# A DEGREE INEQUALITY FOR LIE ALGEBRAS WITH A REGULAR POISSON SEMI-CENTER 

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#### Abstract

For Lie algebras whose Poisson semi-center is a polynomial ring we give a bound for the sum of the degrees of the generating semi-invariants. This bound was previously known in many special cases.


## 1. Introduction

In this paper we work over an algebraically closed base field $k$ of characteristic zero. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A non-zero element $f \in S \mathfrak{g}$ is called a semi-invariant with weight $\chi \in \mathfrak{g}^{*}$ if for all $v \in \mathfrak{g}$ we have

$$
\operatorname{ad}(v)(f)=\chi(v) f
$$

We say that a semi-invariant is proper if $\chi \neq 0$. The $k$-algebra generated by the semi-invariants in $S \mathfrak{g}$ is denoted by $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}$. This ring is called the Poisson semicenter of $S \mathfrak{g}$.

The stabilizer of $x \in \mathfrak{g}^{*}$ is denoted by $\mathfrak{g}_{x}^{*}$. I.e. $\mathfrak{g}_{x}=\{v \in \mathfrak{g} \mid \forall w \in \mathfrak{g}: x([v, w])=$ $0\}$. The minimal value of $\operatorname{dim} \mathfrak{g}_{x}$ is called the index of $\mathfrak{g}$ and is denoted by $i(\mathfrak{g})$. An element $x \in \mathfrak{g}^{*}$ is called regular if $\operatorname{dim} \mathfrak{g}_{x}^{*}=i(\mathfrak{g})$. The regular elements form an open dense subset of $\mathfrak{g}^{*}$ which we denote by $\mathfrak{g}_{\text {reg }}^{*}$.

The following is our main result.
Theorem 1.1. (see Prop. 3.1 and Prop. 5.7 below.) Assume that $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}$ is freely generated by homogeneous elements $f_{1}, \ldots, f_{r}$. Then

$$
\begin{equation*}
\sum_{i=1}^{r} \operatorname{deg} f_{i} \leq \frac{1}{2}(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g})) \tag{1.1}
\end{equation*}
$$

It is well-known that (1.1) holds for semi-simple Lie algebras [5, Thm. 7.3.8] and Frobenius Lie algebras [2, pp. 339-343]. Numerous other special cases are known (e.g. [7, 14]).

For Theorem 1.1 to be valid in the stated generality it is essential that we consider semi-invariants instead of invariants, as the following trivial example by Panyushev shows.
Example 1.2. Let $\mathfrak{g}=k v_{1}+k v_{2}+k v_{3}+k v_{4}$ with non-trivial brackets $\left[v_{1}, v_{2}\right]=v_{2}$, $\left[v_{1}, v_{3}\right]=v_{3},\left[v_{1}, v_{4}\right]=-v_{4}$. Then $\operatorname{dim} \mathfrak{g}=4, i(\mathfrak{g})=2$. The generating invariants are $v_{2} v_{4}$ and $v_{3} v_{4}$. So the sum of their degrees is $2+2=4$ which is strictly bigger than $1 / 2(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g}))=3$. However the generating semi-invariants are $v_{2}, v_{3}, v_{4}$ and the sum of their degrees is 3 , which does not violate the inequality.

[^0]For brevity we will call a Lie algebra coregular if $(S \mathfrak{g})^{\mathfrak{g}}$ is a polynomial ring.
Corollary 1.3. Assume that $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$ and $\mathfrak{g}$ is coregular with center $Z(\mathfrak{g})$. Then

$$
\begin{equation*}
3 i(\mathfrak{g}) \leq \operatorname{dim} \mathfrak{g}+2 \operatorname{dim} Z(\mathfrak{g}) \tag{1.2}
\end{equation*}
$$

Proof. In this situation we have the equality $r=i(\mathfrak{g})$ (see Proposition 4.1 below). The observation that $\operatorname{deg} f_{i} \geq 2$, unless $f_{i} \in Z(\mathfrak{g})$ yields

$$
\operatorname{dim} Z(\mathfrak{g})+2(i(\mathfrak{g})-\operatorname{dim} Z(\mathfrak{g})) \leq \sum_{i=1}^{r} \operatorname{deg} f_{i} \leq \frac{1}{2}(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g}))
$$

which translates into (1.2).
The number on the right hand side of (1.1) occurs frequently in the theory of enveloping algebras. For example it is an upper bound for the transcendence degree of a maximal commutative subfield of the division ring of fractions of $U \mathfrak{g}$ and this bound can be achieved in many cases [11, 16]. Likewise by a result of Sadetov [21] it is the maximum transcendence degree of a Poisson commutative subfield of the field of fractions of $S \mathfrak{g}$.

For the proof of Theorem 1.1 we first reduce to the case that there are no proper semi-invariants (i.e. $\left.(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}\right)$. In this situation one may prove a result which is more precise than Theorem 1.1. Assume first that $\mathfrak{g}$ is non-abelian. Let $B=\left(\left[v_{i}, v_{j}\right]\right)_{i j} \in M_{n}(S \mathfrak{g})$ be the structure matrix of $\mathfrak{g}$ where $v_{1}, \ldots, v_{n}$ is an arbitrary basis of $\mathfrak{g}$. Put $s=\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})$. Then the greatest common divisor of the $s \times s$-minors in $B$ is a semi-invariant in $S \mathfrak{g}$ [2]. Below we will call it the fundamental semi-invariant and we denote its degree by $d(\mathfrak{g})$. If $\mathfrak{g}$ is abelian we put $d(\mathfrak{g})=0$.

Proposition 1.4. (see Prop. 5.7.) Assume that $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$ and $\mathfrak{g}$ is coregular. Then we have

$$
\begin{equation*}
\sum_{i=1}^{r} \operatorname{deg} f_{i}=\frac{1}{2}(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g})-d(\mathfrak{g})) \tag{1.3}
\end{equation*}
$$

Taking into account Propositions 4.1 and 5.1 below, this result may also be deduced from [17, Remark 1.6.3]. Our proof uses the general techniques from $[12,13]$ and is quite different from [17]. We obtain a certain nice complex of length three, consisting of free $S \mathfrak{g}$-modules which, besides implying (1.3), yields some additional information on $\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}$ (see Proposition 1.6 below).

Corollary 1.5. Assume that $(S \mathfrak{g})_{s i}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$ and $\mathfrak{g}$ is coregular. Then (1.1) is an equality if and only if $\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}$ has codimension $\geq 2$.

This follows from the easily verified fact that $d(\mathfrak{g})=0$ if and only if $\operatorname{codim}_{\mathfrak{g}^{*}}\left(\mathfrak{g}^{*} \backslash\right.$ $\left.\mathfrak{g}_{\text {reg }}^{*}\right) \geq 2$.

Proposition 1.4 is false without the assumption $(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$. Counter examples are given by Frobenius Lie algebras. By definition these satisfy $i(\mathfrak{g})=0$ and thus the fundamental semi-invariant is equal to $\operatorname{det} B$. Hence $d(\mathfrak{g})=\operatorname{dim} \mathfrak{g}$ and the righthand side of (1.3) is zero. Since $(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}$ is freely generated by the irreducible factors of det $B[2]$ the lefthand side of (1.3) is never zero. It would be interesting to find a version of Proposition 1.4 which holds in the same generality as Theorem 1.1.

As mentioned above we may use our methods to obtain some additional necessary conditions for coregularity. As $\mathfrak{g}$ acts by derivations on $S \mathfrak{g}$ we have a $S \mathfrak{g}$-linear map

$$
\rho: S \mathfrak{g} \otimes \mathfrak{g} \rightarrow \operatorname{Der}_{k}(S \mathfrak{g})=S \mathfrak{g} \otimes \mathfrak{g}^{*}
$$

Proposition 1.6. (see Prop. 5.4,5.10) Assume that $(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$ and $\mathfrak{g}$ is coregular. Then
(1) $\operatorname{ker} \rho$ is a free $S \mathfrak{g}$-module;
(2) if $\mathfrak{g}$ is not abelian then $\operatorname{codim}\left(\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}\right) \leq 3$.
(3) If $\operatorname{codim}\left(\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}\right)=3$ then $\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}$ is purely of codimension three.

Example 1.7. We illustrate the above results with an easy example. For $n \geq 3$ let $\mathfrak{g}=L(n)$ be the $n$-dimensional standard filiform Lie algebra. $L(n)$ has a basis $v_{1}, \ldots, v_{n}$ and non-trivial brackets $\left[v_{1}, v_{i}\right]=v_{i+1}$ for $i=2, \ldots, n-1$. In this case

$$
(S \mathfrak{g})^{\mathfrak{g}}=k\left[v_{2}, \ldots, v_{n}\right]^{e}
$$

where $e$ is the derivation

$$
e=\sum_{i=2}^{n-1} v_{i+1} \frac{\partial}{\partial v_{i}}
$$

Dixmier verified by direct computation that $L(3), L(4)$ are coregular but $L(5)$ is not [4]. From the classical correspondence between $\mathbb{G}_{a}$-invariants and $\mathrm{SL}_{2}{ }^{-}$ covariants (e.g. [8, §33]) one obtains that $L(n)$ is coregular if and only if $n<5$ (see e.g. [22]).

In order to apply the criteria given above it is advantageous to use the structure matrix $B$ which was already introduced. It is easy to see that the $S \mathfrak{g}$-linear map $\rho$ is represented by the matrix $B$. Furthermore $\mathfrak{g}_{x}=\operatorname{ker} x(B)$. If we write $r(\mathfrak{g})$ for the rank of $B$ over the quotient field of $S \mathfrak{g}$ then $i(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-r(\mathfrak{g})$.

In the case of $L(n)$ the structure matrix looks like

$$
\left(\begin{array}{ccccc}
0 & v_{3} & \cdots & v_{n} & 0 \\
-v_{3} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
-v_{n} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{array}\right)
$$

We deduce $i(\mathfrak{g})=n-2$. Furthermore the fundamental semi-invariant is 1 unless $n=3$ in which case it is $v_{3}^{2}$. Thus

$$
d(\mathfrak{g})= \begin{cases}0 & \text { if } n>3 \\ 2 & \text { if } n=3\end{cases}
$$

Since $\mathfrak{g}$ is nilpotent there are no proper semi-invariants. As $Z(\mathfrak{g})=k v_{n}$ the numerical criterion (1.2) for coregularity becomes

$$
3(n-2) \leq n+2
$$

which holds iff $n \leq 4$. Hence the non-coregularity of $L(n)$ for $n \geq 5$ is detected by (1.2).

We have

$$
\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}=\left\{x \in \mathfrak{g}^{*} \mid x\left(v_{i}\right)=0 \text { for } i=3, \ldots, n\right\}
$$

Thus $\operatorname{codim}\left(\mathfrak{g}^{*} \backslash \mathfrak{g}_{\text {reg }}^{*}\right)=n-2$ and so the fact that $L(n)$ is not coregular for $n \geq 6$ is detected by the numerical criterion Prop. 1.6(2).

If $n=5$ then
$\operatorname{ker} \rho=\left\{\left(A_{1}, \ldots, A_{5}\right) \in k\left[v_{1}, \ldots, v_{5}\right] \mid A_{2} v_{3}+A_{3} v_{4}+A_{4} v_{5}=0, A_{1} v_{3}=A_{1} v_{4}=A_{1} v_{5}=0\right\}$
This kernel is minimally generated by $w_{1}=\left(0, v_{4},-v_{3}, 0,0\right), w_{2}=\left(0,0, v_{5},-v_{4}, 0\right)$, $w_{3}=\left(0, v_{5}, 0,-v_{3}, 0\right)$ and $w_{4}=(0,0,0,0,1)$. These generators are related by $v_{5} w_{1}+v_{3} w_{2}-v_{4} w_{3}=0$ so they are not free. Thus the non-coregularity of $L(5)$ is detected by Prop. 1.6(1) but not by 1.6(2).

Let us now consider $n=3$. In this case the generating invariant is $v_{3}$ and the equality (1.3) becomes $1=(1 / 2)(3+1-2)$.

Assume $n=4$. Now the generating invariants are $v_{4}$ and $v_{2} v_{4}-(1 / 2) v_{3}^{2}$. Then (1.3) becomes $1+2=(1 / 2)(4+2-0)$.

Although not directly related to the content of this paper let us remind the reader that much is known classically about the invariant theory of $\mathrm{SL}_{2}$. This may be translated back into results about $L(n)$. For $\mathfrak{g}=L(7)$ one finds that $(S \mathfrak{g})^{\mathfrak{g}}$ is minimally generated by 23 elements (see [8, §115]). On the other hand the transcendence degree of the fraction field of $(S \mathfrak{g})^{\mathfrak{g}}$ is only 5 .

We refer to [14] for explicit generators of $(S \mathfrak{g})^{\mathfrak{g}}$ for many nilpotent Lie algebras of dimension at most 7 .

We wish to thank Alexander Elashvili for many stimulating discussions around this and related problems.

## 2. Preliminaries

Throughout $\mathfrak{g}$ is a finite dimensional Lie algebra. If $V$ is a finite dimensional representation of $\mathfrak{g}$ then we denote by $(S V)_{\mathrm{si}}^{\mathfrak{g}}$ the ring of semi-invariants in $S V$. Note that if $f$ is a semi-invariant and $g \in S V$ divides $f$ then $g$ is a semi-invariant as well. Thus any semi-invariant in $S V$ is a product of semi-invariants which are irreducible in $S V$.

If $x \in V^{*}$ then $\partial_{x}$ is the derivation of $S V$ such that for $v \in V$ we have $\partial_{x}(v)=$ $x(v)$.

We equip $S \mathfrak{g}$ with the Kostant-Kirillov Poisson bracket of degree -1

$$
\left\{v_{1}, v_{2}\right\}=\left[v_{1}, v_{2}\right] \quad\left(v_{1}, v_{2} \in \mathfrak{g}\right)
$$

If $g \in S \mathfrak{g}$ is a semi-invariant with weight $\chi$ then for all $f \in S \mathfrak{g}$ we have $\{f, g\}=$ $\partial_{\chi}(f) g$. From this we easily deduce the well-known fact that semi-invariants in $S \mathfrak{g}$ Poisson commute.

It will be convenient to introduce the Lie-Rinehart algebra $S V \otimes \mathfrak{g}$ [19]. This is a Lie algebra with Lie bracket

$$
[f \otimes v, g \otimes w]=f v(g) \otimes w-g w(f) \otimes v+f g \otimes[v, w]
$$

Sending $f \otimes v$ to $f v(-)$ defines an $S \mathfrak{g}$-linear Lie algebra homomorphism

$$
\rho: S V \otimes \mathfrak{g} \rightarrow \operatorname{Der}_{k}(S V)
$$

which is called the anchor map. If $\left(v_{i}\right)_{i}$ is a basis for $\mathfrak{g}$ then the kernel of the anchor map is given by the sums $\sum_{i} c_{i} \otimes v_{i}$ such that $\sum_{i} c_{i} v_{i}(w)=0$ for all $w \in V$. Note that this kernel is a Lie ideal (as is any kernel of a homomorphism between Lie algebras).

If we use the identification $\operatorname{Der}_{k}(S V)=S V \otimes V^{*}$ and we choose bases $\left(v_{i}\right)_{i=1}^{n}$, $\left(w_{j}\right)_{j=1}^{m}$ for $\mathfrak{g}$ and $V$ then the anchor map is represented with respect to these bases by the structure matrix $\left(v_{i}\left(w_{j}\right)\right)_{j i} \in M_{m \times n}(S V)$ of $V$.

For convenience we will write $r(V)$ for the rank of the structure matrix of $V$ over the field of fractions of $S V$. If $\mathfrak{g}$ is in doubt we write $r_{\mathfrak{g}}(V)$. There is an open subset $V_{\text {reg }}^{*}$ of $V^{*}$ such that $x \in V_{\text {reg }}^{*}$ iff $\operatorname{dim} \mathfrak{g}_{x}=\operatorname{dim} V-r(V)$ where $\mathfrak{g}_{x}$ denotes the stabilizer of $x$. I.e. $\mathfrak{g}_{x}=\{v \in \mathfrak{g} \mid \forall w \in V: x([v, w])=0\}$.

The fundamental semi-invariant in $S V$ is defined as the greatest common divisor of the $r(V) \times r(V)$ minors in the structure matrix of $S V[2]$, assuming $r(V)>0$. If $C \subset V^{*}$ is defined by the zeroes of the fundamental semi-invariant then we have $C \subset V^{*} \backslash V_{\text {reg }}^{*}$ and the complement of $C \cup V_{\text {reg }}^{*}$ in $V^{*}$ has codimension $\geq 2$. We write $d(V)$ for the degree of the fundamental semi-invariant. If $r(V)=0$ (i.e. the action of $\mathfrak{g}$ on $V$ is trivial) then we put $d(V)=0$. We record the following
Lemma 2.1. If $V=\mathfrak{g}$ then the fundamental semi-invariant in $S \mathfrak{g}$ is the square of the greatest common divisor of the Pfaffians of the principal $r(\mathfrak{g}) \times r(\mathfrak{g})$ minors in the structure matrix of $\mathfrak{g}$.

Proof. According to [10] any $r(V) \times r(V)$-minor can be expressed as a quadratic form in Pfaffians of principal $r(V) \times r(V)$-minors. From this one easily deduces the stated result.

In case $V=\mathfrak{g}$ is the adjoint representation then $r_{\mathfrak{g}}(\mathfrak{g})=r(\mathfrak{g})$ is an even number and we we have $r(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-i(\mathfrak{g})$. We put

$$
\begin{aligned}
c(\mathfrak{g}) & =\frac{1}{2}(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g})) \\
& =\operatorname{dim} \mathfrak{g}-\frac{1}{2} r(\mathfrak{g})
\end{aligned}
$$

## 3. Reduction to the case without proper semi-invariants

The following result which generalizes [2, Thm. 1.19(3)] is the main result of this section.
Proposition 3.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there exists another finite dimensional Lie algebra $\mathfrak{g}^{\prime}$ such that $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=\left(S \mathfrak{g}^{\prime}\right)_{\mathrm{si}}^{\mathfrak{g}^{\prime}}=\left(S \mathfrak{g}^{\prime}\right)^{\mathfrak{g}^{\prime}}$. Moreover $c\left(\mathfrak{g}^{\prime}\right)=c(\mathfrak{g})$.

If $\mathfrak{g}$ is almost algebraic then we may take $\mathfrak{g}^{\prime}$ to be the intersection of the kernels of the non-trivial weights of the semi-invariants in $S \mathfrak{g}$ [1, 2, 7, 18]. This procedure must be modified for non-almost algebraic Lie algebras.
Example 3.2. Let $\mathfrak{g}=k v_{1}+k v_{2}+k v_{3}$ be the Lie algebra with non-trivial brackets $\left[v_{1}, v_{2}\right]=v_{2}+v_{3},\left[v_{1}, v_{3}\right]=v_{3}$. Then $\left.(S \mathfrak{g})\right)_{\mathrm{si}}^{\mathfrak{g}}=k\left[v_{3}\right]$. On the other hand the kernel of the weight of $v_{3}$ is the abelian Lie algebra $k v_{2}+k v_{3}$ whose semi-invariants are $k\left[v_{2}, v_{3}\right]$. So this is different. It turns out that in this case we have to take $\mathfrak{g}^{\prime}=k v_{1}+k v_{2}+k v_{3}$ with non-trivial brackets $\left[v_{1}, v_{2}\right]=v_{3}$.

The proof of Proposition 3.1 will be given after some preparation.
Proposition 3.3. Assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension one. Then one of the inclusions $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{g}} \subset(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}}$ or $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{g}} \subset(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}$ is an equality.

This result is perhaps better appreciated in the following equivalent formulation.

Corollary 3.4. Assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension one. Then either $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}} \subset(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}$ or $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}} \subset(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}}$

Note that both inclusions may be equalities. This happens already for the two dimensional non-abelian Lie algebra.

Proof of Proposition 3.3. Assume that the statement is false. Write $\mathfrak{g}$ as a semidirect product $\mathfrak{h}+k p$. Let $f$ be a semi-invariant with weight $\chi$ in $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}} \backslash(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{g}}$ which is irreducible in $S \mathfrak{h}$ and let $g=\sum_{i=0}^{n} a_{i} p^{i}$ be a semi-invariant with weight $\psi$ in $S \mathfrak{g}$ such that $a_{i} \in S \mathfrak{h}, n>0$ and $a_{n} \neq 0$.

Since $f$ is not a semi-invariant for $\mathfrak{g}$ it is not a semi-invariant for $p$. From the fact that $g$ is a semi-invariant for $p$ we easily deduce that the $a_{i}$ are semi-invariants for $p$. In particular the non-zero $a_{i}$ cannot be divisible by $f$ (since a factor of a semi-invariant for $p$ is a semi-invariant for $p$ ).

We will now obtain a contradiction by computing the Poisson bracket $\{f, g\}$. Since $g$ is a semi-invariant in $S \mathfrak{g}$ with weight $\psi$ we have

$$
\{f, g\}=\partial_{\psi}(f) \sum_{i} a_{i} p^{i}
$$

Since $f$ is a semi-invariant in $S \mathfrak{h}$ with weight $\chi$ we have

$$
\{f, g\}=-\sum_{i} \partial_{\chi}\left(a_{i}\right) f p^{i}+\sum_{i} a_{i} i p^{i-1}\{f, p\}
$$

Assume first $\partial_{\psi}(f) \neq 0$. Using the fact that $\partial_{\psi}(f)$ has lower degree than $f$ and hence is not divisible by $f$ we conclude that $a_{n}$ is divisible by $f$. Since $a_{n} \neq 0$ this is a contradiction.

Assume now $\partial_{\psi}(f)=0$. In that case we obtain from the fact that $f$ does not divide $\{f, p\}$ (since $f$ is not a semi-invariant for $p$ ) that $f$ divides $a_{i}$ for $i>0$. This is again a contradiction.

In the next two propositions we give some conditions under which the ring of semi-invariants for a representation does not change under passage to an ideal of the Lie algebra.

Proposition 3.5. Let $V$ be a finite dimensional representation of $\mathfrak{g}$. Assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ such that $r_{\mathfrak{h}}(V)=r_{\mathfrak{g}}(V)$. Then $(S V)_{\mathrm{si}}^{\mathfrak{h}}=(S V)_{\mathrm{si}}^{\mathfrak{g}}$.

Proof. Assume $r_{\mathfrak{h}}(V)=r_{\mathfrak{g}}(V)$ and $(S V)_{\mathrm{si}}^{\mathfrak{h}} \neq(S V)_{\mathrm{si}}^{\mathfrak{g}}$. Let $f \in(S V)_{\mathrm{si}}^{\mathfrak{h}} \backslash(S V)_{\mathrm{si}}^{\mathfrak{g}}$ be a semi-invariant with weight $\chi$ which is irreducible in $S V$. Let $\left(h_{i}\right)_{i}$ be a basis of $\mathfrak{h}$ and let $p \in \mathfrak{g}-\mathfrak{h}$ be such that $f$ is not a semi-invariant for $p$. Then by elementary linear algebra applied to the structure matrices of $V$ with respect to $\mathfrak{g}$ and $\mathfrak{h}$ there exist $a, b_{i} \in S V$ with $a \neq 0$ such that $\delta=a \otimes p+\sum_{i} b_{i} \otimes h_{i} \in S V \otimes \mathfrak{g}$ has the property that $\rho(\delta)$ acts trivially on $V$. Hence $\delta \in \operatorname{ker} \rho$.

We claim we may choose $\delta$ in such a way that $a$ is not divisible by $f$. Assume on the contrary that $a=f^{n} a^{\prime}, n>0$ such that $f$ does not divide $a^{\prime} \in S V$.

Since ker $\rho$ is an ideal we have that $[1 \otimes p, \delta] \in \operatorname{ker} \rho$ and

$$
[1 \otimes p, \delta]=p(a) \otimes p+\sum_{i} b_{i}^{\prime} \otimes h_{i}
$$

for suitable $b_{i}^{\prime} \in S V$. Then $p(a)=n f^{n-1} p(f) a^{\prime}+f^{n} p\left(a^{\prime}\right)$. Since $p(f)$ is not divisible by $f$ (as it is not a semi-invariant for $p$ ) we see that the highest power of $f$ which
divides $p(a)$ is $f^{n-1}$ (and $p(a) \neq 0$ ). Replacing $\delta$ by $[1 \otimes p, \delta]$ and repeating this procedure we eventually arrive at a $\delta$ such that $a$ is no longer divisible by $f$.

Applying this new $\delta$ to $f$ we get

$$
0=a p(f)+\sum_{i} b_{i} h_{i}(f)=a p(f)+\sum b_{i} \chi\left(h_{i}\right) f
$$

Since neither $a$ nor $p(f)$ is divisible by $f$ we have obtained a contradiction.
Proposition 3.6. Let $V$ be a finite dimensional representation of $\mathfrak{g}$ and assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension one such that $\mathfrak{g}=\mathfrak{h}+k s$ with $s$ acting semi-simply on both $V$ and $\mathfrak{h}$. Then $(S V)_{\mathrm{si}}^{\mathfrak{h}}=(S V)_{\mathrm{si}}^{\mathfrak{g}}$

Proof. Put $S=S V$. We decompose $\mathfrak{h}$ and $S$ according to the $s$-weights (i.e. as $s$-eigenspaces): $\mathfrak{h}=\oplus_{\mu \in k} \mathfrak{h}_{\mu}, S=\oplus_{\lambda \in k} S_{\lambda}$. For $f \in S$ let Supp $f$ be the set of $\lambda$ such that $f_{\lambda} \neq 0$ in the decomposition $f=\sum_{\lambda \in k} f_{\lambda}$ with $f_{\lambda} \in S_{\lambda}$.

Let $f \in S$ be a semi-invariant for $\mathfrak{h}$ with weight $\chi$. Write $f=\sum_{\lambda \in k} f_{\lambda} f_{\lambda} \in S_{\lambda}$. We claim that the $f_{\lambda}$ are semi-invariants for $\mathfrak{h}$, which implies that they are in fact semi-invariants for $\mathfrak{g}=\mathfrak{h}+k s$. Hence $(S V)_{\mathrm{si}}^{\mathfrak{h}} \subset(S V)_{\mathrm{si}}^{\mathfrak{g}}$. Since the other inclusion is obvious we are done.

To prove the claim first assume $\chi=0$. Thus $f \in S^{\mathfrak{h}}$. Pick $h \in \mathfrak{h}_{\mu}$. Then $0=h(f)=\sum_{\lambda \in k} h\left(f_{\lambda}\right)$ with $h\left(f_{\lambda}\right) \in S_{\mu+\lambda}$. Hence $h\left(f_{\lambda}\right)=0$ and thus $f_{\lambda} \in S^{\mathfrak{h}}$. So this case is OK.

Now assume $\chi \neq 0$. We first assert that $\chi\left(\mathfrak{h}_{\mu}\right)=0$ for $\mu \neq 0$. To see this assume there exist $h \in \mathfrak{h}_{\mu}, \mu \neq 0$ such that $\chi(h) \neq 0$. From the equation $h(f)=\chi(h) f$ we deduce $\operatorname{Supp} f=\operatorname{Supp}(h(f)) \subset \mu+\operatorname{Supp} f$ which is impossible if $\mu \neq 0$. So our assertion is correct.

As in the case $\chi=0$ we now deduce for $h \in \mathfrak{h}_{\mu}$ that $h\left(f_{\lambda}\right)=0$ if $\mu \neq 0$ and $h\left(f_{\lambda}\right)=\chi(h) f_{\lambda}$, if $\mu=0$. So the $f_{\lambda}$ 's are semi-invariants in this case also.

Lemma 3.7. Assume that $f \in S \mathfrak{g}$ is a semi-invariant with weight $\chi$ and $\mathfrak{h}=\operatorname{ker} \chi$. Then $c(\mathfrak{g})=c(\mathfrak{h})$.

Proof. We may assume $\mathfrak{g} \neq \mathfrak{h}$, i.e. $\chi$ is non-trivial. Assume $c(\mathfrak{g}) \neq c(\mathfrak{h})$. Choose $p \in \mathfrak{g}$ such that $\chi(p)=1$. Comparing

$$
\begin{aligned}
& c(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-\frac{1}{2} r(\mathfrak{g}) \\
& c(\mathfrak{h})=\operatorname{dim} \mathfrak{g}-\frac{1}{2} r(\mathfrak{h})-1
\end{aligned}
$$

we see that

$$
r(\mathfrak{g}) \neq r(\mathfrak{h})+2
$$

Since the structure matrix of $\mathfrak{g}$ is obtained from that of $\mathfrak{h}$ by adding a row and a column we have $r(\mathfrak{g})-r(\mathfrak{h}) \in\{0,2\}$. We obtain

$$
r(\mathfrak{g})=r(\mathfrak{h})
$$

The proof now parallels that of Lemma 3.5. Let $\left(h_{i}\right)_{i}$ be a basis of $\mathfrak{h}$ and select $a, b_{i} \in S \mathfrak{g}$ with $a \neq 0$ such that $\delta=a \otimes p+\sum_{i} b_{i} \otimes h_{i}$ acts trivially on $p$ and $\left(h_{j}\right)_{j}$. Thus $\delta \in \operatorname{ker} \rho$. In other words $\rho(\delta)$ acts trivially on $S \mathfrak{g}$. But we also find $\rho(\delta)(f)=a f \neq 0$. This is a contradiction.

Proof of Proposition 3.1. We will construct $\mathfrak{g}^{\prime}$ one step at a time. Assume that $\mathfrak{g}$ has a non-trivial weight $\chi$ on $S \mathfrak{g}$. Then we will construct a Lie algebra $\tilde{\mathfrak{g}}$ such that $c(\tilde{\mathfrak{g}})=c(\mathfrak{g}),(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}=(S \tilde{\mathfrak{g}})_{\text {si }}^{\tilde{\mathfrak{g}}}$ and such that either $\operatorname{dim} \tilde{\mathfrak{g}}<\operatorname{dim} \mathfrak{g}$ or $\operatorname{dim} N(\tilde{\mathfrak{g}})>\operatorname{dim} N(\mathfrak{g})$ where $N(-)$ denotes the nil-radical. It is clear that by repeating this procedure we eventually end up with a Lie algebra which has the requested properties.

Let $f \in S \mathfrak{g}$ be an non-zero eigenfunction for $\chi$. Put $\mathfrak{h}=$ ker $\chi$. Since semiinvariants Poisson commute and since the Poisson centralizer of $f$ is equal to $S \mathfrak{h}$ we find (see [2, Cor. 1.15]).

$$
\begin{equation*}
(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}} \subset(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}} \tag{3.1}
\end{equation*}
$$

Choose $c \in \mathfrak{g}$ such that $\chi(c)=1$. Then $\operatorname{ad}(c)=D_{s}+D_{p}$ where $D_{s}, D_{p}$ are two commuting derivations of $\mathfrak{h}$ with $D_{s}$ being semi-simple and $D_{p}$ nilpotent. Let $\mathfrak{j}=\mathfrak{h}+k s+k p$ be the semi-direct product of $\mathfrak{h}$ with an abelian Lie algebra $k s+k p$ such that $\operatorname{ad}(s)$ acts by $D_{s}$ and $\operatorname{ad}(p)$ acts by $D_{p}$. Sending $c$ to $s+p$ yields an embedding $\mathfrak{g} \subset \mathfrak{j}$. Put $\mathfrak{k}=\mathfrak{h}+k p$. Then we have $\mathfrak{j}=\mathfrak{g}+k s=\mathfrak{k}+k s$.

Since $\operatorname{ad}(s)$ acts semi-simply on everything we have by Proposition 3.6

$$
(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{j}}=(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{k}}
$$

and thus by (3.1)

$$
(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=S \mathfrak{h} \cap(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{k}}=(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{k}}
$$

By Proposition 3.3 we have either $(S \mathfrak{h})_{\text {si }}^{\mathfrak{k}}=(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}}$ or $(S \mathfrak{h})_{\text {si }}^{\mathfrak{k}}=(S \mathfrak{k})_{\mathrm{si}}^{\mathfrak{k}}$.
Assume first $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{k}}=(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}}$. Then we put $\tilde{\mathfrak{g}}=\mathfrak{h}$. By Lemma 3.7 we have $c(\tilde{\mathfrak{g}})=c(\mathfrak{g})$. Since $\operatorname{dim} \tilde{\mathfrak{g}}<\operatorname{dim} \mathfrak{g}$ this case is done.

Now assume $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{k}} \neq(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}}$ and thus $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{k}}=(S \mathfrak{k})_{\mathrm{si}}^{\mathfrak{k}}$. In this case we put $\tilde{\mathfrak{g}}=\mathfrak{k}$ and hence we have $\operatorname{dim} \tilde{\mathfrak{g}}=\operatorname{dim} \mathfrak{g}$. By Proposition 3.5 we have $r_{\mathfrak{k}}(\mathfrak{h})>r_{\mathfrak{h}}(\mathfrak{h})$ and hence $r(\mathfrak{k})=r_{\mathfrak{k}}(\mathfrak{k})>r_{\mathfrak{h}}(\mathfrak{h})=r(\mathfrak{h})$.

If $r(\mathfrak{g})=r(\mathfrak{h})$ then $r_{\mathfrak{h}}(\mathfrak{g})=r_{\mathfrak{h}}(\mathfrak{h})$ and hence by Proposition $3.5(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{h}}$. Since $(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}=(S \mathfrak{h})_{\mathrm{si}}^{\mathfrak{k}}$ this is a contradiction. Thus $r(\mathfrak{g})>r(\mathfrak{h})$. Since the ranks involved jump at most by 2 we deduce $r(\mathfrak{g})=r(\mathfrak{k})$ and hence $c(\tilde{\mathfrak{g}})=c(\mathfrak{g})$.

It remains to show that in this case we have $\operatorname{dim} N(\mathfrak{g})<\operatorname{dim} N(\tilde{\mathfrak{g}})$. Since $p$ acts nilpotently we have $N(\tilde{\mathfrak{g}})=k p+N(\mathfrak{h})$. We claim that $N(\mathfrak{g}) \subset N(\mathfrak{h})$ which is sufficient. To prove this claim we need to show that no element of the form $c+n$ for $n \in \mathfrak{h}$ acts nilpotently on $\mathfrak{h}$. Assume such $n$ exists. Then $0=\chi(c+n)=\chi(c)$ and hence $c \in \mathfrak{h}$ which is a contradiction.

## 4. A formula for the transcendence degree of invariants

If $S$ is a commutative domain then we denote its field of fractions by $Q(S)$. In this section we prove the following result.

Proposition 4.1. Let $V$ be a finite dimensional representation of $\mathfrak{g}$ and assume that SV contains no proper semi-invariants. Then

$$
\begin{equation*}
\operatorname{trdeg} Q(S V)^{\mathfrak{g}}=\operatorname{dim} V-r(V) \tag{4.1}
\end{equation*}
$$

In the case that $\mathfrak{g}$ acts algebraically on $V$ the formula (4.1) was proved by Dixmier [5, Lemme 7] (it is a more or less direct consequence of Rosenlicht's theorem [20]). Here we have traded algebraicity for the absence of proper semi-invariants. Both conditions are independent as Example 4.8 below shows.

Let $L / k$ be a finitely generated field extension on which $\mathfrak{g}$ acts by derivations. Put $K=L^{\mathfrak{g}}$. It is clear that $K$ is algebraically closed in $L$. We will say that the action is geometric if the induced map $\rho: L \otimes_{K} \mathfrak{g} \rightarrow \operatorname{Der}_{K}(L)$ is surjective. If $x_{1}, \ldots, x_{n}$ is a transcendence basis of $L / K$ and $\partial_{i}=\partial / \partial x_{i}: L \rightarrow L$ are the corresponding derivations then $\operatorname{Der}_{K}(L)=\sum_{i} L \partial_{i}$. From this we deduce

Lemma 4.2. Let $L$ be as above. Then the action is geometric if one has

$$
\begin{equation*}
\operatorname{trdeg}_{L^{\mathfrak{g}}} L=\operatorname{dim} \mathfrak{g}-\operatorname{dim}_{L} \operatorname{ker} \rho \tag{4.2}
\end{equation*}
$$

Note that $\operatorname{ker} \rho$ can also be computed as $\operatorname{ker}\left(L \otimes_{k} \mathfrak{g} \rightarrow \operatorname{Der}_{k}(L)\right)$. So the number on the right hand side of (4.2) can be computed without knowing $L^{\mathfrak{g}}$.

Lemma 4.3. Let $M \supset L \supset K$ be finitely generated field extensions of $k$ and let $\mathfrak{g}$ be a finite dimensional Lie algebra acting on $M$ such that $K=M^{\mathfrak{g}}$. Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$ and put $L=M^{\mathfrak{h}}$. Assume that the $\mathfrak{h}$-action on $M$ is geometric and likewise for the action $\mathfrak{g} / \mathfrak{h}$ on $L$. Then the action of $\mathfrak{g}$ on $M$ is geometric.

Proof. This follows from the following commutative diagram


Now let $S$ be a $k$-algebra which is an integral domain with finitely generated fraction field. Assume $\mathfrak{g}$ acts on $S$. We say that $\mathfrak{g}$ acts generically geometrically if the induced action on the fraction field is geometric.

If $Y$ is a (possibly singular) variety then we denote the tangent space in a point $y \in Y$ by $T_{Y, y}$. If $Y$ is smooth then $T_{Y, y}$ is the fiber of the tangent bundle $T_{Y}$ of $Y$ at $y$.

Proposition 4.4. Assume that $S / k$ is a finitely generated domain and that $\mathfrak{g}$ acts on $S$. Put $Y=\operatorname{Spec} S$. Let $L$ be the fraction field of $S$. Then the action is generically geometric if and only if

$$
\operatorname{trdeg}_{L^{\mathfrak{g}}} L=\operatorname{dim} \mathfrak{g}-\min _{y \in Y} \operatorname{dim} \mathfrak{g}_{y}
$$

where the minimum is taken over the closed points in $Y$ and $\mathfrak{g}_{y}$ denotes the stabilizer of $y \in Y$, i.e. the kernel of $\mathfrak{g} \rightarrow T_{Y, y}$.

Proof. For technical reasons it is more convenient to work with differentials instead of with vector fields as the sheaf of differentials is always compatible with taking fibers.

There is a canonical pairing $\mathfrak{g} \otimes_{k} \Omega_{Y} \rightarrow \mathcal{O}_{Y}: v \otimes f d g \mapsto f v(g)$ which yields a map of coherent $\mathcal{O}_{Y}$ modules $\rho^{*}: \Omega_{Y} \rightarrow \mathcal{O}_{Y} \otimes \mathfrak{g}^{*}$. Taking the fiber in a point $y \in Y$ one checks $\mathfrak{g}_{y}^{*}=$ coker $\rho_{y}^{*}$. By semi-continuity the dimension of the cokernel of the generic fiber of $\rho^{*}$ is equal to the minimum of the dimensions of the cokernels of the special fibers.

Thus we find

$$
\begin{aligned}
\operatorname{dim}_{L} \operatorname{ker}\left(L \otimes_{k} \mathfrak{g} \rightarrow \operatorname{Der}_{k}(L)\right) & =\operatorname{dim}_{L} \operatorname{coker}\left(\Omega_{L, k} \rightarrow L \otimes_{k} \mathfrak{g}^{*}\right) \\
& =\min _{y \in Y} \operatorname{dim} \mathfrak{g}_{y}^{*} \\
& =\min _{y \in Y} \operatorname{dim} \mathfrak{g}_{y}
\end{aligned}
$$

It now suffices to apply Lemma 4.2.
Lemma 4.5. Assume that $\mathfrak{g}=\operatorname{Lie}(G)$ where $G$ is a connected algebraic group acting rationally on a domain $S$ with finitely generated fraction field. Then the action is generically geometric.

Proof. Let $L$ be the fraction field of $S$. Since $L$ is finitely generated and since $G$ acts rationally on $S$ we may select a finitely generated $G$-invariant subring $S_{0} \subset S$ such that $L$ is the fraction field of $S_{0}$. From here on the proof proceeds as in [3, Lemme 7]. For the benefit of the reader let us repeat the argument. By Rosenlicht's theorem there exists an $G$-invariant open $U \subset Y=\operatorname{Spec} S_{0}$ such that a geometric quotient $U / G$ exists. Shrinking $U$ we may assume that $U$ is smooth. By the properties of a geometric quotient the field of rational functions on $U$ is $L$, the field of rational functions on $U / G$ is $L^{G}$ and the fibers of $U \rightarrow U / G$ are the $G$-orbits. Hence $\operatorname{trdeg} L / L^{G}=\operatorname{dim} U-\operatorname{dim} U / G$ is equal to the dimension of the generic $G$ orbit on $U$ or equivalently on $Y$. This is $\operatorname{dim} G-\min _{y \in Y} \operatorname{dim} G_{y}=\operatorname{dim} \mathfrak{g}-\min _{y \in Y} \operatorname{dim} \mathfrak{g}_{y}$. We may now apply Proposition 4.4 with $S=S_{0}$.

We would like to have a transitivity result as in Lemma 4.3 but one does not always have $Q(S)^{\mathfrak{g}}=Q\left(S^{\mathfrak{g}}\right)$. So we introduce a special situation in which this identity holds.

Let us say that a graded ring $S$ is connected if it is of the form $k+S_{1}+S_{2}+\cdots$ with $\operatorname{dim} S_{i}<\infty$. For technical reasons we do not assume that $S$ is finitely generated. Recall the following.
Lemma 4.6. Let $S$ be a connected graded factorial domain, and assume that $\mathfrak{g}$ acts in a graded way on $S$ without proper semi-invariants. Then $S^{\mathfrak{g}}$ is factorial, and furthermore $Q(S)^{\mathfrak{g}}=Q\left(S^{\mathfrak{g}}\right)$.

The following is a weak version of Rosenlicht's theorem which is also valid for non-algebraic Lie algebras.
Proposition 4.7. Let $S$ be a connected factorial graded domain with finitely generated quotient field and assume that $\mathfrak{g}$ acts in a graded way on $S$ without proper semi-invariants. Then the action is generically geometric.
Proof. We have a filtration by ideals

$$
0=\mathfrak{g}_{0} \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_{n}=\mathfrak{g}
$$

where $\mathfrak{g}_{n} / \mathfrak{g}_{n-1}$ is semi-simple, and the other quotients are abelian. We have a corresponding filtration

$$
S=S^{0} \supset \cdots \supset S^{n-1} \supset S^{n}=S^{\mathfrak{g}}
$$

with $S^{i}=S^{\mathfrak{g}_{i}}$ and hence $S^{i+1}=\left(S^{i}\right)^{\mathfrak{g}_{i+1} / \mathfrak{g}_{i}}$.
By Lemmas 4.6 and 4.3 we may assume that $\mathfrak{g}$ is either semi-simple or abelian. If $\mathfrak{g}$ is semi-simple then it acts algebraically (since $\operatorname{dim} S_{i}<\infty$ ) and hence we may invoke Lemma 4.5. If $\mathfrak{g}$ is abelian then its generalized weights must be zero
(for otherwise we could construct a proper semi-invariant). Hence $\mathfrak{g}$ acts locally nilpotently and hence algebraically. We may again invoke Lemma 4.5.

Proof of Proposition 4.1. Put $L=Q(S V)$. By (4.2) $\operatorname{trdeg}_{L^{\mathfrak{g}}} L=\operatorname{dim} \mathfrak{g}-n$ where $n$ is the dimension of the null space of the structure matrix. The structure matrix has size $\operatorname{dim} V \times \operatorname{dim} \mathfrak{g}$. Thus $\operatorname{trdeg}_{L^{\mathfrak{g}}} L=r(V)$. Hence $\operatorname{trdeg} L^{\mathfrak{g}}=\operatorname{dim} V-r(V)$.

Example 4.8. Let $W$ be a finite dimensional vector space. Then the Heisenberg Lie algebra $\mathfrak{h}$ on $W$ is the vector space $W \oplus W^{*} \oplus k c$ with for all $w, w^{\prime} \in W$, $\phi, \phi^{\prime} \in W^{*}:\left[w, w^{\prime}\right]=0,\left[\phi, \phi^{\prime}\right]=0,[c, w]=0,[c, \phi]=0,[\phi, w]=\phi(w) c$.

For $p \in \operatorname{End}(W)$ let $D_{p}$ be the derivation $\left(p,-p^{*}, 0\right)$ of $\mathfrak{h}$ and let $\mathfrak{g}=\mathfrak{h}+k t$ be the corresponding semi-direct product. Put $S=S \mathfrak{g}$. Assume that $p$ is invertible. Then $\mathfrak{g}$ has a non-degenerate invariant symmetric bilinear form $(-,-)$ whose non-trivial values are given by $(t, c)=1,(\phi, w)=-\phi\left(p^{-1}(w)\right)$. Hence $\mathfrak{g}$ is quadratic. It then follows from [15, Cor. 2.3, Prop. 3.2] that $S \mathfrak{g}$ contains no proper semi-invariants. However if we choose $p$ to be non-diagonalizable then $\mathfrak{g}$ does not act algebraically as $t$ does not have a Jordan decomposition. So the hypotheses of Proposition 4.1 do not imply that $\mathfrak{g}$ acts algebraically.

The formula (4.1) asserts that the transcendence degree of $Q(S)^{\mathfrak{g}}$ should be 2 , as an easy computation shows $r(\mathfrak{g})=\operatorname{dim} \mathfrak{g}-2$.

This example is simple enough to verify directly. As an aside we find that the hypothesis that $p$ is invertible is in fact superfluous. The absence of proper semiinvariants always holds.

Choose a basis $\left(w_{i}\right)_{i}$ for $W$ and assume that $p\left(w_{i}\right)=\sum_{j} p_{i j} w_{j}$. Then one has $p^{*}\left(w_{j}^{*}\right)=\sum_{i} p_{i j} w_{i}^{*}$. Put $z=t c-\sum_{i j} p_{i j} w_{i} w_{j}^{*}$ (this is the Casimir element for the pairing $(-,-)$ in the case that $p$ is invertible). Then $c, z \in S^{\mathfrak{g}}$. We write $S_{c}=k\left[c^{ \pm 1}, z,\left(w_{i} c^{-1}\right)_{i}, w_{j}^{*}\right]$. With respect to these new generators the only nontrivial Poisson brackets are $\left\{w_{j}^{*}, w_{i} c^{-1}\right\}=\delta_{j i}$. A semi-invariant in $S$ generates a principal Poisson ideal in $S$ and hence in $S_{c}$. Computing with the new generators we see that the only principal Poisson ideals in $S_{c}$ are those generated by elements in $k\left[c^{ \pm 1}, z\right]$. Thus $S_{\mathrm{si}}^{\mathfrak{g}}=S \cap k\left[c^{ \pm 1}, z\right]$ which is equal to $k[c, z]$ if $p \neq 0$ and equal to $k[c, t]$ otherwise. Thus we see that $S \mathfrak{g}$ contains no proper semi-invariants. In both cases we find $\operatorname{trdeg} Q(S)^{\mathfrak{g}}=2$ as predicted by (4.1).

## 5. Proofs in the absence of proper semi-invariants

Throughout $V$ is a finite dimensional representation of $\mathfrak{g}$. For convenience we write $S=S V, R=(S V)^{\mathfrak{g}}$ and we let $L$ be the field of fractions of $S$. As we will mostly use geometrical language we also put $Y=\operatorname{Spec} S=V^{*}, X=\operatorname{Spec} R$ and we let $\pi: Y \rightarrow X$ be dual to the inclusion $R \rightarrow S$. If $R$ is finitely generated then the regular locus of $X$ is denoted ${ }^{1}$ by $X_{\mathrm{sm}}$.

The following result is an adaptation of [12] to the case of non-semisimple Lie algebras.

Proposition 5.1. Assume that $(S V)_{\mathrm{si}}^{\mathfrak{g}}=(S V)^{\mathfrak{g}}$ and that $(S V)^{\mathfrak{g}}$ is finitely generated. Let

$$
U=\left\{y \in Y \mid \pi(y) \in X_{\mathrm{sm}} \text { and } \pi \text { is smooth in } y\right\}
$$

Then $\operatorname{codim}_{Y}(Y-U) \geq 2$.

[^1]Proof. The proof is that of [12] with minor adaptations. Without loss of generality we may replace $\mathfrak{g}$ by the algebraic hull of the image of $\mathfrak{g}$ in $\operatorname{End}_{k}(V)$. Let $G \subset \mathrm{GL}(V)$ be the affine connected algebraic group such that Lie $G=\mathfrak{g}$. Then $G$ acts rationally on $S$ and $R=S^{G}$.

Let $E$ be the union of the irreducible divisors in $Y-U$. Then $E$ is $G$-invariant. Since $G$ is connected it follows that $E$ is irreducible. Since $S$ is factorial it follows that $E=V(f)$ for some irreducible $f \in S$.

For $\sigma \in G$ we have that $c_{\sigma}=\sigma(f) f^{-1}$ is a unit in $S$ and hence $c_{\sigma} \in k^{*}$. Thus $f$ is a semi-invariant and hence $f \in R$.

We claim that the map $R / f R \rightarrow S / f S$ is injective. Assume there is some element $\bar{c}$ in the kernel. Then $c=f d$ with $d \in S$. But then $d \in S^{\mathfrak{g}}=R$. Hence $\bar{c}=0$.

Let $D$ be the divisor in $X$ of $f$. Then $E=\pi^{-1}(D)$ and the map $E \rightarrow D$ is a dominant map between irreducible algebraic varieties. Since $X$ is normal $D \cap X_{\mathrm{sm}} \neq \emptyset$.

By generic smoothness there exist dense open $E^{\prime} \subset E, D^{\prime} \subset D \cap X_{\mathrm{sm}}$ such that $E^{\prime}, D^{\prime}$ are regular and $\pi$ restricts to a smooth map $E^{\prime} \rightarrow D^{\prime}$.

Let $y \in E^{\prime}$. We will show that $\pi$ is smooth at $y$, contradicting the fact that $E^{\prime}$ is contained in the non smooth locus of $\pi$. We consider the following commutative diagram of tangent spaces with $x=\pi(y)$


Since $x$ is regular in $D, X$ and $y$ is regular in $E, Y$ the rows are exact. The left most map is surjective since $E^{\prime} \rightarrow D^{\prime}$ is smooth. This implies that the middle map is surjective.

We keep the notations as in the statement of Proposition 5.1 and we assume throughout that $(S V)_{\mathrm{si}}^{\mathfrak{g}}=(S V)^{\mathfrak{g}}$ and that $(S V)^{\mathfrak{g}}$ is finitely generated. This implies in particular that $(S V)^{\mathfrak{g}}$ is factorial and $L^{\mathfrak{g}}=Q\left((S V)^{\mathfrak{g}}\right)$. Furthermore by Propositions 4.7 and 4.4 we have

$$
\begin{equation*}
\operatorname{dim} Y-\operatorname{dim} X=\operatorname{trdeg}_{L^{\mathfrak{g}}} L=\operatorname{dim} \mathfrak{g}-\min _{y \in Y} \operatorname{dim} \mathfrak{g}_{y} \tag{5.1}
\end{equation*}
$$

This leads to the definition

$$
Y^{\prime}=\left\{y \in Y \mid \operatorname{dim} \mathfrak{g}-\operatorname{dim} \mathfrak{g}_{y}<\operatorname{dim} Y-\operatorname{dim} X\right\} \subsetneq Y
$$

and we will also put

$$
\begin{equation*}
W=U \cap\left(Y-Y^{\prime}\right) \tag{5.2}
\end{equation*}
$$

Since the elements of $\mathfrak{g}$ define vector fields on $Y$ which annihilate invariant functions, we have the usual map of vector bundles on $Y$

$$
\rho: \mathcal{O}_{Y} \otimes_{k} \mathfrak{g} \rightarrow T_{Y}
$$

which extends to a complex of vector bundles on $\pi^{-1} X_{\mathrm{sm}}$

$$
\begin{equation*}
\mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \otimes_{k} \mathfrak{g} \xrightarrow{\rho_{\pi^{-1} X_{\mathrm{sm}}}} T_{\pi^{-1} X_{\mathrm{sm}}} \xrightarrow{d \pi_{\pi^{-1} X_{\mathrm{sm}}}} \pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} T_{X_{\mathrm{sm}}} \rightarrow 0 \tag{5.3}
\end{equation*}
$$

We see that $U$ is the locus in $\pi^{-1} X_{\mathrm{sm}}$ where (5.3) is exact at $\pi_{\pi^{-1} X_{\mathrm{sm}}} T_{X_{\mathrm{sm}}}$ and $W$ is the locus where the entire complex is exact. Thus we have an exact sequence

$$
\begin{equation*}
\mathcal{O}_{W} \otimes_{k} \mathfrak{g} \xrightarrow{\rho_{W}} T_{W} \xrightarrow{d \pi_{W}} \pi_{W}^{*} T_{X_{\mathrm{sm}}} \rightarrow 0 \tag{5.4}
\end{equation*}
$$

The following is one of the main results of [12]. For completeness we include the proof in our setting.

Proposition 5.2. One has $W=\left(Y-Y^{\prime}\right) \cap \pi^{-1} X_{\mathrm{sm}}$.
The statement of this Proposition means that if $y \in Y$ is such that $\pi(y)=x$ is regular in $X$ and $\mathfrak{g}_{y}$ has minimal dimension then $\pi$ is smooth in $y$.
Proof of Proposition 5.2. Put $\bar{W}=\left(Y-Y^{\prime}\right) \cap \pi^{-1} X_{\mathrm{sm}}$. Since $W=U \cap\left(Y-Y^{\prime}\right)$ the inclusion $W \subset \bar{W}$ is obvious. To prove the opposite inclusion we look at the complex

$$
\mathcal{O}_{\bar{W}} \otimes_{k} \mathfrak{g} \xrightarrow{\rho_{\bar{W}}} T_{\bar{W}} \xrightarrow{d \pi_{\bar{W}}} \pi_{\bar{W}}^{*} T_{X_{\mathrm{sm}}} \rightarrow 0
$$

Since $\rho_{\bar{W}}$ has constant rank coker $\rho_{\bar{W}}$ is a vector bundle. Furthermore by the exactness of (5.4) we deduce that coker $\rho_{\bar{W}} \rightarrow \pi_{\bar{W}}^{*} T_{X_{\mathrm{sm}}}$ is an isomorphism on $W$.

Since $W \subset \bar{W} \subset Y-Y^{\prime}$ and $Y-Y^{\prime}-W=\left(Y-Y^{\prime}\right) \cap(Y-U)$ has codimension $\geq 2$ in $Y-Y^{\prime}$, the same holds for the codimension of $\bar{W}-W$ in $\bar{W}$. It follows that coker $\rho_{\bar{W}} \rightarrow \pi_{\bar{W}}^{*} T_{X_{\mathrm{sm}}}$ is an isomorphism on the whole of $\bar{W}$. In particular the map $T_{\bar{W}} \rightarrow \pi_{\bar{W}}^{*} T_{X_{\mathrm{sm}}}$ is surjective. This implies that $\bar{W} \subset U$. Hence $\bar{W}=W$.

Lemma 5.3. (see also [13, Lemma 4]) The dual of (5.3) yields an exact sequence

$$
\begin{equation*}
0 \rightarrow \pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} \Omega_{X_{\mathrm{sm}}} \xrightarrow{d \pi_{\pi^{-1} X_{\mathrm{sm}}}^{*}} \Omega_{\pi^{-1} X_{\mathrm{sm}}} \xrightarrow{\rho_{\pi-1}^{*} X_{\mathrm{sm}}} \mathfrak{g}^{*} \otimes \mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \tag{5.5}
\end{equation*}
$$

of vector bundles on $\pi^{-1} X_{\mathrm{sm}}$.
Proof. Let $C=\operatorname{coker} \rho_{\pi^{-1} X_{\mathrm{sm}}}$. Then we have commutative diagram


Since $d \pi_{\pi^{-1} X_{\mathrm{sm}}}$ is surjective on $U$, the same holds for $\beta$. So the vertical sequence is exact on $U$. Furthermore since the upper sequence is exact on $W$ it follows that $K$ is torsion. Dualizing the vertical sequence we obtain $C^{*}\left|U \cong\left(\pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} T_{X_{\mathrm{sm}}}\right)^{*}\right| U$. This extends to an isomorphism $C^{*}=\left(\pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} T_{X_{\mathrm{sm}}}\right)^{*}$.

The lemma now follows by dualizing the lower exact sequence.
We can now prove a generalization of Proposition 1.6(1) (taking into account that if $V=\mathfrak{g}$ we have $\rho^{*}=-\rho$, see below).

Proposition 5.4. Assume that $(S V)_{\text {si }}^{\mathfrak{g}}=(S V)^{\mathfrak{g}}$, and that $(S V)^{\mathfrak{g}}$ is a (necessarily finitely generated) polynomial ring. Then ker $\rho^{*}$ is free.
Proof. This follows immediately from Lemma 5.3, taking into account that $X=$ $X_{\mathrm{sm}}$.

Now we specialize to the case $V=\mathfrak{g}$ but we still assume that the conditions of Proposition 5.1 hold. Our assumption that $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$ implies in particular that $\mathfrak{g}$ is unimodular by [6]. To lighten the notations we silently fix an isomorphism $\wedge^{\operatorname{dim} \mathfrak{g}} \mathfrak{g} \cong k$.

In what follows we need to keep track of the grading. Therefore we introduce an additional 1-dimensional torus $\mathbb{G}_{m}$ which acts with weight $n$ on the degree $n$-part of $S$. Everything we do is equivariant with respect to this torus. If $L$ is the one dimensional representation of $\mathbb{G}_{m}$ corresponding to the identity character then we write ? $(n)$ for $? \otimes L^{n}$.

Taking into account $V=\mathfrak{g}$ we obtain $T_{Y}=\left(\mathcal{O}_{Y} \otimes \mathfrak{g}^{*}\right)(1), \Omega_{Y}=\left(\mathcal{O}_{Y} \otimes \mathfrak{g}\right)(-1)$. In particular $\rho$ may be viewed as a map:

$$
\rho: \mathcal{O}_{Y} \otimes \mathfrak{g} \rightarrow\left(\mathcal{O}_{Y} \otimes \mathfrak{g}^{*}\right)(1)
$$

Since $\rho$ is represented by the structure matrix of $\mathfrak{g}$ which is anti-symmetric we obtain $\rho^{*}=-\rho(-1)$. Concatenating (5.3) with (5.5) we obtain a complex
$0 \rightarrow\left(\pi_{\pi^{-1} X_{\mathrm{sm}}} \Omega_{X_{\mathrm{sm}}}\right)(1) \xrightarrow{d \pi^{*}} \mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \otimes_{k} \mathfrak{g} \xrightarrow{\rho}\left(\mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \otimes \mathfrak{g}^{*}\right)(1) \xrightarrow{d \pi} \pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} T_{X_{\mathrm{sm}}} \rightarrow 0$ which is possibly non-exact at $\left(\mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \otimes \mathfrak{g}^{*}\right)(1)$ and $\pi_{\pi^{-1} X_{\mathrm{sm}}} T_{X_{\mathrm{sm}}}$. The locus where the right non-trivial map is surjective is $U$. The locus where it is exact is precisely $W$.

If $Z$ is a normal variety and $F$ is a coherent torsion free $\mathcal{O}_{Z}$-module then we put $\operatorname{det} F=\left(\wedge^{\mathrm{rk} F} F\right)^{* *}$. The operation $F \mapsto \operatorname{det} F$ is multiplicative on complexes which are exact in codimension $\geq 2$. Furthermore the following is well known.
Lemma 5.5. Let $\rho: F \rightarrow G$ be a map between vector bundles on $Z$ which is generically of rank $r>0$. Let $M=\operatorname{im} \rho$ and let $N \subset G$ be the maximal coherent subsheaf of $G$ containing $M$ such that $N / M$ is torsion. Finally let $I(\rho)$ be the ideal in $\mathcal{O}_{Z}$ locally generated by the $r \times r$ minors in a matrix representation of $\rho$. Then $\operatorname{det} M=(I(\rho) \operatorname{det} N)^{* *}$ as submodules of $\wedge^{r} G$.
Proof. We may reduce to the case that $Z$ is the spectrum of a discrete valuation ring $D, F=D^{p}, G=D^{q}$. In that case $\rho$ may be diagonalized as

$$
\left(\begin{array}{ccccccc}
\pi^{a_{1}} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \pi^{a_{2}} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \pi^{a_{r}} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right)
$$

where $\pi$ is a uniformizing element of $D$. Thus $I(\rho)=\left(\pi^{\sum_{i} a_{i}}\right)$. We find $M=$ $\pi^{a_{1}} D \oplus \cdots \oplus \pi^{a_{r}} D \oplus 0 \oplus \cdots \oplus 0 \subset D^{q}$ and $N=D^{r} \oplus 0 \oplus \cdots \oplus 0 \subset D^{q}$. Let $\left(e_{i}\right)_{i}$ be the standard basis of $D^{q}$. Then $\operatorname{det} M=\pi^{\sum_{i} a_{i}} D e_{1} \wedge \cdots \wedge e_{r}$, $\operatorname{det} N=D e_{1} \wedge \cdots \wedge e_{r}$ and so we have indeed $\operatorname{det} M=I(\rho) \operatorname{det} N$ inside $\wedge^{r} G$.

We put $\omega_{Z}=\operatorname{det} \Omega_{Z}$. This is the so-called dualizing module on $Z$.
Put $M=\operatorname{im} \rho, N=\operatorname{ker} d \pi$ in (5.6). We obtain two exact sequences of torsion free $\mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}}$-modules.

$$
\begin{aligned}
0 \rightarrow & \left(\pi_{\pi^{-1} X_{\mathrm{sm}}} \Omega_{X_{\mathrm{sm}}}\right)(1) \xrightarrow{d \pi^{*}} \mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \otimes_{k} \mathfrak{g} \rightarrow M \rightarrow 0 \\
0 & \rightarrow N \rightarrow\left(\mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \otimes \mathfrak{g}^{*}\right)(1) \xrightarrow{d \pi} \pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} T_{X_{\mathrm{sm}}}
\end{aligned}
$$

where coker $d \pi$ is supported on $\pi^{-1} X_{\mathrm{sm}}-U$ which has codimension $\geq 2$ by Proposition 5.1. We have $M \subset N$ and furthermore the support of $N / M$ is $\pi^{-1} X_{\mathrm{sm}}-W$. Since $\pi_{\pi-1}^{*} X_{\mathrm{sm}} T_{X_{\mathrm{sm}}}$ is torsion free we find that $N$ is the maximal submodule of $\left(\mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}} \otimes \mathfrak{g}^{*}\right)(1)$ containing $M$ such that $N / M$ is torsion. Hence we are in the setting of Lemma 5.5, provided $\mathfrak{g}$ is non-abelian (we want $\operatorname{rk} \rho>0$ ), which we will temporarily assume. Let $f$ be the fundamental semi-invariant (cfr $\S 2$ ) in $S \mathfrak{g}$. Then we have $I(\rho) \subset(f)$ and $(f) / I(\rho)$ is supported in codimension $\geq 2$. Hence in the application of Lemma 5.5 we may replace $I(\rho)$ by $(f)=\mathcal{O}_{Y}(-d(\mathfrak{g}))$.

Taking into account $\operatorname{dim} \mathfrak{g}=\operatorname{dim} Y$ we conclude (using multiplicativity of det)

$$
\begin{aligned}
\operatorname{det} M & =\left(\pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} \omega_{X_{\mathrm{sm}}}\right)^{*}(-\operatorname{dim} X) \\
\operatorname{det} N & =\left(\pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} \omega_{X_{\mathrm{sm}}}\right)(\operatorname{dim} Y)
\end{aligned}
$$

Hence by Lemma 5.5 we obtain

$$
\left(\pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} \omega_{X_{\mathrm{sm}}}\right)^{*}(-\operatorname{dim} X)=\left(\pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} \omega_{X_{\mathrm{sm}}}\right)(\operatorname{dim} Y-d(\mathfrak{g}))
$$

or in other words

$$
\begin{equation*}
\mathcal{O}_{\pi^{-1} X_{\mathrm{sm}}}(-\operatorname{dim} Y-\operatorname{dim} X+d(\mathfrak{g}))=\pi_{\pi^{-1} X_{\mathrm{sm}}}^{*} \omega_{X_{\mathrm{sm}}}^{\otimes 2} \tag{5.7}
\end{equation*}
$$

If $T=k+T_{1}+\cdots+$ is a finitely generated positively graded normal commutative ring and if $\omega_{T} \cong T(-a)$ then we will call $a$ the Gorenstein invariant of $T$ and denote it by $a(T)$. This is for example always defined if $T$ is factorial. ${ }^{2}$

Example 5.6. (1) If $T=k\left[f_{1}, \ldots, f_{r}\right]$ is a graded polynomial ring with homogeneous generators $f_{1}, \ldots, f_{r}$ of strictly positive degree then $a(T)=$ $\sum_{i} \operatorname{deg} f_{i}$.
(2) Similarly if $T=k\left[f_{1}, \ldots, f_{r}\right] /\left(p_{1}, \ldots, p_{s}\right)$ is a homogeneous normal complete intersection then $a(T)=\sum_{i} \operatorname{deg} f_{i}-\sum_{j} \operatorname{deg} p_{j}$.
We can now prove a more general version of Proposition 1.4 in the absence of proper semi-invariants.

Proposition 5.7. Assume that $(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$, and $(S \mathfrak{g})^{\mathfrak{g}}$ is finitely generated. Then $a\left((S \mathfrak{g})^{\mathfrak{g}}\right)$ is defined and is equal to

$$
\begin{equation*}
a\left((S \mathfrak{g})^{\mathfrak{g}}\right)=\frac{1}{2}(\operatorname{dim} \mathfrak{g}+i(\mathfrak{g})-d(\mathfrak{g})) \tag{5.8}
\end{equation*}
$$

Proof. We use the same notations as above. We assume that $\mathfrak{g}$ is non-abelian since otherwise the result is trivial. Since there are no proper semi-invariants, $(S \mathfrak{g})^{\mathfrak{g}}$ is factorial, and hence $a\left((S \mathfrak{g})^{\mathfrak{g}}\right)$ is defined. Put $a=a\left((S \mathfrak{g})^{\mathfrak{g}}\right)$. Thus $\omega_{X}=\mathcal{O}_{X}(-a)$.

[^2]Let $i: \pi^{-1} X_{\mathrm{sm}} \rightarrow Y$ be the inclusion map. Applying $i_{*}$ to (5.7) and using the fact that by Proposition $5.1 \operatorname{codim}_{Y}\left(Y-\pi^{-1} X_{\mathrm{sm}}\right) \geq 2$ and that everything is reflexive, we obtain an equality

$$
\mathcal{O}_{Y}(-\operatorname{dim} Y-\operatorname{dim} X+d(\mathfrak{g}))=\pi^{*} \omega_{X}^{\otimes 2}=\mathcal{O}_{Y}(-2 a)
$$

and hence $2 a=\operatorname{dim} X+\operatorname{dim} Y-d(\mathfrak{g})=i(\mathfrak{g})+\operatorname{dim} \mathfrak{g}-d(\mathfrak{g})$ which yields (5.8).
This result can also be proved using the method exhibited in [17, Remark 1.6.3] as Proposition 5.1 shows that the set $S$ in loc. cit. (which is $U$ in our terminology) is "big" in the sense of [17].

Example 5.8. We apply Proposition 5.7 to a non-coregular example. Let $\mathfrak{g}=L(6)$ (cfr Example 1.7). Using [8, $\S 89, \S 93]$ or the library "ainvar.lib" from Singular [9] we find $(S \mathfrak{g})^{\mathfrak{g}}=k\left[f_{1}, f_{2}, f_{3}, f_{4}, f_{5},\right] /(p)$ where

$$
\begin{aligned}
& f_{1}=v_{5}^{2}-2 v_{4} v_{6} \\
& f_{2}=v_{5}^{3}-3 v_{4} v_{5} v_{6}+3 v_{3} v_{6}^{2} \\
& f_{3}=v_{4}^{2}-2 v_{3} v_{5}+2 v_{2} v_{6} \\
& f_{4}=2 v_{4}^{3}+6 v_{2} v_{5}^{2}+9 v_{3}^{2} v_{6}-12 v_{2} v_{4} v_{6}-6 v_{3} v_{4} v_{5} \\
& f_{5}=v_{6}
\end{aligned}
$$

and

$$
p=f_{4} f_{5}^{3}-3 f_{1} f_{3} f_{5}^{2}+f_{1}^{3}-f_{2}^{2}
$$

According to Example 1.7 we have $\operatorname{dim} \mathfrak{g}=6, i(\mathfrak{g})=4, d(\mathfrak{g})=0$. The equality (5.8) becomes (taking into account Example 1.7(2))

$$
2+3+2+3+1-6=5=(6+4-0) / 2
$$

Remark 5.9. Let $g^{2}$ be the fundamental semi-invariant in $S \mathfrak{g}$ (it is a square by Lemma 2.1). If $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}$ is a polynomial algebra then it seems that in many cases the irreducible factors of $g$ form a subset of the generators of $(S \mathfrak{g})_{\text {si }}^{\mathfrak{g}}$. This is true for Frobenius Lie algebras [2] and also for the examples covered by the methods in [14].

Assume that $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$ and $(S \mathfrak{g})^{\mathfrak{g}}=k\left[f_{1}, \ldots, f_{r}\right]$. If $g=\prod_{i} f_{i}^{\epsilon_{i}}$ then the equality (5.8) becomes

$$
\sum_{i=1}^{r}\left(1+\epsilon_{i}\right) \operatorname{deg} f_{i}=c(\mathfrak{g})
$$

This is similar to a phenomenon observed by Fauquant-Millet and Joseph in [7] that one can sometimes make the inequality (1.1) into an equality by changing the degrees of the $f_{i}$ in some natural way.

We end this section by proving Proposition 1.6(2)(3).
Proposition 5.10. Assume that $(S \mathfrak{g})_{\mathrm{si}}^{\mathfrak{g}}=(S \mathfrak{g})^{\mathfrak{g}}$, $\mathfrak{g}$ is not abelian and $\mathfrak{g}$ is coregular. Then $\operatorname{codim}_{\mathfrak{g}^{*}}\left(\mathfrak{g}^{*}-\mathfrak{g}_{\text {reg }}^{*}\right) \leq 3$. If $\operatorname{codim}_{\mathfrak{g}^{*}}\left(\mathfrak{g}^{*}-\mathfrak{g}_{\text {reg }}^{*}\right)=3$ then $\mathfrak{g}^{*}-\mathfrak{g}_{\text {reg }}^{*}$ is purely of codimension three and is precisely equal to the non-smooth locus of $\pi$.

Proof. Since $X=X_{\mathrm{sm}}$ (5.6) yields a complex of vector bundles on $Y$

$$
\begin{equation*}
0 \rightarrow \pi^{*} \Omega_{X}(1) \xrightarrow{d \pi^{*}} \mathcal{O}_{Y} \otimes_{k} \mathfrak{g} \xrightarrow{\rho}\left(\mathcal{O}_{Y} \otimes \mathfrak{g}^{*}\right)(1) \xrightarrow{d \pi} \pi^{*} T_{X} \rightarrow 0 \tag{5.9}
\end{equation*}
$$

which is exact in $\pi^{*} T_{X}$ at the smooth locus of $\pi$ (denoted by $U$ above). Furthermore the locus in $U$ where it is exact in $\left(\mathcal{O}_{Y} \otimes \mathfrak{g}^{*}\right)(1)$ is $W \stackrel{\text { Prop. }}{=}{ }^{5.2} Y-Y^{\prime}=\mathfrak{g}_{\text {reg }}^{*}$.

Assume $\operatorname{codim}_{\mathfrak{g}^{*}}\left(\mathfrak{g}^{*}-\mathfrak{g}_{\text {reg }}^{*}\right) \geq 4$. Hence the locus where (5.9) is not exact has codimension $\geq 4$. An easy depth computation yields that (5.9) is exact. Thus $\mathfrak{g}^{*}=\mathfrak{g}_{\text {reg }}^{*}$ and hence $0 \in \mathfrak{g}^{*}$ is regular. This is only possible if $\mathfrak{g}$ is abelian, which we had excluded.

Now assume $\operatorname{codim}_{\mathfrak{g}^{*}}\left(\mathfrak{g}^{*}-\mathfrak{g}_{\text {reg }}^{*}\right)=3$. Then by a similar depth computation one finds that (5.9) is exact, except in $\pi^{*} T_{X}$. Thus in particular $W=U$, or in other words $\mathfrak{g}^{*}-\mathfrak{g}_{\text {reg }}^{*}$ coincides with the non-smooth locus of $\pi$. We find that coker $d \pi$ is a Cohen-Macaulay module supported on $\mathfrak{g}^{*}-\mathfrak{g}_{\text {reg }}^{*}$. Since Cohen-Macaulay modules have pure support we are done.

## References

[1] W. Borho, P. Gabriel, and R. Rentschler, Primideale in Einhüllenden auflösbarer LieAlgebren (Beschreibung durch Bahnenräume), Springer-Verlag, Berlin, 1973, Lecture Notes in Mathematics, Vol. 357.
[2] L. Delvaux, E. Nauwelaerts, and A. I. Ooms, On the semicenter of a universal enveloping algebra, J. Algebra 94 (1985), no. 2, 324-346.
[3] J. Dixmier, Sur les représentations unitaires des groupes de Lie nilpotents. II, Bull. Soc. Math. France 85 (1957), 325-388.
[4] ___ Sur les représentations unitaries des groupes de Lie nilpotents. III, Canad. J. Math. 10 (1958), 321-348.
[5] , Enveloping algebras, Grad. Stud. Math., vol. 11, Amer. Math. Soc., Providence, RI, 1996.
[6] J. Dixmier, M. Duflo, and M. Vergne, Sur la représentation coadjointe d'une algèbre de Lie, Compositio Math. 29 (1974), 309-323.
[7] F. Fauquant-Millet and A. Joseph, La somme des faux degrés-un mystère en théorie des invariants, Adv. Math. 217 (2008), no. 4, 1476-1520.
[8] J. H. Grace and A. Young, The algebra of invariants, Cambridge University Press, 1903.
[9] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 3.0, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2005, http://www.singular.uni-kl.de.
[10] P. Heymans, Pfaffians and skew-symmetric matrices, Proc. London Math. Soc. (3) 19 (1969), 730-768.
[11] A. Joseph, Proof of the Gel'fand-Kirillov conjecture for solvable Lie algebras, Proc. Amer. Math. Soc. 45 (1974), 1-10.
[12] F. Knop, Über die Glattheit von Quotientenabbildungen, Manuscripta Math. 56 (1986), no. 4, 419-427.
[13] _, Der kanonische Modul eines Invariantenrings, J. Algebra 127 (1989), 40-54.
[14] A. I. Ooms, Computing invariants and semi-invariants by means of Frobenius Lie algebras, to appear.
[15] , The Frobenius semiradical of a Lie algebra, J. Algebra 273 (2004), no. 1, 274-287.
[16] A. I. Ooms, On certain maximal subfields in the quotient division ring of an enveloping algebra, J. Algebra 230 (2000), no. 2, 694-712.
[17] D. I. Panyushev, On the coadjoint representation of $\mathbb{Z}_{2}$-contractions of reductive Lie algebras, Adv. Math. 213 (2007), no. 1, 380-404.
[18] R. Rentschler and M. Vergne, Sur le semi-centre du corps enveloppant d'une algèbre de Lie, Ann. Sci. École Norm. Sup. (4) 6 (1973), 389-405.
[19] G. S. Rinehart, Differential forms on general commutative algebras, Trans. Amer. Math. Soc. 108 (1963), 195-222.
[20] M. Rosenlicht, A remark on quotient spaces, An. Acad. Brasil. Ci. 35 (1963), 487-489.
[21] S. T. Sadètov, A proof of the Mishchenko-Fomenko conjecture (1981), Dokl. Akad. Nauk 397 (2004), no. 6, 751-754.
[22] A. Tyc, An elementary proof of the Weitzenböck theorem, Colloq. Math. 78 (1998), no. 1, 123-132.

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[^1]:    ${ }^{1}$ Unfortunately the subscript "reg" is already taken by $\mathfrak{g}_{\text {reg }}^{*}$.

[^2]:    ${ }^{2}$ This is a slight abusing of existing terminology as normally the Gorenstein invariant is only defined for Gorenstein rings.

