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Cocommutative Calabi–Yau Hopf algebras and deformations

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ABSTRACT

The Calabi–Yau property of cocommutative Hopf algebras is discussed by using the homological integral, a recently introduced tool for studying infinite dimensional AS-Gorenstein Hopf algebras. It is shown that the skew-group algebra of a universal enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ with a finite subgroup G of automorphisms of $\mathfrak{g}$ is Calabi–Yau if and only if the universal enveloping algebra itself is Calabi–Yau and G is a subgroup of the special linear group $SL(\mathfrak{g})$. The Noetherian cocommutative Calabi–Yau Hopf algebras of dimension not larger than 3 are described. The Calabi–Yau property of Sridharan enveloping algebras of finite dimensional Lie algebras is also discussed. We obtain some equivalent conditions for a Sridharan enveloping algebra to be Calabi–Yau, and then partly answer a question proposed by Berger. We list all the nonisomorphic 3-dimensional Calabi–Yau Sridharan enveloping algebras.

Introduction

We work over a fixed field $\Bbbk$ which is assumed to be algebraically closed and of characteristic zero and is assumed to be the field of complex numbers $\mathbb{C}$ if necessary.

Calabi–Yau algebras are studied in recent years because of their applications in algebraic geometry and mathematical physics. The aim of this paper is to try to understand cocommutative Calabi–Yau Hopf algebras of lower dimensions. We take the Calabi–Yau (CY) property from Ginzburg [9], the definition will be recalled in Section 2. The main tool used in this paper is the homological integral of an AS-Gorenstein Hopf algebra recently introduced by Lu, Wu and Zhang in [16] and extended to

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general AS-Gorenstein algebras (the definition is recalled in Section 1) by Brown and Zhang in [5]. As a consequence of Kostant–Larson's work (cf. [14,25]), we know that any cocommutative Hopf algebra (over an algebraically closed field) is isomorphic to a skew-group (or smash product) algebra of the universal enveloping subalgebra of the primitive elements and the group subalgebra of the group-like elements. Hence it is necessary to discuss how the homological integral works on the skew-group algebras. In Section 1, we discuss finite group actions on AS-Gorenstein algebras. Let A be an AS-Gorenstein algebra and G be a finite group. If there is a G-action on A such that the G-action is compatible with the augmentation map of A, then A is called an augmented G-module algebra. If A is an augmented G-module algebra which is AS-Gorenstein, then we show in Section 1 that the skew-group algebra $A\#\Bbbk G$ is also AS-Gorenstein and the (left) homological integral $\int_A^l$ is a left G-module and the (left) homological integral of $A\#\Bbbk G$ is equal to the G-invariants of the left G-module $\int_A^l \#\Bbbk G$ (Proposition 1.1). A necessary and sufficient condition for a Noetherian Hopf algebra to be CY is given in Section 2. It turns out that a Noetherian CY Hopf algebra must be AS-regular and has bijective antipode. When the AS-Gorenstein Hopf algebra is the universal enveloping algebra of a finite dimensional Lie algebra, we have the following result (Theorem 3.4).

Theorem. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $G \subseteq Aut_{Lie}(\mathfrak{g})$ be a finite group. Then the skew-group algebra $U(\mathfrak{g})\#\Bbbk G$ is CY of dimension d if and only if $U(\mathfrak{g})$ is CY of dimension d and $G \subseteq SL(\mathfrak{g})$.*

Let H be a Hopf algebra, $G(H)$ the group of the group-like elements of H and $P(H)$ the space of the primitive elements of H. Applying the above theorem, we can list all cocommutative CY Hopf algebras H of global dimension not larger than 3 such that $G(H)$ is finite and $P(H)$ is finite dimensional. It is easy to determine the 1-dimensional Noetherian CY cocommutative Hopf algebras with finite group $G(H)$. They are the tensor products of the polynomial algebra $\Bbbk[x]$ with the group algebras of some finite groups. In the 2-dimensional case, such a Hopf algebras H must be isomorphic to a skew-group algebra of the form $H \cong \Bbbk[x, y]\#\Bbbk G$, where G is a finite group and the G-action on $\Bbbk[x, y]$ is induced by a group map $\nu : G \to SL(2, \Bbbk)$ (Theorem 4.2).

For the 3-dimensional case, we show that there are only 4 cases of nonisomorphic 3-dimensional Lie algebras whose universal enveloping algebras are CY (Proposition 4.6). A 3-dimensional Noetherian CY cocommutative Hopf algebra H with finite group $G(H)$ and finite dimensional $P(H)$ is isomorphic to a skew-group algebra of form $U(\mathfrak{g})\#\Bbbk G$, where the Lie algebra $\mathfrak{g}$ is one of the Lie algebras listed in Proposition 4.6 and G is a finite group with a group morphism $\nu : G \to Aut_{Lie}(\mathfrak{g})$ such that $\mathrm{im}(\nu)$ is a subgroup of $SL(\mathfrak{g})$ (Theorem 4.7).

In the last section, we deal with the Sridharan enveloping algebras of finite dimensional Lie algebras. In general, a Sridharan enveloping algebra is no longer a Hopf algebra. However, a Sridharan enveloping algebra is a cocycle deformation of a cocommutative Hopf algebras, and the CY property of the Sridharan enveloping algebras is closely related to that of the universal enveloping algebras. Hence it is proper to include the discussion of Sridharan enveloping algebras in this paper. Sridharan enveloping algebras were introduced in [24] in order to discuss certain representations of Lie algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A Sridharan enveloping algebra is related to a 2-cocycle $f \in Z^2(\mathfrak{g}, \Bbbk)$ of $\mathfrak{g}$, and is usually denoted by $U_f(\mathfrak{g})$ (the definition is recalled in the final section). The class of Sridharan enveloping algebras includes many interesting algebras, such as Weyl algebras. Homological properties, especially the Hochschild (co)homology and cyclic homology, are studied by several authors [24,11,18,19]. Berger proposed at the end of his recent paper [2] a question: to find some necessary and sufficient conditions for a Sridharan enveloping algebra to be CY. We get the following result (Theorem 5.3) which in part answers Berger's question [2].

Theorem. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $f \in Z^2(\mathfrak{g}, \Bbbk)$ be an arbitrary 2-cocycle of $\mathfrak{g}$. The following statements are equivalent.*

(i) *The Sridharan enveloping algebra $U_f(\mathfrak{g})$ is CY of dimension d.*
(ii) *The universal enveloping algebra $U(\mathfrak{g})$ is CY of dimension d.*
(iii) $\dim \mathfrak{g} = d$*, and $\mathfrak{g}$ is unimodular [13], that is, for any $x \in \mathfrak{g}$,* $\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)) = 0$.

We also list all the 3-dimensional CY Sridharan enveloping algebras at the end of the paper (Theorem 5.5). There are exactly 7 classes of nonisomorphic 3-dimensional CY Sridharan enveloping algebras.

1. Homological integrals of skew group algebras

Let A be a left Noetherian augmented algebra with a fixed augmentation map $\varepsilon : A \longrightarrow \Bbbk$. Recall that A is said to be *left Artin–Schelter Gorenstein* (AS-Gorenstein for short, cf. [5]), if

(i) $\operatorname{inj\,dim}_A A = d < \infty$,
(ii) $\dim \operatorname{Ext}_A^d({}_A\Bbbk, {}_AA) = 1$ and $\operatorname{Ext}_A^i({}_A\Bbbk, {}_AA) = 0$ for all $i \neq d$.

If A is a right Noetherian augmented algebra with a fixed augmentation map, and the right versions of (i) and (ii) above hold, then A is said to be *right AS-Gorenstein*. If A is both left and right AS-Gorenstein (relative to the same augmentation map ε), then we say that A is *AS-Gorenstein*. Furthermore, if $\operatorname{gl\,dim} A < \infty$, then A is called an *AS-regular* algebra.

The concept of a homological integral was first introduced in [16] for an AS-Gorenstein Hopf algebra as a generalization of the classical concept of an integral for a finite dimensional Hopf algebra. The concept was further extended to a general AS-Gorenstein algebra in [5]. It seems that the homological integral is a useful tool to study infinite dimensional noncocommutative algebras. Let A be a left AS-Gorenstein algebra. Then $\operatorname{Ext}_A^d({}_A\Bbbk, {}_AA)$ is a one-dimensional right A-module. Any nonzero element in $\operatorname{Ext}_A^d({}_A\Bbbk, {}_AA)$ is called a *left homological integral* of A. Write $\int_A^l$ for $\operatorname{Ext}_A^d({}_A\Bbbk, {}_AA)$, and call it, by a slight abuse of terminology, *the homological integral of* A. Similarly, if A is a right AS-Gorenstein algebra, any nonzero element of the one-dimensional left module $\operatorname{Ext}_A^d(\Bbbk_A, A_A)$ is called a *right homological integral* of A. We denote it by $\int_A^r$.

Let G be a finite group. A left G-module algebra A is called *an augmented G-module algebra* if A has an augmentation map ε and ε is a G-map. For an augmented left G-module algebra A, the skew group algebra $A\#\Bbbk G$ is also an augmented algebra with the augmentation map $\epsilon : A\#\Bbbk G \longrightarrow \Bbbk$ defined by $a\#g \mapsto \varepsilon(a)$ for all $a \in A$ and $g \in G$. With this augmentation map, $\Bbbk$ is naturally an $A\#\Bbbk G$-$A\#\Bbbk G$-bimodule. Since G is a finite group, $A\#\Bbbk G$ is a left Noetherian algebra if A is left Noetherian.

Let M and N be left $A\#\Bbbk G$-modules. Then $\operatorname{Hom}_A(M, N)$ is a left G-module with the adjoint action:

$$g \rightharpoonup f(m) = g \cdot f\left(g^{-1} \cdot m\right),$$

for $g \in G$, $f \in \operatorname{Hom}_A(M, N)$ and $m \in M$. For a left G-module X, let $X^G = \{x \in X \mid g \cdot x = x, \text{ for all } g \in G\}$ be the set of G-invariant elements. It is easy to see that

$$\operatorname{Hom}_{A\#\Bbbk G}(M, N) = \operatorname{Hom}_A(M, N)^G. \tag{1}$$

Since G is finite, the functor $(-)^G$ is exact. It follows that a left $A\#\Bbbk G$-module Q is projective (resp. injective) if and only if it is projective (resp. injective) as an A-module. Also the G-module structure on $\operatorname{Hom}_A(M, N)$ can be extended to the extension groups $\operatorname{Ext}_A^i(M, N)$, and the isomorphism (1) can be extended to the following isomorphisms (cf. [17])

$$\operatorname{Ext}^i_{A\#\Bbbk G}(M, N) \cong \operatorname{Ext}^i_A(M, N)^G, \quad \text{for all } i \geqslant 0. \tag{2}$$

Now let A be a left AS-Gorenstein algebra of $\operatorname{inj\,dim}_A A = d$ and consider the one-dimensional module ${}_{A\#\Bbbk G}\Bbbk$ with the module structure defined by the augmentation map ϵ. Let

$$\cdots \longrightarrow P^{-n} \xrightarrow{\partial^{-n}} \cdots \xrightarrow{\partial^{-2}} P^{-1} \xrightarrow{\partial^{-1}} P^0 \longrightarrow {}_{A\#\Bbbk G}\Bbbk \longrightarrow 0$$

be a finitely generated projective resolution of the $A\#\Bbbk G$-module ${}_{A\#\Bbbk G}\Bbbk$. Then the resolution can be regarded as a projective resolution of the A-module ${}_A\Bbbk$. Applying the functor $\mathrm{Hom}_A(-, A\#\Bbbk G)$ to the projective resolution above, we obtain a complex

$$\begin{aligned}\cdots \longleftarrow \mathrm{Hom}_A\big(P^{-n}, A\#\Bbbk G\big) \longleftarrow \cdots \longleftarrow \mathrm{Hom}_A\big(P^{-1}, A\#\Bbbk G\big)\\ \longleftarrow \mathrm{Hom}_A\big(P^{0}, A\#\Bbbk G\big) \longleftarrow 0.\end{aligned} \tag{3}$$

Since G is a finite group, there are natural isomorphisms of vector spaces, for $n \geqslant 0$,

$$\varphi^n : \mathrm{Hom}_A\big(P^{-n}, A\big) \otimes \Bbbk G \longrightarrow \mathrm{Hom}_A\big(P^{-n}, A\#\Bbbk G\big) \tag{4}$$

defined by $\varphi^n(f \otimes g)(p) = f(p)\#g$ for $f \in \mathrm{Hom}_A(P^{-n}, A)$, $g \in G$ and $p \in P^{-n}$.

Let Y_A be an A-module. The tensor space $Y \otimes \Bbbk G$ is a right $A\#\Bbbk G$-module defined by

$$(y \otimes g) \cdot (a\#h) = y \cdot (ga) \otimes gh, \quad \text{for } y \in Y,\ g, h \in G \text{ and } a \in A.$$

We write this right $A\#\Bbbk G$-module as $Y\#\Bbbk G$. Since $\mathrm{Hom}_A(P^{-n}, A)$ is a right A-module, $\mathrm{Hom}_A(P^{-n}, A) \otimes \Bbbk G$ is a right $A\#\Bbbk G$-module. $\mathrm{Hom}_A(P^{-n}, A\#\Bbbk G)$ is also a right $A\#\Bbbk G$-module. Thus the natural isomorphisms in (4) are in fact right $A\#\Bbbk G$-module isomorphisms

$$\varphi^n : \mathrm{Hom}_A\big(P^{-n}, A\big)\#\Bbbk G \longrightarrow \mathrm{Hom}_A\big(P^{-n}, A\#\Bbbk G\big). \tag{5}$$

Observe that A is a left $A\#\Bbbk G$-module and $\mathrm{Hom}_A(P^{-n}, A)$ is a left G-module. With the diagonal G-action, $\mathrm{Hom}_A(P^{-n}, A)\#\Bbbk G$ becomes a left G-module. On the other side, $\mathrm{Hom}_A(P^{-n}, A\#\Bbbk G)$ is also a left G-module with the adjoint G-action. Thus it is not hard to see that both $\mathrm{Hom}_A(P^{-n}, A)\#\Bbbk G$ and $\mathrm{Hom}_A(P^{-n}, A\#\Bbbk G)$ are left G- and right $A\#\Bbbk G$-bimodules, and the isomorphisms φ^n in (5) are isomorphisms of G-$A\#\Bbbk G$-bimodules.

Now one may check that the complex (3) is a complex of G-$A\#\Bbbk G$-bimodules, and it is isomorphic to the following complex of G-$A\#\Bbbk G$-bimodules

$$\begin{aligned}\cdots \longleftarrow \mathrm{Hom}_A\big(P^{-n}, A\big)\#\Bbbk G \overset{(\partial^{-n})^*\otimes id}{\longleftarrow} \cdots \overset{(\partial^{-2})^*\otimes id}{\longleftarrow} \quad \mathrm{Hom}_A\big(P^{-1}, A\big)\#\Bbbk G\\ \overset{(\partial^{-1})^*\otimes \mathrm{id}}{\longleftarrow} \mathrm{Hom}_A\big(P^{0}, A\big)\#\Bbbk G \longleftarrow 0.\end{aligned}$$

By taking the cohomologies of the complex (3) and those of the complex above we arrive at isomorphisms of G-$A\#\Bbbk G$-bimodules:

$$\mathrm{Ext}^i_A({}_A\Bbbk, A\#\Bbbk G) \cong \mathrm{Ext}^i_A({}_A\Bbbk, {}_AA)\#\Bbbk G \tag{6}$$

for all $i \geqslant 0$. Hence by (2), we have right $A\#\Bbbk G$-module isomorphisms

$$\begin{aligned}\mathrm{Ext}^i_{A\#\Bbbk G}({}_{A\#\Bbbk G}\Bbbk, A\#\Bbbk G) &\cong \mathrm{Ext}^i_A({}_A\Bbbk, A\#\Bbbk G)^G\\ &\cong \big(\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA)\#\Bbbk G\big)^G\end{aligned} \tag{7}$$

for all $i \geqslant 0$.

Summarizing the above we arrive at the following results.

Proposition 1.1. *Let G be a finite group, A an augmented left G-module algebra. Assume A is a left AS-Gorenstein algebra with* $\mathrm{inj\,dim}\ {}_AA = d$. *Then the following statements hold.*

(i) *The left homological integral* $\int_A^l$ *is a one-dimensional left* G*-module, and the* G*-action is compatible with the right* A*-module structure of* $\int_A^l$*, that is; for* $g \in G$, $a \in A$ *and* $t \in \int_A^l$, $g(ta) = (gt)(ga)$.

(ii) $A\#kG$ *is left AS-Gorenstein of* $\operatorname{inj\,dim}_{A\#\Bbbk G}(A\#\Bbbk G) = d$*, and as right* $A\#\Bbbk G$*-modules*

$$\int\limits_{A\#\Bbbk G}^{l} \cong \left(\int\limits_{A}^{l} \#\Bbbk G\right)^G,$$

where the G *acts on* $\int_A^l \#\Bbbk G$ *diagonally.*

Proof. The statement (i) is evident, and the isomorphism in (ii) is a direct consequence of the isomorphisms in (7) if $A\#\Bbbk G$ is left AS-Gorenstein.

What remains to be shown is that the left injective dimension of $A\#\Bbbk G$ is d, and $\dim \operatorname{Ext}^d_{A\#\Bbbk G}({}_{A\#\Bbbk G}\Bbbk, A\#\Bbbk G) = 1$. Let

$$0 \longrightarrow A \longrightarrow Q^0 \xrightarrow{\delta^0} Q^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{d-1}} Q^d \xrightarrow{\delta^d} \cdots \tag{8}$$

be an injective resolution of ${}_{A\#\Bbbk G}A$. Since G is a finite group, all the Q^i's are injective as left A-modules. Hence $coker\,\delta^{d-1}$ is injective as an A-module by the assumption that ${}_AA$ has injective dimension d, which in turn implies that $coker\,\delta^{d-1}$ is injective as an $A\#\Bbbk G$-module. Thus we may assume that the resolution (8) ends at the dth position. Now tensoring (8) with $\Bbbk G$, we obtain an exact sequence

$$0 \longrightarrow A \otimes \Bbbk G \longrightarrow Q^0 \otimes \Bbbk G \longrightarrow Q^1 \otimes \Bbbk G \longrightarrow \cdots \longrightarrow Q^d \otimes \Bbbk G \longrightarrow 0.$$

For a left $A\#\Bbbk G$-module M, the space $M \otimes \Bbbk G$ is a left $A\#\Bbbk G$-module defined by

$$(a\#g) \cdot (m \otimes h) = a(gm) \otimes gh.$$

Since Q^i is injective as an A-module, $Q^i \otimes \Bbbk G$ is injective as an $A\#\Bbbk G$-module for all $i \geqslant 0$. Therefore we obtain that the injective dimension of ${}_{A\#\Bbbk G}A\#\Bbbk G$ is not larger than d. We claim that $\operatorname{Ext}^d_{A\#\Bbbk G}({}_{A\#\Bbbk G}\Bbbk, A\#\Bbbk G) \cong (\int_A^l \#\Bbbk G)^G \neq 0$. Assume that α is a nonzero element in $\int_A^l$. Since $\dim \int_A^l = 1$, there is an algebra map $\pi : \Bbbk G \longrightarrow \Bbbk$ such that $g \cdot \alpha = \pi(g)\alpha$ for all $g \in G$. Let π^{-1} be the convolution inverse of π in the dual Hopf algebra $\Bbbk G^*$. Then π^{-1} is an algebra map from $\Bbbk G$ to $\Bbbk$. Hence π^{-1} defines a one-dimensional G-module. Since G is a finite group, we have that there is an element $0 \neq t \in \Bbbk G$ such that $gt = \pi^{-1}(g)t$ for all $g \in G$. Now for $g \in G$, $g \cdot (\alpha\#t) = (g \cdot \alpha)\#gt = \pi(g)\pi^{-1}(g)\alpha\#t = \alpha\#t$. The claim follows. Therefore $\operatorname{inj\,dim}_{A\#\Bbbk G}A\#\Bbbk G = d$. If $0 \neq t' \in \Bbbk G$ is another element such that $g \cdot (\alpha\#t') = \alpha\#t'$ for all $g \in G$, then we get $gt' = \pi(g)^{-1}t' = \pi^{-1}(g)t'$ for all $g \in G$. Then we must have $t' = kt$ for some $k \in \Bbbk$. Otherwise, there would be two one-dimensional G-modules that are isomorphic to each other in the decomposition of the regular G-module, which certainly contradicts the well-known result that in the decomposition of the regular representation of a finite group the multiplicity of an irreducible representation equals its dimension (cf. [22, Sec. 2.4]). Hence $\dim \operatorname{Ext}^d_{A\#\Bbbk G}({}_{A\#\Bbbk G}\Bbbk, A\#\Bbbk G) = 1$. □

We certainly should not expect that $\int_{A\#\Bbbk G}^l \cong \Bbbk\alpha\#t$ such that $\alpha \in \int_A^l$ and t is an integral of $\Bbbk G$, see the following example.

Example 1.2. Let G be a cyclic group of order p. Let λ be a generator of G. Assume that A is an augmented G-module algebra and A is left AS-Gorenstein. Then $A\#\Bbbk G$ is left AS-Gorenstein. In fact, if G acts on $\int_A^l$ trivially, that is, $\lambda \cdot \alpha = \alpha$ where $0 \neq \alpha \in \int_A^l$, then $\operatorname{Ext}^d_{A\#\Bbbk G}({}_{A\#\Bbbk G}\Bbbk, A\#\Bbbk G) = \Bbbk\alpha\#t$ where

$t = \frac{1}{p}\sum_{i=0}^{p-1}\lambda^i$. Now suppose G acts on $\int_A^l$ nontrivially. Then $\lambda \cdot \alpha = \omega\alpha$, where $\omega \in \Bbbk$ is a pth root of the unit. Assume that $t' = x_0\lambda^0 + x_1\lambda + \cdots + x_{p-1}\lambda^{p-1}$ is such that $\lambda \cdot (\alpha \# t') = \alpha \# t'$. Then we obtain

$$\begin{cases} x_0 = \omega x_{p-1} \\ x_1 = \omega x_0 \\ \quad \vdots \\ x_{p-1} = \omega x_{p-2}. \end{cases}$$

Obviously, the linear equations above have an 1-dimensional solution space with a basis given by $(x_0, x_1, \ldots, x_{p-2}, x_{p-1}) = (\omega, \omega^2, \ldots, \omega^{p-1}, 1)$. Let $t' = \omega\lambda^0 + \omega^2\lambda + \cdots + \omega^{p-1}\lambda^{p-2} + \lambda^{p-1}$. Then one can check that

$$\operatorname{Ext}^d_{A\#\Bbbk G}({}_{A\#\Bbbk G}\Bbbk, A\#\Bbbk G) \cong \left(\int_A^l \#\Bbbk G\right)^G = \Bbbk\alpha\#t'.$$

Let A be an augmented left G-module algebra. If A is right AS-Gorenstein, we want to know what $\int_{A\#\Bbbk G}^r$ looks like. The right version of Proposition 1.1 also holds, but it is more complicated. The algebra A can be viewed as an augmented right G-module algebra through the right G-action: $a \cdot g = g^{-1}a$ for $a \in A$ and $g \in G$. We have the skew group algebra $\Bbbk G\#A$ defined in the usual way. There is an algebra isomorphism $\theta : A\#\Bbbk G \longrightarrow \Bbbk G\#A$ by $a\#g \mapsto g\#g^{-1}a$. Moreover, θ is compatible with the augmentation maps of $A\#\Bbbk G$ and $\Bbbk G\#A$ respectively. Now we can deal with right $A\#\Bbbk G$-modules as right $\Bbbk G\#A$-modules. Let M and N be right $\Bbbk G\#A$-modules. $\operatorname{Hom}_A(M, N)$ is a right G-module through the G-action: $(f \leftharpoonup g)(m) = f(mg^{-1})g$ for $f \in \operatorname{Hom}_A(M, N)$, $g \in G$ and $m \in M$. Also we have $\operatorname{Hom}_{\Bbbk G\#A}(M, N) = \operatorname{Hom}_A(M, N)^G$. Similar to Proposition 1.1, we have:

Proposition 1.3. *Let G be a finite group, A be an augmented left G-module algebra. Assume A is a right AS-Gorenstein algebra with* $\operatorname{inj\,dim} A_A = d$. *Then the following statements hold.*

(i) *The right homological integral $\int_A^r$ is a 1-dimensional right G-module, and the G-action is compatible with the left A-module structure of $\int_A^r$, that is; for $g \in G$, $a \in A$ and $t \in \int_A^r$, $(at) \cdot g = (a \cdot g)(t \cdot g) = (g^{-1}a)(t \cdot g)$.*
(ii) *$A\#kG$ is right AS-Gorenstein and* $\operatorname{inj\,dim}(A\#\Bbbk G)_{A\#\Bbbk G} = d$, *also as left $A\#\Bbbk G$-modules*:

$$\int_{A\#\Bbbk G}^r \cong \left(\Bbbk G \otimes \int_A^r\right)^G,$$

where the left $A\#\Bbbk G$-action on $\Bbbk G \otimes \int_A^r$ is given by $(a\#g) \cdot (h \otimes \alpha) = gh \otimes (h^{-1}g^{-1}a)\alpha$ for $g, h \in G$, $a \in A, \alpha \in \int_A^r$, and the right G-action on $\Bbbk G \otimes \int_A^l$ is diagonal.

2. Homological integrals of Calabi–Yau Hopf algebras

In this section we study Noetherian CY Hopf algebras. We show that a Noetherian CY Hopf algebra has trivial homological integrals, and its antipode must be bijective.

Let A be an algebra. Recall that A is said to be a *Calabi–Yau algebra of dimension d* (cf. [9,3]) if (i) A is homologically smooth, that is; A has a bounded resolution of finitely generated projective A-A-bimodules, (ii) $\operatorname{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq d$ and $\operatorname{Ext}^d_{A^e}(A, A^e) \cong A$ as A-A-bimodules, where $A^e = A \otimes A^{op}$ is the enveloping algebra of A. In what follows, Calabi–Yau is abbreviated to CY for short.

Let A be an algebra, $\sigma : A \to A$ an algebra morphism, and M a right A-module. Denote by M^σ the right A-module twisted by the algebra morphism σ. If N is an A-A-bimodule, we denote by ${}^1N^\sigma$ the bimodule whose right A-action is twisted by σ.

Let H be a Hopf algebra with antipode S. We write $\Bbbk$ or $\Bbbk_H$ for the trivial module defined by the counit of H. Let M be an H-H-bimodule. Denote M^{ad} the left adjoint H-module defined by $h \cdot m = \sum_{(h)} h_{(1)} m S(h_{(2)})$ for $h \in H$ and $m \in M$.

Let $D : H \to H \otimes H^{op}$ be the map defined by $D = (1 \otimes S) \circ \Delta$. Then D is an algebra morphism, and $H \otimes H^{op}$ is a free left (and a free right) H-module through D (see [5, Sect. 2]). We write $L(H \otimes H^{op})$ (resp. $R(H \otimes H^{op})$) for the left (resp. right) H-module defined through D. Let ${}_{\bullet}H \otimes H$ be the left H-module defined by the left multiplication of H to the left factor, and $H_{\bullet} \otimes H$ be the free right H-module defined by the right multiplication of H to the left factor. Then $L(H \otimes H^{op}) \cong {}_{\bullet}H \otimes H$ and $R(H \otimes H^{op}) \cong H_{\bullet} \otimes H$. The isomorphisms are given as follows:

$$\varphi : L\big(H \otimes H^{op}\big) \to {}_{\bullet}H \otimes H, \qquad g \otimes h \mapsto \sum_{(g)} g_{(1)} \otimes h S^2(g_{(2)}), \tag{9}$$

with its inverse

$$\phi : {}_{\bullet}H \otimes H \to L\big(H \otimes H^{op}\big), \qquad g \otimes h \mapsto \sum_{(g)} g_{(1)} \otimes h S(g_{(2)}); \tag{10}$$

and

$$\psi : R\big(H \otimes H^{op}\big) \to H_{\bullet} \otimes H, \qquad g \otimes h \mapsto \sum_{(g)} g_{(1)} \otimes g_{(2)} h, \tag{11}$$

with its inverse

$$\xi : H_{\bullet} \otimes H \to R\big(H \otimes H^{op}\big), \qquad g \otimes h \mapsto \sum_{(g)} g_{(1)} \otimes S(g_{(2)}) h. \tag{12}$$

Clearly, $L(H \otimes H^{op})$ is an H-H^e-bimodule and $R(H \otimes H^{op})$ is an H^e-H-bimodule. Let M be an H-H-bimodule. Then one has $M^{\mathrm{ad}} \cong \mathrm{Hom}_{H^e}(R(H \otimes H^{op}), M)$. By [5, Lemma 2.2], the functor $(-)^{\mathrm{ad}}$ preserves injective modules. This property implies the key fact that the Hochschild cohomology of a bimodule M over a Hopf algebra H can be computed through the extension groups of the trivial module ${}_H\Bbbk$ by M^{ad}, see [5, Lemma 2.4] or [10, Prop. 5.6]:

Lemma 2.1. *Let H be a Hopf algebra, and M be an H-H-bimodule. Then we have $\mathrm{Ext}^i_{H^e}(H, M) \cong \mathrm{Ext}^i_H({}_H\Bbbk, M^{\mathrm{ad}})$ for all i.*

Let H be an AS-Gorenstein Hopf algebra. The left homological integrals $\int_H^l$ of H is a one-dimensional right H-module, and the H-module structure is defined through an algebra morphism $\pi : H \to \Bbbk$. We have an algebra automorphism $\nu : H \to H$ defined by $\nu(h) = \sum_{(h)} \pi(h_{(1)}) h_{(2)}$ for $h \in H$. Then, as right H-modules, $\int_H^l \cong \Bbbk^{\nu}$. The following corollary is proved in [5, Prop. 4.5]. We include the proof for the completeness here. Notice that the hypothesis that the antipode is bijective is dropped.

Corollary 2.2. *Let H be a Noetherian AS-Gorenstein Hopf algebra with injective dimension d. Then $\mathrm{Ext}^i_{H^e}(H, H^e) = 0$ for $i \neq d$ and $\mathrm{Ext}^d_{H^e}(H, H^e) = {}^1H^{S^2\nu}$.*

Proof. The proof is just a slight modification of that of [5, Prop. 4.5]. Since H is noetherian, we have

$$\begin{aligned}\mathrm{Ext}^i_{H^e}\big(H, H^e\big) &\cong \mathrm{Ext}^i_H\big({}_H\Bbbk, L\big(H \otimes H^{op}\big)\big)\\ &\cong \mathrm{Ext}^i_H({}_H\Bbbk, H) \otimes_H L\big(H \otimes H^{op}\big).\end{aligned}$$

Hence $\mathrm{Ext}^i_{H^e}(H, H^e) = 0$ for $i \neq d$, and

$$\mathrm{Ext}^d_{H^e}\big(H, H^e\big) \cong \Bbbk^\nu \otimes_H L\big(H \otimes H^{op}\big) \overset{(a)}{\cong} \Bbbk^\nu \otimes_H ({}_\bullet H \otimes H) \overset{(b)}{\cong} {}^1H^{S^2\nu},$$

where ν is the algebra automorphism of H corresponding to the left homological integrals $\int^l_H$. The isomorphism (a) is given by the isomorphism φ constructed through the map (9) above; and the isomorphism (b) holds because the right H^e-module structure on ${}_\bullet H \otimes H$ induced by the isomorphism φ is given as $(g \otimes h) \cdot (x \otimes y) = \sum_{(x)} gx_{(1)} \otimes yhS^2(x_{(2)})$ for $g, h, x, y \in H$. $\square$

Now we arrive at the main result of this section. Recall from [16] that an AS-Gorenstein Hopf algebra is *unimodular* if $\int^l_H \cong \Bbbk_H$ as right H-modules.

Theorem 2.3. *Let H be a Noetherian Hopf algebra. Then H is CY of dimension d if and only if*

(i) *H is AS-regular with global dimension* $\mathrm{gl\,dim}(H) = d$ *and unimodular,*
(ii) S^2 *is an inner automorphism of H.*

Proof. Suppose that H is CY of dimension d. By [12, Lemma 4.1] the triangulated category $D^b_{fd}(H)$ is a CY category of dimension d, where $D^b_{fd}(H)$ is the full triangulated subcategory of the derived category of H formed by complexes whose homology is of finite total dimension. Hence $\mathrm{gl\,dim}\, H = d$. It is well known that, for $i \geqslant 0$,

$$\mathrm{Ext}^i_H({}_H\Bbbk, H) \cong \mathrm{Ext}^i_{H^e}\big(H, \mathrm{Hom}_\Bbbk({}_H\Bbbk, H)\big) \cong \mathrm{Ext}^i_{H^e}(H, H \otimes \Bbbk_H),$$

where the left H^e-bimodule structure on $H \otimes \Bbbk_H$ is given by the left multiplication of H on the first factor and the right H-action on the trivial module $\Bbbk_H$. Since H is CY of dimension d, we may choose a finitely generated projective resolution of the H^e-module H as follows

$$P^\bullet : 0 \longrightarrow P^{-d} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow H \longrightarrow 0.$$

Then we have isomorphisms of complexes

$$\begin{aligned} \mathrm{Hom}_{H^e}\big(P^\bullet, H \otimes \Bbbk_H\big) &\cong \mathrm{Hom}_{H^e}\big(P^\bullet, H^e\big) \otimes_{H^e} (H \otimes \Bbbk_H) \\ &\cong \Bbbk \otimes_H \mathrm{Hom}_{H^e}\big(P^\bullet, H^e\big) \otimes_H H \\ &\cong \Bbbk \otimes_H \mathrm{Hom}_{H^e}\big(P^\bullet, H^e\big). \end{aligned} \tag{13}$$

Let $Q^\bullet := \mathrm{Hom}_{H^e}(P^\bullet, H^e)$. Since H is CY of dimension d, $Q^\bullet[d]$ is a projective resolution of the H^e-module H. Hence we have

$$\Bbbk \otimes_H \mathrm{Hom}_{H^e}\big(P^\bullet, H^e\big) = \Bbbk \otimes_H Q^\bullet \overset{\simeq}{\longrightarrow} \Bbbk_H[-d],$$

where the second map is a quasi-isomorphism of right H-modules. Note that the isomorphisms in (13) are also right H-module morphisms. By taking the cohomology of the complexes in (13), we obtain the statement (i).

According to Part (i) and Corollary 2.2, $\mathrm{Ext}^d_{H^e}(H, H^e) \cong {}^1H^{S^2}$. On the other hand, the CY property of H implies $\mathrm{Ext}^d_{H^e}(H, H^e) \cong H$ as H-H-bimodules. Hence H and ${}^1H^{S^2}$ are isomorphic as H-H-bimodules. Therefore S^2 must be an inner automorphism.

Conversely, since H is Noetherian and of finite global dimension, by [5, Lemma 5.2] H is homologically smooth. The assertion (i) and Corollary 2.2 insure $\mathrm{Ext}^i_{H^e}(H, H^e) = 0$ and $\mathrm{Ext}^d_{H^e}(H, H^e) \cong {}^1H^{S^2}$. The assertion (ii) tells us that ${}^1H^{S^2}$ is isomorphic to H as an H-H-bimodule. □

3. Group actions on universal enveloping algebras

In this section we consider the universal enveloping Hopf algebra of a Lie algebra and study the CY property of its smash product Hopf algebra. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Recall from [5, Prop. 6.3] that $U(\mathfrak{g})$ is an AS-regular Hopf algebra. Now let G be a finite group. We say that $\mathfrak{g}$ is a left *G-module Lie algebra* if there is a G-action on $\mathfrak{g}$ such that $\mathfrak{g}$ is a left G-module and $g[x, y] = [gx, gy]$ for all $g \in G$ and $x, y \in \mathfrak{g}$. If $\mathfrak{g}$ is a left G-module Lie algebra, then $U(\mathfrak{g})$ is an augmented left G-module algebra. For a Lie algebra $\mathfrak{g}$, we write $Aut_{Lie}(\mathfrak{g})$ to be the group of Lie algebra automorphisms. If $\mathfrak{g}$ is a left G-module Lie algebra, write the associated group morphism as $\nu : G \longrightarrow Aut_{Lie}(\mathfrak{g})$.

Assume $\dim \mathfrak{g} = d$. Consider the Chevalley–Eilenberg resolution of the trivial $U(\mathfrak{g})$-module (cf. [6, Ch. 8] or [15, Ch. 10]):

$$0 \longrightarrow U(\mathfrak{g}) \otimes \wedge^d \mathfrak{g} \xrightarrow{\partial^d} \cdots \xrightarrow{\partial^3} U(\mathfrak{g}) \otimes \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{\partial^2} U(\mathfrak{g}) \otimes \mathfrak{g} \xrightarrow{\partial^1} U(\mathfrak{g}) \longrightarrow {}_{U(\mathfrak{g})}\Bbbk \longrightarrow 0, \quad (14)$$

where for $x_1, \ldots, x_n \in \mathfrak{g}$,

$$\partial^n(1 \otimes x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n + \sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j} 1 \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n.$$

Since $\mathfrak{g}$ is a left G-module, $\wedge^n \mathfrak{g}$ is a left G-module with the diagonal action. Thus $U(\mathfrak{g}) \otimes \wedge^n \mathfrak{g}$ is a left $U(\mathfrak{g})\#\Bbbk G$-module. It is not hard to check that the differentials in the resolution above are also left G-module maps. Hence the resolution above is in fact a projective resolution of the left $U(\mathfrak{g})\#\Bbbk G$-module ${}_{U(\mathfrak{g})\#\Bbbk G}\Bbbk$.

Lemma 3.1. *Let G be a finite group, and $\mathfrak{g}$ be a G-module Lie algebra of dimension d. Then $U(\mathfrak{g})$ is AS-regular of global dimension d and as left G-modules $\int^l_{U(\mathfrak{g})} \cong \wedge^d \mathfrak{g}^*$, where left G-module action on $\mathfrak{g}^*$ is defined by $(g \cdot \beta)(x) = \beta(g^{-1}x)$ for $g \in G$, $\beta \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$, and G acts on $\wedge^d \mathfrak{g}^*$ diagonally.*

Proof. Since $\mathfrak{g}$ is of dimension d, the universal enveloping algebra $U(\mathfrak{g})$ has global dimension d (cf. [6, Ch. VIII]). The AS-regularity of $U(\mathfrak{g})$ is proved in [5, Prop. 6.3]. Applying the functor $\mathrm{Hom}_{U(\mathfrak{g})}(-, U(\mathfrak{g}))$ to the projective resolution (14) of ${}_{U(\mathfrak{g})}\Bbbk$ above, we obtain that $\int^l_{U(\mathfrak{g})}$ is the homology at the final position of the following complex of left G- and right $U(\mathfrak{g})$-modules (warning: they are not G-$U(\mathfrak{g})$-bimodules)

$$0 \longrightarrow \mathrm{Hom}_{U(\mathfrak{g})}\big(U(\mathfrak{g}), U(\mathfrak{g})\big) \xrightarrow{\partial^{1*}} \mathrm{Hom}_{U(\mathfrak{g})}\big(U(\mathfrak{g}) \otimes \mathfrak{g}, U(\mathfrak{g})\big) \xrightarrow{\partial^{2*}} \cdots \xrightarrow{\partial^{d*}} \mathrm{Hom}_{U(\mathfrak{g})}\big(U(\mathfrak{g}) \otimes \wedge^d \mathfrak{g}, U(\mathfrak{g})\big) \longrightarrow 0,$$

which is isomorphic to the following complex of left G- and right $U(\mathfrak{g})$-modules (also not bimodules)

$$0 \longrightarrow U(\mathfrak{g}) \xrightarrow{\delta^0} \mathfrak{g}^* \otimes U(\mathfrak{g}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{d-1}} \wedge^d \mathfrak{g}^* \otimes U(\mathfrak{g}) \longrightarrow 0. \quad (15)$$

Note that the left G-action on $\wedge^i \mathfrak{g}^* \otimes U(\mathfrak{g})$ is diagonal and $\wedge^i \mathfrak{g}^* \otimes U(\mathfrak{g})$ as a right $U(\mathfrak{g})$-module is free. The differential δ^{i-1} is induced by ∂^{i*} (for $i \geqslant 1$) through the obvious isomorphisms $\mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes \wedge^i \mathfrak{g}, U(\mathfrak{g})) \cong \wedge^i \mathfrak{g}^* \otimes U(\mathfrak{g})$.

Now let $\{x_1, \ldots, x_d\}$ be a basis of $\mathfrak{g}$ and $\{x_1^*, \ldots, x_d^*\}$ the dual basis of $\mathfrak{g}^*$. Note that the differentials in the complex (15) are also right $U(\mathfrak{g})$-module morphisms. The image of the element $\alpha = x_1^* \wedge \cdots \wedge x_d^* \otimes 1$ in the dth cohomology is nonzero. Otherwise it would imply that the dth cohomology is zero. For $\beta \in \wedge^d \mathfrak{g}^* \otimes U(\mathfrak{g})$, let $\overline{\beta}$ be the image of β in the dth cohomology. Now for $g \in G$, $g \cdot \overline{\alpha} = \overline{g \cdot \alpha} = g \cdot (x_1^* \wedge \cdots \wedge x_d^*) \otimes 1 = \omega \overline{\alpha}$, for some nonzero element $\omega \in k$. Thus we obtain an isomorphism of left G-modules: $\int_{U(\mathfrak{g})}^l \xrightarrow{\cong} \wedge^d \mathfrak{g}^*$ sending $\overline{\alpha}$ to $x_1^* \wedge \ldots \wedge x_d^*$. □

Let G be a group, $\mathfrak{g}$ a left G-module Lie algebra. It is well known that $U(\mathfrak{g})\#\Bbbk G$ is a cocommutative Hopf algebra with the coproduct and counit given by those of $U(\mathfrak{g})$ and of $\Bbbk G$.

Lemma 3.2. *Let G be a finite group. If $\mathfrak{g}$ is a finite dimensional G-module Lie algebra, then $U(\mathfrak{g})\#\Bbbk G$ is an AS-regular algebra.*

Proof. Since $\mathfrak{g}$ is finite dimensional, $U(\mathfrak{g})$ is an AS-regular algebra. Hence $U(\mathfrak{g})\#\Bbbk G$ has finite global dimension. Now the statement follows from Propositions 1.1 and 1.3. □

Lemma 3.3. *Let G be a finite group, and $\mathfrak{g}$ a finite dimensional left G-module Lie algebra. If $U(\mathfrak{g})\#\Bbbk G$ is CY of dimension d, then $U(\mathfrak{g})$ is CY of dimension d.*

Proof. Write B for $U(\mathfrak{g})\#\Bbbk G$. By Lemma 3.2, B is AS-regular of global dimension d. By Proposition 1.1, $\int_B^l \cong (\int_{U(\mathfrak{g})}^l \#\Bbbk G)^G$. Choose a basis $\alpha\#t$ of $\int_B^l$ with $\alpha \in \int_{U(\mathfrak{g})}^l$ and $t \in \Bbbk G$. Since B is a Noetherian cocommutative Hopf algebra, by Theorem 2.3 the right B-module action on $\int_B^l$ is trivial. It follows that $\alpha\#t = (\alpha\#t) \cdot (1\#g) = \alpha\#tg$ for all $g \in G$. This implies $tg = t$ for all $g \in G$ and hence t is an integral of $\Bbbk G$. We may now assume $t = \sum_{g \in G} g$. For $a \in U(\mathfrak{g})$, we have

$$\varepsilon(a)\alpha\#t = (\alpha\#t) \cdot (a\#1) = \sum_{(t)} \alpha \cdot (t_{(1)}a)\#t_{(2)} = \sum_{g \in G} \alpha \cdot (ga)\#g,$$

which forces $\alpha \cdot (ga) = \varepsilon(a)\alpha$ for all $g \in G$. Replacing a by $g^{-1}a$, we obtain $\alpha \cdot a = \varepsilon(g^{-1}a)\alpha = \varepsilon(a)\alpha$ for all $a \in U(\mathfrak{g})$. Therefore, the right $U(\mathfrak{g})$-action on the integral space $\int_{U(\mathfrak{g})}^l$ is trivial. Now the result follows from Theorem 2.3. □

Theorem 3.4. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $G \subseteq Aut_{Lie}(\mathfrak{g})$ a finite group. Then the skew group algebra $U(\mathfrak{g})\#\Bbbk G$ is a CY algebra of dimension d if and only if $U(\mathfrak{g})$ is a CY algebra of dimension d and $G \subseteq SL(\mathfrak{g})$.*

Proof. Suppose $G \subseteq SL(\mathfrak{g})$ and $U(\mathfrak{g})$ is CY. As before, write B for $U(\mathfrak{g})\#\Bbbk G$. Since B is a cocommutative Hopf algebra, by Theorem 2.3 we only need to show that $\int_B^l \cong \Bbbk_B$ as right B-modules, where $\Bbbk_B$ is the trivial right B-module. From Proposition 1.1, we have $\int_B^L \cong (\int_{U(\mathfrak{g})}^l \#\Bbbk G)^G$. By Lemma 3.1, $\int_{U(\mathfrak{g})}^l \cong \wedge^d \mathfrak{g}^*$ as left G-modules. Let $\{x_1^*, \ldots, x_d^*\}$ be a basis of $\mathfrak{g}^*$. We have $g \cdot (x_1^* \wedge \cdots \wedge x_d^*) = \det(g)^{-1} x_1^* \wedge \ldots \wedge x_d^* = x_1^* \wedge \cdots \wedge x_d^*$ for $g \in G \subseteq SL(\mathfrak{g})$. If α is a basis of $\int_{U(\mathfrak{g})}^l$, then $g \cdot \alpha = \alpha$ for all $g \in G$. Assume that $\alpha\#t$ is an element in $(\int_{U(\mathfrak{g})}^l \#\Bbbk G)^G$ for some $t \in \Bbbk G$. Then $\alpha\#t = g \cdot (\alpha\#t) = g \cdot \alpha\#gt = \alpha\#gt$ for all $g \in G$. So we have $gt = t$ for all $g \in G$. Hence t must be an integral of $\Bbbk G$. Now we may assume that $\alpha\#t$ is a basis of $\int_B^l$ with t a nonzero integral of $\Bbbk G$. By assumption, $U(\mathfrak{g})$ is a cocommutative CY Hopf algebra. It follows from Theorem 2.3 that the right $U(\mathfrak{g})$-module structure on $\int_{U(\mathfrak{g})}^l$ is trivial. Now for $a \in U(\mathfrak{g})$ and $g \in G$, $(\alpha \otimes t) \cdot (a\#g) = \sum_{(t)} \alpha \cdot (t_{(1)}a) \otimes t_{(2)}g = \sum_{(t)} \alpha \otimes \varepsilon(t_{(1)}a)t_{(2)}g = \alpha \otimes \varepsilon(a)tg = \varepsilon(a)\alpha \otimes t = \varepsilon_B(a\#g)\alpha \otimes t$. Thus the right B-module structure of $\int_B^l$ is trivial. Therefore B is a CY Hopf algebra of dimension d.

Conversely, assume B is a CY algebra. By Lemma 3.3, $U(\mathfrak{g})$ is CY. By the proof of Lemma 3.3, we may assume that $\alpha \otimes t$ is a basis of $\int_B^l$ with t an integral of $\Bbbk G$ and $\alpha \in \int_{U(\mathfrak{g})}^l$. Note that $\int_B^l \cong (\int_{U(\mathfrak{g})}^l \#\Bbbk G)^G$. Hence for $g \in G$, $\alpha \otimes t = g \cdot (\alpha \otimes t) = g \cdot \alpha \otimes gt = g \cdot \alpha \otimes t$. We get $g \cdot \alpha = \alpha$. This implies that the left G-action on $\int_{U(\mathfrak{g})}^l = \wedge^d \mathfrak{g}^*$ is trivial. Thus we have $\det(g) = 1$ for all $g \in G$, i.e., $G \subseteq SL(\mathfrak{g})$. □

Remark 3.5. If the Lie algebra $\mathfrak{g}$ is abelian, then $U(\mathfrak{g})$ is a polynomial algebra. In this case, the CY property of the skew group algebra $U(\mathfrak{g})\#\Bbbk G$ has been shown in [8, Example 24].

Now let G be an arbitrary finite group, $\mathfrak{g}$ a finite dimensional G-module Lie algebra. As before, we let $\nu : G \longrightarrow Aut_{Lie}(\mathfrak{g})$ be the associated group map. From the proof of the theorem above, we obtain

Corollary 3.6. *Let G and $\mathfrak{g}$ be as above. Then $U(\mathfrak{g})\#\Bbbk G$ is CY if and only if $U(\mathfrak{g})$ is CY and* $\operatorname{im}(\nu) \subseteq SL(\mathfrak{g})$.

4. Cocommutative CY Hopf algebras of low dimensions

Let A be an augmented algebra with a fixed augmentation map $\varepsilon : A \longrightarrow \Bbbk$. If A is a CY algebra of dimension d, then by [12, Lemma 4.1] the shift functor $[d]$ of the triangulated category $D^b_{fd}(A)$ is a (graded) Serre functor (see the appendix of [4]), where $D^b_{fd}(A)$ is the full triangulated subcategory of the derived category of A consisting of complexes with finite dimensional total cohomologies. Hence $\operatorname{Ext}^d_A({}_A\Bbbk, {}_A\Bbbk) \cong \operatorname{Ext}^0_A({}_A\Bbbk, {}_A\Bbbk) \cong \Bbbk$.

Now let $A = U(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ of dimension d. Tensoring with $\Bbbk_A$ the Chevalley–Eilenberg resolution of ${}_A\Bbbk$, we obtain the following complex:

$$0 \longrightarrow \wedge^d \mathfrak{g} \xrightarrow{\delta^d} \cdots \xrightarrow{\delta^3} \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{\delta^2} \mathfrak{g} \xrightarrow{\delta^1} \Bbbk \longrightarrow 0, \tag{16}$$

where the differential is given as, for $2 \leqslant n \leqslant d$, and $x_1, \ldots, x_n \in \mathfrak{g}$,

$$\delta^n(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leqslant i<j \leqslant n} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n,$$

and $\delta^1 = 0$. The nth homology of the complex above is $\operatorname{Tor}^A_n(\Bbbk_A, {}_A\Bbbk)$.

The following lemma can be deduced from [5, Proposition 6.3].

Lemma 4.1. *Let $\mathfrak{g}$ be a Lie algebra of dimension d. The following are equivalent.*

(i) *$U(\mathfrak{g})$ is a CY algebra.*
(ii) $\operatorname{Ext}^d_A({}_A\Bbbk_A\Bbbk) \neq 0$.
(iii) *The differential δ^d in the complex* (16) *is zero.*
(iv) $\operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x)) = 0$ *for all* $x \in \mathfrak{g}$.

Proof. (i) ⇒ (ii) ⇒ (iii) are obvious. (iv) ⇒ (i) follows from [5, Proposition 6.3]. We just need to show that (iii) ⇒ (iv). Assume $\delta^d = 0$. Let $\{x_1, \ldots, x_d\}$ be a basis of $\mathfrak{g}$. Then

$$\begin{aligned} 0 &= \delta^d(x_1 \wedge \cdots \wedge x_d) \\ &= \sum_{1 \leqslant i<j \leqslant d} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_d \\ &= \sum_{i=1}^{d} (-1)^i \operatorname{tr}\big(\operatorname{ad}_{\mathfrak{g}}(x_i)\big) x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_d. \end{aligned}$$

We have $\operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x_i)) = 0$ for all $1 \leqslant i \leqslant d$. □

Let H be a Hopf algebra. Denote by $P(H)$ the space of all primitive elements of H and denote by $G(H)$ the group of all group-like elements of H.

Theorem 4.2. *Let H be a cocommutative Hopf algebra such that $\dim P(H) < \infty$ and $G(H)$ is finite. Then H is CY of dimension 2 if and only if there is a finite group G and a group map $\nu : G \to SL(2, \Bbbk)$ such that $H \cong \Bbbk[x, y]\#\Bbbk G$, where the G-action on $\Bbbk[x, y]$ is given by ν.*

Proof. The sufficiency follows from Corollary 3.6. For the necessity, it is well known that a cocommutative Hopf algebra H over an algebraic closed field is isomorphic to $U(\mathfrak{g})\#\Bbbk G$, where $\mathfrak{g} = P(H)$ is the Lie algebra of all the primitive elements of H and $G = G(H)$ is the group of the group-like elements of H (cf. [14,25]). Since H is CY of dimension 2, by Lemma 3.3, $U(\mathfrak{g})$ is CY of dimension 2. Hence we get the global dimension of $U(\mathfrak{g})$ is 2, which implies $\dim(\mathfrak{g}) = 2$. By Lemma 4.1, $\mathfrak{g}$ must be abelian. Hence $U(\mathfrak{g}) \cong \Bbbk[x, y]$. The rest of the proof follows from Corollary 3.6. □

Remark 4.3. Let $K = \mathrm{im}(\nu) \subseteq SL(2, \Bbbk)$ and $N = \ker(\nu)$. Then N is a normal subgroup of G. There is a weak H-action on the group algebra $\Bbbk N$ (cf. [1]), and there is a map $\sigma : K \times K \longrightarrow \Bbbk N$ so that $\Bbbk G$ is isomorphic to the crossed product of $\Bbbk N$ and K, that is; $\Bbbk G \cong \Bbbk N\#_\sigma \Bbbk K$ (cf. [1] or [20, Ch. 4]). Since $K \subseteq SL(2, \Bbbk)$, K acts naturally on $\Bbbk[x, y]$. So K acts on $\Bbbk[x, y] \otimes \Bbbk N$ diagonally. The map $\sigma : K \times K \longrightarrow \Bbbk N$ may be extended to $K \times K \longrightarrow \Bbbk[x, y] \otimes \Bbbk N$ (the map is also denoted by σ). Then one can check a CY cocommutative pointed Hopf algebra as in Theorem 4.2 has the form $H \cong (\Bbbk[x, y] \otimes \Bbbk N)\#_\sigma \Bbbk K$.

Next we discuss 3-dimensional CY cocommutative Hopf algebras. We know that a cocommutative Hopf algebra is the skew-group algebra of a universal enveloping algebra of a Lie algebra and a group algebra. Let us now deal with 3-dimensional Lie algebras. By Lemma 4.1, we may list all 3-dimensional Lie algebras whose universal enveloping algebra is CY, since the 3-dimensional Lie algebras are classified completely. However, let us first get a view of the Lie bracket of such Lie algebras over an arbitrary basis. Let $\mathfrak{g}$ be a 3-dimensional vector space with a basis $\{x, y, z\}$. Define a bracket on $\mathfrak{g}$ as follows:

$$\begin{aligned} [x, y] &= ax + by + wz, \\ [x, z] &= cx + vy - bz, \\ [y, z] &= ux - cy + az, \end{aligned} \tag{17}$$

where $a, b, c, u, v, w \in \Bbbk$. A direct verification shows that $\mathfrak{g}$ is a Lie algebra.

Lemma 4.4. *With the bracket defined above, $\mathfrak{g}$ is a Lie algebra.*

Now we have the following easy but useful result.

Proposition 4.5. *Let $\mathfrak{g}$ be a 3-dimensional Lie algebra, and $\{x, y, z\}$ be a basis of $\mathfrak{g}$. Then $U(\mathfrak{g})$ is a CY algebra if and only if the Lie bracket is given by (17).*

Proof. This is an immediate consequence of Theorems 3.2 and 3.6 and Lemma 5.8 of [3]. □

Let $\mathfrak{g}$ be a 3-dimensional vector space. Fix a basis $\{x, y, z\}$ of $\mathfrak{g}$. Proposition 4.5 states that, given a sextuple $(a, b, c, u, v, w) \in \Bbbk^6$, there is a Lie bracket on $\mathfrak{g}$ defined by this sextuple via (17) so that the universal enveloping algebra of $\mathfrak{g}$ is a 3-dimensional CY algebra. Moreover, any 3-dimensional CY universal enveloping algebra is obtained in this way. We list below all 3-dimensional Lie algebras whose universal enveloping algebras are CY.

Proposition 4.6. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $U(\mathfrak{g})$ is CY of dimension 3 if and only if $\mathfrak{g}$ is isomorphic to one of the following Lie algebras*:

(i) *The 3-dimensional simple Lie algebra $\mathfrak{sl}(2, \Bbbk)$;*
(ii) *$\mathfrak{g}$ has a basis $\{x, y, z\}$ such that $[x, y] = y$, $[x, z] = -z$ and $[y, z] = 0$;*
(iii) *The Heisenberg algebra, that is; $\mathfrak{g}$ has a basis $\{x, y, z\}$ such that $[x, y] = z$ and $[x, z] = [y, z] = 0$;*
(iv) *The 3-dimensional abelian Lie algebra.*

Proof. Note that by Proposition 4.5 the universal enveloping algebras of the Lie algebras listed above are CY of dimension 3. We show that they are the only possible cases. We divide the 3-dimensional Lie algebras into 4 classes:

Case 1: $\dim[\mathfrak{g}, \mathfrak{g}] = 3$, that is, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$;
Case 2: $\dim[\mathfrak{g}, \mathfrak{g}] = 2$;
Case 3: $\dim[\mathfrak{g}, \mathfrak{g}] = 1$;
Case 4: $\dim[\mathfrak{g}, \mathfrak{g}] = 0$ or $\mathfrak{g}$ is abelian.

Case 1. If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then it is well known that $\mathfrak{g} \cong \mathfrak{sl}(2, \Bbbk)$. This gives us the Lie algebra (i).

Case 2. Assume that the Lie algebra $\mathfrak{g}$ has $\dim[\mathfrak{g}, \mathfrak{g}] = 2$. We choose a proper basis $\{x, y, z\}$ for $\mathfrak{g}$ so that $\mathfrak{g}$ satisfies (cf. [7]):

(a) $[x, y] = y$, $[x, z] = \mu z$ and $[y, z] = 0$, where $0 \neq \mu \in \mathbb{C}$; or
(b) $[x, y] = y$, $[x, z] = y + z$ and $[y, z] = 0$.

Since $\mathfrak{g}$ is CY, it follows from Proposition 4.5 that the Lie bracket of $\mathfrak{g}$ must satisfy the relations in (17). In the case (a), we must have $\mu = -1$. So $\mathfrak{g}$ is the Lie algebra given by (ii). Since the defining relations in the case (b) do not satisfy (17), the Lie bracket defined in (b) does not define a Lie algebra with CY universal enveloping algebra.

Case 3. Assume that the Lie algebra $\mathfrak{g}$ has $\dim[\mathfrak{g}, \mathfrak{g}] = 1$. Similar to Case 2, by choosing a proper basis $\{x, y, z\}$, we see that $\mathfrak{g}$ is determined by either of the following two cases:

(a) $[\mathfrak{g}, \mathfrak{g}]$ is contained in the center of $\mathfrak{g}$. In this case, $\mathfrak{g}$ is the Heisenberg algebra: $[x, y] = z$, and $[x, z] = [y, z] = 0$.
(b) $[\mathfrak{g}, \mathfrak{g}]$ is not contained in the center of $\mathfrak{g}$. In this case, we have: $[x, y] = y$ and $[x, z] = [y, z] = 0$.

Clearly, the Lie algebra defined by the case (b) does not satisfy the relations in (17), and hence its universal enveloping algebra can not be CY. Therefore, we have only the Heisenberg Lie algebra (iii).

Case 4. When a 3-dimensional Lie algebra is abelian, then its universal enveloping algebra is certainly CY. This yields the class (iv). □

Now, similar to Theorem 4.2, we may write down all possible 3-dimensional Noetherian CY cocommutative Hopf algebras with a finite number of group-like elements.

Theorem 4.7. *Let H be a cocommutative Hopf algebra such that $\dim P(H) < \infty$ and $G(H)$ is finite. Then H is CY of dimension 3 if and only if $H \cong U(\mathfrak{g})\#\Bbbk G$, where $\mathfrak{g}$ is one of the 3-dimensional Lie algebras listed in Proposition 4.6 and G is a finite group with a group morphism $\nu : G \to Aut_{Lie}(\mathfrak{g})$ such that $\mathrm{im}(\nu)$ is also a subgroup of $SL(\mathfrak{g})$.*

Proof. The proof is similar to that of Theorem 4.2. □

Remark 4.8. The cocommutative Hopf algebra discussed in this section possesses finite number of group-like elements. If the group of group-like elements is infinite, then the situation becomes very complicated. For an infinite group, it is hard to determine when the group algebra is CY, even in the low dimensional cases. However, there are some examples of CY group algebras of low dimensions (see [16]). If G is a finitely generated group such that $\Bbbk G$ is Noetherian and is of GK-dimension 1, then $G \cong \mathbb{Z}$ (cf. [16, Prop. 8.2]). In this case, $\Bbbk G$ is CY of dimension 1. Example 8.5 of [16] provides us an example of Noetherian affine CY group algebra of dimension 2.

5. Sridharan enveloping algebras

In this section, we discuss the CY property of a Sridharan enveloping algebra of a finite dimensional Lie algebra. In general, a Sridharan enveloping algebra is no longer a Hopf algebra, but a cocycle deformation of a cocommutative Hopf algebra or a Poincaré–Birkhoff–Witt (PBW) deformation of a polynomial algebra (cf. [24,18,19]). We will see that the CY property of a Sridharan enveloping algebra is closely related to the CY property of a universal enveloping algebra. The class of Sridharan algebras contains many interesting algebras, such as Weyl algebras. Many homological properties of Sridharan enveloping algebras have been discussed in [24,11,18,19], especially the Hochscheld (co)homology and the cyclic homology. In [18], Nuss listed all nonisomorphic Sridharan enveloping algebras of 3-dimensional Lie algebras. Based on these results, we obtain in this section some equivalent conditions for a Sridharan enveloping algebra of a finite dimensional Lie algebra to be CY. We then list all possible nonisomorphic 3-dimensional CY Sridharan enveloping algebras, and partly answer a question proposed by Berger at the end of [2].

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and let $f \in Z^2(\mathfrak{g}, \Bbbk)$ be an arbitrary 2-cocycle, that is; $f : \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ such that

$$f(x,x) = 0 \quad \text{and} \quad f\big(x,[y,z]\big) + f\big(y,[z,x]\big) + f\big(z,[x,y]\big) = 0$$

for all $x, y, z \in \mathfrak{g}$. The Sridharan enveloping algebra of $\mathfrak{g}$ is defined to be the associative algebra $U_f(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the two-side ideal of $T(\mathfrak{g})$ generated by the elements

$$x \otimes y - y \otimes x - [x,y] - f(x,y), \quad \text{for all } x, y \in \mathfrak{g}.$$

For $x \in \mathfrak{g}$, we still denote by x its image in $U_f(\mathfrak{g})$. Clearly, $U_f(\mathfrak{g})$ is a filtered algebra with the associated graded algebra $gr(U_f(\mathfrak{g}))$ being a polynomial algebra. By [24, Cor. 3.3], there is one to one correspondence between the group of the algebra automorphisms $\theta : U_f(\mathfrak{g}) \to U_f(\mathfrak{g})$ such that the associated graded map $gr(\theta)$ is the identity, and the group $Z^1(\mathfrak{g}, \Bbbk)$ of the first cocycles. Thus given a 1-cocycle $h \in Z^1(\mathfrak{g}, \Bbbk)$, there is an algebra automorphism $\xi_f : U_f(\mathfrak{g}) \longrightarrow U_f(\mathfrak{g})$, for any 2-cocycle $f \in Z^2(\mathfrak{g}, \Bbbk)$, defined by

$$\xi_f(x) = x + h(x) \tag{18}$$

for all $x \in \mathfrak{g}$. When $f = 0$, the map (18) defines an algebra automorphism $\xi : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$. Clearly ξ_f has an inverse given by $\xi_f^{-1}(x) = x - h(x)$ for all $x \in \mathfrak{g}$.

In the sequel, we fix a 2-cocycle $f \in Z^2(\mathfrak{g}, \Bbbk)$ and let $A = U_f(\mathfrak{g})$, and $A^e = A \otimes A^{op}$. Define a linear map $D : \mathfrak{g} \longrightarrow A^e$ by $D(x) = x \otimes 1 - 1 \otimes x$ for all $x \in \mathfrak{g}$. By [24], D induces an algebra morphism $U(\mathfrak{g}) \longrightarrow A^e$, still denoted by D. Thus A^e can be viewed both as a left and as a right $U(\mathfrak{g})$-module. Now let $\xi : U(\mathfrak{g}) \to U(\mathfrak{g})$ and $\xi_f : A \to A$ be defined by (18). Since ${}^{\xi_f}A^1 \otimes A^{op}$ is a left A^e-module, it is also a left $U(\mathfrak{g})$-module. We have the following isomorphisms.

Lemma 5.1. *As left $U(\mathfrak{g})$- and right A^e-bimodules,*

$$ {}^{\xi}\big(A \otimes A^{op}\big) \cong {}^{1}A^{\xi_f^{-1}} \otimes A^{op},$$

$$ {}^{\xi^{-1}}\big(A \otimes A^{op}\big) \cong {}^{1}A^{\xi_f} \otimes A^{op}.$$

Proof. It is easy to check that the following diagram of algebra morphisms is commutative.

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{D} & A \otimes A^{op} \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi_f \otimes id} \\ U(\mathfrak{g}) & \xrightarrow{D} & A \otimes A^{op}. \end{array}$$

It follows that ${}^{\xi}(A \otimes A^{op}) \cong {}^{\xi_f}A^1 \otimes A^{op}$ as left $U(\mathfrak{g})$- and right A^e-bimodules. On the other hand, we have the $U(\mathfrak{g})$-A^e-bimodule isomorphism $\xi_f^{-1} \otimes id : {}^{\xi_f}A^1 \otimes A^{op} \longrightarrow {}^1A^{\xi_f^{-1}} \otimes A^{op}$. The composite of the aforementioned two isomorphisms gives us the desired isomorphism.

The second isomorphism in the lemma can be proved similarly. □

As $U(\mathfrak{g})$ is a Hopf algebra, the space $\Bbbk$ is a trivial $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule. Thus $\Bbbk^{\xi}$ is a right $U(\mathfrak{g})$-module twisted by the automorphism ξ.

Lemma 5.2. *As right A^e-modules, $\Bbbk^{\xi} \otimes_{U(\mathfrak{g})} A^e \cong {}^1A^{\xi_f}$.*

Proof. Consider the exact sequence of right $U(\mathfrak{g})$-modules:

$$0 \longrightarrow I \longrightarrow U(\mathfrak{g}) \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0.$$

Applying the functor $- \otimes_{U(\mathfrak{g})} {}^1U(\mathfrak{g})^{\xi}$, we obtain the following exact sequence of right $U(\mathfrak{g})$-modules:

$$0 \longrightarrow I^{\xi} \longrightarrow U(\mathfrak{g})^{\xi} \longrightarrow \Bbbk^{\xi} \longrightarrow 0.$$

By [24, Prop. 5.2], A^e is a free $U(\mathfrak{g})$-module on both sides. Tensoring the above exact sequence with A^e, we obtain the following exact sequence of right A^e-modules:

$$0 \longrightarrow I^{\xi} \otimes_{U(\mathfrak{g})} A^e \longrightarrow U(\mathfrak{g})^{\xi} \otimes_{U(\mathfrak{g})} A^e \longrightarrow \Bbbk^{\xi} \otimes_{U(\mathfrak{g})} A^e \longrightarrow 0,$$

which is isomorphic to the following sequence of right A^e-modules:

$$0 \longrightarrow I \otimes_{U(\mathfrak{g})} {}^{\xi^{-1}}\big(A^e\big) \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} {}^{\xi^{-1}}\big(A^e\big) \longrightarrow \Bbbk \otimes_{U(\mathfrak{g})} {}^{\xi^{-1}}\big(A^e\big) \longrightarrow 0.$$

By Lemma 5.1, the sequence above is isomorphic to the following exact sequence of right A^e-modules:

$$0 \longrightarrow I \otimes_{U(\mathfrak{g})} \big({}^1A^{\xi_f} \otimes A^{op}\big) \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \big({}^1A^{\xi_f} \otimes A^{op}\big) \longrightarrow \Bbbk \otimes_{U(\mathfrak{g})} \big({}^1A^{\xi_f} \otimes A^{op}\big) \longrightarrow 0.$$

Hence we obtain the following right A^e-module isomorphisms.

$$\Bbbk^{\xi} \otimes_{U(\mathfrak{g})} A^e \cong \Bbbk \otimes_{U(\mathfrak{g})} \big({}^1A^{\xi_f} \otimes A^{op}\big) \cong \frac{{}^1A^{\xi_f} \otimes A^{op}}{D(I)({}^1A^{\xi_f} \otimes A^{op})}.$$

On the other hand, by a right version of the proof of [24, Prop. 5.3] we have the following exact sequence of right A^e-modules:

$$0 \longrightarrow I \otimes_{U(\mathfrak{g})} \big(A \otimes A^{op}\big) \longrightarrow A \otimes A^{op} \longrightarrow A \longrightarrow 0.$$

Tensoring it with ${}^1A^{\xi_f}$ over A, we obtain an exact sequence of right A^e-modules:

$$0 \longrightarrow I \otimes_{U(\mathfrak{g})} \left(A \otimes A^{op}\right) \otimes_A {}^1A^{\xi_f} \longrightarrow \left(A \otimes A^{op}\right) \otimes_A {}^1A^{\xi_f} \longrightarrow A \otimes_A {}^1A^{\xi_f} \longrightarrow 0,$$

which is isomorphic to

$$0 \longrightarrow I \otimes_{U(\mathfrak{g})} \left({}^1A^{\xi_f} \otimes A^{op}\right) \longrightarrow {}^1A^{\xi_f} \otimes A^{op} \longrightarrow {}^1A^{\xi_f} \longrightarrow 0.$$

Therefore as right A^e-modules $\frac{{}^1A^{\xi_f} \otimes A^{op}}{D(I)({}^1A^{\xi_f} \otimes A^{op})} \cong {}^1A^{\xi_f}$. The proof is then complete. □

Theorem 5.3. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then for any 2-cocycle $f \in Z^2(\mathfrak{g}, \Bbbk)$, the following statements are equivalent.*

(i) *The Sridharan enveloping algebra $U_f(\mathfrak{g})$ is CY of dimension d.*
(ii) *The universal enveloping algebra $U(\mathfrak{g})$ is CY of dimension d.*
(iii) *$\dim \mathfrak{g} = d$ and $\mathfrak{g}$ is unimodular [13], that is, for any $x \in \mathfrak{g}$, $\operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x)) = 0$.*

Proof. Following Lemma 4.1 it is sufficient to show that (i) and (ii) are equivalent. We show first (ii) ⇒ (i). Assume that $U(\mathfrak{g})$ is CY of dimension d. Then $\dim(\mathfrak{g}) = d$. Note that $U(\mathfrak{g})$ is a cocommutative Hopf algebra. By Theorem 2.3, $\mathrm{RHom}_{U(\mathfrak{g})}(\Bbbk, U(\mathfrak{g})) \cong \Bbbk[d]$ as objects in the derived category of complexes of right $U(\mathfrak{g})$-modules, where $\Bbbk$ is the trivial right $U(\mathfrak{g})$-module. Once again we write A for $U_f(\mathfrak{g})$. Recall that A^e is a free $U(\mathfrak{g})$-module. Now let $P^\bullet$ be the Chevalley–Eilenberg resolution of the trivial left $U(\mathfrak{g})$-module $\Bbbk$. Then $A^e \otimes_{U(\mathfrak{g})} P^\bullet$ is the standard resolution of A as a left A^e-module (also see [11, Prop. 3]). It follows that we have the following isomorphisms in the derived category $D^\circ(A^e)$ of complexes of right A^e-modules:

$$\begin{aligned}
\mathrm{RHom}_{A^e}(A, A^e) &\cong \mathrm{Hom}_{A^e}\left(A^e \otimes_{U(\mathfrak{g})} P^\bullet, A^e\right) \\
&\cong \mathrm{Hom}_{U(\mathfrak{g})}\left(P^\bullet, A^e\right) \\
&\cong \mathrm{Hom}_{U(\mathfrak{g})}\left(P^\bullet, U(\mathfrak{g})\right) \otimes_{U(\mathfrak{g})} A^e \\
&\cong \mathrm{RHom}_{U(\mathfrak{g})}\left(\Bbbk, U(\mathfrak{g})\right) \otimes_{U(\mathfrak{g})} A^e \\
&\cong \Bbbk[-d] \otimes_{U(\mathfrak{g})} A^e \\
&\overset{(a)}{\cong} A[-d],
\end{aligned}$$

where the isomorphism (a) follows from the right version of the proof of [24, Prop. 5.3]. Therefore $U_f(\mathfrak{g}) = A$ is a CY algebra of dimension d.

(i) ⇒ (ii). Assume that $A = U_f(\mathfrak{g})$ is CY of dimension d. The first four isomorphisms above are still valid, and thus we have

$$\mathrm{RHom}_{A^e}\left(A, A^e\right) \cong \mathrm{RHom}_{U(\mathfrak{g})}\left(\Bbbk, U(\mathfrak{g})\right) \otimes_{U(\mathfrak{g})} A^e \tag{19}$$

as right A^e-modules. Since A is CY of dimension d and A^e is a free left $U(\mathfrak{g})$-module, $H^i\,\mathrm{RHom}_{U(\mathfrak{g})}(\Bbbk, U(\mathfrak{g})) = 0$ for $i \neq d$ and $H^d\,\mathrm{RHom}_{U(\mathfrak{g})}(\Bbbk, U(\mathfrak{g})) \neq 0$. Hence $\mathrm{RHom}_{U(\mathfrak{g})}(\Bbbk, U(\mathfrak{g})) \cong \Bbbk^\xi[-d]$, where ξ is, by [5, Prop. 6.3], the algebra automorphism $\xi : U(\mathfrak{g}) \to U(\mathfrak{g})$ defined by $\xi(x) = x + \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x))$ for all $x \in \mathfrak{g}$. Let $h = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(-)) : \mathfrak{g} \to \Bbbk$. Then $h \in Z^1(\mathfrak{g}, \Bbbk)$. Now the 1-cocycle h also defines an algebra automorphism $\xi_f : A \to A$. Combining the isomorphisms in (19) and the isomorphism in Lemma 5.2, we obtain the following isomorphisms:

$$A[-d] \cong \Bbbk^\xi[-d] \otimes_{U(\mathfrak{g})} A^e \cong {}^1A^{\xi_f}.$$

Thus we have an A-A-bimodule isomorphism: $A \cong {}^1A^{\xi_f}$. It follows that the automorphism $\xi_f : A \to A$ must be inner. That is, $\xi_f(a) = u^{-1}au$ for some unit $u \in A$. It is easy to see that $u \in \Bbbk$. Therefore $\xi_f = id$ and $h = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(-)) = 0$. By Lemma 4.1, $U(\mathfrak{g})$ is CY. Moreover $U(\mathfrak{g})$ is of dimension d. □

The proof of Theorem 5.3 yields a more general fact about rigid dualizing complexes (for the definition, see [26]) of Sridharan enveloping algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $f \in Z^2(\mathfrak{g}, \Bbbk)$ a 2-cocycle. Let $A = U_f(\mathfrak{g})$ as before. By [26, Cor. 8.7] or [27, Prop. 1.1], the rigid dualizing complex R of A exists. Moreover, R is invertible and $R^{-1} = \operatorname{RHom}_{A^e}(A, A^e)$. Notice that the linear map $h = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(-)) : \mathfrak{g} \to \Bbbk$ is a 1-cocycle of $\mathfrak{g}$. As early pointed out at the beginning of this section, h defines both an isomorphism ξ on $U(\mathfrak{g})$ and an isomorphism ξ_f on $U_f(\mathfrak{g})$. Now we have the following corollary which generalizes [27, Theorem A] to Sridharan enveloping algebras.

Corollary 5.4. *Let $\mathfrak{g}$ be a Lie algebra of dimension d, and $f \in Z^2(\mathfrak{g}, \Bbbk)$ a 2-cocycle. Then the rigid dualizing complex of the Sridharan enveloping algebra $U_f(\mathfrak{g})$ is ${}^1U_f(\mathfrak{g})^{\zeta_f}[d]$, where $\zeta_f : U_f(\mathfrak{g}) \to U_f(\mathfrak{g})$ is an algebra automorphism, and is defined by*

$$\zeta_f(x) = x - \operatorname{tr}\big(\operatorname{ad}_{\mathfrak{g}}(x)\big), \quad \text{for all } x \in \mathfrak{g}.$$

Proof. Let $A = U_f(\mathfrak{g})$. By [5, Prop. 6.3], $\operatorname{RHom}_{U(\mathfrak{g})}(\Bbbk, U(\mathfrak{g})) \cong \Bbbk^{\xi}[-d]$. Following the isomorphisms in (19), we have

$$\operatorname{RHom}_{A^e}\big(A, A^e\big) \cong \Bbbk^{\xi}[-d] \otimes_{U(\mathfrak{g})} A^e.$$

By Lemma 5.2, $\operatorname{RHom}_{A^e}(A, A^e) \cong {}^1A^{\xi_f}[-d]$. Therefore the rigid dualizing complex of A is $R = {}^1A^{\xi_f^{-1}}[d]$. Write ζ_f for ξ_f^{-1}, We obtain the desired result. □

Now we focus on 3-dimensional CY Sridharan enveloping algebras. By Theorem 5.3, such an algebra must be constructed from a 3-dimensional Lie algebra. Combining Proposition 4.6, Theorem 5.3 and [18, Theorem 1.3], we may list all possible nonisomorphic 3-dimensional CY Sridharan enveloping algebras.

Theorem 5.5. *Let $U_f(\mathfrak{g})$ be a Sridharan enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$. Then $U_f(\mathfrak{g})$ is CY of dimension 3 if and only if $U_f(\mathfrak{g})$ is isomorphic to $\Bbbk\langle x, y, z\rangle/(R)$ with the commuting relations R listed in the following table*:

Case	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$
1	z	$-2x$	$2y$
2	y	$-z$	0
3	z	0	0
4	0	0	0
5	y	$-z$	1
6	z	1	0
7	1	0	0

where $\{x, y\} = xy - yx$.

Note that in the above table the cases 1–4 give the CY universal enveloping algebras listed in Proposition 4.6.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $f \in Z^2(\mathfrak{g}, \Bbbk)$ a 2-cocycle. The Sridharan enveloping algebra $U_f(\mathfrak{g})$ is a PBW-deformation of the polynomial algebra $\Bbbk[x_1, \dots, x_n]$ where $n = \dim \mathfrak{g}$ (cf. [24,3,21]). Conversely, a PBW-deformation of a polynomial algebra is exactly a Sridharan enveloping algebra (cf. [24,19]). It is shown in [3, Theorem 3.6] that if a PBW-deformation of a 3-dimensional graded CY algebra is defined by a potential, then the deformed algebra is also CY of dimension 3. Whether the converse is true or not is not shown in [3]. However, a CY Sridharan enveloping algebra $U_f(\mathfrak{g})$ of a 3-dimensional Lie algebra is always defined by a potential. In fact, all

3-dimensional Lie algebras whose Sridharan enveloping algebras are CY are listed in Theorem 5.5. It is easy to check that the defining relations of these algebras satisfy the condition in [3, Theorem 3.2]. Hence any 3-dimensional CY Sridharan enveloping algebra A is defined by a potential. That is; $A \cong \Bbbk\langle x, y, z\rangle/(\frac{\partial\Phi}{\partial_x}, \frac{\partial\Phi}{\partial_y}, \frac{\partial\Phi}{\partial_z})$, where $\Phi \in \Bbbk\langle x, y, z\rangle/[\Bbbk\langle x, y, z\rangle, \Bbbk\langle x, y, z\rangle]$ is a potential.

In fact, we can write down the potentials corresponding to the Sridharan enveloping algebras of the Lie algebras in Theorem 5.5 respectively:

(1) $\Phi = xyz - yxz - \frac{1}{2}z^2 - 2xy$;
(2) $\Phi = xyz - yxz - yz$;
(3) $\Phi = xyz - yxz - \frac{1}{2}z^2$;
(4) $\Phi = xyz - yxz$;
(5) $\Phi = xyz - yxz - yz - x$;
(6) $\Phi = xyz - yxz - \frac{1}{2}z^2 - y$;
(7) $\Phi = xyz - yxz - z$.

Note that the potential in the case (1) of the list is proportional to the Casimir element of the Lie algebra $\mathfrak{sl}(2, \Bbbk)$ (we thank the referee point out it to us). So, a PBW-deformation A of the polynomial algebra $\Bbbk[x, y, z]$ is CY if and only if A is defined by a potential. This phenomenon does not occur accidentally. Travis Schedler shows in [23] that any 3-dimensional CY PBW-deformation of a 3-dimensional graded CY algebra (associated to a finite quiver) must be defined by a potential. Combining with the results of [3], we then obtain that a PBW-deformation A of a 3-dimensional graded CY algebra (associated to a finite quiver) is CY of dimension 3 if and only if A is defined by a potential.

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