No more replicating portfolios : a simple convex combination to understand the risk-neutral valuation method for the multi-step binomial valuation of a call option.

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Abstract

This paper covers the valuation, from beginning to implementation, of a European call option on a stock using the multi-step binomial model in a risk-neutral world. The aim is to introduce this model in a simple but rather unconventional way. The usual presentation of the risk-neutral valuation, see Hull (2009), among others, relies on replicating portfolios. For most practitioners, this technique looks rather mysterious. We present a new transparent analysis requiring no replicating portfolios. The new finding to understand why the risk-neutral pricing is consistent with investors being risk-averse is the notion of a convex combination.

Key words : investments, stock, Black-Scholes, volatility

Our discussion is summarized as follows. We start by considering a one-step binomial valuation model using the risk-neutral principle. Next, we explain how, involving the no-arbitrage principle, risk-neutral valuation makes no assumption of risk neutrality. The added value of this paper is the use of a convex combination instead of replicating portfolios to explain why the model is not ignoring the risk. This is the core of the text. We then show how the method can be generalized to a multi-step binomial valuation model. In the next section we describe how historical data can be used to estimate the parameters in the model. Finally, we present a real case to illustrate the model.

1. Call options in a nutshell

A call option is a contract that gives the holder the right, without the obligation, to buy an asset on or before a specified date for a guaranteed price K. The price K in the contract is called the exercise price, the date in the contract is known as the exercise date or expiration date, the purchase price of the option is called the premium. In the financial world, the underlying asset is generally a share of stock. A European option can be exercised only on the exercise date, an American option can be exercised at any time during its life.
In this paper we limit ourselves to a European call option on a share of stock paying no dividends and ignoring transaction costs and taxes.
We can describe the value of the call option on the exercise date in terms of the stock price at expiration and the exercise price. Let C_t be the value of the call option on the exercise date, S_t the stock price at expiration and K the exercise price. Evidently the option will only be exercised if $S_t > K$. Hence,

$$C_t(S_t) = \max\{S_t - K, 0\}$$

It is much more difficult to compute the current value of the call option. In this paper the popular binomial valuation model is used, but there are alternatives, e.g. the Black-Scholes model.

2. One-step binomial valuation model

Suppose the stock's current price is S_0, and one period later the price will either increase to uS_0 or fall to dS_0 with $0 < d < 1+r < u$ and r the risk-free interest rate per period and generally $d < 1$. We do not know the probabilities of the stock price moving up or down. The risk-free interest rate r is the rate you can earn by leaving money in risk-free assets such as government bonds. What about the inequities $d < 1+r < u$? No-arbitrage arguments give the answer. We defer a detailed discussion to the next section. Let's calculate the current value C_0 of a call option on this stock. Suppose an exercise price K and one period time to exercise date.

Table 1 : stock price and call option value

Stock price		Call option value	
Time 0	time 1	Time 0	time 1
	uS_0		$\max\{uS_0 - K, 0\}$
S_0		C_0 ?	
	dS_0		$\max\{dS_0 - K, 0\}$

We use an important general principle known as risk-neutral option valuation.

The risk-neutral principle states :

> The current value C_0 of a call option is obtained by discounting, at the risk-free interest rate, the expected option value at expiration, computed in a risk-neutral world.

Recall that this concept is based on the absence of arbitrage opportunities and will be discussed in the next section.

Return to the problem :

Step 1 : At expiration, the expected stock price in a risk-neutral world must equal the stock price invested at the risk-free interest rate r.

$$(1+r)S_0 = puS_0 + (1-p)dS_0$$

Solve for p :

$$p = \frac{1+r-d}{u-d} \qquad 0 < p < 1$$

This probability p is referred to as the risk-neutral probability.

Step 2 : Use this p to compute the expected value at expiration of a call option on the stock.

$$E(C_t) = p \cdot \max\{uS_0 - K, 0\} + (1-p) \cdot \max\{dS_0 - K, 0\}$$

Step 3 : The current value C_0 of the call option is obtained by discounting $E(C_t)$ at the risk-free interest rate r.

$$C_0 = \frac{p \cdot \max\{uS_0 - K, 0\} + (1-p) \cdot \max\{dS_0 - K, 0\}}{1+r}$$

3. A simple convex combination to understand why the risk-neutral principle is not ignoring the risk

Because the binomial valuation model uses the risk-neutral probability p and the risk-free interest rate r, this approach suggests that we are ignoring the risk. It is worth exploring why the resulting formulas are not just correct in a risk-neutral world, but in other worlds as well.

In contrast to the usual approach, we present an analysis requiring no replicating portfolios.

For a clear discussion of how a replicating portfolio can be used to explain why the risk-neutral principle is not ignoring the risk, we refer to the famous book by Hull (2009). The general approach adopted by Hull is similar to that in the important seminal paper by Cox, Ross and Rubinstein (1979). Numerous other authors have attempted to describe this finding, such as Stampfli and Goodman (2001), Capinsky and Zastawniak (2003) and McDonald (2003). For a simple, extended illustration of the notion of replication, we refer to Smart, Megginson and Gitman (2004) and Bodie, Kane and Marcus (2005). For a mathematically thorough discussion, see Etheridge (2002) and Ross (2003), among others.

Now, suppose an asset's current price is A_0 and the price can increase one period later to uA_0 with probability q or fall to dA_0 with probability 1-q. The expected price of the asset one period later is

$$E(A_1) = quA_0 + (1-q)dA_0$$

A risk-averse investor requires compensation for risk taking. So he wants a larger expected price as risk increases. Consequently, in a risky world, the asset's current price is

$$A_0 = \frac{quA_0 + (1-q)dA_0}{1+R}$$

with R = r + r' the risk-adjusted interest rate, r the risk-free interest rate and r' the risk premium on the risky asset. An increase in an asset's risk decreases its current price.

In the absence of arbitrage opportunities there is a value p ($0 < p < 1$) that can be substituted for q to modify R into the risk-free interest rate r. Here is a way to illustrate this proposition.

The assumption of no arbitrage requires $d < 1+r < u$. We verify these conditions.
Suppose that $u \leq 1+r$. An investor who invested in government bonds would be certain to make more profit than investors holding stock. No one would want to buy stock. Suppose $1+r \leq d$. An investment in the stock financed by debt would lead to a certain profit. The stock would be a great buy. No one would want to buy the debt. A real market would not support such stock behaviors. Consequently, a complete market with no arbitrage requires $d < 1+r < u$.

Using elementary mathematics, we can find values p ($0 < p < 1$) and 1 - p such that

$$1+r = pu + (1-p)d$$

In words, 1 + r is a convex combination of u and d.

Multiplying this expression by A_0 and dividing by 1 + r yields the formula

$$A_0 = \frac{puA_0 + (1-p)dA_0}{1+r}$$

Clearly, an asset can be priced using the risk-neutral probability p and discounting at the risk-free interest rate r. Generally, the artificial risk-neutral probability p is not equal to the probability of an up movement. The probability p only yields an asset return equivalent to the riskless return. Hence, it is easy to understand that the relationship between the current option value and the underlying stock in a risky world is the same as it would be in a risk-neutral world. Risk-neutral valuation and no-arbitrage arguments are equivalent and lead to the same option values. Risk-neutral valuation uses the assumption of no arbitrage, but makes no assumption of risk neutrality. It turns out that for the purpose of valuation of call options the relevant probability is the abstract risk-neutral probability p. The procedure described in this section, requiring no replicating portfolios, is unconventional. The key is a convex combination.

4. Multi-step binomial valuation model

A stock that can take one of only two possible prices at expiration is not realistic. We can however generalize the one-step model to incorporate more realistic assumptions. Suppose that the life of an option on a stock is divided into n subintervals. We assume the stock price starts at S_0, and in each period the stock price S_k can increase to uS_k or decrease to dS_k with $0 < d < 1+r < u$ and r the risk-free interest rate per period (subinterval) and generally $d < 1$. Let's calculate the current value C_0 of a call option on this stock.

Table 2 : stock price tree

Time 0	time 1	time 2	time 3	…	time n
			u^3S_0	…	
		u^2S_0			
	uS_0		u^2dS_0	…	
S_0		udS_0			$u^kd^{n-k}S_0$
	dS_0		ud^2S_0	…	
		d^2S_0			
			d^3S_0	…	

At any node the structure is identical. First, solve for p : $1+r = pu + (1-p)d$
to find the risk-neutral probability p. The price of the stock at expiration for k up movements and n - k down movements is $S_0u^kd^{n-k}$ every such outcome having probability $p^k(1-p)^{n-k}$. There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ paths through the tree leading to $S_0u^kd^{n-k}$. Therefore the probability that the stock price is $S_0u^kd^{n-k}$ at expiration is $\binom{n}{k}p^k(1-p)^{n-k}$.

This analysis says that if the random variable x denotes the number of up movements, then x is a binomial random variable with parameters n and p.

It follows that the expected value at expiration of a call option on the stock is :

$$E(C_t) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \max\{S_0 u^k d^{n-k} - K, 0\}$$

The value of the call option today is :

$$C_0 = \frac{1}{(1+r)^n} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \max\{S_0 u^k d^{n-k} - K, 0\}$$

The summation starts with the smallest m such that $S_0 u^m d^{n-m} > K$.
Consequently, the formula for the current value C_0 of a call option on a stock can be simplified to :

$$C_0 = \frac{1}{(1+r)^n} \sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \left(S_0 u^k d^{n-k} - K\right)$$

The first study to clearly show this finding is the seminal paper by Cox, Ross and Rubinstein (1979).

5. Adjusting the model to real stock data

How do we choose the parameters u and d in the binomial valuation model ?
We illustrate a conventional method for estimating these values from historical stock prices $S_0^h, S_1^h, ..., S_N^h$. We assume Δt for the tree (from one node to another node) equals the Δt for the data set. If Δt for the tree does not equal the Δt for the data set, see Stampfli and Goodman (2001).
If we consider the ratio $\frac{S_k}{S_{k-1}}$ in the binomial model as a random variable x then x is a Bernoulli random variable with values u and d.

Table 3 : Bernoulli random variable

x	x_1	x_2
probability	p	1-p

x	u	d
probability	p	1-p

In this section, the probabilities p and 1 - p are the probabilities of the stock price moving up or down.

The mean for a Bernoulli random variable x is

$$E(x) = px_1 + (1-p)x_2$$

and the variance is

$$E((x-\mu)^2) = p(1-p)(x_1 - x_2)^2$$

Hence, for $x = \frac{S_k}{S_{k-1}}$ is

$$\mu = pu + (1-p)d$$

$$\sigma^2 = p(1-p)(u-d)^2$$

In a simple model, we set p = 1/2 :

$$\mu = \frac{u+d}{2}$$

$$\sigma = \frac{u-d}{2}$$

and determine u and d :

$$u = \mu + \sigma$$

$$d = \mu - \sigma$$

Reasonable point estimators of μ and σ^2 are the sample mean and sample variance computed from the historical real-world data $S_0^h, S_1^h, \ldots, S_N^h$:

$$\bar{x} = \frac{\sum_{k=1}^{N} \frac{S_k^h}{S_{k-1}^h}}{N}$$

$$s^2 = \frac{\sum_{k=1}^{N} \left(\frac{S_k^h}{S_{k-1}^h}\right)^2}{N-1} - \frac{N}{N-1}\bar{x}^2$$

The value μ - 1 is known as the drift parameter, the parameter σ as the volatility.

6. Applying the model : the Fiction stock case

To illustrate the model just discussed, we present a real-world example. We want to use the binomial valuation model to calculate the value of a 4-months call option on a Fiction stock, listed on New-Money Stock Exchange. We assume Δt for the tree equals one month. The exercise price is assumed to be the stock's current price of 87.8 euro (the price on June 30, 2009). The monthly risk-free interest rate is 0.2466% (3% per annum). Using the sequence of 11 monthly historical stock prices we first estimate the parameters μ and σ to compute u and d and then calculate the current value of a call option on this stock.

A spreadsheet program is useful.

In this illustration we have only 11 data. Evidently in practice we want much more monthly historical stock prices.

Table 4 : Fiction stock

date (2008-2009)	closing price S_k^h (euro)	$\frac{S_k^h}{S_{k-1}^h}$	
July 31	69.9		
August 29	66.25	0.947783	
Sept 30	67.1	1.012830	
Oct 31	69.5	1.035768	
Nov 28	69.15	0.994964	
Dec 31	74.5	1.077368	
Jan 30	79	1.060403	mean :
Feb 27	83	1.050633	1.023917
March 31	81	0.975904	
Apr 30	82.25	1.015432	standard deviation :
May 29	87.85	1.068085	0.042332

$$\mu \approx \overline{x} = 1.023917$$

$$\sigma \approx s = 0.042332$$

$$u = \mu + \sigma = 1.066249$$

$$d = \mu - \sigma = 0.981585$$

Next, we fill in the tree.

Table 5 : Fiction : stock price tree and call option values

Time 0	time 1	time 2	time 3	time 4	probability	call option value time 4
				113.48	p^4	25.68
			106.43			
		99.82		104.47	$4p^3(1-p)$	16.67
	93.62		97.98			
87.8		91.89		96.18	$6p^2(1-p)^2$	8.38
	86.18		90.20			
		84.60		88.54	$4p(1-p)^3$	0.74
			83.04			
				81.51	$(1-p)^4$	0

This tree allows us to compute the current value of a call option on the stock.

Step 1 : From one node to the following period, the expected stock price in a risk-neutral world must equal the stock price invested at the risk-free interest rate r.

$$1 + r = pu + (1 - p)d$$

Solve for p :

$$p = \frac{1 + r - d}{u - d} = 0.24664$$

This probability p is the risk-neutral probability.

Step 2 : Use this p to compute the expected value at expiration of a call option on the stock.

$$E(C_t) = p^4 \cdot (25.68) + 4p^3(1-p) \cdot (16.67) + 6p^2(1-p)^2 \cdot (8.38) + 4p(1-p)^3 \cdot (0.74) + (1-p)^4 \cdot 0$$

$$E(C_t) = 2.90$$

Step 3 : The current value C_0 of the call option is obtained by discounting $E(C_t)$ at the risk-free interest rate r.

$$C_0 = \frac{E(Ct)}{(1.002466)^4} = 2.87$$

The current value of a call option on a Fiction stock is 2.87 euro.

7. Conclusion

In this paper, we have presented a framework, from beginning to implementation, how the valuation of a European call option on a stock works using the multi-step binomial model in a risk-neutral world. The key to understand why this risk-neutral principle is not ignoring the risk is the notion of a convex combination. Therefore, our approach, requiring no replicating portfolios, is unconventional.

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