E} e^{-pX} = -ap + \frac{\sigma^2}{2}p^2 + \int\limits_R \left(e^{-xp} - 1 + xp\, \mathbf{I}_{\{|x|\le 1\}}\right) \Pi(dx), \ \Re(p) = 0,$$

where $a \in \mathbf{R}$, $\sigma > 0$ and the Lévy measure Π is a Radon measure on $\mathbf{R}$ such that $\Pi(\{0\}) = 0$ and

$$\int\limits_R \min\left(1, x^2\right) \Pi(dx) < \infty.$$

Definition 2.2.2. *A stochastic process $\{X_t;\ t \ge 0\}$, $X_0 = 0$ defined on a filtered probability space $(\Omega,\ \mathfrak{F}, \{\mathcal{F}_t\},\ \mathbf{P})$ is said to be a Lévy process if it has stationary and independent increments, and its paths are $\mathbf{P}$-almost surely right continuous with left limits (*cadlag*).*

It follows from the definition of the Lévy process that for every $t > 0$ the random variable X_t can be written as a sum of n independent variables distributed as $X_{\frac{t}{n}}$:

$$X_t = X_{\frac{t}{n}} + (X_{\frac{2t}{n}} - X_{\frac{t}{n}}) + \cdots + (X_t - X_{\frac{(n-1)t}{n}})$$

Hence, the distribution of X_t is infinitely divisible. Conversely, every family of infinitely divisible distributions with Laplace transform of the form $e^{tk(p)}$ can regulate a Lévy process. It is then possible to construct a Markov process

$X = \{X_t;\ t \geq 0\}$ with stationary increments such that $X_0 = 0$ and $\mathbf{E}\, e^{-pX_t} = e^{tk(p)}$, $\quad \Re(p) = 0$ (Kyprianou (2006, p.5)).

Let $\{X_t;\ t \geq 0\}$, $X_0 = 0$ be the real-valued Lévy process defined on the filtered probability space $(\Omega,\ \mathfrak{F},\ \{\mathcal{F}_t\},\ \mathbf{P})$. Then the Laplace transform of the increments of the process is of the form $\mathbf{E}e^{-p(X_t - X_0)} = e^{tk(p)}$, $\Re(p) = 0$. The function $k(p)$ is called the Laplace exponent, and it is given by means of the Lévy-Khintchine representation:

$$k(p) = -ap + \frac{\sigma^2}{2}p^2 + \int\limits_R \left(e^{-xp} - 1 + xp\,\mathbf{I}_{\{|x| \leq 1\}}\right)\Pi(dx), \quad \Re(p) = 0, \quad (2.2.1)$$

where $\alpha \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$, and $\Pi(dx)$ is a measure of the jumps such that

$$\int\limits_R \min\left(1, x^2\right)\Pi(dx) < \infty.$$

Observe, that this process is a strong Markov process (Dynkin (1965, p.99)). Recall that a Markov process $X = (x_t,\ \zeta,\ \mathcal{F}_t,\ \mathbf{P}_x)$ is given by

(i) $\zeta(w)$ on Ω taking values in $[0, \infty)$

(ii) a function $x(t, w) = x_t(w)$ defined for $w \in \Omega$, $\quad t \in [0, \zeta(w)]$ and taking values in the state space $(E, \mathcal{B})$

(iii) for each $t \geq 0$ a sigma-algebra $\mathcal{F}_t$ on the space $\Omega_t = \{w : \zeta(w) > t\}$

(iv) for $x \in E$ a function $\mathbf{P}_x(A)$ on some sigma-algebra $\mathcal{F}^0$ on the space Ω containing $\mathcal{F}_t$ for all $t \geq 0$.

Definition 2.2.3. *The real-valued function $\tau(w)$ is called a Markov time if*

(i) $0 \leq \tau(w) \leq \zeta(w)$

(ii) for each $t \geq 0$ $\{\tau \leq t < \zeta\} \in \mathcal{F}_t$

Definition 2.2.4. *A measurable Markov process $X = (x_t,\ \zeta,\ \mathcal{F}_t,\ P_x)$ is called a strong Markov process on the state space $(E, \mathcal{B})$ if for any Markov time τ and for any $t \geq 0$ $x \in E$, $\Gamma \in \mathcal{B}$*

$$\mathbf{P}_x[x_{\tau+t} \in \Gamma / \mathcal{F}_\tau] = \mathbf{P}_{x_\tau}[x_t \in \Gamma] \quad (a.s.\, \Omega_\tau, \mathbf{P}_x).$$

In order to solve the two-boundary problems for the process X_t, we will require the integral transforms of the one-boundary characteristics of the process. Denote by

$$X_t^+ = \sup_{u \le t} X_u, \qquad X_t^- = \inf_{u \le t} X_u,$$

the running supremum and infimum of the process, and by $\nu_s \sim \exp(s)$ an exponential random variable with expectation $1/s$, independent of the process. The following identity (due to Spitzer (1964) and Rogozin (1966)) plays an important role for solving boundary problems:

$$\mathbf{E}e^{-pX_{\nu_s}} = \frac{s}{s - k(p)} = \mathbf{E}e^{-pX_{\nu_s}^+}\mathbf{E}e^{-pX_{\nu_s}^-}, \qquad \Re(p) = 0, \tag{2.2.2}$$

where

$$\mathbf{E}e^{-pX_{\nu_s}^\pm} = \exp\left(\int_0^\infty \frac{1}{t} e^{-st}\mathbf{E}\left[e^{-pX_t} - 1; \pm X_t > 0\right] dt\right), \quad \pm\Re(p) \ge 0.$$

For all $x \ge 0$ define

$$\tau^x = \inf\{t : X_t > x\},\ T^x = X_{\tau^x} - x, \quad \tau_x = \inf\{t : X_t < -x\},\ T_x = -X_{\tau_x} - x$$

the first passage time of the positive (negative $-x$) level x and the value of the overshoot through this level. We use the convention that $\inf\{\emptyset\} = \infty$, and on the events $\{\tau^x = \infty\}$, $\{\tau_x = \infty\}$ we assume that $T^x = \infty$, $T_x = \infty$ respectively.

Here and in the sequel we will use the following notation

$$f^x(du, s) = \mathbf{E}\left[e^{-s\tau^x}; T^x \in du, \tau^x < \infty\right], \quad f^x(s) = \mathbf{E}\left[e^{-s\tau^x}; \tau^x < \infty\right],$$
$$f_x(du, s) = \mathbf{E}\left[e^{-s\tau_x}; T_x \in du, \tau_x < \infty\right], \qquad f_x(s) = \mathbf{E}\left[e^{-s\tau_x}; \tau_x < \infty\right].$$

Lemma 2.2.1. *Let $\{X_t;\ t \ge 0\}$ be the real-valued Lévy process whose Laplace exponent is given by (2.2.1). Then*

(i) *the integral transforms of the joint distributions of $\{\tau^x, T^x\}$, $\{\tau_x, T_x\}$ are such that*

$$\mathbf{E}e^{-s\tau^x - pT^x} = \left(\mathbf{E}e^{-pX_{\nu_s}^+}\right)^{-1}\mathbf{E}\left[e^{-p(X_{\nu_s}^+ - x)}; X_{\nu_s}^+ > x\right], \qquad \Re(p) \ge 0,$$
$$\mathbf{E}e^{-s\tau_x - pT_x} = \left(\mathbf{E}e^{pX_{\nu_s}^-}\right)^{-1}\mathbf{E}\left[e^{p(X_{\nu_s}^- + y)}; -X_{\nu_s}^- > x\right], \qquad \Re(p) \ge 0; \tag{2.2.3}$$

the integral transforms of the joint distribution of $\{X_{\nu_s}, X^{\pm}_{\nu_s}\}$ *obey the equalities for* $s \geq 0$

$$\mathbf{E}\left[e^{-pX_{\nu_s}}; X^{+}_{\nu_s} \leq x\right] = \mathbf{E}e^{-pX^{-}_{\nu_s}}\mathbf{E}\left[e^{-pX^{+}_{\nu_s}}; X^{+}_{\nu_s} \leq x\right], \quad \Re(p) \leq 0, \tag{2.2.4}$$

$$\mathbf{E}\left[e^{-pX_{\nu_s}}; X^{-}_{\nu_s} \geq -x\right] = \mathbf{E}e^{-pX^{+}_{\nu_s}}\mathbf{E}\left[e^{-pX^{-}_{\nu_s}}; X^{-}_{\nu_s} \geq -x\right], \quad \Re(p) \geq 0.$$

Proof. The integral transforms of these joint distributions were determined by Pecherskii and Rogozin (1969), see also Darling *et al.* (1972). We sketch a brief proof of the formulae (2.2.3), (2.2.4), applying the Spitzer-Rogozin factorization identity (2.2.2) and probabilistic reasoning. The total probability law combined with the strong Markov property of the process allow us to write the following equation for $\Re(p) = 0$

$$\mathbf{E}e^{-pX_{\nu_s}} = \mathbf{E}\left[e^{-pX_{\nu_s}}; X^{+}_{\nu_s} \leq x\right] + \mathbf{E}e^{-s\tau^x - pX_{\tau^x}}\mathbf{E}e^{-pX_{\nu_s}}. \tag{2.2.5}$$

Observe, that the increments of the process $\{X_t;\ t \geq 0\}$ on the exponential interval $[0,\ \nu_s]$ are realized either on the sample paths which do not cross the upper level x (the first term of the right-hand side) or on the sample paths which do cross the upper level x, and then the evaluation of the process on $[0, \nu_s]$ is its probabilistic copy (the second term in the right-hand side). In accordance with the total probability law and taking into account that $\{X^{+}_{t} \leq x\} = \{\tau^x > t\}$, we can write for $\Re(p) = 0$

$$\begin{aligned}\mathbf{E}e^{-pX_t} &= \mathbf{E}\left[e^{-pX_t}; \tau^x \geq t\right] + \mathbf{E}\left[e^{-pX_t}; \tau^x < t\right] \\ &= \mathbf{E}\left[e^{-pX_t}; X^{+}_{t} \leq x\right] + \mathbf{E}\left[e^{-pX_{\tau^x}}e^{-p\theta_{\tau^x}X_{t-\tau^x}}; \tau^x < t\right],\end{aligned} \tag{2.2.6}$$

where θ_t is a shift operator (Gihman and Skorokhod (1973, p.432)). Since τ^x is a Markov time, the increments of the process $\theta_{\tau^x}X_{t-\tau^x}$ do not depend on the sigma-algebra $\mathfrak{F}_{\tau^x}$, generated by the events $\{X(u) < v\} \cap \{\tau^x > u\}$ for all u, v. Hence

$$\mathbf{E}\left[e^{-pX_{\tau^x}}e^{-p\theta_{\tau^x}X_{t-\tau^x}}; \tau^x < t\right] = \int_0^t \mathbf{E}\left[e^{-pX_u}; \tau^x \in du\right]\mathbf{E}e^{-pX_{t-u}}.$$

Substituting the right-hand side of this formula into (2.2.6), we find

$$\mathbf{E}e^{-pX_t} = \mathbf{E}\left[e^{-pX_t}; X_t^+ \leq x\right] + \int_0^t \mathbf{E}\left[e^{-pX_u}; \tau^x \in du\right] \mathbf{E}e^{-pX_{t-u}}.$$

Multiplying the latter equality by the density $s\,e^{-st}$ of the random variable ν_s, and integrating it with respect to $t \geq 0$, we obtain the equality (2.2.5). In view of the identity (2.2.2) we rewrite (2.2.5) as follows:

$$\left(\mathbf{E}e^{-pX_{\nu_s}^-}\right)^{-1} \mathbf{E}\left[e^{-p(X_{\nu_s}-x)}; X_{\nu_s}^+ \leq x\right] - \mathbf{E}\left[e^{-p(X_{\nu_s}^+-x)}; X_{\nu_s}^+ \leq x\right] =$$
$$= \mathbf{E}\left[e^{-p(X_{\nu_s}^+-x)}; X_{\nu_s}^+ > x\right] - \mathbf{E}e^{-pX_{\nu_s}^+}\mathbf{E}e^{-s\tau^x - pT^x}, \quad \Re(p) = 0.$$

The function which enters the left-hand side of the latter equality is analytic in $\mathrm{Re}\, p < 0$, and continuous including the boundary $\mathrm{Re}\, p = 0$. By means of this equality it is analytically extended to an analytic function in $\Re(p) < 0$, remaining bounded. Hence, in view of the Liouville theorem this function is a constant with respect to p, say $C(s)$. In order to find this constant, we compute the limit as $p \to \infty$ which yields $C(s) = 0$. This standard factorization reasoning (see Borovkov (1972, p.115)) yields two formulae

$$\mathbf{E}e^{-s\tau^x - pT^x} = \left(\mathbf{E}e^{-pX_{\nu_s}^+}\right)^{-1} \mathbf{E}\left[e^{-p(X_{\nu_s}^+-x)}; X_{\nu_s}^+ > x\right], \qquad \Re(p) \geq 0$$
$$\mathbf{E}\left[e^{-pX_{\nu_s}}; X_{\nu_s}^+ \leq x\right] = \mathbf{E}e^{-pX_{\nu_s}^-}\mathbf{E}\left[e^{-pX_{\nu_s}^+}; X_{\nu_s}^+ \leq x\right], \qquad \Re(p) \leq 0$$

for the integral transforms of the joint distribution of $\{\tau^x, T^x\}$, and of $\{X_{\nu_s}, X_{\nu_s}^+\}$. Applying the first formula to the dual process $\{-X_t,;\, t \geq 0\}$, we will derive the second equality (2.2.3) of the lemma. Applying the second formula to the dual process $\{-X_t;\, t \geq 0\}$, we obtain the second equality of (2.2.4). ▲

2.2.1 Spectrally one-sided case

We will now derive the one-boundary characteristics for a Lévy process which has only positive jumps (this means that the Lévy measure Π has no mass on $(-\infty, 0)$). Let $\{X_t;\, t \geq 0\}$, $X_0 = 0$ be the spectrally positive Lévy process whose Laplace exponent is given by

$$k(p) = \frac{1}{2}\,p^2\sigma^2 - \alpha p + \int_0^\infty \left(e^{-xp} - 1 + xp\,\mathbf{I}_{\{x\le 1\}}\right)\Pi(dx), \quad \Re(p) \ge 0. \quad (2.2.7)$$

We will exclude subordinators (processes with non-decreasing paths) from the consideration. In this case the integral transforms of the distributions of $X^+_{\nu_s}$, $X^-_{\nu_s}$ are of the following form (see for instance Zolotarev (1964), Borovkov (1972) or Bertoin (1996a)):

$$\mathbf{E}e^{-pX^-_{\nu_s}} = \frac{c(s)}{c(s)-p}, \quad \Re(p) \le 0, \quad \mathbf{E}e^{-pX^+_{\nu_s}} = \frac{s}{c(s)}\frac{p-c(s)}{k(p)-s}, \quad \Re(p) \ge 0, \tag{2.2.8}$$

where $c(s) > 0$ is a unique solution of the characteristic equation $k(p) - s = 0$, $s > 0$ in $\Re(p) > 0$. It follows from (2.2.3) that the integral transforms of the joint distributions of $\{\tau^x, T^x\}$, $\{\tau_x, T_x\}$ are such that ($\Re(p), \Re(z) \ge 0$)

$$\begin{aligned} &\mathbf{E}\left[e^{-s\tau_x}; T_x \in du\right] = e^{-xc(s)}\,\delta(u)\,du, \\ &\int_0^\infty e^{-px}\mathbf{E}e^{-s\tau^x - zX_{\tau^x}}\,dx = \frac{1}{p}\left(1 - \frac{p+z-c(s)}{k(p+z)-s}\frac{k(z)-s}{z-c(s)}\right), \end{aligned} \tag{2.2.9}$$

where $\delta(u)$ is the delta function, whose fundamental property is as follows: $\int\limits_{-\infty}^{\infty} f(u)\,\delta(u-a)\,du = f(a)$.

One of the well-studied and the most applied spectrally one-sided Lévy processes is a compound Poisson process with a linear drift. Its Laplace exponent is such that

$$k(p) = \alpha p + c\left(\mathbf{E}e^{-p\eta} - 1\right), \qquad \alpha, \eta > 0 \quad \Re(p) \ge 0, \tag{2.2.10}$$

where η is the size of the positive jumps, c is the rate of the jumps, and $-\alpha$ is a coefficient of the negative drift. That is to say that

$$X_t = \sum_{k=0}^{N(t)} \eta_k - \alpha\,t, \quad t \ge 0$$

where $\eta_0 = 0$, $\eta_k \sim \eta$ are positive independent identically distributed variables, $\{N(t); t \ge 0\}$, $N(0) = 0$ is an ordinary Poisson process with parameter c, independent from $\{\eta_k;\ k \ge 0\}$. Such process and its generalizations serve for modeling risk processes, storage processes etc.

2.2.2 Scale function

In this subsection we introduce a so called scale function (or resolvent function in Ukrainian and Russian literature) which plays a key role in the theory of spectrally one-sided Lévy processes. In the context of ruin theory, scale functions appeared in Zolotarev (1964), Takacs (1967) and then later in Suprun and Shurenkov (1975), Suprun (1976), Suprun and Shurenkov (1986), Korolyuk (1975) and Pistorius (2004). The importance of scale functions as a class with which one may express a whole range of fluctuation identities for spectrally negative Lévy processes became apparent in Chaumont (1996), Bertoin (1996a) and a number of other articles (see for example Doney (2005), Alili and Kyprianou (2005) and Lambert (2000)). In addition, the scale functions often appear in martingale relations (see Chan *et al.* (2007)). They also found applications in queueing theory (see for instance Dube *et al.* (2004), Bekker *et al.* (2008)), in risk theory and optimal control Avram *et al.* (2006), Loeffen (2008), Renaud and Zhou (2007).. Motivated by their wide applications, Borovskikh (1979) and Borovskikh and Korolyuk (1981) studied their asymptotic properties. More recently, Chan *et al.* (2007) studied smoothness of the scale functions. Hubalek and Kyprianou (2007) described a parametric family of scale functions explicitly. They constructed a spectrally negative Lévy process having a particular pre-determined Wiener-Hopf factorization. Kyprianou and Rivero (2008) employed the approach proposed in Hubalek and Kyprianou (2007) and combined it with methods of the potential analysis of subordinators. It appears that the theory of special Bernstein functions is closely related to the theory of scale functions. The authors found pairs of spectrally negative Lévy processes whose scale functions are conjugate to one another in an appropriate sense. They also proposed new explicit examples of conjugate pairs of scale functions. Surya (2008) developed a robust numerical method to compute the scale function of a general spectrally negative Lévy process. The method is based on the Esscher transform of a measure under which the scale function is determined.

Notion of resolvent goes back to 60s of the last century. Takács (1966) determined the probabilities of the first exit from an interval $[-y, x]$ by a semi-continuous process without Gaussian component:

$$\mathbf{P}[A_y] = \frac{W(x)}{W(B)}, \quad \mathbf{P}[A^x] = 1 - \frac{W(x)}{W(B)}, \qquad B = x + y.$$

The function $W(x)$, $x \geq 0$ was defined by its Laplace transform:

$$\int_0^\infty e^{-px} W(x)\, dx = \frac{1}{k(p)}, \qquad \Re(p) > c,$$

where $c \geq 0$ is a unique root of the equation $k(p) = 0$ in the semi-plane $\Re(p) \geq 0$. Its modification, the function $W^s(x)$, $x \geq 0$, whose Laplace transform is given as follows

$$\int_0^\infty e^{-px} W^s(x)\, dx = \frac{1}{k(p) - s}, \qquad \Re(p) > c(s), \quad s \geq 0$$

is known as a scale function or the resolvent. Here $c(s) > 0$, $s > 0$ is the unique positive solution of the equation $k(p) - s = 0$ in the semi-plane $Re\, p > 0$. Korolyuk (1975) introduced the term *resolvent* and *potential* for the functions $W^s(x)$, $x \geq 0$ and $W(x)$, $x \geq 0$, and the notation $R_s(x), R_s$ in case of the compound Poisson process with positive jumps and negative drift with Laplace exponent (2.2.10). Later the function $W^s(x)$, $x \geq 0$, was introduced in Suprun and Shurenkov (1975), Suprun (1976), Borovskikh (1979) for the spectrally one-sided Lévy processes. The authors also used notation and terminology as in Korolyuk (1975). Here and in the sequel we will adopt the following definition and notation.

Definition 2.2.5. *The function $R_s(x) : [0, \infty) \to [0, \infty)$, $x \geq 0$, given by*

$$R_s(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{xp} \, \frac{1}{k(p) - s}\, dp, \qquad \gamma > c(s) \quad s \geq 0, \tag{2.2.11}$$

is called the scale function of the spectrally one-sided Lévy process. Here $c(s)$ is the unique root of the equation $k(p) - s = 0$ in the semi-plane $\Re(p) > 0$.

2.2.3 Compound Poisson process with two-sided jumps

Another important example of a Lévy process is the compound Poisson process $\{X_t;\, t \geq 0\}$ whose Laplace exponent is given by

$$k(p) = c \int_0^\infty \left(e^{-px} - 1\right) dF(x), \qquad \Re(p) = 0, \tag{2.2.12}$$

where ξ is a jump size, $F(x) = \mathbf{P}[\xi \leq x]$, $x \in \mathbb{R}$ is the cumulative distribution function of the jumps, $c > 0$ is the rate of the jumps.

That is to say that

$$X(t) = \sum_{k=0}^{N(t)} \xi_k, \quad t \geq 0$$

where $\xi_0 = 0$, $\xi_k \sim \xi$ are independent identically distributed variables with distribution function $F(x)$; $\{N(t); t \geq 0\}$, $N(0) = 0$ is an ordinary Poisson process withe parameter c, independent from $\{\xi_k;\, k \geq 0\}$. In particular, if the distribution function is of the form

$$F(x) = ae^{x\lambda}\mathbf{I}_{\{x\leq 0\}} + (a + (1-a)\mathbf{P}[\eta < x])\mathbf{I}_{\{x>0\}}, \qquad a \in (0,1), \quad \lambda > 0,$$

where the r.v. $\eta \in (0, \infty)$, then

$$k(p) = a_1 \frac{p}{\lambda - p} + a_2\left(\mathbf{E}\, e^{-p\eta} - 1\right), \quad \Re(p) = 0, \qquad (2.2.13)$$

where $a_1 = ac$, $a_2 = (1-a)c$, $c > 0$, $a \in (0,1)$.

Note, that inter-arrival times of the jumps of the process X_t are exponentially distributed with parameter c. With probability $1 - a$ there occur positive jumps of size η, and with probability a there occur negative jumps of value $-\gamma$, where γ is exponentially distributed with parameter λ. Here and in the sequel we will call such process the compound Poisson process with a negative exponential component. The first term of (2.2.13) is the simplest case of a rational function, while the second term is nothing but the Laplace exponent of a compound Poisson process with positive jumps of value η and intensity of jumps a_2. It is a well-known fact (see for instance Borovkov (1972)), that in this case the characteristic equation $k(p) - s = 0$, $s > 0$ has a unique root $c(s) \in (0, \lambda)$ in the semi-plane $\Re(p) > 0$. In this case the integral transforms of $X^+_{\nu_s}$, $X^-_{\nu_s}$ are such that

$$\mathbf{E}\, e^{-pX^-_{\nu_s}} = \frac{c(s)}{\lambda} \frac{\lambda - p}{c(s) - p}, \qquad \Re(p) \leq 0, \qquad (2.2.14)$$

$$\mathbf{E}\, e^{-pX^+_{\nu_s}} = \frac{s\lambda}{c(s)} (p - c(s))\mathbb{R}(p, s), \qquad \Re(p) \geq 0, \qquad (2.2.15)$$

where

$$\mathbb{R}(p,s) = \frac{1}{(\lambda - p)(k(p) - s)}, \qquad \Re(p) \geq 0, \quad p \neq c(s). \qquad (2.2.16)$$

It follows from the first two equalities of (2.2.3) and the formulae (2.2.14), (2.2.15) that the integral transforms of the joint distributions of $\{\tau^x, T^x\}$, $\{\tau_x, T_x\}$ are given by

$$\mathbf{E}\left[e^{-s\tau_x}; T_x \in du\right] = (\lambda - c(s))\, e^{-xc(s)-\lambda u}\, du = \mathbf{E}e^{-s\tau_x}\mathbf{P}[\gamma \in du], \quad (2.2.17)$$

$$\int_0^\infty e^{-px}\, \mathbf{E}e^{-s\tau^x - zX_{\tau^x}}\, dx = \frac{1}{p}\left(1 - \frac{p+z-c(s)}{z-c(s)}\, \frac{\mathbb{R}(p+z,s)}{\mathbb{R}(z,s)}\right), \qquad (2.2.18)$$

where $\Re(p) > 0$, $\Re(z) \geq 0$. Note that the random variables τ_x, T_x are independent and for all $x \geq 0$ the value of the overshoot T_x is exponentially distributed with parameter λ. This property is a characteristic feature of the process introduced. Observe that the function $\mathbb{R}(p,s)$ is analytic in the semi-plane $\Re(p) > c(s)$ with respect to p, and $\lim\limits_{p\to\infty} \mathbb{R}(p,s) = 0$. Therefore, it allows a representation in the form of an absolutely convergent Laplace integral (Ditkin and Prudnikov (1966, p.71)):

$$\mathbb{R}(p,s) = \int_0^\infty e^{-px} R_s(x)\, dx, \qquad \Re(p) > c(s). \qquad (2.2.19)$$

We will call the function $R_x(s)$, $x \geq 0$ the resolvent of the compound Poisson process with a negative exponential component. We assume that $R_s(x) = 0$, for $x < 0$. Note, that $R_s(0) = \lim\limits_{p\to\infty} p\,\mathbb{R}(p,s) = (c+s)^{-1}$, ($p$ is real) and

$$\mathbf{P}[X^-_{\nu_s} = 0] = \frac{c(s)}{\lambda}, \qquad \mathbf{P}[X^+_{\nu_s} = 0] = \frac{s\lambda}{c(s)(s+c)}.$$

It follows from (2.2.15) that

$$\mathbb{R}(p,s) = \frac{c(s)}{s\lambda}\, \frac{1}{p - c(s)}\, \mathbf{E}\, e^{-pX^+_{\nu_s}}, \qquad \Re(p) > c(s). \qquad (2.2.20)$$

The functions which enter the right-hand side of (2.2.20), are the Laplace transforms of certain functions for $\Re(p) > c(s)$. Therefore, the original functions of the left-hand side and the right-hand side of (2.2.20) coincide, and

$$R_s(x) = \frac{c(s)}{s\lambda} \int_{-0}^{x} e^{c(s)(x-u)} d\,\mathbf{P}[X^+_{\nu_s} < u], \qquad x \geq 0, \qquad (2.2.21)$$

which is the resolvent representation of the compound Poisson process with a negative exponential component. The representation (2.2.21) implies that $R_s(x)$, $x \geq 0$ is a positive, monotone, continuous, increasing function of an exponential order, i.e. there exists $0 < A(s) < \infty$ such that for all $x \geq 0$ $R_s(x) < A(s) \exp\{xc(s)\}$, $A(s) < \infty$. Therefore,

$$\int_0^\infty R_s(x) e^{-\alpha x} dx < \infty, \qquad \alpha > c(s).$$

Moreover, in the neighborhood of any $x \geq 0$ the function $R_s(x)$ has bounded variation. Hence, the inversion formula Ditkin and Prudnikov (1966, p. 68) is valid:

$$R_s(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{xp} \mathbb{R}(p, s)\, dp, \qquad \alpha > c(s). \tag{2.2.22}$$

The latter formula can serve as a definition of the resolvent of the compound Poisson process with exponential component. This definition is useful for solving two-boundary problems, since it allows to invert the Laplace transforms that appear in the analytic expressions for the functionals studied. In order to illustrate this, we give some examples:

$$\begin{aligned} f^x(s) &= 1 - \frac{s\lambda}{c(s)} R_s(x) + s\lambda S_s(x), \\ \tilde{f}^x(c(s)) &= \mathbf{E}\left[e^{-s\tau^x - c(s)T^x}; \tau^x < \infty \right] = e^{xc(s)} - R_s(x) r(s), \end{aligned} \tag{2.2.23}$$

where

$$S_s(x) = \int_0^x R_s(u)\, du \quad r(s) = \frac{d}{dp} \mathbb{R}(p, s)^{-1} \Big|_{p = c(s)}.$$

The relations (2.2.23) can be derived from formula (2.2.18) for $z = 0$ and $z = c(s)$ respectively and from the definition of the resolvent (2.2.22).

The knowledge of the one-boundary characteristics of the process allows to solve a more complicated problem, i.e. a two-sided exit problem, which is the topic of the next section.

2.3 First exit from the interval

The first two-boundary problem we are going to consider is determining the integral transform of the joint distribution of the first exit time and the value of the overshoot through the boundary at this instant. This joint distribution will play a key role for solving other two-boundary problems. Let $B > 0$ be fixed, $x \in [0, B]$, $y = B - x$, $X_0 = 0$. Denote by

$$\chi = \inf\{t > 0 : X_t \notin [-y, x]\}$$

the first exit time from the interval $[-y, x]$ by the process. Note, that χ is a Markov time of the process (Skorokhod (1964)), and that $P[\chi < \infty] = 1$. Observe, that the exit from the interval can occur either through the upper boundary x, or through the lower boundary $-y$. In view of this remark we introduce the following events

$\mathfrak{A}^x = \{X_\chi > x\}$, i.e. the exit occurs through the upper boundary;

$\mathfrak{A}_y = \{X_\chi < -y\}$, i.e. the process exits the interval through the lower boundary. Denote by

$$T = (X_\chi - x)\mathbf{I}_{\mathfrak{A}^x} + (-X_\chi - y)\mathbf{I}_{\mathfrak{A}_y} \in \mathbb{R}_+, \qquad P[\mathfrak{A}^x + \mathfrak{A}_y] = 1$$

the value of the overshoot through the boundary at the instant of the first exit from the interval. Here $\mathbf{I}_A = \mathbf{I}_A(\omega)$ is the indicator of the set A.

We now give a short review of the existing literature related to the study of the joint distribution of the first exit time χ and the value of the overshoot T.

Ito and McKean (1965) derived the Laplace transforms of the first exit time χ from the interval by the Wiener process. Takacs (1967) determined the exit probabilities for the spectrally one-sided Lévy process without Gaussian component ($\sigma = 0$ in the defining formula (2.2.7) of Laplace exponent) in terms of the scale function. Emery (1973) obtained the exit probabilities for the spectrally one-sided Lévy process (2.2.7) in terms of the scale functions. For the general Lévy process with Laplace exponent (2.2.1) Gihman and Skorokhod (1973) (p.450) determined the joint distribution of $\{X_t^-, X_t, X_t^+\}$, where $X_t^+ = \sup_{u \le t} X_u$, $X_t^- = \inf_{u \le t} X_u$. For determining the joint distribution of

χ, $\{\chi, T\}$ Shurenkov, for instance, suggested to use the formulae of Dynkin (1965, p.191) which are valid for any homogeneous Markov process. Utilizing this idea, Suprun and Shurenkov (1975) obtained the representations for the Laplace transform of the distribution of the first exit time χ in terms of the scale functions. ($s \geq 0$)

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] = \frac{R_s(x)}{R_s(B)}, \tag{2.3.1}$$
$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^x\right] = 1 - \frac{R_s(x)}{R_s(B)} - s\frac{R_s(x)}{R_s(B)}\int_0^B R_s(u)\,du + s\int_0^x R_s(u)\,du,$$

For the compound Poisson process with negative drift whose Laplace exponent is given by (2.2.10), the representations (2.3.1) were found by Korolyuk (1975). Korolyuk and Shurenkov (1977) determined main boundary functionals for the random walks defined on a Markov chain. In particular, they introduced a matrix resolvent and found matrix analogues of the formulae (2.3.1). Shurenkov (1978) redetermined the Laplace transforms of the joint distribution of $\{\chi, X_\chi\}$ in terms of the joint distribution of $\{X_t^-, X_t, X_t^+\}$ and the measure $\Pi(A)$, for the spectrally one-sided Lévy processes:

$$\mathbf{E}\left[e^{-s\chi}; X_\chi < l\right] = \int_{-y}^{x}\left[\frac{R_s(y)}{R_s(B)}R_s(x-u) - R_s(-u)\right]\Pi([l-y-u, -\infty))\,du, \tag{2.3.2}$$

where $s > 0$, $l < -y$,

$$\frac{R_s(y)}{R_s(B)}R_s(x-u)du - R_s(-u)\,du = \int_0^\infty e^{-st}\mathbf{P}[-y \leq X_t^-,\, X_t \in du,\, X_t^+ \leq x]\,dt.$$

In the same article the weak convergence was established for the distribution of the overshoot through the boundary. To prove (2.3.2), the author employed the Dynkin's formulae, Dynkin (1965, p.191). Kemperman (1963) derived the factorization identities for random walks, and Pecherskii (1974) proved these identities for Lévy processes. It is worth mentioning that in Kemperman (1963), Pecherskii (1974), Shurenkov (1978) the joint distribution of $\{\chi, T\}$ was determined in terms of the joint distribution of $\{X_t^-, X_t, X_t^+\}$ and the measure $\Pi(A)$. This fact makes it difficult to employ these formulae for solving other two-boundary problems. Our contribution is that we use one-boundary functionals of the process, which allows to determine other characteristics of the process.

We now present the main result of this section. For all $x, u \geq 0$ we define

$$f^x(du,s) = \mathbf{E}\left[e^{-s\tau^x}; T^x \in du\right], \qquad f_x(du,s) = \mathbf{E}\left[e^{-s\tau_x}; T_x \in du\right].$$

For $x \in [0,B]$, $y = B - x$ we will use the following notation

$$F^x(du,s) = f^x(du,s) - \int_0^\infty f_y(dv,s) f^{v+B}(du,s),$$
$$F_y(du,s) = f_y(du,s) - \int_0^\infty f^x(dv,s) f_{v+B}(du,s).$$

Theorem 2.3.1 (Kadankov and Kadankova (2005b)). *Let $\{X_t; t \geq 0\}$ $X(0) = 0$ be the real-valued Lévy process with Laplace exponent (2.2.1), $B > 0$ be fixed and $x \in [0,B]$, $y = B - x$. Then the integral transforms of the joint distribution of $\{\chi, T\}$ satisfy the following formulae for $s > 0$*

$$V^x(du,s) = \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}^x\right] = F^x(du,s) + \int_0^\infty F^x(dv,s)\mathfrak{K}_+^s(v,du),$$
$$V_y(du,s) = \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}_y\right] = F_y(du,s) + \int_0^\infty F_y(dv,s)\mathfrak{K}_-^s(v,du), \tag{2.3.3}$$

where

$$\mathfrak{K}_\pm^s(v,du) = \sum_{n\in\mathbb{N}} K_\pm^{(n)}(v,du,s), \qquad v \geq 0 \tag{2.3.4}$$

are the Neumann series of the successive iterations, $\mathbb{N} = \{1,2,...\}$;

$$K_\pm^{(1)}(v,du,s) = K_\pm(v,du,s),$$
$$K_\pm^{(n+1)}(v,du,s) = \int_0^\infty K_\pm^{(n)}(v,dl,s)K_\pm(l,du,s), \qquad n \in \mathbb{N}, \tag{2.3.5}$$

are the successive iterations of the kernels $K_\pm(v,du,s)$, which are defined as follows:

$$K_+(v,du,s) = \int_0^\infty f_{v+B}(dl,s) f^{l+B}(du,s),$$
$$K_-(v,du,s) = \int_0^\infty f^{v+B}(dl,s) f_{l+B}(du,s). \tag{2.3.6}$$

Proof. In view of the total probability law combined with the strong Markov property of the process we can write the following system for the functions $V^x(du,s)$, $V_y(du,s)$:

$$f^x(du,s) = V^x(du,s) + \int_0^\infty V_y(dv,s) f^{v+B}(du,s),$$
$$f_y(du,s) = V_y(dv,s) + \int_0^\infty V^x(dv,s) f_{v+B}(du,s). \qquad (2.3.7)$$

The first equation reflects the fact the first passage time of the upper boundary x can be realized either on the sample paths of the process X_t which do not intersect the lower boundary $-y$ (the first term in the right-hand side) or on the sample paths which do intersect the lower boundary $-y$ and then intersect the upper boundary x (the second term in the right-hand side). We also give a brief explanation of this equation. It is obvious that

$$\mathbf{E}\left[e^{-s\tau^x}; T^x \in du, \tau^x < \tau_y\right] = \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}^x\right],$$
$$\mathbf{E}\left[e^{-s\tau_y}; T_y \in du, \tau_y < \tau^x\right] = \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}_y\right].$$

In view of the total probability law the following chain of equalities is valid:

$$\mathbf{E}\left[e^{-s\tau^x}; T^x \in du\right] = \mathbf{E}\left[e^{-s\tau^x}; T^x \in du, \tau^x < \tau_y\right] + \mathbf{E}\left[e^{-s\tau^x}; T^x \in du, \tau_y < \tau^x\right]$$
$$= \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}^x\right] + \mathbf{E}\left[e^{-s\chi} e^{-s\tau^{B+T}}; T^{B+T} \in du, \mathfrak{A}_y\right]. \qquad (2.3.8)$$

Since χ is a Markov time of the process, then the random variables τ^{B+T}, T^{B+T} do not depend on the sigma algebra $\mathfrak{F}_\chi$, generated by the events $\{X(u) < v\} \cap \{\chi > u\}$ for all u, v. Hence

$$\mathbf{E}\left[e^{-s\chi} e^{-s\tau^{B+T}}; T^{B+T} \in du, \mathfrak{A}_y\right] = \int_0^\infty V_y(dv,s) \mathbf{E}\left[e^{-s\tau^{v+B}}; T^{v+B} \in du\right].$$

Substituting the right-hand side of the latter equality in (2.3.8), we get the first formula of (2.3.7). The second equality can be verified analogously. Let us turn now to the system of the integral equations (2.3.7). It is analogous to a system of linear equations with two unknowns and can be solved by a substitution method. In view of this remark we substitute the expression for the function $V_y(du,s)$ from the second equation into the first one, which yields

$$V^x(du,s) = F^x(du,s) + \int_{l=0}^\infty \int_{v=0}^\infty V^x(dv,s) f_{v+B}(dl,s) f^{l+B}(du,s). \qquad (2.3.9)$$

Changing the order of integration in the second term of the equation (2.3.9), we obtain

$$V^x(du,s) = F^x(du,s) + \int_0^\infty V^x(dv,s)K_+(v,du,s) \tag{2.3.10}$$

i.e. a linear integral equation for the function $V^x(du,s)$. The kernel $K_+(v,du,s)$ of this equation obeys the following inequality for $v,u \in \mathbb{R}_+,\ s > s_0 > 0$

$$\begin{aligned} K_+(v,du,s) &= \int_0^\infty \mathbf{E}\left[e^{-s\tau_{v+B}}; T_{v+B} \in dl\right] \mathbf{E}\left[e^{-s\tau^{l+B}}; T^{l+B} \in du\right] \\ &\le \int_0^\infty E\left[e^{-s\tau_{v+B}}; T_{v+B} \in dl\right] \mathbf{E}e^{-s\tau^{l+B}} \\ &\le \mathbf{E}e^{-s\tau^B} \int_0^\infty \mathbf{E}\left[e^{-s\tau_{v+B}}; T_{v+B} \in dl\right] \\ &\le \mathbf{E}e^{-s\tau^B}\mathbf{E}e^{-s\tau_B} \le \lambda < 1, \qquad\qquad s > s_0 > 0, \end{aligned}$$

where

$$\lambda = \mathbf{E}e^{-s_0\tau^B}\mathbf{E}e^{-s_0\tau_B} < 1, \qquad s_0 > 0.$$

To derive this chain of inequalities, we used the relation $\mathbf{E}e^{-s\tau^{v+B}} \le \mathbf{E}e^{-s\tau^B}$, $v \ge 0$, which is the result of the following chain of relations

$$\begin{aligned} \mathbf{E}e^{-s\tau^{v+B}} &= \mathbf{E}\left[e^{-s\tau^B}; T^B > v\right] + \int_0^v \mathbf{E}\left[e^{-s\tau^B}; T^B \in du\right] \mathbf{E}e^{-s\tau^{v-u}} \\ &\le \mathbf{E}\left[e^{-s\tau^B}; T^B > v\right] + \mathbf{E}\left[e^{-s\tau^B}; T^B \le v\right] = \mathbf{E}e^{-s\tau^B}. \end{aligned}$$

It can be shown similarly that $\mathbf{E}e^{-s\tau_{v+B}} \le \mathbf{E}e^{-s\tau_B}$, $v \ge 0$. For the sequence of the n-th iterations

$$K_+^{(1)}(v,du,s) = K_+(v,du,s), \quad K_+^{(n+1)}(v,du,s) = \int_0^\infty K_+^{(n)}(v,dl,s)\, K_+(l,du,s),$$

of the kernel $K_+(v,du,s)$, we deduce by means of the mathematical induction that for all $v,u \in \mathbb{R}_+$, $\quad s > s_0 > 0$ $K_+^{(n)}(v,du,s) < \lambda^n$, $n \in \mathbb{N}$. Thus, the series

$$\mathfrak{K}_+^s(v,du) = \sum_{n\in\mathbb{N}} K_+^{(n)}(v,du,s) < \lambda\,(1-\lambda)^{-1}$$

of the successive iterations of the kernel $K_+(v,du,s)$ converges uniformly for all $v,u \in \mathbb{R}_+$, $s > s_0 > 0$. Thus, we can apply the method of successive iterations

(Petrovskii (1965)) to solve the integral equation (2.3.10). This yields

$$V^x(du, s) = F^x(du, s) + \int_0^\infty F^x(dv, s)\mathfrak{K}_+^s(v, du), \qquad s > s_0.$$

Letting $s \to 0$, we get the first equality of the theorem. The second equality can be verified analogously. ▲

2.3.1 First exit from the interval by a spectrally positive Lévy process

Let $\{X_t; t \geq 0\}$ be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7). In order to determine the joint distribution of $\{\chi, T\}$, we apply the results of Theorem 2.3.1. In this case the Neumann series (2.3.4) are the geometric series. We now state a corollary of Theorem 2.3.1.

Corollary 2.3.1. *Let $\{X_t; t \geq 0\}$ be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7), $s > 0$. Then*

(i) *the integral transforms of the joint distribution of $\{\chi, T\}$ satisfy the equalities:*

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] = e^{-yc(s)} \, \frac{1 - G_x^s(c(s))}{1 - G_B^s(c(s))}, \tag{2.3.11}$$

$$V^x(du, s) = \mathbf{E}\left[e^{-s\tau^x}; T^x \in du\right] - \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] \mathbf{E}\left[e^{-s\tau^B}; T^B \in du\right],$$

where

$$G_x^s(z) = \mathbf{E}\left[e^{-s\tau^x - zX_{\tau^x}}; \tau^x < \infty\right] = 1 - e^{-xc(s)} R_s(x) r(s), \quad x \geq 0;$$

(ii) *the integral transform of the distribution of the first exit time is such that*

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] = \frac{R_s(x)}{R_B(s)}, \tag{2.3.12}$$

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^x\right] = 1 - \frac{R_s(x)}{R_B(s)} - s\frac{R_s(x)}{R_B(s)} \int_0^B R_s(u)\, du + s \int_0^x R_s(u)\, du,$$

where $R_s(x)$ is the scale function (2.2.11) of the spectrally positive Lévy process.

Proof. It is worth mentioning that these results are not new. Formulae (2.3.12) were obtained by means of the resolvent methods in Suprun (1976) and Suprun and Shurenkov (1975). In case of the Poisson process with positive jumps and negative drift (2.2.10) formulae (2.3.12) were derived in Korolyuk (1975). Equalities (2.3.11) were obtained in Kadankov and Kadankova (2004) and in Kadankova (2003a) after solving the system (2.3.7) for a spectrally positive Lévy process. These results can also be found in the monograph Kyprianou (2006) (see also Pistorius (2004) and the references therein).

Note, that it is the knowledge of the distribution of the value of the overshoot through the boundary which allows us to solve other two-boundary problems for the spectrally one-sided Lévy processes (see next sections).

We now verify equalities (2.3.11) by applying the results of the theorem. For the spectrally positive Lévy process the function $F_y(du, s)$ and the successive iterations $K_-^{(n)}(v, du, s)$, $n \in \mathbb{N}$ are easy to calculate:

$$F_y(du, s) = e^{-yc(s)} \left(1 - G_x^s(c(s))\right) \delta(u)\, du,$$
$$K_-^{(n)}(v, du, s) = e^{vc(s)} G_{v+B}^s(c(s)) \;\left(G_B^s(c(s))\right)^{n-1} \delta(u)\, du, \qquad n \in \mathbb{N},$$

where $\delta(\cdot)$ is the delta function, $\int_{-\infty}^{\infty} f(u)\; \delta(u-a)\, du = f(a)$. The Neumann series of the successive iterations $K_-(v, du, s)$ is a geometric progression, whose sum is equal to:

$$\mathfrak{K}_-^s(v, du) = \sum_{n \in \mathbb{N}} K_-^{(n)}(v, du, s) = e^{vc(s)} \frac{G_{v+B}^s(c(s))}{1 - G_B^s(c(s))}\; \delta(u)\, du,$$

Substituting the expressions for the functions $F_y(du, s)$, $\mathfrak{K}_-^s(v, du)$ into the second formula of the theorem, we find that

$$V_y(du, s) = \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}_y\right] = e^{-yc(s)}\, \frac{1 - G_x^s(c(s))}{1 - G_B^s(c(s))}\, \delta(u)\, du. \qquad (2.3.13)$$

Integrating (2.3.13) with respect to $u \in \mathbb{R}_+$ yields the first formula of (2.3.11). The function $F^x(du, s)$ and the successive iterations $K_+^{(n)}(v, du, s)$ are given as follows:

$$F^x(du, s) = \mathbf{E}\left[e^{-s\tau^x}; T^x \in du\right] - e^{-yc(s)}\, \mathbf{E}\left[e^{-s\tau^B}; T^B \in du\right],$$
$$K_+^{(n)}(v, du, s) = e^{-c(s)(v+B)} \;\left(G_B^s(c(s))\right)^{n-1}\; \mathbf{E}\left[e^{-s\tau^B}; T^B \in du\right], \qquad n \in \mathbb{N}.$$

The Neumann series of the successive iterations of the kernel $K_+(v, du, s)$ is a geometric progression, whose sum is equal to:

$$\mathfrak{K}_+^s(v, du) = \sum_{n\in\mathbb{N}} K_+^{(n)}(v, du, s) = \frac{e^{-c(s)(v+B)}}{1 - G_B^s(c(s))}\,\mathbf{E}\left[e^{-s\tau^B}; T^B \in du\right].$$

Substituting the expression for the functions $F^x(du, s)$, $\mathfrak{K}_+^s(v, du)$ into the first formula of the theorem, we obtain

$$V^x(du, s) = \mathbf{E}\left[e^{-s\tau^x}; T^x \in du\right] - e^{-yc(s)}\frac{1 - G_x^s(c(s))}{1 - G_B^s(c(s))}\mathbf{E}\left[e^{-s\tau^B}; T^B \in du\right] \tag{2.3.14}$$

i.e. the second formula of (2.3.11). Integrating the equalities (2.3.13), (2.3.14) with respect to $u \in \mathbb{R}_+$, we get

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] = e^{-yc(s)}\,\frac{1 - G_x^s(c(s))}{1 - G_B^s(c(s))},$$
$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^x\right] = \mathbf{E}e^{-s\tau^x} - e^{-yc(s)}\,\frac{1 - G_x^s(c(s))}{1 - G_B^s(c(s))}\,\mathbf{E}e^{-s\tau^B}.$$

Employing the definition of the scale function (2.2.11) and the integral transform of the joint distribution of $\{\tau^x, T^x\}$ (2.2.9), we derive the representation of the functions $G_x^s(c(s))$, $\mathbf{E}e^{-s\tau^x}$ in terms of the scale function:

$$G_x^s(c(s)) = 1 - r(s)e^{-xc(s)}R_s(x), \quad \mathbf{E}e^{-s\tau^x} = 1 - \frac{s}{c(s)}R_s(x) + s\int_0^x R_s(u)\,du,$$

where $r(s) = \frac{d}{dp}k(p)\Big|_{p=c(s)}$. Substituting these expressions into the latter equalities, we derive

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] = \frac{R_s(x)}{R_B(s)},$$
$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^x\right] = 1 - \frac{R_s(x)}{R_B(s)} - s\frac{R_s(x)}{R_B(s)}\int_0^B R_s(u)\,du + s\int_0^x R_s(u)\,du$$

i.e. the formulae (2.3.12) of the corollary. ▲

2.3.2 Exit by a standard Wiener process

Let $\{w_t;\ t \geq 0\}$ be a standard Wiener process whose Laplace exponent is $k(p) = \sigma^2 p^2/2$. In this case $\mathbf{P}[T^x = T_x = 0] = 1$,

$$\mathbf{E}\left[e^{-s\tau^x}; T^x \in du\right] = e^{-x\sqrt{2s}/\sigma}\delta(u)\,du = \mathbf{E}\left[e^{-s\tau_x}; T_x \in du\right],$$

and the formulae of Theorem 2.3.1 have a very simple form.

Corollary 2.3.2. *Let $\{w_t;\ t \geq 0\}$, $w_0 = 0$ be the standard real-valued Wiener process, $x \in [0, B]$, $y = B - y$, and*

$$\chi = \inf\{t > 0 : w_t \notin [-y, x]\}, \qquad \mathfrak{A}^x = \{w_\chi = x\}, \quad \mathfrak{A}_y = \{w_\chi = -y\}$$

be the first exit time from the interval $[-y, x]$ and the events on which it can take place. Then

(i) *the Laplace transforms of χ are such that*

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^x\right] = \frac{\sinh\left(y\sqrt{2s}/\sigma\right)}{\sinh\left(B\sqrt{2s}/\sigma\right)}, \qquad \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] = \frac{\sinh\left(x\sqrt{2s}/\sigma\right)}{\sinh\left(B\sqrt{2s}/\sigma\right)},$$
$$\mathbf{E}e^{-s\chi} = \cosh\left(\frac{x-y}{2}\sqrt{2s}/\sigma\right) \Big/ \cosh\left(\frac{B}{2}\sqrt{2s}/\sigma\right); \tag{2.3.15}$$

(ii) *the distribution of χ admits the following representation:*

$$\mathbf{P}\left[\chi \in dt; \mathfrak{A}^x\right] = \frac{\pi\sigma^2}{B^2}\sum_{k\in\mathbb{N}} k \exp\left(-\frac{t}{2}\left(k\pi\sigma/B\right)^2\right)\sin\left(\frac{x}{B}k\pi\right)dt,$$
$$\mathbf{P}\left[\chi \in dt; \mathfrak{A}_y\right] = \frac{\pi\sigma^2}{B^2}\sum_{k\in\mathbb{N}} k \exp\left(-\frac{t}{2}\left(k\pi\sigma/B\right)^2\right)\sin\left(\frac{y}{B}k\pi\right)dt, \tag{2.3.16}$$

$$\mathbf{P}\left[\chi \in dt\right] = \frac{2\pi\sigma^2}{B^2}\sum_{k=0}^{\infty}(2k+1)e^{-\frac{t}{2}((2k+1)\pi\sigma/B)^2}\sin\left(\frac{x}{B}(2k+1)\pi\right)dt;$$

(iii) *the moments of χ can be calculated as follows:*

$$\mathbf{E}\chi^n = \frac{1}{(2n-1)!!}\left(\frac{B}{2\sigma}\right)^{2n}\sum_{k=0}^{n}(-1)^k\left(\frac{x-y}{B}\right)^{2k}\binom{2n}{2k}E_{n-k}, \quad n \in \mathbb{N}. \tag{2.3.17}$$

In particular, for $x = y = B/2$

$$\mathbf{E}\chi^n = \frac{1}{(2n-1)!!}\left(\frac{B}{2\sigma}\right)^{2n} E_n, \qquad n \in \mathbb{N}^+,$$

where $E_1 = 1$, $E_2 = 5, \ldots$ are the Euler numbers, defined by their generating function:

$$\sec x - 1 \equiv \frac{E_1 x^2}{2!} + \frac{E_2 x^4}{4!} + \frac{E_3 x^6}{6!} + \ldots \tag{2.3.18}$$

$$n!! = \begin{cases} n \times (n-2) \times \ldots 5 \times 3 \times 1, & n \textit{ is odd;} \\ n \times (n-2) \times \ldots 6 \times 4 \times 2, & n \textit{ is even;} \\ 1, & n\text{=-1,0.} \end{cases}$$

Proof. For the Wiener process we have that

$$K_\pm(v, du, s) = e^{-(v+2B)\sqrt{2s}/\sigma}\delta(u)\,du, \quad \mathfrak{K}^s_\pm(v, du) = \frac{e^{-(v+2B)\sqrt{2s}/\sigma}}{1 - e^{-2B\sqrt{2s}/\sigma}}\,\delta(u)\,du.$$

Inserting these expressions into the equalities (2.3.3), we derive formulae (2.3.15) of the corollary. It is worth mentioning that formulae (2.3.15) were derived in Ito and McKean (1965). In order to determine the functions

$$\mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^x\right] = \mathbf{E}\left[e^{-s\chi}; \tau^x < \tau_y\right], \qquad \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] = \mathbf{E}\left[e^{-s\chi}; \tau_y < \tau^x\right],$$

the authors of Ito and McKean (1965) derived a system of the equations similar to (2.3.7), for the case of the Wiener process. They also found that

$$\mathbf{P}\left[\chi \in dt; \mathfrak{A}^x\right] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, \frac{e^{y\sqrt{2s}/\sigma} - e^{-y\sqrt{2s}/\sigma}}{e^{B\sqrt{2s}/\sigma} - e^{-B\sqrt{2s}/\sigma}} ds, \quad \gamma > 0.$$

Observe that the integrand in the later expression has simple poles in

$$s_k = -1/2\,\left(k\pi\sigma/B\right)^2, \quad k \in \mathbb{Z}^+ = \{0, 1, \ldots\}.$$

Thus, choosing an appropriate integration contour (see Ditkin and Prudnikov (1966)), we evaluate the latter integral, which results into the first equality of (2.3.16). The second and the third formula of (2.3.16) follow from the first one. It is worth mentioning that the distributions (2.3.16) are the limit distributions for the corresponding distributions of the first exit time for the Lévy processes and random walks. Moreover, the formulae (2.3.16) are the asymptotic expansions of the probabilities which enter their left-hand sides. The formula (2.3.17) can be derived from the series expansion of the functions $\cosh x$ and $(\cosh x)^{-1}$. ▲

Remark 2.3.1. *As a byproduct we also state the formula for the following series* $(x, y > 0, \quad x + y = B)$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \cos\left(\frac{x-y}{2B}(2k+1)\pi\right) = \frac{\pi^{2n+1}}{2^{2n+2}} \sum_{k=0}^{n} (-1)^k \left(\frac{x-y}{B}\right)^{2k} \frac{E_{n-k}}{(2k)!(2n-2k)!},$$

which can be derived from (2.3.17) and the third formula of (2.3.16). In particular, for $x = y$, *we obtain*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}} \frac{1}{(2n)!} E_n, \qquad E_0 = 1, \quad n \in \mathbb{Z}^+,$$

where E_n, $n \geq 0$ *are the Euler numbers defined by (2.3.18).*

2.3.3 Exit from the interval by a compound Poisson process with arbitrary positive and exponential negative jumps

In this section we will apply the results of Theorem 2.3.1, to determine the joint distribution of the first exit time and the value of the overshoot by the compound Poisson process with Laplace exponent (2.2.13).

Corollary 2.3.3 (Kadankov and Kadankova (2006))**.** *Let* $\{X_t;\, t \geq 0\}$ *be the compound Poisson process with Laplace exponent (2.2.13). Then for* $s > 0$:

(i) *the integral transforms of the joint distribution of* $\{\chi, T\}$ *satisfy the equalities:*

$$V_y(du, s) = f_y(s)\left(1 - \mathbf{E}e^{-s\tau^x - c(s)X_{\tau^x}}\right)(1 - T(s))^{-1}\lambda e^{-\lambda u} du,$$

$$V^x(du, s) = f^x(du, s) - \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_y\right] \mathbf{E} f^{\gamma + B}(du, s), \tag{2.3.19}$$

where $f_y(s) = \mathbf{E}e^{-s\tau_y}$,

$$\mathbf{E}f^{\gamma+B}(du,s) = \lambda\int_0^\infty e^{-\lambda v} f^{v+B}(du,s)\,dv,$$
$$T(s) = f_B(s)\int_0^\infty \mathbf{E}f^{\gamma+B}(du,s)e^{-uc(s)};$$

(ii) *the Laplace transforms of* χ *are such that*

$$V_y(s) = \mathbf{E}\left[e^{-s\chi};\mathfrak{A}_y\right] = f_y(s)\left(1-\mathbf{E}e^{-s\tau^x - c(s)X_{\tau^x}}\right)(1-T(s))^{-1},$$
$$V^x(s) = \mathbf{E}\left[e^{-s\chi};\mathfrak{A}^x\right] = f^x(s) - \mathbf{E}\left[e^{-s\chi};\mathfrak{A}_y\right]\mathbf{E}f^{\gamma+B}(s), \qquad (2.3.20)$$

where

$$f^x(s) = \mathbf{E}e^{-s\tau^x}, \qquad \mathbf{E}f^{\gamma+B}(s) = \lambda\int_0^\infty e^{-\lambda v} f^{v+B}(s)\,dv;$$

(iii) *the Laplace transform of* χ *admits the following representation:*

$$V_y(du,s) = \frac{R_s(x)}{R_s^\lambda(B)}\lambda e^{-\lambda u}\,du, \qquad V_y(s) = \frac{R_s(x)}{R_s^\lambda(B)}, \qquad (2.3.21)$$

$$V^x(s) = 1 - \frac{R_s(x)}{R_s^\lambda(B)}\left[1 + s\lambda S_s^\lambda(B)\right] + s\lambda S_s(x), \qquad (2.3.22)$$

$$\int_0^\infty e^{-st}\mathbf{P}[\chi > t]\,dt = \lambda\frac{R_s(x)}{R_s^\lambda(B)}S_s^\lambda(B) - \lambda S_s(x), \qquad (2.3.23)$$

where $R_s(x)$, $x \ge 0$ *is the scale function (2.2.22) of the compound Poisson process with negative exponential component,* $S_s(x) = \int_0^x R_s(u)\,du$,

$$R_s^\lambda(B) = \lambda\int_0^\infty e^{-\lambda v}R_s(v+B)\,du, \quad S_s^\lambda(B) = \lambda\int_0^\infty e^{-\lambda v}S_s(v+B)\,du.$$

Proof. To verify the statements of the corollary, we will use the results of Theorem 2.3.1. Employing the equalities (2.2.17), (2.2.18) and the definitions (2.3.6) of the kernels $K_\pm(v,du,s)$, we find that

$$K_-(v,du,s) = f_B(s)\mathbf{E}\left[e^{-s\tau^{v+B} - c(s)T^{v+B}}\right]\lambda e^{-\lambda u}du,$$
$$K_+(v,du,s) = e^{-c(s)v}f_B(s)\mathbf{E}f^{\gamma+B}(du,s).$$

Employing these equalities, mathematical induction and the formula (2.3.5), we determine the successive iterations $K_{\pm}^{(n)}(v,du,s)$, $n \in \mathbb{N}$ of the kernels $K_{\pm}(v,du,s)$:

$$K_{-}^{(n)}(v,du,s) = f_B(s)\mathbf{E}\left[e^{-s\tau^{v+B}-c(s)T^{v+B}}\right] T(s)^{n-1}\lambda e^{-\lambda u}\, du,$$
$$K_{+}^{(n)}(v,du,s) = e^{-vc(s)} f_B(s) T(s)^{n-1}\mathbf{E} f^{\gamma+B}(du,s).$$

The series $\mathfrak{K}_{\pm}^{s}(v,du)$ of the successive iterations $K_{\pm}^{(n)}(v,du,s)$ are the geometrical progressions, whose sums are equal to:

$$\mathfrak{K}_{-}^{s}(v,du) = f_B(s)\mathbf{E}\left[e^{-s\tau^{v+B}-c(s)T^{v+B}}\right](1-T(s))^{-1}\lambda e^{-\lambda u} du,$$
$$\mathfrak{K}_{+}^{s}(v,du) = e^{-vc(s)} f_B(s)\,(1-T(s))^{-1}\,\mathbf{E} f^{\gamma+B}(du,s).$$

Inserting the expressions for the functions $\mathfrak{K}_{\pm}^{s}(v,du)$ into (2.3.4) of Theorem 2.3.1, we obtain the equalities (2.3.19) of the corollary. Integrating (2.3.19) with respect to $u \geq 0$, we get (2.3.20). Now, using the definition (2.2.22) of the scale function and the formulae (2.2.17), (2.2.18), we find the following representations for the functions $f^x(s)$, $\mathbf{E}e^{-s\tau^x - c(s)X_{\tau^x}}$:

$$f^x(s) = 1 - \frac{s\lambda}{c(s)} R_s(x) + s\lambda S_x(s),$$
$$\mathbf{E}\left[e^{-s\tau^x - c(s)X_{\tau^x}}\right] = 1 - e^{-xc(s)} R_s(x) r(s), \tag{2.3.24}$$

where $r(s) = \frac{d}{dp}\mathbb{R}(p,s)^{-1}\Big|_{p=c(s)}$. Substituting these expressions into (2.3.20), we derive the representations (2.3.21)-(2.3.23) of the corollary. It is worth mentioning that the representations similar to (2.3.21)-(2.3.23) were derived for random walks with negative geometrical jumps in Kadankova (2003b). ▲

Another part of our research is studying the asymptotic behavior of the two-boundary characteristics of the process. One of the results is given in the following lemma. Denote $\mu = \mathbf{E}\eta$, $d^2 = \mathbf{E}\eta^2$.

Lemma 2.3.1. *Let $\{X_t;\ t \geq 0\}$ be the compound Poisson process whose Laplace exponent is given by (2.2.13). Suppose, the following conditions are satisfied:*

$$\mathbf{E}X_1 = a_1/\lambda - a_2\mu = 0, \quad \mathbf{E}X_1^2 = 2a_1/\lambda^2 + a_2 d^2 = \sigma^2 < \infty. \tag{2.3.25}$$

Then for all $x \geq 0$ the following limit equalities hold:

$$\lim_{B\to\infty} \frac{1}{B} R_{s/B^2}(xB) = \frac{1}{\lambda\sigma}\sqrt{\frac{2}{s}} \sinh\left(x\sqrt{2s}/\sigma\right),$$
$$\lim_{B\to\infty} \frac{1}{B} R^{\lambda}_{s/B^2}(xB) = \frac{1}{\lambda\sigma}\sqrt{\frac{2}{s}} \sinh\left(x\sqrt{2s}/\sigma\right), \qquad (2.3.26)$$

$$\lim_{B\to\infty} \frac{1}{B^2} S_{s/B^2}(xB) = \frac{1}{\lambda s}\left[\cosh\left(x\sqrt{2s}/\sigma\right) - 1\right],$$
$$\lim_{B\to\infty} \frac{1}{B^2} S^{\lambda}_{s/B^2}(xB) = \frac{1}{\lambda s}\left[\cosh\left(x\sqrt{2s}/\sigma\right) - 1\right]. \qquad (2.3.27)$$

Proof. The proof of the lemma is based on using asymptotic properties of the scale function of the process, which were studied for spectrally one-sided Lévy processes in Takacs (1967), Shurenkov (1978), Suprun and Shurenkov (1981), Suprun and Shurenkov (1989), and more profoundly in Borovskikh (1979), Borovskikh and Korolyuk (1981), see also Kyprianou (2006). In the proof we will use the explicit form of the Laplace exponent (2.2.13) of the process and the following limiting equality

$$\mathbf{E}e^{-p\eta} = 1 - \mu p + \frac{1}{2}\sigma^2 p^2 + o(p^2),$$

which is valid for small values of $p > 0$. Let us verify the first equality of (2.3.26). Under the assumptions (2.3.25) the function $\mathbb{R}(p,s)$ (defined by (2.2.16)) admits the following chain of equalities for $B \to \infty$

$$\mathbb{R}(p/B, s/B^2) = \left(a_1 p/B + (\lambda - p/B)\left[a_2(d^2p^2/2B^2 - \mu p/B) - s/B^2 + o(1/B^2\right]\right)^{-1}$$
$$= \frac{1}{\lambda}\left(\frac{1}{B^2}(\sigma^2p^2/2 - s) + o(1/B^2)\right)^{-1}, \quad p, s > 0.$$

Hence

$$\lim_{B\to\infty} \frac{1}{B^2}\mathbb{R}(p/B, s/B^2) = \frac{\lambda^{-1}}{\sigma^2p^2/2 - s} = \frac{1}{\lambda\sigma}\sqrt{\frac{2}{s}}\int_0^\infty e^{-px}\sinh\left(x\sqrt{2s}/\sigma\right)dx.$$

Since

$$\frac{1}{B^2}\mathbb{R}(p/B, s/B^2) = \frac{1}{B^2}\int_0^\infty e^{-up/B}R_{s/B^2}(u)\,du = \frac{1}{B}\int_0^\infty e^{-xp}R_{s/B^2}(xB)\,dx,$$

then

$$\lim_{B\to\infty} \frac{1}{B} R_{s/B^2}(xB) = \frac{1}{\lambda\sigma}\sqrt{\frac{2}{s}} \sinh\left(x\sqrt{2s}/\sigma\right).$$

The second equality of (2.3.26) can be verified analogously. We are now ready to verify the first equality of (2.3.27). The function

$$\mathbb{S}(p,s) = \int_0^\infty e^{-xp} S_s(x)\, dx = \frac{1}{p}\mathbb{R}(p,s), \quad p,s > 0$$

obeys the following expansion under the assumptions (2.3.25) for large enough values of B :

$$\mathbb{S}(p/B, s/B^2) = \frac{B}{\lambda p}\left(\frac{1}{B^2}(\sigma^2 p^2/2 - s) + o(1/B^2)\right)^{-1}, \quad p,s > 0.$$

It follows from the latter equality that

$$\lim_{B\to\infty} \frac{1}{B^3}\mathbb{S}(p/B, s/B^2) = \frac{1}{\lambda p}\, \frac{1}{\sigma^2 p^2/2 - s} = \frac{1}{\lambda s}\int_0^\infty e^{-px}\left[\cosh\left(x\sqrt{2s}/\sigma\right) - 1\right] dx.$$

Since

$$\frac{1}{B^3}\mathbb{S}(p/B, s/B^2) = \frac{1}{B^2}\int_0^\infty e^{-xp} S_{s/B^2}(xB)\, dx,$$

then

$$\lim_{B\to\infty} \frac{1}{B^2} S_{s/B^2}(xB) = \frac{1}{\lambda s}\left[\cosh\left(x\sqrt{2s}/\sigma\right) - 1\right].$$

The second formula of (2.3.27) can be established analogously. ▲

Corollary 2.3.4. *Let $\{X_t;\ t \geq 0\}$ be the compound Poisson process with negative exponential component whose Laplace exponent is given by (2.2.13), $x, y > 0$, $x + y = 1$, and let*

$$\chi(B) = inf\{t : X_t \notin [-yB, xB]\}$$

be the first exit time from the interval $[-yB, xB]$. Under the assumptions (2.3.25) the following relations hold:

$$\lim_{B\to\infty} \mathbf{P}\left[\frac{\chi(B)}{B^2}\in dt, \mathfrak{A}^{xB}\right] = \pi\sigma^2 \sum_{k\in\mathbb{N}} k e^{-\frac{t}{2}(k\pi\sigma)^2} \sin(xk\pi)\, dt,$$

$$\lim_{B\to\infty} \mathbf{P}\left[\frac{\chi(B)}{B^2}\in dt, \mathfrak{A}_{yB}\right] = \pi\sigma^2 \sum_{k\in\mathbb{N}} k e^{-\frac{t}{2}(k\pi\sigma)^2} \sin(yk\pi)\, dt, \tag{2.3.28}$$

$$\lim_{B\to\infty} \mathbf{P}\left[\frac{\chi(B)}{B^2}\in dt\right] = 2\pi\sigma^2 \sum_{k=0}^{\infty} (2k+1) e^{-\frac{t}{2}((2k+1)\pi\sigma)^2} \sin(x(2k+1)\pi)\, dt.$$

Proof. Let x, y be such that $x + y = 1$. The formula (2.3.21) implies that

$$\mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}_{yB}\right] = \frac{R_{s/B^2}(xB)}{R^{\lambda}_{s/B^2}(B)},$$

It follows from the limiting equalities (2.3.26) that

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}_{yB}\right] = \frac{\sinh\left(x\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)} = \mathbf{E}\left[e^{-s\chi^*}; \mathfrak{A}_y\right],$$

where $\chi^* = \inf\{t : w_t \notin [-y, x]\}$ is the first exit time form the interval by the Wiener process with dispersion σ^2. In view of the formulae (2.3.16) for $B = 1$ we have:

$$\lim_{B\to\infty} \mathbf{P}\left[\frac{\chi(B)}{B^2}\in dt, \mathfrak{A}_{yB}\right] = \mathbf{P}\left[\chi^* \in dt, \mathfrak{A}_y\right] = \pi\sigma^2 \sum_{k\in\mathbb{N}} k e^{-\frac{t}{2}(k\pi\sigma)^2} \sin(yk\pi)\, dt.$$

The formula (2.3.22) implies that

$$\mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}^{xB}\right] = 1 - \frac{R_{s/B^2}(xB)}{R^{\lambda}_{s/B^2}(B)}\left[1 + \frac{s\lambda}{B^2} S^{\lambda}_{s/B^2}(B)\right] + \frac{s\lambda}{B^2} S_{s/B^2}(xB).$$

Calculating the limits in the right-hand side of the latter equality as $B \to \infty$, we get

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}^{xB}\right] = \frac{\sinh\left(y\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)} = \mathbf{E}\left[e^{-s\chi^*}; \mathfrak{A}^x\right],$$

This equality and the formula (2.3.16) imply for $B = 1$ that

$$\lim_{B\to\infty} P\left[\frac{\chi(B)}{B^2}\in dt, \mathfrak{A}^{xB}\right] = P\left[\chi^* \in dt, \mathfrak{A}^x\right] = \pi\sigma^2 \sum_{k\in\mathbb{N}} k e^{-\frac{t}{2}(k\pi\sigma)^2} \sin(xk\pi)\, dt.$$

The third equality of (2.3.28) immediately follows from the other two. Hence, we established the weak convergence of the distribution of $\chi(B)/B^2$ to the distribution of χ^*.▲ It is worth noting that Doney and Maller (2002) studied the asymptotic behaviour of a Lévy process at the instant of the first exit. See also Bertoin (1997) for the asymptotic behaviour of the first exit time by a spectrally one-sided Lévy process.

2.4 Supremum, infimum and the position of the process

The aim of this section is to determine the joint distribution of the supremum, infimum and the value of the Lévy process. We will derive the integral transforms of this joint distribution in terms of the joint distribution of the first exit time and the value of the overshoot through the boundary.

Let $\{X_t;\ t \geq 0\}$, $X_0 = 0$ be the Lévy process with Laplace exponent (2.2.1), $x, y \geq 0$. We will determine the function

$$Q_p^s(-y, x) = \int_{-y}^{x} e^{-up}\mathbf{P}\left[-y \leq X_{\nu_s}^-,\ X_{\nu_s} \in du,\ X_{\nu_s}^+ \leq x\right] = \mathbf{E}\left[e^{-pX_{\nu_s}}; \chi > \nu_s\right].$$

It is worth mentioning that the joint distribution of $\{X_{\nu_s}^-, X_{\nu_s}, X_{\nu_s}^+\}$ was obtained in the monograph Skorokhod (1964), in terms of the distributions of X_t, $\{\tau^x, T^x\}$ and $\{\tau_y, T_y\}$. In order to derive the equations for the unknown function, the authors used probabilistic reasoning based on the inclusion-exclusion formula. The method of the successive iterations was applied to solve the integral equations. Our contribution is that we find the joint distribution of $\{X_{\nu_s}^-, X_{\nu_s}, X_{\nu_s}^+\}$ in terms of the distribution $\{\chi, T\}$ and of $\{X_{\nu_s}, X_{\nu_s}^{\pm}\}$, which are given by (2.2.4). The following statement is true.

Theorem 2.4.1 (Kadankov and Kadankova (2005b)). *Let $\{X_t; t \geq 0\}$, $X_0 = 0$ be the Lévy process whose Laplace exponent is given by (2.2.1), $x, y \geq 0$, $B = x + y$. Then the integral transform $Q_p^s(-y, x)$ of the joint distribution of $\{X_{\nu_s}^-, X_{\nu_s}, X_{\nu_s}^+\}$ admits the following representation:*

$$\begin{aligned} Q_p^s(-y, x) &= U^x(s, p) - e^{yp}\int_0^\infty e^{vp}V_y(dv, s)U^{v+B}(s, p), \\ Q_p^s(-y, x) &= U_y(s, p) - e^{-xp}\int_0^\infty e^{-vp}V^x(dv, s)U_{v+B}(s, p), \end{aligned} \tag{2.4.1}$$

where

$$U^x(s, p) = \mathbf{E}\left[e^{-pX_{\nu_s}}; X_{\nu_s}^+ \leq x\right] = \mathbf{E}e^{-pX_{\nu_s}^-}\mathbf{E}\left[e^{-pX_{\nu_s}^+}; X_{\nu_s}^+ \leq x\right], \quad \Re(p) \leq 0,$$

$$U_y(s, p) = \mathbf{E}\left[e^{-pX_{\nu_s}}; X_{\nu_s}^- \geq -y\right] = \mathbf{E}e^{-pX_{\nu_s}^+}\mathbf{E}\left[e^{-pX_{\nu_s}^-}; X_{\nu_s}^- \geq -y\right], \quad \Re(p) \geq 0.$$

In particular

$$\mathbf{P}[\chi > \nu_s] = \mathbf{P}\left[X^+_{\nu_s} \le x\right] - \int_0^\infty \mathbf{E}\left[e^{-s\chi}; T \in dv, \mathfrak{A}_y\right] \mathbf{P}\left[X^+_{\nu_s} \le v + B\right] \tag{2.4.2}$$

$$= \mathbf{P}\left[X^-_{\nu_s} \ge -y\right] - \int_0^\infty \mathbf{E}\left[e^{-s\chi}; T \in dv, \mathfrak{A}^x\right] \mathbf{P}\left[X^-_{\nu_s} \ge -v - B)\right].$$

Proof. We start by verifying the first formula of (2.4.1). Observe, that the increments of the process X_t on the exponential time interval $[0, \nu_s]$ given that the upper boundary x is not exceeded, can be realized either on the sample paths of the process which do not intersect the lower boundary $-y$, or on the sample paths which do intersect the lower boundary $-y$, and then do not exceed the upper boundary. In view of this remark, employing the total probability law combined with the strong Markov property and the spacial homogeneity of the process, we can write the following equation:

$$\mathbf{E}\left[e^{-pX_{\nu_s}}; X^+_{\nu_s} \le x\right] = \mathbf{E}\left[e^{-pX_{\nu_s}}; \chi > \nu_s\right] +$$
$$+ \int_0^\infty \mathbf{E}\left[e^{-s\chi}; T \in dv, \mathfrak{A}_y\right] e^{(v+y)p} \mathbf{E}\left[e^{-pX_{\nu_s}}; X^+_{\nu_s} \le v + B\right], \quad \Re(p) \le 0. \tag{2.4.3}$$

Let us give more explantation, how we derived this equation. It is obvious that the event $\{X^+_t \le x\}$ is equivalent to the event $\{\tau^x > t\}$. Then in accordance with the total probability law we write

$$\mathbf{E}\left[e^{-pX_t}; X^+_t \le x\right] = \mathbf{E}\left[e^{-pX_t}; X^+_t \le x, \tau_y > t\right] + \mathbf{E}\left[e^{-pX_t}; \tau^x > t, \tau_y \le t\right] =$$
$$\mathbf{E}\left[e^{-pX_t}; X^-_t \ge -y, X^+_t \le x\right] + \mathbf{E}\left[e^{p(y+T)} e^{-p\theta_\chi X_{t-\chi}}; \chi \le t, \tau^{B+T} > t - \chi, \mathfrak{A}_y\right],$$

where θ_t is a shift operator. Since χ is a Markov time of the process, then the increments of the process $\theta_\chi X_{t-\chi}$ and τ^{B+T} do not depend on the sigma algebra $\mathfrak{F}_\chi$. Hence,

$$\mathbf{E}\left[e^{p(y+T)} e^{-p\theta_\chi X_{t-\chi}}; \chi \le t, \tau^{B+T} > t - \chi, \mathfrak{A}_y\right] =$$
$$= e^{py} \int_0^t \int_0^\infty e^{pv} \mathbf{P}\left[\chi \in du, T \in dv, \mathfrak{A}_y\right] \mathbf{E}\left[e^{-pX_{t-u}}; \tau^{v+B} > t - u\right].$$

Substituting the right-hand side of this equation into the previous equality, we find that

$$\mathbf{E}\left[e^{-pX_t}; X_t^+ \le x\right] = \mathbf{E}\left[e^{-pX_t}; X_t^- \ge -y, X_t^+ \le x\right] +$$
$$+ e^{py} \int_0^t \int_0^\infty e^{pv} \mathbf{P}\left[\chi \in du, T \in dv, \mathfrak{A}_y\right] \mathbf{E}\left[e^{-pX_{t-u}}; X_{t-u}^+ \le v + B\right].$$

Multiplying this equality by the density $s\,e^{-st}$ of ν_s and integrating it with respect to $t \ge 0$, we obtain the formula (2.4.3). The equality (2.4.3) implies the first formula of (2.4.1). The second formula of (2.4.1) can be verified analogously. The equality (2.4.2) follows from (2.4.1) for $p = 0$. ▲

2.4.1 Supremum, infimum and the value of the spectrally positive Lévy process

Let $\{X_t;\, t \ge 0\}$, $X_0 = 0$ be the spectrally positive Lévy process with Laplace exponent (2.2.7). The integral transforms of the joint distribution of $\{X_{\nu_s}^-, X_{\nu_s}, X_{\nu_s}^+\}$ can be derived from (2.4.1) in terms of the scale function of the process. For $-y \le \alpha < \beta \le x$ denote

$$\tilde{Q}_{\alpha,\beta}^s(-y, x) = \int_0^\infty e^{-st} \mathbf{P}\left[-y \le X_t^-,\ X_t \in [\alpha, \beta],\ X_t^+ \le x\right] dt. \tag{2.4.4}$$

Corollary 2.4.1. *Let $\{X_t;\, t \ge 0\}$, $X_0 = 0$ be the spectrally positive Lévy process. Then the integral transform $Q_p^s(-y, x)$ of the joint distribution of $\{X_t^-, X_t, X_t^+\}$ is such that*

$$Q_p^s(-y, x) = U^x(s, p) - e^{py} \frac{R_s(x)}{R_s(B)} U^B(s, p), \qquad B = x + y, \tag{2.4.5}$$

$$\tilde{Q}_{\alpha,\beta}^s(-y, x) = \frac{R_s(x)}{R_s(B)} \int_\alpha^\beta R_s(u + y)\, du - \int_{\max\{0,\alpha\}}^{\max\{0,\beta\}} R_s(u)\, du, \tag{2.4.6}$$

where

$$U^x(s, p) = \mathbf{E}\left[e^{-pX_{\nu_s}}; X_{\nu_s}^+ \le x\right] = \frac{c(s)}{c(s) - p} \mathbf{E}\left[e^{-pX_{\nu_s}^+}; X_{\nu_s}^+ \le x\right], \tag{2.4.7}$$

Proof. It follows from the corollary 2.3.1 that

$$V_y(du,s) = \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}_y\right] = \frac{R_s(x)}{R_s(B)}\,\delta(u)\,du.$$

Inserting the expression for $V_y(du,s)$ into the first formula of (2.4.1) of Theorem 2.4.1, we get the equality (2.4.5) of the corollary.

Now we will determine the function $U^x(s,p)$, $x \geq 0$. Employing formulae (2.2.8) for the integral transforms of the distributions of $X^-_{\nu_s}$, $X^+_{\nu_s}$, and the equality (2.2.4), we obtain the integral transform of the joint distribution of $\{X_{\nu_s}, X^+_{\nu_s}\}$:

$$\int_0^\infty e^{-zx} U^x(s,p)\,dx = \frac{s}{c(s)-p}\left(1 - \frac{c(s)-p}{z}\right)\frac{1}{k(p+z)-s}, \qquad \Re(z) \geq 0.$$

Taking into account the defining formula (2.2.11) of the scale function, we invert the Laplace transforms with respect to z which enter the right-hand side of the latter equality. This yields

$$U^x(s,p) = \frac{s}{c(s)-p} e^{-xp} R_s(x) - s\int_0^x e^{-up} R_s(u)\,du, \qquad x \geq 0$$

i.e. the representation of $U^x(s,p)$, $x \geq 0$ in terms of the scale function. Substituting the expression for $U^x(s,p)$ into (2.4.5), we find that

$$Q^s_p(-y,x) = s\int_{-y}^x e^{-up}\left[\frac{R_s(x)}{R_s(B)} R_s(u+y) - R_s(u)\right] du$$

where $R_s(x) = 0$ for $x < 0$. Inverting the Laplace transforms in the right-hand side of the latter equality, we get

$$\mathbf{P}\left[-y \leq X^-_{\nu_s},\ X_{\nu_s} \in du,\ X^+_{\nu_s} \leq x\right] = s\frac{R_s(x)}{R_s(B)} R_s(u+y)\,du - sR_s(u)\,du \tag{2.4.8}$$

This density of the joint distribution of $\{X^-_{\nu_s}, X_{\nu_s}, X^+_{\nu_s}\}$ was determined (by employing Dynkin's formula) in Suprun (1976), Shurenkov (1978). Integrating the equality (2.4.8) over the interval $[\alpha,\beta]$, we obtain (2.4.6). ▲

Corollary 2.4.2 (Kadankova (2004))**.** *Let* $\{w_t;\, t \geq 0\}$, $w_0 = 0$ *be the Wiener process whose Laplace exponent is such that* $k(p) = \sigma^2 p^2/2$. *Denote by* $w^-_t =$

$\inf\limits_{u\le t} w_u$ *the running infimum and by* $w_t^+ = \sup\limits_{u\le t} w_u$ *the running supremum of the process. Then the joint distribution*

$$\bar q^t_{\alpha,\beta}(-y,x) = \mathbf{P}\left[-y \le w_t^-,\ w_t \in [\alpha,\beta],\ w_t^+ \le x\right], \quad -y \le \alpha < \beta \le x$$

obeys the following equality for all $x, y \ge 0$ $(B = x + y)$

$$\bar q^t_{\alpha,\beta}(-y,x) = \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\exp\left(-\frac{t}{2}(\pi\nu\sigma/B)^2\right) \times \sin\left(\frac{x}{B}\pi\nu\right)\sin\left(\frac{2x-\alpha-\beta}{2B}\pi\nu\right)\sin\left(\frac{\beta-\alpha}{2B}\pi\nu\right). \qquad (2.4.9)$$

Proof. Firstly, we calculate the scale function of the Wiener process.

$$R_s(x) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{px}\,\frac{dp}{\frac{1}{2}\,p^2\sigma^2 - s} = \frac{1}{\sigma}\sqrt{\frac{2}{s}}\ \sinh\left(\frac{x}{\sigma}\sqrt{2s}\right), \qquad (2.4.10)$$

where $\gamma > \sqrt{2s}/\sigma$, $\sinh u = \left(e^u - e^{-u}\right)/2$. Inserting the expression for the scale function (2.4.10) into (2.4.6), we obtain

$$\tilde Q^s_{\alpha,\beta}(-y,x) = \frac{1}{s}\,\frac{\sinh\frac{x}{\sigma}\sqrt{2s}}{\sinh\frac{B}{\sigma}\sqrt{2s}}\left[\cosh\left(\frac{y+\beta}{\sigma}\sqrt{2s}\right) - \cosh\left(\frac{y+\alpha}{\sigma}\sqrt{2s}\right)\right] - \frac{1}{s}\left[\cosh\left(\frac{\beta^+}{\sigma}\sqrt{2s}\right) - \cosh\left(\frac{\alpha^+}{\sigma}\sqrt{2s}\right)\right], \qquad (2.4.11)$$

where $\alpha^+ = \max\{0,\alpha\}$, $\beta^+ = \max\{0,\beta\}$, $\cosh u = \left(e^u + e^{-u}\right)/2$. We have to distinguish three cases, namely $0 \le \alpha < \beta$, $\alpha < 0 < \beta$, $\alpha < \beta \le 0$. Then the formula (2.4.11) implies that

$$\tilde Q^s_{\alpha,\beta}(-y,x) = \frac{2}{s}\,\frac{\sinh\frac{y}{\sigma}\sqrt{2s}}{\sinh\frac{B}{\sigma}\sqrt{2s}}\ \sinh\left(\frac{2x-\alpha-\beta}{2\sigma}\sqrt{2s}\right)\sinh\left(\frac{\beta-\alpha}{2\sigma}\sqrt{2s}\right), \quad 0 \le \alpha < \beta;$$

$$\tilde Q^s_{\alpha,\beta}(-y,x) = \frac{2}{s}\,\frac{\sinh\frac{y}{\sigma}\sqrt{2s}}{\sinh\frac{B}{\sigma}\sqrt{2s}}\ \sinh\left(\frac{2x-\beta}{2\sigma}\sqrt{2s}\right)\sinh\left(\frac{\beta}{2\sigma}\sqrt{2s}\right) + \frac{2}{s}\,\frac{\sinh\frac{x}{\sigma}\sqrt{2s}}{\sinh\frac{B}{\sigma}\sqrt{2s}}\ \sinh\left(\frac{2y+\alpha}{2\sigma}\sqrt{2s}\right)\sinh\left(\frac{-\alpha}{2\sigma}\sqrt{2s}\right), \quad \alpha < 0 < \beta; \qquad (2.4.12)$$

$$\tilde Q^s_{\alpha,\beta}(-y,x) = \frac{2}{s}\,\frac{\sinh\frac{x}{\sigma}\sqrt{2s}}{\sinh\frac{B}{\sigma}\sqrt{2s}}\ \sinh\left(\frac{2y+\alpha+\beta}{2\sigma}\sqrt{2s}\right)\sinh\left(\frac{\beta-\alpha}{2\sigma}\sqrt{2s}\right), \quad \alpha < \beta \le 0.$$

Let us consider the first equality of (2.4.12). Employing the inversion formula (Doutsch (1960)), we get

$$\bar{q}^{t}_{\alpha,\beta}(-y,x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\tilde{Q}^{s}_{\alpha,\beta}(-y,x)\,ds, \qquad \gamma>0.$$

The integrand is analytic in the entire complex plane apart from the points

$$s_\nu = -\frac{1}{2}\left(\frac{\pi\nu\sigma}{B}\right)^2, \qquad \nu\in\mathbb{N},$$

where it has simple poles. Hence, choosing an appropriate integration contour (Doutsch (1960)), we calculate the residues in these poles. This yields

$$\begin{aligned}\bar{q}^{t}_{\alpha,\beta}(-y,x) &= \sum_{\nu\in\mathbb{N}} Res_{s=s_\nu} e^{st}\,\tilde{Q}^{s}_{\alpha,\beta}(-y,x) \\ &= \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}e^{-\frac{1}{2}t(\pi\nu\sigma/B)^2}\sin\left(\frac{x}{B}\nu\pi\right)\sin\left(\frac{2x-\alpha-\beta}{2B}\nu\pi\right)\sin\left(\frac{\beta-\alpha}{2B}\nu\pi\right).\end{aligned} \tag{2.4.13}$$

In the same vain we establish that for all three cases of (2.4.12) that the equality (2.4.13) holds. ▲

Remark 2.4.1. *It is worth mentioning that the distribution (2.4.13) is the limit distribution for the corresponding distribution of Lévy processes and random walks. Note, that the right-hand side of this equality is the asymptotic expansion of the probability which enters its left-hand side. This distribution admits the following representation:*

$$\begin{aligned}&\mathbf{P}\left[-y\le\inf_{u\le t} w_u,\ w_t\in[\alpha,\beta],\ \sup_{u\le t} w_u\le x\right] = \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\alpha}^{\beta}\left(\sum_{k=-\infty}^{\infty} e^{-(u-2k(x+y))^2/2t\sigma^2} - \sum_{k=-\infty}^{\infty} e^{-(u-2x-2k(x+y))^2/2t\sigma^2}\right)du,\end{aligned}$$

which is due to Lévy (1948).

We now establish the asymptotic results for the joint distribution (2.4.4).

Corollary 2.4.3 (Kadankova (2004))**.** *Let $\{X_t;\ t\ge 0\}$, $X_0=0$ be the spectrally positive Lévy process with Laplace exponent (2.2.7). Assume that*

$\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = \sigma^2 < \infty$, $x, y > 0$, $x + y = 1$, $-y \le \alpha < \beta \le x$, *and denote by*

$$q^{tB^2}_{\alpha B,\beta B}(-yB, xB) = \mathbf{P}\left[-yB \le X^-_{tB^2},\, X_{tB^2} \in [\alpha B, \beta B],\, X^+_{tB^2} \le xB\right]$$

the joint distribution of $\{X^+, X, X^-\}$ *re-scaled with respect to time by the coefficient* B^2, *and with respect to space by the coefficient* B. *Then this re-scaled joint distribution* $q^{tB^2}_{\alpha B,\beta B}(-yB, xB)$ *converges weakly as* $B \to \infty$ *to the joint distribution of the infimum, supremum and the value of the Wiener process* w_t *with Laplace exponent* $k(p) = \sigma^2 p^2/2$. *Besides, the following equality holds:*

$$\begin{aligned}
&\lim_{B\to\infty} q^{tB^2}_{\alpha B,\beta B}(-yB, xB) = \bar{q}^{t}_{\alpha,\beta}(-y, x) = \\
&= \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}e^{-\frac{1}{2}t(\pi\nu\sigma)^2}\sin(x\pi\nu)\sin\left(\frac{2x-\alpha-\beta}{2}\pi\nu\right)\sin\left(\frac{\beta-\alpha}{2}\pi\nu\right).
\end{aligned} \tag{2.4.14}$$

Proof. We suppose that the assumptions of Corollary 2.4.3 are satisfied. The following chain of the equalities is obvious

$$\begin{aligned}
&\frac{s}{B^2}\tilde{Q}^{s/B^2}_{\alpha B,\beta B}(-yB, xB) = \\
&\quad = \frac{s}{B^2}\int_0^\infty e^{-us/B^2}\mathbf{P}\left[-yB \le X^-_u,\ X_u \in [\alpha B, \beta B],\ X^+_u \le xB\right] du \\
&\quad = s\int_0^\infty e^{-st}\mathbf{P}\left[-yB \le X^-_{tB^2},\ X(tB^2) \in [\alpha B, \beta B],\ X^+_{tB^2} \le xB\right] dt \\
&\quad = s\int_0^\infty e^{-st} q^{tB^2}_{\alpha B,\beta B}(-yB, xB)\, dt.
\end{aligned}$$

Thus, formula (2.4.6) of Corollary 2.4.1 implies that

$$\begin{aligned}
&\lim_{B\to\infty} s\int_0^\infty e^{-st} q^{tB^2}_{\alpha B,\beta B}(-yB, xB)\, dt = \lim_{B\to\infty}\frac{s}{B^2}\tilde{Q}^{s/B^2}_{\alpha B,\beta B}(-yB, xB) = \\
&= \lim_{B\to\infty}\frac{s}{B^2}\left(\frac{R_{s/B^2}(xB)}{R_{s/B^2}(B)}\int\limits_{B(y+\alpha)}^{B(y+\beta)} R_{s/B^2}(u)\, du - \int\limits_{B\max\{0,\alpha\}}^{B\max\{0,\beta\}} R_{s/B^2}(u)\, du\right).
\end{aligned} \tag{2.4.15}$$

We will use the asymptotic properties of the scale function of the spectrally positive Lévy process (Borovskikh (1979), Borovskikh and Korolyuk (1981)).

We will need the following relations:

$$\lim_{B\to\infty}\frac{1}{B}R_{s/B^2}(xB)=\frac{1}{\sigma}\sqrt{\frac{2}{s}}\ \sinh\left(\frac{x}{\sigma}\sqrt{2s}\right),$$
$$\lim_{B\to\infty}\frac{s}{B^2}\int_0^{xB}R_{s/B^2}(u)\,du=\cosh\left(\frac{x}{\sigma}\sqrt{2s}\right)-1.$$

Taking into account these equalities, we calculate the limit in the right-hand side of (2.4.15), which yields

$$\lim_{B\to\infty}\int_0^\infty e^{-st}q^{tB^2}_{\alpha B,\beta B}(-yB,xB)\,dt=\frac{1}{s}\left[\cosh\left(\frac{\alpha^+}{\sigma}\sqrt{2s}\right)-\cosh\left(\frac{\beta^+}{\sigma}\sqrt{2s}\right)\right]$$
$$+\frac{1}{s}\,\frac{\sinh\frac{x}{\sigma}\sqrt{2s}}{\sinh\frac{1}{\sigma}\sqrt{2s}}\left[\cosh\left(\frac{y+\beta}{\sigma}\sqrt{2s}\right)-\cosh\left(\frac{y+\alpha}{\sigma}\sqrt{2s}\right)\right].$$

The right-hand side of this equality coincides with the right-hand side of (2.4.11) for $B=1$. Hence, the left-hand sides also coincide and

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\mathbf{P}\left[-yB\le X^-_{tB^2},\ X_{tB^2}\in[\alpha B,\beta B],\ X^+_{tB^2}\le xB\right]dt=$$
$$=\int_0^\infty e^{-st}\mathbf{P}\left[-y\le\inf_{u\le t}w_u,\ w_t\in[\alpha,\beta],\ \sup_{u\le t}w_u\le x\right]dt.$$

where $x,y>0$, $x+y=1$. Thus we established the weak convergence of the joint distribution $q^{tB^2}_{\alpha B,\beta B}(-yB,xB)$ to the corresponding distribution of the Wiener process as $B\to\infty$. The equality (2.4.14) follows from (2.4.9) for $B=1$. ▲

2.4.2 Supremum, infimum and the value of a compound Poisson process with jumps of both signs

Let $\{X_t;\ t\ge 0\}$, $X_0=0$ be the compound Poisson process whose Laplace exponent is given by (2.2.13). We will now apply the results of Theorem 2.4.1 to determine the joint distribution of $\{X^+_{\nu_s},X_{\nu_s},X^-_{\nu_s}\}$ for this process.

Corollary 2.4.4 (Kadankov and Kadankova (2006))**.** *Let* $\{X_t;t\ge 0\}$, $X_0=0$ *be the compound Poisson process whose Laplace exponent is given by (2.2.13),* $x,y\ge 0$, $B=x+y$*. Then the joint distribution of* $\{X^-_{\nu_s},X_{\nu_s},X^+_{\nu_s}\}$ *obeys the* *formula:*

$$\tilde{Q}^s_u(-y,x) = \mathbf{P}\left[-y \le \inf_{t\le\nu_s} X_t,\ X_{\nu_s} \le u,\ \sup_{t\le\nu_s} X_t \le x\right] = \tag{2.4.16}$$

$$= s\lambda \frac{R_s(x)}{R_s^\lambda(B)} \int\limits_0^{u+y} R_s(v)\,dv - s\lambda \int\limits_0^{u} R_s(v)\,dv + sR_s(u), \qquad u \in [-y,x].$$

where $R_s(v) = 0$ *for* $v < 0$.

Proof. Firstly, we find the expression for $U^x(s,p)$, $x \ge 0$, in case of the compound Poisson process with negative exponential component. Employing formulae (2.2.14), (2.2.15) for the integral transforms of $X^-_{\nu_s}$, $X^+_{\nu_s}$, and the equality (2.2.4), we find the integral transform of the joint distribution of $\{X_{\nu_s}, X^+_{\nu_s}\}$:

$$\int_0^\infty e^{-zx} U^x(s,p)\,dx = s\frac{\lambda - p}{c(s)-p}\left(1 - \frac{c(s)-p}{z}\right)\mathbb{R}(p+z,s), \qquad \Re(z) \ge 0.$$

Taking into the account the defining formula of the scale function (2.2.22), we invert the Laplace transforms in the right-hand side of the latter equality, which yields

$$U^x(s,p) = s\frac{\lambda-p}{c(s)-p} e^{-xp} R_s(x) - s(\lambda - p)\int_0^x e^{-up} R_s(u)\,du, \qquad x \ge 0$$

i.e. the representation of $U^x(s,x)$, $x \ge 0$ in terms of the scale function. It follows from (2.3.11) and from Corollary 2.3.3 that

$$V_y(du,s) = \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}_y\right] = \frac{R_s(x)}{R_s^\lambda(B)}\,\lambda e^{-\lambda u} du.$$

Inserting the expression for the functions $U^x(s,p)$, $x \ge 0$, $V_y(du,s)$ into the first formula of (2.4.1) of Theorem 2.4.1, we obtain

$$Q^s_p(-y,x) = s(p-\lambda)\int_0^x e^{-up} R_s(u)\,du + se^{-xp} R_s(x)$$
$$+ s\lambda \frac{R_s(x)}{R_s^\lambda(B)} \int_0^B e^{-p(u-y)} R_s(u)\,du.$$

It is not difficult to derive the equality

$$\int_{-y}^{x} e^{-up}\mathbf{P}\left[X_{\nu_s} \le u, \chi > \nu_s\right] du = \frac{1}{p}\left(Q_p^s(-y,x) - \mathbf{P}\left[\chi > \nu_s\right] e^{-xp}\right). \quad (2.4.17)$$

Now taking into account the obvious identity $\lambda S_s^\lambda(B) = R_s^\lambda(B) + \lambda S_s(B)$, we derive from the formula (2.3.23) the expression for $\mathbf{P}\left[\chi > \nu_s\right]$

$$\mathbf{P}\left[\chi > \nu_s\right] = sR_s(x) + s\lambda\frac{R_s(x)}{R_s^\lambda(B)}\int_0^B R_s(u)\, du - s\lambda\int_0^x R_u(s)\, du.$$

Inserting the expressions for $Q_p^s(-y,x)$, $\mathbf{P}\left[\chi > \nu_s\right]$ into (2.4.17) and performing some manipulations, we get

$$\int_{-y}^{x} e^{-pu}\mathbf{P}\left[-y \le \inf_{t\le\nu_s} X_t,\ X_{\nu_s} \le u,\ \sup_{t\le\nu_s} X_t \le x\right] du =$$
$$= \int_{-y}^{x} e^{-pu}\left(s\lambda\,\frac{R_s(x)}{R_s^\lambda(B)}\int_0^{u+y} R_s(v)\, dv - s\lambda\int_0^{u} R_s(v)\, dv + sR_s(u)\right) du,$$

where $R_s(u) = 0$ for $u < 0$. The latter formula is the equality of the Laplace transforms. Hence, the original functions coincide as well, which yields the formula (2.4.16). ▲

Corollary 2.4.5. *Let* $\{X_t; t \ge 0\}$, $X(0) = 0$ *be the compound Poisson process whose Laplace exponent is give by (2.2.13). Assume that* $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = \sigma^2 < \infty$, $x, y > 0$, $x + y = 1$, $u \in [-y, x]$. *Denote by*

$$q_{uB}^{tB^2}(-yB, xB) = \mathbf{P}\left[-yB \le X_{tB^2}^-,\ X_{tB^2} \le uB,\ X_{tB^2}^+ \le xB\right]$$

the joint distribution of $\{X^+, X, X^-\}$ *re-scaled with respect to time by parameter* B^2, *and with respect to space by parameter* B. *Then this re-scaled distribution* $q_{uB}^{tB^2}(-yB, xB)$ *converges weakly as* $B \to \infty$ *to the joint distribution of the supremum, infimum and the value of the Wiener process* w_t *whose Laplace transform is of the form* $k(p) = \sigma^2 p^2/2$. *In addition, the following relation is valid:*

$$\lim_{B\to\infty} q_{uB}^{tB^2}(-yB, xB) = \bar{q}_u^t(-y, x) =$$
$$= \frac{4}{\pi}\sum_{k\in\mathbb{N}} \frac{(-1)^{k-1}}{k}\, e^{-\frac{t}{2}(\pi k\sigma)^2} \sin(xk\pi)\left[\sin\left(\frac{y+u}{2}\, k\pi\right)\right]^2. \quad (2.4.18)$$

Proof. We assume that the conditions imposed in the Corollary 2.4.5 are satisfied. We start proof by writing the obvious chain of the equalities.

$$\frac{s}{B^2}\,\tilde{Q}^{s/B^2}_{uB}(-yB,xB) = $$
$$= \frac{s}{B^2}\int_0^\infty e^{-vs/B^2}\mathbf{P}\left[-yB \le X_v^-,\ X_v \le uB,\ X_v^+ \le xB\right]dv$$
$$= s\int_0^\infty e^{-st}\mathbf{P}\left[-yB \le X^-_{tB^2},\ X(tB^2) \le uB,\ X^+_{tB^2} \le xB\right]dt$$
$$= s\int_0^\infty e^{-st}q^{tB^2}_{uB}(-yB,xB)\,dt.$$

Taking into account the formulae (2.4.16) of Corollary 2.4.4, we derive

$$\lim_{B\to\infty} s\int_0^\infty e^{-st}q^{tB^2}_{uB}(-yB,xB)\,dt = \lim_{B\to\infty}\frac{s}{B^2}\tilde{Q}^{s/B^2}_{uB}(-yB,xB) = \tag{2.4.19}$$
$$= \lim_{B\to\infty}\frac{s\lambda}{B^2}\left(\frac{R_{s/B^2}(xB)}{R^\lambda_{s/B^2}(B)}\int\limits_0^{(u+y)B}R_{s/B^2}(v)\,dv - \int\limits_0^{uB}R_{s/B^2}(v)\,dv + \frac{1}{\lambda}R_{s/B^2}(uB)\right).$$

In order to calculate the limits in the right-hand side of (2.4.19), we employ the limit equalities (2.3.26), (2.3.27) of Lemma 2.3.1. This step implies that

$$\lim_{B\to\infty}\int_0^\infty e^{-st}q^{tB^2}_{uB}(-yB,xB)\,dt = \frac{1}{s}\left[\cosh\left(u^+\sqrt{2s}/\sigma\right) - 1\right]$$
$$+\frac{1}{s}\,\frac{\sinh\left(x\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)}\left[\cosh\left((y+u)\sqrt{2s}/\sigma\right) - 1\right].$$

$u^+ = \max\{0,u\}$. The right-hand side of this equality coincides with the right-hand side of (2.4.11) for $B = 1$, $\alpha = -y$, $\beta = u$. Thus, the left-hand sides coincide as well and

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\mathbf{P}\left[-yB \le X^-_{tB^2},\ X_{tB^2} \le uB,\ X^+_{tB^2} \le xB\right]dt =$$
$$= \int_0^\infty e^{-st}\mathbf{P}\left[-y \le \inf_{u\le t} w_u,\ w_t \le u,\ \sup_{u\le t} w_u \le x\right]dt,$$

where $x, y > 0$, $x + y = 1$. Hence, we established the weak convergence of the distribution $q^{tB^2}_{uB}(-yB,xB)$ to the corresponding distribution of the Wiener process as $B \to \infty$. The formula (2.4.18) follows from (2.4.9) for $B = 1$, $\alpha = -y$, $\beta = u$. ▲

2.5 Intersections of the interval

Let $\{X_t;\ t \geq 0\}$, $X_0 = 0$ be a general Lévy process. Following the notation from Lotov and Orlova (2004), we introduce the random sequences ($k \in \mathbb{N}$)

$$i_0^{\pm} = 0, \quad i_k^{-} = \inf\{t > i_{k-1}^{+} : X_t < -y\}; \quad i_k^{+} = \inf\{t > i_k^{-} : X_t > x\},$$

where $\inf\{\emptyset\} = \infty$. Denote by

$$\alpha_t^{+} = \max\{k \in \mathbb{Z}^{+} : i_k^{+} \leq t\} \in \mathbb{Z}^{+},$$

the number of the up-crossings of the interval $[-y, x]$ by the process X_t on the time interval $[0, t]$. Similarly, we introduce

$$\tilde{i}_0^{\pm} = 0, \quad \tilde{i}_k^{+} = \inf\{t > \tilde{i}_{k-1}^{-} : X_t > x\}; \quad \tilde{i}_k^{-} = \inf\{t > \tilde{i}_k^{+} : X_t < -y\}.$$

Define

$$\alpha_t^{-} = \max\{k \in \mathbb{Z}^{+} : \tilde{i}_k^{-} \leq n\} \in \mathbb{Z}^{+},$$

i.e. the number of the down-crossings of the interval $[-y, x]$ by the process X_t on the time interval $[0, t]$. For all $k, l \in \mathbb{Z}^{+}$, $|k - l| \leq 1$ introduce the following notation:

$$p_k^l(t) = \mathbf{P}\left[\alpha_t^{+} = k, \alpha_t^{-} = l\right], \quad p_k^{\pm}(t) = \mathbf{P}\left[\alpha_t^{\pm} = k\right], \quad p_k(t) = \mathbf{P}\left[\alpha_t = k\right], \tag{2.5.1}$$

where $\alpha_t = \alpha_t^{+} + \alpha_t^{-}$ is the total number of the intersections of the interval $[-y, x]$ by the process X_t. Here and in the sequel we will denote the length of the interval $B = x + y$.

It is worth mentioning that the distribution $p_k^{\pm}(\nu_s)$, $k \in \mathbb{Z}^{+}$ was found in Lotov and Orlova (2004) for random walks, by using the projectors applied to the factorization components of the random walk. Our contribution is that we obtain the distributions (2.5.1) in terms of the joint distribution of $\{\chi, T\}$ and of the successive iterations $K_{\pm}^{(n)}(v, du, s)$, $n \in \mathbb{N}$ (2.3.5).

Let $v \geq 0$, $a, b \in [0,1]$, and introduce the generating functions

$$h^v(s,a,b) = \mathbf{E}\left[a^{\alpha^+_{\nu_s}} b^{\alpha^-_{\nu_s}} / X_0 = x + v\right],$$
$$h(s,a,b) = \mathbf{E}\left[a^{\alpha^+_{\nu_s}} b^{\alpha^-_{\nu_s}} / X_0 = 0\right],$$
$$h_v(s,a,b) = \mathbf{E}\left[a^{\alpha^+_{\nu_s}} b^{\alpha^-_{\nu_s}} / X_0 = -(y + v)\right].$$

Theorem 2.5.1 (Kadankov (2005))**.** *Let $\{X_t;\ t \geq 0\}$, $X_0 = 0$ be the general Lévy process. Then*

(i) *the generating functions of the joint distribution of $\{\alpha^+_{\nu_s}, \alpha^-_{\nu_s}\}$ are given as follows:*

$$h^v(s,a,b) = \int_0^\infty \mathfrak{K}^s_+(v,du,a,b)\,(1 - f_{u+B}(s)) + b\int_0^\infty \mathfrak{K}^s_+(v,du,a,b)\int_0^\infty f_{u+B}(dl,s)\left(1 - f^{l+B}(s)\right), \tag{2.5.2}$$

$$h_v(s,a,b) = \int_0^\infty \mathfrak{K}^s_-(v,du,a,b)\,\left(1 - f^{u+B}(s)\right) + a\int_0^\infty \mathfrak{K}^s_-(v,du,a,b)\int_0^\infty f^{u+B}(dl,s)\,(1 - f_{l+B}(s)), \tag{2.5.3}$$

$$h(s,a,b) = 1 - \mathbf{E}e^{-s\chi} + \int_0^\infty V^x(dv,s)h^v(a,b,s) + \int_0^\infty V_y(dv,s)h_v(a,b,s), \tag{2.5.4}$$

where $f^x(s) = \mathbf{E}e^{-s\tau^x}$, $f_x(s) = \mathbf{E}e^{-s\tau_x}$,

$$\mathfrak{K}^s_\pm(v,du,a,b) = \delta(u-v)\,du + \sum_{n\in\mathbb{N}}(ab)^n K^{(n)}_\pm(v,du,s);$$

(ii) *the joint distribution*

$$\tilde{p}^l_k(s) = p^l_k(\nu_s), \qquad k,l \in \mathbb{Z}^+, \quad |k - l| \leq 1$$

of $\{\alpha^+_{\nu_s}, \alpha^-_{\nu_s}\}$ *obeys the relations for* $n \in \mathbb{Z}^+$

$$\tilde{p}_n^{n+1}(s) = \int\limits_0^\infty V^x(dv,s) \int\limits_0^\infty K_+^{(n)}(v,du,s) \int\limits_0^\infty f_{u+B}(dl,s)(1-f^{l+B}(s)),$$

$$\tilde{p}_{n+1}^{n}(s) = \int_0^\infty V_y(dv,s) \int_0^\infty K_-^{(n)}(v,du,s) \int_0^\infty f^{u+B}(dl,s)\left(1-f_{l+B}(s)\right), \tag{2.5.5}$$

$$\begin{aligned}\tilde{p}_n^{n}(s) = \mathbf{I}_{\{n=0\}} &\left(1 - \int_0^\infty V^x(dv,s) f_{v+B}(s) - \int_0^\infty V_y(dv,s) f^{v+B}(s)\right) \\ &+ \mathbf{I}_{\{n\in\mathbb{N}\}} \int_0^\infty V^x(dv,s) \int_0^\infty K_+^{(n)}(v,du,s)\left(1-f_{u+B}(s)\right) \\ &+ \mathbf{I}_{\{n\in\mathbb{N}\}} \int_0^\infty V_y(dv,s) \int_0^\infty K_-^{(n)}(v,du,s)\left(1-f^{u+B}(s)\right);\end{aligned}$$

(iii) *the distributions* $\tilde{p}_n^{\pm}(s) = \mathbf{P}\left[\alpha^{\pm}_{\nu_s} = n\right]$, $n \in \mathbb{Z}^+$ *are such that*

$$\begin{aligned}\tilde{p}_0^{+}(s) &= 1 - \int_0^\infty V_y(dv,s) f^{v+B}(s) - \int_0^\infty V^x(dv,s) K_+^{(1)}(v,s), \\ \tilde{p}_n^{+}(s) &= \int_0^\infty V^x(dv,s)\left[K_+^{(n)}(v,s) - K_+^{(n+1)}(v,s)\right] \\ &+ \int_0^\infty V_y(dv,s) \int_0^\infty \left[K_-^{(n-1)}(v,du,s) - K_-^{(n)}(v,du,s)\right] f^{u+B}(s), \quad n\in\mathbb{N},\end{aligned} \tag{2.5.6}$$

$$\begin{aligned}\tilde{p}_0^{-}(s) &= 1 - \int_0^\infty V^x(dv,s) f_{v+B}(s) - \int_0^\infty V_y(dv,s) K_-^{(1)}(v,s) \\ \tilde{p}_n^{-}(s) &= \int_0^\infty V_y(dv,s)\left[K_-^{(n)}(v,s) - K_-^{(n+1)}(v,s)\right] \\ &+ \int_0^\infty V^x(dv,s) \int_0^\infty \left[K_+^{(n-1)}(v,du,s) - K_+^{(n)}(v,du,s)\right] f_{u+B}(s), \quad n\in\mathbb{N},\end{aligned} \tag{2.5.7}$$

where $K_+^{(0)}(v,du,s) = \delta(u-v)\,du$,

$$K_{\pm}^{(n)}(v,s) = \int_0^\infty K_{\pm}^{(n)}(v,du,s), \quad n\in\mathbb{N};$$

(iiii) *for the distributions* $\tilde{p}_n(s) = \mathbf{P}\left[\alpha_{\nu_s} = n\right]$, $n \in \mathbb{Z}^+$ *the following equalities hold:*

$$\tilde{p}_0(s) = 1 - \int_0^\infty V^x(dv,s) f_{v+B}(s) - \int_0^\infty V_y(dv,s) f^{v+B}(s),$$

$$\tilde{p}_{2n}(s) = \int_0^\infty V^x(dv,s) \left[K_+^{(n)}(v,s) - \int_0^\infty K_+^{(n)}(v,du,s) f_{u+B}(s) \right]$$
$$+ \int_0^\infty V_y(dv,s) \left[K_-^{(n)}(v,s) - \int_0^\infty K_-^{(n)}(v,du,s) f^{u+B}(s) \right], \quad n \in \mathbb{N}, \tag{2.5.8}$$

$$\tilde{p}_{2n+1}(s) = \int_0^\infty V^x(dv,s) \left[\int_0^\infty K_+^{(n)}(v,du,s) f_{u+B}(s) - K_+^{(n+1)}(v,s) \right]$$
$$+ \int_0^\infty V_y(dv,s) \left[\int_0^\infty K_-^{(n)}(v,du,s) f^{u+B}(s) - K_-^{(n+1)}(v,s) \right], \quad n \in \mathbb{Z}^+. \tag{2.5.9}$$

Proof. To verify the formulae of the theorem, we will follow the reasoning similar to those from the previous sections. That is to say, we will set up a system of equations for unknown functions and will use the solution of the two-sided exit problem. The total probability law combined with the spatial and time homogeneity of the process X_t and Markov property of τ^x, τ_x, χ allow us to write the following system of the equations for the functions $h^v(s,a,b)$, $h_v(s,a,b)$, $h(s,a,b)$:

$$h^v(s,a,b) = 1 - \mathbf{E}e^{-s\tau_{v+B}} + b \int_0^\infty f_{v+B}(du,s) h_u(s,a,b),$$
$$h_u(s,a,b) = 1 - \mathbf{E}e^{-s\tau^{u+B}} + a \int_0^\infty f^{u+B}(dv,s) h^v(s,a,b), \tag{2.5.10}$$
$$h(s,a,b) = 1 - \mathbf{E}e^{-s\chi} + \int_0^\infty V^x(dv,s) h^v(s,a,b) + \int_0^\infty V_y(du,s) h_u(s,a,b).$$

Substituting the expression for the function $h_u(s,a,b)$ from the second equation into the first one, we find that

$$h^v(s,a,b) = 1 - f_{v+B}(s) + b \int_0^\infty f_{v+B}(du,s) \left(1 - f^{u+B}(s)\right)$$
$$+ ab \int_0^\infty K_+(v,du,s) h^u(s,a,b), \qquad v \geq 0,$$

i.e. a linear integral equation for the function $h^v(s,a,b)$. Employing the method of successive iterations (Petrovskii (1965)), we obtain

$$h^v(s,a,b) = \int_0^\infty \mathfrak{K}_+^s(v,du,a,b)\left(1-\mathbf{E}e^{-s\tau_{u+B}}\right) + b\int_0^\infty \mathfrak{K}_+^s(v,du,a,b)\int_0^\infty f_{u+B}(dl,s)\left(1-\mathbf{E}e^{-s\tau^{l+B}}\right)$$

i.e. the formula (2.5.2) of Theorem 2.5.1. The function $h_u(s,a,b)$ can be determined analogously, and the function $h(s,a,b)$ is then determined by means of the equality (2.5.8). Comparing the coefficients of $a^m b^n$, $m,n\in\mathbb{Z}^+$, $|m-n|\le 1$ in both sides of the first part of Theorem 2.5.1, we get the equalities of the second part of 2.5.1.

We now turn to proving (2.5.6). Letting $b=1$ in (2.5.2) implies for the generating function

$$h^v(s,a,1) = \mathbf{E}\left[a^{\alpha^+_{\nu_s}}/X_0 = x+v\right], \qquad a\in[0,1],$$

of the joint distribution $\mathbf{P}\left[\alpha^+_{\nu_s} = n/X_0 = x+v\right]$ that

$$h^v(s,a,1) = 1 - K_+^{(1)}(v,s) + \sum_{n\in\mathbb{N}} a^n\left[K_+^{(n)}(v,s) - K_+^{(n+1)}(v,s)\right]. \qquad (2.5.11)$$

Similarly for the generating function $h_v(s,a,1)$ of the distribution $\mathbf{P}\left[\alpha^+_{\nu_s} = n/X_0 = -v-y\right]$ in view of (2.5.3) for $b=1$, we find

$$h_v(s,a,1) = 1 - f^{v+B}(s) + \sum_{n\in\mathbb{N}} a^n \int_0^\infty \left[K_-^{(n-1)}(v,du,s) - K_-^{(n)}(v,du,s)\right] f^{u+B}(s). \quad (2.5.12)$$

Inserting the right-hand sides of the formulae (2.5.11), (2.5.12) into the third equality of the system (2.5.10), we derive for the generating function $h(s,a,1)$ of the distribution $\mathbf{P}\left[\alpha^+_{\nu_s} = n\right]$ the following relation:

$$\begin{aligned} h(s,a,1) = 1 &- \int_0^\infty V_y(dv,s) f^{v+B}(s) - \int_0^\infty V^x(dv,s) K_+^{(1)}(v,s) \\ &+ \sum_{n\in\mathbb{N}} a^n \int_0^\infty V^x(dv,s)\left[K_+^{(n)}(v,s) - K_+^{(n+1)}(v,s)\right] \\ &+ \sum_{n\in\mathbb{N}} a^n \int_0^\infty V_y(dv,s)\int_0^\infty \left[K_-^{(n-1)}(v,du,s) - K_-^{(n)}(v,du,s)\right] f^{u+B}(s). \end{aligned}$$

Comparing the coefficients of $a^m b^n$, $m,n \in \mathbb{Z}^+$, $|m-n| \le 1$ in both sides of the latter equality yields the formulae (2.5.6). The equalities (2.5.7) can be verified analogously. We now verify (2.5.8), (2.5.9). Letting $b = a$ in (2.5.2) yields for the generating function

$$h^v(s,a,a) = \mathbf{E}\left[a^{\alpha_{\nu_s}}/X_0 = x+v\right], \qquad a \in [0,1],$$

of the distribution $\mathbf{P}\left[\alpha_{\nu_s} = n/X_0 = x+v\right]$ the following relation:

$$\begin{aligned} h^v(s,a,a) = \sum_{n\in\mathbb{Z}^+} a^{2n} K_+^{(n)}(v,s) - a \sum_{n\in\mathbb{Z}^+} a^{2n} K_+^{(n+1)}(v,s) \\ - (1-a) \sum_{n\in\mathbb{Z}^+} a^{2n} \int_0^\infty K_+^{(n)}(v,du,s) f_{u+B}(s). \end{aligned} \tag{2.5.13}$$

Similarly for the generating function $h_v(s,a,a)$ of the distribution $\mathbf{P}\left[\alpha_{\nu_s} = n/X_0 = -v-y\right]$ it follows from (2.5.3) for $b=1$ that

$$\begin{aligned} h_v(s,a,a) = \sum_{n\in\mathbb{Z}^+} a^{2n} K_-^{(n)}(v,s) - a \sum_{n\in\mathbb{Z}^+} a^{2n} K_-^{(n+1)}(v,s) \\ - (1-a) \sum_{n\in\mathbb{Z}^+} a^{2n} \int_0^\infty K_-^{(n)}(v,du,s) f^{u+B}(s). \end{aligned} \tag{2.5.14}$$

Inserting the right-hand sides of (2.5.13), (2.5.14) into (2.5.4) of Theorem 2.5.1, and then comparing the coefficients of a^{2n}, a^{2n+1}, $n \in \mathbb{Z}^+$, we derive the formulae (2.5.8), (2.5.9). ▲

2.5.1 Intersections by a spectrally positive Lévy process

Let $\{X_t;\, t \ge 0\}$, $X_0 = 0$ be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7). Recall, that in this case the lower boundary is reached continuously, i.e. $\mathbf{P}\left[T_x = 0\right] = 1$. The integral transforms of τ_x, $\{\tau^x, T^x\}$ are such that

$$\begin{aligned} &\mathbf{E}\left[e^{-s\tau_x}; T_x \in du\right] = e^{-xc(s)}\delta(u)\, du, \\ &\int_0^\infty e^{-px} \mathbf{E} e^{-s\tau^x - zX_{\tau^x}}\, dx = \frac{1}{p}\left(1 - \frac{p+z-c(s)}{k(p+z)-s}\, \frac{k(z)-s}{z-c(s)}\right), \end{aligned} \tag{2.5.15}$$

where $c(s) > 0$, $s > 0$ is a unique solution of the equation $k(p) - s = 0$ in the semi-plane $\Re(p) > 0$, and $\delta(\cdot)$ is the delta function. The following corollary from Theorem 2.5.1 is true.

Corollary 2.5.1 (Kadankova (2003a)). *Let $\{X_t;\, t \geq 0\}$, $X_0 = 0$ be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7). Then*

(i) *the joint distribution of $\{\alpha^+_{\nu_s}, \alpha^-_{\nu_s}\}$ obeys the following equalities for $n \in \mathbb{Z}^+$*

$$\begin{aligned} \tilde{p}_n^{n+1}(s) &= \left[1 - f^B(s)\right]\left(e^{-yc(s)} - \frac{R_s(x)}{R_s(B)}\right) G^s_B(c(s))^n, \\ \tilde{p}^n_{n+1}(s) &= \left[f^B(s) - G^s_B(c(s))\right] \frac{R_s(x)}{R_s(B)} G^s_B(c(s))^n, \\ \tilde{p}^n_n(s) &= \mathbf{I}_{\{n=0\}}\left(1 - e^{-yc(s)}\right) + \left[1 - f^B(s)\right] \frac{R_s(x)}{R_s(B)} G^s_B(c(s))^n \\ &\quad + \mathbf{I}_{\{n\in\mathbb{N}\}}\left[f^B(s) - G^s_B(c(s))\right]\left(e^{-yc(s)} - \frac{R_s(x)}{R_s(B)}\right) G^s_B(c(s))^{n-1}, \end{aligned} \tag{2.5.16}$$

where $R_s(x)$, $x \geq 0$ is the scale function (2.2.11), and

$$G^s_x(c(s)) = 1 - r(s)e^{-xc(s)}R_s(x), \quad f^x(s) = 1 - \frac{s}{c(s)}R_s(x) + s\int_0^x R_s(u)\,du;$$

(ii) *the distributions of $\alpha^+_{\nu_s}$, $\alpha^-_{\nu_s}$ are given by*

$$\begin{aligned} \tilde{p}^+_0(s) &= 1 - e^{-yc(s)} f^B(s), \\ \tilde{p}^+_n(s) &= e^{-yc(s)} f^B(s)\left[1 - G^s_B(c(s))\right] G^s_B(c(s))^{n-1}, \quad n \in \mathbb{N}, \end{aligned} \tag{2.5.17}$$

$$\begin{aligned} \tilde{p}^-_0(s) &= 1 - e^{-yc(s)} G^s_x(c(s)), \\ \tilde{p}^-_n(s) &= e^{-yc(s)} G^s_x(c(s))\left[1 - G^s_B(c(s))\right] G^s_B(c(s))^{n-1}, \quad n \in \mathbb{N}; \end{aligned} \tag{2.5.18}$$

(iii) *the distribution of* α_{ν_s} *is such that*

$$\begin{aligned}
\tilde{p}_0(s) &= 1 - f^B(s)\frac{R_s(x)}{R_s(B)} - \left(e^{-yc(s)} - \frac{R_s(x)}{R_s(B)}\right),\\
\tilde{p}_{2n}(s) &= \frac{R_s(x)}{R_s(B)}\left[1 - f^B(s)\right] G_B^s(c(s))^{n-1}\\
&\quad + \left[f^B(s) - G_B^s(c(s))\right]\left(e^{-yc(s)} - \frac{R_s(x)}{R_s(B)}\right) G_B^s(c(s))^{n-1}, \quad n \in \mathbb{N},\\
\tilde{p}_{2n+1}(s) &= \frac{R_s(x)}{R_s(B)}\left[f^B(s) - G_B^s(c(s))\right] G_B^s(c(s))^{n}\\
&\quad + \left[1 - f^B(s)\right]\left(e^{-yc(s)} - \frac{R_s(x)}{R_s(B)}\right) G_B^s(c(s))^{n}, \quad n \in \mathbb{Z}^+.
\end{aligned} \tag{2.5.19}$$

Proof. It is worth noticing that formulae (2.5.16) of the corollary were obtained in Kadankova (2003a) as the result of solving the system (2.5.10) for the case when X_t is a spectrally positive Lévy process. Let us verify the equalities (2.5.16)–(2.5.19) of Corollary 2.5.1.

In Section 2.3.1 we already determined the successive iterations

$$\begin{aligned}
K_-^{(n)}(v, du, s) &= e^{vc(s)} G_{v+B}^s(c(s))\, G_B^s(c(s))^{n-1}\, \delta(u)\, du, \quad n \in \mathbb{N},\\
K_+^{(n)}(v, du, s) &= e^{-c(s)(v+B)} G_B^s(c(s))^{n-1} f^B(du, s), \quad n \in \mathbb{N},
\end{aligned} \tag{2.5.20}$$

and the integral transforms of the joint distribution of $\{\chi, T\}$

$$\begin{aligned}
V_y(du, s) &= \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}_y\right] = \frac{R_s(x)}{R_s(B)}\, \delta(u)\, du,\\
V^x(du, s) &= \mathbf{E}\left[e^{-s\chi}; T \in du, \mathfrak{A}^x\right] = f^x(du, s) - \frac{R_s(x)}{R_s(B)} f^B(du, s).
\end{aligned} \tag{2.5.21}$$

Inserting the expressions for the functions $K_\pm^{(n)}(v, du, s)$, $V_y(du, s)$, $V^x(du, s)$ into the formulae (2.5.15)–(2.5.9) of Theorem 2.5.1, we derive the formulae of the corollary. ▲

We now apply the results of Theorem 2.5.1 to determine the distributions (2.5.1) for the standard Wiener process.

Corollary 2.5.2. *Let* $\{w_t;\ t \geq 0\}$ *be the Wiener process whose Laplace exponent is of the form* $k(p) = \sigma^2 p^2/2$. *Then*

(i) *the joint distribution of* $\{\alpha^+_{\nu_s}, \alpha^-_{\nu_s}\}$ *obeys the equalities for* $n \in \mathbb{Z}^+$

$$\begin{aligned}
\tilde{p}^{n+1}_n(s) &= \frac{e^{y\sqrt{2s}/\sigma} - e^{-y\sqrt{2s}/\sigma}}{1+e^{-B\sqrt{2s}/\sigma}}\, e^{-2B(n+1)\sqrt{2s}/\sigma}, \qquad (2.5.22)\\
\tilde{p}^{n}_{n+1}(s) &= \frac{e^{x\sqrt{2s}/\sigma} - e^{-x\sqrt{2s}/\sigma}}{1+e^{-B\sqrt{2s}/\sigma}}\, e^{-2B(n+1)\sqrt{2s}/\sigma},\\
\tilde{p}^{0}_0(s) &= 1 - \frac{e^{x\sqrt{2s}/\sigma} + e^{y\sqrt{2s}/\sigma}}{1+e^{-B\sqrt{2s}/\sigma}}\, e^{-2B\sqrt{2s}/\sigma},\\
\tilde{p}^{n}_n(s) &= \frac{e^{x\sqrt{2s}/\sigma} + e^{y\sqrt{2s}/\sigma}}{1+e^{-B\sqrt{2s}/\sigma}}\, \left(1 - e^{-B\sqrt{2s}/\sigma}\right) e^{-B(2n+1)\sqrt{2s}/\sigma}, \quad n \in \mathbb{N};
\end{aligned}$$

(ii) *the distributions of* $\alpha^+_{\nu_s}$, $\alpha^-_{\nu_s}$ *are such that*

$$\begin{aligned}
\tilde{p}^+_0(s) &= 1 - e^{-(y+B)\sqrt{2s}/\sigma},\\
\tilde{p}^+_n(s) &= \left(1 - e^{-2B\sqrt{2s}/\sigma}\right) e^{-y-B(2n-1)\sqrt{2s}/\sigma}, \quad n \in \mathbb{N},\\
\tilde{p}^-_0(s) &= 1 - e^{-(x+B)\sqrt{2s}/\sigma},\\
\tilde{p}^-_n(s) &= \left(1 - e^{-2B\sqrt{2s}/\sigma}\right) e^{-x-B(2n-1)\sqrt{2s}/\sigma}, \quad n \in \mathbb{N};
\end{aligned}$$

(iii) *the distribution of* α_{ν_s} *is given as follows:*

$$\begin{aligned}
\tilde{p}_0(s) &= 1 - \frac{e^{x\sqrt{2s}/\sigma} + e^{y\sqrt{2s}/\sigma}}{1+e^{-B\sqrt{2s}/\sigma}}\, e^{-2B\sqrt{2s}/\sigma},\\
\tilde{p}_n(s) &= \frac{e^{x\sqrt{2s}/\sigma} + e^{y\sqrt{2s}/\sigma}}{1+e^{-B\sqrt{2s}/\sigma}}\, \left(1 - e^{-B\sqrt{2s}/\sigma}\right) e^{-B(n+1)\sqrt{2s}/\sigma} \quad n \in \mathbb{N}.
\end{aligned}$$

Proof. For the Wiener process we can easily calculate

$$\begin{aligned}
&f^x(s) = f_x(s) = e^{-x\sqrt{2s}/\sigma}, \qquad K^{(n)}_{\pm}(v, du, s) = e^{-(v+2Bn)\sqrt{2s}/\sigma}\delta(u)\, du,\\
&V^x(du, s) = \frac{\sinh\left(y\sqrt{2s}/\sigma\right)}{\sinh\left(B\sqrt{2s}/\sigma\right)}\, \delta(u)\, du, \quad V_y(du, s) = \frac{\sinh\left(x\sqrt{2s}/\sigma\right)}{\sinh\left(B\sqrt{2s}/\sigma\right)}\, \delta(u)\, du.
\end{aligned}$$

Substituting the expressions for these functions into the equalities of Theorem 2.5.1, we get the formulae of the corollary. ▲

Observe, that the Laplace transforms in the formulae of the corollary can be easily inverted, and, hence, we can determine the distributions themselves.

Denote

$$\mu_t(a,b) = \mathbf{P}[w_t \in (a,b)] = \frac{1}{\sigma\sqrt{2\pi t}} \int\limits_a^b e^{-u^2/2t\sigma^2}\, du, \quad \mu_t(a) = \mu_t(a,\infty).$$

Corollary 2.5.3. *Let $\{w_t;\, t \geq 0\}$ be the Wiener process whose Laplace exponent is of the form $k(p) = \sigma^2 p^2/2$. Then*

(i) *the joint distribution of $\{\alpha_t^+, \alpha_t^-\}$ admits the following expansion for $n \in \mathbb{Z}^+$*

$$p_n^{n+1}(t) = 2 \sum_{k \geq 2n+2} (-1)^k \mu_t(-y+kB, y+kB),$$
$$p_{n+1}^{n}(t) = 2 \sum_{k \geq 2n+2)} (-1)^k \mu_t(-x+kB, x+kB), \tag{2.5.23}$$

$$p_0^0(t) = 1 - 2 \sum_{k \in \mathbb{N}} (-1)^{k-1} \left[\mu_t(x+kB) + \mu_t(y+kB)\right],$$
$$p_n^n(t) = 2 \sum_{k \geq 2n+1} (-1)^{k-1} \left[\mu_t(-x+kB, x+kB) + \mu_t(-y+kB, y+kB)\right];$$

(ii) *the distributions of the number of the up-crossings α_t^+ and of the down-crossings α_t^- obey the formulae for $n \in \mathbb{N}$*

$$p_0^+(t) = 1 - 2\mu_t(y+B), \quad \mathbf{P}\left[\alpha_t^+ \geq n\right] = 2\mu_t(y+(2n-1)B),$$
$$p_0^-(t) = 1 - 2\mu_t(x+B), \quad \mathbf{P}\left[\alpha_t^- \geq n\right] = 2\mu_t(x+(2n-1)B); \tag{2.5.24}$$

(iii) *the distribution of the total number of intersections α_t is such that for $n \in \mathbb{N}$*

$$p_0(t) = 1 - 2 \sum_{k \in \mathbb{N}} (-1)^{k-1} \left[\mu_t(x+kB) + \mu_t(y+kB)\right],$$
$$\mathbf{P}\left[\alpha_t \geq n\right] = 2 \sum_{k \geq n} (-1)^{k-n} \left[\mu_t(x+kB) + \mu_t(y+kB)\right]. \tag{2.5.25}$$

Proof. Let us verify the first formula of (2.5.23). Taking into account the expansion

$$\left(1 + e^{-B\sqrt{2s}/\sigma}\right)^{-1} = \sum_{k \in \mathbb{Z}^+} (-1)^k e^{-kB\sqrt{2s}/\sigma},$$

we derive from (2.5.22) that

$$\int_0^\infty e^{-st} p_n^{n+1}(t)\,dt = \sum_{k\in\mathbb{Z}^+} (-1)^k \frac{1}{s} e^{-(-y+kB+2(n+1)B)\sqrt{2s}/\sigma} - \sum_{k\in\mathbb{Z}^+} (-1)^k \frac{1}{s} e^{-(y+kB+2(n+1)B)\sqrt{2s}/\sigma}. \tag{2.5.26}$$

In order to invert the latter integral, we will employ the following well-known identity for the Wiener process $\mathbf{P}\left[\tau^x \le t\right] = 2\mathbf{P}\left[w_t > x\right], x \ge 0$ and the inversion formula:

$$\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{1}{s}\, e^{-u\sqrt{2s}/\sigma} ds = 2\mathbf{P}[w_t > u], \qquad \gamma > 0.$$

This formula combined with (2.5.26) yields

$$\begin{aligned} p_n^{n+1}(t) &= 2\sum_{k\in\mathbb{Z}^+} (-1)^k \mu_t(-y + kB + 2(n+1)B) - \\ &- 2\sum_{k\in\mathbb{Z}^+} (-1)^k \mu_t(y + kB + 2(n+1)B) = \\ &= 2\sum_{k\ge 2n+2} (-1)^k \mu_t(-y + kB, y + kB) \end{aligned}$$

i.e. the fist equality of (2.5.23). Other equalities can be proven analogously. ▲

It is worth mentioning that the distribution of the total number of the intersections $p_n(t)$, $n \in \mathbb{Z}^+$ was found in Androchuk (2001) for a symmetric Bernoulli random walk and for a symmetric Wiener process. The author used the symmetry principle and the reflection principle to obtain formulae similar to (2.5.25).

Remark 2.5.1. *Taking into account the equalities (2.3.16), we can derive the* *following expansions for the density of* χ

$$\mathbf{P}\left[\chi \in dt; \mathfrak{A}^x\right] = \frac{\pi\sigma^2}{B^2} \sum_{k\in\mathbb{N}} k\, e^{-\frac{t}{2}(k\pi\sigma/B)^2} \sin\left(\frac{x}{B}k\pi\right) dt,$$

$$\mathbf{P}\left[\chi \in dt; \mathfrak{A}_y\right] = \frac{\pi\sigma^2}{B^2} \sum_{k\in\mathbb{N}} k\, e^{-\frac{t}{2}(k\pi\sigma/B)^2} \sin\left(\frac{y}{B}k\pi\right) dt, \tag{2.5.27}$$

$$\mathbf{P}\left[\chi \in dt\right] = \frac{2\pi\sigma^2}{B^2} \sum_{k=0}^{\infty} (2k+1) e^{-\frac{t}{2}((2k+1)\pi\sigma/B)^2} \sin\left(\frac{x}{B}(2k+1)\pi\right) dt.$$

The formulae (2.5.23), (2.5.25) re-written in terms of the density of χ *take a more compact form. For instance, the joint distribution of* $\{\alpha_t^+, \alpha_t^-\}$ *admits the following representation for* $n \in \mathbb{Z}^+$

$$p_n^{n+1}(t) = 2\int_0^t \mathbf{P}\left[\chi \in du; \mathfrak{A}^x\right] \mu_{t-u}((2n+1)B, 2(n+1)B),$$

$$p_{n+1}^{n}(t) = 2\int_0^t \mathbf{P}\left[\chi \in du; \mathfrak{A}_y\right] \mu_{t-u}((2n+1)B, 2(n+1)B),$$

$$p_0^0(t) = 1 - 2\int_0^t \mathbf{P}\left[\chi \in du\right] \mu_{t-u}(B),$$

$$p_n^n(t) = 2\int_0^t \mathbf{P}\left[\chi \in du\right] \mu_{t-u}(2nB, (2n+1)B), \quad n \in \mathbb{N}. \tag{2.5.28}$$

The distribution of α_t *is such that*

$$\mathbf{P}\left[\alpha_t = 0\right] = 1 - 2\int_0^t \mathbf{P}\left[\chi \in du\right] \mathbf{P}\left[w_{t-u} > B\right],$$

$$\mathbf{P}\left[\alpha_t \geq n\right] = 2\int_0^t \mathbf{P}\left[\chi \in du\right] \mathbf{P}\left[w_{t-u} > nB\right], \quad n \in \mathbb{N}. \tag{2.5.29}$$

Now we study the asymptotic behavior of the number of intersections. Denote by

$$\alpha^+ = \lim_{t\to\infty} \alpha_t^+, \qquad \alpha^- = \lim_{t\to\infty} \alpha_t^-$$

the number of the upward and downward intersections of the interval $[-y, x]$ by the process $\{X_t;\, t \geq 0\}$ with an infinite time horizon respectively. For all $k, l \in \mathbb{Z}^+$, $|k-l| \leq 1$ introduce the notation

$$p_k^l = \mathbf{P}\left[\alpha^+ = k, \alpha^- = l\right], \quad p_k^{\pm} = \mathbf{P}\left[\alpha^{\pm} = k\right], \quad p_k = \mathbf{P}\left[\alpha = k\right],$$

where $\alpha = \alpha^+ + \alpha^-$ is the total number of intersections of the interval $[-y, x]$ by the process $\{X_t;\ t \geq 0\}$, $X_0 = 0$. Denote $m = \mathbf{E}[X_1] = -k'(0)$.

Corollary 2.5.4. *Let $\{X_t; t \geq 0\}$ be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7). Then*

(i) *if $m > 0$ the joint distribution of $\{\alpha^+, \alpha^-\}$ satisfies the formulae:*

$$p_{n+1}^n = \frac{R(x)}{R(B)}[1 - G(B)]G(B)^n, \quad p_n^{n+1} = 0, \qquad n \in \mathbb{Z}+, \tag{2.5.30}$$
$$p_0^0 = 1 - e^{-yc}, \quad p_n^n = \left(e^{-yc} - \frac{R(x)}{R(B)}\right)(1 - G(B))G(B)^{n-1}, \quad n \in \mathbb{N},$$

where $c = \lim_{s \to 0} c(s)$,

$$R(x) = \lim_{s \to 0} R^s(x), \quad G(x) = 1 - e^{-xc}k'(c)R(x);$$

(ii) *if $m < 0$ then the joint distribution of $\{\alpha^+, \alpha^-\}$ admits the representations for $n \in \mathbb{Z}+$*

$$p_n^n = \mathbf{P}[X^+ < x]\ (\mathbf{P}[X^+ > B])^n, \qquad p_{n+1}^n = 0,$$
$$p_n^{n+1} = \mathbf{P}[x < X^+ < B](\mathbf{P}[X^+ > B])^n, \tag{2.5.31}$$

where

$$X^+ = \sup_{t \geq 0} X_t, \qquad \mathbf{P}[X^+ < x] = -mR(x).$$

Proof. Assume that $m > 0$. Then

$$\lim_{s \to 0} \mathbf{E}e^{-s\tau^x} = \mathbf{P}[\tau^x < \infty] = 1, \qquad \lim_{s \to 0} c(s) = c > 0.$$

Calculating the limits in (2.5.16) as $s \to 0$, we derive (2.5.30) of the corollary. Note that the function $R(x)$, $x \geq 0$ determined by its Laplace transform

$$\int_0^\infty e^{-px}R(x)dx = k(p)^{-1}, \qquad \Re(p) > c,$$

enters the right-hand sides of the latter equalities. Let $m < 0$. Then

$$\lim_{s \to 0} c(s) = 0, \qquad \lim_{s \to 0} \mathbf{E}e^{-s\tau^x} = \mathbf{P}[X^+ > x] < 1.$$

The equality (2.2.8) implies that

$$\mathbf{E}e^{-pX^+} = -m\frac{p}{k(p)} \quad \Rightarrow \quad \mathbf{P}[X^+ \le x] = -mR(x).$$

Calculating the limits in (2.5.16) as $s \to 0$, we derive formulae (2.5.31) of the corollary. Taking into account formulae (2.5.17)–(2.5.19), one can derive for $\alpha^{\pm}$, α the equalities similar to (2.5.30), (2.5.31). ▲

It is worth mentioning that the distributions of the number of the intersections for the Wiener process (2.5.23)–(2.5.25) serve as the limiting distributions for the corresponding distributions for Lévy processes and random walks. As an example we state such limit equalities for a spectrally positive Lévy process.

Corollary 2.5.5. *Let* $\{X_t;\ t \ge 0\}$, $X_0 = 0$ *be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7). Assume that* $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = \sigma^2 < \infty$ *and* $x, y > 0$, $x + y = 1$, $B > 0$. *Denote by* $\alpha^+_{tB^2}(B)$ *the number of the upward intersections of the interval* $[-yB, xB]$ *by the process* X *on the time interval* $[0, tB^2]$; *by* $\alpha^-_{tB^2}(B)$ *the number of the downward intersections of the interval* $[-yB, xB]$ *by the process* X *on the time interval* $[0, tB^2]$; *and let*

$$p_l^k(t, B) = \mathbf{P}\left[\alpha^+_{tB^2}(B) = k, \alpha^-_{tB^2}(B) = l\right], \qquad k, l \in \mathbb{Z}^+.$$

Then this distribution is such that

$$\lim_{B\to\infty} p_n^{n+1}(t, B) = 2\sum_{k\ge 2(n+1)} (-1)^k \mu_t(-y + k, y + k) = \bar{p}_n^{n+1}(t), \tag{2.5.32}$$

$$\lim_{B\to\infty} p_{n+1}^{n}(t, B) = 2\sum_{k\ge 2(n+1)} (-1)^k \mu_t(-x + k, x + k) = \bar{p}_n^{n+1}(t),$$

$$\lim_{B\to\infty} p_0^0(t, B) = 1 - 2\sum_{k\in\mathbb{N}} (-1)^{k-1}\left[\mu_t(x + k) + \mu_t(y + k)\right] = \bar{p}_0^0(t),$$

$$\lim_{B\to\infty} p_n^n(t, B) = 2\sum_{k\ge 2n+1} (-1)^{k-1}\left[\mu_t(-x + k, x + k) + \mu_t(-y + k, y + k)\right] = \bar{p}_n^n(t),$$

where

$$\bar{p}_l^k(t) = \mathbf{P}\left[\bar{\alpha}^+(t) = k, \bar{\alpha}^-(t) = l\right], \qquad k, l \in \mathbb{Z}^+,$$

$\bar{\alpha}^+(t)$, $\bar{\alpha}^-(t)$ *are the number of the upward and downward intersections respectively of the interval* $[-y, x]$, $x, y > 0$, $x + y = 1$ *by the Wiener process with the Laplace exponent* $k(p) = \sigma^2 p^2/2$.

Proof. Suppose the conditions of the corollary are met. Consider the first equality of (2.5.16) and employ it for the case when $[-yB, xB]$ as $s \to s/B^2$. This yields

$$\frac{s}{B^2}\int_0^\infty e^{-su/B^2} p_n^{n+1}(u)\,du =$$
$$= \left[1 - f^B(s/B^2)\right]\left(e^{-yBc(s/B^2)} - \frac{R_{s/B^2}(xB)}{R_{s/B^2}(B)}\right) G_B^{s/B^2}(c(s/B^2))^n. \quad (2.5.33)$$

Since

$$\frac{s}{B^2}\int_0^\infty e^{-su/B^2} p_n^{n+1}(u)\,du =$$
$$= s\int_0^\infty e^{-st}\mathbf{P}\left[\alpha^+_{tB^2}(B) = n, \alpha^-_{tB^2}(B) = n+1\right]dt = s\int_0^\infty e^{-st} p_n^{n+1}(t, B)\,dt,$$

then

$$\lim_{B\to\infty}\int_0^\infty e^{-st} p_n^{n+1}(t, B)\,dt = \quad (2.5.34)$$
$$= \frac{1}{s}\lim_{B\to\infty}\left[1 - f^B(s/B^2)\right]\left(e^{-yBc(s/B^2)} - \frac{R_{s/B^2}(xB)}{R_{s/B^2}(B)}\right) G_B^{s/B^2}(c(s/B^2))^n.$$

We will now use the asymptotic properties of the scale function and we will employ the following identity:

$$\lim_{B\to\infty}\frac{1}{B}R_{s/B^2}(xB) = \frac{1}{\sigma}\sqrt{\frac{2}{s}}\,\sinh\left(\frac{x}{\sigma}\sqrt{2s}\right), \quad \lim_{B\to\infty} Bc(s/B^2) = \sqrt{2s}/\sigma,$$
$$\lim_{B\to\infty}\frac{s}{B^2}\int_0^{xB} R_{s/B^2}(u)\,du = \cosh\left(\frac{x}{\sigma}\sqrt{2s}\right) - 1, \quad x \in [0, 1].$$

Taking into account these equalities, we find that

$$\lim_{B\to\infty} G_{xB}^{s/B^2}(c(s/B^2)) = e^{-2x\sqrt{2s}/\sigma}, \quad x \in [0, 1].$$

Calculating the limits in the right-hand side of (2.5.34), we obtain

$$\lim_{B\to\infty}\int_0^\infty e^{-st} p_n^{n+1}(t, B)\,dt = \frac{1}{s}\,\frac{\sinh(y\sqrt{2s}/\sigma)}{\sinh(\sqrt{2s}/\sigma)}\left(1 - e^{-\sqrt{2s}/\sigma}\right)e^{-(2n+1)\sqrt{2s}/\sigma}$$
$$= \frac{1}{s}\left(e^{-(-y+2(n+1))\sqrt{2s}/\sigma} - e^{-(y+2(n+1))\sqrt{2s}/\sigma}\right)\left(1 + e^{-\sqrt{2s}/\sigma}\right)^{-1}. \quad (2.5.35)$$

The right-hand side of this equality coincides with the right-hand side of the first formula of (2.5.22) for $B = 1$. This means that the left-hand sides of these formulae should coincide as well. This results to the following equality:

$$\lim_{B\to\infty}\int_0^\infty e^{-st}p_n^{n+1}(t,B)\,dt = \int_0^\infty e^{-st}\bar{p}_n^{n+1}(t)\,dt.$$

By means of this equality we established the weak convergence of the distribution $p_n^{n+1}(t,B)$ to the distribution $\bar{p}_n^{n+1}(t)$ of the symmetric Wiener process in case when $x, y > 0$, $x + y = 1$. Further, it follows from (2.5.22), (2.5.28) for $B = 1$ that

$$\begin{aligned}\bar{p}_n^{n+1}(t) &= 2\sum_{k\geq 2(n+1)}(-1)^k\mu_t(-y+k, y+k) = \\ &= 2\pi\sigma^2\sum_{k\in\mathbb{N}}\int_0^t k\,e^{-\frac{u}{2}(k\pi\sigma)^2}\sin(yk\pi)\,\mu_{t-u}(2n+1,2n+2)\,du\end{aligned}$$

i.e. the first formula of (2.5.32). The second formula can be verified analogously. ▲

2.5.2 Intersections of the interval by a compound Poisson process with arbitrary positive and exponential negative jumps

Corollary 2.5.6 (Kadankov and Kadankova (2006))**.** *Let $\{X_t; t \geq 0\}$, $X_0 = 0$ be the compound Poisson process with Laplace exponent (2.2.13). Then*

(i) *the joint distribution of $\{\alpha^+_{\nu_s}, \alpha^-_{\nu_s}\}$ obeys the formulae $n \in \mathbb{Z}^+$*

$$\begin{aligned}\tilde{p}_n^{n+1}(s) &= \left[1 - \mathbf{E}f^{\gamma+B}(s)\right]\left(f_y(s) - \frac{R_s(x)}{R_s^\lambda(B)}\right)T(s)^n,\\ \tilde{p}^n_{n+1}(s) &= \left[\mathbf{E}f^{\gamma+B}(s) - T(s)\right]\frac{R_s(x)}{R_s^\lambda(B)}T(s)^n, \qquad (2.5.36)\\ \tilde{p}_n^n(s) &= \mathbf{I}_{\{n=0\}}\left(1 - f_y(s)\right) + \left[1 - \mathbf{E}f^{\gamma+B}(s)\right]\frac{R_s(x)}{R_s^\lambda(B)}T(s)^n\\ &\quad + \mathbf{I}_{\{n\in\mathbb{N}\}}\left[\mathbf{E}f^{\gamma+B}(s) - T(s)\right]\left(f_y(s) - \frac{R_s(x)}{R_s^\lambda(B)}\right)T(s)^{n-1},\end{aligned}$$

where $R_s(x)$, $x \geq 0$ *is the scale function of the process (2.2.22),*

$$\mathbf{E}f^{\gamma+B}(du,s) = \lambda \int_0^\infty e^{-\lambda v} f^{v+B}(du,s)\,dv = 1 - \frac{s\lambda}{c(s)} R_s^\lambda(B) + s\lambda S_s^\lambda(B),$$

$$T(s) = f_B(s) \int_0^\infty \mathbf{E}f^{\gamma+B}(du,s) e^{-uc(s)} = 1 - f_B(s) R_s^\lambda(B) r(s); \quad (2.5.37)$$

(ii) *the distributions of* $\alpha^+_{\nu_s}$, $\alpha^-_{\nu_s}$ *are such that*

$$\begin{aligned}
\tilde{p}_0^+(s) &= 1 - f_y(s)\mathbf{E}f^{\gamma+B}(s),\\
\tilde{p}_n^+(s) &= f_y(s)\mathbf{E}f^{\gamma+B}(s)\,[1 - T(s)]\,T(s)^{n-1}, \quad n \in \mathbb{N},
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_0^-(s) &= 1 - \tilde{f}^x(c(s)),\\
\tilde{p}_n^-(s) &= \tilde{f}^x(c(s))\,(1 - T(s))\,T(s)^{n-1}, \quad n \in \mathbb{N},
\end{aligned} \quad (2.5.38)$$

where

$$\tilde{f}^x(c(s)) = \int_0^\infty e^{-uc(s)} f^x(du,s) = e^{xc(s)} - R_s(x) r(s);$$

(iii) *the distribution of* α_{ν_s} *admits the following representation:*

$$\begin{aligned}
\tilde{p}_0(s) &= 1 - \mathbf{E}f^{\gamma+B}(s)\frac{R_s(x)}{R_s^\lambda(B)} - \left(f_y(s) - \frac{R_s(x)}{R_s^\lambda(B)}\right),\\
\tilde{p}_{2n}(s) &= \frac{R_s(x)}{R_s^\lambda(B)}\left[1 - \mathbf{E}f^{\gamma+B}(s)\right] T(s)^{n-1}\\
&\quad + \left[\mathbf{E}f^{\gamma+B}(s) - T(s)\right]\left(f_y(s) - \frac{R_s(x)}{R_s^\lambda(B)}\right) T(s)^{n-1}, \quad n \in \mathbb{N},\\
\tilde{p}_{2n+1}(s) &= \frac{R_s(x)}{R_s^\lambda(B)}\left[\mathbf{E}f^{\gamma+B}(s) - T(s)\right] T(s)^{n}\\
&\quad + \left[1 - \mathbf{E}f^{\gamma+B}(s)\right]\left(f_y(s) - \frac{R_s(x)}{R_s^\lambda(B)}\right) T(s)^{n}, \quad n \in \mathbb{Z}^+.
\end{aligned} \quad (2.5.39)$$

Proof. In Section 2.3.3 we already determined the integral transforms of the joint distribution of $\{\chi, T\}$:

$$\begin{aligned}
V_y(du,s) &= \frac{R_s(x)}{R_s^\lambda(B)}\lambda e^{-\lambda u}du,\\
V^x(du,s) &= f^x(du,s) - \frac{R_s(x)}{R_s^\lambda(B)}\mathbf{E}f^{\gamma+B}(du,s),
\end{aligned} \quad (2.5.40)$$

and the successive iterations

$$K_-^{(n)}(v,du,s) = f_B(s)\tilde f^{v+B}(c(s))T(s)^{n-1}\lambda e^{-\lambda u}\,du,$$
$$K_+^{(n)}(v,du,s) = e^{-vc(s)}f_B(s)T(s)^{n-1}\mathbf{E}f^{\gamma+B}(du,s),$$

where

$$\tilde f^x(c(s)) = \int_0^\infty f^x(du,s)e^{-uc(s)} = e^{xc(s)} - R_s(x)r(s),$$
$$T(s) = f_B(s)\int_0^\infty \mathbf{E}f^{\gamma+B}(du,s)e^{-uc(s)} = 1 - f_B(s)R_s^\lambda(B)r(s). \qquad (2.5.41)$$

Substituting the expressions for $K_\pm^{(n)}(v,du,s)$, $V_y(du,s)$, $V^x(du,s)$ into the formulae (2.5.5)–(2.5.9) of Theorem 2.5.1 we derive the equalities of the corollary. ▲

Corollary 2.5.7. *Let* $\{X_t;\ t\ge 0\}$, $X_0=0$ *be the compound Poisson whose Laplace exponent is given by (2.2.13). Suppose that* $\mathbf{E}X_1=0$, $\mathbf{E}X_1^2=\sigma^2<\infty$ *and* $x,y>0$, $x+y=1$, $B>0$. *Denote by*

$\alpha^+_{tB^2}(B)$ *the number of the upward intersections of the interval* $[-yB,xB]$ *by the process* X *on the time interval* $[0,tB^2]$;

$\alpha^-_{tB^2}(B)$ *the number of the downward intersections of the interval* $[-yB,xB]$ *by the process* X *on the time interval* $[0,tB^2]$; *and*

$$p_l^k(t,B) = \mathbf{P}\left[\alpha^+_{tB^2}(B)=k, \alpha^-_{tB^2}(B)=l\right], \qquad k,l\in\mathbb{Z}^+.$$

Then this distribution satisfies the following equalities:

$$\lim_{B\to\infty} p_n^{n+1}(t,B) = 2\sum_{k\ge 2(n+1)}(-1)^k\mu_t(-y+k,y+k) = \bar p_n^{n+1}(t), \qquad (2.5.42)$$
$$\lim_{B\to\infty} p_{n+1}^{n}(t,B) = 2\sum_{k\ge 2(n+1)}(-1)^k\mu_t(-x+k,x+k) = \bar p_n^{n+1}(t),$$

$$\lim_{B\to\infty} p_0^0(t,B) = 1-2\sum_{k\in\mathbb{N}}(-1)^{k-1}\left[\mu_t(x+k)+\mu_t(y+k)\right] = \bar p_0^0(t),$$
$$\lim_{B\to\infty} p_n^n(t,B) = 2\sum_{k\ge 2n+1}(-1)^{k-1}\left[\mu_t(-x+k,x+k)+\mu_t(-y+k,y+k)\right] = \bar p_n^n(t),$$

where

$$\bar{p}_l^k(t) = \mathbf{P}\left[\bar{\alpha}^+(t) = k, \bar{\alpha}^-(t) = l\right], \qquad k, l \in \mathbb{Z}^+,$$

$\bar{\alpha}^+(t)$, $\bar{\alpha}^-(t)$ *are the numbers of the upward and downward intersections of the interval* $[-y, x]$, $x, y > 0$, $x + y = 1$ *by the Wiener process with the Laplace exponent of the form* $k(p) = \sigma^2 p^2/2$.

Proof. Suppose that the assumptions of the corollary are met. Then taking into account (2.5.37) and the equalities (2.3.26), (2.3.27) when $B \to \infty$, we find that

$$\mathbf{E} f^{\gamma+B}\left(\frac{s}{B^2}\right) = 1 - \frac{s\lambda}{c(s/B^2)B^2} R^\lambda_{s/B^2}(B) + \frac{s\lambda}{B^2} S^\lambda_{s/B^2}(B) \to e^{-\sqrt{2s}/\sigma}. \tag{2.5.43}$$

The Laplace exponent of the process can be written as follows for small $p > 0$

$$k(p) = \frac{1}{2}\sigma^2 p^2 + o(p^2).$$

In view of the latter equality and the identity $k(c(s)) = s$, we derive

$$\lim_{B\to\infty} Bc(s/B^2) = \sqrt{2s}/\sigma, \quad \lim_{B\to\infty} r(s/B^2) = \lambda\sigma\sqrt{2s}. \tag{2.5.44}$$

Then the function $T(s)$ is such that (when $B \to \infty$)

$$T\left(\frac{s}{B^2}\right) = 1 - f_B(s/B^2) R^\lambda_{s/B^2}(B) r(s/B^2) \to e^{-2\sqrt{2s}/\sigma}. \tag{2.5.45}$$

Let us consider the equality (2.5.36) for the case when $[-yB, xB]$ and $s \to s/B^2$. This yields

$$\frac{s}{B^2}\int_0^\infty e^{-su/B^2} p_n^{n+1}(u)\,du = s\int_0^\infty e^{-st} p_n^{n+1}(t, B)\,dt =$$
$$= \left[1 - \mathbf{E} f^{\gamma+B}(s/B^2)\right]\left(f_{yB}(s/B^2) - \frac{R_{s/B^2}(xB)}{R^\lambda_{s/B^2}(B)}\right)\left(T(s/B^2)\right)^n.$$

Hence,

$$\lim_{B\to\infty}\int_0^\infty e^{-st} p_n^{n+1}(t, B)\,dt = \frac{e^{y\sqrt{2s}/\sigma} - e^{-y\sqrt{2s}/\sigma}}{1 + e^{-\sqrt{2s}/\sigma}}\, e^{-2(n+1)\sqrt{2s}/\sigma}.$$

The right-hand side of the latter equality coincides with right-hand side of (2.5.22) for $B = 1$. Then, necessarily the left-hand sides of these equalities coincide as well, so that we derive

$$\lim_{B\to\infty} \int_0^\infty e^{-st} p_n^{n+1}(t,B)\,dt = \int_0^\infty e^{-st} \bar{p}_n^{n+1}(t)\,dt.$$

By means of the latter equality we established the weak convergence of the distribution $p_n^{n+1}(t,B)$ to the distribution $\bar{p}_n^{n+1}(t)$ of the Wiener process for the case when $x, y > 0$, $x + y = 1$. It follows from (2.5.22), (2.5.28) for $B = 1$ that

$$\bar{p}_n^{n+1}(t) = 2 \sum_{k\geq 2(n+1)} (-1)^k \mu_t(-y+k, y+k) =$$
$$= 2\pi\sigma^2 \sum_{k\in\mathbb{N}} \int_0^t k\, e^{-\frac{u}{2}(k\pi\sigma)^2} \sin(yk\pi)\, \mu_{t-u}(2n+1, 2n+2)\,du$$

i.e. the first formula of (2.5.42). Other formulae can be verified analogously. ▲

2.6 Occupation time of an interval

Let $\{X_t;\, t \geq 0\}$, $X_0 = 0$ be a general Lévy process. Fix $B > 0$ and introduce the indicator functions:

$$g(x) = \mathbf{I}_{\{x \in [0,B]\}}, \quad g^*(x) = 1 - g(x), \qquad x \in \mathbb{R}.$$

Define the following random variables

$$\sigma_y(t) = \int_0^t g(y + X_u)\, du, \quad \sigma_y^*(t) = t - \sigma_y(t)$$

i.e. the sojourn occupation time inside the interval $[0, B]$ by the process $y + X_u$, $u \in [0, t]$ till the time t, and the total occupation time outside the interval $[0, B]$ by the process $y + X_u$, $u \in [0, t]$ till time t. This section is concerned with determining the integral transforms of the joint distribution of $\{\sigma_y(t), \sigma_y^*(t)\}$ denoted as follows:

$$C_y^s(a, b) = s \int_0^\infty e^{-st} \mathbf{E} e^{-a\sigma_y(t) - b\sigma_y^*(t)}\, dt, \quad a, b \geq 0.$$

We will focus on the compound Poisson process with positive jumps and negative drift, the spectrally one-sided Lévy process and on the compound Poisson with arbitrary positive jumps and exponential negative jumps. For a symmetric Wiener process we are able to invert analytically the Laplace transform $C_y^s(a, b)$ with respect to s, a, b, which results into the distributions of the random variables $\sigma_y(t), \sigma_y^*(t)$. These distributions appear to be the limiting distributions for the corresponding distributions of the spectrally one-sided Lévy process. Under the conditions $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 < \infty$ and after re-scaling time and space, we will prove weak convergence of the distributions of $\sigma_y(t), \sigma_y^*(t)$ to the corresponding distributions of the Wiener process.

2.6.1 Compound Poisson process with linear drift

Let $\{X_t; t \geq 0\}$, $X_0 = 0$ be the compound Poisson process with positive jumps and negative drift whose Laplace exponent is such that

$$k(p) = \alpha p + \mu(\mathbf{E}^{-p\varkappa} - 1), \qquad \alpha, \mu \geq 0, \quad \varkappa \in (0, \infty), \tag{2.6.1}$$

where μ is the intensity of he jumps, $\varkappa$ is the jump size, and α is a coefficient of the drift. Introduce the scale function of the process by means of its Laplace transform $R_s(x) : [0, \infty) \to [0, \infty)$, $x \geq 0$

$$R_s(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{xp} \, \frac{1}{k(p) - s} \, dp, \qquad \gamma > c(s) \quad s \geq 0, \tag{2.6.2}$$

where $c(s) > 0$ is a unique root of the characteristic equation $k(p) - s = 0$ in the semi-plane $\Re(p) > 0$. Per definition we set $R_s(x) = 0$ for all $x < 0$. Observe, that

$$R_s(0) = \lim_{p \to \infty} p(k(p) - s)^{-1} = 1/\alpha > 0.$$

In order to determine the Laplace transforms $C_y^s(a, b)$, we first need to calculate the auxiliary function which is defined below. Let $y \in \mathbb{R}_+$, $\tau_y = \inf\{t > 0 : y + X_t < 0\}$, $\sigma_y = \sigma_y(\tau_y)$, $\sigma_y^* = \sigma_y^*(\tau_y)$. Denote by

$$D_y^s(a, b) = \mathbf{E}\left[e^{-s\tau_y - a\sigma_y - b\sigma_y^*}, \tau_y < \infty \right], \qquad a, b, s \geq 0$$

the integral transform of the joint distribution of $\{\tau_y, \sigma_y, \sigma_y^*\}$, where σ_y is the duration of stay inside the interval $[0, B]$ by the process $y + X_{(\cdot)}$ until the first passage time τ_y of the lower boundary 0, and $\sigma_y^* = \tau_y - \sigma_y$.

Lemma 2.6.1. *Let $\{X_t;\ t \geq 0\}$, $X_0 = 0$ be the compound Poisson process whose Laplace exponent is given by (2.6.1). Then the integral transform of the joint distribution of $\{\tau_y,\ \sigma_y,\ \sigma_y^*\}$ are such that:*

$$D_y^s(a, b) = \frac{U_{B-y}^s(a, b)}{U_B^s(a, b)} \, e^{-yc(s+b)}, \quad y \geq 0, \tag{2.6.3}$$

where $U_x^s(a, b) = 1$ for $x < 0$,

$$U_x^s(a, b) = 1 + (a - b) \int_0^x e^{-uc(s+b)} \, R_{s+a}(u) du, \quad x \geq 0. \tag{2.6.4}$$

Proof. Let $y \in [0, B]$, $x = B - y$ and denote by

$$\chi = \inf\{t : y + X_t \notin [0, B]\}, \qquad T = (X_\chi - x)\mathbf{I}_{\mathfrak{A}^x} + (-y - X_\chi)\mathbf{I}_{\mathfrak{A}_y}$$

the first exit time from the interval and the value of the overshoot through the boundary at this instant. Note that $\mathfrak{A}^x = \{X_\chi > x\}$, $\mathfrak{A}_y = \{X_\chi < -y\}$

are the events on which the exit can take place. Taking into account spatial homogeneity of the process, the Markov property of χ, and employing the total probability law, we can write for the function $D_y^s(a,b)$, $y \in \mathbb{R}_+$ the following equations:

$$\begin{aligned} D_y^s(a,b) &= \mathbf{E}\left[e^{-(s+a)\chi}; \mathfrak{A}_y\right] \\ &\quad + \int_0^\infty \mathbf{E}\left[e^{-(s+a)\chi}; T \in du, \mathfrak{A}^x\right] e^{-uc(s+b)} D_B^s(a,b), \quad y \in [0,B], \\ D_y^s(a,b) &= D_B^s(a,b) e^{-(y-B)c(s+b)}, \quad y > B. \end{aligned} \tag{2.6.5}$$

Letting $z = c(s+b)$ in (2.2.18), we derive for the function $G_x^s(z) = \mathbf{E} e^{-s\tau^x - zX_{\tau^x}}$, $x \in \mathbb{R}_+$ the following representation

$$\begin{aligned} G_x^{s+a}(c(s+b)) = 1 - \frac{a-b}{c(s+a)-c(s+b)}\, R_{s+a}(x)\, e^{-xc(s+b)} \\ + (a-b)\int_0^x e^{-uc(s+b)} R_{s+a}(u)\, du. \end{aligned} \tag{2.6.6}$$

Employing the latter equality and (2.3.11), (2.3.12), we get for the function $\tilde{V}^x(z,s) = \mathbf{E}\left[e^{-s\chi - zX_\chi}; \mathfrak{A}^x\right]$ for $z = c(s+b)$, $s \to s+a$

$$\tilde{V}^x(c(s+b), s+a) = U_x^s(a,b) e^{xc(s+b)} - \frac{R_{s+a}(x)}{R_{s+a}(B)} U_B^s(a,b) e^{Bc(s+b)}. \tag{2.6.7}$$

Substituting the expression for the function $\tilde{V}^x(z,s)$ into the first equation of (2.6.5), we get $(x = B - y)$

$$\begin{aligned} D_y^s(a,b) &= \frac{R_{s+a}(x)}{R_{s+a}(B)} \\ &+ \left[U_x^s(a,b) e^{xc(s+b)} - \frac{R_{s+a}(x)}{R_{s+a}(B)} U_B^s(a,b) e^{Bc(s+b)}\right] D_B^s(a,b), \quad y \in [0,B]. \end{aligned}$$

Letting $y = B$ in this equality, we find that

$$D_B^s(a,b) = \frac{1}{U_B^s(a,b)} e^{-Bc(s+b)}.$$

Substituting the latter expression for $D_B^s(a,b)$ into (2.6.5), we derive the formula (2.6.3) of Lemma 2.6.1. ▲

The next step is to determine the auxiliary function

$$Q_y^s(a,b) = \mathbf{E}\left[e^{-a\sigma_y(\nu_s) - b\sigma_y^*(\nu_s)}; \tau_y > \nu_s\right], \quad y \geq 0.$$

Lemma 2.6.2. *Let $\{X_t;\, t \geq 0\}$ be the compound Poisson process with Laplace exponent (2.6.1). Then the integral transform of the joint distribution of $\{\sigma_y,\ \sigma_y^*\}$ on the event $\{\tau_y > \nu_s\}$ is such that*

$$Q_y^s(a,b) = \frac{s}{s+b}\left(u_{B-y}^s(a,b) - u_B^s(a,b) D_y^s(a,b)\right), \quad y \geq 0, \tag{2.6.8}$$

where $u_x^s(a,b) = 1$ for $x < 0$,

$$u_x^s(a,b) = 1 + (a-b)\int_0^x R_{s+a}(u)\, du, \quad x \geq 0, \tag{2.6.9}$$

and the function $D_y^s(a,b)$, $y \geq 0$ is given by (2.6.3).

Proof. Employing the total probability law, homogeneity of the process and Markov property of χ, we write for $Q_y^s(a,b)$, $y \in \mathbb{R}_+$ the following system of equations:

$$\begin{aligned} Q_y^s(a,b) &= \frac{s}{s+a}\left(1 - \mathbf{E}e^{-(s+a)\chi}\right) \\ &\quad + \frac{s}{s+b}\int_0^\infty \mathbf{E}\left[e^{-(s+a)\chi}; T \in du, \mathfrak{A}^x\right]\left(1 - e^{-uc(s+b)}\right) \\ &\quad + \int_0^\infty \mathbf{E}\left[e^{-(s+a)\chi}; T \in du, \mathfrak{A}^x\right] Q_B^s(a,b) e^{-uc(s+b)}, \quad y \in [0,B], \\ Q_y^s(a,b) &= \frac{s}{s+b}\left(1 - e^{-(y-B)c(s+b)}\right) + Q_B^s(a,b)\, e^{-(y-B)c(s+b)}, \quad y > B. \end{aligned} \tag{2.6.10}$$

Taking into account (2.6.7) and (2.3.11), (2.3.12), we re-write the first equation of (2.6.10) as follows ($x = B - y$)

$$\begin{aligned} Q_y^s(a,b) &= \frac{s}{s+b}\left(u_x^s(a,b) - U_x^s(a,b) e^{xc(s+b)}\right) \\ &- \frac{s}{s+b}\left(u_B^s(a,b) - U_B^s(a,b) e^{Bc(s+b)}\right)\frac{R_{s+a}(x)}{R_{s+a}(B)} \\ &+ Q_B^s(a,b)\left(U_x^s(a,b) e^{xc(s+b)} - \frac{R_{s+a}(x)}{R_{s+a}(B)}\, U_B^s(a,b) e^{Bc(s+b)}\right), \quad y \in [0,B]. \end{aligned} \tag{2.6.11}$$

Letting $y = B$ in the latter equality, we find

$$Q_B^s(a,b) = \frac{s}{s+b} - \frac{s}{s+b}\frac{u_B^s(a,b)}{U_B^s(a,b)} e^{-Bc(s+b)}.$$

Inserting this expression for the function $Q_B^s(a,b)$ into (2.6.10), we derive the second formula (2.6.8) of Lemma 2.6.2. ▲

Thus, we found the auxiliary functions $D^s_y(a,b)$, $Q^s_y(a,b)$, $y\in\mathbb{R}_+$ (formulae 2.6.3, 2.6.8), and we are now able to determine the function of interest:

$$C^s_y(a,b)=s\int_0^\infty e^{-st}\mathbf{E}e^{-a\sigma_y(t)-b\sigma^*_y(t)}\,dt,\quad a,b\geq 0.$$

Denote $\tau^y=\inf\{t>0:X_t>y\}$, $T^y=X_{\tau^y}-y$, $y\geq 0$,

$$f^y(dv,s)=\mathbf{E}\left[e^{-s\tau^y};T^y\in dv,\tau^y<\infty\right],\quad f^y(s)=\mathbf{E}\left[e^{-s\tau^y};\tau^y<\infty\right].$$

Theorem 2.6.1. *Let $\{X_t;\ t\geq 0\}$ be the compound Poisson process with Laplace exponent (2.6.1), $u^s_x(a,b)=U^s_x(a,b)=1$, for $x<0$,*

$$U^s_x(a,b)=1+(a-b)\int_0^x e^{-uc(s+b)}\,R_{s+a}(u)du,\quad x\geq 0,$$
$$u^s_x(a,b)=1+(a-b)\int_0^x R_{s+a}(u)\,du,\quad x\geq 0.$$

Then the following equalities are valid for the integral transforms of the joint distribution of $\{\sigma_y(t),\sigma^*_y(t)\}$, $y\in\mathbb{R}$

$$\begin{aligned}C^s_y(a,b)&=\frac{s}{s+b}\left(u^s_{B-y}(a,b)-C(B)U^s_{B-y}(a,b)e^{-yc(s+b)}\right),\quad y\in\mathbb{R}_+,\\ C^s_{-y}(a,b)&=\frac{s}{s+b}\left(1+(a-b)\int_0^B f^y(du,s+b)\int_0^{B-u}R_{s+a}(v)\,dv\right)\\ &\quad-\frac{s}{s+b}\,C(B)\int_0^\infty f^y(dv,s+b)U^s_{B-v}(a,b)e^{-vc(s+b)},\quad y>0,\end{aligned}\tag{2.6.12}$$

where $k'(c(s+b))=\frac{d}{dp}k(p)\Big|_{p=c(s)}$,

$$C(B)=\frac{a-b}{c(s+b)}\,\frac{U^s_B(a,b)\,e^{Bc(s+b)}-u^s_B(a,b)}{k'(c(s+b))+(a-b)\int_0^B U^s_x(a,b)\,dx}.$$

Proof. Again, in view of the total probability law, spatial homogeneity of the process and Markov property of χ, we write the system of equations for the function $C^s_y(a,b)$, $y\in\mathbb{R}$

$$\begin{aligned}C^s_y(a,b)&=Q^s_y(a,b)+D^s_y(a,b)\,C^s_{0-}(a,b),\quad y\geq 0,\\ C^s_{-y}(a,b)&=\frac{s}{s+b}\,(1-f^y(s+b))+\int_0^\infty f^y(dv,s+b)C^s_v(a,b),\quad y>0,\end{aligned}\tag{2.6.13}$$

where $C^s_{0-}(a,b) = \lim_{y\to 0} C^s_{-y}(a,b)$. Inserting the expression for the function $C^s_y(a,b)$, $y \ge 0$ from the second equation into the first one implies that

$$C^s_{-y}(a,b) = \frac{s}{s+b}\left(1 - f^y(s+b)\right) + \int_0^\infty f^y(dv, s+b) Q^s_v(a,b)$$
$$+ C^s_{0-}(a,b) \int_0^\infty f^y(dv, s+b) D^s_v(a,b), \quad y > 0.$$

Letting $y \to 0$ in this equation, we derive a linear equation for the function $C^s_{0-}(a,b)$, solving which yields

$$C^s_{0-}(a,b) = \frac{\frac{s}{s+b}\big(1 - f^0(s+b)\big) + \int_0^\infty f^0(dv, s+b) Q^s_v(a,b)}{1 - \int_0^\infty f^0(dv, s+b) D^s_v(a,b)}.$$

Using the equalities (2.6.3), (2.6.8) and performing some calculations, we find

$$C^s_{0-}(a,b) = \frac{s}{s+b}\left(u^s_B(a,b) - U^s_B(a,b) C(B)\right). \tag{2.6.14}$$

While performing the calculations, we also used the relations:

$$\int_0^B f^0(dv, s+b) e^{-vc(s+b)} \int_0^{B-v} e^{-uc(s+b)} R_{s+a}(u)\, du =$$
$$= \int_0^B e^{-uc(s+b)} R_{s+a}(u)\, du + \frac{1}{\alpha}\int_0^B U^s_x(a,b)\, dx,$$
$$\int_0^B f^0(dv, s+b) \int_0^{B-v} R_{s+a}(u)\, du = \int_0^B R_{s+a}(u)\, du -$$
$$- \frac{1}{\alpha c(s+b)}\left[e^{Bc(s+b)} U^s_B(a,b) - u^s_B(a,b)\right],$$

which follow from (2.2.17, 2.2.18) and the definition of the resolvent (2.6.2). Substituting the expression for the function $C^s_{0-}(a,b)$ (2.6.14) into (2.6.13), we get formulae (2.6.12) of Theorem 2.6.1. ▲

2.6.2 Spectrally one-sided Lévy process

Let $\{X_t; t \ge 0\}$, $X_0 = 0$ be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7). In this section we will assume that $\sigma > 0$. In this case $\lim_{x\to 0} R_s(x) = \lim_{p\to\infty} p(k(p) - s)^{-1} = 0$, $p > 0$. It is well-known (see Pistorius (2004)), that the function $R'_s(x) = \frac{d}{dx} R_s(x)$, $x > 0$ is continuous and

$$\lim_{x\to 0} R'_s(x) = \lim_{p\to\infty} p^2 (k(p) - s)^{-1} = \sigma^{-2}, \quad x, p > 0.$$

The function $C_y^s(a,b)$, $y \in \mathbb{R}$ obeys the integro-differential equation of the second kind (Skorokhod (1964)), and $\frac{d}{dy}C_y^s(a,b)$, $y \in \mathbb{R}$ is a continuous function with respect to y. Note, that the functions $\frac{d}{dy}D_y^s(a,b)$, $\frac{d}{dy}Q_y^s(a,b)$ possess the same properties for $y > 0$.

Let $y \in \mathbb{R}_+$, $\tau_y = \inf\{t > 0 : \ y + X_t < 0\}$, $\sigma_y = \sigma_y(\tau_y)$, $\sigma_y^* = \sigma_y^*(\tau_y)$,

$$D_y^s(a,b) = \mathbf{E}\left[e^{-s\tau_y - a\sigma_y - b\sigma_y^*}, \tau_y < \infty\right], \qquad a,b,s \geq 0.$$

Lemma 2.6.3. *Let $\{X_t;\, t \geq 0\}$, $X_0 = 0$ be the spectrally positive Lévy process whose Laplace exponent is given by (2.2.7). Then the integral transform of the joint distribution of $\{\tau_y,\ \sigma_y,\ \sigma_y^*\}$ is such that*

$$D_y^s(a,b) = \frac{U_{B-y}^s(a,b)}{U_B^s(a,b)}\, e^{-yc(s+b)}, \quad y \geq 0, \tag{2.6.15}$$

where $U_x^s(a,b) = 1$, for $x < 0$,

$$U_x^s(a,b) = 1 + (a-b)\int_0^x e^{-uc(s+b)}\, R_{s+a}(u)du, \quad x \geq 0,$$

$R_s(x)$, $x \geq 0$ is the scale function (2.2.11) of the process.

Corollary 2.6.1. *Let $\{X_t; t \geq 0\}$ be the standard Wiener process with Laplace exponent $k(p) = \frac{1}{2}\, p^2$ and $y \in [0,B]$, $x = B - y$. Then*

(i) *the integral transform of the joint distribution of $\{\tau_y, \sigma_y, \sigma_y^*\}$ is given by*

$$D_y^s(a,b) = \frac{\sqrt{s+a}\cosh(x\sqrt{2(s+a)}) + \sqrt{s+b}\sinh(x\sqrt{2(s+a)})}{\sqrt{s+a}\cosh(B\sqrt{2(s+a)}) + \sqrt{s+b}\sinh(B\sqrt{2(s+a)})}, \tag{2.6.16}$$

where $\sinh x$, $\cosh x$ *are the hyperbolic sine and cosine;*

(ii) *the Laplace transforms of σ_y, σ_y^* are such that*

$$\mathbf{E}e^{-a\sigma_y} = \frac{\cosh x\sqrt{2a}}{\cosh B\sqrt{2a}}, \quad \mathbf{E}e^{-b\sigma_y^*} = \frac{1 + x\sqrt{2b}}{1 + B\sqrt{2b}};$$

(iii) *the following equalities are valid for the random variables* σ_y, σ_y^*

$$\mathbf{P}[\sigma_y > t] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\left(-t\frac{(2n+1)^2\pi^2}{8B^2}\right)\cos\frac{x\pi}{2B}(2n+1), \tag{2.6.17}$$

$$\mathbf{P}[\sigma_y^* = 0] = \frac{x}{B}, \qquad \mathbf{P}[\sigma_y^* > t] = \frac{y}{B}e^{t/2B^2}\sqrt{\frac{2}{\pi}}\int_{\sqrt{t}/B}^{\infty} e^{-u^2/2}\,du; \tag{2.6.18}$$

(iv) σ_y^* *does not have moments, but the moments of the random variable* σ_y *are given as follows:*

$$\mathbf{E}\sigma_y^n = \frac{B^{2n}}{(2n-1)!!}\sum_{k=0}^{n}(-1)^k\left(\frac{x}{B}\right)^{2k}\binom{2n}{2k}E_{n-k}, \quad n\in\mathbb{N}, \tag{2.6.19}$$

where E_n, $n\in\mathbb{N}$ *are the Euler numbers defined by (2.3.18).*

Proof. Let $y\in[0,B]$, $x = B-y$ and denote by

$$\chi = \inf\{t : y + X_t \notin [0,B]\}, \qquad T = (X_\chi - x)\mathbf{I}_{\mathfrak{A}^x} + (-y - X_\chi)\mathbf{I}_{\mathfrak{A}_y}$$

the first exit time from the interval and the value of the overshoot at this instant. Here $\mathfrak{A}^x = \{X_\chi > x\}$, $\mathfrak{A}_y = \{X_\chi < -y\}$ are the events on which the exit can take place. It is not difficult to derive the following system of equations for the function $D_y^s(a,b)$, $y\in\mathbb{R}_+$

$$\begin{aligned} D_y^s(a,b) &= \mathbf{E}\left[e^{-(s+a)\chi};\mathfrak{A}_y\right] \\ &\quad + \int_0^\infty \mathbf{E}\left[e^{-(s+a)\chi}; T\in du, \mathfrak{A}^x\right] e^{-uc(s+b)}D_B^s(a,b), \quad y\in[0,B], \\ D_y^s(a,b) &= D_B^s(a,b)e^{-(y-B)c(s+b)}, \quad y > B. \end{aligned} \tag{2.6.20}$$

Here we used the Markov property of χ, spatial homogeneity of the process and total probability law. Let us explain this system. The first equation means that the sojourn occupation inside the interval $[0,B]$ till the first passage time of the lower boundary occurs either on the sample paths of the process which do not intersect the upper boundary, or on the paths which exit the interval $[0, B]$ through the upper boundary. The second equation is evident. In view of (2.2.9) we can write the following representation for the function $G_x^s(z) = \mathbf{E}e^{-s\tau^x - zX_{\tau^x}}$, $x \in \mathbb{R}_+$ for $z = c(s+b)$

$$G_x^{s+a}(c(s+b)) = 1 - \frac{a-b}{c(s+a) - c(s+b)}\, R_{s+a}(x)\, e^{-xc(s+b)} \\ + (a-b) \int_0^x e^{-uc(s+b)} R_{s+a}(u)\, du.$$

Taking into account the latter expression and (2.3.11), (2.3.12), we find for the function $\tilde{V}^x(z,s) = \mathbf{E}\left[e^{-s\chi - zX_\chi}; \mathfrak{A}^x\right]$ for $z = c(s+b)$, $s \to s+a$ that

$$\tilde{V}^x(c(s+b), s+a) = U_x^s(a,b)e^{xc(s+b)} - \frac{R_{s+a}(x)}{R_{s+a}(B)} U_B^s(a,b)e^{Bc(s+b)}. \quad (2.6.21)$$

Inserting the expression for the function $\tilde{V}^x(z,s)$ into (2.6.20), we get ($x = B - y$)

$$D_y^s(a,b) = \frac{R_{s+a}(B-y)}{R_{s+a}(B)} + \\ + \left[U_{B-y}^s(a,b)e^{(B-y)c(s+b)} - \frac{R_{s+a}(x)}{R_{s+a}(B)} U_B^s(a,b)e^{Bc(s+b)}\right] D_B^s(a,b), \quad y \in [0,B], \\ D_y^s(a,b) = D_B^s(a,b)e^{-(y-B)c(s+b)}, \quad y > B. \quad (2.6.22)$$

Differentiating the equalities (2.6.22) and letting $y = B$, we obtain the following system of equations:

$$-\frac{d}{dy} D_y^s(a,b)\Big|_{y=B} = \frac{R'_{s+a}(0)}{R_{s+a}(B)} + c(s+b)D_B^s(a,b) - \\ - D_B^s(a,b)\, \frac{R'_{s+a}(0)}{R_{s+a}(B)} U_B^s(a,b)e^{Bc(s+b)}, \\ \frac{d}{dy} D_y^s(a,b)\Big|_{y=B} = -c(s+b)D_B^s(a,b),$$

where $R'_{s+a}(0) = \frac{d}{dx} R_{s+a}(x)\big|_{x\to 0} = \sigma^{-2} > 0$. Solving this system, we find

$$D_B^s(a,b) = U_B^s(a,b)^{-1} \exp\{-Bc(s+b)\}.$$

Substituting this expression into (2.6.22), we derive (2.6.15). Thus, Lemma 2.6.3 is proved. We now verify the statements of Corollary 2.6.1. Using the definition (2.2.11), we now calculate the scale function of the standard Wiener process. For $\gamma > c(s) = \sqrt{2s}$ we have

$$R_s(x) = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{xp} \, \frac{1}{\frac{1}{2}\, p^2 - s} \, dp = \sqrt{\frac{2}{s}} \sinh\left(x\sqrt{2s}\right). \tag{2.6.23}$$

Substituting this expression for the scale function into (2.6.15) and performing necessary calculations, we obtain formula (2.6.16) of Corollary 2.6.1.

Further, letting step by step in (2.6.16) $s = 0, \quad b = 0$ and then $s = 0, \quad a = 0$, we derive the Laplace transforms of the random variables $\sigma_y, \ \sigma_y^*$

$$\mathbf{E}e^{-a\sigma_y} = \frac{\cosh x\sqrt{2a}}{\cosh B\sqrt{2a}}, \qquad \mathbf{E}e^{-b\sigma_y^*} = \frac{1 + x\sqrt{2b}}{1 + B\sqrt{2b}}, \quad x = B - y.$$

Calculating the contour integrals (Ditkin and Kuznecov (1951))

$$\frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{at} \, \frac{1}{a} \left(1 - \frac{\cosh x\sqrt{2a}}{\cosh B\sqrt{2a}}\right) \, da,$$

$$\frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{bt} \, \frac{1}{b} \left(1 - \frac{1 + x\sqrt{2b}}{1 + B\sqrt{2b}}\right) \, db, \ (\gamma > 0),$$

we find the distributions of σ_y, σ_y^*

$$\mathbf{P}\left[\sigma_y > t\right] = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \, \exp\left(-t\frac{(2n+1)^2\pi^2}{8B^2}\right) \cos\frac{x\pi}{2B}(2n+1), \tag{2.6.24}$$

$$\mathbf{P}\left[\sigma_y^* = 0\right] = \frac{x}{B}, \quad \mathbf{P}\left[\sigma_y^* > t\right] = \frac{y}{B} \, e^{t/2B^2} \, \sqrt{\frac{2}{\pi}} \int_{\sqrt{t}/B}^{\infty} \, e^{-u^2/2} \, du. \tag{2.6.25}$$

The formula (2.6.24) is also the asymptotic expansion for the probability which enters its left-hand side. For instance, restricting us only to the first term of the series of (2.6.24), we can write for $t \to \infty$

$$\mathbf{P}\left[\sigma_B > t\right] = \frac{4}{\pi} \, \exp\left(-t\frac{\pi^2}{8B^2}\right) + o\left(e^{-t\frac{\pi^2}{8B^2}}\right).$$

Using the asymptotic expansion of the integral (Dwight (2005)), we get from the formula (2.6.25) as $t \to \infty$

$$\mathbf{P}\left[\sigma_y^* > t\right] = \frac{y}{\sqrt{2\pi t}}\left(1 + \sum_{k=1}^{n} (-1)^k \frac{B^{2k}}{t^k}\,(2k-1)!!\right) + o\left(\frac{1}{t^{n+\frac{1}{2}}}\right), \quad y \in [0, B].$$

In order to determine the moments of the random variable σ_y (2.6.25), whose Laplace transform is given by

$$\mathbf{E}e^{-a\sigma_y} = \frac{\cosh x\sqrt{2a}}{\cosh B\sqrt{2a}}, \quad x = B - y,$$

it is necessary to use the series expansion (Dwight (2005)) of the function $\operatorname{sech} = \cosh^{-1}$. ▲

Now we will determine the auxiliary function

$$Q_y^s(a, b) = \mathbf{E}\left[e^{-a\sigma_y(\nu_s) - b\sigma_y^*(\nu_s)}; \tau_y > \nu_s\right], \quad y \geq 0.$$

Lemma 2.6.4. *Let* $\{X_t;\, t \geq 0\}$, $X_0 = 0$ *be the spectrally positive Lévy process with Laplace exponent (2.2.7). Then for the integral transform of the joint distribution of* $\{\sigma_y,\ \sigma_y^*\}$ *on the event* $\{\tau_y > \nu_s\}$ *the following equalities are valid:*

$$Q_y^s(a, b) = \frac{s}{s + b}\left(u_{B-y}^s(a, b) - u_B^s(a, b) D_y^s(a, b)\right), \quad y \geq 0, \qquad (2.6.26)$$

where $u_x^s(a, b) = 1$ *for* $x < 0$

$$u_x^s(a, b) = 1 + (a - b)\int_0^x R_{s+a}(u)\, du, \quad x \geq 0,$$

the functions $D_y^s(a, b)$, $y \geq 0$ *are determined by the equalities* (2.6.15) *of Lemma 2.6.3.*

Proof. According to the total probability law and Markov property of χ, spatial homogeneity of the process, we can write the following system of equations for the function $Q_y^s(a,b)$, $y \in \mathbb{R}_+$

$$
\begin{aligned}
Q_y^s(a,b) &= \frac{s}{s+a}\left(1-\mathbf{E}e^{-(s+a)\chi}\right)\\
&+\frac{s}{s+b}\int_0^\infty \mathbf{E}\left[e^{-(s+a)\chi}; T\in du, \mathfrak{A}^x\right]\left(1-e^{-uc(s+b)}\right) \qquad (2.6.27)\\
&+\int_0^\infty \mathbf{E}\left[e^{-(s+a)\chi}; T\in du, \mathfrak{A}^x\right] Q_B^s(a,b)e^{-uc(s+b)}, \quad y\in[0,B],\\
Q_y^s(a,b) &= \frac{s}{s+b}\left(1-e^{-(y-B)c(s+b)}\right)+Q_B^s(a,b)\; e^{-(y-B)c(s+b)}, \quad y>B.
\end{aligned}
$$

The three summands which enter the right-hand side of the first equation mean that the total time of staying inside the interval $[0,B]$ (outside the interval $[0,B]$) by the process without intersection of the lower boundary can be realized either on the sample paths of the process which do not leave the interval $[0,B]$, or on the paths which do leave the interval through the upper boundary and never return back, or on the paths which do leave the interval through the upper boundary and then return inside. The second equation is evident. Employing (2.6.21) and (2.3.11), (2.3.12), we re-write the system (2.6.27) as follows:

$$
\begin{aligned}
&Q_y^s(a,b) = \frac{s}{s+b}\left(u_{B-y}^s(a,b)-U_{B-y}^s(a,b)e^{(B-y)c(s+b)}\right)\\
&-\frac{s}{s+b}\left(u_B^s(a,b)-U_B^s(a,b)e^{Bc(s+b)}\right)\frac{R_{s+a}(B-y)}{R_{s+a}(B)} \qquad (2.6.28)\\
&+Q_B^s(a,b)\left(U_{B-y}^s(a,b)e^{(B-y)c(s+b)}-\frac{R_{s+a}(B-y)}{R_{s+a}(B)}\,U_B^s(a,b)e^{Bc(s+b)}\right),\; y\in[0,B],\\
&Q_y^s(a,b) = \frac{s}{s+b}\left(1-e^{-(y-B)c(s+b)}\right)+Q_B^s(a,b)\; e^{-(y-B)c(s+b)}, \quad y>B.
\end{aligned}
$$

Differentiating the equalities (2.6.28) and then letting $y=B$ yields the following system:

$$
\begin{aligned}
\frac{d}{dy}Q_y^s(a,b)\Big|_{y=B} &= \frac{s}{s+b}\left(c(s+b)+\left(u_B^s(a,b)-U_B^s(a,b)e^{Bc(s+b)}\right)\frac{R'_{s+a}(0)}{R_{s+a}(B)}\right)\\
&\quad -Q_B^s(a,b)\left(c(s+b)-U_B^s(a,b)e^{Bc(s+b)}\frac{R'_{s+a}(0)}{R_{s+a}(B)}\right),\\
\frac{d}{dy}Q_y^s(a,b)\Big|_{y=B} &= \frac{s}{s+b}\,c(s+b)-Q_B^s(a,b)c(s+b).
\end{aligned}
$$

Solving this system, we find that

$$Q_B^s(a,b) = \frac{s}{s+b}\left(1 - \frac{u_B^s(a,b)}{U_B^s(a,b)} e^{-Bc(s+b)}\right).$$

Substituting this expression for the function $Q_B^s(a,b)$ in the equality (2.6.28), we obtain the formulae (2.6.24) of the Lemma 2.6.4. ▲

Hence, the auxiliary functions $D_y^s(a,b)$, $Q_y^s(a,b)$, $y \in \mathbb{R}_+$ are determined by (2.6.15), (2.6.16), and we are now able to find the function

$$C_y^s(a,b) = s\int_0^\infty e^{-st}\mathbf{E}e^{-a\sigma_y(t)-b\sigma_y^*(t)}\,dt, \quad a,b \ge 0.$$

Denote $\tau^y = \inf\{t>0 : X_t > y\}$, $T^y = X_{\tau^y} - y$, $y \ge 0$,

$$f^y(dv,s) = \mathbf{E}\left[e^{-s\tau^y}; T^y \in dv, \tau^y < \infty\right], \quad f^y(s) = \mathbf{E}\left[e^{-s\tau^y}; \tau^y < \infty\right].$$

Theorem 2.6.2 (Kadankov and Kadankova (2004))**.** *Let* $\{X_t;\ t \ge 0\}$, $X_0 = 0$ *be the spectrally positive Lévy process with Laplace exponent (2.2.7) and* $\{R_s(x)\}_{x\ge 0}$ *be the scale function of the process (2.2.11). Denote*

$$U_x^s(a,b) = 1 + (a-b)\int_0^x e^{-uc(s+b)}\,R_{s+a}(u)du, \quad x \ge 0,$$
$$u_x^s(a,b) = 1 + (a-b)\int_0^x R_{s+a}(u)\,du, \quad x \ge 0,$$

$u_x^s(a,b) = U_x^s(a,b) = 1$, *for* $x < 0$.

Then the integral transforms of the joint distribution of $\{\sigma_y(t), \sigma_y^*(t)\}$, $y \in \mathbb{R}$ *are such that*

$$C_y^s(a,b) = \frac{s}{s+b}\left(u_{B-y}^s(a,b) - C(B)U_{B-y}^s(a,b)e^{-yc(s+b)}\right), \quad y \in \mathbb{R}_+,$$
$$C_{-y}^s(a,b) = \frac{s}{s+b}\left(1 + (a-b)\int_0^B f^y(du,s+b)\int_0^{B-u} R_{s+a}(v)\,dv\right) \tag{2.6.29}$$
$$- \frac{s}{s+b}\,C(B)\int_0^\infty f^y(dv,s+b)U_{B-v}^s(a,b)e^{-vc(s+b)}, \quad y > 0,$$

where $r(s,b) = k'(c(s+b)) = \frac{d}{dp}k(p)\Big|_{p=c(s)}$,

$$C(B) = \frac{a-b}{c(s+b)}\;\frac{U_B^s(a,b)\,e^{Bc(s+b)} - u_B^s(a,b)}{r(s,b) + (a-b)\int_0^B U_x^s(a,b)\,dx}.$$

Proof. One can find that the function $C^s_y(a,b)$, $y \in \mathbb{R}$ obeys to the following system of the equations:

$$C^s_y(a,b) = Q^s_y(a,b) + D^s_y(a,b)\, C^s_{0-}(a,b), \quad y \ge 0, \tag{2.6.30}$$
$$C^s_{-y}(a,b) = \frac{s}{s+b}\,(1 - f^y(s+b)) + \int_0^\infty f^y(dv, s+b) C^s_v(a,b), \quad y > 0,$$

where $C^s_{0-}(a,b) = \lim_{y\to 0} C^s_{-y}(a,b)$. We again used the total probability law, spatial homogeneity of the process and Markov property of χ. Now inserting the expression for the function $C^s_y(a,b)$ $y \ge 0$ from the first equation into the second one, we find that

$$C^s_{-y}(a,b) = \frac{s}{s+b}\,(1 - f^y(s+b)) + \int_0^\infty f^y(dv, s+b) Q^s_v(a,b)$$
$$+ C^s_{0-}(a,b) \int_0^\infty f^y(dv, s+b) D^s_v(a,b), \quad y > 0. \tag{2.6.31}$$

It follows from (2.6.15), (2.6.26) that

$$\frac{d}{dy} D^s_y(a,b)\bigg|_{y\to 0} = -c(s+b) - \frac{a-b}{U^s_B(a,b)} R_{s+a}(B) e^{-Bc(s+b)},$$
$$\frac{d}{dy} Q^s_y(a,b)\bigg|_{y\to 0} = -\frac{s}{s+b}(a-b) R_{s+a}(B) +$$
$$+ \frac{s}{s+b} u^s_B(a,b) \left(c(s+b) + \frac{a-b}{U^s_B(a,b)} R_{s+a}(B) e^{-Bc(s+b)} \right). \tag{2.6.32}$$

Taking into account (2.2.9), (2.6.21) and the definition of the scale function (2.2.11), we calculate the following limits:

$$\frac{d}{dy} \int_0^B f^y(dv, s+b) \int_0^{B-v} R_{s+a}(u)\, du \bigg|_{y\to 0} =$$
$$= R_{s+a}(B) - \sigma^{-2} \left[U^s_B(a,b) e^{Bc(s+b)} - u^s_B(a,b) \right] c(s+b)^{-1},$$
$$\frac{d}{dy} \int_0^B f^y(dv, s+b) e^{-vc(s+b)} \int_0^{B-v} e^{-uc(s+b)} R_{s+a}(u)\, du \bigg|_{y\to 0} = R_{s+a}(B) e^{-Bc(s+b)}$$
$$+ c(s+b) \int_0^B e^{-uc(s+b)} R_{s+a}(u)\, du - \frac{1}{\sigma^2} \int_0^B U^s_x(a,b)\, dx,$$
$$\frac{d}{dy} \int_0^\infty f^y(dv, s+b) e^{-vc(s+b)} \bigg|_{y\to 0} = c(s+b) - r(s,b)\sigma^{-2}. \tag{2.6.33}$$

Since the function $\frac{d}{dy}C^s_y(a,b)$, $y\in\mathbb{R}$ is continuous, we have

$$\frac{d}{dy}C^s_y(a,b)\bigg|_{y\to 0}=-\frac{d}{dy}C^s_{-y}(a,b)\bigg|_{y\to 0}.$$

Employing the latter equality, we calculate the limits (2.6.32), (2.6.33). Then differentiating the first equality of (2.6.30) and the formula (2.6.31), we obtain an equation with respect to the function $C^s_{0-}(a,b)$:

$$0=\frac{s}{s+b}A^s(a,b)+B^s(a,b)C^s_{0-}(a,b), \tag{2.6.34}$$

where

$$\begin{aligned}A^s(a,b)&=\frac{u^s_B(a,b)}{U^s_B(a,b)}\left(r(s,b)+(a-b)\int_0^B U^s_x(a,b)\,dx\right)\sigma^{-2}\\&\quad-\frac{a-b}{c(s+b)}\left(U^s_B(a,b)e^{Bc(s+b)}-u^s_B(a,b)\right)\sigma^{-2},\\B^s(a,b)&=\frac{\sigma^{-2}}{U^s_B(a,b)}\left(r(s,b)+(a-b)\int_0^B U^s_x(a,b)\,dx\right).\end{aligned}$$

It follows from (2.6.34) that

$$C^s_{0-}(a,b)=\frac{s}{s+b}u^s_B(a,b)-\frac{s}{s+b}U^s_B(a,b)C(B).$$

Inserting the expression for $C^s_{0-}(a,b)$ into the first equality of (2.6.30), we get the formulae (2.6.31) of Theorem 2.6.2. ▲

Remark 2.6.1. *It is worth noting that the formulae (2.6.15), (2.6.26), (2.6.29) for the integral transforms of the distributions $\sigma_y(t)$, $y\in\mathbb{R}$ for the spectrally positive Lévy processes are identical to the formulae (2.6.3), (2.6.8), (2.6.12) for the compound Poisson process. The reason is as follows. A spectrally one-sided Lévy process can be approximated by a sequence of compound Poisson processes with a drift (Gihman and Skorokhod (1973, ch.4,§2)). This fact assures the weak convergence of boundary functionals of the approximating Poisson processes to the corresponding boundary functionals of the spectrally one-sided Lévy process.*

2.6.3 Standard Wiener process

Now we apply the results obtained in the previous sections to determine the distribution of the sojourn occupation inside the fixed interval by a standard Wiener process $\{w(t);\ t \geq 0\}$.

Corollary 2.6.2. *Let $\{w(t); t \geq 0\}$ be the standard Wiener process with the Laplace exponent $k(p) = \frac{1}{2}p^2$. Then*

(i) *the integral transform*

$$C_y^s(a,b) = s\int_0^\infty e^{-st}\mathbf{E}\left[e^{-a\sigma_y(t)-b\sigma_y^*(t)}\right]dt, \quad y \in \mathbb{R}$$

of the joint distribution of $\{\sigma_y(t), \sigma_y^*(t)\}$ satisfies the following equalities:

$$C_y^s(a,b) = \frac{s}{s+b}\left(1 - \frac{a-b}{\sqrt{s+a}}\frac{\sinh(\frac{B}{\sqrt{2}}\sqrt{s+a})\exp\{-(y-B)\sqrt{2(s+b)}\}}{\sqrt{s+a}\sinh(\frac{B}{\sqrt{2}}\sqrt{s+a}) + \sqrt{s+b}\cosh(\frac{B}{\sqrt{2}}\sqrt{s+a})}\right),$$
$$y > B, \tag{2.6.35}$$

$$C_y^s(a,b) = \frac{s}{s+a}\left(1 + \frac{a-b}{\sqrt{s+b}}\frac{\cosh(\frac{B-2y}{\sqrt{2}}\sqrt{s+a})}{\sqrt{s+a}\sinh(\frac{B}{\sqrt{2}}\sqrt{s+a}) + \sqrt{s+b}\cosh(\frac{B}{\sqrt{2}}\sqrt{s+a})}\right),$$
$$y \in [0,B], \tag{2.6.36}$$

$$C_y^s(a,b) = \frac{s}{s+b}\left(1 - \frac{a-b}{\sqrt{s+a}}\frac{\sinh(\frac{B}{\sqrt{2}}\sqrt{s+a})\ \exp\{y\sqrt{2(s+b)}\}}{\sqrt{s+a}\sinh(\frac{B}{\sqrt{2}}\sqrt{s+a}) + \sqrt{s+b}\cosh(\frac{B}{\sqrt{2}}\sqrt{s+a})}\right),$$
$$y < 0; \tag{2.6.37}$$

(ii) *the expectations of $\sigma_y(t)$, $\sigma_y^*(t)$ are such that*

$$\begin{aligned}
\mathbf{E}\sigma_y^*(t) &= \int_0^t \mathbf{P}[w(u) > y]\,du + \int_0^t \mathbf{P}[w(u) > B-y]\,du,\\
\mathbf{E}\sigma_y(t) &= t - \mathbf{E}\sigma_y^*(t), \quad y \in [0,B], \qquad (2.6.38)\\
\mathbf{E}\sigma_y(t) &= \int_0^t \mathbf{P}[w(u) > y-B]\,du - \int_0^t \mathbf{P}[w(u) > y]\,du,\\
\mathbf{E}\sigma_y^*(t) &= t - \mathbf{E}\sigma_y(t), \quad y > B,
\end{aligned}$$

where

$$\int_0^t \mathbf{P}[w(u) > x]\, du = \frac{t + x^2}{\sqrt{2\pi t}} \int_x^\infty e^{-\frac{u^2}{2t}}\, du - \frac{x\sqrt{t}}{\sqrt{2\pi}} e^{-\frac{x^2}{2t}}.$$

It is worth noting that for $b = 0$ the formulae (2.6.35)$-$(2.6.37) are given in Fusai (2000). Inverting the Laplace transforms in the equalities (2.6.35)$-$(2.6.37) yields the distributions of $\sigma_y(t)$, $\sigma_y^*(t)$. The following statement contains the equalities for these distributions in the case when $y \in [0, B]$.

Theorem 2.6.3 (Kadankov and Kadankova (2004)). *Let* $\{w(t); t \geq 0\}$ *be the standard Wiener process,*

$$\sigma_y(t) = \int_0^t g(\, y + w(u))\, du, \qquad \sigma_y^*(t) = \int_0^t g^*(\, y + w(u))\, du, \quad y \in [0, B]$$

be the sojourn occupation times inside and outside the interval $[0, B]$ *by the process* $y + w(\cdot)$ *till time* t*. Then the distributions of* $\sigma_y(t)$, $\sigma_y^*(t)$ *are such that*

$$\begin{aligned}
\mathbf{P}\left[\sigma_y(t) < u\right] &= \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^u \exp\left(-\frac{(y + nB)^2}{2v}\right) \mathbf{K}_n\left(\frac{t-u}{t-v}\right) d_v \arcsin\sqrt{\frac{v}{t}} \\
&+ \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^u \exp\left(-\frac{(x + nB)^2}{2v}\right) \mathbf{K}_n\left(\frac{t-u}{t-v}\right) d_v \arcsin\sqrt{\frac{v}{t}}, \\
& u \in (0, t), \\
\mathbf{P}\left[\sigma_y^*(t) < u\right] &= 1 - \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^{t-u} e^{-(y+nB)^2/2v}\, \mathbf{K}_n\left(\frac{u}{t-v}\right) d_v \arcsin\sqrt{\frac{v}{t}} \\
&- \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^{t-u} e^{-(x+nB)^2/2v}\, \mathbf{K}_n\left(\frac{u}{t-v}\right)\, d_v \arcsin\sqrt{\frac{v}{t}},
\end{aligned}$$

where $x = B - y$, $u \in (0, t)$ *and*

$$\mathbf{K}_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^k\, (1 - x)^{n-k}, \qquad x \in [0, 1].$$

In particular,

$$
\begin{aligned}
\mathbf{P}\left[\sigma_y^*(t)=0\right] &= \mathbf{P}\left[\sigma_y(t)=t\right] \\
&= 1-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^t \exp\left(-\frac{(y+nB)^2}{2v}\right) d_v \arcsin\sqrt{\frac{v}{t}} \\
&- \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^t \exp\left(-\frac{(x+nB)^2}{2v}\right)\ d_v \arcsin\sqrt{\frac{v}{t}}.
\end{aligned}
$$

For the proof we refer to Kadankov and Kadankova (2004).

2.6.4 Compound Poisson process with arbitrary positive and exponential negative jumps

Theorem 2.6.4 (Kadankova and Veraverbeke (2005)). *Let* $\{X_t;\, t\ge 0\}$, $X_0 = 0$ *be the compound Poisson process with Laplace exponent (2.2.13),*

$$
\begin{aligned}
&v_a^s(x) = 1 + a\lambda\int_0^x R_{s+a}(u)\,du, \quad x\ge 0; \quad v_a^s(x)=1,\ x<0, \\
&V_a^s(x) = 1 + a(\lambda - c(s))\int_0^x e^{-uc(s)}R_{s+a}(u)\,du, \quad x\ge 0; \quad V_a^s(x)=1,\ x<0.
\end{aligned}
$$

Then the integral transform $C_a^s(y)$ *of* $\sigma_y(t)$, $y\in\mathbb{R}$ *satisfies the equalities for* $B>0,\ s>0,\ a\ge 0$

$$
\begin{aligned}
&C_a^s(y) = v_a^s(B-y) - aR_{s+a}(B-y) + D_a^s(y)C^*(B), && y\ge 0, \quad (2.6.39)\\
&C_a^s(-y) = 1 - \mathbf{E}e^{-s\tau^y} + \int_0^\infty \mathbf{E}\left[e^{-s\tau^y}; T^y\in du\right] C_a^s(u), && y>0,
\end{aligned}
$$

where $r(c(s),s) = \frac{d}{dp}(\lambda-p)(k(p)-s)\Big|_{p=c(s)}$,

$$
C^*(B) = \frac{\frac{a\lambda}{c(s)}\left(v_a^s(B) - V_a^s(B)e^{c(s)B}\right)V_a^s(B)}{r(c(s),s) + a(\lambda-c(s))\int\limits_0^B \left(V_a^s(x) - ae^{-xc(s)}R_{s+a}(x)\right)\,dx}.
$$

The proof is quit technical and can be found in Kadankova and Veraverbeke (2005).

Chapter 3

Difference of two compound renewal processes

3.1 Introduction

The methodology developed in the previous chapter can be applied to another class of stochastic processes. This chapter deals with the difference of the compound Poisson process and the compound renewal process. Such processes have proven to be appropriate models in many applied fields of probability theory, such as telecommunication networks, cash management, computer networks, and in particular, in queueing theory. In Chapter 4 we will illustrate how our results are applied for certain queueing systems with finite buffer.

For this class of stochastic processes we determine several two-boundary characteristics. The corresponding results are obtained by means of employing similar methods as for Lévy processes and random walks. Note, that the process under consideration is not a Markov process in general. In order to employ the method from Chapter 2, we introduce an auxiliary two-component Markov process by adding to the original process a linear component (called age process).

We start with a short overview of the existing literature related to our problem.

Distributions of the one-boundary functionals for the difference of renewal processes have been studied by Lindley (1952), Prabhu (1980) and Cohen (1982). The two-sided exit problem for such processes is closely related to the G|G|1 type queueing models. A summary of known results for the G|G|1 type model can be found in Cohen (1982). The joint distributions of the one-boundary functionals for the difference of compound renewal processes have been considered by Nasirova (1984) in terms of the solutions of linear integral equations. Special cases of the difference of renewal processes have been studied by many authors: Pirdzhanov (1990), Ezhov (1993), Bratiychuk and Pirdzhanov (1991), Ezhov and Kadankov (2002). One-boundary functionals for the difference of two compound Poisson processes with drifts and with various jump distributions have been studied by Perry *et al.* (2002), Perry *et al.* (2005). Two-boundary problems for the difference of two compound Poisson processes with exponential negative jumps were solved in Kadankov and Kadankova (2006).

This chapter is structured as follows. Firstly, we introduce the process and state some auxiliary results. In Section 3 we derive the integral transforms of the one-boundary characteristics of the process. Then the two-sided exit problem is solved for the difference of the compound Poisson process and the compound renewal process. The Laplace transforms of the joint distribution of the first exit time, the value of the overshoot and the value of the age process at this instant are determined (Section 4). In Section 5 we determine the joint distribution of the infimum, supremum and the value of the process in terms of the integral transforms of the one-boundary characteristics of the process. The results obtained are applied for a particular case of this process, namely, for the difference of the compound Poisson process and the renewal process whose jumps are geometrically distributed. The advantage is that these results are in a closed form, in terms of the resolvent sequences of the process, so that we can derive the exit probabilities.

In this case the limit distributions of the one-boundary and two-boundary functionals are found when $\rho = 1$, $\sigma^2 < \infty$. The weak convergence is established to the distributions of the corresponding functionals of the Wiener process with dispersion σ^2 in Section 6. The two final sections deal with the reflected processes. We determine the distribution of the first passage time of the lower boundary, distribution of the increments of the process and asymptotic behavior of the process. We consider the reflections generated by infimum (Section 7) and by supremum (Section 8) of the process. These processes serve to study the main characteristics of queueing systems with batch arrivals and finite buffer (see Chapter 4).

We also determined the Laplace transforms of the distribution of the number of intersections of the interval and investigated the limit behavior of this functional. But this part of the study is not included in this thesis, and we refer to Kadankov *et al.* (2009) for the results.

The results of this chapter can be found in the following articles:

> Kadankov and Kadankova (2007). Two-boundary problems for semi-Markov walk with a linear drift. *Random Oper. and Stoch. Equ. (ROSE)*, 15(3), 223-251.
>
> Kadankov and Kadankova (2008c). A two-sided exit problem for a difference of a compound Poisson process and a compound renewal process with a discrete phase space. *Stochastic Models*, 24(1), 152-172.
>
> Kadankov *et al.* (2009). Intersections of an interval by a difference of a compound Poisson process and a compound renewal process. *Stochastic Models*, 25(2), 270-300.

3.2 Definitions and auxiliary results

Let $\varkappa, \delta \in \mathbb{N} = \{1, 2, \dots\}$ be independent integer random variables, and $\eta \in (0, \infty)$ be a positive random variable independent of $\varkappa, \delta$ with the distribution function $F(x) = \mathbf{P}[\eta \le x]$, $x \ge 0$. We will assume that $\mathbf{E}\varkappa$, $\mathbf{E}\delta$, $\mathbf{E}\eta < \infty$. Introduce the sequences $\{\eta, \eta'_n\}$, $\{\varkappa, \varkappa'_n\}$, $\{\delta, \delta'_n\}$, $n \in \mathbb{N}$ of independent identically distributed (inside of each sequence) variables and define the renewal sequences

$$\begin{aligned} &\eta_0(x) = 0, \quad \eta_1(x) = \eta_x, \quad \eta_{n+1}(x) = \eta_x + \eta'_1 + \dots + \eta'_n, \qquad n \in \mathbb{N}, \quad (3.2.1)\\ &\varkappa_0 = 0, \quad \varkappa_n = \varkappa'_1 + \dots + \varkappa'_n; \qquad \delta_0 = 0, \quad \delta_n = \delta'_1 + \dots + \delta'_n; \qquad n \in \mathbb{N}, \end{aligned}$$

where $\eta_x \in (0, \infty)$ is the random variable with the following distribution function

$$F_x(u) = \mathbf{P}[\eta_x \le u] = [F(x+u) - F(x)](1 - F(x))^{-1}, \qquad u \ge 0.$$

Denote by $\{\pi(t)\}_{t \ge 0} \in \mathbb{Z}^+ = \{0, 1, \dots\}$ a compound Poisson process with the generating function of the form

$$\mathbf{E}\,\theta^{\pi(t)} = e^{tk(\theta)}, \qquad k(\theta) = \mu\left(\mathbf{E}\,\theta^{\varkappa} - 1\right), \quad |\theta| \le 1,$$

where $\mu > 0$ is the intensity of the jumps and $\varkappa$ is a jump size. For all $t \ge 0$ define a delayed renewal process generated by the random sequence $\{\eta_n(x)\}_{n \in \mathbb{Z}^+}$:

$$N_x(t) = \max\{n \in \mathbb{Z}^+ : \eta_n(x) \le t\} \in \mathbb{Z}^+, \qquad x \ge 0. \qquad (3.2.2)$$

Introduce a right-continuous step process for all $x \ge 0$

$$D_x(t) = \pi(t) - \delta_{N_x(t)} \in \mathbb{Z}, \qquad t \ge 0; \quad D_x(0) = 0. \qquad (3.2.3)$$

We will call the process $\{D_x(t)\}_{t \ge 0}$ a difference of the compound Poisson process and the compound renewal process. Observe, that this process is not a Markov process in general. To be able to apply the method from the previous chapter, we require a supplementary process. Therefore, for all $t \geq 0$ introduce a right-continuous linear process

$$\eta_x^+(t) = \begin{cases} t + x, & 0 \leq t < \eta_x, \\ t - \eta_{N_x(t)}(x), & t \geq \eta_x \end{cases} \in \mathbb{R}_+ = [0, \infty), \qquad x \in \mathbb{R}_+. \quad (3.2.4)$$

The process $\{\eta_x^+(t)\}_{t\geq 0}$ increases linearly on the intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{Z}^+$, it is killed to zero at the points $\eta_n(x)$, $n \in \mathbb{N}$, and the value of the process at the instant $t_0 \geq \eta_x$ is equal to the time elapsed from the moment of the last renewal of the renewal process (3.2.2) till t_0. We will call the process (3.2.4) a linear component. Note, that this linear component is sometimes referred to as the age process (age since the last renewal). By adding this linear component to the process $\{D_x(t)\}_{t\geq 0}$, we obtain a right-continuous Markov process

$$\{X_t\}_{t\geq 0} = \left\{D_x(t),\, \eta_x^+(t)\right\}_{t\geq 0} \in \mathbb{Z} \times \mathbb{R}_+, \qquad X_0 = \{0, x\}, \quad x \in \mathbb{R}_+, \ (3.2.5)$$

which governs the process $\{D_x(t)\}_{t\geq 0}$. Note, that the process defined in (3.2.5) is homogeneous with respect to the first component (Ezhov and Skorokhod (1969)). It means that the transient probability of the process

$$P_{k,x}^t(A, y) = P\left[D_x(t) \in A, \eta_x^+(t) \leq y / \, D_x(0) = k, \eta_x^+(0) = x\right]$$

enjoys the following property for all $k \in \mathbb{Z}$, $y \geq 0$:

$$P_{k,x}^t(A, y) = P_{0,x}^t(A_{-k}, y),$$

where A is a subset of $\mathbb{Z}$, A_{-k} is the subset of integers k' such that $k' + k \in A$. This property will be constantly used when setting up the equations. If $X_{t_0} = \{k, u\}$, $k \in \mathbb{Z}$, $u \geq 0$, then the evolution of the process $\{X_t\}_{t\geq t_0}$ in the sequel does not depend on the value k of the first component. The first jump of the process $\{\pi(t)\}_{t\geq t_0}$ (which is distributed as $\varkappa$) will occur after an exponential period of time with parameter μ. The first renewal instant of the process $\{N_x(t)\}_{t\geq t_0}$ (with jump size distributed as δ) will take place after elapsing of time η_u.

To determine the two-boundary characteristics of the original process, we first derive the one-boundary characteristics of the process $\{X_t\}_{t\geq 0}$. We now define an auxiliary right-continuous step process which will be used in the sequel. Introduce a generating sequence as follows ($n \in \mathbb{N}$)

$$\xi_0(x) = 0, \quad \xi_1(x) = \pi(\eta_x) - \delta, \quad \xi_{n+1}(x) = \xi_1(x) + \sum_{i=1}^{n} \xi_i', \quad \xi_n = \xi_n(0),$$

where $\xi = \pi(\eta) - \delta \in \mathbb{Z}$, $\{\xi, \xi_n'\}$, $n \in \mathbb{N}$ is a sequence of i.i.d. random variables. Define the process

$$\{S_x(t)\}_{t\geq 0} = \left\{\xi_{N_x(t)}(x)\right\}_{t\geq 0} \in \mathbb{Z}, \qquad S_x(0) = 0, \quad x \in \mathbb{R}_+. \tag{3.2.6}$$

The sample paths of this process are the constants on the time intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{Z}^+$, and there occur jumps at the instants $\eta_n(x)$, $n \in \mathbb{N}$. These jumps have the same distribution as $\xi \doteq \pi(\eta) - \delta$, where $n \in \{2, 3, \dots\}$, and $\xi_1(x) \doteq \pi(\eta_x) - \delta$ for $n = 1$. Here and in the sequel we will call the process $\{S_x(t)\}_{t\geq 0}$ a semi-Markov random walk generated by the sequences $\{\eta_n(x)\}$, $\{\xi_n(x)\}$, $n \in \mathbb{Z}^+$. This name originated from Nasirova (1984).

For all $x \in \mathbb{R}_+$, $|\theta| \leq 1$ denote

$$\tilde{f}_x(s) = \mathbf{E}e^{-s\eta_x}, \quad \tilde{f}(s) = \tilde{f}_0(s), \quad \tilde{f}_x(s,\theta) = \tilde{f}_x(s - k(\theta)) = \mathbf{E}\left[e^{-s\eta_x}\theta^{\pi(\eta_x)}\right].$$

The following statement relates the Laplace transforms of the processes $N_x(t)$, $D_x(t)$, $\eta_x^+(t)$ and $S_x(t)$, $t \geq 0$.

Lemma 3.2.1. *Let $N_x(t)$, $D_x(t)$, $\eta_x^+(t)$, $S_x(t)$, $t \geq 0$ be the random processes defined by formulae (3.2.2)-(3.2.6), and $\nu_s \sim \exp(s)$ be an exponential random variable independent of these processes. Then for all $x \in \mathbb{R}_+$, $s > 0$, $|a| \leq 1$, $|\theta|, |b| = 1$, $p \geq 0$ the following equality holds:*

$$E_x^s(a,b,\theta,p) = \mathbf{E}a^{N_x(\nu_s)}b^{D_x(\nu_s)}\theta^{S_x(\nu_s)}e^{-p\eta_x^+(\nu_s)} = \frac{se^{-px}}{s+p-k(b)}(1 - \tilde{f}_x(s+p,b))$$
$$+ \frac{sa}{s+p-k(b)}\,\tilde{f}_x(s,\theta b)\mathbf{E}(\theta b)^{-\delta}\frac{1-\tilde{f}(s+p,b)}{1-a\tilde{f}(s,\theta)\mathbf{E}(\theta b)^{-\delta}}. \tag{3.2.7}$$

In particular, for all $x \in \mathbb{R}_+$, $s > 0$, the following formula is valid:

$$\mathbf{E}\theta^{S_x(\nu_s)} = 1 - \tilde{f}_x(s) + \tilde{f}_x(s,\theta)\mathbf{E}\theta^{-\delta}\frac{1 - \tilde{f}(s)}{1 - \tilde{f}(s,\theta)\mathbf{E}\theta^{-\delta}}, \qquad |\theta| = 1. \qquad (3.2.8)$$

Proof. It is not difficult to establish that the mathematical expectation $E_x^s(a,b,\theta,p)$ obeys the following equation for $x \in \mathbb{R}_+$, $s > 0$, $|a| \le 1$, $|\theta|, |b| = 1$, $p \ge 0$

$$E_x^s(a,b,\theta,p) = s\frac{1 - \tilde{f}_x(s+p,b)}{s+p-k(b)}e^{-px} + a\tilde{f}_x(s,\theta b)\mathbf{E}(\theta b)^{-\delta}E_0^s(a,b,\theta,p).$$

In order to write this equation, we used the total probability law, independence of the random variables δ and $\eta_1(x)$, homogeneity of the process $\{X_t\}_{t\ge0}$ with respect to the first component and the fact that the random time $\eta_1(x)$ is a Markov time. Setting $x = 0$ in this equation, we get

$$E_0^s(a,b,\theta,p) = \frac{s}{s+p-k(b)}\,\frac{1 - \tilde{f}(s+p,\theta b)}{1 - a\tilde{f}(s,\theta b)\mathbf{E}(\theta b)^{-\delta}}.$$

Substituting the expression for $E_0^s(a,b,\theta,p)$ into the previous equality, we get (3.2.7). Formula (3.2.8) follows from (3.2.7) for $p = 0$, $a = b = 1$. ▲

To determine the one-boundary characteristics of the process, we require the one-boundary functionals of the semi-Markov walk $\{S_x(t)\}_{t\ge0}$ (3.2.6). We now formally define them. For all $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$ denote by

$$\tilde{\tau}^k(x) = \inf\{t : S_x(t) > k\}, \qquad \tilde{T}^k(x) = S_x(\tilde{\tau}^k(x)) - k \in \mathbb{N},$$

$\tilde{\tau}^k{=}\tilde{\tau}^k(0)$, $\tilde{T}^k{=}\tilde{T}^k(0)$, the instant of the first overshoot of the upper level k by the process $\{S_x(t)\}_{t\ge0}$ and the value of the overshoot through this level; and by

$$\tilde{\tau}_k(x) = \inf\{t : S_x(t) < -k\}, \qquad \tilde{T}_k(x) = -S_x(\tilde{\tau}_k(x)) - k \in \mathbb{N},$$

$\tilde{\tau}_k{=}\tilde{\tau}_k(0)$, $\tilde{T}_k{=}\tilde{T}_k(0)$, the instant of the first overshoot of the lower level $-k$ by the process $\{S_x(t)\}_{t\ge0}$ and the value of the overshoot at this instant. Observe, that the random variables $\tilde{\tau}^k(x)$, $\tilde{\tau}_k(x)$ take their values from a countable set $\{\eta_n(x),\, n \in \mathbb{N}\}$, and they are Markov times of the process $\{S_x(t)\}_{t\ge0}$. We now formulate and prove some results for the one-boundary characteristics of the semi-Markov walk $\{S_x(t)\}_{t\geq 0}$ which appear to be analogous to the results for usual random walks and Lévy processes (due to Spitzer (1964), Rogozin and Pecherskii (1966), Pecherskii and Rogozin (1969)).

Lemma 3.2.2. *Let $\{S_0(t)\}_{t\geq 0} \in \mathbb{Z}$ be the semi-Markov walk (3.2.6), and*

$$S_t^+ = \sup_{u\leq t} S_0(u), \qquad S_t^- = \inf_{u\leq t} S_0(u), \qquad u, t \geq 0$$

be the running supremum and the infimum of the process $\{S_0(t)\}_{t\geq 0}$, $s > 0$. Then

(i) *the following identity (Spitzer (1964), Rogozin(1966)) is valid for the semi-Markov walk $\{S_0(t)\}_{t\geq 0}$*

$$\mathbf{E}\theta^{S_0(\nu_s)} = \frac{1 - \tilde{f}(s)}{1 - \tilde{f}(s,\theta)\,\mathbf{E}\theta^{-\delta}} = \mathbf{E}\theta^{S_{\nu_s}^+}\,\mathbf{E}\theta^{S_{\nu_s}^-}, \qquad |\theta| = 1, \qquad (3.2.9)$$

where the random variables $-S_{\nu_s}^-, S_{\nu_s}^+ \in \mathbb{Z}^+$ are infinitely divisible and their Laplace transforms are given as follows:

$$\mathbf{E}\theta^{S_{\nu_s}^\pm} = \exp\left\{\sum_{n\in\mathbb{N}} \frac{1}{n}\mathbf{E}\left[e^{-s\eta_n}\left(\theta^{\xi_n} - 1\right); \pm\xi_n > 0\right]\right\}, \quad |\theta|^{\pm 1} \leq 1; \qquad (3.2.10)$$

(ii) *the Laplace transforms of the joint distributions of $\{\tilde{\tau}^k, \tilde{T}^k\}$, $\{\tilde{\tau}_k, \tilde{T}_k\}$, $k \in \mathbb{Z}^+$ of the semi-Markov walk $\{S_0(t)\}_{t\geq 0}$ obey the following equalities (Pecherskii and Rogozin (1969)) ($|\theta| \leq 1$)*

$$\mathbf{E}\left[e^{-s\tilde{\tau}^k}\theta^{\tilde{T}^k}; \tilde{\tau}^k < \infty\right] = \left(\mathbf{E}\,\theta^{S_{\nu_s}^+}\right)^{-1}\mathbf{E}\left[\theta^{S_{\nu_s}^+ - k}; S_{\nu_s}^+ > k\right],$$
$$\mathbf{E}\left[e^{-s\tilde{\tau}_k}\theta^{\tilde{T}_k};\ \tilde{\tau}_k < \infty\right] = \left(\mathbf{E}\,\theta^{-S_{\nu_s}^-}\right)^{-1}\mathbf{E}\left[\theta^{-S_{\nu_s}^- - k};\ S_{\nu_s}^- < -k\right]; \quad (3.2.11)$$

(iii) *the integral transforms of the joint distributions of $\{S_0(\nu_s),\ S_{\nu_s}^\pm\}$ are such that for all $k \in \mathbb{Z}^+$*

$$\mathbf{E}\left[\theta^{S_0(\nu_s)};\ S_{\nu_s}^+ \leq k\right] = \mathbf{E}\theta^{S_{\nu_s}^-}\mathbf{E}\left[\theta^{S_{\nu_s}^+}; S_{\nu_s}^+ \leq k\right], \qquad |\theta| \geq 1,$$
$$\mathbf{E}\left[\theta^{S_0(\nu_s)}; S_{\nu_s}^- \geq -k\right] = \mathbf{E}\theta^{S_{\nu_s}^+}\mathbf{E}\left[\theta^{S_{\nu_s}^-}; S_{\nu_s}^- \geq -k\right], \quad |\theta| \leq 1. \quad (3.2.12)$$

The proof is based on the factorization method (Borovskikh (1979)) and it is given in Kadankov and Kadankova (2005b).

We now consider a particular case, when the random variable $\delta \in \mathbb{N}$ is geometrically distributed with parameter $\lambda \in [0,1)$:

$$\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}, \quad n \in \mathbb{N}, \qquad \mathbf{E}\theta^{-\delta} = \frac{1-\lambda}{\theta-\lambda}, \quad |\theta| \geq 1. \qquad (3.2.13)$$

This assumption means that the process $\{D_x(t)\}_{t\geq 0}$ has geometrically distributed negative jumps at time instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$. Here and in the sequel we will denote the distribution (3.2.13) as follows $\delta \sim ge(\lambda)$. In this case all characteristics of the process can be expressed in terms of a certain function which we introduce in the next section (see the defining formulae (3.3.4), (3.3.5)).

Lemma 3.2.3 (Kadankov and Kadankova (2008c)). *Let $\tilde{f}(s) = \mathbf{E}e^{-s\eta}$. Then for $s > 0$ the equation*

$$\theta - \lambda = (1-\lambda)\tilde{f}(s - k(\theta)), \qquad |\theta| < 1 \qquad (3.2.14)$$

has a unique solution $c(s)$ inside the circle $|\theta| < 1$. This solution is positive and $c(s) \in (\lambda, 1)$. If $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$, $\rho = \mu(1-\lambda)\mathbf{E}[\varkappa]\,\mathbf{E}[\eta]$, then for $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (\lambda, 1)$; and for $\rho \leq 1$, $\lim_{s\to 0} c(s) = 1$.

Corollary 3.2.1. *Let $\delta \sim ge(\lambda)$, $s > 0$, $k \in \mathbb{Z}^+$. Then*

(i) *the generating functions of $S^-_{\nu_s}$, $S^+_{\nu_s}$ are such that*

$$\mathbf{E}\theta^{S^-_{\nu_s}} = \frac{1-c(s)}{1-\lambda}\,\frac{1-\lambda/\theta}{1-c(s)/\theta}, \qquad |\theta| \geq 1,$$
$$\mathbf{E}\theta^{S^+_{\nu_s}} = \frac{1-\lambda}{1-c(s)}\,\frac{(1-\tilde{f}(s))(\theta-c(s))}{\theta-\lambda-(1-\lambda)\tilde{f}(s,\theta)}, \qquad |\theta| \leq 1; \qquad (3.2.15)$$

(i) *the generating functions of the joint distributions of $\{\tilde{\tau}_k, \tilde{T}_k\}$, $\{\tilde{\tau}^k, \tilde{T}^k\}$*

are given as follows:

$$\mathbf{E}\left[e^{-s\tilde{\tau}_k}; \tilde{T}_k = m\right] = (c(s) - \lambda)c(s)^k \lambda^{m-1} = \mathbf{E}e^{-s\tilde{\tau}_k}(1-\lambda)\lambda^{m-1},$$

$$\sum_{k\in\mathbb{Z}^+} \theta^k \mathbf{E}\left[e^{-s\tilde{\tau}^k} z^{\tilde{T}^k}\right] = \frac{1}{1-\theta/z}\left[1 - \frac{\theta - c(s)}{z - c(s)}\,\frac{(1-\lambda)\tilde{f}(s,z) + \lambda - z}{(1-\lambda)\tilde{f}(s,\theta) + \lambda - \theta}\right]. \tag{3.2.16}$$

Proof. The equalities (3.2.15) can be derived from (3.2.14) after taking into account the fact that the factorization expansion is unique. Formulae (3.2.16) follow from (3.2.11) of Lemma 3.2.2 and from (3.2.15). The first equality of (3.2.16) implies that the random variable $\tilde{T}_k$ is independent from $\tilde{\tau}_k$ for all $k \in \mathbb{Z}^+$, and it is geometrically distributed with parameter λ. ▲

3.3 One-boundary functionals of the process

In this section we will determine the one-boundary characteristics of the process $\{D_x(t)\}_{t\geq 0}$ generated by the first overshoot time of a fixed level. Let $X_0 = \{0, x\}$, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. Define

$$\tau_k(x) = \inf\{t : D_x(t) < -k\}, \qquad T_k(x) = -D_x(\tau_k(x)) - k, \qquad \inf\{\emptyset\} = \infty$$

i.e. the first overshoot time of the negative level $-k$ by the process $\{D_x(t)\}_{t\geq 0}$ and the value of the overshoot at this instant. We will use the convention that on the event $\{\tau_k(x) = \infty\}$ $T_k(x) = \infty$. Denote $\mathfrak{B}_k(x) = \{\tau_k(x) < \infty\}$,

$$f_k(x, m, s) = \mathbf{E}\left[e^{-s\tau_k(x)}; T_k(x) = m, \mathfrak{B}_k(x)\right], \quad m \in \mathbb{N}.$$

Lemma 3.3.1. *The generating function* $\tilde{f}_k(x, \theta, s) \overset{def}{=} \sum_{m\in\mathbb{N}} \theta^m f_k(x, m, s)$ *is such that*

$$\begin{aligned}
\tilde{f}_k(0, \theta, s) &= \left(\mathbf{E}\,\theta^{-S^-_{\nu_s}}\right)^{-1} \mathbf{E}\left[\theta^{-S^-_{\nu_s}-k};\, S^-_{\nu_s} < -k\right],\\
\tilde{f}_k(x, \theta, s) &= \mathbf{E}\left[e^{-s\eta_x}\theta^{-\xi_1(x)-k}; \xi_1(x) < -k\right] +\\
&\quad + \sum_{i\in\mathbb{Z}^+} \mathbf{E}\left[e^{-s\eta_x}; \xi_1(x) = i - k\right] \tilde{f}_i(0, \theta, s),
\end{aligned} \tag{3.3.1}$$

where $\xi_1(x) \doteq \pi(\eta_x) - \delta$ *and* $S^-_{\nu_s} = \inf_{t\leq\nu_s} S_0(t)$ *is the running infimum of the semi-Markov walk* $\{S_0(t)\}_{t\geq 0}$.

Proof. Observe, that the processes $D_x(t)$, $S_x(t)$, $t \geq 0$ do not decrease on the intervals $[\eta_n(x), \eta_{n+1}(x))$. It follows from the definitions of these processes (3.2.3), (3.2.6) that the negative jumps of $D_x(t)$, $S_x(t)$, $t \geq 0$ can only occur at instants $\{\eta_n(x),\, n \in \mathbb{N}\}$. It also follows from (3.2.3), (3.2.6) that

$$D_x(\eta_n(x)) = S_x(\eta_n(x)) = \pi(\eta_n(x)) - \delta_n, \qquad n \in \mathbb{Z}^+.$$

Thus, the first overshoot time $\tilde{\tau}_k(x)$ of the negative level $-k$ and the value of the overshoot $\tilde{T}_k(x)$ through this level by the semi-Markov walk $\{S_x(t)\}_{t\geq 0}$ coincide in distribution with the first overshoot time $\tau_k(x)$ and the value of the overshoot $T_k(x)$ by the process $\{D_x(t)\}_{t\geq 0}$. The first equality of (3.3.1) follows straightforwardly from the second formula of (3.2.11). In order to derive the second formula of (3.3.1), we used the total probability formula, the Markov property of $\eta_1(x) \doteq \eta_x$ and homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component. ▲

Corollary 3.3.1. *Let $\delta \sim ge(\lambda)$, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$, $m \in \mathbb{N}$, $s > 0$. Then*

(i) *the Laplace transform of the joint distribution of $\{\tau_k(x), T_k(x)\}$ satisfies the following equality:*

$$f_k(x, m, s) = \tilde{f}_x(s - k(c(s)))\, c(s)^k\, (1-\lambda)\lambda^{m-1}, \tag{3.3.2}$$

where $c(s) \in (\lambda, 1)$ is the unique solution of the equation (3.2.14) inside the circle $|\theta| < 1$, $\tilde{f}_x(s) = \mathbf{E}\, e^{-s\eta_x}$, $\tilde{f}(s) = \mathbf{E}\, e^{-s\eta} = \tilde{f}_0(s)$;

(ii) *if $\rho > 1$, then $\mathbf{P}[\tau_k(x) < \infty] = \tilde{f}_x(-k(c))\, c^k < 1$, and $\tau_k(x)$ is a defective random variable for all $k \in \mathbb{Z}^+$, $x \geq 0$; if $\rho \leq 1$, then $\mathbf{P}[\tau_k(x) < \infty] = 1$, and $\tau_k(x)$ is a proper variable for all $k \in \mathbb{Z}^+$, $x \geq 0$.*

Proof. It is not difficult to assure that

$$\mathbf{P}[\pi(t) - \delta = i] = (1-\lambda)\mathbf{E}\left[\lambda^{\pi(t)-(i+1)}; \pi(t) > i\right], \qquad i \in \mathbb{Z}.$$

Substituting the expression for $\tilde{f}_i(0, m, s)$ from (3.2.16) into the second formula of (3.3.1), after some calculations we obtain (3.3.2). ▲

A more difficult problem is determining the generating function of the joint distribution of the upper-boundary functionals of the process $\{D_x(t)\}_{t\geq 0}$. Let $X_0 = \{0, x\}$, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. Denote by

$$\tau^k(x) = \inf\{t : D_x(t) > k\}, \qquad \inf\{\emptyset\} = \infty$$

the instant of the first overshoot of the upper level k by the process $\{D_x(t)\}_{t\geq 0}$. On the event $\mathfrak{B}^k(x) = \{\tau^k(x) < \infty\}$ define

$$l_x^k = \eta_x^+(\tau^k(x)), \qquad T^k(x) = D_x(\tau^k(x)) - k$$

the value of the linear component and the value of the overshoot at the instant of the first overshoot of the upper level. On the event $\{\tau^k(x) = \infty\}$ we set per definition $l_x^k = T^k(x) = \infty$. Introduce the following notation for the mathematical expectations $s > 0$, $k \in \mathbb{Z}^+$, $m \in \mathbb{N}$, $x \in \mathbb{R}_+$

$$f^k(x, dl, m, s) = \mathbf{E}\left[e^{-s\tau^k(x)}; l_x^k \in dl, T^k(x) = m, \mathfrak{B}^k(x)\right],$$

and for the generating functions ($|\theta| < 1$)

$$\Phi_\theta^s(x, dl, m) = \sum_{k\in\mathbb{Z}^+} \theta^k f^k(x, dl, m, s), \quad \Phi_\theta^s(x) = \int_0^\infty \sum_{m\in\mathbb{N}} \Phi_\theta^s(x, dl, m).$$

Let $k \in \mathbb{Z}^+$ and

$$\hat{\tau}^k = \inf\{t : \pi(t) > k\}, \quad \hat{T}^k = \pi(\hat{\tau}^k) - k$$

be the first crossing time through the upper level k by the compound Poisson process $\{\pi(t)\}_{t\geq 0}$ and the value of the overshoot at this instant. Denote by

$$\rho_k(t) = \mathbf{P}[\pi(t) = k], \quad \sum_{k\in\mathbb{Z}^+} \theta^k \rho_k(t) = \mathbf{E}\,\theta^{\pi(t)} = e^{tk(\theta)}, \quad |\theta| \leq 1,$$

$$p_k^m(dt) = \mathbf{P}[\hat{\tau}^k \in dt, \hat{T}^k = m] = \mu \sum_{i=0}^{k} \rho_i(t)\mathbf{P}[\varkappa = k - i + m]\,dt, \quad m \in \mathbb{N}.$$

For the Laurent series $L(\theta) = \sum\limits_{k=-\infty}^{\infty} a_k\theta^k$, $|\theta| = 1$ such that $\sum\limits_{k=-\infty}^{\infty} |a_k| < \infty$ we now introduce the projectors (Kemperman (1963)) in the following way:

$$\mathfrak{P}_\theta^+[L(\theta)] = \sum_{k=0}^{\infty} a_k\theta^k, \quad |\theta| \leq 1, \qquad \mathfrak{P}_\theta^-[L(\theta)] = \sum_{-\infty}^{-1} a_k\theta^k, \quad |\theta| \geq 1.$$

Theorem 3.3.1. *Let $\{D_x(t)\}_{t\geq 0}$ be the difference of the compound Poisson process and the compound renewal process (3.2.3), $S_{\nu_s}^+ = \sup\limits_{t\leq\nu_s} S_0(t)$, $S_{\nu_s}^- = \inf\limits_{t\leq\nu_s} S_0(t)$ be the supremum and infimum (3.2.10) of the semi-Markov walk $\{S_0(t)\}_{t\geq 0}$ on the time interval $[0, \nu_s]$. Then the generating function of the Laplace transform of the joint distribution of $\{\tau^k(x), l_x^k, T^k(x)\}$ satisfies the following formula on the event $\mathfrak{B}^k(x)$*

$$\Phi_\theta^s(x, dl, m) = e^{-s(l-x)}\frac{1 - F(l)}{1 - F(x)}\,\mathbf{I}_{\{l>x\}}\Pi_\theta^m(d(l - x)) +$$
$$+ e^{-sl}[1 - F(l)]\,\frac{\tilde{f}_x(s, \theta)}{1 - \tilde{f}(s)}\,\mathbf{E}\theta^{S_{\nu_s}^+}\mathfrak{P}_\theta^+\left[\mathbf{E}\theta^{-\delta+S_{\nu_s}^-}\Pi_\theta^m(dl)\right], \quad (3.3.3)$$

where $\Pi_\theta^m(dl) = \sum_{k\in\mathbb{Z}^+} \theta^k p_k^m(dl) = \mu e^{lk(\theta)}\mathbf{E}\left[\theta^{\varkappa-m};\varkappa \geq m\right]dl$; *and in particular,*

$$\Phi_\theta^s(x) = \frac{1-\tilde{\Pi}_\theta^s}{1-\theta} + \frac{\tilde{f}_x(s,\theta)}{1-\tilde{f}(s)}\,\mathbf{E}\theta^{S_{\nu_s}^+}\mathfrak{P}_\theta^+\left[\left(\mathbf{E}\theta^{-\delta}-1\right)\mathbf{E}\theta^{S_{\nu_s}^-}\frac{1-\tilde{\Pi}_\theta^s}{1-\theta}\right],$$

where $\tilde{\Pi}_\theta^s = \mathbf{E}\theta^{\pi(\nu_s)} = s/(s-k(\theta))$.

Proof of the theorem can be found in Kadankov *et al.* (2009).

We now introduce a sequence which will play a crucial role in the sequel. The idea to employ this sequence for semi-continuous random walks and semi-continuous Lévy processes was due to Takács (1967). Since the function

$$\tilde{f}_x(s-k(\theta)) = \mathbf{E}\left[e^{-s\eta_x}\theta^{\pi(\eta_x)}\right] = \sum_{i\in\mathbb{Z}^+}\theta^i\int_0^\infty e^{-st}\mathbf{P}[\eta_x\in dt,\,\pi(t)=i],\quad |\theta|\leq 1,$$

is analytic inside the unit circle for all $s, x \geq 0$, we have that the function

$$\mathbb{Q}_\theta^s(x) = \frac{(1-\lambda)\tilde{f}_x(s-k(\theta))}{\left((1-\lambda)\tilde{f}(s-k(\theta))+\lambda-\theta\right)},\quad s,x\geq 0 \tag{3.3.4}$$

is analytic on the open set $|\theta| < c(s)$. In this region it can be represented as a power series

$$\mathbb{Q}_\theta^s(x) = \sum_{k\in\mathbb{Z}^+}\theta^k Q_k^s(x),\quad s,x\geq 0.$$

The coefficients of this expansion can be calculated by means of the inversion formula:

$$Q_k^s(x) = \frac{1}{2\pi i}\oint_{|\theta|=\alpha}\frac{1}{\theta^{k+1}}\,\frac{(1-\lambda)\tilde{f}_x(s-k(\theta))}{(1-\lambda)\tilde{f}(s-k(\theta))+\lambda-\theta}\,d\theta,\quad \alpha\in(0,c(s)). \tag{3.3.5}$$

We will call the sequence $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$, defined by the formula (3.3.5), the resolvent sequence of the process $\{D_x(t)\}_{t\geq 0}$. The following result expresses the one-boundary characteristics of the process in terms of the resolvent sequence.

Corollary 3.3.2 (Kadankov and Kadankova (2008c))**.** *Let* $\delta \sim ge(\lambda)$, $\{D_x(t)\}_{t\geq 0}$ *be the difference of the compound Poisson process and the compound renewal process (3.2.3) whose jumps are geometrically distributed,* $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. *Then*

(i) *the Laplace transforms of the joint distribution of* $\{\tau^k(x), l_x^k, T^k(x)\}$ *satisfy the following equality:*

$$f^k(x, dl, m, s) = e^{-s(l-x)} \frac{1-F(l)}{1-F(x)} \mathbf{I}\{l > x\} p_k^m(d(l-x))$$

$$+ \Phi_\lambda^s(0, dl, m)\, Q_k^s(x) - e^{-sl}\, [1-F(l)] \sum_{i=0}^{k} Q_i^s(x)\, p_{k-i}^m(dl), \qquad (3.3.6)$$

where $\Phi_\lambda^s(0, dl, m) = e^{-sl}[1-F(l)] \sum_{k\in\mathbb{Z}^+} c(s)^k p_k^m(dl)$;

(ii) *for the Laplace transform of the first crossing time through the upper level* k *by the process* $\{D_x(t)\}_{t\geq 0}$ *for all* $k \in \mathbb{Z}^+$, $s, x \in \mathbb{R}_+$ *the following formula holds:*

$$\mathbf{E}e^{-s\tau^k(x)} = 1 - \frac{s}{s-k(c(s))} \frac{Q_k^s(x)}{1-\lambda} + \sum_{i=0}^{k} \tilde{\rho}_i(s) \left[\frac{Q_{k-i}^s(x)}{1-\lambda} - 1\right], \quad (3.3.7)$$

where $\tilde{\rho}_k(s) = s\int_0^\infty e^{-st} \rho_k(t)\, dt$;

(iii) *for* $\mathbf{E}[\varkappa]$, $\mathbf{E}[\eta] < \infty$ *and* $\rho < 1$, $\tau^k(x)$ *is a defective random variable and*

$$\mathbf{P}[\tau^k(x) < \infty] = 1 - (1-\rho)(1-\lambda)^{-1} Q_k(x) < 1,$$

where $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$ *is the resolvent sequence of the process* $\{D_x(t)\}_{t\geq 0}$, *given by (3.3.9) for* $s = 0$:

$$Q_k(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{d\theta}{\theta^{k+1}} \frac{(1-\lambda)\tilde{f}_x(-k(\theta))}{(1-\lambda)\tilde{f}(-k(\theta)) + \lambda - \theta}, \quad \alpha \in (0, c(0)); \quad (3.3.8)$$

if $\rho \geq 1$, *then for all* $k \in \mathbb{Z}^+$, $x \in \mathbb{R}_+$ $\tau^k(x)$ *is a proper random variable.*

Proof. In case of $\delta \sim ge(\lambda)$ formulae (3.2.13), (3.2.15) imply that

$$\mathfrak{P}_\theta^+ \left[\mathbf{E}\theta^{-\delta + S_{\nu_s}^-} \Pi_\theta^m(dl)\right] = \mathfrak{P}_\theta^+ \left[\frac{1-c(s)}{\theta - c(s)} \Pi_\theta^m(dl)\right] = \frac{1-c(s)}{\theta - c(s)} \left(\Pi_\theta^m(dl) - \Pi_{c(s)}^m(dl)\right).$$

Substituting this projector and $\mathbf{E}\theta^{S_{\nu_s}^+}$ (3.2.15) into (3.3.3) yields

$$\Phi_\theta^s(x, dl, m) = e^{-s(l-x)} \frac{1-F(l)}{1-F(x)} \mathbf{I}_{\{l\geq x\}} \Pi_\theta^m(d(l-x)) -$$

$$- e^{-sl}[1-F(l)] \frac{(1-\lambda)\tilde{f}_x(s,\theta)}{(1-\lambda)\tilde{f}(s,\theta) + \lambda - \theta} \left(\Pi_\theta^m(dl) - \Pi_{c(s)}^m(dl)\right). \qquad (3.3.9)$$

Employing the definition of the resolvent sequence (3.3.5) and comparing the coefficients of θ^k, $k \in \mathbb{Z}^+$ in both sides of this equality, we get formula (3.3.6) of Corollary 3.3.2. Analogously we calculate

$$\mathfrak{P}_\theta^+ \left[\left(\mathbf{E}\theta^\delta - 1 \right) \mathbf{E}\theta^{S^-_{\nu_s}} \frac{1 - \mathbf{E}\theta^{\pi(\nu_s)}}{1-\theta} \right] = \frac{1-c(s)}{1-\lambda} \frac{\mathbf{E}\left[c(s)^{\pi(\nu_s)} - \theta^{\pi(\nu_s)} \right]}{\theta - c(s)},$$

$$\Phi_\theta^s(x) = \frac{1 - \mathbf{E}\theta^{\pi(\nu_s)}}{1-\theta} - \frac{\tilde{f}_x(s,\theta)}{(1-\lambda)\tilde{f}(s,\theta) + \lambda - \theta} \mathbf{E}\left[c(s)^{\pi(\nu_s)} - \theta^{\pi(\nu_s)} \right]. \quad (3.3.10)$$

Taking into account the definition of the resolvent sequence (3.3.5), we compare the coefficients of θ^k, $k \in \mathbb{Z}^+$ in both sides of this equality, which yields the formula (3.3.7). Note that the formulae of the corollary were obtained by other methods in Kadankov and Kadankova (2008c). ▲

Remark 3.3.1. *Along with expression (3.3.8) there exists another way to calculate $Q_k(x)$, which is more applicable from practical point of view. We will now derive the recurrent formula for $Q_k(x)$. It follows from (3.3.4) for $s, \theta = 0$ that*

$$Q_0(x) = (1-\lambda)(\lambda + (1-\lambda)f_0)^{-1} f_0(x),$$

where for all $k \in \mathbb{Z}^+$

$$f_k(x) = \mathbf{P}\left[\pi(\eta_x) = k \right] = \int_0^\infty \mathbf{P}[\eta_x \in dt,\, \pi(t) = k], \quad f_k = f_k(0).$$

Again, it follows from (3.3.4) for $s = 0$ that

$$(1-\lambda)\tilde{f}_x(-k(\theta)) = (1-\lambda)\tilde{f}(-k(\theta))\mathbb{Q}_\theta(x) + (\lambda - \theta)\mathbb{Q}_\theta(x).$$

Comparing the coefficients of θ^k, $k \in \mathbb{N}$ in both sides implies that

$$(1-\lambda)f_k(x) = (1-\lambda)\sum_{i=0}^{k} Q_i(x) f_{k-i} + \lambda Q_k(x) - Q_{k-1}(x).$$

Combining like terms yields

$$\left(\lambda + (1-\lambda)f_0\right) Q_k(x) = (1-\lambda)f_k(x) + Q_{k-1}(x) - (1-\lambda)\sum_{i=0}^{k-1} Q_i(x) f_{k-i}.$$

The latter formula is a recurrent relation which allows to calculate successively the terms $Q_k(x)$ given the previous terms $Q_0(x), \dots, Q_{k-1}(x)$. For instance, given the expression for $Q_0(x)$ one finds that

$$Q_1(x) = \frac{1-\lambda}{\lambda + (1-\lambda) f_0} \left[f_1(x) + \frac{1-(1-\lambda)f_0}{\lambda + (1-\lambda) f_0} \, f_0(x) \right].$$

We state another lemma which is essential in our derivations. For all $x \geq 0$, $k \in \mathbb{Z}^+$ denote by

$$i_x^k = \inf\{t > \tau^k(x) : \, D_x(t) < k - B\}, \qquad I_x^k = k - B - D_x(i_x^k) \in \mathbb{N}$$

the first time of the downward intersection of the interval $[k-B, k]$ by the process $\{D_x(t)\}_{t \geq 0}$ after the first passage time of the upper boundary k, and the value of the overshoot through the lower boundary $k - B$ at this instant. As mentioned before, we use the convention that $\inf\{\emptyset\} \stackrel{\text{def}}{=} \infty$ and $I_x^k = \infty$ on the event $\{i_x^k = \infty\}$. It is obvious that $\eta_x^+(i_x^k) = 0$.

Lemma 3.3.2. *Let $\delta \sim ge(\lambda)$, $\{D_x(t)\}_{t \geq 0}$ be the difference of the compound Poisson process and the compound renewal process (3.2.3), $\{Q_k^s(x)\}_{k \in \mathbb{Z}^+}$, $\{Q_k(x)\}_{k \in \mathbb{Z}^+}$, $x \geq 0$ be the resolvent sequences of the process given by (3.3.5), (3.3.8). Then*

(i) *the Laplace transforms of the joint distribution of $\{i_x^k, I_x^k\}$*

$$\varphi_x^k(m,s) = \mathbf{E}\left[e^{-s i_x^k};\, I_x^k = m, i_x^k < \infty\right]$$

satisfy the following equality for all $k \in \mathbb{Z}^+$, $x \geq 0$

$$\varphi_x^k(m,s) = \left[c_x(s) c(s)^{B-k} - c(s)^{B+1} r(s) \frac{Q_k^s(x)}{1-\lambda}\right] (1-\lambda)\lambda^{m-1}, \quad (3.3.11)$$

where $c_x(s) = \tilde{f}_x(s - k(c(s)))$, $r(s) = 1 + (1-\lambda) k'(c(s)) \tilde{f}'(s - k(c(s)))$;

(ii) *if $\rho < 1$, then i_x^k is a defective random variable, and*

$$\mathbf{P}\left[i_x^k < \infty\right] = 1 - (1-\rho)(1-\lambda)^{-1} Q_k(x);$$

if $\rho > 1$, then i_x^k is a defective random variable and

$$\mathbf{P}\left[i_x^k < \infty\right] = \tilde{f}_x(-k(c)) c^{B-k} - c^{B+1} \left[(1-\lambda)^{-1} + k'(c) \tilde{f}'(-k(c)\right] Q_k(x);$$

if $\rho = 1$, then i_x^k a proper random variable.

Proof. Observe, that the following equality holds:

$$\varphi_x^k(m,s) = \sum_{i\in\mathbb{N}} \int_0^\infty f^k(x,dl,i,s) f_{i+B}(l,m,s).$$

To obtain this formula, it is necessary to use the total probability law, space homogeneity of the process $\{D_x(t)\}_{t\geq 0}$ and the Markov property of the stopping time $\tau^k(x)$. Employing formula (3.3.2), it follows from the latter equality that

$$\begin{aligned} \tilde{\varphi}_x^\theta(m,s) &= \sum_{k\in\mathbb{Z}^+} \theta^k \varphi_x^k(m,s) \\ &= \int_0^\infty \tilde{\Phi}_\theta^s(x,dl,c(s)) \tilde{f}_l(s-k(c(s))) c(s)^B (1-\lambda)\lambda^{m-1}, \qquad (3.3.12) \end{aligned}$$

where $|\theta|<1$,

$$\tilde{\Phi}_\theta^s(x,dl,z) = \sum_{m\in\mathbb{N}} z^m \Phi_\theta^s(x,dl,m), \qquad |z|\leq 1$$

is the generating function of the Laplace transform $f^k(x,dl,m,s)$ of the joint distribution of $\{\tau^k(x),\eta^k(x),T^k(x)\}$. From (3.3.6) we have

$$\begin{aligned} \tilde{\Phi}_\theta^s(x,dl,z) &= e^{-s(l-x)} \frac{1-F(l)}{1-F(x)} \mathbf{I}\{l>x\} \tilde{\Pi}_\theta^z(d(l-x)) \\ &\quad + e^{-sl}[1-F(l)]\mathbb{Q}_\theta^s(x) \left(\tilde{\Pi}_{c(s)}^z(dl) - \tilde{\Pi}_\theta^z(dl)\right), \quad |\theta|<1, \quad (3.3.13) \end{aligned}$$

where

$$\tilde{\Pi}_\theta^z(dl) = \sum_{m\in\mathbb{N}} z^m \Pi_\theta^m(dl) = e^{lk(\theta)} \frac{k(z)-k(\theta)}{1-\theta/z}\, dl, \qquad |z|\leq 1.$$

Substituting the expression for the function $\tilde{\Phi}_\theta^s(x,dl,z)$ for $z=c(s)$ into (3.3.12), and integrating it with respect to $l\geq 0$, we find

$$\tilde{\varphi}_x^\theta(m,s) = \left[\frac{\tilde{f}_x(s-k(c(s)))}{1-\theta/c(s)}\, c(s)^B - c(s)^{B+1}\, r(s)(1-\lambda)^{-1}\mathbb{Q}_\theta^s(x)\right](1-\lambda)\lambda^{m-1},$$

where $|\theta|<1$, $r(s) = 1+(1-\lambda)k'(c(s))\tilde{f}'(s-k(c(s)))$. Comparing the coefficients of θ^k, $k\in\mathbb{Z}^+$ in both sides of this equality, we get for all $s,x\geq 0$

$$\varphi_x^k(m,s) = [c_x(s)c(s)^{B-k} - c(s)^{B+1} r(s)(1-\lambda)^{-1} Q_k^s(x)](1-\lambda)\lambda^{m-1},$$

which is (3.3.11). Letting $s \to 0$ and taking into account Lemma 3.3.1, we obtain the statements of the second part of Lemma 3.3.2. ▲

Denote by $D_t^+(x) = \sup_{u \le t} D_x(u)$, $t \ge 0$ the running supremum of the process. Denote by

$$\tilde{U}^{\theta}_{s,z}(x,p) = \sum_{k \in \mathbb{Z}^+} \theta^k \mathbf{E}\left[z^{D_x(\nu_s)} e^{-p\eta_x^+(\nu_s)}; D^+_{\nu_s}(x) \le k \right], \quad |\theta| < 1,\ |z| \ge 1$$

the generating function of the joint distribution of the value of the process, its linear component and its running supremum.

Theorem 3.3.2. *Let* $\{D_x(t)\}_{t \ge 0}$ *be the difference of the compound Poisson process and the compound renewal process (3.2.3),* $S^+_{\nu_s} = \sup_{t \le \nu_s} S_0(t)$, $S^-_{\nu_s} = \inf_{t \le \nu_s} S_0(t)$ *be the supremum and infimum (3.2.10) of the semi-Markov walk,* $\{S_0(t)\}_{t \ge 0}$ *on the time interval* $[0, \nu_s]$, $\nu_s \sim \exp(s)$ *be an exponential random variable independent of these processes. Then the generating function of the Laplace transform of the joint distribution of* $\{D_x(\nu_s), \eta_x^+(\nu_s), D^+_{\nu_s}(x)\}$ *satisfies the following formula:*

$$\tilde{U}^{\theta}_{s,z}(x,p) = \frac{F^s_{z\theta}(x,p)}{1-\theta} + \frac{\tilde{f}_x(s, z\theta)}{1 - \tilde{f}(s)} \mathbf{E}(z\theta)^{S^+_{\nu_s}} \mathfrak{P}^+_{\theta} \left[\frac{F^s_{z\theta}(0,p)}{1-\theta} \mathbf{E}(z\theta)^{S^-_{\nu_s} - \delta} \right], \tag{3.3.14}$$

where

$$F^s_{\theta}(x,p) = s\frac{1 - \tilde{f}_x(s + p - k(\theta))}{s + p - k(\theta)} e^{-xp}.$$

In particular, for $p = 0$ *the generating function* $\tilde{U}^{\theta}_{s,z}(x) = \tilde{U}^{\theta}_{s,z}(x,0)$ *of the joint distribution of* $\{D_x(\nu_s), D^+_{\nu_s}(x)\}$ *is such that*

$$\tilde{U}^{\theta}_{s,z}(x) = \frac{\tilde{\Pi}^s_{z\theta}}{1-\theta} + \frac{\tilde{f}_x(s, z\theta)}{1 - \tilde{f}(s)} \mathbf{E}(z\theta)^{S^+_{\nu_s}} \mathfrak{P}^+_{\theta} \left[\frac{\tilde{\Pi}^s_{z\theta}}{1-\theta} \left(\mathbf{E}(z\theta)^{-\delta} - 1 \right) \mathbf{E}(z\theta)^{S^-_{\nu_s}} \right], \tag{3.3.15}$$

where $\tilde{\Pi}^s_{\theta} = \mathbf{E}\theta^{\pi(\nu_s)} = s/(s - k(\theta))$.

Corollary 3.3.3. *Let* $\delta \sim ge(\lambda)$, $\{D_x(t)\}_{t \ge 0}$ *be the difference of the compound Poisson process and the compound renewal process (3.2.3) whose jumps are geometrically distributed,* $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. *Then*

(i) *the generating function of the joint distribution of* $\{D_x(\nu_s), D^+_{\nu_s}(x)\}$ *is such that*

$$\tilde{U}^{\theta}_{s,z}(x) = \frac{\tilde{\Pi}^s_{z\theta}}{1-\theta} - \frac{\tilde{\Pi}^s_{z\theta}}{1-\lambda}\frac{1-z\theta}{1-\theta}\mathbb{Q}^s_{z\theta}(x) + \frac{\tilde{\Pi}^s_{c(s)}}{1-\lambda}\frac{1-c(s)}{1-c(s)/z}\mathbb{Q}^s_{z\theta}(x), \tag{3.3.16}$$

where $\mathbb{Q}^s_{\theta}(x)$ *is the generating function of the resolvent sequence (3.3.4);*

(ii) *the generating function*

$$U^k_{s,z}(x) = \sum_{i\le k} z^k \mathbf{P}\left[D_x(\nu_s) = i, D^+_{\nu_s}(x) \le k\right], \quad |z| \ge 1$$

of the joint distribution $\{D_x(\nu_s), D^+_{\nu_s}(x)\}$ *satisfies the following equality for* $k \in \mathbb{Z}^+$

$$U^k_{s,z}(x) = \sum_{i=0}^{k} z^i [A^s_{i-1}(x) - A^s_i(x)] + \frac{\tilde{\Pi}^s_{c(s)}}{1-\lambda}\frac{1-c(s)}{1-c(s)/z} z^k Q^s_k(x), \tag{3.3.17}$$

where $A^s_k(x) = 0$ *for* $k < 0$ *and*

$$A^s_k(x) = \sum_{i=0}^{k} \tilde{\rho}_i(s)\left[\frac{Q^s_{k-i}(x)}{1-\lambda} - 1\right], \quad k \in \mathbb{Z}^+;$$

(iii) *the distribution* $u^k_i(x,s) = \mathbf{P}\left[D_x(\nu_s) = i, D^+_{\nu_s}(x) \le k\right]$ *is such that for* $k \in \mathbb{Z}^+$, $i \le k$

$$u^k_i(x,s) = A^s_{i-1}(x) - A^s_i(x) + \frac{1-c(s)}{1-\lambda} c(s)^{k-i} Q^s_k(x)\tilde{\Pi}^s_{c(s)}. \tag{3.3.18}$$

Proof The function $U^k_{s,z}(x,p)$ obeys the following equation for all $x \ge 0$, $k \in \mathbb{Z}^+$, $|z| \ge 1$, $p \ge 0$

$$U^k_{s,z}(x,p) = s\int_0^{\infty} e^{-st}e^{-p(x+t)}\mathbf{P}\left[\eta_x > t\right]\sum_{i=0}^{k} z^i \rho_i(t)\,dt +$$
$$+ \sum_{i=0}^{k}\mathbf{E}\left[e^{-s\eta_x}; \pi(\eta_x) = i\right] z^i \sum_{j\in\mathbb{N}} \mathbf{P}[\delta = j] z^{-j} U^{k-i+j}_{s,z}(0,p),$$

where $\rho_i(t) = \mathbf{P}[\pi(t) = i]$. Multiplying the latter equation by θ^k, $|\theta| < 1$ and then summing over $k \in \mathbb{Z}^+$, we get an equation

$$\tilde{U}^{\theta}_{s,z}(x,p) = \frac{F^s_{z\theta}(x,p)}{1-\theta} + \tilde{f}_x(s - k(z\theta))\mathfrak{P}^+_\theta \left[\tilde{U}^{\theta}_{s,z}(0,p)\mathbf{E}(z\theta)^{-\delta}\right]. \tag{3.3.19}$$

To solve this equation, we will proceed as follows. Setting in (3.3.19) $x = 0$ and multiplying by $\mathbf{E}(z\theta)^{-\delta}$, we get for the auxiliary function $\mathbf{I}^+_\theta = \mathfrak{P}^+_\theta \left[\tilde{U}^{\theta}_{s,z}(0,p)\mathbf{E}(z\theta)^{-\delta}\right]$ the following equation:

$$\mathbf{I}^+_\theta \left(1 - \tilde{f}(s, z\theta)\mathbf{E}(z\theta)^{-\delta}\right) = \frac{F^s_{z\theta}(x,p)}{1-\theta}\mathbf{E}(z\theta)^{-\delta} - \mathbf{I}^-_\theta, \quad |\theta| = 1,$$

where $\mathbf{I}^-_\theta = \mathfrak{P}^-_\theta \left[\tilde{U}^{\theta}_{s,z}(0,p)\mathbf{E}(z\theta)^{-\delta}\right]$. Employing the factorization identity (3.2.9), we rewrite this equation in the following form:

$$\mathbf{I}^+_\theta \left(\mathbf{E}(z\theta)^{S^+_{\nu_s}}\right)^{-1} - \frac{1}{1-\tilde{f}(s)}\mathfrak{P}^+_\theta \left[\frac{F^s_{z\theta}(0,p)}{1-\theta}\mathbf{E}(z\theta)^{S^-_{\nu_s}-\delta}\right] =$$
$$= \frac{1}{1-\tilde{f}(s)}\mathfrak{P}^-_\theta \left[\frac{F^s_{z\theta}(0,p)}{1-\theta}\mathbf{E}(z\theta)^{S^-_{\nu_s}-\delta}\right] - \frac{\mathbf{I}^-_\theta}{1-\tilde{f}(s)}\mathbf{E}(z\theta)^{S^-_{\nu_s}}, \quad |\theta| = 1.$$

In view of the factorization reasoning we find that

$$\mathbf{I}^+_\theta = \frac{1}{1-\tilde{f}(s)}\mathbf{E}(z\theta)^{S^+_{\nu_s}}\mathfrak{P}^+_\theta \left[\frac{F^s_{z\theta}(0,p)}{1-\theta}\mathbf{E}(z\theta)^{S^-_{\nu_s}-\delta}\right].$$

Substituting the expression for the projector $\mathbf{I}^+_\theta$ into (3.3.19), we obtain (3.3.14) of Theorem 3.3.2. The equality (3.3.15) follows from the previous one.

To verify the statements of the corollary, we have to calculate the projector which enters the right-hand side of (3.3.15). To do this, we need the following lemmas.

Lemma 3.3.3. *Let $a \in \mathbb{C}$, $|a| \le 1$, $A(\theta)$ be an analytic function for $|\theta| < 1$ and continuous on $|\theta| = 1$. Then the following decomposition is valid for $|\theta| \in [a, 1]$, $\theta \ne a$*

$$\frac{A(\theta)}{1 - a/\theta} = \frac{aA(a)}{\theta - a} + \frac{\theta A(\theta) - aA(a)}{\theta - a}$$

where $[\theta A(\theta) - aA(a)](\theta - a)^{-1}$ *is a bounded analytic function for* $|\theta| \leq 1$, $aA(a)(\theta - a)^{-1}$ *is a bounded analytic function for* $|\theta| > a$. *Moreover, the following equalities hold:*

$$\mathfrak{P}_\theta^+ \left[\frac{A(\theta)}{1 - a/\theta} \right] = \frac{\theta A(\theta) - aA(a)}{\theta - a}, \qquad \mathfrak{P}_\theta^- \left[\frac{A(\theta)}{1 - a/\theta} \right] = \frac{aA(a)}{\theta - a}. \tag{3.3.20}$$

Lemma 3.3.4. *Let* $a \in \mathbb{C}$, $|a| \leq 1$, $A(\theta)$ *be an analytic function for* $|\theta| > 1$ *and continuous for* $|\theta| = 1$. *Then the following decomposition is valid for* $|\theta| \in [1, 1/a]$, $\theta \neq 1/a$

$$\frac{A(\theta)}{1 - a\theta} = \frac{A(1/a)}{1 - a\theta} + \frac{A(\theta) - A(1/a)}{1 - a\theta}$$

where $A(1/a)(1 - a\theta)^{-1}$ *is a bounded analytic function for* $|\theta| < 1/a$, $[A(\theta) - A(1/a)](1 - a\theta)^{-1}$ *is a bounded analytic function for* $|\theta| \geq 1$. *In particular,*

$$\mathfrak{P}_\theta^+ \left[\frac{A(\theta)}{1 - a\theta} \right] = \frac{A(1/a)}{1 - a\theta}, \qquad \mathfrak{P}_\theta^- \left[\frac{A(\theta)}{1 - a\theta} \right] = \frac{A(\theta) - A(1/a)}{1 - a\theta}. \tag{3.3.21}$$

It follows from (3.2.13), (3.2.15) that

$$\left(\mathbf{E}(z\theta)^{-\delta} - 1 \right) \mathbf{E}(z\theta)^{S_{\nu_s}^-} = \frac{1 - c(s)}{1 - \lambda} \frac{1 - z\theta}{z\theta - c(s)}$$

Employing (3.3.20), (3.3.21), we evaluate

$$\mathfrak{P}_\theta^+ \left[\frac{\tilde{\Pi}_{z\theta}^s}{1 - \theta} \frac{1 - z\theta}{z\theta - c(s)} \right] = \frac{1}{z\theta - c(s)} \left[\frac{1 - z\theta}{1 - \theta} \tilde{\Pi}_{z\theta}^s - \frac{1 - c(s)}{1 - c(s)/z} \tilde{\Pi}_{c(s)}^s \right]$$

Inserting the expression for the projector, the expression (3.2.15) for the function $\mathbf{E}(z\theta)^{S_{\nu_s}^+}$ into (3.3.15), we derive

$$\tilde{U}_{s,z}^\theta(x) = \frac{\tilde{\Pi}_{z\theta}^s}{1 - \theta} - \frac{\tilde{\Pi}_{z\theta}^s}{1 - \lambda} \frac{1 - z\theta}{1 - \theta} \mathbb{Q}_{z\theta}^s(x) + \frac{\tilde{\Pi}_{c(s)}^s}{1 - \lambda} \frac{1 - c(s)}{1 - c(s)/z} \mathbb{Q}_{z\theta}^s(x)$$

where $\mathbb{Q}_\theta^s(x)$ is the generating function of the resolvent sequence (3.3.4). Comparing the coefficients of θ^k, $k \in \mathbb{Z}+$, in both sides, we get (3.3.17). Comparing the coefficients of z^i, $i \leq k$ in (3.3.17), we derive (3.3.18). ▲

3.4 Exit from the interval

Knowledge of the one-boundary characteristics of the process enables us to solve the two-sided exit problem, which is considered below.

Let $B \in \mathbb{Z}^+$ be fixed, $k \in \{0, \dots, B\}$, $r = B - k$, $X_0 = \{0, x\}$, $x \geq 0$, and introduce the random variable

$$\chi_r^k(x) = \inf\{t : D_x(t) \notin [-r, k]\} \overset{\text{def}}{=} \chi$$

the first exit time from the interval $[-r, k]$ by the process $\{D_x(t)\}_{t \geq 0}$. This random variable takes values from a countable set, and it is a Markov time of the process $\{X_t\}_{t \geq 0}$. Exit from the interval can occur either through the upper boundary k, or through the lower boundary $-r$. In view of this remark we introduce the events

$\mathfrak{A}^k = \{D_x(\chi) > k\}$ i.e. the process $\{D_x(t)\}_{t \geq 0}$ exits the interval $[-r, k]$ through the upper boundary k;

$\mathfrak{A}_r = \{D_x(\chi) < -r\}$ i.e. the process $\{D_x(t)\}_{t \geq 0}$ exits the interval $[-r, k]$ through the lower boundary $-r$. Denote by

$$T = (D_x(\chi) - k)\mathbf{I}_{\mathfrak{A}^k} + (-D_x(\chi) - r)\mathbf{I}_{\mathfrak{A}_r}, \quad L = \eta_x^+(\chi)\mathbf{I}_{\mathfrak{A}^k} + 0 \cdot \mathbf{I}_{\mathfrak{A}_r}, \quad \mathbf{P}[\mathfrak{A}^k + \mathfrak{A}_r] = 1$$

the value of the overshoot through the boundaries of the interval $[-r, k]$ by the process $\{D_x(t)\}_{t \geq 0}$ and the value of the age process. Here $\mathbf{I}_{\mathfrak{A}} = \mathbf{I}_{\mathfrak{A}}(\omega)$ is the indicator function of the event $\mathfrak{A}$. For all $k \in \mathbb{Z}^+$, $m \in \mathbb{N}$, $x \geq 0$ denote

$$F^k(x, dl, m, s) = f^k(x, dl, m, s) - \sum_{i \in \mathbb{N}} f_r(x, i, s) f^{i+B}(0, dl, m, s),$$

$$F_r(x, m, s) = f_r(x, m, s) - \sum_{i \in \mathbb{N}} \int_0^\infty f^k(x, dl, i, s) f_{i+B}(l, m, s).$$

Theorem 3.4.1 (Kadankov and Kadankova (2008c)). *Let $\{D_x(t)\}_{t \geq 0}$ be the difference of the compound Poisson process and the compound renewal process (3.2.3) and $B \in \mathbb{Z}^+$, $k \in \{0, \dots, B\}$, $r = B - k$, $X_0 = \{0, x\}$, $x \geq 0$. Then the Laplace transforms*

$$V^k(x, dl, m, s) = \mathbf{E}\left[e^{-s\chi}; L \in dl, T = m, \mathfrak{A}^k\right], \quad V_r(x, m, s) = \mathbf{E}\left[e^{-s\chi}; T = m, \mathfrak{A}_r\right]$$

of the joint distribution of $\{\chi, L, T\}$ *satisfy the following formulae for* $s > 0$, $m \in \mathbb{N}$

$$V^k(x, dl, m, s) = F^k(x, dl, m, s) + \sum_{i\in\mathbb{N}} \int_0^\infty F^k(x, d\nu, i, s)\mathfrak{K}^+_{\nu,i}(dl, m, s),$$
$$V_r(x, m, s) = F_r(x, m, s) + \sum_{i\in\mathbb{N}} F_r(x, i, s)\, \mathfrak{K}^-_i(m, s), \tag{3.4.1}$$

where

$$\mathfrak{K}^+_{\nu,i}(dl, m, s) = \sum_{n\in\mathbb{N}} K^+_{\nu,i}(dl, m, s)^{*(n)}, \quad \mathfrak{K}^-_i(m, s) = \sum_{n\in\mathbb{N}} K^-_i(m, s)^{*(n)} \tag{3.4.2}$$

are the uniformly convergent series of the iterations, and

$$K^+_{\nu,i}(dl, m, s)^{*(1)} \stackrel{\text{def}}{=} K^+_{\nu,i}(dl, m, s), \qquad K^-_i(m, s)^{*(1)} \stackrel{\text{def}}{=} K^-_i(m, s),$$
$$K^+_{\nu,i}(dl, m, s)^{*(n+1)} = \sum_{j\in\mathbb{N}} \int_0^\infty K^+_{\nu,i}(du, j, s)K^+_{u,j}(dl, m, s)^{*(n)}, \quad n \in \mathbb{N}$$
$$K^-_i(m, s)^{*(n+1)} = \sum_{j\in\mathbb{N}} K^-_i(j, s)\, K^-_j(m, s)^{*(n)}, \quad n \in \mathbb{N} \tag{3.4.3}$$

are the successive iterations of the kernels $K^+_{\nu,i}(dl, m, s)$, $K^-_i(m, s)$, *which are given by the following defining formulae:*

$$K^+_{\nu,i}(dl, m, s) = \sum_{j\in\mathbb{N}} f_{i+B}(\nu, j, s)\, f^{j+B}(0, dl, m, s),$$
$$K^-_i(m, s) = \sum_{j\in\mathbb{N}} \int_0^\infty f^{i+B}(0, du, j, s)\, f_{j+B}(u, m, s). \tag{3.4.4}$$

Proof.

Following Kadankov and Kadankova (2005b), we derive a system of equations with respect to the functions $V^k(x, dl, m, s)$, $V_r(x, m, s)$, $s > 0$, $m \in \mathbb{N}$

$$f^k(x, dl, m, s) = V^k(x, dl, m, s) + \sum_{i\in\mathbb{N}} V_r(x, i, s)\, f^{i+B}(0, dl, m, s),$$
$$f_r(x, m, s) = V_r(x, m, s) + \sum_{i\in\mathbb{N}} \int_0^\infty V^k(x, d\nu, i, s)\, f_{i+B}(\nu, m, s). \tag{3.4.5}$$

In order to write this system, we have used the law of total probability, homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component, Markov property of the random variables $\tau^k(x)$, $\tau_k(x)$, χ and the following reasoning. The first equation of the system (3.4.5) stresses the fact that the first overshoot of the upper level k by the process $\{D_x(t)\}_{t\geq 0}$ (expression on the left-hand side of the equation) can be realized either on sample paths which do not intersect the lower level $-r$ (the first term on the right-hand side of the equation) or on the sample paths which do intersect the level $-r$ and then later on intersect the level k (the second term on the right-hand side of the equation). The second equation of the system is written analogously. This system is similar to a system of linear equations with two unknown variables. Substituting the expression for the function $V^k(x, dl, m, s)$ from the first equation of system (3.4.5) into the second one, we get

$$V_r(x,m,s) = f_r(x,m,s) - \sum_{i\in\mathbb{N}} \int_0^\infty f^k(x, d\nu, i, s)\, f_{i+B}(\nu, m, s) + \\ + \sum_{j\in\mathbb{N}} \int_0^\infty \sum_{i\in\mathbb{N}} V_r(x,i,s)\, f^{i+B}(0, du, j, s)\, f_{j+B}(u, m, s).$$

Changing the order of summation in the third term on the right-hand of the latter equation, we obtain a discrete analog of the Fredholm integral equation of the second kind for the function $V_r(x,m,s)$

$$V_r(x,m,s) = F_r(x,m,s) + \sum_{i\in\mathbb{N}} V_r(x,i,s)\, K_i^-(m,s), \tag{3.4.6}$$

where the function

$$K_i^-(m,s) = \sum_{j\in\mathbb{N}} \int_0^\infty f^{i+B}(0, du, j, s)\, f_{j+B}(u, m, s)$$

is the kernel of this equation. As mentioned above, $\tau^k(x) \in \{\eta_n^*,\, n \in \mathbb{N}\}$, $\tau_k(x) \in \{\eta_n(x),\, n \in \mathbb{N}\}$, where η_n^* is the instant of the n-th jump of the compound Poisson process. It follows from the properties of the compound Poisson process that for all $k \in \mathbb{Z}+$, $x, s \geq 0$ the following inequalities hold:

$$\mathbf{E}e^{-s\tau^k(x)} \leq \frac{\mu}{s+\mu}, \qquad \mathbf{E}e^{-s\tau_k(x)} \leq \mathbf{E}e^{-s\eta_x}.$$

Therefore, the kernel of the equation (3.4.6) enjoys the following property for all $i, m \in \mathbb{N}$, $s > 0$

$$K_i^-(m,s) \leq \sum_{j\in\mathbb{N}} \int_0^\infty f^{i+B}(0,du,j,s)\, \mathbf{E} e^{-s\eta_u}$$
$$\leq \sum_{j\in\mathbb{N}} \int_0^\infty f^{i+B}(0,du,j,s) \leq \frac{\mu}{s+\mu} < 1.$$

Using the method of mathematical induction and the latter bound for the kernel, one can show that for all $i, m \in \mathbb{N}$, $K_i^-(m,s)^{*(n)} \leq \mu^n (s+\mu)^{-n}$, $n \in \mathbb{N}$. Hence, the series

$$\mathfrak{K}_i^-(m,s) = \sum_{n\in\mathbb{N}} K_i^-(m,s)^{*(n)} \leq \frac{\mu}{s} < \infty, \qquad s > 0$$

converges uniformly for all $i, m \in \mathbb{N}$. Utilizing the method of successive iterations (Petrovskii (1965)) to solve (3.4.6), we get the second equality of Theorem 3.4.1. Substituting the expression for the function $V_r(x,m,s)$ from the second equation into the first one, we find

$$V^k(x,dl,m,s) = f^k(x,dl,m,s) - \sum_{i\in\mathbb{N}} f_r(x,i,s)\, f^{i+B}(0,dl,m,s) +$$
$$+ \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \int_0^\infty V^k(x,d\nu,i,s)\, f_{i+B}(\nu,j,s)\, f^{j+B}(0,dl,m,s).$$

Changing the order of summation in the third term in the right-hand side of this equation, we obtain for the function $V^k(x,dl,m,s)$

$$V^k(x,dl,m,s) = F^k(x,dl,m,s) + \sum_{i\in\mathbb{N}} \int_0^\infty V^k(x,d\nu,i,s)\, K_{\nu,i}^+(dl,m,s), \tag{3.4.7}$$

i.e. a discrete analog of a linear integral equation. The kernel of this equation is given by

$$K_{\nu,i}^+(dl,m,s) = \sum_{j\in\mathbb{N}} f_{i+B}(\nu,j,s)\, f^{j+B}(0,dl,m,s),$$

and for all $\nu, l > 0$ $i, m \in \mathbb{N}$, $s > 0$ it satisfies the following inequality

$$K_{\nu,i}^+(dl,m,s) \leq \frac{\mu}{s+\mu} \sum_{j\in\mathbb{N}} f_{i+B}(\nu,j,s) \leq \frac{\mu}{s+\mu}\, \mathbf{E} e^{-s\tau_{i+B}(\nu)} \leq \frac{\mu}{s+\mu} < 1.$$

Hence, the series of the successive iterations

$$\mathfrak{K}^{+}_{\nu,i}(dl,m,s)=\sum_{n\in\mathbb{N}}K^{+}_{\nu,i}(dl,m,s)^{*(n)}\le\frac{\mu}{s}<\infty$$

converges uniformly for all $\nu, l>0$ $i,m\in\mathbb{N}$. Employing the method of successive iterations (Petrovskii (1965)) for the equation (3.4.7), we derive the first equality of Theorem 3.4.1. ▲

Corollary 3.4.1 (Kadankov and Kadankova (2008c))**.** *Let $\{D_x(t)\}_{t\ge 0}$ be the difference of the compound Poisson process and the renewal process (3.2.3), $\delta\sim ge(\lambda)$, $\{Q^s_k(x)\}_{k\in\mathbb{Z}^+}$, $x\ge 0$ be the resolvent sequence of the process given by (3.3.5), $Q^s_k\stackrel{\text{def}}{=}Q^s_k(0)$. Then*

(i) *the Laplace transforms of the joint distribution of $\{\chi,L,T\}$ satisfy the following equalities for all $x,s\ge 0$, $m\in\mathbb{N}$*

$$\begin{aligned}V_r(x,m,s)&=\frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}}(1-\lambda)\lambda^{m-1},\\ V^k(x,dl,m,s)&=f^k(x,dl,m,s)-\frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}}\,\mathbf{E}f^{\delta+B}(0,dl,m,s),\end{aligned}\qquad(3.4.8)$$

where the function $f^k(x,dl,m,s)$ is given by (3.3.6),

$$\begin{aligned}\mathbf{E}\,Q^s_{\delta+B}&=\sum_{k\in\mathbb{N}}(1-\lambda)\lambda^{k-1}Q^s_{k+B},\\ \mathbf{E}f^{\delta+B}(0,dl,m,s)&=\sum_{k\in\mathbb{N}}(1-\lambda)\lambda^{k-1}f^{k+B}(0,dl,m,s);\end{aligned}$$

(ii) *for the Laplace transforms of the first exit time χ the following formulae hold:*

$$\begin{aligned}\mathbf{E}\left[e^{-s\chi};\mathfrak{A}_r\right]&=\frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}},\\ \mathbf{E}\left[e^{-s\chi};\mathfrak{A}^k\right]&=1-\frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}}+A^s_k(x)-\frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}}\mathbf{E}\,A^s_{\delta+B}(0),\end{aligned}\qquad(3.4.9)$$

where

$$A^s_k(x)=\sum_{i=0}^{k}\tilde{\rho}_i(s)\left[\frac{Q^s_{k-i}(x)}{1-\lambda}-1\right],\quad \tilde{\rho}_i(s)=s\int_0^\infty e^{-st}\mathbf{P}[\pi(t)=i]\,dt;$$

(iii) *the probabilities of the exit from the interval through the upper and the lower boundary by the process* $\{D_x(t)\}_{t\geq 0}$ *are such that*

$$\mathbf{P}[\mathfrak{A}_r] = \frac{Q_k(x)}{\mathbf{E}\,Q_{\delta+B}}, \qquad \mathbf{P}[\mathfrak{A}^k] = 1 - \frac{Q_k(x)}{\mathbf{E}\,Q_{\delta+B}},$$

where the resolvent sequence of the process $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$, $Q_k \stackrel{\text{def}}{=} Q_k(0)$ *is defined by (3.3.8).*

Proof. Proof of Corollary 3.4.1 follows straightforwardly from Theorem 3.4.1. We clarify the formulae of Corollary 3.4.1 by employing equalities (3.4.1) of Theorem 3.4.1 which take a simple form for the case the renewal process has geometrically distributed jumps. To apply (3.4.1), we have to calculate the kernels (3.4.4), the successive iterations (3.4.3) and the series of the successive iterations (3.4.2) for the process $\{D_x(t)\}_{t\geq 0}$ in case when $\delta \sim ge(\lambda)$. Let us verify the first formula of (3.4.8). Employing Lemma 3.3.2 and (3.4.2)-(3.4.4), for all $i, m, n \in \mathbb{N}$, one can derive

$$\begin{aligned}
&K_i^-(m,s) = \varphi_0^{i+B}(m,s),\\
&K_i^-(m,s)^{*(n)} = \varphi_0^{i+B}(m,s)\left(\mathbf{E}\varphi_0^{\delta+B}(s)\right)^{n-1},\\
&\mathfrak{K}_i^-(m,s) = \varphi_0^{i+B}(m,s)\left(1-\mathbf{E}\varphi_0^{\delta+B}(s)\right)^{-1},\\
&F_r(x,m,s) = c_x(s)c(s)^r(1-\lambda)\lambda^{m-1} - \varphi_x^k(m,s),
\end{aligned} \tag{3.4.10}$$

where $c_x(s) = \tilde{f}_x(s - k(c(s)))$, and the mathematical expectations $\{\varphi_x^k(m,s), k \in \mathbb{Z}^+\}$, $x \geq 0$ are given by (3.3.11),

$$\varphi_x^k(s) = \mathbf{E}\left[e^{-si_x^k}; i_x^k < \infty\right] = \sum_{m\in\mathbb{N}} \varphi_x^k(m,s).$$

Substituting the expressions (3.4.10) into the second equality of (3.4.1), we obtain the first formula (3.4.8) of Corollary 3.4.1.

Let us verify the second formula of (3.4.8). Employing the statements of (3.3.2) and (3.4.2)-(3.4.4), we can calculate for all $s, \nu, x \geq 0$, $i, m, n \in \mathbb{N}$ that

$$K^{+}_{\nu,i}(dl, m, s) = c_{\nu}(s) c(s)^{i+B} \mathbf{E}\, f^{\delta+B}(0, dl, m, s),$$
$$K^{+}_{\nu,i}(dl, m, s)^{*(n)} = c_{\nu}(s) c(s)^{i+B} \left(\mathbf{E}\, \varphi_0^{\delta+B}(s)\right)^{n-1} \mathbf{E}\, f^{\delta+B}(0, dl, m, s),$$
$$\mathfrak{K}^{+}_{\nu,i}(dl, m, s) = c_{\nu}(s) c(s)^{i+B} \left(1 - \mathbf{E}\, \varphi_0^{\delta+B}(s)\right)^{-1} \mathbf{E}\, f^{\delta+B}(0, dl, m, s),$$
$$F^{k}(x, dl, m, s) = f^{k}(x, dl, m, s) - c_x(s) c(s)^{r} \mathbf{E}\, f^{\delta+B}(0, dl, m, s), \qquad (3.4.11)$$

where the mathematical expectations $f^k(x, dl, m, s)$, $k \in \mathbb{Z}^+$, $x \geq 0$ are given by (3.3.6). Substituting the expressions (3.4.11) into the first equality of (3.4.1), we obtain the second formula of (3.4.8) of Corollary 3.4.1.

Summing both sides of the first equality of (3.4.8) with respect to $m \in \mathbb{N}$, and summing the second equality of (3.4.8) over $m \in \mathbb{N}$, and then integrating it with respect to $l \geq 0$, we get the formulae (3.4.9). Letting $s \to 0$ in both sides of these equalities, we derive the probabilities of the exit from the interval by the process $\{D_x(t)\}_{t\geq 0}$. ▲

Let us stress the following fact. If we set parameter $\lambda = 0$ in the geometrical distribution $\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}$, $n \in \mathbb{N}$, $\lambda \in [0, 1)$, then $\mathbf{P}[\delta = 1] = 1$. In other words it means that the process $\{D_x(t)\}_{t\geq 0}$ has unit negative jumps at the instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$, and $\delta_{N_x(t)} = N_x(t)$. Then, it follows from (3.2.3) that

$$D_x(t) = \pi(t) - N_x(t) \in \mathbb{Z}, \qquad t \geq 0. \qquad (3.4.12)$$

We will call this process a difference of the compound Poisson process and a simple renewal process. Setting the parameter $\lambda = 0$ in the statements of Lemma 3.2.3 leads to the following result.

Lemma 3.4.1. *For $s > 0$ the equation $\theta = \tilde{f}(s - k(\theta))$ has a unique solution $c(s)$ inside the circle $|\theta| < 1$. This solution is positive, $c(s) \in (0, 1)$. If $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$, $\rho = \mu \mathbf{E}[\varkappa]\, \mathbf{E}[\eta]$, then for $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (0, 1)$; and for $\rho \leq 1$, $\lim_{s\to 0} c(s) = 1$.*

Statements of Corollaries 3.3.1, 3.3.1, 3.3.2 and 3.3.3 can be reformulated in a similar way. Letting $\lambda = 0$ in the defining formula (3.3.5) for all $s, x \geq 0$ we get

$$Q_k^s(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{1}{\theta^{k+1}} \frac{\tilde{f}_x(s-k(\theta))}{\tilde{f}(s-k(\theta))-\theta}\, d\theta, \quad \alpha \in (0, c(s)) \tag{3.4.13}$$

a resolvent sequence of the process $\{D_x(t)\}_{t\geq 0}$, which is given by (3.4.12). This resolvent sequence has been introduced in Kadankov (1985). Setting $\lambda = 0$ in (3.3.6), (3.3.7), we obtain

$$\begin{aligned} f^k(x, dl, m, s) &= e^{-s(l-x)} \frac{1-F(l)}{1-F(x)} \mathbf{I}\{l > x\} p_k^m(d(l-x)) \\ &\quad + \Phi_0^s(dl, m)\, Q_k^s(x) - e^{-sl}\,[1-F(l)] \sum_{i=0}^{k} Q_i^s(x)\, p_{k-i}^m(dl), \\ \mathbf{E}e^{-s\tau^k(x)} &= 1 - \frac{s}{s-k(c(s))}\, Q_k^s(x) + \sum_{i=0}^{k} \tilde{\rho}_i(s)\, \left[Q_{k-i}^s(x) - 1\right] \end{aligned} \tag{3.4.14}$$

the Laplace transforms of the upper one-boundary functionals of the process $\{D_x(t)\}_{t\geq 0}$ (3.4.12), where

$$\Phi_0^s(dl, m) = e^{-sl}[1-F(l)] \sum_{k\in\mathbb{Z}^+} c(s)^k p_k^m(dl), \quad \tilde{\rho}_i(s) = s\int_0^\infty e^{-st}\mathbf{P}[\pi(t) = i]\, dt.$$

We have introduced the auxiliary functions and the resolvent sequence of the process (3.4.12), therefore we can state the following result.

Corollary 3.4.2. *Let $\{D_x(t)\}_{t\geq 0}$ be the difference of the compound Poisson process and the renewal process (3.4.12), $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$ be the resolvent sequence of the process given by (3.4.13), $Q_k^s \stackrel{\text{def}}{=} Q_k^s(0)$. Then*

(i) *the Laplace transforms $V_r(x, m, s)$, $V^k(x, dl, m, s)$ of the joint distribution of $\{\chi, L, T\}$ satisfy the following equalities for all $x, s \geq 0$, $m \in \mathbb{N}$*

$$\begin{aligned} V_r(x, m, s) &= \frac{Q_k^s(x)}{Q_{B+1}^s}\, \delta_{m1}, \\ V^k(x, dl, m, s) &= f^k(x, dl, m, s) - \frac{Q_k^s(x)}{Q_{B+1}^s} f^{B+1}(0, dl, m, s), \end{aligned}$$

where δ_{ij} is the Kronecker symbol and the function $f^k(x, dl, m, s)$ is given by (3.4.14);

(ii) *for the Laplace transforms of the first exit time* χ *from the interval by the process* $\{D_x(t)\}_{t\geq 0}$ *the formulae hold:*

$$\mathbf{E}\left[e^{-s\chi};\mathfrak{A}_r\right] = \frac{Q_k^s(x)}{Q_{B+1}^s}, \qquad \mathbf{E}\left[e^{-s\chi};\mathfrak{A}^k\right] = 1-$$

$$-\frac{Q_k^s(x)}{Q_{B+1}^s} + \sum_{i=0}^{k} \tilde{\rho}_i(s)[Q_{k-i}^s(x) - 1] - \frac{Q_k^s(x)}{Q_{B+1}^s}\sum_{i=0}^{B} \tilde{\rho}_i(s)[Q_{B+1-i}^s - 1];$$

(iii) *the exit probabilities from the interval by the process* $\{D_x(t)\}_{t\geq 0}$ *are such that*

$$\mathbf{P}[\mathfrak{A}_r] = \frac{Q_k(x)}{Q_{B+1}}, \qquad \mathbf{P}[\mathfrak{A}^k] = 1 - \frac{Q_k(x)}{Q_{B+1}},$$

where the resolvent sequence of the process $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$, $Q_k \stackrel{\text{def}}{=} Q_k(0)$ *is given by (3.4.13) for* $s = 0$.

To prove the corollary, one has to put $\lambda = 0$ in the statements of Corollary 3.4.1.

3.5 Infimum, supremum and the value of the process

Let $D_x^+(t) = \sup_{[0,t]} D_x(\cdot)$, $D_x^-(t) = \inf_{[0,t]} D_x(\cdot)$ be the running supremum and infimum of the process. In this section we will determine the generating function

$$\tilde{q}_x^s(-r, z, k) = \mathbf{E}\left[-r \le D_x^-(\nu_s), z^{D_x(\nu_s)}, D_x^+(\nu_s) \le k\right] = \mathbf{E}\left[z^{D_x(\nu_s)}; \chi_r^k(x) > \nu_s\right]$$
$$= \sum_{i=-r}^{k} z^i \mathbf{P}\left[-r \le D_x^-(\nu_s),\, D_x(\nu_s) = i,\, D_x^+(\nu_s) \le k\right], \quad k, r \in \mathbb{Z}^+$$

of the joint distribution of the infimum, supremum and the value of the process. In order to do this, we will follow the approach proposed for Lévy processes in Kadankov and Kadankova (2005b).

Theorem 3.5.1. *Let $\{D_x(t)\}_{t\ge 0}$ be the difference of the compound Poisson process and the compound renewal process (3.2.3), $\nu_s \sim \exp(s)$, $s > 0$ be the exponential random variable independent of the process, $k, r \in \mathbb{Z}^+$, $B = r + k$. Then the generating function of the joint distribution of $\{D_x^-(\nu_s), D_x(\nu_s), D_x^+(\nu_s)\}$ is such that*

$$\tilde{q}_x^s(-r, z, k) = E_k^+(x, z, s) - z^{-r} \sum_{i\in\mathbb{N}} z^{-i} \mathbf{E}\left[e^{-s\chi}; T = i, \mathfrak{A}_r\right] E_{i+B}^+(0, z, s),$$
$$\tilde{q}_x^s(-r, z, k) = E_r^-(x, z, s) - z^k \sum_{i\in\mathbb{N}} \int_0^\infty z^i \mathbf{E}\left[e^{-s\chi}; L \in dl, T = i, \mathfrak{A}^k\right] E_{i+B}^-(l, z, s),$$
$$(3.5.1)$$

where

$$E_k^+(x, z, s) = \mathbf{E}\left[z^{D_x(\nu_s)}; D_x^+(\nu_s) \le k\right], \quad E_r^-(x, z, s) = \mathbf{E}\left[z^{D_x(\nu_s)}; D_x^-(\nu_s) \ge -r\right].$$

In particular, the following relation is true:

$$\mathbf{P}[\chi > \nu_s] = \mathbf{P}[D_x^+(\nu_s) \le k] - \int_0^\infty \mathbf{E}[e^{-s\chi}; T = i, \mathfrak{A}_r]\mathbf{P}[D_{\nu_s}^+(0) \le i + B] =$$
$$(3.5.2)$$
$$= \mathbf{P}[D_x^-(\nu_s) \ge -r] - \sum_{i\in\mathbb{N}} \int_0^\infty \mathbf{E}[e^{-s\chi}; L \in dl, T = i, \mathfrak{A}^k]\mathbf{P}[D_{\nu_s}^-(l) \ge -i - B].$$

Proof. According to the total probability law, the Markov property of χ, homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component and the properties of ν_s, we write

$$\mathbf{E}\left[z^{D_x(\nu_s)}; D_x^+(\nu_s) \leq k\right] = \mathbf{E}\left[z^{D_x(\nu_s)}; D_x^-(\nu_s) \geq -r, D_x^+(\nu_s) \leq k\right]$$
$$+ z^{-r} \sum_{i\in\mathbb{N}} z^{-i} \mathbf{E}\left[e^{-s\chi}; T = i, \mathfrak{A}_r\right] \mathbf{E}\left[z^{D_0(\nu_s)}; D_0^+(\nu_s) \leq i + B\right].$$

This equation shows that the increments of the process $\{D_x(t)\}_{t\geq 0}$ on the time interval $[0, \nu_s]$ given that no intersection of the upper level $k \in \mathbb{Z}^+$ occurs (the left-hand side of the equation) are realized either on the samples paths which do not intersect the lower level $-r \leq 0$, (the first term on the right-hand side) or on the sample paths which do intersect the lower level $-r$ and further do not cross the upper level k (the second term on the right-hand side). A more detailed derivation of this equation for Lévy processes and random walks is given in Kadankov and Kadankova (2005b). One can note, that the first formula of (3.5.1) follows immediately from the latter equation. The second formula of (3.5.1) follows from the equation

$$\mathbf{E}\left[z^{D_x(\nu_s)}; D_x^-(\nu_s) \geq -r\right] = \mathbf{E}\left[z^{D_x(\nu_s)}; D_x^-(\nu_s) \geq -r, D_x^+(\nu_s) \leq k\right]$$
$$+ z^k \sum_{i\in\mathbb{N}} \int_0^\infty z^i \mathbf{E}\left[e^{-s\chi}; L \in dl, T = i, \mathfrak{A}^k\right] \mathbf{E}\left[z^{D_l(\nu_s)}; D_l^-(\nu_s) \geq -i - B\right].$$

The identity (3.5.2) of Theorem 3.5.1 can be obtained from (3.5.1) by letting $z = 1$. This remark completes the proof. ▲

We now assume that $\delta \sim ge(\lambda)$. In this case, we first determine the joint distributions of $\{D_x(\nu_s), D_x^+(\nu_s)\}$ and of $\{D_x^-(\nu_s), D_x(\nu_s), D_x^+(\nu_s)\}$ in terms of the resolvent sequence. Secondly, we will consider the asymptotic behavior of these functionals. Finally, we study the joint distribution of $\{D_x(\nu_s), D_x^-(\nu_s)\}$.

Lemma 3.5.1. *Let $k \in \mathbb{Z}^+$ and $E_k^+(x, z, s) = \mathbf{E}\left[z^{D_x(\nu_s)}; D_x^+(\nu_s) \leq k\right]$, $|z| \geq 1$ be the generating function of the joint distribution of $\{D_x(\nu_s), D_x^+(\nu_s)\}$. Then*

(i) *the generating function* $E_k^+(x,z,s)$ *is such that*

$$E_k^+(x,z,s) = z^k A_x^k(s) + (1-z)\sum_{i=0}^{k-1} z^i A_x^i(s) + z^k Q_k^s(x)\mathbb{E}_{\lambda/z}^s(0,z), \tag{3.5.3}$$

where

$$\mathbb{E}_{\lambda/z}^s(0,z) = \frac{s(1-\lambda)^{-1}}{s-k(c(s))}\,\frac{1-c(s)}{1-c(s)/z};$$

(ii) *the joint distribution* $\mathfrak{E}_k^+(x,u,s) = \mathbf{P}\left[D_x(\nu_s) \le u, D_x^+(\nu_s) \le k\right]$, $u \in]-\infty,\, k]$ *satisfies the following equality:*

$$\mathfrak{E}_k^+(x,u,s) = A_x^u(s) + \frac{s(1-\lambda)^{-1}}{s-k(c(s))}\,c(s)^{k-u}Q_k^s(x), \quad A_x^u(s)=0,\ u<0; \tag{3.5.4}$$

(iii) *under the condition* (A)

(A) $\rho = (1-\lambda)\mu\mathbf{E}\eta\mathbf{E}\varkappa = 1, \quad \sigma^2 = \mu\left[\mathbf{E}\varkappa(\varkappa-1) + \frac{\mathbf{E}\varkappa\mathbf{E}\eta^2}{(1-\lambda)(\mathbf{E}\eta)^2}\right] < \infty,$

the following limiting equality holds as $B \to \infty$, $k > 0$

$$\mathbf{P}[D_x(tB^2) \le [uB], D_x^+(tB^2) \le [kB]] \to \frac{1}{\sigma\sqrt{2\pi t}}\int_{-u}^{2k-u} e^{-v^2/2\sigma^2 t}dv,$$

where $[a]$ *is the integer part of the number* a, $u \le k$.

Proof. In view of the total probability law, homogeneity of the process X_t with respect to the first component, Markov property of $\eta_1(x)$ we can write for the function $E_k^+(x,z,s)$, $k \in \mathbb{Z}^+$, $x \ge 0$ the following equation:

$$E_k^+(x,z,s) = s\int_0^\infty e^{-st}\mathbf{P}\left[\eta_x > t\right]\mathbf{E}\left[z^{\pi(t)}; \pi(t) \le k\right]dt +$$
$$+\int_0^\infty e^{-su}\sum_{v=0}^{k}\mathbf{P}\left[\eta_x \in du, \pi(u)=v\right]z^v\sum_{r=1}^{\infty}(1-\lambda)\lambda^{r-1}z^{-r}E_{k-v+r}^+(0,z,s).$$

Introduce the generating function $\mathbb{E}_\theta^s(x,z) = \sum\limits_{k\in\mathbb{Z}^+}\theta^k E_k^+(x,z,s)$, $|\theta|<1$. Multiplying the latter equation by θ^k and summing over $k \in \mathbb{Z}^+$, we derive the following equation for the function $\mathbb{E}^s_\theta(x,z)$

$$\mathbb{E}^s_\theta(x,z) = \frac{s}{1-\theta}\,\frac{1-\tilde{f}_x(s-k(z\theta))}{s-k(z\theta)} + \tag{3.5.5}$$
$$+\tilde{f}_x(s-k(z\theta))\frac{(1-\lambda)}{\lambda-z\theta}\,\Big[\mathbb{E}^s_{\lambda/z}(0,z)-\mathbb{E}^s_\theta(0,z)\Big], \quad |\theta|<1,\ |z|\geq 1.$$

Letting $x=0$ in 3.5.5 yields

$$\mathbb{E}^s_\theta(0,z) = \frac{z\theta-\lambda}{(1-\lambda)\tilde{f}(s-k(z\theta))+\lambda-z\theta}\times$$
$$\times\left[\tilde{f}(s-k(z\theta))\frac{1-\lambda}{z\theta-\lambda}\,\mathbb{E}^s_{\lambda/z}(0,z) - \frac{s}{1-\theta}\frac{1-\tilde{f}(s-k(z\theta))}{s-k(z\theta)}\right].$$

The function which enters the left-hand side of this equation, is analytic in $|\theta|<1$. In view of Lemma 3.2.14 the denominator on the right-hand side has a simple zero in $\theta = c(s)/z$. Hence, the nominator on right-hand side should also have the simple zero. Letting $\theta = c(s)/z$ in the nominator, we find the function $\mathbb{E}^s_{\lambda/z}(0,z)$

$$\mathbb{E}^s_{\lambda/z}(0,z) = \frac{s(1-\lambda)^{-1}}{s-k(c(s))}\,\frac{1-c(s)}{1-c(s)/z}, \qquad |z|\geq 1.$$

Employing the definition of the resolvent (3.3.4) and substituting the expression for $\mathbb{E}^s_\theta(0,z)$ into (3.5.5), we get

$$\mathbb{E}^s_\theta(x,z) = \mathbb{A}^{z\theta}_x(s) + \theta\,\frac{1-z}{1-\theta}\,\mathbb{A}^{z\theta}_x(s) + \mathbb{Q}^s_{z\theta}(x)\mathbb{E}^s_{\lambda/z}(0,z), \tag{3.5.6}$$

where

$$\mathbb{A}^{z\theta}_x(s) = \sum_{k=0}^{\infty}(z\theta)^k A^k_x(s) = \frac{s}{s-k(z\theta)}\left(\frac{1}{1-z\theta} - \frac{1}{1-\lambda}\,\mathbb{Q}^s_{z\theta}(x)\right).$$

Using the definition of the resolvent (3.3.5) and comparing the coefficients of θ^k, $k\in\mathbb{Z}^+$ in both sides of (3.5.6) implies that

$$E^+_k(x,z,s) = z^k A^k_x(s) + (1-z)\sum_{i=0}^{k-1} z^i A^i_x(s) + z^k Q^s_k(x)\mathbb{E}^s_{\lambda/z}(0,z),$$

i.e. the equality (3.5.3) of the lemma. Comparing the coefficients of z^i, $i \le k$ in both sides of the latter equality, we find

$$\mathbf{P}[D_x(\nu_s) = i,\ D_x^+(\nu_s) \le k] =$$
$$= A_x^i(s) - A_x^{i-1}(s) + \frac{s}{s-k(c(s))}\,\frac{1-c(s)}{1-\lambda}\,c(s)^{k-i}Q_k^s(x), \qquad i \le k,$$

where $A_x^i(s) = 0$, for $i < 0$. The latter formula implies (3.5.4). We now verify the third statement of the lemma. Denote
$\tilde{e}_k^t(x,u,B) = \mathbf{P}\left[D_x(tB^2) \le [uB], D_x^+(tB^2) \le [kB]\right]$. It is clear that

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\tilde{e}_k^t(x,u,B)\,dt = \frac{1}{s}\lim_{B\to\infty}\mathfrak{E}_{[kB]}^{s/B^2}(x,[uB]).$$

To proceed further, we need the following limiting equalities (see Kadankov *et al.* (2009))

$$\begin{aligned}
&c(s/B^2) = 1 - B^{-1}\sqrt{2s}/\sigma + o(B^{-1}),\\
&\lim_{B\to\infty} B^{-1}Q_{[kB]}^{s/B^2}(x) = \frac{2\sinh\left(k\sqrt{2s}/\sigma\right)}{\sigma\sqrt{2s}\mathbf{E}\eta} = \lim_{B\to\infty} B^{-1}\mathbf{E}Q_{\delta+[kB]}^{s/B^2},\\
&\lim_{B\to\infty} A_x^{[kB]}(s/B^2) = 1 - \cosh\left(k\sqrt{2s}/\sigma\right) = \lim_{B\to\infty} A_0^{\delta+[kB]}(s/B^2).
\end{aligned} \qquad (3.5.7)$$

In view of these equalities and of the formula (3.5.4) we derive

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\tilde{e}_k^t(x,u,B)\,dt = s^{-1}\mathbf{I}_{\{u<0\}}\left(e^{u\sqrt{2s}/\sigma}/2 - e^{-(2k-u)\sqrt{2s}/\sigma}/2\right)$$
$$+\, s^{-1}\mathbf{I}_{\{u\in[0,k]\}}\left(1 - e^{-u\sqrt{2s}/\sigma}/2 - e^{-(2k-u)\sqrt{2s}/\sigma}/2\right), \quad u \le k.$$

Denote by $\{w_t;:\ t \ge 0\}$ the symmetric Wiener process with the dispersion σ and by $\tau^a = \inf\{t : w_t \ge a\}$ the first passage time of the level $a \in \mathbb{R}_+$. The Lévy formula $\mathbf{P}\left[\tau \le t\right] = 2\mathbf{P}\left[w_t \ge a\right]$ implies for the Laplace transforms that

$$\frac{1}{s}\,e^{-a\sqrt{2s}/\sigma} = 2\int_0^\infty e^{-st}\mathbf{P}\left[w_t \ge a\right]dt.$$

Employing the latter formula to invert the Laplace transforms in the previous equality, we derive the second limiting formula of the lemma. ▲

Let $k, r \in \mathbb{Z}^+$, $u \in [-r, k]$ and denote by

$$\tilde{e}^t_{r,k}(x,u) = \mathbf{P}\left[-r \le D^-_x(t),\, D^-_x(t) \le u,\, D^+_x(t) \le k\right] = \mathbf{P}\left[D_x(t) \le u,\, \chi^B_x(r) > t\right],$$
$$\mathfrak{E}^s_{r,k}(x,u) = \tilde{e}^{\nu_s}_{r,k}(x,u) = s\int_0^\infty e^{-st}\tilde{e}^t_{r,k}(x,u)\,dt$$

the joint distribution of $\{D^-_x(t), D_x(t), D^+_x(t)\}$ and its Laplace transform.

Theorem 3.5.2. *Let $\nu_s \sim \exp(s)$ be an exponential random variable independent of the process $D_x(t)$, $B = r + k$. Then*

(i) *the joint distribution of $\{D^-_x(\nu_s), D_x(\nu_s), D^+_x(\nu_s)\}$ is such that*

$$\mathfrak{E}^s_{r,k}(x,u) = A^u_x(s) - \frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}}\,\mathbf{E}A^{\delta+r+u}_x(s), \quad u \in [-r,k] \tag{3.5.8}$$

where $\mathbf{E}A^{\delta+r+u}_x(s) = (1-\lambda)\sum_{i=1}^{\infty}\lambda^{i-1}A^{i+r+u}_x(s)$;

(ii) *under the condition (A) and $r \in (0,1)$, $k = 1 - r$ the joint distribution $\tilde{e}^{tB^2}_{[rB],[kB]}(x,[uB])$ weakly converges as $B \to \infty$ to the joint distribution*

$$\mathbf{P}\left[-r \le \inf_{v \le t} w_v,\, w_t \le u,\, \sup_{v\le t} w_v \le k\right], \quad u \in [-r,k]$$

of the infimum, the supremum and the value of the symmetric Wiener process with the dispersion σ. In addition, the following limiting equality holds

$$\lim_{B\to\infty} \tilde{e}^{tB^2}_{[rB],[kB]}(x,[uB]) = \frac{4}{\pi}\sum_{n\in\mathbb{N}}\frac{e^{-\frac{t}{2}(\pi n\sigma)^2}}{n}\,\sin(r\pi n)\sin^2\left(\frac{r+u}{2}\,n\pi\right). \tag{3.5.9}$$

Proof. The total probability law, homogeneity of the process X_t with respect to the first component, Markov property of $\chi^B_r(x)$ for all $k, r \in \mathbb{Z}^+$, $x \ge 0$ imply the following equation for $|z| \ge 1$

$$\mathbf{E}\left[z^{D_x(\nu_s)}; D^+_x(\nu_s) \le k\right] = \mathbf{E}\left[z^{D_x(\nu_s)}; \chi^B_r(x) > \nu_s\right] +$$
$$+\sum_{i=1}^{\infty}\mathbf{E}\left[e^{-s\chi^B_r(x)}; T = i, \mathfrak{A}_r\right] z^{-(r+i)}\mathbf{E}\left[z^{D_x(\nu_s)}; D^+_x(\nu_s) \le i + B\right], \tag{3.5.10}$$

where $B = k + r$. Let us briefly explain this equation. The increments of the process $D_x(t)$ on the interval $[0, \nu_s]$ without intersection of the level k (the left-hand side) can be realized either on the sample paths of the process which do not cross the negative level $-r$ (the first term of the right-hand side) or on the sample paths which do cross the level $-r$ and then the further evolution of the process is nothing but its probabilistic copy on $[0, \nu_s]$ (the second term). In view of (3.5.10) and (3.4.8), (3.5.3) we find for the function $E^s_{r,k}(x,z) = \mathbf{E}\left[z^{D_x(\nu_s)}; \chi^B_r(x) > \nu_s\right]$ that

$$E^s_{r,k}(x,z) = E^+_k(x,z,s) - \frac{Q^s_k(x)}{\mathbf{E}\, Q^s_{\delta+B}}(1-\lambda)\sum_{i\in\mathbb{N}} \lambda^{i-1} z^{-(r+i)} E^+_{i+B}(0,z,s) =$$

$$= z^k A^k_x(s) + (1-z)\sum_{i=0}^{k-1} z^i A^i_x(s) +$$

$$+ z^k \frac{Q^s_k(x)}{\hat{Q}^s_B(\lambda)} \, \frac{(1-\lambda)\check{A}^B_0(s,\lambda) - (1-z)(\lambda/z)^{B+1}\check{A}^B_0(s,z)}{1-\lambda/z}, \quad (3.5.11)$$

where $\check{A}^B_0(s,z) = \sum\limits_{i=0}^{B} z^i A^i_0(s)$, $\hat{Q}^s_B(\lambda) = \sum\limits_{i=B+1}^{\infty} \lambda^i Q^s_i(0)$. The formula (3.4.9) yields

$$\mathbf{P}\left[\chi^B_r(x) > \nu_s\right] = A^k_x(s) + \frac{Q^s_k(x)}{\hat{Q}^s_B(\lambda)}\, \check{A}^B_x(s,\lambda).$$

It is not difficult to derive the following equality

$$\sum_{u=-r}^{k} z^u \mathfrak{E}^s_{r,k}(x,u) = \frac{1}{1-z}\left(E^s_{r,k}(x,z) - z^{k+1}\mathbf{P}\left[\chi^B_r(x) > \nu_s\right]\right).$$

The right-hand side of (3.5.11) implies that

$$\sum_{u=-r}^{k} z^u \mathfrak{E}^s_{r,k}(x,u) = \sum_{u=0}^{k} z^u A^u_x(s) + z^k \frac{Q^s_k(x)}{\hat{Q}^s_B(\lambda)} \sum_{i=0}^{B} (\lambda/z)^i \sum_{j=0}^{B-i} \lambda^j A^j_0(s).$$

Comparing the coefficients of z^u, $u \in [-r, k]$, we find

$$\mathfrak{E}^s_{r,k}(x,u) = A^u_x(s) + \frac{Q^s_k(x)}{\hat{Q}^s_B(\lambda)}\, \lambda^{k-u} \sum_{i=0}^{r+u} \lambda^i A^i_0(s).$$

Since

$$\mathbf{E} Q^s_{\delta+B} = (1-\lambda)\lambda^{-B-1}\hat{Q}^s_B(\lambda), \qquad \sum_{i=0}^{\infty} \lambda^i A^i_0(s) = 0,$$

one can see that the previous equality is the formula (3.5.8). Let us verify (3.5.9). It is clear that

$$s\int_0^\infty e^{-st}\tilde{e}^{tB^2}_{[rB],[kB]}(x,[uB])\,dt=\mathfrak{E}^{s/B^2}_{[rB],[kB]}(x,[uB]),\qquad k\in(0,1)\quad r=1-k,$$

where the function $\mathfrak{E}^s_{r,k}(x,u)$ is determined by (3.5.8). Thus,

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\tilde{e}^{tB^2}_{[rB],[kB]}(x,[uB])\,dt=\frac{1}{s}\lim_{B\to\infty}\mathfrak{E}^{s/B^2}_{[rB],[kB]}(x,[uB])\overset{\text{def}}{=}e^*(s)=$$
$$=\frac{1}{s}\left[1-\cosh\left(\frac{u^+}{\sigma}\sqrt{2s}\right)\right]+\frac{1}{s}\,\frac{\sinh k\sqrt{2s}/\sigma}{\sinh\sqrt{2s}/\sigma}\left[\cosh\left(\frac{r+u}{\sigma}\sqrt{2s}\right)-1\right],\tag{3.5.12}$$

where $u^+=\max(0,u)$. In order to compute this limit, we used the formulae (3.5.7). Note, that the inversion of the Laplace transform in the right-hand side of (3.5.12) this equality was found in Kadankova (2004) and resulted into the following formula ($\alpha>0$)

$$\mathbf{P}\left[-r\le\inf_{v\le t}w_v,\ w_t\le u,\ \sup_{v\le t}w_v\le k\right]=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{st}e^*(s)\,ds=$$
$$=\frac{4}{\pi}\sum_{n\in\mathbb{N}}\frac{e^{-\frac{t}{2}(\pi n\sigma)^2}}{n}\,\sin(r\pi n)\sin^2\left(\frac{r+u}{2}\,n\pi\right),\qquad u\in[-r,k].$$

Therefore, we established the weak convergence of the joint distribution $\tilde{e}^t_x(u,B)$ as $B\to\infty$ to the corresponding distribution of the Wiener process and also verified the formula (3.5.9). ▲

Now we consider the joint distribution of the position of the process and its infimum $\{D^-_x(\cdot),D_x(\cdot)\}$.

Theorem 3.5.3. *Denote by* $E^-_r(x,z,s)=\mathbf{E}\left[z^{D_x(\nu_s)};D^-_x(\nu_s)\ge -r\right]$, $r\in\mathbb{Z}^+$,$|z|\le 1$ *the generating function of the joint distribution of* $\{D^-_x(\cdot),D_x(\cdot)\}$. *Then*

(i) *The function* $E^-_r(x,z,s)$ *is such that*

$$E^-_r(x,z,s)=(1-z)\mathbb{A}^z_x(s)+(1-z)f_r(x,s)\frac{1-\lambda}{\lambda-z}\,\mathbb{A}^z_0(s)z^{-r},\tag{3.5.13}$$

where $f_r(x,s) = \mathbf{E}\left[e^{-s\tau_r(x)}; \mathfrak{B}_r(x)\right] = c_x(s)c(s)^r$, $c_x(s) = \tilde{f}_x(s - k(c(s)))$,

$$\mathbb{A}_x^z(s) = \sum_{k\in\mathbb{Z}+} z^k A_x^k(s) = \frac{s}{s-k(z)}\left(\frac{1}{1-z} - \frac{1}{1-\lambda}\,\mathbb{Q}_z^s(x)\right);$$

(ii) *the joint distribution* $\mathbb{E}_r^-(x,u,s) = \mathbf{P}\left[D_x(\nu_s) \geq u, D_x^-(\nu_s) \geq -r\right]$, $u \in [-r,\infty[$ *satisfies the following equality:*

$$\mathbb{E}_r^-(x,u,s) = 1 - A_x^{u-1}(s) - f_r(x,s)\left(1 - \mathbf{E}A_0^{u+r+\delta-1}(s)\right), \quad (3.5.14)$$

where $\mathbf{E}A_x^{k+\delta}(s) = (1-\lambda)\sum_{i\in\mathbb{N}} \lambda^{i-1} A_x^{k+i}(s)$, $A_x^u(s) = 0$ *for* $u < 0$;

(iii) *under the condition* (A),

(A) $\rho = (1-\lambda)\mu\mathbf{E}\eta\mathbf{E}\varkappa = 1, \quad \sigma^2 = \mu\left[\mathbf{E}\varkappa(\varkappa-1) + \frac{\mathbf{E}\varkappa\mathbf{E}\eta^2}{(1-\lambda)(\mathbf{E}\eta)^2}\right] < \infty$

the following limiting equality holds as $B \to \infty$, $r > 0$

$$\mathbf{P}[D_x(tB^2) \geq [uB], D_x^-(tB^2) \geq [-rB]] \to \frac{1}{\sigma\sqrt{2\pi t}}\int_u^{u+2r} e^{-v^2/2\sigma^2 t}dv,$$

where $[a]$ *is integer part of the number* a, $u \in [-r,\infty)$.

Proof. In accordance with the total probability law and the Markov property of $\tau_r(x)$ for all $r \in \mathbb{Z}^+$ we can write the following equation

$$\mathbf{E}z^{D_x(\nu_s)} = \mathbf{E}\left[z^{D_x(\nu_s)}; D_x^-(\nu_s) \geq -r\right] + f_r(x,s)\frac{1-\lambda}{z-\lambda}\,z^{-r}\mathbf{E}z^{D_0(\nu_s)}, \quad |z| = 1.$$

To write this equation, we used the path decomposition principle. It means that the increments of the process $D_x(\nu_s)$ on the interval $[0,\nu_s]$ are realized either on the sample paths which do not cross the lower level $-r$, or on the sample paths which do intersect the level $-r$, and further evolution of the process is its probabilistic replica on $[0,\nu_s]$. From the latter equation we find that

$$E_r^-(x,z,s) = \mathbb{D}_x^s(z) - f_r(x,s)\frac{1-\lambda}{z-\lambda}\,z^{-r}\mathbb{D}_0^s(z).$$

Observe, that the function which enters the left-hand side of the equation is analytic in $|z| \le 1$. Therefore, the right-hand side is also an analytic function for all $|z| \le 1$. Employing (3.2.8) and the definition of resolvent (3.3.4), we get

$$E_r^-(x,z,s) = (1-z)\mathbb{A}_x^z(s) + (1-z) f_r(x,s)\frac{1-\lambda}{\lambda - z}\,\mathbb{A}_0^z(s) z^{-r}, \quad |z| \le 1.$$

It is not difficult to establish the following equality

$$\sum_{u=-r}^{\infty} z^u \mathbb{E}_r^-(x,u,s) = \frac{1-f_r(x,s)}{1-z}\,z^{-r} - \frac{z}{1-z}\,E_r^-(x,z,s), \quad |z| \le 1.$$

It follows from this and the previous equality that

$$\sum_{u=-r}^{\infty} z^u \mathbb{E}_r^-(x,u,s) = \frac{1-f_r(x,s)}{1-z}\,z^{-r} - z\mathbb{A}_x^z(s) - f_r(x,s)\frac{1-\lambda}{\lambda - z}\,\mathbb{A}_0^z(s) z^{-r+1}.$$

Comparing the coefficients of z^u, $u \in [-r,\infty[$ and taking into account that $\mathbb{A}_x^\lambda(s) = 0$, we obtain (3.5.14).

Denote $\tilde{e}_r^t(x,u,B) = \mathbf{P}\left[D_x(tB^2) \ge [uB], D_x^-(tB^2) \ge [-rB]\right]$, $r > 0$, $u \ge -r$. It is clear that

$$\lim_{B\to\infty} \int_0^\infty e^{-st}\tilde{e}_k^t(x,u,B)\,dt = \frac{1}{s}\lim_{B\to\infty} \mathbb{E}_{[kB]}^{s/B^2}(x,[uB]).$$

Employing the limiting equalities (3.5.7) and also (3.5.14), we find that

$$\lim_{B\to\infty} \int_0^\infty e^{-st}\tilde{e}_k^t(x,u,B)\,dt = s^{-1}\mathbf{I}_{\{u>0\}}\left(e^{-u\sqrt{2s}/\sigma}/2 - e^{-(2r+u)\sqrt{2s}/\sigma}/2\right) +$$
$$+ s^{-1}\mathbf{I}_{\{u\in[-r,0]\}}\left(1 - e^{u\sqrt{2s}/\sigma}/2 - e^{-(2r+u)\sqrt{2s}/\sigma}/2\right), \quad u \ge -r.$$

Denote by $\{w_t;\ t \ge 0\}$ the symmetric Wiener process with dispersion σ and by $\tau^a = \inf\{t : w_t \ge a\}$ the first passage time of the level $a \in \mathbb{R}_+$. The Lévy identity $\mathbf{P}[\tau \le t] = 2\mathbf{P}[w_t \ge a]$ implies the following relation for the Laplace transforms:

$$\frac{1}{s}\,e^{-a\sqrt{2s}/\sigma} = 2\int_0^\infty e^{-st}\mathbf{P}[w_t \ge a]\,dt.$$

Using this formula to invert the Laplace transforms in the previous equality, we obtain the limiting equality of the theorem. ▲

3.6 Asymptotic results

In this section we will prove weak convergence of the distribution of the first exit time for the difference of the compound Poisson process and the compound renewal process to the corresponding distribution of the Wiener process under certain conditions. Here and in the sequel we will assume that $\delta \sim ge(\lambda)$, and that the following conditions are satisfied:

(A) $\rho = (1-\lambda)\mu \mathbf{E}\eta \mathbf{E}\varkappa = 1, \quad \sigma^2 = \mu \left[\mathbf{E}\varkappa(\varkappa - 1) + \frac{\mathbf{E}\varkappa \mathbf{E}\eta^2}{(1-\lambda)(\mathbf{E}\eta)^2} \right] < \infty.$

We first state some auxiliary results and then formulate and prove the theorem. In the sequel we will require the following expansions:

$$\tilde{f}_x(s) = 1 - s\mathbf{E}\eta_x + \frac{1}{2}s^2 \mathbf{E}\eta_x^2 + o(s^2), \quad x \in \mathbb{R}_+,$$
$$\mathbf{E}e^{-p\varkappa} = 1 - p\mathbf{E}\varkappa + \frac{1}{2}p^2\mathbf{E}\varkappa^2 + o(p^2), \tag{3.6.1}$$

which are valid for small s, p.

Lemma 3.6.1. *Let $x, k \in \mathbb{R}_+$, $s > 0$. The following limiting equalities hold:*

$$\lim_{B\to\infty} \mathbf{E}e^{-s\tau_{[kB]}(x)/B^2} = \lim_{B\to\infty} \mathbf{E}e^{-s\tau^{[kB]}(x)/B^2} = e^{-k\sqrt{2s}/\sigma}, \tag{3.6.2}$$

where $[a]$ stands for the integer part of the number a;

$$\lim_{B\to\infty} \frac{1}{B} Q^{s/B^2}_{[kB]}(x) = \lim_{B\to\infty} \frac{1}{B}\mathbf{E}Q^{s/B^2}_{[kB]+\delta}(x) = \frac{1}{\mathbf{E}\eta}\frac{2}{\sigma\sqrt{2s}}\sinh(k\sqrt{2s}/\sigma). \tag{3.6.3}$$

Proof. It follows from (3.6.1) and (3.2.14) that the following representation is valid for $c(s/B^2)$ as $B \to \infty$

$$c(s/B^2) = 1 - \frac{1}{B}\sqrt{2s}/\sigma + o\left(\frac{1}{B}\right). \tag{3.6.4}$$

Formula (3.3.2) and this asymptotic equality imply the first part of (3.6.2). We now focus on the asymptotic properties of the resolvent sequence $\{Q^s_k(x)\}_{k\in\mathbb{Z}^+}$. It follows from the definition (3.3.4) for $\theta = e^{-p}$, $p > -ln\, c(s)$ that

$$\mathbb{Q}^s_\theta(x) = \sum_{k\in\mathbb{Z}^+} \theta^k Q^s_k(x)\Big|_{\theta=e^{-p}} = \int_0^\infty e^{-p[k]} Q^s_{[k]}(x)\, dk$$
$$= \int_0^\infty e^{\{k\}p} e^{-pk} Q^s_{[k]}(x)\, dk = \mathbb{Q}^s_{e^{-p}}(x), \quad p > -ln\, c(s),$$

where $\{a\} = a - [a]$ is the fractional part of the number a. It is clear that

$$\mathfrak{Q}_p^s(x) \le \mathbb{Q}_{e^{-p}}^s(x) \le e^p \mathfrak{Q}_p^s(x), \tag{3.6.5}$$

where $\mathfrak{Q}_p^s(x) = \int_0^\infty e^{-pk} Q_{[k]}^s(x)\, dk, \quad p > -ln\, c(s)$ is the Laplace transform of the function $Q_{[k]}^s(x)$, $k \in \mathbb{R}_+$. The definition (3.3.5) and (3.6.1) imply for $p > \sqrt{2s}/\sigma$ that

$$\begin{aligned}\lim_{B\to\infty} \frac{1}{B^2} \mathbb{Q}_{e^{-p/B}}^{s/B^2}(x) &= \lim_{B\to\infty} \frac{1}{B^2} \frac{(1-\lambda)\tilde{f}_x(s/B^2 - k(e^{-p/B}))}{(1-\lambda)\tilde{f}(s/B^2 - k(e^{-p/B})) + \lambda - e^{-p/B}} \\ &= \frac{1}{\mathbf{E}\eta} \frac{1}{\frac{1}{2}p^2\sigma^2 - s}, \qquad p > \sqrt{2s}/\sigma.\end{aligned}$$

It follows from the chain (3.6.5) that

$$\lim_{B\to\infty} \frac{1}{B^2} \mathfrak{Q}_{p/B}^{s/B^2}(x) = \lim_{B\to\infty} \frac{1}{B^2} \mathbb{Q}_{e^{-p/B}}^{s/B^2}(x) = \frac{1}{\mathbf{E}\eta} \frac{1}{\frac{1}{2}p^2\sigma^2 - s}, \quad p > \sqrt{2s}/\sigma.$$

Inverting the Laplace transforms in both sides, we obtain

$$\lim_{B\to\infty} \frac{1}{B} Q_{[kB]}^{s/B^2}(x) = \frac{1}{\mathbf{E}\eta} \frac{2}{\sigma\sqrt{2s}} \sinh(k\sqrt{2s}/\sigma),$$

i.e. the first part of (3.6.3). It is not difficult to derive the following representation:

$$\tilde{Q}_\theta^s = \sum_{k\in\mathbb{Z}^+} \theta^k \mathbf{E} Q_{k+\delta}^s = \frac{1-\lambda}{(1-\lambda)\tilde{f}(s - k(c(s))) + \lambda - \theta}, \quad \theta \in (0, c(s)).$$

The latter equality and (3.6.1) imply that for $p > \sqrt{2s}/\sigma$

$$\lim_{B\to\infty} \frac{1}{B^2} \tilde{Q}_{e^{-p/B}}^{s/B^2} = \lim_{B\to\infty} \frac{1}{B^2} \int_0^\infty e^{-p[k]/B} \mathbf{E} Q_{[k]+\delta}^{s/B^2}\, dk = \frac{1}{\mathbf{E}\eta} \frac{1}{\frac{1}{2}p^2\sigma^2 - s}.$$

This asymptotic equality implies the second part of (3.6.3). We now establish the second part of (3.6.2). Taking into account formulae (3.3.10), (3.6.1), we derive

$$\lim_{B\to\infty} \frac{1}{B} \Phi_{e^{-p/B}}^{s/B^2}(x) = \lim_{B\to\infty} \frac{1}{B} \int_0^\infty e^{-p[k]/B} \mathbf{E} e^{-s\tau^{[k]}(x)/B^2}\, dk = \frac{1}{p + \sqrt{2s}/\sigma}.$$

It is obvious that $\tilde{\Phi}^s_p(x) \le \Phi^s_{e^{-p}}(x) \le e^p \tilde{\Phi}^s_p(x)$, where

$$\tilde{\Phi}^s_p(x) = \int_0^\infty e^{-pk} \mathbf{E} e^{-s\tau^{[k]}(x)}\, dk \quad p > -ln\, c(s)$$

is the Laplace transform of the function $\mathbf{E}e^{-s\tau^{[k]}(x)}$, $k \in \mathbb{R}_+$. Hence,

$$\lim_{B\to\infty} \frac{1}{B} \tilde{\Phi}^{s/B^2}_{p/B}(x) = \lim_{B\to\infty} \frac{1}{B} \Phi^{s/B^2}_{e^{-p/B}}(x)$$

and

$$\lim_{B\to\infty} \frac{1}{B} \int\limits_0^\infty e^{\frac{-pk}{B}} \mathbf{E} e^{-s\tau^{[k]}(x)/B^2}\, dk = \lim_{B\to\infty} \int\limits_0^\infty e^{-pk} \mathbf{E} e^{-s\tau^{[kB]}(x)/B^2}\, dk = \frac{1}{p + \sqrt{2s}/\sigma}.$$

The latter equality implies the second part of (3.6.2). ▲

Denote by $\{w_t;\, t \ge 0\}$ a standard Wiener process, $\mathbf{E}[w_1] = 0$, $\mathbf{Var}[w_1] = \sigma^2 > 0$, and let

$$\chi^* = \inf\{t :\, w_t \notin (-r, k)\}, \qquad k \in (0,1), \quad r = 1-k,$$

denote the first exit time from the interval $(-r, k)$ by the process w_t. It is well-known (see for instance Ito and McKean (1965)) that the Laplace transforms of χ^* are such that

$$\mathbf{E}\left[e^{-s\chi^*}; A^k\right] = \frac{\sinh\left(r\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)}, \qquad \mathbf{E}\left[e^{-s\chi^*}; A_r\right] = \frac{\sinh\left(k\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)},$$

where $A^k = \{w_{\chi^*} = k\}$, $A_r = \{w_{\chi^*} = -r\}$ are the events denoting the exit from the interval $(-r, k)$ through the upper boundary k and through the lower boundary $-r$.

Theorem 3.6.1 (Kadankov *et al.* (2009))**.** *Let $\{D_x(t)\}_{t\ge 0}$ be the difference of the compound Poisson process and the compound renewal process (3.2.3), $\delta \sim ge(\lambda)$. Assume that the conditions (A) are satisfied, and*

$$\chi(B) = \inf\{t : D_x(t) \notin [-rB, kB]\}, \qquad k \in (0,1), \quad r = 1-k, \quad B \in \mathbb{R}_+,$$

$\mathfrak{A}^k(B) = \{D_x(\chi(B)) > kB\}$, $\mathfrak{A}_r(B) = \{D_x(\chi(B)) < -rB\}$. *Then the following limiting equalities hold for* $B \to \infty$

$$\mathbf{P}\left[\frac{\chi(B)}{B^2} \in dt; \mathfrak{A}^k(B)\right] \to \mathbf{P}\left[\chi^* \in dt; A^k\right] = \pi\sigma^2 \sum_{n\in\mathbb{N}} ne^{-\frac{t}{2}(\sigma\pi n)^2} \sin(k\pi n)\, dt,$$
$$\mathbf{P}\left[\frac{\chi(B)}{B^2} \in dt; \mathfrak{A}_r(B)\right] \to \mathbf{P}\left[\chi^* \in dt; A_r\right] = \pi\sigma^2 \sum_{n\in\mathbb{N}} ne^{-\frac{t}{2}(\sigma\pi\nu)^2} \sin(r\pi n)\, dt. \tag{3.6.6}$$

The limiting exit probabilities admit the following representations for $B \to \infty$

$$\mathbf{P}\left[\mathfrak{A}^k(B)\right] \to \frac{2}{\pi}\sum_{n\in\mathbb{N}} \frac{\sin(k\pi n)}{n} = r, \quad \mathbf{P}\left[\mathfrak{A}_r(B)\right] \to \frac{2}{\pi}\sum_{n\in\mathbb{N}} \frac{\sin(r\pi n)}{n} = k.$$

Proof. The first formula of (3.4.8) and (3.6.3) imply that

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}_r(B)\right] = \lim_{B\to\infty} \frac{Q^{s/B^2}_{[kB]}(x)}{\mathbf{E}Q^{s/B^2}_{[B]+\delta}} = \frac{\sinh\left(k\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)} = \mathbf{E}\left[e^{-s\chi^*}; A_r\right].$$

Inverting the Laplace transform in the right-hand side of this equality, we derive the first equality of (3.6.6). Taking into account the definition of the function $A^s_k(x)$ (3.4.9), we have

$$\mathbb{A}^s_\theta(x) = \sum_{k\in\mathbb{Z}^+} \theta^k A^s_k(x) = \frac{s}{s-k(\theta)}\left(\frac{\tilde{f}_x(s-k(\theta))}{(1-\lambda)\tilde{f}(s-k(\theta))+\lambda-\theta} - \frac{1}{1-\theta}\right),$$

where $\theta \in (0, c(s))$. The latter equality and (3.6.1) imply for $p > \sqrt{2s}/\sigma$ that

$$\lim_{B\to\infty} \frac{1}{B}\mathbb{A}^{s/B^2}_{e^{-p/B}}(x) = \lim_{B\to\infty} \frac{1}{B}\int_0^\infty e^{\{k\}p/B} e^{-pk/B} A^{s/B^2}_{[k]}(x)\, dk = \frac{1}{p}\,\frac{s}{\frac{1}{2}p^2\sigma^2 - s}.$$

It is clear that $\mathfrak{A}^s_p(x) \le \mathbb{A}^s_{e^{-p}}(x) \le e^p\mathfrak{A}^s_p(x)$, where $\mathfrak{A}^s_p(x) = \int_0^\infty e^{-kp} A^s_{[k]}(x)\, dk$ is the Laplace transform of the function $A^s_{[k]}(x)$, $k \in \mathbb{R}_+$. Hence, $\lim\limits_{B\to\infty} \frac{1}{B}\mathfrak{A}^{s/B^2}_{p/B}(x) = \lim\limits_{B\to\infty} \frac{1}{B}\mathbb{A}^{s/B^2}_{e^{-p/B}}(x)$ and, thus,

$$\lim_{B\to\infty} \frac{1}{B}\mathfrak{A}^{s/B^2}_{p/B}(x) = \lim_{B\to\infty} \int_0^\infty e^{-kp} A^{s/B^2}_{[kB]}(x)\, dk = \frac{1}{p}\,\frac{s}{\frac{1}{2}p^2\sigma^2 - s}.$$

Inverting the Laplace transforms in both sides, we get

$$\lim_{B\to\infty} A_{[kB]}^{s/B^2}(x) = \cosh\left(k\sqrt{2s}/\sigma\right) - 1, \quad p > \sqrt{2s}/\sigma.$$

Analogously we derive that for all $k \in \mathbb{R}_+$

$$\lim_{B\to\infty} \mathbf{E} A_{[kB]+\delta}^{s/B^2}(0) = \cosh\left(k\sqrt{2s}/\sigma\right) - 1, \quad p > \sqrt{2s}/\sigma,$$

where $\mathbf{E}\, A_{[u]+\delta}^{s} = \sum\limits_{i\in\mathbb{N}} (1-\lambda)\lambda^{i-1} A_{[u]+i}^{s}(0)$. It follows from the second formula of (3.4.9) and from the two latter asymptotic equations that

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}^k(B)\right] = \cosh\left(k\sqrt{2s}/\sigma\right) - \frac{\sinh\left(k\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)}\cosh\left(\sqrt{2s}/\sigma\right)$$
$$= \frac{\sinh\left(r\sqrt{2s}/\sigma\right)}{\sinh\left(\sqrt{2s}/\sigma\right)} = \mathbf{E}\left[e^{-s\chi^*}; A^k\right].$$

Inverting the Laplace transforms in both sides, we obtain the second equality of (3.6.6). It is worth noting that by means of (3.6.6) we established the weak convergence of $\chi(B)/B^2$ to χ^* as $\to \infty$.

Remark 3.6.1. *It is worth noting that all the characteristics considered in this chapter were determined for another class of stochastic processes, namely for a semi-Markov work with a linear drift. The approach and methodology are similar to the present case, therefore we do not present these results in this framework, but refer to the corresponding articles:*

Kadankov, V.F., Kadankova, T. (2007). Two-boundary problems fora semi-Markov walk with a linear drift. Random Oper. and Stoch. Equ. **15**, *223-251.*

Kadankov, V., Kadankova, T. (2008). Intersections of the interval and reflections for a semi-Markov walk with linear drift. Random Oper. and Stoch. Equ. (submitted).

3.7 Reflections from the boundary generated by the supremum

This and the next section are concerned with the processes reflected at the boundaries. Our motivation to consider such processes comes from applications in inventory and queueing theory (see Chapter 4). We will determine one-boundary characteristics of these processes and also the distribution of their increments. We now move to the formal definition of reflections.

Denote by $D_x^r(t) = r + D_x(t)$, $t \geq 0$ the process starting from $r \in \mathbb{Z}$ when $\eta_x^+(0) = x \geq 0$. Let $B \in \mathbb{Z}^+$ and for all $t \geq 0$ we define a right-continuous process reflected at the boundary B as follows:

$$\overline{D}_r^B(x,t) = D_x^r(t) - \max\left\{0, \sup_{[0,t]} D_x^r(\cdot) - B\right\} \in]-\infty, B], \quad r \in]-\infty, B]. \tag{3.7.1}$$

The first reflection from the upper boundary B of the process $\overline{D}_r^B(x,t)$ takes place at time $\tau^{B-r}(x)$. Then the process stays at this boundary for some random time η_l, where $l = \eta_x^+(\tau^{B-r}(x))$. At the instant $t = \tau^{B-r}(x) + \eta_l$ the process is reflected to a random state $B - \delta$. In the sequel the evolution of the process $\overline{D}_r^B(x,t)$ is a probabilistic copy of its evolution on $[0, \tau^{B-r}(x) + \eta_l)$. It is worth noting that reflections from the boundaries reflected by infimum (supremum) were introduced by Lévy for a standard Wiener process. Applying the symmetry principle and the mirror reflection principle, Lévy determined the distributions of the boundary functionals of the reflected standard Wiener process. We will show that these distributions are the limit distributions for the reflected process after an appropriate scaling of time and space.

The reflected spectrally one-sided Lévy processes generated by the infimum (supremum) of the process were considered in Avram *et al.* (2004), Nguyen-Ngoc and Yor (2005). An interesting application in queueing theory for the spectrally one-sided Lévy process reflected by its infimum was given in Bekker *et al.* (2008). Note, that boundary functionals were studied in Lotov and Khodzhibaev (1998b) for the reflected Lévy process generated by its infimum (supremum). It is worth noting that in this article the asymptotic expansions for the distributions of the characteristics of the process were determined for the reflected Lévy processes obeying the two-boundary Cramer's conditions.

3.7.1 Passage of the lower boundary

We now define the boundary functionals for the process (3.7.1). For $r \in [0, B]$ denote

$$\overline{\tau}_r^B(x) = \inf\{t : \overline{D}_r^B(x,t) < 0\} \stackrel{\text{def}}{=} \overline{\tau}, \qquad \overline{T}_r^B(x) = -\overline{D}_r^B(\overline{\tau}) \stackrel{\text{def}}{=} \overline{T}, \quad r \in [0, B]$$

the first crossing time of the lower level 0 by the process $\overline{D}_r^B(x,t)$ and the value of the overshoot at this instant. The following statement is true.

Theorem 3.7.1. *Let $\{\overline{D}_r^B(x,t)\}_{t\geq 0}$ be the reflected processes defined by (3.7.1), $B \in \mathbb{Z}^+$, $r \in [0, B]$ and*

$$V^k(x, dl, m, s) = \mathbf{E}\left[e^{-s\chi}; L \in dl, T = m, \mathfrak{A}^k\right], \; V_r(x, m, s) = \mathbf{E}\left[e^{-s\chi}; T = m, \mathfrak{A}_r\right]$$

be the Laplace transforms of the joint distribution of $\{\chi_r^B(x), L, T\}$ of the process $\{D_x(t)\}_{t\geq 0}$ given by Theorem 3.4.1. Then

(i) *if $\delta \in \mathbb{N}$ (an arbitrarily distributed non-negative variable), then the Laplace transform of the joint distribution of $\{\overline{\tau}, \overline{T}\}$ is such that $(m \in \mathbb{N})$*

$$\overline{v}_r^s(x, m) = \mathbf{E}\left[e^{-s\overline{\tau}_r^B(x)}; \overline{T} = m\right] = V_r(x, m, s) + \frac{a^{B-r}(x)}{1 - A(0)}\left[\mathbf{P}[\delta = m + B] + \sum_{i=1}^{B}\mathbf{P}[\delta = i]V_{B-i}(0, m, s)\right], \qquad (3.7.2)$$

where $V^k(x, dl, s) = \sum_{m=1}^{\infty} V^k(x, dl, m, s)$,

$$a^k(x) = \int_0^{\infty} V^k(x, dl, s)\tilde{f}_l(s), \qquad A(x) = \sum_{k=1}^{B}\mathbf{P}\left[\delta = k\right] a^k(x);$$

(ii) *if $\delta \sim ge(\lambda)$, then the following equalities hold:*

$$\overline{v}_r^s(x, m) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{B-r-1}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E}\, S_{\delta+B-1}^s}(1 - \lambda)\lambda^{m-1}, \quad r \in [0, B], \qquad (3.7.3)$$

where $S_k^s(x) = \sum_{i=0}^{k} Q_i^s(x)$, $\mathbf{E}\, S_{\delta+B-1}^s = (1-\lambda)\sum_{i=1}^{\infty}\lambda^{i-1}S_{i-1+B}^s(0)$; *the random variable* $\overline{\tau}_r^B(x)$ *is proper* ($\mathbf{P}\left[\overline{\tau}_r^B(x) < \infty\right] = 1$) *and*

$$\mathbf{E}\overline{\tau}_r^B(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta\left[\mathbf{E}S_{\delta+B-1} - S_{B-r-1}(x)\right] < \infty, \qquad (3.7.4)$$

where $S_k(x) = S_k^0(x)$, $\mathbf{E}S_{\delta+B} = \mathbf{E}S_{\delta+B}^0$;

(iii) *under the conditions* (A) *the following equality is valid*

$$\lim_{B\to\infty}\mathbf{E}e^{-s\overline{\tau}_{[rB]}^B(x)/B^2} = \frac{\cosh\left(k\sqrt{2s}/\sigma\right)}{\cosh\left(\sqrt{2s}/\sigma\right)}, \quad r\in(0,1),\ k = 1-r.$$

Proof. Let us verify the formula (3.7.2). It follows from the definition of the process $\overline{D}_r^B(x,t)$ (3.7.1), the total probability law and the Markov property of χ, $\eta_n(x)$ that the following system of linear integral equations holds:

$$\overline{v}_r^s(x,m) = V_r(x,m,s) + \int_0^\infty V^k(x,dl,s)\overline{v}_B^s(l,m),$$

$$\overline{v}_B^s(x,m) = \tilde{f}_x(s)\mathbf{P}[\delta = m+B] + \tilde{f}_x(s)\sum_{r=1}^{B}\mathbf{P}[\delta = r]\overline{v}_{B-r}^s(0,m).$$

This system is similar to a system of linear equations with two unknowns, and it can be solved analogously. Substituting the expression for $\overline{v}_B^s(x,m)$ from the second equation into the first one, we find that

$$\overline{v}_r^s(x,m) = V_r(x,m,s) + a^k(x)\mathbf{P}[\delta = m+B] + a^k(x)\sum_{r=1}^{B}\mathbf{P}[\delta = r]\overline{v}_{B-r}^s(0,m).$$

Letting $x = 0$ in the latter equation yields after calculations

$$\sum_{r=1}^{B}\mathbf{P}[\delta = r]\overline{v}_{B-r}^s(0,m) = -\mathbf{P}[\delta = m+B]$$
$$+\left[\mathbf{P}[\delta = m+B] + \sum_{k=1}^{B}\mathbf{P}[\delta = k]V_{B-k}(0,du,s)\right](1-A(0))^{-1}.$$

Inserting the right-hand side of this quality in the previous one, we get (3.7.2). In case when $\delta\sim ge(\lambda)$, formula (3.7.2) takes a more simple form. The first formula of (3.4.8) and (3.7.2) imply that $\overline{T}_r^B(x)\sim ge(\lambda)$ for any $r\in\overline{0,B}$.

Summing over $m \in \mathbb{N}$ in both sides of (3.7.2), we find for the function $\overline{v}_r^s(x) = \mathbf{E}e^{-s\overline{\tau}_r^B(x)}$ that

$$\overline{v}_r^s(x) = V_r(x,s) + \frac{a^{B-r}(x)}{1-A(0)}\left[\lambda^k + (1-\lambda)\sum_{i=1}^{B}\lambda^{i-1}V_{B-i}(0,s)\right]. \qquad (3.7.5)$$

Now we calculate $a^k(x)$, $A(0)$ in case when $\delta \sim ge(\lambda)$. Employing formulae (3.3.6), (3.4.8) and performing the necessary calculations, we find that

$$a^k(x) = \tilde{f}_x(s) + (1-\tilde{f}(s))S_{k-1}^s(x) - \frac{Q_k^s(x)}{\mathbf{E}\,Q_{\delta+B}^s}\left[\tilde{f}(s) + (1-\tilde{f}(s))\mathbf{E}S_{\delta+B-1}^s\right],$$

$$1 - A(0) = \frac{1-\lambda}{\lambda\mathbf{E}\,Q_{\delta+B}^s}\left[1 - Q_0^s(0)\right]\left[\tilde{f}(s) + (1-\tilde{f}(s))\mathbf{E}S_{\delta+B-1}^s\right],$$

$$\lambda^k + (1-\lambda)\sum_{i=1}^{B}\lambda^{i-1}V_{B-i}(0,s) = \frac{1-\lambda}{\lambda\mathbf{E}\,Q_{\delta+B}^s}\left[1 - Q_0^s(0)\right].$$

Substituting the right-hand sides of these equalities into (3.7.5) and taking into account that $\overline{T}_r^B(x) \sim ge(\lambda)$, we derive (3.7.3). The equality (3.7.4) follows from the following relation $\mathbf{E}\overline{\tau}_r^B(x) = -\left.\frac{d}{ds}\overline{v}_r^s(x)\right|_{s=0}$.

To verify the limiting formula of the theorem, we will employ equalities (3.5.7) which were obtained in Kadankov *et al.* (2009). On can also derive that for all $k > 0$

$$\lim_{B\to\infty} B^{-2}\,S_{[kB]}^{s/B^2}(x) = \frac{1}{s\mathbf{E}\eta}\left(\cosh\left(k\sqrt{2s}/\sigma\right) - 1\right) = \lim_{B\to\infty} B^{-2}\,\mathbf{E}S_{[kB]+\delta}^{s/B^2}. \qquad (3.7.6)$$

Letting $B \to \infty$, we have $\tilde{f}_x(s/B^2) = 1 - \mathbf{E}\eta_x s/B^2 + o(s/B^2)$, which implies for $r \in (0,1)$ that

$$\lim_{B\to\infty}\mathbf{E}e^{-s\overline{\tau}_{[rB]}^B(x)/B^2} = \frac{1 + \left(\cosh\left(k\sqrt{2s}/\sigma\right) - 1\right)}{1 + \left(\cosh\left(\sqrt{2s}/\sigma\right) - 1\right)} = \frac{\cosh\left(k\sqrt{2s}/\sigma\right)}{\cosh\left(\sqrt{2s}/\sigma\right)}, \quad k = 1 - r.$$

▲

3.7.2 Increments of the process reflected in its supremum

Define $\overline{D}_0^k(x,t) = D_x(t) - \max\left\{0, \sup_{[0,t]} D_x(\cdot) - k\right\} \in]-\infty, k]$, the process reflected from the upper boundary $k \in \mathbb{Z}^+$ generated by its supremum.

Theorem 3.7.2. *Let $\{\overline{D}_0^k(x,t)\}_{t\geq 0}$ be the process reflected from the upper boundary and $\overline{p}_k^s(x,u) = \mathbf{P}\left[\overline{D}_0^k(x,\nu_s) \leq u\right]$, $u \in]-\infty, k]$ be the distribution of its increments on the exponential interval $[0,\nu_s]$. Then*

(i) *for all $k \in \mathbb{Z}^+$ $\overline{p}_k^s(x,k) = 1$, and for $u \in]-\infty, k-1]$, $x \geq 0$ the following equality holds*

$$\overline{p}_k^s(x,u) = A_x^u(s) + c(s)^{k-u-1} F(s) \left(\frac{\tilde{f}_x(s)}{1-\tilde{f}(s)} + S_{k-1}^s(x) \right), \qquad (3.7.7)$$

where $A_x^u(s) = 0$, for $u < 0$, $F(s) = s(1-c(s))/(1-\lambda)(s-k(c(s)))$;

(ii) *under the condition (A) for $k > 0$, $u \leq k$ the following relation is valid:*

$$\lim_{B\to\infty} \mathbf{P}\left[\overline{D}_0^{[kB]}(x,tB^2) \leq [uB]\right] = 1 - \frac{1}{\sigma\sqrt{2\pi t}} \int_u^{2k-u} e^{-v^2/2\sigma^2 t} dv; \qquad (3.7.8)$$

(iii) *if $\rho > 1$, then the ergodic distribution $p_k(u) = \lim_{t\to\infty} \mathbf{P}\left[\overline{D}_0^k(x,t) \leq u\right]$ exists, and*

$$p_k(u) = \frac{\mathbf{E}\varkappa}{\rho} \frac{1-c}{1-\mathbf{E}c^{\varkappa}} c^{k-u-1}, \quad u \in]-\infty, k-1], \qquad c = \lim_{s\to 0} c(s) \in (\lambda, 1).$$

Proof. Define the generating function of the increments of the process

$$\overline{P}_k^s(x,z) = \mathbf{E} z^{\overline{D}_0^k(x,\nu_s)} = \sum_{-\infty}^{k} z^i \mathbf{P}\left[\overline{D}_0^k(x,\nu_s) = i\right], \quad |z| \geq 1, \ k \in \mathbb{Z}^+.$$

It is obvious that $\overline{P}_k^s(x,1) = \mathbf{P}\left[\overline{D}_0^k(x,\nu_s) \leq k\right] = 1$. In accordance with the total probability law and the definition of the process $\overline{D}_0^k(x,t)$ we can write

$$\overline{P}_k^s(x,z) = E_k^+(x,z,s) + z^k \int_0^\infty f_x^k(dl)(1-\tilde{f}_l(s)) + z^k \int_0^\infty f_x^k(dl)\tilde{f}_l(s)(1-\lambda) \sum_{i=1}^\infty \lambda^{i-1} z^{-i} \overline{P}_i^s(0,z), \qquad (3.7.9)$$

where the function $E_k^+(x,z,s) = \mathbf{E}\left[e^{-zD_x(\nu_s)}; \tau^k(x) > \nu_s\right]$ is given by (3.5.3), and the function $f_x^k(dl) = \mathbf{E}\left[e^{-s\tau^k(x)}; \eta^k(x) \in dl\right]$ is determined by (3.3.6).

This equation means the following. The sample paths on which the increments of the process $D_x(t)$ occur can be decomposed into three parts: 1) the sample paths which do not intersect the upper boundary k (the first term on the right-hand side); 2) the sample paths which do intersect the upper boundary and stay there (the second term); 3) the sample paths cross the upper boundary and then they are reflected (the third term). After some calculations which we skip, one can see that formula (3.3.6) implies that

$$\int_0^\infty f_x^k(dl)\tilde{f}_l(s) = \tilde{f}_x(s) + (1-\tilde{f}(s))S_k^s(x) - \frac{1-\tilde{f}(s)}{1-c(s)}\, Q_k^s(x).$$

Letting $x = 0$ in (3.7.9) and taking into account the latter equality, we derive

$$(1-\lambda)\sum_{i=1}^\infty \lambda^{i-1}z^{-i}\overline{P}_i^s(0,z) = \frac{F(s)}{1-\tilde{f}(s)}\,\frac{1/z-1}{1-c(s)/z}, \qquad |z| \geq 1.$$

Substituting the right-hand of this equality and the expression (3.5.3) for $E_k^+(x,z,s)$ into (3.7.9), we find that ($|z| \geq 1$)

$$\overline{P}_k^s(x,z) = z^k + (1-z)\sum_{i=0}^{k-1} z^i A_x^i(s) + z^k F(s)\frac{1/z-1}{1-c(s)/z}\left(\frac{\tilde{f}_x(s)}{1-\tilde{f}(s)} + S_{k-1}^s(x)\right).$$

Comparing the coefficients of z^i, $i \in \{k, k-1, \dots\}$, we get

$$\mathbf{P}\left[\overline{D}_0^k(x,\nu_s) = k\right] = 1 - A_x^{k-1}(s) - F(s)\left(\frac{\tilde{f}_x(s)}{1-\tilde{f}(s)} + S_{k-1}^s(x)\right),$$

$$\mathbf{P}\left[\overline{D}_0^k(x,\nu_s) = i\right] = A_x^i(s) - A_x^{i-1}(s) +$$

$$+ F(s)c(s)^{k-i-1}(1-c(s))\left(\frac{\tilde{f}_x(s)}{1-\tilde{f}(s)} + S_{k-1}^s(x)\right), \qquad i < k.$$

One can assure that the second formula implies the equality (3.7.7) of the theorem. Let us verify (3.7.8). It follows from the first formula of (3.5.7) that

$$F(s/B^2) = \frac{s}{B^2}\,\mathbf{E}\eta + o(B^{-2}), \qquad \lim_{B\to\infty} c(s/B^2)^{[B(k-u)]-1} = e^{-(k-u)\sqrt{2s}/\sigma}.$$

Denote $\tilde{p}_k^t(x,u,B) = \mathbf{P}\left[\overline{D}_0^{[kB]}(x,tB^2) \leq [uB]\right]$, $k > 0$. Then in view of (3.5.7), (3.7.6) we have:

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\tilde{p}_k^t(x,u,B)\,dt = \frac{1}{s}\lim_{B\to\infty}\overline{p}_{[kB]}^{s/B^2}(x,[uB]) =$$
$$= \frac{1}{s}\left(1-\cosh\left(u^+\sqrt{2s}/\sigma\right)+e^{-(k-u)\sqrt{2s}/\sigma}\cosh\left(k\sqrt{2s}/\sigma\right)\right) =$$
$$= \frac{1}{s}\,\mathbf{I}_{\{u<0\}}\left(e^{u\sqrt{2s}/\sigma}/2+e^{-(2k-u)\sqrt{2s}/\sigma}/2\right)+$$
$$+\frac{1}{s}\,\mathbf{I}_{\{u\in[0,k]\}}\left(1-e^{-u\sqrt{2s}/\sigma}/2+e^{-(2k-u)\sqrt{2s}/\sigma}/2\right),\quad u\le k,$$

where $u^+=\max\{0,u\}$. Employing the formula $\frac{1}{s}\,e^{-a\sqrt{2s}/\sigma}=2\int_0^\infty e^{-st}\mathbf{P}\left[w_t\ge a\right]dt$, to invert the Laplace transform, we derive the limiting equality of the theorem.

For $\rho>1$ the mathematical expectation of $\tau^k(x)$ is finite. It follows from (3.3.7) that

$$\mathbf{E}\tau^k(x)=\frac{Q_k(x)}{(1-\lambda)k(c)}+\sum_{i=0}^{k}\rho_i\left[1-\frac{Q_{k-i}(x)}{1-\lambda}\right]<\infty,$$

where $\rho_i=\lim_{s\to0}s^{-1}\tilde{\rho}_i(s)=\int_0^\infty\mathbf{P}\left[\pi(t)=i\right]dt<\infty$. Moreover, the process $\overline{D}_0^k(x,t)$ is of regenerative type (Kuznetsov *et al.* (1983)). The instants of the passages of the upper boundary are the regeneration times. Hence, there exists the ergodic distribution of the process (Kuznetsov *et al.* (1983)) $p_k(u)=\lim_{t\to\infty}\mathbf{P}\left[\overline{D}_0^k(x,t)\le u\right]$. To determine this distribution, it suffices to apply to (3.7.7) the Tauberian theorem: $p_k(u)=\lim_{s\to0}\overline{p}_k^s(x,u)$. ▲

We will now determine the joint distribution of the increments of the process and the first exit time from the interval $[-r,k]$. Let $\{\overline{D}_0^k(x,t)\}_{t\ge0}$ be the process reflected from the upper boundary. Define for $r,k\in\mathbb{Z}^+$

$$\overline{\tau}_{r,k}(x)=\inf\{t:\overline{D}_0^k(x,t)<-r\}\stackrel{\text{def}}{=}\overline{\tau},\qquad \overline{T}_{r,k}(x)=-\overline{D}_0^k(x,\overline{\tau})-r\stackrel{\text{def}}{=}\overline{T},$$

the first exit time from the interval $[-r,k]$ by the process $\overline{D}_0^k(x,t)$ and the value of the overshoot through the lower boundary $-r$. Since X_t is homogeneous with respect to the first component, then $\{\overline{\tau}_{r,k}(x),\overline{T}_{r,k}(x)\}$ are identically distributed as $\{\overline{\tau}_r^B(x),\overline{T}_r^B(x)\}$, $B=k+r$, and their joint distribution is determined by (3.7.3).

Theorem 3.7.3. *Let* $\{\overline{D}_0^k(x,t)\}_{t\ge 0}$ *be the process reflected from the upper boundary,* $\overline{p}_{r,k}^s(x,u) = \mathbf{P}\left[\overline{D}_0^k(x,\nu_s) \le u;\ \overline{\tau}_{r,k}(x) > \nu_s\right]$, $u \in [-r,k]$ *be the distribution of the increments of the process on the interval* $[0,\nu_s]$ *on the event* $\{\overline{\tau}_{r,k}(x) > \nu_s\}$.

(i) *the distribution of the increments is such that for all* $r,k \in \mathbb{Z}^+$

$$\overline{p}_{r,k}^s(x,k) = 1 - \frac{\tilde{f}_x(s) + (1-\tilde{f}(s))S_{k-1}^s(x)}{\tilde{f}(s) + (1-\tilde{f}(s))\mathbf{E}\,S_{\delta+B-1}^s}, \quad B = k + r, \tag{3.7.10}$$

$$\overline{p}_{r,k}^s(x,u) = A_x^u(s) - \frac{\tilde{f}_x(s) + (1-\tilde{f}(s))S_{k-1}^s(x)}{\tilde{f}(s) + (1-\tilde{f}(s))\mathbf{E}\,S_{\delta+B-1}^s}\,\mathbf{E}A_0^{\delta+u+r}(s),\ u \in [-r,k-1];$$

(ii) *under the condition* (A) *the following limiting equality holds:*

$$\lim_{B\to\infty}\mathbf{P}\left[\overline{D}_0^{[kB]}(x,tB^2) \le [uB];\ \overline{\tau}_{[rB],[kB]}(x) > tB^2\right] \stackrel{\text{def}}{=} p(t) = \tag{3.7.11}$$

$$= \frac{4}{\pi}\sum_{n\in\mathbb{Z}^+}\frac{e^{-\frac{t}{2}(\pi(n+\frac{1}{2})\sigma)^2}}{n+\frac{1}{2}}\sin\left(r\left(n+\frac{1}{2}\right)\pi\right)\sin^2\left(\frac{r+u}{2}\left(n+\frac{1}{2}\right)\pi\right),$$

where $r \in (0,1),\ k = 1-r,\ u \in [-r,k]$.

Proof. In accordance with the total probability law, homogeneity of the process X_t with respect to the first component, Markov property of $\overline{\tau}_{r,k}(x)$ and the properties of the exponential variable ν_s we can write

$$\overline{p}_k^s(x,u) = \overline{p}_{r,k}^s(x,u) + \overline{v}_r^s(x)(1-\lambda)\sum_{i=1}^{\infty}\lambda^{i-1}\overline{p}_{i+B}^s(0,u+r+i), \quad u \in]-\infty,k],$$

where the function $\overline{p}_k^s(x,u) = \mathbf{P}\left[\overline{D}_0^k(x,\nu_s) \le u\right]$ is determined in Theorem 3.7.2. This equation means that the increments of the process $\overline{D}_0^k(x,\nu_s)$ are realized either on the sample paths which do not exit the interval $[-r,k]$, or on the sample paths which do exit the interval and the further evaluation of the process is its probabilistic replica on $[0,\nu_s]$. Substituting the expression for the function $\overline{p}_k^s(x,u)$ into (3.7.7), after necessary calculations we derive (3.7.10). For $r \in (0,1),\ k = 1-r,\ u \in [-r,k]$ denote

$$\overline{p}_{r,k}^t(x,u,B) = \mathbf{P}\left[\overline{D}_0^{[kB]}(x,tB^2) \le [uB];\ \overline{\tau}_{[rB],[kB]}(x) > tB^2\right].$$

Employing the third formula of (3.5.7), the limiting equality of Theorem 3.7.1, we find

$$\frac{1}{s}\lim_{B\to\infty}\overline{p}^{s/B^2}_{[kB]}(x,[uB]) = \lim_{B\to\infty}\int_0^\infty e^{-st}\overline{p}^{\,t}_{r,k}(x,u,B)\,dt = \tag{3.7.12}$$

$$= \frac{1-\cosh(u^+\sqrt{2s}/\sigma)}{s} + \frac{1}{s}\,\frac{\cosh(k\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}\Big(\cosh((u+r)\sqrt{2s}/\sigma)-1\Big) \stackrel{\text{def}}{=} p^*(s),$$

where $u^+ = \max\{0,u\}$. When $u \in [-r,0]$ we derive from this formula that

$$p^*(s) = \frac{2}{s}\,\frac{\cosh(k\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}\sinh^2\left(\frac{r+u}{2}\sqrt{2s}/\sigma\right), \qquad u\in[-r,0].$$

It is clear that $s=0$ is not a singular point (pole or point of branching) of the function $p^*(s)$. In the semi-plane $\Re(s)<0$ this function has simple poles in

$$s_n = -\frac{1}{2}\sigma^2\pi^2\left(n+\frac{1}{2}\right)^2, \qquad n\in\mathbb{Z}^+,$$

and it is analytic in the whole plane apart from these points. Hence, for $\alpha>0$

$$p(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}p^*(s)\,ds = \sum_{n\in\mathbb{Z}^+}\operatorname{Res}_{s=s_n}p^*(s).$$

Calculating the residues of the function $p^*(s)$ in s_n, we obtain the right-hand side of the formula (3.7.11) for $u\in[-r,0]$. On can see that the first term in the right-hand side of (3.7.12) is analytic in the whole plane for $u\in(0,k]$. Applying the inversion formula, we find that the contour integral of this term is equal to zero. The second term of (3.7.12) is the same also for $u\in[-r,0]$. Thus, the formula (3.7.11) holds for $u\in[-r,k]$. ▲

3.8 Reflections from the boundary generated by the infimum

Denote by $D^r_x(t) = r + D_x(t)$, $t\ge 0$ the process starting from $r\in\mathbb{Z}$ when $\eta^+_x(0) = x \ge 0$. Let $r\in\mathbb{Z}^+$, and for all $t\ge0$ we define a right-continuous process reflected at the boundary 0 as follows:

$$\underline{D}^r_0(x,t) = D^r_x(t) - \min\left\{0, \inf_{[0,t]} D^r_x(\cdot)\right\} \in \mathbb{Z}^+, \qquad \underline{D}^0_r(x,0) = r. \tag{3.8.1}$$

The process $D^r_x(t)$ is reflected from the lower boundary 0 by its infimum Observe, that the first hitting of the boundary 0 by the process $\underline{D}^r_0(x,t)$ occurs at the instant $\tau_r(x)$. Subsequent time periods between the hitting times are identically distributed as $\tau_0(0)$.

3.8.1 Passage of the upper boundary

We will now determine boundary functionals for the process (3.8.1). For $B \in \mathbb{Z}^+$, $r \in [0,B]$ denote

$$\underline{\tau}^B_r(x) = \inf\{t : \underline{D}^r_0(x,t) > B\} \stackrel{\text{def}}{=} \underline{\tau}, \qquad \underline{T}^k_r(x) = \underline{D}^0_r(\underline{\tau}) - B, \quad \underline{L}^k_r(x) = \eta^+_x(\underline{\tau})$$

the first crossing time of the upper level B by the process $\underline{D}^r_0(x,t)$, the value of the overshoot and the value of the linear component at this instant.

Theorem 3.8.1. *Let $\underline{D}^r_0(x,t)_{\{t\geq 0\}}$ be the process reflected at its infimum (3.8.1). Then*

(i) *the Laplace transform $\underline{v}^r_x(dl,m,s) = \mathbf{E}\left[e^{-s\underline{\tau}^B_r(x)}; \underline{L} \in dl, \underline{T} = m\right]$ of the joint distribution of $\{\underline{\tau}^B_r(x), \underline{L}, \underline{T}\}$ satisfies the following equality for $s > 0$*

$$\underline{v}^r_x(dl,m,s) = V^k(x,dl,m,s) + \frac{V_r(x,s)}{1 - V_0(0,s)} V^B(0,dl,m,s), \qquad (3.8.2)$$

where $k = B-r$, $V_r(x,s) = \sum\limits_{m\in\mathbb{N}}^{\infty} V_r(x,m,s)$, and the functions $V^k(x,dl,m,s)$, $V_r(x,m,s)$ are determined by (3.4.8); in particular

$$\underline{v}^r_x(s) = \mathbf{E} e^{-s\underline{\tau}^B_r(x)} = 1 - A^k_x(s) + Q^s_k(x) \frac{\mathbf{E} A^{\delta+B}_0(s) - A^B_0(s)}{\mathbf{E}\, Q^s_{\delta+B} - Q^s_B}; \qquad (3.8.3)$$

(ii) *under the condition (A) the following equality is valid:*

$$\lim_{B\to\infty} \mathbf{P}\left[\underline{\tau}^B_{rB}(x)/B^2 > t\right] = \frac{4}{\pi} \sum_{n\in\mathbb{Z}^+} \frac{e^{-\frac{t}{2}(\pi(n+\frac{1}{2})\sigma)^2}}{2n+1} \sin\left(k\,(2n+1)\,\pi/2\right),$$

where $r \in (0,1)$, $k = 1-r$;

(iii) *the random variable* $\underline{\tau}_r^B(x)$ *is proper* ($\mathbf{P}\left[\underline{\tau}_r^B(x) < \infty\right] = 1$), *and*

$$\mathbf{E}\underline{\tau}_r^B(x) = A_x^k + Q_k(x)\frac{\mathbf{E}A_0^{\delta+B} - A_0^B}{\mathbf{E}\,Q_{\delta+B} - Q_B(0)} < \infty,$$

where $Q_k(x) = Q_k^0(x)$, $\mathbf{E}\,Q_{\delta+B} = \mathbf{E}\,Q_{\delta+B}^0$,

$$A_x^k = \sum_{i=0}^{k} \rho_i^* \left[1 - (1-\lambda)^{-1} Q_{k-i}(x)\right], \quad \rho_i^* = \int_0^\infty \mathbf{P}\left[\pi(t) = i\right] dt.$$

Let us verify formula (3.8.2). It follows from the definition of the process $\underline{D}_0^r(x,t)$ (3.8.1), the total probability law and the Markov property of χ that the following equation is valid:

$$\underline{v}_x^r(dl, m, s) = V^k(x, dl, m, s) + V_r(x, s)\underline{v}_0^0(dl, m, s).$$

Letting $x = r = 0$ in this equation, we find that

$$\underline{v}_0^0(dl, m, s) = V^B(0, dl, m, s)\left(1 - V_0(0, s)\right)^{-1}.$$

Substituting the expression for the function $\overline{v}_0^0(dl, du, s)$ into the previous equation, we get formula (3.8.2). Formula (3.8.3) follows from (3.8.2) and (3.4.9).

Lemma 3.8.1. *Under the condition* (A) *and* $k > 0$ *the following limiting equalities hold:*

$$\begin{aligned}
&\lim_{B\to\infty} B^{-2} S_{[kB]}^{s/B^2}(x) = \frac{1}{s\mathbf{E}\eta}\left(\cosh\left(k\sqrt{2s}/\sigma\right) - 1\right) = \lim_{B\to\infty} B^{-2}\mathbf{E}S_{\delta+[kB]}^{s/B^2},\\
&\lim_{B\to\infty}\left[\mathbf{E}Q_{[\delta+kB]}^{s/B^2} - Q_{[kB]}^{s/B^2}\right] = \frac{2\mu\mathbf{E}\varkappa}{\sigma^2}\cosh\left(k\sqrt{2s}/\sigma\right), \qquad (3.8.4)\\
&\lim_{B\to\infty} B\left[A_0^{[kB]}(s/B^2) - \mathbf{E}A_0^{[\delta+kB]}(s/B^2)\right] = \frac{\sqrt{2s}}{(1-\lambda)\sigma}\sinh\left(k\sqrt{2s}/\sigma\right).
\end{aligned}$$

Proof of the lemma can be found in Appendix. Formulae (3.8.3), (3.5.7) and the latter equalities imply for $B \to \infty$, $k \in (0,1)$, $r = 1-k$, that

$$\begin{aligned}
&\mathbf{E}e^{-s\,\underline{\tau}_{rB}^B(x)/B^2} \to \cosh\left(k\sqrt{2s}/\sigma\right) -\\
&-\frac{2\sinh\left(k\sqrt{2s}/\sigma\right)}{\sigma\sqrt{2s}\mathbf{E}\eta}\frac{\sqrt{2s}}{(1-\lambda)\sigma}\sinh\left(\sqrt{2s}/\sigma\right)\frac{\sigma^2}{2\mu\varkappa}/\cosh\left(\sqrt{2s}/\sigma\right) = \frac{\cosh\left(r\sqrt{2s}/\sigma\right)}{\cosh\left(\sqrt{2s}/\sigma\right)}.
\end{aligned}$$

It is obvious that

$$\int_0^\infty e^{-st} \lim_{B\to\infty} \mathbf{P}\left[\underline{\tau}^B_{rB}(x)/B^2 > t\right] dt = \frac{1}{s} - \frac{1}{s}\,\frac{\cosh\left(r\sqrt{2s}/\sigma\right)}{\cosh\left(\sqrt{2s}/\sigma\right)}.$$

Inverting the Laplace transforms in both sides of the latter equality, we get the limiting equality of the theorem.

3.8.2 Increments of the process reflected in its infimum

Let $r \in \mathbb{Z}^+$, and denote by $\underline{D}^0_{-r}(x,t) = D_x(t) - \min\left\{0, \inf_{[0,t]} D_x(\cdot) + r\right\} \in [-r,\infty[$, the process reflected from the lower boundary $-r$. We remind that the reflections are generated by its infimum. Introduce

$$\underline{P}^s_r(x,z) = \mathbf{E} z^{\underline{D}^0_{-r}(x,\nu_s)},\ |z| \le 1, \quad \underline{p}^s_r(x,u) = \mathbf{P}\left[\underline{D}^0_{-r}(x,\nu_s) \ge u\right],\ u \in [-r,\infty[$$

the generating function of the increments of the process on the exponential time interval $[0,\nu_s]$. The following statement holds.

Theorem 3.8.2. *Let $\{\underline{D}^0_{-r}(x,t)\}_{t\ge0}$ be the process reflected from the lower boundary. Then*

(i) *the following equalities are valid for $r \in \mathbb{Z}^+$, $x \ge 0$, $u \in [-r,\infty[$*

$$\underline{P}^s_r(x,z) = (1-z)\left[\mathbb{A}^z_x(s) + z^{-r} f_r(x,s)\frac{1-\lambda}{1-c(s)}\,\frac{1-z}{\lambda - z}\,\mathbb{A}^z_0(s)\right], \quad (3.8.5)$$

$$\underline{p}^s_r(x,u) = 1 - A^{u-1}_x(s) - \frac{1-\lambda}{1-c(s)}\, f_r(x,s)\left[A^{u+r-1}_0(s) - \mathbf{E}A^{\delta+u+r-1}_0(s)\right],$$

where $A^u_x(s) = 0$, for $u < 0$;

(ii) *Under the condition (A), for $r > 0$, $u \ge -r$*

$$\lim_{B\to\infty} \mathbf{P}\left[\underline{D}^0_{[-rB]}(x,tB^2) \ge [uB]\right] = 1 - \frac{1}{\sigma\sqrt{2\pi t}}\int_{-u}^{u+2r} e^{-v^2/2\sigma^2 t} dv; \quad (3.8.6)$$

(iii) *if* $\rho = (1-\lambda)\mu\mathbf{E}\eta\mathbf{E}\varkappa < 1$, *then the ergodic distribution exists* $\underline{p}_r(u) = \lim_{t\to\infty} \mathbf{P}\left[\underline{D}^0_{-r}(x,t) \geq u\right]$, *and the limiting equality holds:*

$$\underline{p}_r(u) = 1 - \frac{1-\rho}{\mathbf{E}\eta}\left[A_0^{u+r-1} - \mathbf{E}A_0^{\delta+u+r-1}\right],$$

where $A_x^k = \sum_{i=0}^{k} \rho_i^*\left[1-(1-\lambda)^{-1}Q_{k-i}(x)\right], \quad \rho_i^* = \int_0^\infty \mathbf{P}\left[\pi(t)=i\right]dt.$

Proof. In view of the total probability law, the Markov property of $\tau_r(x)$, homogeneity property of the process with respect to the first component, we can write for the function $\underline{P}^s_r(x,z)$ the following equation:

$$\underline{P}^s_r(x,z) = E^-_r(x,z,s) + \frac{sf_r(x,s)z^{-r}}{s+\mu} + \frac{\mu f_r(x,s)z^{-r}}{1-\tilde{f}(s+\mu)}\int_0^\infty \sum_{i\in\mathbb{N}} a_i(dl)\underline{P}^s_i(l,z)z^i,$$

$$a_i(dl) = e^{-l(s+\mu)}[1-F(l)]\mathbf{P}\left[\varkappa = i\right]dl, \tag{3.8.7}$$

where the generating function $E^-_r(x,z,s) = \mathbf{E}\left[z^{D_x(\nu_s)}; D^-_x(\nu_s) \geq -r\right]$, $|z|\leq 1$ is determined by (3.5.13).

This equation reflects the following fact. The increments of the process $\{\underline{D}^0_{-r}(x,t)\}$ can take place either on the sample paths which do not hit the lower boundary $-r$, (the first term on the right-hand side) or on the sample paths which hit the lower boundary $-r$ and stay there (the second term) or, finally, on the sample paths which hit the boundary $-r$, and then they are reflected from the boundary (the third term). Denote

$$X(s,z) = \frac{\mu}{1-\tilde{f}(s+\mu)}\int_0^\infty \sum_{i\in\mathbb{N}} a_i(dl)\underline{P}^s_i(l,z)z^i.$$

Setting $x=0$ in (3.8.7) and performing necessary calculations, we get

$$X(s,z) = \frac{1-\lambda}{1-c(s)}\left[\mu\int_0^\infty \sum_{r\in\mathbb{N}} a_r(dx)E^-_r(x,z,s)z^r + s\,\frac{1-\tilde{f}(s+\mu)}{s+\mu}\right] - \frac{s}{s+\mu}.$$

Inserting this expression for the function $X(s,z)$ into (3.8.7) yields

$$\underline{P}^s_r(x,z) = E^-_r(x,z,s) + \\ + f_r(x,s)z^{-r}\frac{1-\lambda}{1-c(s)}\left[\mu\int_0^\infty \sum_{r\in\mathbb{N}} a_r(dx)E^-_r(x,z,s)z^r + s\,\frac{1-\tilde{f}(s+\mu)}{s+\mu}\right].$$

Inserting the expression (3.5.13) for the function $E_r^-(x,z,s)$ into the latter equality and performing necessary calculations, we find that

$$\underline{P}_r^s(x,z) = (1-z)\left[\mathbb{A}_x^z(s) + z^{-r} f_r(x,s)\frac{1-\lambda}{1-c(s)}\,\frac{1-z}{\lambda - z}\,\mathbb{A}_0^z(s)\right].$$

It is not difficult to derive the following relation:

$$\hat{p}_r^s(x,z) = \sum_{u=-r}^{\infty} z^u \underline{p}_r^s(x,u) = \frac{z^{-r}}{1-z} - \frac{z}{1-z}\,\underline{P}_r^s(x,z), \quad |z| \le 1.$$

This relation and the previous equality imply that

$$\hat{p}_r^s(x,z) = \frac{z^{-r}}{1-z} - z\mathbb{A}_x^z(s) - z^{-r+1} f_r(x,s)\frac{1-\lambda}{1-c(s)}\,\frac{1-z}{\lambda - z}\,\mathbb{A}_0^z(s).$$

Comparing the coefficients of z^u, $u[-r,\infty[$, we get the second formula of (3.8.5). We now verify (3.8.6). The first formula of (3.8.5) and (3.3.2) imply that

$$\lim_{B\to\infty} f_{[rB]}(x, s/B^2) = e^{-r\sqrt{2s}/\sigma}, \qquad r > 0.$$

Denote $\tilde{p}_r^t(x,u,B) = \mathbf{P}\left[\underline{D}^0_{[-rB]}(x, tB^2) \ge [uB]\right]$, $r > 0$, $u \ge -r$. It is obvious that

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\tilde{p}_k^t(x,u,B)\,dt = \frac{1}{s}\lim_{B\to\infty}\underline{p}_{[rB]}^{s/B^2}(x,[uB]).$$

Employing (3.5.7) and Lemma 3.8.1, we obtain

$$\lim_{B\to\infty}\int_0^\infty e^{-st}\tilde{p}_k^t(x,u,B)\,dt = s^{-1}\mathbf{I}_{\{u>0\}}\left(e^{-u\sqrt{2s}/\sigma}/2 + e^{-(2r+u)\sqrt{2s}/\sigma}/2\right) +$$
$$+\, s^{-1}\mathbf{I}_{\{u\in[-r,0]\}}\left(1 - e^{u\sqrt{2s}/\sigma}/2 + e^{-(2r+u)\sqrt{2s}/\sigma}/2\right), \quad u \ge -r.$$

In view of the relation $s^{-1}\,e^{-a\sqrt{2s}/\sigma} = 2\int_0^\infty e^{-st}\mathbf{P}\left[w_t \ge a\right] dt$ we invert the Laplace transforms in the right-hand side of the latter equality, which yields the limiting equality of the theorem. For $\rho < 1$ mathematical expectation of $\tau_r(x)$ is finite. It follows from (3.3.2) that

$$\mathbf{E}\tau_r(x) = \left[\mathbf{E}\eta_x + r(1-\lambda)\mathbf{E}\eta\right](1-\rho)^{-1} < \infty.$$

Moreover, the process $\underline{D}^0_{-r}(x,t)$ is of regenerative type (Kuznetsov *et al.* (1983)). The instants of the passages of the lower boundary are the regeneration times. Hence, there exists the ergodic distribution of the process

(Kuznetsov *et al.* (1983)) $p_k(u) = \lim_{t\to\infty} \mathbf{P}\left[\underline{D}_0^k(x,t) \le u\right]$. To determine this distribution, it suffices to apply the Tauberian theorem to formula (3.8.5). Here we denoted $p_k(u) = \lim_{s\to 0} \underline{p}_k^s(x,u)$. ▲

Let $\{\underline{D}^0_{-r}(x,t)\}_{t\ge 0}$ be the process reflected from the lower boundary $-r$. Introduce the following random variables

$$\underline{\tau}_{r,k}(x) = \inf\{t : \underline{D}^0_{-r}(x,t) > k\} = \underline{\tau},\ \underline{T}_{r,k}(x) = \underline{D}^0_{-r}(x,\underline{\tau}) - k,\ \underline{L}_{r,k}(x) = \eta_x^+(\underline{\tau})$$

i.e. the first exit time from the interval $[-r,k]$, by the process $\underline{D}^0_{-r}(x,t)$, the value of the overshoot through the upper boundary k and the value of the linear component at this instant, $r,k \in \mathbb{Z}^+$.. Since the process X_t is homogeneous with respect to the first component, then the random variables $\{\underline{\tau}_{r,k}(x), \underline{T}_{r,k}(x), \underline{L}_{r,k}(x)\}$ are identically distributed as $\{\underline{\tau}_r^B(x), \underline{T}_r^B(x), \underline{L}_r(x)\}$, $B = k + r$ and, hence, their joint distribution is given by (3.8.2).

Theorem 3.8.3. *Let $\{\underline{D}^0_{-r}(x,t)\}_{t\ge 0}$ be the process reflected from the lower boundary $-r$. Denote by $\underline{p}^s_{r,k}(x,u) = \mathbf{P}\left[\underline{D}^0_{-r}(x,\nu_s) \le u;\ \underline{\tau}_{r,k}(x) > \nu_s\right]$, $u \in [-r,k]$ the distribution of the increments of the process on the exponential interval $[0,\nu_s]$, $s > 0$ on the event $\{\underline{\tau}_{r,k}(x) > \nu_s\}$. Then*

(i) *the following equality is valid for $r,k \in \mathbb{Z}^+$, $u \in [-r,k]$*

$$\underline{p}^s_{r,k}(x,u) = A_x^u(s) - Q_k^s(x)\,\frac{\mathbf{E}A_0^{\delta+u+r}(s) - A_0^{u+r}(s)}{\mathbf{E}Q^s_{B+\delta} - Q^s_B}; \tag{3.8.8}$$

(ii) *under the condition (A) the limiting equality holds:*

$$\lim_{B\to\infty} \mathbf{P}\left[\underline{D}_x^{[kB]}(tB^2) \le [uB];\ \underline{\tau}_{[rB],[kB]}(x) > tB^2\right] \stackrel{\text{def}}{=} p(t) = \tag{3.8.9}$$
$$= \frac{4}{\pi} \sum_{n\in\mathbb{Z}^+} \frac{e^{-\frac{t}{2}(\pi(n+\frac{1}{2})\sigma)^2}}{2n+1} \sin\left((r+u)\left(n+\frac{1}{2}\right)\pi\right) \cos\left(r\left(n+\frac{1}{2}\right)\pi\right),$$

where $r \in (0,1)$, $k = 1 - r$, $u \in [-r,k]$.

Proof. Introduce the generating function

$$\underline{P}^s_{r,k}(x,z) = \mathbf{E}\left[z^{\underline{D}^0_{-r}(x,\nu_s)};\ \underline{\tau}_{r,k}(x) > \nu_s\right].$$

According to the total probability law, the homogeneity property of the process X_t with respect to the first component, Markov property of $\underline{\tau}_{r,k}(x)$, properties of the exponential variable ν_s we can write the following equation:

$$\underline{P}^s_r(x,z) = \underline{P}^s_{r,k}(x,z) + \int_0^\infty \sum_{m\in\mathbb{N}} \underline{v}^r_x(dl,m,s)\underline{P}^s_{m+B}(l,z)z^{m+k}, \quad |z| \le 1,$$

where $\underline{v}^r_x(dl,m,s) = \mathbf{E}\left[e^{-s\underline{\tau}^B_r(x)}; \underline{L} \in dl, \underline{T} = m\right]$ is determined by (3.8.2), and $\underline{P}^s_r(x,z) = \mathbf{E}\left[z^{\underline{D}^0_{-r}(x,\nu_s)}\right]$ is found in Theorem 3.8.2 by (3.8.5). This equation is written using the path decomposition principle. The increments of the process $\underline{D}^0_{-r}(x,\nu_s)$ can be realized on one of the following self-excluding events: 1) the sample paths do not cross the upper boundary k, 2) the sample paths do intersect the upper boundary and then the evolution of the process is just a probabilistic replica of the process on $[0,\nu_s]$. Inserting the expression (3.8.5) for the function $\underline{P}^s_r(x,z)$ into the latter equation, we find that

$$\begin{aligned}\frac{1}{1-z}\,\underline{P}^s_{r,k}(x,z) &= \mathbb{A}^z_x(s) - z^k \int_0^\infty \tilde{v}^r_x(dl,z,s)\mathbb{A}^z_l(s) + \\ &+ z^{-r}\mathbb{A}^z_0(s)\frac{1-\lambda}{1-c(s)}\frac{1-z}{\lambda-z}\left[f_r(x,s) - \sum_{m\in\mathbb{N}}\int_0^\infty \underline{v}^r_x(dl,m,s)f_{m+B}(l,s)\right],\end{aligned} \tag{3.8.10}$$

where $\tilde{v}^r_x(dl,z,s) = \mathbf{E}\left[e^{-s\underline{\tau}^B_r(x)}z^{\underline{T}}; \underline{L} \in dl\right]$. Performing necessary calculations and taking into account (3.3.2) and (3.8.2) yields

$$\sum_{m\in\mathbb{N}}\int_0^\infty \underline{v}^r_x(dl,m,s)f_{m+B}(l,s) = f_r(x,s) - \frac{1-c(s)}{1-\lambda}\,\frac{Q^s_k(x)}{\mathbf{E}Q^s_{B+\delta} - Q^s_B}.$$

In view of this equality and (3.8.10), we find that

$$\begin{aligned}\frac{\underline{P}^s_{r,k}(x,z) - z^{k+1}(1-\underline{v}^r_x(s))}{1-z} &= \mathbb{A}^z_x(s) - z^k\int_0^\infty \tilde{v}^r_x(dl,z,s)\mathbb{A}^z_l(s) - \\ - \frac{z^{k+1}(1-\underline{v}^r_x(s))}{1-z} &+ z^{-r}\mathbb{A}^z_0(s)\frac{1-z}{\lambda-z}\,\frac{Q^s_k(x)}{\mathbf{E}Q^s_{B+\delta} - Q^s_B}, \quad |z|\le 1,\end{aligned} \tag{3.8.11}$$

where $\tilde{v}^r_x(dl,z,s) = \mathbf{E}\left[e^{-s\underline{\tau}^B_r(x)}z^{\underline{T}}; \underline{L}\in dl,\right]$. Then it is easily verified that

$$\frac{1}{1-z}\,\underline{P}^s_{r,k}(x,z) - \frac{z^{k+1}}{1-z}(1-\underline{v}^r_x(s)) = \sum_{u=-r}^{k} z^u \underline{p}^s_{r,k}(x,u).$$

Comparing the coefficients of z^u, $u\in[-r,k]$ in both sides of (3.8.11), we obtain (3.8.8). For $r\in(0,1)$, $k=1-r$, $u\in[-r,k]$ denote

$$\overline{p}^t_{r,k}(x,u,B) = \mathbf{P}\left[\underline{D}^{[kB]}_x(tB^2)\le[uB];\ \underline{\tau}_{[rB],[kB]}(x) > tB^2\right].$$

Employing formula (3.8.8) and the limiting equality of Lemma 3.8.1, we find that

$$\frac{1}{s}\lim_{B\to\infty}\underline{p}^{s/B^2}_{[rB],[kB]}(x,[uB]) = \lim_{B\to\infty}\int_0^\infty e^{-st}\overline{p}^t_{r,k}(x,u,B)\,dt = \tag{3.8.12}$$
$$= \frac{1-\cosh(u^+\sqrt{2s}/\sigma)}{s} + \frac{1}{s}\,\frac{\sinh(k\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}\,\sinh\left((u+r)\sqrt{2s}/\sigma\right) \stackrel{\text{def}}{=} p^*(s),$$

where $u^+=\max\{0,u\}$. When $u\in[-r,0]$ we derive from this formula that

$$p^*(s) = \frac{1}{s}\,\frac{\sinh(k\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}\,\sinh\left((u+r)\sqrt{2s}/\sigma\right), \qquad u\in[-r,0].$$

It is clear that $s=0$ is not a singular point (pole or point of branching) of the function $p^*(s)$. In the semi-plane $\Re(s)<0$ this function has simple poles in

$$s_n = -\frac{1}{2}\,\sigma^2\pi^2\left(n+\frac{1}{2}\right)^2, \qquad n\in\mathbb{Z}^+,$$

and it is analytic in the whole plane apart from these points. Hence, for $\alpha>0$

$$p(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}p^*(s)\,ds = \sum_{n\in\mathbb{Z}^+}\mathrm{Res}_{s=s_n}p^*(s).$$

Calculating the residues of the function $p^*(s)$ in s_n, we obtain the right-hand side of the formula (3.8.9) for $u\in[-r,0]$. On can see that the first term on the right-hand side of (3.8.12) is analytic in the whole plane for $u\in(0,k]$. Applying the inversion formula, we find that the contour integral of this term is equal to zero. The second term of (3.8.12) is the same also for $u\in[-r,0]$. Thus, the formula (3.8.9) holds for $u\in[-r,k]$. ▲

Chapter 4

Applications for queueing systems

4.1 Introduction

In this chapter we illustrate the applications of our results in queueing theory. Namely, we will study the queueing system with batch arrivals and finite buffer. In the $M^{\varkappa}|G^{\delta}|1|B$ system, for instance, customers arrive in batches of random size $\varkappa$ according to a Poisson process. Service time η has general distribution and during the service cycle $\min\{\delta, r\}$ customers are served, where r is an initial number of the customers in the buffer. We consider partial rejection, meaning that if an overflow of buffer occurs due to the arrival of a batch of customers, the amount of work brought by this batch is only partially admitted to the buffer, up to the limit of the free buffer space just before the arrival. The rest is rejected and therefore, is lost.

In many telecommunication systems, it is frequently observed that the server processes the packets in groups of random size. For example, in ATM (Asynchronous Transfer Mode) networks with multiple input links where each link may serve messages that consist of several packets. Besides applications in telecommunication systems, batch service queues have a wide range of applications in several areas including transportation systems and automatic manufacturing systems. In order to mathematically investigate the packet loss and delay properties of a telecommunication network or a component of such a network, one needs to study the characteristics of an appropriate queueing system which express such phenomena.

One of the crucial performance issues of the single-server queue with finite buffer (waiting room) is losses, namely, customers (packets, cells, jobs) that were not allowed to enter the system due to the buffer overflow. This issue is especially important in the analysis of telecommunication networks. Therefore, our target is to determine the most important performance measures of the queueing systems with batch arrivals and finite buffer. More precisely, we consider the $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ and $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ queueing systems (see a rigorous description of such systems below) and their modifications. Such systems also serve as adequate models to study loss sales, cash management, transmission of traffic, internet servers networks, telecommunications (packet losses, packet delays), etc. A queueing model of a communication network, for instance, typically includes one or more sources which send packets (customers) into the network (service station); these packets are then transmitted (served) over a network link if the link is free, or stored temporarily in a buffer memory (queue) if the link is busy transmitting another packet. Given such a queueing model, the question is how to evaluate performance measures of interest, such as the buffer overflow (or packet loss) probability, number of lost customers, etc.

The evolution of the number of customers in the aforementioned systems is described by a process with two reflecting boundaries. In the general case this process is a difference of two renewal processes. Reflections from the upper boundary are generated by the supremum (infimum) of the process. Reflections from the lower boundary govern the server's behavior.

In general such processes are not Markovian, but by adding a complementary linear component (in some literature called age process), we obtain a Markov process, which describes the functioning of the queueing system. Studying the main characteristics of the system results in investigating the two-boundary functionals of the governing process. For the queueing systems of $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$,

$G^{\delta}|M^{\varkappa}|1|B$ type the governing process is the difference of the compound Poisson process and the compound renewal process complemented with the linear component.

Before introducing the model, we give a brief review of related results. Analysis of the single-server queues with finite waiting room goes back to the articles of Truslove (1975), Ohsone (1981) and others. A huge amount of literature exists on the study of the single-server queue with all its variants. It is worth mentioning that the $M^X|G|1$ queue with finite waiting room was studied for both the partially rejected model and the totally rejected model in Baba (1984). The author found the asymptotic distribution of the number of customers at an arbitrary moment and immediately after a departure. Finite dams with Poisson arrivals and the $M|G|1$ queue with impatient customers were studied in Lee *et al.* (2001).

First passage times of the level by Lévy processes in context of queues were considered in Dube *et al.* (2004), where the explicit characterization of the Laplace transform of the busy period distribution was found for a finite capacity $M|G|1$ queue, see also Perry *et al.* (2000). In regard to finding the buffer overflow time, the closest results are presented in Asmussen *et al.* (2002). The authors considered the system where arrivals are modeled by a Markov modulated Poisson process (MMPP) and service time is exponential. Previous works on the overflow period were concentrated on simple Poisson arrivals (De Boer *et al.* (2001), Chydzinski (2004)), batch Poisson arrivals Chydzinski (2006), or renewal arrivals (Fakinos (1982) Pacheco and Ribeiro (2008)). See also Chaudhry and Zhao (1994) for a discrete Geom(n)|Geom(n)|1|N model. Recently, Chydzinski (2007b) carried out the study on the distribution of the first buffer overflow time in a queue with batch Markovian arrival process (BMAP), general service distribution, and finite buffer of size b ($BMAP|G|1|b$). The main result represents the explicit formula for the Laplace transform of the distribution of the first buffer overflow time.

In recent years there has been a great interest in analyzing various queueing models with MAP (Markov Arrival Process) as input process or MSP (Markov service process). MAP is used to represent correlated traffic arising in modern telecommunication networks. In systems with Markov arrival or service processes (MAP, BMAP, or BMSP) and their modifications, it is common to use the supplementary variable methods and/or embedded Markov chains. For the method of supplementary variable we refer, for instance, to Choi *et al.* (1998), where the authors considered the $MAP|G|1$ queueing system with infinite capacity. They derived the double transform of the queue length and the remaining service time of the customer in service in the steady state. See also Gupta and Vijaya Laxmi (2001), where the distributions of the number of customers in the $MAP|G^{a,b}|1|N$ queue at arbitrary, post-departure and pre-arrival epochs have been obtained using both the supplementary variable and the embedded Markov chain techniques. Gupta and Banik (2006) derived explicit analytic expressions for the steady-state length distribution of the $GI|MSP|1$ queue with finite as well as infinite buffer. For the embedded Markov chains techniques we refer to Dukhovny (1996), where the generating function of the steady-state probabilities of the chain for the bulk systems of $GI|M|1$ type was found. De Boer *et al.* (2001) studied the stationary distribution of the remaining service time upon reaching some target level in an $M|G|1$ system. The asymptotic analysis of the $G|MSP|1|r$ queue has been carried out by Bocharov *et al.* (2003). See also Chaplygin (2003) for the stationary analysis of the $GI|BMSP|1$ queue and Kim and Kim (2007) for the asymptotic behavior of the loss probability as the buffer size tends to infinity. For the $MAP|G|1|b$ system Chydzinski (2007a) found distribution of the overflow period in transient and stationary regimes and the distribution of the number of cells lost during the overflow interval. Recently Banik *et al.* (2008) considered a finite-buffer single-server queue $GI|BMSP|1|N$. The steady-state distribution of number of customers in the system at pre-arrival and arbitrary epochs has been obtained.

As one can see, the majority of the recent literature is devoted mainly to queue size and workload, most of the times in the steady state case. However, recently it was shown that steady-state parameters do not reflect the reality. A detailed discussion of the drawbacks of steady-state parameters when used as a quality of service criteria in telecommunication networks may be found in Schwefel *et al.* (2001). To our knowledge there are no results employing two-boundary characteristics of the aforementioned governing process to determine main performance measures. Thus, our contribution is that we apply direct probability methods and supplementary variable approach to determine several transient measure performances. More specific, we find the Laplace transforms and the expectations of the busy period, the time of the first loss of the customer, and the number of customers in the system in transient and stationary regimes. To our knowledge, there are no results on transient distributions for the models of $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ or $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ type.

The remainder of this chapter is structured as follows. Section 4.2 introduces the $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ system. Then we study the main characteristics of the system such as the busy period (Subsection 4.2.1), the time of the first loss of the customer (Subsection 4.2.2) and the number of customers in the system at arbitrary time (Subsection 4.2.3). In Subsection 4.2.4 we consider a special case, when the governing process $\{D_x(t)\}_{t\geq 0}$ has unit negative jumps at the instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$ and $\delta_{N_x(t)} = N_x(t)$. It means that the customers arrive not in batches but one-by-one. In this case we obtain more tractable results. Section 4.3 deals with the $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ system. We determine the busy period (Subsection 4.3.1), the time of the first loss (Subsection 4.3.1), the distribution of the number of customers in the system (Subsection 4.3.1) and the virtual waiting time (Subsection 4.3.1). We also treat a partial case separately in Subsection 4.3.5.

The results of this chapter appeared in the following articles:

> Kadankov, V. and Kadankova,T. (2008) Exit problems for the difference of a compound Poisson process and a compound renewal process. *Queueing Systems*, 59, 271-296.
>
> Kadankov, V. and Kadankova,T. (2008) Busy period, virtual waiting time and number of customers in $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ system. (submitted).
>
> Kadankov, V. and Kadankova,T. (2008) Busy period, time of the first loss of a customer and the number of customers in $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ system. (submitted).

4.2 Description of $M^{\varkappa}|G^{\delta}|1|B$ system

In this section we will introduce the governing process and describe how the queueing system $M^{\varkappa}|G^{\delta}|1|B$ functions. Customers arrive in batches of random size $\varkappa$ according to a Poisson process, i.e. the inter arrival times are exponential (with parameter μ). If upon arrival of a batch the server is busy, then these customers join the waiting room (buffer) of size $B+1$. If the number of customers in the batch exceeds the number of the vacant places in the buffer, then the amount of work brought by this batch is only partially admitted to the buffer, up to the limit of the free buffer space. This means that the customers are partially rejected. A service cycle lasts a random time η (arbitrarily distributed). After the cycle is completed, the buffer is reduced by $\min\{r,\delta\}$ and the new service cycle starts. If the waiting room becomes empty, then the server stays idle up to the arrival of a new batch.

It appears, that the reflected process (3.7.1) introduced in the previous chapter serves to describe the number of the customers in the buffer, and the age process (3.2.4) stands for the time elapsed since the start of the service cycle. Let us give a formal definition of the governing process. In order to determine main measure performances of this system, we will employ a two-component Markov process, that is defined below. Let $B \in \mathbb{Z}^{+}$, $r \in [0, B+1]$, $x \geq 0$. Introduce

$$Y_{r,x}(t) \stackrel{def}{=} \{d_{r,x}(t), \eta_{r,x}(t)\} \in [0, B+1] \times \mathbb{R}_{+}, \quad Y_{r,x}(0) = (r, x),$$

where $d_{r,x}(t)$ is the number of the customers in the waiting room at time t;

$$\eta_{r,x}(t) = \begin{cases} \text{time elapsed since the start of the service cycle up to t, if } d_{r,x}(t) > 0, \\ \eta_{r,x}(t) = 0 \text{ with probability } 1, \text{ if } d_{r,x}(t) = 0 \end{cases}$$

by means of the following recurrent equations:

$$Y_{r,x}(t) = \begin{cases} \left(\overline{D}_{r}^{B+1}(x,t), \eta_{x}^{+}(t)\right), & 0 \leq t < \tilde{\tau}_{r}^{B+1}(x), \\ Y_{0,0}(t - \tilde{\tau}_{r}^{B+1}(x)), & t \geq \tilde{\tau}_{r}^{B+1}(x), \end{cases} \quad , \quad r \in [1, B+1], \tag{4.2.1}$$

$$Y_{0,0}(t) = \begin{cases} (0,0), \quad 0 \le t < \tilde{\mu} \sim \exp(\mu), \\ Y_{r,0}(t-\tilde{\mu}) \text{ with probabilty } \mathbf{P}[\varkappa = r], \quad r \in [1,B], \quad t \ge \tilde{\mu}, \\ Y_{B+1,0}(t-\tilde{\mu}) \text{ with probabilty } \mathbf{P}[\varkappa \ge B+1], \quad t \ge \tilde{\mu}, \end{cases} \tag{4.2.2}$$

where $\overline{D}_r^{B+1}(x,t)$ is the process reflected at the upper boundary (3.7.1), $\tilde{\tau}_r^{B+1}(x) = \inf\{t : \overline{D}_r^{B+1}(x,t) < 1\}$, $r \in [1, B+1]$.

Remark 4.2.1. *Since the process X_t (3.2.5) is homogeneous with respect to the first component, then the random variable $\tilde{\tau}_r^{B+1}(x)$ is identically distributed as $\overline{\tau}_{r-1}^{B}(x)$ (3.7.3) and, hence,*

$$\tilde{v}_r^s(x) = \mathbf{E}\left[e^{-s\tilde{\tau}_r^{B+1}(x)}; \tilde{\tau}_r^{B+1}(x) < \infty\right] = \overline{v}_{r-1}^s(x), \quad r \in [1, B+1].$$

The process $Y_{r,x}(t)_{\{t\ge 0\}}$ serves as a stochastic model of the functioning of the $\text{M}^{\varkappa}|\text{G}^{\delta}|1|\text{B}$, ($\delta \sim ge(\lambda)$) system, which has the following properties:

(i) The customers arrive in groups (batch arrivals) according to the Poisson process with intensity $\mu > 0$. The number of the customers in each group is represented by the random variable $\varkappa \in \mathbb{N}$.

(ii) The system has a finite waiting room (buffer) whose size equals $B+1 < \infty$. Suppose that upon the arrival of a new group of customers of size $\varkappa$ it finds $r \in [0, B+1]$ occupied space in the waiting room. Then $\min\{k, \varkappa\}$ joins the queue, and loss of size $\max\{0, \varkappa - k\}$ occurs, where $k = B + 1 - r$ is the size of empty space in the waiting room (partial rejection);

(iii) The duration of service completion is arbitrary distributed as $\eta > 0$. Suppose, that at a time t the service cycle is accomplished. Then the occupied space in the buffer is reduced by $\min\{r, \delta\}$, where $r \in [1, B+1]$ is the value of occupied spaces in the waiting room at a time $t-0$. If at the instant of the service completion $r - \min\{r, \delta\} > 0$, then a new service cycle starts. If at the instant of the service completion $r - \min\{r, \delta\} = 0$, then the new service cycle starts upon arrival of a new customer (after exponential time with parameter $\mu > 0$).

For all $t \geq 0$ the event $\{Y_{r,x}(t) = (i, y)\}$, $i \in [1, B+1]$, $y \geq 0$ means that at time t there are i occupied places in the waiting room, and y stands for the time elapsed since the beginning of the service cycle. Here (r, x) is an initial state of the system.

The event $\{Y_{r,x}(t) = (0, 0)\}$ means that at a time t the waiting room is empty and the server is idle. The system stays in the $(0, 0)$ state for an exponential period of time (with parameter μ.)

Therefore, $d_{r,x}(t)$ is the number of the customers in the waiting room at time t. If $d_{r,x}(t) > 0$, then $\eta_{r,x}(t)$ is the time elapsed since the last start of the service cycle up to time t. If $d_{r,x}(t) = 0$, then $\mathbf{P}[\eta_{r,x}(t) = 0] = 1$. For this system we will study several important performance measures which is the topic of the next sections.

4.2.1 Busy period of the system

Assume that at a time $t_0 = 0$ system is in the state (r, x). Here $r \in [1, B+1]$ is the number of customers in the waiting room, and $x \geq 0$ is time elapsed since the duration of the current service cycle. Introduce the random variable

$$b_r(x) = \inf\{t : d_{r,x}(t) = 0\}$$

i.e. the instant at which the system for the first time becomes empty. Clearly, the interval $[0, b_r(x)]$ is a busy period of (r, x) type. Thus, determining the distribution of the busy period translates to the first passage time problem of the governing process.

Theorem 4.2.1. *Let $b_r^s(x) = \mathbf{E}\left[e^{-sb_r(x)}; b_r(x) < \infty\right]$ be the Laplace transform of the busy period of (r, x) type. Then the following equalities are valid:*

$$b_r^s(x) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{B-r}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E}\, S_{\delta+B-1}^s}, \quad x \geq 0, \quad r \in [1, B+1], \tag{4.2.3}$$

where

$$S_k^s(x) = \sum_{i=0}^{k} Q_i^s(x), \quad \mathbf{E}\, S_{\delta+B-1}^s = (1-\lambda)\sum_{i=1}^{\infty} \lambda^{i-1} S_{i-1+B}^s(0).$$

Observe, that the random variable $b_r(x)$ *is proper* ($\mathbf{P}[b_r(x) < \infty] = 1$), *and*

$$\mathbf{E}b_r(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta\left[\mathbf{E}S_{\delta+B-1} - S_{B-r}(x)\right] < \infty, \tag{4.2.4}$$

where $S_k(x) = S_k^0(x)$, $\mathbf{E}S_{\delta+B} = \mathbf{E}S^0_{\delta+B}$.

These formulae follow straightforwardly from Theorem 3.7.1 and Remark 4.2.1.

4.2.2 Time of the first loss of a customer

Suppose that the system starts functioning from the state (r, x) and denote by $l_r(x)$ the time of the first loss of a customer (a group of customers).

Theorem 4.2.2. *Let* $l_r^s(x) = \mathbf{E}\left[e^{-sl_r(x)}; l_r(x) < \infty\right]$ *be the Laplace transform of* $l_r(x)$. *Then the following relation is valid:*

$$l_0^s(0) = 1 - \mathbf{E}A_0^{\delta+B}(s) + \mathbf{E}Q^s_{\delta+B}\frac{\mathbf{E}A_0^{\delta+B}(s) - \frac{\mu}{s+\mu}\tilde{A}(s) - \frac{s}{s+\mu}}{\mathbf{E}Q^s_{\delta+B} - \frac{\mu}{s+\mu}\,\tilde{Q}(s)},$$

$$l_r^s(x) = 1 - A_x^k(s) + Q_k^s(x)\frac{\mathbf{E}A_0^{\delta+B}(s) - \frac{\mu}{s+\mu}\tilde{A}(s) - \frac{s}{s+\mu}}{\mathbf{E}Q^s\delta + B - \frac{\mu}{s+\mu}\,\tilde{Q}(s)}, \tag{4.2.5}$$

where $r \in [1, B+1]$, $k = B + 1 - r$,

$$\tilde{Q}(s) = \sum_{i=1}^{B+1}\mathbf{P}[\varkappa = i]Q^s_{B+1-i}, \qquad \tilde{A}(s) = \sum_{i=1}^{B+1}\mathbf{P}[\varkappa = i]A_0^{B+1-i}(s).$$

Note, that the random variables $l_0(0)$, $l_r(x)$ *are proper, and they have finite mathematical expectations.*

Proof. This functional can be found by employing the first exit problem for the governing process. Let $r \in [1, B+1]$, $x \geq 0$. Denote by $\tilde{\chi}_r^{B+1}(x) = \inf\{t : r + D_x(t) \notin [1, B+1]\}$ the first exit time from the interval $[1, B+1]$ by the process $r + D_x(t)$. Since the process $\{X_t\}_{t\geq 0} = \{D_x(t),\, \eta_x^+(t)\}_{t\geq 0}$ (3.2.5) is homogeneous with respect to the first component, then the random variable $\tilde{\chi}_r^{B+1}(x)$ is identically distributed as $\chi_{r-1}^B(x)$. Besides, its Laplace transform is determined by the formula of Corollary 3.4.1. In accordance with the definition of the process $Y_{r,x}(t)$ we can write the following system of the equations for the functions $l_r^s(x)$, $l_0^s(0)$:

$$
\begin{aligned}
l_r^s(x) &= V^{B+1-r}(x,s) + V_{r-1}(x,s)\, l_0^s(0), \qquad r \in [1, B+1],\ x \geq 0, \\
l_0^s(0) &= \frac{\mu}{s+\mu}\hat{a}_{B+1} + \frac{\mu}{s+\mu}\sum_{i=1}^{B+1} a_i l_i^s(0),
\end{aligned} \tag{4.2.6}
$$

where $a_i = \mathbf{P}[\varkappa = i]$, $\hat{a}_i = \mathbf{P}[\varkappa > i]$, and in view of (3.4.9)

$$
\begin{aligned}
V_{r-1}(x,s) &= \frac{Q^s_{B+1-r}(x)}{\mathbf{E}\, Q^s_{\delta+B}}, \\
V^{B+1-r}(x,s) &= 1 - A_x^{B+1-r}(s) - \frac{Q^s_{B+1-r}(x)}{\mathbf{E}\, Q^s_{\delta+B}} \left(1 - \mathbf{E}\, A_0^{\delta+B}(s)\right).
\end{aligned} \tag{4.2.7}
$$

Substituting the right-hand side of the first equation of (4.2.6) for $x = 0$ into the second one, we get

$$
l_0^s(0) = \frac{\mu}{s+\mu}\,\hat{a}_{B+1} + \frac{\mu}{s+\mu}\sum_{i=1}^{B+1} a_i V^{B+1-i}(0,s) + \frac{\mu}{s+\mu}\sum_{i=1}^{B+1} a_i V_{i-1}(0,s) l_0^s(0).
$$

The latter equation yields

$$
l_0^s(0) = \frac{\mu}{s+\mu}\left(\hat{a}_{B+1} + \sum_{i=1}^{B+1} a_i V^{B+1-i}(0,s)\right)\left(1 - \frac{\mu}{s+\mu}\sum_{i=1}^{B+1} a_i V_{i-1}(0,s)\right)^{-1}.
$$

Taking into account the first formula of (4.2.6) and (4.2.7), we derive the equalities (4.2.5) of the theorem. ▲

4.2.3 Number of customers in the system

Let $\nu_s \sim \exp(s)$ be the exponential random variable with parameter $s > 0$. Introduce the transient probabilities of the process $d_{r,x}(t)_{\{t\geq 0\}}$:

$$
\begin{aligned}
&q^s_{r,x}(u) = \mathbf{P}[d_{r,x}(\nu_s) \leq u], \quad q^s_{0,0}(u) = \mathbf{P}[d_{0,0}(\nu_s) \leq u], \quad r, u \in [1, B+1], \\
&q^s_{r,x}(0) = \mathbf{P}[d_{r,x}(\nu_s) = 0], \quad q^s_{0,0}(0) = \mathbf{P}[d_{0,0}(\nu_s) = 0].
\end{aligned}
$$

Denote $\tilde{b}(s) = \hat{a}_B\, b^s_{B+1}(0) + \sum_{i=1}^{B} a_i\, b_i^s(0)$.

Theorem 4.2.3. *The distribution of the number of customers at time* ν_s *is such that*

$$q^s_{0,0}(u) = \mathbf{E}A_0^{\delta+u-1}(s) + \frac{C_u(s,\lambda)}{s+\mu-\mu\tilde{b}(s)}, \qquad q^s_{0,0}(B+1) = 1 - \frac{s}{s+\mu-\mu\tilde{b}(s)},$$
$$q^s_{r,x}(u) = A_x^{u-r}(s) + b^s_r(x)\frac{C_u(s,\lambda)}{s+\mu-\mu\,\tilde{b}(s)}, \qquad q^s_{r,x}(B+1) = 1 - \frac{sb^s_r(x)}{s+\mu-\mu\tilde{b}(s)},$$
$$q^s_{0,0}(0) = \frac{s}{s+\mu-\mu\tilde{b}(s)}, \qquad q^s_{r,x}(0) = \frac{s\,b^s_r(x)}{s+\mu-\mu\,\tilde{b}(s)},$$

where

$$C_u(s,\lambda) = \frac{sQ^s_u}{1-\lambda} - s + \lambda(s+\mu)\left(A_0^u(s) - \mathbf{E}A_0^{\delta+u}(s)\right).$$

Corollary 4.2.1. *Let* $\pi_i = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) = i]$, $i \in [0, B+1]$ *be the stationary distribution of the number of customers in the* $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ *system. Then*

$$\pi_0 = \left[1 + \mu\mathbf{E}\eta\left(\frac{\lambda}{1-\lambda}\mathbf{E}Q_{\delta+B} + \sum_{i=0}^{B}\hat{a}_i Q_{B-i}\right)\right]^{-1},$$
$$\pi_{B+1} = 1 - \pi_0(1 + C_B(\lambda)), \qquad \pi_i = \pi_0\left(C_i(\lambda) - C_{i-1}(\lambda)\right), \quad i \in [1, B],$$

where $\rho_i = \lim_{s\to 0} s^{-1}\tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}\left[\pi(t) = i\right] dt < \infty$, $C_0(\lambda) = 0$,

$$C_u(\lambda) = \frac{Q_u}{1-\lambda} - 1 + \lambda\mu\left(A_0^u - \mathbf{E}A_0^{\delta+u}\right), \quad A_0^u = \lim_{s\to 0}\frac{1}{s}A_0^u(s) = \sum_{i=0}^{u}\rho_i\left[1 - \frac{Q_{u-i}}{1-\lambda}\right].$$

Proof. By $\tilde{p}^s_{r,x}(u) = \mathbf{P}\left[d_{r,x}(\nu_s) \le u; b_r(x) > \nu_s\right]$, $r, u \in [1, B+1]$ denote the transient probability of the process $d_{r,x}(\nu_s)$ on the event $\{b_r(x) > \nu_s\}$ (meaning that the server is busy). Taking into account the homogeneity of the process X_t (3.2.5) with respect to the first component, the definition of the process $d_{r,x}(t)$ and the formulae (3.7.10) of Theorem 3.7.3, we derive

$$\tilde{p}^s_{r,x}(B+1) = 1 - b^s_r(x), \qquad \tilde{p}^s_{r,x}(u) = A_x^{u-r}(s) - b^s_r(x)\mathbf{E}A_0^{\delta+u-1}(s), \quad u \in [1, B],$$

where the function $b^s_r(x)$ is determined by (4.2.3).

In accordance with the definition of the process $Y_{r,x}(t)$ we can write the following equations for the functions $q^s_{r,x}(u)$, $q^s_{0,0}(u)$ for $u \in [1, B]$

$$\begin{aligned} q^s_{r,x}(u) &= \tilde{p}^s_{r,x}(u) + b^s_r(x) q^s_{0,0}(u), \\ q^s_{0,0}(u) &= \frac{\mu}{s+\mu} \left[\hat{a}_B\, q^s_{B+1,0}(u) + \sum_{i=1}^{B} a_i\, q^s_{i,0}(u) \right]. \end{aligned} \tag{4.2.8}$$

Substituting the right-hand side of the second equation into the first one, we get

$$q^s_{r,x}(u) = \tilde{p}^s_{r,x}(u) + b^s_r(x) \frac{\mu}{s+\mu}\, \tilde{q}(s,u),$$

where $\tilde{q}(s,u) = \hat{a}_B\, q^s_{B+1,0}(u) + \sum_{i=1}^{B} a_i\, q^s_{i,0}(u)$. Letting $x = 0$ in the latter equality implies that

$$\begin{aligned} \hat{a}_B\, q^s_{B+1,0}(u) &= \hat{a}_B\, \tilde{p}^s_{B+1,0}(u) + \hat{a}_B\, b^s_{B+1}(0) \frac{\mu}{s+\mu}\, \tilde{q}(s,u), \\ \sum_{i=1}^{B} a_i\, q^s_{i,0}(u) &= \sum_{i=1}^{B} a_i\, \tilde{p}^s_{i,0}(u) + \sum_{i=1}^{B} a_i\, b^s_i(0) \frac{\mu}{s+\mu}\, \tilde{q}(s,u). \end{aligned}$$

Adding these equalities, we obtain the function $\tilde{q}(s,u)$

$$\tilde{q}(s,u) = \tilde{p}(s,u) \left(1 - \frac{\mu}{s+\mu}\, \tilde{b}(s) \right)^{-1}, \quad u \in [1, B],$$

where $\tilde{b}(s) = \hat{a}_B\, b^s_{B+1}(0) + \sum_{i=1}^{B} a_i\, b^s_i(0)$,

$$\tilde{p}(s,u) = \hat{a}_B\, \tilde{p}^s_{B+1,0}(u) + \sum_{i=1}^{B} a_i\, \tilde{p}^s_{i,0}(u) = \sum_{i=1}^{u} a_i\, A_0^{u-i}(s) - \tilde{b}(s) \mathbf{E} A_0^{\delta+u-1}(s).$$

Substituting the expression for the function $\tilde{q}(s,u)$ into (4.2.8), we find that

$$\begin{aligned} q^s_{0,0}(u) &= \mathbf{E} A_0^{\delta+u-1}(s) + \frac{C_u(s,\lambda)}{s+\mu-\mu\tilde{b}(s)}, \\ q^s_{r,x}(u) &= A_x^{u-r}(s) + b^s_r(x) \frac{C_u(s,\lambda)}{s+\mu-\mu\,\tilde{b}(s)}, \end{aligned} \tag{4.2.9}$$

where

$$C_u(s,\lambda) = \frac{sQ^s_u}{1-\lambda} - s + \lambda(s+\mu) \left(A_0^u(s) - \mathbf{E} A_0^{\delta+u}(s) \right).$$

To derive the latter equalities, we used the following relation

$$\mu \sum_{i=1}^{u} a_i\, A_0^{u-i}(s) - (s+\mu) A_0^u(s) = \frac{s}{1-\lambda}\, Q_u^s - s, \qquad u \in [1,\infty],$$

which follows from the definition of the function $A_x^u(s)$. If $u = B+1$, then $\tilde{p}^s_{r,x}(B+1) = 1 - b_r^s(x)$, $\tilde{p}(s, B+1) = 1 - \tilde{b}(s)$, and the following formulae are valid:

$$q_{0,0}^s(B+1) = 1 - \frac{s}{s+\mu-\mu\tilde{b}(s)}, \qquad q_{r,x}^s(B+1) = 1 - \frac{s b_r^s(x)}{s+\mu-\mu\tilde{b}(s)}. \tag{4.2.10}$$

Taking into account the definition of the process $Y_{r,x}(t)$, we can write the following equations for the functions $q_{r,x}^s(0)$, $q_{0,0}^s(0)$

$$\begin{aligned} q_{r,x}^s(0) &= b_r^s(x) q_{0,0}^s(0), \\ q_{0,0}^s(0) &= \frac{s}{s+\mu} + \frac{\mu}{s+\mu}\left[\hat{a}_B q_{B+1,0}^s(0) + \sum_{i=1}^{B} a_i q_{i,0}^s(0)\right]. \end{aligned}$$

Solving this system yields

$$q_{0,0}^s(0) = \frac{s}{s+\mu-\mu\tilde{b}(s)}, \qquad q_{r,x}^s(0) = \frac{s\, b_r^s(x)}{s+\mu-\mu\,\tilde{b}(s)}. \tag{4.2.11}$$

Observe that $\lim_{s\to 0} A_x^u(s) = \lim_{s\to 0} \mathbf{E} A_0^{\delta+u}(s) = 0$, $\lim_{s\to 0} b_r^s(x) = 1$. It follows from (4.2.9)–(4.2.11) and properties of the Laplace transforms that

$$\begin{aligned} &\lim_{s\to 0} q_{r,x}^s(u) = \lim_{s\to 0} q_{0,0}^s(u) = q(u) = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) \le u], \qquad u \in [1, B+1], \\ &\lim_{s\to 0} q_{r,x}^s(0) = \lim_{s\to 0} q_{0,0}^s(0) = q(0) = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) = 0]. \end{aligned}$$

The formulae (4.2.11) imply that

$$q(0) = \lim_{s\to 0} \frac{s}{s+\mu-\mu\tilde{b}(s)} = \left[1 + \mu \mathbf{E}\eta \left(\frac{\lambda}{1-\lambda}\mathbf{E} Q_{\delta+B} + \sum_{i=0}^{B} \hat{a}_i Q_{B-i}\right)\right]^{-1}.$$

In view of (4.2.9), (4.2.10) we find

$$q(B+1) = 1 - q(0), \qquad q(u) = q(0) C_u(\lambda), \quad u \in [1, B],$$

where $\rho_i = \lim_{s\to 0} s^{-1}\tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}\left[\pi(t) = i\right] dt < \infty$,

$$C_u(\lambda) = \frac{Q_u}{1-\lambda} - 1 + \lambda\mu\left(A_0^u - \mathbf{E}A_0^{\delta+u}\right), \quad A_0^u = \lim_{s\to 0}\frac{1}{s}A_0^u(s) = \sum_{i=0}^{u}\rho_i\left[1 - \frac{Q_{u-i}}{1-\lambda}\right].$$

▲

4.2.4 Special case: M$^{\varkappa}$|G|1|B system

We will now consider a more simple and also more tractable queueing system, where the customers are served one by one, and service time is distributed as η. Then the governing process is the difference of the compound Poisson process and the simple renewal process, i.e. the process $\{D_x(t)\}_{t\geq 0}$ has unit negative jumps at the times instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$ and $\delta_{N_x(t)} = N_x(t)$. Obviously, we can apply the results of previous sections to study the characteristics of the M$^{\varkappa}$|G|1|B system ($\mathbf{P}[\delta = 1] = 1$) with finite buffer. To illustrate this, we now will determine the distribution of the busy period, the number of customers in the system, time of the first loss of a customer. In this case we also find the distribution of the number of lost customers at time of the first loss.

Corollary 4.2.2. *Let $\mathbf{P}[\delta = 1] = 1$, $b_r^s(x) = \mathbf{E}\left[e^{-sb_r(x)}; b_r(x) < \infty\right]$ be the Laplace transform of the busy period of (r, x) type of the* M$^{\varkappa}$|G|1|B *system. Then the following relation is valid:*

$$b_r^s(x) = \frac{\tilde{f}_x(s) + (1-\tilde{f}(s))S_{B-r}^s(x)}{\tilde{f}(s) + (1-\tilde{f}(s))S_B^s}, \quad x \geq 0, \quad r \in [1, B+1],$$

where $S_k^s(x) = \sum_{i=0}^k Q_i^s(x)$, $S_k^s(x) = 0$, for $k < 0$. The random variable $b_r(x)$ is proper and

$$\mathbf{E}b_r(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta\left[S_B - S_{B-r}(x)\right] < \infty,$$

where $S_k(x) = S_k^0(x)$.

These formulae were derived in Kadankov (1985). To prove the corollary, it suffices to set $\lambda = 0$ in the equalities of Theorem 4.2.1.

Corollary 4.2.3. *Let* $\mathbf{P}[\delta = 1] = 1$, $l_r(x)$ *be the time of the first loss of the batch of customers in the system* $\mathrm{M}^{\varkappa}|\mathrm{G}|1|\mathrm{B}$, *and* $l_r^s(x) = \mathbf{E}\left[e^{-sl_r(x)}; l_r(x) < \infty\right]$ *be the Laplace transform of* $l_r(x)$. *Then the following formula holds:*

$$l_r^s(x) = 1 - A_x^k(s) - Q_k^s(x)\frac{s}{s+\mu}\left(1 - \frac{\mu}{s+\mu}\tilde{Q}(s)/Q_{B+1}^s\right)^{-1},$$

where $r \in [0, B+1]$, $k = B+1-r$, $\tilde{Q}(s) = \sum_{i=1}^{B+1}\mathbf{P}[\varkappa = i]Q_{B+1-i}^s$. *The random variable* $l_r(x)$ *is proper and has finite mathematical expectation.*

The function $l_r^s(x)$ was found in Kadankov (1985) in a different form. Note, that Bratiychuk (2000) also studied this functional employing the potential method. The author derived an analytic expression in terms of the triple superposition of the series and operator functionals.

Corollary 4.2.4. *The distribution of the number of customers in the system* $\mathrm{M}^{\varkappa}|\mathrm{G}|1|\mathrm{B}$ *at time* ν_s *is such that for* $r \in [0, B+1]$, $u \in [1, B+1]$

$$q_{r,x}^s(u) = A_x^{u-r}(s) + b_r^s(x)\frac{s(Q_u^s - 1)}{s+\mu-\mu\,\tilde{b}(s)}, \quad q_{r,x}^s(B+1) = 1 - \frac{sb_r^s(x)}{s+\mu-\mu\tilde{b}(s)},$$

$$q_{r,x}^s(0,0) = \frac{s\,b_r^s(x)}{s+\mu-\mu\,\tilde{b}(s)}, \qquad b_0^s(x) \stackrel{\mathrm{def}}{=} 1.$$

Corollary 4.2.5. *Let* $\pi_i = \lim_{t\to\infty}\mathbf{P}[d_{(\cdot)}(t) = i]$, $i \in [0, B+1]$ *be the stationary distribution of the number of customers in the system* $\mathrm{M}^{\varkappa}|\mathrm{G}|1|\mathrm{B}$. *Then*

$$\pi_0 = \left(1 + \mu\mathbf{E}\eta\sum_{i=0}^{B}\hat{a}_iQ_{B-i}\right)^{-1},$$

$$\pi_{B+1} = 1 - \pi_0 Q_B, \qquad \pi_i = \pi_0(Q_i - Q_{i-1}), \quad i \in [1, B],$$

where the resolvent sequence of the process $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $Q_k \stackrel{\mathrm{def}}{=} Q_k(0)$, $Q_0 = 1$ *is given by (3.4.13) for* $s = 0$.

To prove Corollaries 4.2.3–4.2.5, it suffices to set $\lambda = 0$ in the equalities of Corollaries 4.2.2, 4.2.1 and Theorem 4.2.3. Note, that the formulae of Corollary 4.2.5 were obtained in Kadankov (1985).

Suppose that the system starts functioning from the state (r, x), $r \in [0, B+1]$. Denote by $i_{r,x}$ the number of the lost customers at the time of the first loss $l_r(x)$.

Corollary 4.2.6. *The generating function* $L^s_{r,x}(z) = \mathbf{E}\left[e^{-sl_r(x)}z^{i_{r,x}}\right]$ *of the joint distribution of* $\{l_r(x), i_{r,x}\}$ *is such that*

$$L^s_{r,x}(z) = \frac{\mu}{s}\sum_{i=0}^{k} E^i(z)\left[A_x^{k-i}(s) - A_x^{k-i-1}(s)\right] + \\ + \mu\,\frac{Q^s_k(x)}{Q^s_{B+1}}\,\frac{\sum\limits_{i=0}^{B+1} E^i(z)\left[Q^s_{B+1-i} - Q^s_{B-i}\right]}{s+\mu-\mu\,\tilde{Q}(s)/Q^s_{B+1}}, \qquad k = B+1-r, \quad (4.2.12)$$

$$\mathbf{E}\left[e^{-sl_r(x)}; i_{r,x} = n\right] = \frac{\mu}{s}\left[a_n A^k_x(s) + \sum_{i=1}^{k}(a_{n+i} - a_{n+i-1})A_x^{k-i}(s)\right] + \\ + \mu\,\frac{Q^s_k(x)}{Q^s_{B+1}}\,\frac{a_n Q^s_{B+1} + \sum\limits_{i=1}^{B+1}(a_{n+i} - a_{n+i-1})Q^s_{B+1-i}}{s+\mu-\mu\,\tilde{Q}(s)/Q^s_{B+1}}, \qquad n \in \mathbb{N},$$

where $E^i(z) = \mathbf{E}\left[z^{\varkappa - i}; \varkappa > i\right],\ i \in \mathbb{Z}^+$.

Proof. Conditioning on the first exit time, we can write the following system of equations for $L^s_{r,x}(z)$:

$$L^s_{r,x}(z) = \tilde{V}^k(x,z,s) + V_{r-1}(x,s)L^s_{0,0}(z), \quad r \in [1, B+1]$$
$$L^s_{0,0}(z) = \frac{\mu}{s+\mu}\,E^{B+1}(z) + \frac{\mu}{s+\mu}\sum_{r=1}^{B+1} a_r L^s_{r,0}(z), \qquad (4.2.13)$$

where

$$\tilde{V}^k(x,z,s) = \mathbf{E}\left[e^{-s\chi^B_{r-1}(x)}z^T; \mathfrak{A}^k\right] = \tilde{f}^k(x,z,s) - \frac{Q^s_k(x)}{Q^s_{B+1}}\,\tilde{f}^{B+1}(0,z,s). \qquad (4.2.14)$$

The equality (3.3.6) implies for the function $\tilde{f}^k(x,z,s) = \mathbf{E}\left[e^{-s\tau^k(x)}z^{T^k(x)}; \mathfrak{B}^k(x)\right]$ that

$$\tilde{f}^k(x,z,s) = \frac{\mu}{s}\sum_{i=0}^{k} E^i(z)\left[A_x^{k-i}(s) - A_x^{k-i-1}(s)\right] + Q^s_k(x)F(c(s),z), \quad (4.2.15)$$

where $F(c(s),z) = \frac{1-c(s)}{1-c(s)/z}\,\frac{k(z)-k(c(s))}{s-k(c(s))}$. Solving the system (4.2.13) and taking into account (4.2.14) for all $r \in [0, B+1]$, $x \geq 0$, we get

$$L^s_{r,x}(z) = \tilde{f}^k(x,z,s) + \mu\,\frac{Q^s_k(x)}{Q^s_{B+1}}\,\frac{E^{B+1}(z) + \sum\limits_{i=1}^{B+1} a_i \tilde{f}^{B+1-i}(0,z,s) - \tilde{f}^{B+1}(0,z,s)}{s+\mu-\mu\,\tilde{Q}(s)/Q^s_{B+1}}.$$

The first formula of the corollary follows from the latter equality and from (4.2.15). Comparing the coefficients of z^n, $n \in \mathbb{N}$ in both sides of (4.2.12), we derive the second formula of the corollary. ▲

4.3 Description of $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ system

In this section we will study the queueing system with batch arrivals of random size δ. Time arrivals are the renewal instants of a compound renewal process $N_x(t)_{\{t\geq 0\}}$. Service time is exponential, and during the service cycle the amount $\varkappa$ is processed. Before considering the queueing system of interest, we stress the following facts.

Remark 4.3.1. *Let $B \in \mathbb{Z}^+$ be fixed, $k \in [0,B]$, $r = B-k$. Introduce the process*

$$\overline{D}^{B+1}_{k+1}(x,t) = -\underline{D}^0_{-r}(x,t) + k + 1 \in]-\infty, B+1], \quad \overline{D}^{B+1}_{k+1}(x,0) = k+1. \tag{4.3.1}$$

This process is reflected at the upper boundary $B+1$, generated by the infimum of the process $D_x(t)$. Introduce the following random variable

$$\begin{aligned}\overline{\tau}_{k+1}(x) &= \inf\{t : \overline{D}^{B+1}_{k+1}(x,t) < 1\} = \inf\{t : \underline{D}^0_{-r}(x,t) > k\} = \\ &= \inf\{t : \underline{D}^r_0(x,t) > B\} = \underline{\tau}^B_r(x).\end{aligned}$$

This defining chain of stochastic equalities implies that $\overline{\tau}_{k+1}(x)$ is identically distributed as $\underline{\tau}^B_r(x)$, and, hence, in view of (3.8.3)

$$\overline{v}^k_x(s) = \mathbf{E}e^{-s\overline{\tau}_k(x)} = 1 - A^{k-1}_x(s) + Q^s_{k-1}(x)\frac{\mathbf{E}A^{\delta+B}_0(s) - A^B_0(s)}{\mathbf{E}\,Q^s_{\delta+B} - Q^s_B}, \quad k \in [1, B+1].$$

Remark 4.3.2. *Let* $u \in [1, B+1]$. *Denote by*

$$\overline{p}^s_{k+1,x}(u) = \mathbf{P}\left[\overline{D}^{B+1}_{k+1}(x,\nu_s) \geq u; \overline{\tau}_{k+1}(x) > \nu_s\right]$$

the Laplace transform of the increments of the process $\overline{D}^{B+1}_{k+1}(x,t)$ *on the event* $\{\overline{\tau}_{k+1}(x) > t\}$. *Definition of the process and Remark 4.3.1 imply that*

$$\mathbf{P}\left[\overline{D}^{B+1}_{k+1}(x,\nu_s) \geq u;\ \overline{\tau}_{k+1}(x) > \nu_s\right] = \mathbf{P}\left[\underline{D}^{0}_{-r}(x,\nu_s) \leq k+1-u;\ \underline{\tau}^{B}_{r}(x) > \nu_s\right].$$

It follows from the latter equality and from (3.8.8) that for $k, u \in [1, B+1]$

$$\overline{p}^s_{k,x}(u) = A^{k-u}_x(s) - Q^s_{k-1}(x)\,\frac{\mathbf{E}A^{\delta+B+1-u}_0(s) - A^{B+1-u}_0(s)}{\mathbf{E}Q^s_{B+\delta} - Q^s_B}. \tag{4.3.2}$$

We now introduce the governing process of the system. Let $B \in \mathbb{Z}^+$, $k \in [0, B+1]$, $x \geq 0$. Define the two-component Markov process

$$Y_{k,x}(t) = \{d_{k,x}(t), \eta^+_x(t)\} \in [0, B+1] \times \mathbb{R}_+, \quad Y_{k,x}(0) = (k,x)$$

by means of the following stochastic recurrent equalities:

$$Y_{k,x}(t) = \begin{cases} \left(\overline{D}^{B+1}_k(x,t), \eta^+_x(t)\right), & 0 \leq t < \overline{\tau}_k(x), \\ Y_{0,\eta^+_x(\overline{\tau}_k(x))}(t - \overline{\tau}_k(x)), & t \geq \overline{\tau}_k(x), \end{cases} \quad k \in [1, B+1],$$

$$Y_{0,x}(t) = \begin{cases} (0, \eta^+_x(t)), \quad 0 \leq t < \eta_x, \\ Y_{k,0}(t - \eta_x) : (1-\lambda)\lambda^{k-1}, \quad k \in [1, B], \quad t \geq \eta_x, \\ Y_{B+1,0}(t - \eta_x) : \lambda^B, \quad t \geq \eta_x. \end{cases}$$

The process $Y_{k,x}(t)_{\{t \geq 0\}}$ serves as a mathematical model of the functioning of the $G^\delta|M^\varkappa|1|B$ system with ($\delta \sim ge(\lambda)$). Let us describe how this system works.

(i) Customers arrive into the system in batches according to the renewal process $N_x(t)_{\{t \geq 0\}}$. The number of customers in every batch is a random variable distributed as $\delta \sim ge(\lambda) \in \mathbb{N}$.

(ii) The system has a finite buffer whose size equals $B+1<\infty$. Suppose that upon the arrival of a new customer of size δ, it finds $k\in[0,B+1]$ occupied spaces in the waiting room. Then $\min\{r,\delta\}$ joins the queue, and a loss of size $\max\{0,\delta-r\}$ occurs, where $r=B+1-k$ is the size of empty space in the waiting room (partial rejection);

(iii) The duration of service completion is exponentially distributed with parameter $\mu>0$. Suppose, that at a time t the service cycle is accomplished. Then the occupied space in the buffer is reduced by $\min\{k,\varkappa\}$, where $k\in[1,B+1]$ is the value of occupied space in the waiting room at a time $t-0$. If at the instant of the service completion $k-\min\{k,\varkappa\}>0$, then a new service cycle starts. If at the instant of the service completion $k-\min\{k,\varkappa\}=0$, then the new service cycle starts upon arrival of a new customer.

For all $t\geq 0$ the event $\{Y_{k,x}(t)=(i,y)\}$, $i\in[1,B+1]$, $y\geq 0$ means that at the time t there are i customers in the waiting room, and that time y has elapsed since the last arrival up to time t. We assume that (k,x) is an initial state of the system.

The event $\{Y_{k,x}(t)=(0,y)\}$ means that at time t the waiting room is empty and the system is idle, and y time has elapsed since the last customers arrival (up to time t). Hence, η_y is duration of the idle period (state $(0,y)$).

Thus, $d_{k,x}(t)$ is the number of customers in the buffer at time t, $\eta_x^+(t)$ is the time elapsed since the last arrival of the batch up to time t. The definition of the process $Y_{k,x}(t)$ (homogeneity of the process X_t (3.2.5) with respect to the first component) implies that the linear component $\eta_x^+(t)$ does not depend on k.

4.3.1 Busy period of the system

Suppose that the system starts functioning at time $t_0=0$ from the state (k,x), where $k\in[1,B+1]$ is the number of customers in the waiting room, $x\geq 0$ is time elapsed since the last arrival up to time $t_0 = 0$ Denote by

$$b_k(x) = \inf\{t : d_{k,x}(t) = 0\}, \quad \eta(x) = \eta_x^+(b_k(x))$$

the instant at which the system becomes empty for the first time and the value of the linear component at time $b_k(x)$. Hence, the interval $[0, b_k(x)]$ is a busy period of (k, x) type.

Theorem 4.3.1. *Let* $b_k^s(x) = \mathbf{E}\left[e^{-sb_k(x)}; b_k(x) < \infty\right]$ *be the Laplace transform of the busy period of* (k, x) *type. Then*

(i) *the following equality holds:*

$$b_k^s(x) = 1 - A_x^{k-1}(s) + Q_{k-1}^s(x)\frac{\mathbf{E}A_0^{\delta+B}(s) - A_0^B(s)}{\mathbf{E}\,Q_{\delta+B}^s - Q_B^s}, \quad k \in [1, B+1], \tag{4.3.3}$$

the random variable $b_k(x)$ *is proper* ($\mathbf{P}\,[b_k(x) < \infty] = 1$), *and its mathematical expectation is given by*

$$\mathbf{E}b_k(x) = A_x^{k-1} - Q_{k-1}(x)\frac{\mathbf{E}A_0^{\delta+B} - A_0^B}{\mathbf{E}\,Q_{\delta+B} - Q_B} < \infty, \tag{4.3.4}$$

(ii) *the Laplace transform* $b_k^s(x, dy) = \mathbf{E}\left[e^{-sb_k(x)};\ \eta(x) \in dy\right]$ *of the joint distribution of* $\{b_k(x), \eta(x)\}$ *is such that* $k \in [1, B+1]$

$$b_k^s(x, dy) = f^{k-1}(x, dy, s) - Q_{k-1}^s(x)\frac{\mathbf{E}f^{\delta+B}(0, dy, s) - f^B(0, dy, s)}{\mathbf{E}\,Q_{\delta+B}^s - Q_B^s}, \tag{4.3.5}$$

where the function $f^k(x, dy, s) = \mathbf{E}\left[e^{-s\tau^k(x)};\ T^k(x) \in dy, \mathfrak{B}^k(x)\right]$, $k \in \mathbb{Z}^+$ *is determined by (3.8.8).*

Proof. The formula (4.3.3) follows straightforwardly from (3.8.3) and Remark 4.3.1. The equalities (3.8.2), (3.4.8) imply (4.3.5). ▲

4.3.2 Time of the first loss of a customer

Suppose that the initial state of the system is (k, x), $k \in [0, B+1]$, $x \geq 0$. Introduce the following variables: $l_k(x)$ the time of the first loss of a customer (group of customers); $i_{k,x}(t)$ the number of lost customers on the time interval $[0, t]$; and $i_{k,x} = i_{k,x}(l_k(x))$ the number of lost customers at time $l_k(x)$.

Theorem 4.3.2. *Let $l_k^s(x) = \mathbf{E}\left[e^{-sl_k(x)}; l_k(x) < \infty\right]$ be the Laplace transform of $l_k(x)$. Then the following relations hold:*

$$l_k^s(x) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{k-1}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E}\, S_{\delta+B}^s}, \qquad k \in [0, B+1],$$
$$l_k^s(x, m) = \mathbf{E}\left[e^{-sl_k(x)}; i_{k,x} = m\right] = l_k^s(x)(1 - \lambda)\lambda^{m-1}, \quad m \in \mathbb{N}, \qquad (4.3.6)$$

where $S_k^s(x) = \sum_{i=0}^k Q_i^s(x)$, $S_k^s(x) = 0$ for $k < 0$. The random variable $l_k(x)$ is proper with finite mathematical expectation:

$$\mathbf{E}l_k(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta\left[\mathbf{E}S_{\delta+B} - S_{k-1}(x)\right] < \infty,$$

where $S_k(x) = S_k^0(x)$, $\mathbf{E}S_{\delta+B} = \mathbf{E}S_{\delta+B}^0$.

Proof. The functions $l_k^s(x)$, $k \in [1, B+1]$, $l_0^s(y)$ obey the following system of equations:

$$l_k^s(x) = V_{B+1-k}(x, s) + \int_0^\infty V^{k-1}(x, dy, s) l_0^s(y),$$
$$l_0^s(y) = \tilde{f}_y(s)\lambda^{B+1} + \tilde{f}_y(s)\sum_{k=1}^{B+1}(1 - \lambda)\lambda^{k-1} l_k^s(0), \qquad (4.3.7)$$

where the functions $V_r(x, s)$, $V^k(x, dy, s)$ are given by (3.4.8), (3.4.9). Substituting the expression for the function $l_0^s(y)$ from the second equation into the first one, we get:

$$l_k^s(x) = V_{B+1-k}(x, s) + \lambda^{B+1}\int_0^\infty V^{k-1}(x, dy, s)\tilde{f}_y(s) +$$
$$+ \int_0^\infty V^{k-1}(x, dy, s)\tilde{f}_y(s)\sum_{k=1}^{B+1}(1 - \lambda)\lambda^{k-1} l_k^s(0).$$

Letting $x = 0$ in the latter equation, we find for the function $X(s) = \sum_{k=1}^{B+1}(1-\lambda)\lambda^{k-1}l_k^s(0)$ that

$$X(s) + \lambda^{B+1} = \frac{\lambda^{B+1} + \check{V}_B(\lambda, s)}{1 - \int_0^\infty \check{V}^B(\lambda, dy, s)\tilde{f}_y(s)}, \tag{4.3.8}$$

where $\check{V}_B(\lambda, s) = (1-\lambda)\sum_{k=1}^{B+1}\lambda^{k-1}V_{B+1-k}(0, s)$,

$$\check{V}^B(\lambda, dy, s) = (1-\lambda)\sum_{k=1}^{B+1}\lambda^{k-1}V^{k-1}(0, dy, s).$$

Employing the formulae (3.3.6), (3.4.8), (3.4.9) and performing necessary calculations, we obtain:

$$\lambda^{B+1} + \check{V}_B(\lambda, s) = (1-\lambda)\left(\mathbf{E}\, Q_{\delta+B}^s\right)^{-1},$$
$$1 - \int_0^\infty \check{V}^B(\lambda, dy, s)\tilde{f}_y(s) = (1-\lambda)S_B(\lambda, s)\left(\mathbf{E}\, Q_{\delta+B}^s\right)^{-1},$$

where $S_B(\lambda, s) = \tilde{f}(s) + (1-\tilde{f}(s))\mathbf{E}\, S_{\delta+B}^s$. These equalities, formula (4.3.8) and the second equality of (4.3.7) imply that

$$l_0^s(y) = \tilde{f}_y(s)S_B(\lambda, s)^{-1}.$$

Inserting the right-hand side of this equality into the second equality of (4.3.7), we get

$$l_k^s(x) = \frac{Q_{k-1}^s(x)}{\mathbf{E}\, Q_{\delta+B}^s} + S_B(\lambda, s)^{-1}\int_0^\infty V^{k-1}(x, dy, s)\tilde{f}_y(s), \qquad k \in [1, B+1].$$

In view of (3.3.6), (3.4.8), (3.4.9) we find that

$$\int_0^\infty V^{k-1}(x, dy, s)\tilde{f}_y(s) = \tilde{f}_x(s) + (1-\tilde{f}(s))S_{k-1}^s(x) - \frac{Q_{k-1}^s(x)}{\mathbf{E}\, Q_{\delta+B}^s}S_B(\lambda, s).$$

The latter and the previous equality imply the first equality of (4.3.6). We now verify the second equality. Observe, that $i_{k,x} \sim ge(\lambda)$. This can be formally derived from the first formula of (3.4.8) and from the following system of equations:

$$l_k^s(x, m) = V_{B+1-k}(x, s)(1-\lambda)\lambda^{m-1} + \int_0^\infty V^{k-1}(x, dy, s)l_0^s(y, m),$$
$$l_0^s(y, m) = \tilde{f}_y(s)(1-\lambda)\lambda^{B+m} + \tilde{f}_y(s)\sum_{k=1}^{B+1}(1-\lambda)\lambda^{k-1}l_k^s(0, m).$$

To solve this system, one can apply a similar reasoning as for the the system (4.3.7). ▲

Theorem 4.3.3. *Let* $\nu_s \sim \exp(s)$ *be an exponential variable with parameter* $s > 0$, *independent from the process* $Y_{k,x}(t)$. *Denote by* $I^s_{k,x}(n) = \mathbf{P}\left[i_{k,x}(\nu_s) = n\right]$, $n \in \mathbb{Z}^+$ *the distribution of the number of lost customers in the time interval* $[0, \nu_s]$. *For all* $k \in [1, B+1]$, $x \geq 0$ *the following equalities are valid:*

$$\begin{aligned} &I^s_{k,x}(0) = 1 - l^s_k(x), \\ &I^s_{k,x}(n) = l^s_k(x)(1-\lambda)(1 - l^s_{B+1}(0))(\lambda + (1-\lambda) l^s_{B+1}(0))^{n-1}, \qquad n \in \mathbb{N}. \end{aligned} \tag{4.3.9}$$

Proof. Let $\tilde{I}^s_{k,x}(z) = \mathbf{E}\left[z^{i_{k,x}(\nu_s)}\right]$, $|z| \leq 1$ be the generating function of the distribution of the number of lost customers. Then it obeys to the following equation:

$$\tilde{I}^s_{k,x}(z) = 1 - l^s_k(x) + \tilde{L}^s_{k,x}(z) \tilde{I}^s_{B+1,0}(z), \tag{4.3.10}$$

where (4.3.6)

$$\tilde{L}^s_{k,x}(z) = \mathbf{E}\left[e^{-s l_k(x)} z^{i_{k,x}}\right] = l^s_k(x) z \, \frac{1-\lambda}{1 - z\lambda}.$$

Letting $k = B+1$, $x = 0$ in (4.3.10), we find that

$$\tilde{I}^s_{B+1,0}(z) = \left(1 - l^s_{B+1}(0)\right)\left(1 - \tilde{L}^s_{B+1,0}(z)\right)^{-1}.$$

Inserting the right-hand side of this equality into (4.3.10) implies

$$\tilde{I}^s_{k,x}(z) = 1 - l^s_k(x) \frac{1-z}{1 - z\lambda - z(1-\lambda) l^s_{B+1}(0)}.$$

Comparing the coefficients of z^n, $n \in \mathbb{Z}^+$, we obtain (4.3.9) of the theorem. ▲

4.3.3 Number of customers in the system

Let $\nu_s \sim \exp(s)$ be an exponential r.v. with parameter $s > 0$. Introduce the transient probabilities of the process $d_{k,x}(t)_{\{t \geq 0\}}$, $k \in [0, B+1]$, $x \geq 0$

$$q^s_{k,x}(0) = \mathbf{P}\left[d_{k,x}(\nu_s) = 0\right], \qquad q^s_{k,x}(u) = \mathbf{P}\left[d_{k,x}(\nu_s) \geq u\right], \quad u \in [1, B+1].$$

Theorem 4.3.4. *The distribution of the number of customers in the system at time ν_s is such that*

$$q^s_{k,x}(0) = 1 - A^{k-1}_x(s) - \left(A^B_0(s) - \mathbf{E}A^{\delta+B}_0(s)\right)\frac{1-\lambda}{\mathbf{E}Q^s_{\delta+B}}\,C^s_{k-1}(x),$$
$$q^s_{k,x}(u) = A^{k-u}_x(s) + \left(A^{B+1-u}_0(s) - \mathbf{E}A^{\delta+B+1-u}_0(s)\right)\frac{1-\lambda}{\mathbf{E}Q^s_{\delta+B}}\,C^s_{k-1}(x), \tag{4.3.11}$$

where $A^k_x(s) = S^s_k(x) = 0$ *for* $k < 0$,

$$C^s_k(x) = \tilde{f}_x(s)(1-\tilde{f}(s))^{-1} + S^s_k(x).$$

We will now explore the asymptotic behavior of the number of customers in the system. In the sequel we will assume that the condition (A) is satisfied:

$$\rho = (1-\lambda)\mu\mathbf{E}\eta\mathbf{E}\varkappa = 1, \quad \sigma^2 = \mu\left[\mathbf{E}\varkappa(\varkappa-1) + \frac{\mathbf{E}\varkappa\mathbf{E}\eta^2}{(1-\lambda)(\mathbf{E}\eta)^2}\right] < \infty$$

Then the following limiting equality holds:

$$\lim_{B\to\infty}\mathbf{P}\left[d_{[kB],x}(tB^2) \geq [uB]\right] \stackrel{\text{def}}{=} q(t) =$$
$$=1 - u - \frac{2}{\pi}\sum_{n\in\mathbb{N}}\frac{e^{-\frac{t}{2}(\pi\sigma n)^2}}{n}\sin(k\pi n)\sin(u\pi n), \quad k, u \in (0,1). \tag{4.3.12}$$

Corollary 4.3.1. *Let* $q_0 = \lim_{t\to\infty}\mathbf{P}[d_{(\cdot)}(t) = 0]$, $q_u = \lim_{t\to\infty}\mathbf{P}[d_{(\cdot)}(t) \geq u]$, $u \in [1, B+1]$ *be the stationary distribution of the number of customers in the* $\mathrm{G}^\delta|\mathrm{M}^\varkappa|1|\mathrm{B}$ *system. Then*

$$q_0 = 1 - \frac{1-\lambda}{\mathbf{E}\eta}\left(A^B_0 - \mathbf{E}A^{\delta+B}_0\right)(\mathbf{E}Q_{\delta+B})^{-1},$$
$$q_u = \frac{1-\lambda}{\mathbf{E}\eta}\left(A^{B+1-u}_0 - \mathbf{E}A^{\delta+B+1-u}_0\right)(\mathbf{E}Q_{\delta+B})^{-1},$$

where $\rho_i = \lim_{s\to 0} s^{-1}\tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}\left[\pi(t) = i\right]dt < \infty$, $Q_k = Q^0_k$,

$$A^u_0 = \lim_{s\to 0}\frac{1}{s}A^u_0(s) = \sum_{i=0}^{u}\rho_i\left[1 - \frac{Q_{u-i}}{1-\lambda}\right].$$

Proof. In view of the definition of the process $Y_{k,x}(t)$, Remark 4.3.2, we can write the following equations for the functions $q^s_{k,x}(u)$, $q^s_{0,x}(u)$ for $u \in [1, B+1]$

$$q^s_{k,x}(u) = p^s_{k,x}(u) + \int_0^\infty b^s_k(x, dy) q^s_{0,y}(u), \quad k \in \overline{1, B+1},$$
$$q^s_{0,y}(u) = \tilde{f}_y(s) \left[\lambda^B q^s_{B+1,0}(u) + (1-\lambda) \sum_{k=1}^{B} \lambda^{k-1} q^s_{k,0}(u) \right], \tag{4.3.13}$$

where the function $b^s_k(x, dy) = \mathbf{E}\left[e^{-sb_k(x)};\ \eta(x) \in dy\right]$ is given by (4.3.5). Inserting the right-hand side of the second equation into the first one, we get

$$q^s_{k,x}(u) = p^s_{k,x}(u) + \int_0^\infty b^s_k(x, dy) \tilde{f}_y(s) q^s_B(\lambda, u),$$

where $q^s_B(\lambda, u) = \lambda^B q^s_{B+1,0}(u) + (1-\lambda) \sum_{k=1}^{B} \lambda^{k-1} q^s_{k,0}(u)$. After some manipulations, we get

$$q^s_B(\lambda, u) = p^s_B(\lambda, u)(1 - \tilde{b}(s, \lambda))^{-1},$$

where $p^s_B(\lambda, u) = \lambda^B p^s_{B+1,0}(u) + (1-\lambda) \sum_{k=1}^{B} \lambda^{k-1} p^s_{k,0}(u)$,

$$\tilde{b}(s, \lambda) = \lambda^B \int_0^\infty b^s_{B+1}(0, dy) \tilde{f}_y(s) + (1-\lambda) \sum_{k=1}^{B} \lambda^{k-1} \int_0^\infty b^s_k(0, dy) \tilde{f}_y(s).$$

Employing (3.3.6), (4.3.2), (4.3.5) and performing necessary calculations, we obtain

$$\int_0^\infty b^s_k(x, dy) \tilde{f}_y(s) = (1 - \tilde{f}(s)) \left(C^s_{k-1}(x) - \frac{Q^s_{k-1}(x)}{1-\lambda} \frac{\mathbf{E} Q^s_{\delta+B}}{\mathbf{E} Q^s_{\delta+B} - Q^s_B} \right),$$
$$1 - \tilde{b}(s, \lambda) = (1 - \tilde{f}(s)) \frac{\mathbf{E} Q_{\delta+B}}{\mathbf{E} Q^s_{\delta+B} - Q^s_B},$$
$$p^s_B(\lambda, u) = (1-\lambda) \frac{A_0^{B+1-u}(s) - \mathbf{E} A_0^{\delta+B+1-u}(s)}{\mathbf{E} Q^s_{\delta+B} - Q^s_B},$$
$$q^s_B(\lambda, u) = \frac{1-\lambda}{1-\tilde{f}(s)} \left(A_0^{B+1-u}(s) - \mathbf{E} A_0^{\delta+B+1-u}(s) \right) (\mathbf{E} Q^s_{\delta+B})^{-1}.$$

In view of these equalities and the equations of the system (4.3.13) we derive the second formula of (4.3.11). Taking into account the definition of the process $Y_{k,x}(t)$, Remark 4.3.2, we find for the functions $q^s_{k,x}(0)$, $q^s_{0,x}(0)$ that

$$q^s_{k,x}(0) = \int_0^\infty b^s_k(x, dy) q^s_{0,y}(0),$$

$$q^s_{0,y}(0) = 1 - \tilde{f}_y(s) + \tilde{f}_y(s) \left[\lambda^B q^s_{B+1,0}(0) + (1-\lambda) \sum_{k=1}^{B} \lambda^{k-1} q^s_{k,0}(0) \right]. \quad (4.3.14)$$

Inserting the right-hand side of the second equation into the first one, we get

$$q^s_{k,x}(0) = \int_0^\infty b^s_k(x, dy)(1 - \tilde{f}_y(s)) + \int_0^\infty b^s_k(x, dy) \tilde{f}_y(s) q^s_B(\lambda), \qquad (4.3.15)$$

where $q^s_B(\lambda) = \lambda^B q^s_{B+1,0}(0) + (1-\lambda) \sum_{k=1}^{B} \lambda^{k-1} q^s_{k,0}(0)$. After some transformations of the latter equation, we obtain

$$q^s_B(\lambda) = 1 - (1 - b(s, \lambda))(1 - \tilde{b}(s, \lambda))^{-1},$$

where

$$b(s, \lambda) = \lambda^B b^s_{B+1}(0) + (1-\lambda) \sum_{k=1}^{B} \lambda^{k-1} b^s_k(0) = 1 - (1-\lambda) \frac{A^B_0(s) - \mathbf{E} A^{B+\delta}_0(s)}{\mathbf{E} Q^s_{\delta+B} - Q^s_B}.$$

The latter and the previous equality imply that

$$q^s_B(\lambda) = 1 - \frac{1-\lambda}{1 - \tilde{f}(s)} \left(A^{B+}_0(s) - \mathbf{E} A^{\delta+B}_0(s) \right) (\mathbf{E} Q^s_{\delta+B})^{-1}.$$

Inserting the right-hand side of this equality into (4.3.15) yields the first equality of (4.3.11).

For $k, u \in (0,1)$ denote $q^t_k(x, u, B) = \mathbf{P}\left[d_{[kB],x}(tB^2) \ge [uB] \right]$. Employing the second formula of (4.3.11), the limiting equality (3.5.7), (3.8.4), we find that

$$\frac{1}{s} \lim_{B\to\infty} q^{s/B^2}_{[kB],x}([uB]) = \lim_{B\to\infty} \int_0^\infty e^{-st} q^t_k(x, u, B)\, dt = \qquad (4.3.16)$$

$$= \frac{1 - \cosh((k-u)^+ \sqrt{2s}/\sigma)}{s} + \frac{1}{s} \frac{\cosh(k\sqrt{2s}/\sigma)}{\sinh(\sqrt{2s}/\sigma)} \sinh((1-u)\sqrt{2s}/\sigma) \stackrel{\text{def}}{=} q^*(s),$$

where $u^+ = \max\{0, u\}$. When $u \in [k, 1)$ we derive from this formula that

$$q^*(s) = \frac{1}{s} \frac{\cosh(k\sqrt{2s}/\sigma)}{\sinh(\sqrt{2s}/\sigma)} \sinh((1-u)\sqrt{2s}/\sigma), \qquad u \in [k, 1).$$

It is clear that $s_0 = 0$ is a simple pole of the function $q^*(s)$. In the semi-plane $\Re(s) < 0$ this function has simple poles in $s_n = -\frac{1}{2}(\sigma\pi n)^2$, $n \in \mathbb{N}$, and it is analytic in the whole plane apart from these points. Hence, for $\alpha > 0$

$$q(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} q^*(s)\, ds = \sum_{n\in\mathbb{Z}^+} \mathrm{Res}_{s=s_n} q^*(s).$$

Calculating the residues of the function $q^*(s)$ in s_n, we obtain the right-hand side of the formula (4.3.12) for $u \in [k, 1)$. On can see that the first term in the right-hand side of (4.3.16) is analytic in the whole plane for $u \in (0, k]$. Applying the inversion formula, we find that the contour integral of this term is equal to zero. The second term of (4.3.16) is the same also for $u \in [k, 1)$. Thus, the formula (4.3.12) holds for $u \in (0, 1)$.

Observe, that $\lim_{s\to 0} A_x^u(s) = \lim_{s\to 0} \mathbf{E}A_0^{\delta+u}(s) = 0$, $\lim_{s\to 0} b_k^s(x) = 1$. It follows from (4.3.11) and the properties of Laplace transforms that

$$\lim_{s\to 0} q_{k,x}^s(u) = \lim_{s\to 0} q_{0,x}^s(u) = q_u = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) \ge u], \qquad u \in [1, B+1],$$
$$\lim_{s\to 0} q_{k,x}^s(0) = \lim_{s\to 0} q_{0,x}^s(0) = q_0 = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) = 0].$$

Calculating the limits in the right-hand sides of (4.3.11) as $s \to 0$ yields the equalities of Corollary 4.3.1. ▲

4.3.4 Virtual waiting time

Suppose that at time $t_0 = 0$ the system is at the state (k, x), $k \in [0, B+1]$, $x \ge 0$. Denote by $W_{k,x}(t)$ the time required to serve the customers present in the system at time t. Formally, this random variable can be determined in the following way. Let $\tilde{\tau}(k) = \inf\{t : \pi(t) \ge k\}$, $k \in \mathbb{Z}^+$. Then

$$W_{k,x}(t) = \tilde{\tau}(d_{k,x}(t)), \qquad \mathbf{E}e^{-pW_{k,x}(t)} = \sum_{i=0}^{B+1} \mathbf{P}\left[d_{k,x}(t) = i\right] \mathbf{E}e^{-p\tilde{\tau}(i)}, \quad p > 0.$$

Corollary 4.3.2. *Let $k \in [0, B+1]$, $x \ge 0$, $\nu_s \sim \exp(s)$. The following equality* *holds for all* $v \geq 0$

$$\mathbf{P}\left[W_{k,x}(\nu_s) \leq v\right] = 1 - \sum_{i=0}^{B} \mathbf{P}\left[\pi(v) = i\right] q^s_{k,x}(i+1),$$

$$\lim_{t\to\infty} \mathbf{P}\left[W_{k,x}(t) \leq v\right] = 1 - \sum_{i=0}^{B} \mathbf{P}\left[\pi(v) = i\right] q_{i+1},$$

where the distributions $q^s_{k,x}(u)$, q_u, $u \in [0, B{+}1]$ *are given by (4.3.11), (4.3.12).*

Proof. It is clear that $\mathbf{P}[\tilde{\tau}(k) > t] = \mathbf{P}[\pi(t) < k]$, and, hence,

$$\mathbf{E}e^{-p\tilde{\tau}(k)} = 1 - \mathbf{P}[\pi(\nu_p) < k] = 1 - \sum_{i=0}^{k-1} \tilde{\rho}_i(p),$$

where $\tilde{\rho}_i(p) = p\int_0^\infty e^{-pv}\mathbf{P}\left[\pi(v) = i\right] dv$. Then

$$\mathbf{E}e^{-pW_{k,x}(\nu_s)} = 1 - \sum_{i=0}^{B+1} \mathbf{P}\left[d_{k,x}(\nu_s) = i\right] \sum_{j=0}^{i-1} \tilde{\rho}_j(p) = 1 - \sum_{i=0}^{B} \tilde{\rho}_i(p) q^s_{k,x}(i+1).$$

The right-hand side of this equality implies the formulae of Corollary 4.3.2. ▲

4.3.5 Special case: G|M$^\varkappa$|1|B system

The process $\{D_x(t)\}_{t\geq 0}$ governing the G|M$^\varkappa$|1|B system has unit negative jumps at the times instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$ and $\delta_{N_x(t)} = N_x(t)$. It means that the customers arrive one by one at the renewal times of the process $N_x(t)$. We will now determine the distributions of the busy period, time of the first loss of a customer, number of the lost customers at time of the first loss, virtual waiting time.

Corollary 4.3.3. *Let* $\mathbf{P}[\delta = 1] = 1$, $b^s_k(x) = \mathbf{E}\left[e^{-sb_r(x)}; b_r(x) < \infty\right]$ *be the Laplace transform of the duration of the busy period of* (k, x) *type of the* G|M$^\varkappa$|1|B *system. Then*

(i) *the following equality holds:*

$$b^s_k(x) = 1 - A^{k-1}_x(s) + Q^s_{k-1}(x)\frac{A^{B+1}_0(s) - A^B_0(s)}{Q^s_{B+1} - Q^s_B}, \quad k \in [1, B+1],$$

the random variable $b_k(x)$ *is proper* ($\mathbf{P}\left[b_k(x) < \infty\right] = 1$) *with finite mathematical expectation*

$$\mathbf{E}b_k(x) = A_x^{k-1} - Q_{k-1}(x)\frac{A_0^{B+1} - A_0^B}{Q_{B+1} - Q_B} < \infty;$$

(ii) *the Laplace transform* $b_k^s(x, dy) = \mathbf{E}\left[e^{-sb_k(x)};\ \eta(x) \in dy\right]$ *of the joint distribution of* $\{b_k(x), \eta(x)\}$ *is such that for* $k \in [1, B+1]$

$$b_k^s(x, dy) = f^{k-1}(x, dy, s) - Q_{k-1}^s(x)\frac{f^{B+1}(0, dy, s) - f^B(0, dy, s)}{Q_{B+1}^s - Q_B^s},$$

where the function $f^k(x, dy, s) = \mathbf{E}\left[e^{-s\tau^k(x)};\ T^k(x) \in dy, \mathfrak{B}^k(x)\right]$, $k \in \mathbb{Z}^+$ *is determined by* ***(3.3.7)***.

Corollary 4.3.4. *Let* $\mathbf{P}[\delta = 1] = 1$, *and let* $l_k(x)$ *be the time of the first loss of a customer in the* G|M$^\varkappa$|1|B *system,* $i_{k,x}(\nu_s)$ *be the number of lost customers in the time interval* $[0, \nu_s]$. *Then*

(i) *the Laplace transform* $l_k^s(x) = \mathbf{E}\left[e^{-sl_k(x)}; l_k(x) < \infty\right]$ *of* $l_k(x)$ *is such that*

$$l_k^s(x) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_{k-1}^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))S_{B+1}^s}, \qquad k \in [0, B+1],$$

where $S_k^s(x) = \sum_{i=0}^k Q_i^s(x)$, $S_k^s(x) = 0$ *for* $k < 0$. *The random variable* $l_k(x)$ *is proper with finite mathematical expectation*

$$\mathbf{E}l_k(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta\left[S_{B+1} - S_{k-1}(x)\right] < \infty;$$

(ii) *the distribution* $I_{k,x}^s(n) = \mathbf{P}\left[i_{k,x}(\nu_s) = n\right]$, $n \in \mathbb{Z}^+$ *of the number of lost customers on the time interval* $[0, \nu_s]$ *obeys the equality:*

$$I_{k,x}^s(n) = \mathbf{I}_{\{n=0\}}(1 - l_k^s(x)) + \mathbf{I}_{\{n\in\mathbb{N}\}}l_k^s(x)(1 - l_{B+1}^s(0))(l_{B+1}^s(0))^{n-1}.$$

Corollary 4.3.5. *The distribution of the number of customers in the system* G|M$^\varkappa$|1|B *at time* ν_s *is such that*

$$q_{k,x}^s(0) = 1 - A_x^{k-1}(s) - \left(A_0^B(s) - A_0^{B+1}(s)\right)C_{k-1}^s(x)/Q_{B+1}^s,$$
$$q_{k,x}^s(u) = A_x^{k-u}(s) + \left(A_0^{B+1-u}(s) - A_0^{B+2-u}(s)\right)C_{k-1}^s(x)/Q_{B+1}^s,$$

where $A_x^k(s) = S_k^s(x) = 0$ *for* $k < 0$, $C_k^s(x) = \tilde{f}_x(s)(1-\tilde{f}(s))^{-1} + S_k^s(x)$.
Let $q_0 = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) = 0]$, $q_u = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) \geq u]$, $u \in [1, B+1]$ *be the stationary distribution of the number of customers in the* G|M$^{\varkappa}$|1|B *system. Then*

$$q_0 = 1 - \frac{1}{\mathbf{E}\eta}\left(A_0^B - A_0^{B+1}\right)/Q_{B+1},$$
$$q_u = \frac{1}{\mathbf{E}\eta}\left(A_0^{B+1-u} - A_0^{B+2-u}\right)/Q_{B+1},$$
$$\lim_{t\to\infty} \mathbf{P}\left[W_{k,x}(t) \leq v\right] = 1 - \frac{1}{\mathbf{E}\eta}\sum_{i=0}^{B} \mathbf{P}\left[\pi(v) = i\right]\left(A_0^{B-i} - A_0^{B+1-i}\right)/Q_{B+1},$$

where $\rho_i = \lim_{s\to 0} s^{-1}\tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}\left[\pi(t) = i\right] dt < \infty$, $Q_k = Q_k^0$,

$$A_0^u = \lim_{s\to 0}\frac{1}{s}\, A_0^u(s) = \sum_{i=0}^{u} \rho_i\left[1 - Q_{u-i}\right].$$

In order to prove Corollaries 4.3.3–4.3.5, it suffices to set $\lambda = 0$ in the formulae of Theorem 4.3.4 and of Corollaries 4.3.1–4.3.3.

Chapter 5

Concluding remarks and further research

Both in applied sciences and in the academic world Lévy processes are considered as a valuable object. For this reason, studying this class of stochastic processes and problems of their modeling receive much attention. Despite the numerous works, there still remain a lot of open problems. Some of these issues have been addressed in this thesis.

One part of this dissertation is concerned with so called one- and two-sided exit problems for Lévy processes. This is a problem of determining the law of the first passage of a level (the first exit time from a fixed interval) by the process. Next, we studied other important characteristics such as position of the process at the first exit time, the value of the overshoot through the level, the sojourn time spent inside the interval, the number of intersections of the interval, etc. We proposed a general approach for deriving the Laplace transforms of the mentioned characteristics for the general Lévy processes. Additionally, asymptotic analysis of these characteristics was performed. As a result, we established the weak convergence of these functionals to the corresponding functionals of the Wiener process (whose two-boundary functionals are well known). In Chapter 2 we extended the existing solutions for particular classes of Lévy processes to the most general case. The methodology is mainly based on a probabilistic approach, use of the one-boundary functionals of the process and the theory of Fredholm equations of the second kind. It is worth mentioning that an essential part of the results is given in closed form, in terms of the scale functions of the process.

From a computational point of view, the scale function approach can rely on a large number of works on Laplace transform methods (Lee (2004), Duffie *et al.* (2000)) and integro-differential equations (Kythe and Puri (2002)). Rogers (2000) and Surya (2008), for instance, provided robust methods for numerically computing scale functions. In recent literature scale functions proved to play a substantial role in optimal barrier strategies, (see: Zhou (2005), Renaud and Zhou (2007), Kyprianou and Palmowski (2007), Albrecher *et al.* (2008), Kyprianou and Loeffen (2008) and Kyprianou *et al.* (2008b)).

From a practical point of view, the results obtained can be applied for modeling options prices, probability of ruin of an insurance company, number of customers in the queue, overflow of a dam, number of lost packages. Other applications are in physics (Kramer's problem) and biology (neuron threshold models see Van Kampen (1992)).

The aforementioned methodology proved to be efficient for more complicated classes of stochastic processes, such as a semi-Markov walk and a difference of compound renewal processes. In Chapter 3 we studied several two-boundary characteristics for the difference of a compound Poisson process and a compound renewal process. Additionally, we considered processes reflected at their infimum (supremum) which serve as governing processes in various applications.

The final part of this thesis deals with applications of the results obtained in queueing theory. It is concerned with determining important performance measures of single server queueing systems with batch arrivals and finite buffer. One of the crucial performance issues of the single-server queue with finite buffer is losses, namely, customers (packets, cells, jobs) that were not allowed to enter the system due to the buffer overflow. This issue is especially important in the analysis of telecommunication networks. The overflow interval plays an important role in the design of forward error correction (FEC) algorithms in packet networks. Therefore, knowledge of the distribution of the time of the first loss, as well as the number of lost customers, the busy period play a solid role. We obtained the Laplace transforms of these characteristics in terms of the resolvent sequences and their modifications.

This research has risen a number of open problems. Despite the universal nature, our approach leads to cumbersome expressions for the quantities of interest. This is the price to pay for generality. Although we have presented the exact formulae for the double Laplace transforms of the major two-boundary functionals, still the question arises how to invert the Laplace transforms obtained. Recently a lot of articles on the inversions techniques have appeared. See for instance, Abate *et al.* (1996), Whitt (1999) Den Iseger (2006), Abate *et al.* (1998). This brings us to continuation of the research in numerical direction.

From a practical point of view, the estimation of the parameters of the aforementioned queueing models is needed. Another direction of future work is to study governing processes for the oscillating queueing systems. Motivation to consider oscillating queueing systems stems from the following. In the basic model of a single server queue it is desired to have the average service time shorter than the average inter-arrival time, so that the queue length tends to be smaller. In many situations we do not have this convenience. For instance, it might be too expensive to consistently use a high performing server. Hence, problem of a long queue could be resolved by designing an oscillating queueing system. To our knowledge, there are few works related to this topic. We mention Bekker *et al.* (2008), for instance, who considered oscillating spectrally one-sided Lévy processes and Chydzinski (2002) who studied the $M|GG|1$ oscillating queueing system. The author used the potential method to determine the steady-state distribution of the length of the queue. Bekker *et al.* (2009) studied Lévy processes with adaptable exponent. Examples of such models are queueing models in which the service speed or customer arrival rate changes depending on the workload level, and dam models in which the release rate depends on the buffer content.

Let us verify the equalities (3.8.4) of Lemma 3.8.1. Suppose that the condition (A) (given in Section (3.5) of Chapter 3) is satisfied. Then for $s, p \to 0$ the following expansions hold

$$\tilde{f}_x(s) = 1 - s\mathbf{E}\eta_x + \frac{1}{2}s^2\mathbf{E}\eta_x^2 + o(s^2), \quad x \geq 0,$$
$$\mathbf{E}e^{-p\varkappa} = 1 - p\mathbf{E}\varkappa + \frac{1}{2}p^2\mathbf{E}\varkappa^2 + o(p^2). \tag{5.0.1}$$

We now derive the asymptotic expansions for the function $S_{[kB]}^{s/B^2}(x)$ as $B \to \infty$. The generating function

$$\mathbb{S}_\theta^s(x) = \sum_{k \in \mathbb{Z}^+} \theta^k \sum_{i=0}^{k} Q_i^s(x) = \frac{1}{1-\theta} \frac{(1-\lambda)\tilde{f}_x(s - k(\theta))}{(1-\lambda)\tilde{f}(s-k(\theta)) + \lambda - \theta} \quad \theta \in (0, c(s)).$$

is such that for $\theta = e^{-p}$, $p > -ln\, c(s)$

$$\mathbb{S}_\theta^s(x) = \sum_{k \in \mathbb{Z}^+} \theta^k S_k^s(x)\Big|_{\theta = e^{-p}} = \int_0^\infty e^{-p[k]} S_{[k]}^s(x)\, dk =$$
$$= \int_0^\infty e^{\{k\}p} e^{-pk} S_{[k]}^s(x)\, dk = \mathbb{S}_{e^{-p}}^s(x), \qquad p > -ln\, c(s),$$

where $\{a\}$ is the fractional part of the number a. By $\mathfrak{S}_p^s(x) = \int_0^\infty e^{-pk} S_{[k]}^s(x)\, dk$, $p > -ln\, c(s)$ denote the Laplace transform of the function $S_{[k]}^s(x)$. It is clear that

$$\mathfrak{S}_p^s(x) \leq \mathbb{S}_{e^{-p}}^s(x) \leq e^p \mathfrak{S}_p^s(x). \tag{5.0.2}$$

Employing the limiting equalities (5.0.1) and the definition of the function $\mathbb{S}_\theta^s(x)$, we obtain

$$\lim_{B\to\infty} \mathbb{S}_{e^{-p/B}}^{s/B^2}(x) B^{-3} = \lim_{B\to\infty} \frac{B^{-3}}{1 - e^{-p/B}} \frac{(1-\lambda)\tilde{f}_x(s/B^2 - k(e^{-p/B}))}{(1-\lambda)\tilde{f}(s/B^2 - k(e^{-p/B})) + \lambda - e^{-p/B}} =$$
$$= \frac{1}{sp\mathbf{E}\eta} \frac{1}{\frac{1}{2}p^2\sigma^2 - s}, \qquad p > \sqrt{2s}/\sigma.$$

It follows from the chain (5.0.2) that

$$\lim_{B\to\infty} \frac{1}{B^3} \mathfrak{S}_{p/B}^{s/B^2}(x) = \lim_{B\to\infty} \frac{1}{B^3} \mathbb{S}_{e^{-p/B}}^{s/B^2}(x) = \frac{1}{sp\,\mathbf{E}\eta} \frac{1}{\frac{1}{2}p^2\sigma^2 - s}. \tag{5.0.3}$$

Inverting the Laplace transforms (with respect to p) in both sides, we obtain

$$\lim_{B\to\infty}\frac{1}{B^2}\,S^{s/B^2}_{[kB]}(x)=\frac{1}{s\mathbf{E}\eta}\left(\cosh(k\sqrt{2s}/\sigma-1\right).$$

In order to invert the Laplace transforms, we have calculated the residues in the right-hand side of (5.0.3) in the simple poles $p=0,\pm\sqrt{2s}/\sigma$. The second part of the first equality (3.8.4) can be verified analogously:

$$\lim_{B\to\infty}B^{-2}\mathbf{E}S^{s/B^2}_{\delta+[kB]}=\frac{1}{s\mathbf{E}\eta}\left(\cosh(k\sqrt{2s}/\sigma-1\right).$$

We will now verify the second formula of (3.8.4). Denote $q^s_k=\mathbf{E}Q^s_{\delta+k}-Q^s_k$, $k\in\mathbb{Z}^+$. Employing (3.3.4), (3.3.5), we determine the generating function of this sequence

$$\tilde q^s_\theta=\sum_{k\in\mathbb{Z}^+}\theta^k q^s_k=\frac{(1-\lambda)(1-\tilde f(s-k(\theta)))}{(1-\lambda)\tilde f(s-k(\theta))+\lambda-\theta},\qquad \theta\in(0,c(s)).$$

This generating function is such that for $\theta=e^{-p}$, $p>-ln\,c(s)$

$$\tilde q^s_{e^{-p}}=\int_0^\infty e^{-p[k]}q^s_{[k]}\,dk=\int_0^\infty e^{\{k\}p}e^{-pk}q^s_{[k]}\,dk.$$

By $\mathfrak{q}^s_p=\int_0^\infty e^{-pk}q^s_{[k]}\,dk,\quad p>-ln\,c(s)$ denote the Laplace transform of the function $q^s_{[k]}$. It is clear that

$$\mathfrak{q}^s_p\le\tilde q^s_{e^{-p}}\le e^p\mathfrak{q}^s_p. \tag{5.0.4}$$

In view of the limiting equalities (5.0.1) and the definition of the function $\tilde q^s_\theta$, we find

$$\lim_{B\to\infty}\tilde q^{s/B^2}_{e^{-p/B}}B^{-1}=\lim_{B\to\infty}\frac{(1-\lambda)(1-\tilde f(s/B^2-k(e^{-p/B})))B^{-1}}{(1-\lambda)\tilde f(s/B^2-k(e^{-p/B}))+\lambda-e^{-p/B}}=\frac{p\,\mu\mathbf{E}\varkappa}{\frac12p^2\sigma^2-s}.$$

It follows from the chain (5.0.4) that

$$\lim_{B\to\infty}\frac{1}{B}\,\mathfrak{q}^{s/B^2}_{p/B}=\lim_{B\to\infty}\frac{1}{B}\,\tilde q^{s/B^2}_{e^{-p/B}}=\frac{p\,\mu\mathbf{E}\varkappa}{\frac12p^2\sigma^2-s}.$$

Inverting the Laplace transforms (with respect to p) in both sides, we obtain

$$\lim_{B\to\infty}q^{s/B^2}_{[kB]}=\lim_{B\to\infty}\left[\mathbf{E}Q^{s/B^2}_{[\delta+kB]}-Q^{s/B^2}_{[kB]}\right]=\frac{\mu\mathbf{E}\varkappa}{\sigma^2}\cosh\left(k\sqrt{2s}/\sigma\right).$$

The third formula of (3.8.4) can be verified analogously.

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In dit proefschrift bestuderen we de zogenaamde tweezijdige overschrijdingsproblemen voor verschillende klassen van stochastische processen. Deze stochastische processen dienen als mathematische modellen voor verschillende fenomenen. Meer specifiek houden we ons bezig met Lévy processen en met verschillen van samengestelde vernieuwingsprocessen. Een Lévy proces is een stochastisch proces dat onafhankelijke, gelijk verdeelde toenamen heeft en waarvan de paden rechtscontinu zijn en linker limieten hebben. De klasse van Lévy processen heeft een zeer rijke structuur, wat onder andere blijkt uit het feit dat de klasse in één op één verhouding staat met de klasse van oneindig deelbare verdelingen.

De rijke structuur van deze processen en hun eigenschappen laten allerlei toepassingen toe in verschillende gebieden zoals risicotheorie, financiële wiskunde, wachtlijntheorie, biologische wetenschappen en zo voort. In het bijzonder, zijn Lévy processen uitstekend geschikt voor het modelleren van de prijsbepaling van bijvoorbeeld opties in financiële wiskunde.

Deze thesis kan onderverdeeld worden in drie delen. Na een inleiding beschouwen we (in Hoofdstuk 2) een aantal karakteristieken voor algemene Lévy processen en hun speciale deelklassen. We bestuderen het zogenaamde het eerste passage probleem: wat is de kansverdeling van het eerste tijdstip waarop het Lévy proces een bepaald niveau overschrijdt of een interval verlaat en zijn positie op dat ogenblik? Dezelfde vraag beantwoorden we voor bepaalde deelklassen zoals Lévy processen met sprongen in één richting en het samengesteld Poisson proces met sprongen in beide richtingen.

De gevonden resultaten stellen ons in staat andere karakteristieken van het proces te bepalen. Aldus vinden wij de Laplace getransformeerde van de gezamenlijke verdeling van infimum, supremum en positie van het proces op een willekeurige tijdstip, het aantal kruisingen van het interval, de totale verblijfsduur binnen het interval, het aantal binnenkomsten in het interval. Ook geven wij de beschrijving van het ergodisch gedrag van deze karakteristieken. We geven een wiskundig bewijs voor de zwakke convergentie van deze verdelingen naar de corresponderende verdelingen van het Wiener proces. Daarbij bepalen we bijvoorbeeld hoe groot bij benadering de kans is dat zo'n Lévy proces pas na zeer lange tijd een eindig interval verlaat.

Vanuit praktisch oogpunt zijn de verkregen uitdrukkingen bruikbaar voor het bestuderen van andere klassen van stochastische processen en toepassingen ervan: modelleren van het prijsgedrag van opties, het aantal verloren pakketten, het aantal wachtende klanten in de wachtlijn, de kans op overstroming van een dijk, de kans op ruïne van een verzekeringsmaatschappij en zo voort.

In de loop van het werk aan de bovengenoemde hoofdproblemen hebben wij de volgende probabilistische aanpak gevolgd. Voor de Laplace getransformeerde van de gezamenlijke verdeling van het eerste uitgangstijd van het interval en de waarde van overschrijding stellen wij een stelsel voor van lineaire integraalvergelijkingen. De oplossing ervan wordt gevonden met behulp van de theorie van Fredholm vergelijkingen van de tweede soort en de methode van opeenvolgende iteraties. Het belangrijkste aspect hiervan is het gebruik van de Laplace getransformeerde van de eenvoudige karakteristieken van het proces (gezamenlijke verdeling van de eerste passagetijd en de waarde van de overschrijding van een bepaald niveau). Deze zijn te bepalen via de Wiener-Hopf factoren. In de meest algemene situatie is het niet evident om oplossingen te vinden in gesloten vorm, maar voor bepaalde subklassen krijgen we de uitdrukkingen in termen van zogenaamde schaalfuncties. Aan de hand van deze oplossing kunnen we andere belangrijke karakteristieken van het proces bepalen.

Vertrekkend van deze methodologie, hebben we onze bevindingen uitgebreid naar het verschil van samengestelde vernieuwingsprocessen en semi-Markov wandelingen. In Hoofdstuk 3 wordt de bestaande methode uitgebreid voor het verschil van samengestelde vernieuwingsprocessen. Zulke processen kunnen gebruikt worden als mathematische modellen van de wachtlijnsystemen.

Hoofdstuk 4 handelt over toepassingen van de gevonden resultaten in de wachtlijntheorie. We illustreren dit door het bepalen van de bezige periode van het systeem, het aantal klanten en het tijdstip waarop we het eerste verlies van klanten ervaren voor $M^{\varkappa}|G^{\delta}|1|B$ en $G^{\delta}|M^{\varkappa}|1|B$ wachtlijnsystemen (zie Hoofdstuk 4 voor de formele beschrijving). Bij de inrichting van computersystemen, communicatienetwerken en productiesystemen kent de wachtlijntheorie een belangrijke toepassing. In de telecommunicatie treden wachtrijen op bij telefoonoproepen die op een vrije lijn in de centrale wachten, digitale berichten die op een vrij communicatiekanaal wachten, en zo voort.

We bestuderen wachtlijnmodellen met bulk aankomsten, één verwerkingseenheid en een eindige buffer. (Bulk aankomst betekent dat op elk aankomsttijdstip klanten als groep het systeem binnenkomen). Het aankomstproces wordt dan door zowel de verdeling van de tussenaankomsttijden, als de verdeling van de grootte van de bulk bepaald. De capaciteit van de bufferruimte geeft aan hoeveel klanten tegelijk in het systeem kunnen aanwezig zijn. Bij een eindige capaciteit kan overflow optreden en bijgevolg verlies van klanten, pakketten, e-mails en zo voort. Tijdstip van overflow is dus een essentiële performantie karakteristiek van het systeem. Gemotiveerd door dit feit hebben wij onder ander de verdeling van het tijdstip van het eerste verlies van de klanten verkregen. In dit kader bewijzen we dat bepalen van deze karakteristieken zich vertaalt in het oplossen van een tweezijdige overschrijdingsprobleem voor het proces dat het gedrag van het wachtlijn regeert. Dit proces is een modificatie van een twee componenten Markov proces dat in Hoofdstuk 3 beschouwd wordt.

Tenslotte, in Hooofdstuk 5, bespreken we verdere uitbreidingen en voortzetting van ons onderzoek. De volgende stap is de implementatie van concrete wachtlijnmodellen en de schatting van de parameters van het model. Voor de numerieke toepassingen is verder onderzoek noodzakelijk.