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Gevrey properties of the asymptotic critical wave speed in a family of scalar reaction–diffusion equations

Peter De Maesschalck [a], Nikola Popović [b,*]

[a] Universiteit Hasselt, Campus Diepenbeek, Agoralaan Gebouw D, B-3590 Diepenbeek, Belgium

[b] University of Edinburgh, School of Mathematics and Maxwell Institute for Mathematical Sciences, James Clerk Maxwell Building, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom

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ABSTRACT

We consider front propagation in a family of scalar reaction–diffusion equations in the asymptotic limit where the polynomial degree of the potential function tends to infinity. We investigate the Gevrey properties of the corresponding critical propagation speed, proving that the formal series expansion for that speed is Gevrey-1 with respect to the inverse of the degree. Moreover, we discuss the question of optimal truncation. Finally, we present a reliable numerical algorithm for evaluating the coefficients in the expansion with arbitrary precision and to any desired order, and we illustrate that algorithm by calculating explicitly the first ten coefficients. Our analysis builds on results obtained previously in [F. Dumortier, N. Popović, T.J. Kaper, The asymptotic critical wave speed in a family of scalar reaction–diffusion equations, J. Math. Anal. Appl. 326 (2) (2007) 1007–1023], and makes use of the blow-up technique in combination with geometric singular perturbation theory and complex analysis, while the numerical evaluation of the coefficients in the expansion for the critical speed is based on rigorous interval arithmetic.

1. Introduction

The general family of scalar reaction–diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f_m(u), \tag{1}$$

with $f_m(u) = 2u^m(1-u)$ and $m \geqslant 1$ real, has been studied extensively as a 'bridge' [26] between the classical Fisher–Kolmogorov–Petrowskii–Piscounov (FKPP) equation [10,15], which is obtained for $m = 1$ in (1), and the family of non-degenerate bistable cubic equations with potential $f(u) = u(1-u)(u-a)$ [3,14,21], where $a \in (0, \frac{1}{2})$ is a real parameter. Moreover, it has found numerous applications in the biological [21] and physical [3] sciences, especially when $m = 1$ or $m = 2$. (In the latter case, Eq. (1) is also known as the Zeldovich equation.)

Of particular interest to us here are traveling front solutions that connect the two rest states at $u = 0$ and $u = 1$ in (1). Reverting to a co-moving frame by introducing the traveling wave variable $\xi = x - ct$, where c is the front propagation speed, we denote the corresponding front by $U(\xi) = u(x,t)$; thus, we find

$$U'' + cU' + 2U^m(1-U) = 0 \tag{2}$$

* Corresponding author.
E-mail address: Nikola.Popovic@ed.ac.uk (N. Popović).

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for the traveling wave equation corresponding to (1), with

$$\lim_{\xi\to\infty} U(\xi)=0 \quad \text{and} \quad \lim_{\xi\to-\infty} U(\xi)=1. \tag{3}$$

(Here, the prime denotes differentiation with respect to ξ.) As is well known [2,3], for each $m \geqslant 1$, there exists a so-called 'critical' wave speed $c_{\mathrm{crit}}(m)$ such that Eq. (2) supports front solutions which satisfy the condition in (3) when $c \geqslant c_{\mathrm{crit}}$. The speed c_{crit} is critical in the sense that it separates fronts of different decay rates at the zero rest state: for $m=1$, the front solution corresponding to $c_{\mathrm{crit}} = 2\sqrt{2}$ decays at an algebro-exponential rate as $\xi \to \infty$, whereas the decay is strictly exponential for $c > c_{\mathrm{crit}}$. By contrast, when $m > 1$, the front that propagates with speed c_{crit} decays exponentially, while the decay is merely algebraic in ξ for $c > c_{\mathrm{crit}}$.

The family of equations in (1) has been studied in detail in the regimes where m is close to 1 and 2, using geometric singular perturbation theory [24] and matched asymptotics [20,27]. Finally, the large-m limit in (1), first introduced as a model for a δ-distribution potential centered about $u=1$, which was considered in [23,27] as well as in [22] via the method of matched asymptotic expansions, was analyzed in full rigor in [7]. In particular, it was proven there that the critical wave speed $c_{\mathrm{crit}}(m)$ for (2) is $\mathcal{C}^\infty$-smooth in m^{-1}, as well as that

$$c_{\mathrm{crit}}(m) = \frac{c_1}{m} + \frac{c_2}{m^2} + \mathcal{O}\big(m^{-3}\big) \quad \text{as } m \to \infty, \tag{4}$$

where $c_1 = 2$ and c_2 is defined as

$$c_2 = \lim_{w_0\to\infty} \int_0^{w_0} \left[\frac{\omega^2 \mathrm{e}^{-\omega}}{\sqrt{1-(1+\omega)\mathrm{e}^{-\omega}}} - \frac{\omega^3}{2}\mathrm{e}^{-\omega}\right] d\omega \approx -0.31191;$$

see [7, Theorem 1.1]. At the same time, the approach developed in [7] – which relied on a combination of geometric singular perturbation theory [9,13] and the blow-up technique (geometric desingularization) [6,16] – yielded an alternative (constructive) proof for the existence and uniqueness of the corresponding traveling front solutions; in particular, it allowed for the regularization of the neutrally stable zero rest state in the singular limit as $m \to \infty$ in (1).

The motivation in [7] for studying (2) in the large-m limit was twofold: first, it was confirmed that $c_{\mathrm{crit}}(m)$ is monotonically decreasing in m, as predicted on formal and numerical grounds in [22,27]. Second, and perhaps more importantly, the approximation for $c_{\mathrm{crit}}(m)$ provided by (4) was shown to agree well with the numerically obtained front speed over a wide range of m-values, down to $m=2$; cf. [27, Fig. 3(a)]. (To state it differently, the large-m asymptotics of c_{crit} seems to remain accurate even for finite values of m.)

In this article, we investigate the structure of the series expansion for the critical wave speed $c_{\mathrm{crit}}(m)$ in (4) in more detail. As indicated already in [7, Remark 10], that expansion can be expected to have Gevrey properties [1,4]. Here, we confirm this expectation; more precisely, we prove that the asymptotics in (4) is, in fact, Gevrey-1 with respect to the (small) parameter m^{-1}, *i.e.*, that the k-th order coefficient c_k in the expansion for c_{crit} will grow at most like $k!$; a precise definition can be found in Eq. (6) below. Moreover, we determine the optimal truncation point in that expansion, and we obtain a bound on the error incurred by the resulting truncation; to the best of our knowledge, no comparable results have been obtained before. Our study is based on the geometric framework that was established in [7], complemented by techniques from complex analysis and Gevrey asymptotics. (We remark that the blow-up technique has been applied previously in the derivation of Gevrey-type expansions; see, *e.g.*, [18,19] for details.) Specifically, our main result can be expressed as follows:

Theorem 1. *For $m \geqslant m_0$, with $m_0 > 0$ sufficiently large, the function $c_{\mathrm{crit}}(m)$ has a formal power series expansion of the form*

$$c_{\mathrm{crit}}(m) \sim \sum_{k=1}^{\infty} \frac{c_k}{m^k}, \tag{5}$$

with $c_1 = 2$. Moreover, the expansion in (5) *is Gevrey-*1 *with respect to m^{-1}, i.e., there exists a constant $A > 0$ such that, for $k = 1, 2, 3, \ldots,$*

$$|c_k| \leqslant A B^k k!, \quad \text{with } B \gtrsim (\ln 2)^{-1} \approx 1.44270. \tag{6}$$

*Finally, the function $c_{\mathrm{crit}}(m)$ is well approximated by this Gevrey-*1 *series, in the sense that*

$$\left| c_{\mathrm{crit}}(m) - \sum_{k=1}^{[\frac{m}{B}]} \frac{c_k}{m^k} \right| \leqslant A\sqrt{2\pi}\left(\frac{m}{B}\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{m}{B}}\big(1+\mathcal{O}\big(Bm^{-1}\big)\big). \tag{7}$$

(Here, $[\frac{m}{B}]$ denotes the integer that is nearest to $\frac{m}{B}$.)

Our second result in this article concerns the accurate numerical evaluation of the coefficients c_k in the formal series expansion for c_{crit} in (5). In other words, we explicitly extend the leading-order expansion in (4), as found in [7,23]: while

Table 1
The coefficients c_k in (5) for $k = 1, \dots, 10$.

c_1	2.0	c_6	−0.2691796252
c_2	−0.3119086360	c_7	0.3478753430
c_3	0.6762845522	c_8	−0.2473705415
c_4	−0.2941414626	c_9	0.2959031591
c_5	0.4425500680	c_{10}	−0.2309020840

only the coefficient c_2 was calculated there, we present an algorithm to evaluate c_k numerically with arbitrary precision and for any $k \geqslant 1$, thus obtaining a uniform approximation for $c_{\text{crit}}(m)$ to any desired degree of accuracy. Thus, for $k = 1, \dots, 10$, the coefficients c_k in (5) are as given in Table 1, up to 10 digits' precision. The evaluation of these coefficients is computer-assisted, and will be outlined in Section 4; in particular, we remark that it makes substantial use of the Gevrey character of the expansion for c_{crit} and, specifically, of the asymptotic bound on the growth of c_k given in (6).

This article is organized as follows. In Section 2, we review the geometric framework established in [7]; in Section 3, we prove our main result, Theorem 1; in Section 4, we outline the derivation of Table 1; finally, in Section 5, we discuss and interpret our findings.

2. Geometric framework

In this section, we retrace some of the analysis from [7], as required for our purposes. First, we note that it is useful to recast (2) into Liénard form, which yields

$$U' = V - cU, \qquad V' = -2U^m(1 - U); \tag{8}$$

then, front solutions connecting the rest states at $U = 1$ and $U = 0$ in (2) correspond to heteroclinic connections between the two equilibrium points $Q^- = (1, c)$ and $Q^+ = (0, 0)$ of (8). In particular, the point Q^+ is a saddle-node for $c > 0$ (with eigenvalues $-c$ and 0) and fully degenerate (with a double zero eigenvalue) when $c = 0$; cf. [7, Lemma 2.1]. Hence, for any $m > 1$, the critical speed $c_{\text{crit}}(m) > 0$ is determined by the condition that the unstable manifold $\mathcal{W}^u(Q^-)$ of the hyperbolic saddle equilibrium at Q^- coincides with the strong stable manifold $\mathcal{W}^s(Q^+)$ of Q^+; by contrast, for $c > c_{\text{crit}}(m)$, $\mathcal{W}^u(Q^-)$ approaches Q^+ on a center manifold. We refer the reader to [7,24] for a more complete discussion of critical wave speed phenomena from a geometric point of view.

Following [7], we define $\varepsilon = m^{-1}$ for m large, and we consider the limit as $\varepsilon \to 0$. After a preliminary rescaling via $V = \varepsilon\widetilde{V}$, $c = \varepsilon\tilde{c}$, and $\xi = \frac{\tilde{\xi}}{\varepsilon}$, see [7, Section 2.1], the equations in (8) read

$$\dot{U} = \widetilde{V} - \tilde{c}U, \qquad \dot{\widetilde{V}} = -\frac{2}{\varepsilon^2}U^{\frac{1}{\varepsilon}}(1 - U), \tag{9}$$

where the overdot denotes differentiation with respect to $\tilde{\xi}$.

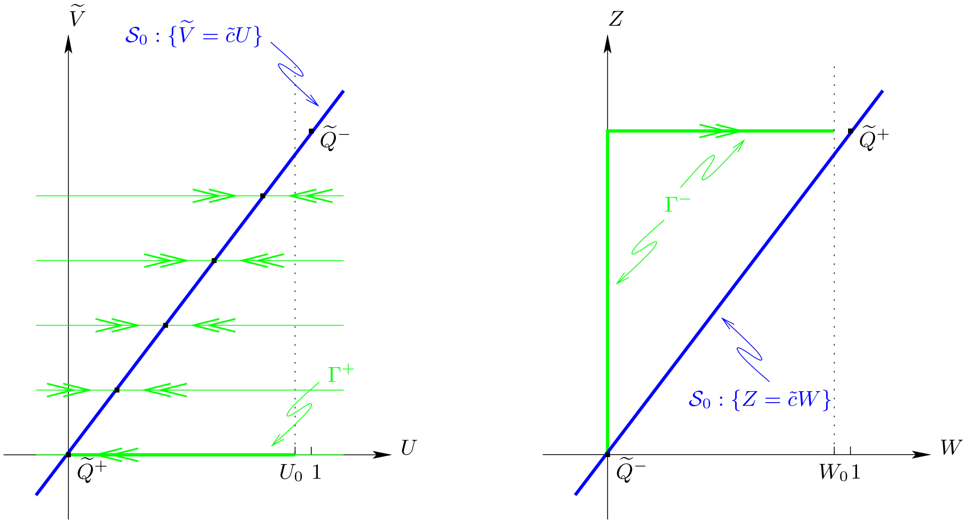
The analysis of (2) in [7] then proceeds by decomposing the phase space of the equivalent, rescaled first-order system (9) into two distinct regions, the 'outer region' (with $0 \leqslant U < 1$) and the 'inner region' (where $U \approx 1$). The large-m asymptotics of $c_{\text{crit}}(m)$ in (4) is hence obtained by constructing a solution for (9) that is uniformly valid on $[0, 1]$. In particular, when $\varepsilon = 0$, a singular orbit Γ can be defined as the unique heteroclinic connection between the equilibrium points $\widetilde{Q}^- = (1, \tilde{c})$ and $\widetilde{Q}^+ = (0, 0)$ of (9). The construction of Γ is outlined below; details can be found in [7, Section 2].

2.1. The 'outer problem'

Since $U^{\frac{1}{\varepsilon}} = e^{\frac{1}{\varepsilon}\ln U}$, the right-hand side in (9) is exponentially small in ε for $U \in [0, U_0]$, with $U_0 < 1$ constant. Correspondingly, the dynamics in this outer region is governed by

$$\dot{U} = \widetilde{V} - \tilde{c}U, \qquad \dot{\widetilde{V}} = 0, \tag{10}$$

where $\tilde{c} = \tilde{c}(\varepsilon)$ now. The line $\mathcal{S}_0 := \{(U, \widetilde{V}) \mid \widetilde{V} = \tilde{c}(0)U,\ U \in [0, U_0]\}$ is invariant for (10) and normally attracting, as $\tilde{c} > 0$; hence, $\mathcal{S}_0$ will persist, by standard theory [8,9], as the line $\mathcal{S}_\varepsilon := \{(U, \widetilde{V}) \mid \widetilde{V} = \tilde{c}(\varepsilon)U,\ U \in [0, U_0]\}$, for $\varepsilon > 0$ sufficiently small. (Here, we note that $\widetilde{Q}^+$ lies on $\mathcal{S}_\varepsilon$ for any value of ε.) Similarly, the fast foliation $\mathcal{F}_0$, which consists of axis-parallel fibers $\{\widetilde{V} = \widetilde{V}_0\}$ for $\widetilde{V}_0$ constant, will persist as a foliation $\mathcal{F}_\varepsilon$ whose fibers are exponentially close (in ε) to those of $\mathcal{F}_0$. In particular, the fiber Γ^+: $\{\widetilde{V} = 0\}$ gives the leading-order strong stable manifold $\mathcal{W}^s(\widetilde{Q}^+)$ of the origin $\widetilde{Q}^+$; see Fig. 1(a) for an illustration.



(a) The 'outer problem' (10).

(b) The 'inner problem' (11).

Fig. 1. The geometry for $\varepsilon = 0$.

2.2. The 'inner problem'

For $U \approx 1$, *i.e.*, close to the point $\widetilde{Q}^-$, the contribution from the right-hand side in (9) remains significant even as $\varepsilon \to 0$. (In fact, the rapid variation of $f_m(U)$ signals the existence of a boundary layer in this inner region.) Translating $\widetilde{Q}^-$ to the origin by introducing the new variables $W = 1 - U$ and $Z = -(\widetilde{V} - \tilde{c})$ in (9), we find

$$\dot{W} = Z - \tilde{c}W, \qquad \dot{Z} = \frac{2}{\varepsilon^2}(1 - W)^{\frac{1}{\varepsilon}} W. \tag{11}$$

Even though the right-hand side in (11) is undefined in the (non-uniform) limit as $(W, \varepsilon) \to (0, 0)$, it was shown in [7] that the corresponding singular dynamics can still be obtained using geometric desingularization, or blow-up [6,16]. Heuristically, that dynamics is described by the singular orbit $\Gamma^- := \{(0, Z) \mid Z \in [0, \tilde{c}]\} \cup \{(W, \tilde{c}) \mid W \in [0, W_0]\}$, with $W_0 = 1 - U_0$. Geometrically speaking, Γ^- consists of a portion of the Z-axis which represents the boundary layer at $W = 0$ (to lowest order), as well as of a segment of $\{Z = \tilde{c}\}$ that corresponds to the fiber Γ^+; cf. Fig. 1(b).

2.3. The blow-up transformation for (11)

As in [7, Section 3], the dynamics of the inner problem in a neighborhood of $(W, \varepsilon) = (0, 0)$ is desingularized via the cylindrical blow-up transformation

$$W = \bar{r}\bar{w}, \qquad Z = \bar{z}, \quad \text{and} \quad \varepsilon = \bar{r}\bar{\varepsilon}, \tag{12}$$

which maps the Z-axis to the quarter-cylinder $\mathbb{S}^1_+ \times [0, z_0]$. (Here, $(\bar{w}, \bar{\varepsilon}) \in \mathbb{S}^1_+ = \{(\bar{w}, \bar{\varepsilon}) \mid \bar{w}^2 + \bar{\varepsilon}^2 = 1,\ \bar{w}, \bar{\varepsilon} \geqslant 0\}$, $\bar{z} \in [0, z_0]$, and $\bar{r} \in [0, r_0]$, with $z_0 > 2$ fixed and r_0 positive and sufficiently small.) For details on the blow-up technique, the reader is again referred to [7] and the references therein.

The blown-up vector field corresponding to the equations in (11) is best studied in two coordinate charts: the dynamics in the inner region is covered by a 'rescaling chart' K_2, which is defined by $\bar{\varepsilon} = 1$; the transition between the inner and outer regions, which will be termed the 'intermediate region,' is naturally described in a 'phase-directional chart' K_1, with $\bar{w} = 1$ in (12). For future reference, we note that the coordinate change $\kappa_{21} : K_2 \to K_1$ on the domain of overlap between these charts is given by

$$r_1 = r_2 w_2, \qquad z_1 = z_2, \quad \text{and} \quad \varepsilon_1 = w_2^{-1}. \tag{13}$$

The geometry in blown-up coordinates is illustrated in Fig. 2.

Remark 1. Given any object $\square$, we will denote the corresponding blown-up object by $\bar{\square}$; in chart K_j $(j = 1, 2)$, the same object will appear as $\square_j$.

2.3.1. Dynamics in chart K_2

In K_2, the blow-up transformation in (12) is given by

$$W = r_2 w_2, \qquad Z = z_2, \quad \text{and} \quad \varepsilon = r_2,$$

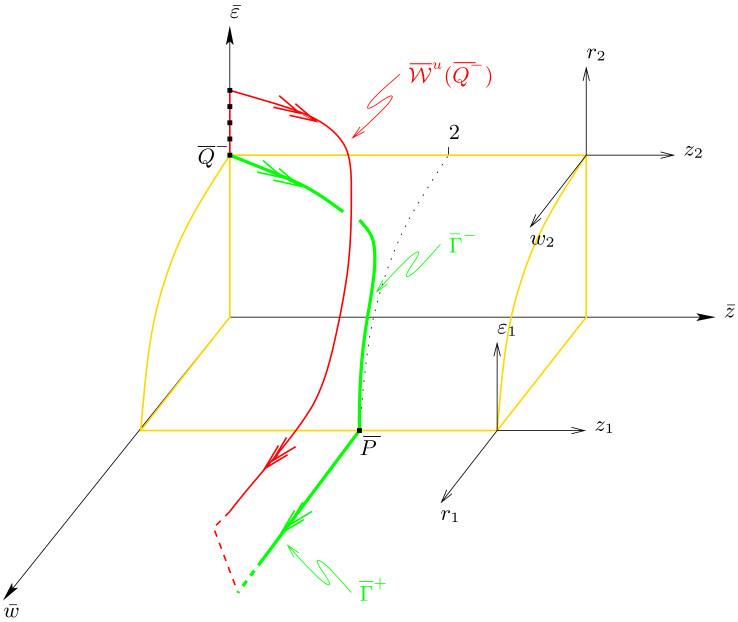


Fig. 2. The geometry of (11) after blow-up.

which we substitute into (11) to obtain the dynamics in this chart. The resulting equations – after desingularization, *i.e.*, after multiplication of the right-hand sides by a factor of r_2 – read

$$w_2' = z_2 - r_2\tilde{c}w_2, \qquad z_2' = f(w_2, r_2), \qquad r_2' = 0, \tag{14}$$

where the function f is defined as

$$f(w_2, r_2) = 2w_2(1 - r_2w_2)^{\frac{1}{r_2}} = 2w_2 \exp\left[\frac{1}{r_2}\ln(1 - r_2w_2)\right]. \tag{15}$$

The only finite equilibrium for the $(\tilde{c}, r_2)$-family of vector fields in (14) is the origin $\widetilde{Q}_2^-$, which is a hyperbolic saddle point for $\tilde{c} > 0$ and $r_2 \in [0, r_0]$ sufficiently small; see [7, Lemma 3.2]. We note that $\widetilde{Q}_2^-$ corresponds to the origin in (W, Z)-space or, alternatively, to the saddle equilibrium at $\widetilde{Q}^- = (1, \tilde{c})$ in the original $(U, \widetilde{V})$-coordinates.

The singular limit of $r_2 = 0$ in (14) is described by the integrable system

$$w_2' = z_2, \qquad z_2' = 2w_2 e^{-w_2}$$

or, equivalently, by the equation $z_2\frac{dz_2}{dw_2} = 2w_2e^{-w_2}$; the unique solution satisfying $z_2(0) = 0$ and $z_2 \to 2$ as $w_2 \to \infty$ is given by

$$z_2(w_2) = 2\sqrt{1 - (1 + w_2)e^{-w_2}}. \tag{16}$$

The corresponding orbit, which we denote by Γ_2^-, approximates the unstable manifold $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ of $\widetilde{Q}_2^-$ to lowest order. Hence, Γ_2^- represents the portion of the singular orbit Γ^- (before blow-up) that is located in chart K_2; see Fig. 3(a) for an illustration.

2.3.2. Dynamics in chart K_1

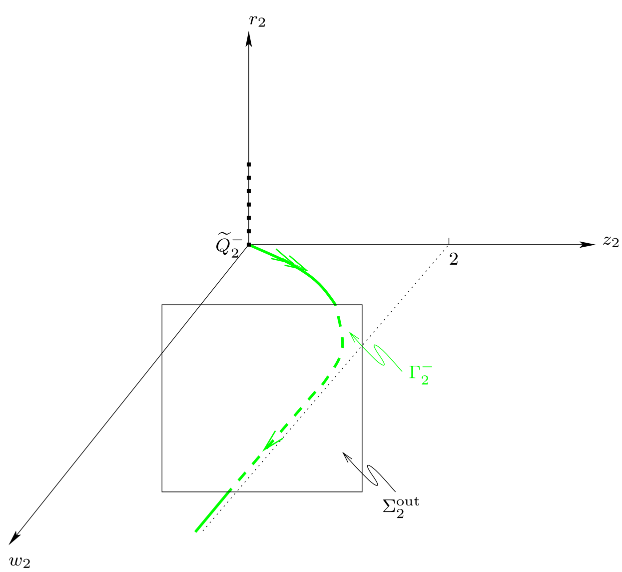
In chart K_1, the transformation in (12) reduces to

$$W = r_1, \qquad Z = z_1, \quad \text{and} \quad \varepsilon = r_1\varepsilon_1,$$

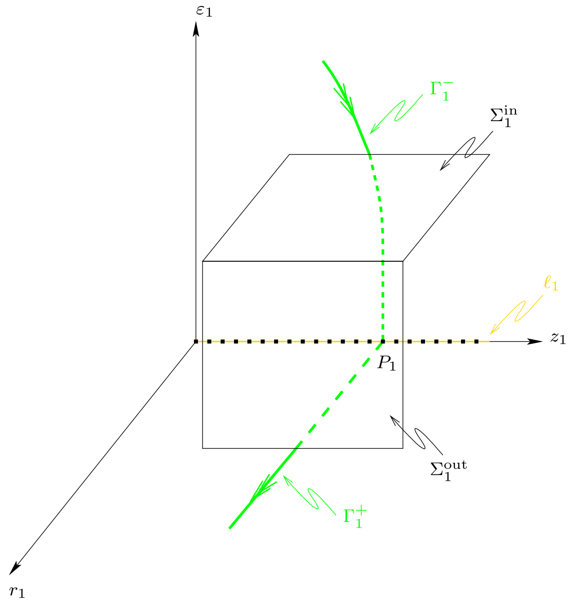
which implies

$$r_1' = r_1(z_1 - r_1\tilde{c}), \qquad z_1' = \frac{2}{\varepsilon_1^2}\exp\left[\frac{1}{r_1\varepsilon_1}\ln(1 - r_1)\right], \qquad \varepsilon_1' = -\varepsilon_1(z_1 - r_1\tilde{c}) \tag{17}$$

after desingularization (multiplication by r_1); in particular, (17) extends to a $\mathcal{C}^\infty$-smooth vector field as $\varepsilon_1 \to 0$ [7]. For r_1 small, all equilibria of (17) are located on the line $\ell_1 = \{(0, z_1, 0) \mid z_1 \in [0, z_0]\}$. Given κ_{21}, as defined in (13), and the expression for Γ_2^- in (16), we find $z_1(\varepsilon_1) = 2\sqrt{1 - (1 + \frac{1}{\varepsilon_1})e^{-\frac{1}{\varepsilon_1}}}$ for the portion Γ_1^- of the singular orbit $\overline{\Gamma}$ that lies on



(a) The 'rescaling chart' K_2.



(b) The 'phase-directional chart' K_1.

Fig. 3. The dynamics of the blown-up vector field.

the (invariant) blow-up locus $\{r_1 = 0\}$ in K_1. Since $z_1 \to 2$ as $\varepsilon_1 \to 0$, it follows that $\Gamma_1^- \to P_1 = (0, 2, 0) \in \ell_1$ in that limit, which also shows $\tilde{c} \sim 2$, to lowest order in ε; cf. [7, Lemma 4.1]. The geometry in chart K_1 is illustrated in Fig. 3(b).

2.3.3. *Regularity of the transition in K_1*

Following [7, Section 3.3], we define two sections Σ_1^{in} and Σ_1^{out} in chart K_1 via

$$\Sigma_1^{\text{in}} = \bigl\{ \bigl(r_1^{\text{in}}, z_1^{\text{in}}, \delta\bigr) \bigm| r_1^{\text{in}} \in [0, \rho],\ \bigl|z_1^{\text{in}} - 2\bigr| \leqslant \alpha \bigr\} \quad \text{and}$$
$$\Sigma_1^{\text{out}} = \bigl\{ \bigl(\rho, z_1^{\text{out}}, \varepsilon_1^{\text{out}}\bigr) \bigm| \bigl|z_1^{\text{out}} - 2\bigr| \leqslant \alpha,\ \varepsilon_1^{\text{out}} \in [0, \delta] \bigr\}, \tag{18}$$

where δ, ρ, and α are small and positive constants; see again Fig. 3(b). (Here, we note that Σ_1^{in} corresponds, under the change of coordinates κ_{21} defined in (13), to a section Σ_2^{out} for the flow of (14); recall Fig. 3(a).) Let $\Pi_1 : \Sigma_1^{\text{in}} \to \Sigma_1^{\text{out}}$ denote the corresponding transition map that is induced by the flow of (17); the following result on the regularity of Π_1 can be found in [7]:

Proposition 1. *(See [7, Proposition 3.4].) The map*

$$\Pi_1 : \begin{cases} \Sigma_1^{\text{in}} \to \Sigma_1^{\text{out}}, \\ (\varepsilon\delta^{-1}, z_1^{\text{in}}, \delta) \mapsto (\rho, z_1^{\text{out}}, \varepsilon\rho^{-1}) \end{cases}$$

is C^∞-smooth in z_1^{in}, as well as in the parameters ε and $\tilde{c}$.

Remark 2. It was conjectured in [7, Remark 7] that the transition map Π_1 is 'infinitely close' to the identity, as the right-hand side in (17) goes to zero as $\varepsilon \to 0$, along with its derivatives. This conjecture appears to be true only in the following, more restrictive formulation: Π_1 tends towards the identity exponentially fast (in ε_1) as $\delta \to 0$ in the definition of Σ_1^{in}, recall (18); however, that convergence is not uniform in $r_1 \in [0, \rho]$, *i.e.*, for $r_1 \to 0$ and ε_1 fixed.

3. Proof of Theorem 1

As in the proof of [7, Theorem 1.1], the critical wave speed c_{crit} defined in Theorem 1 is obtained in the intersection of two invariant manifolds in an appropriately defined section in phase space: specifically, the unstable manifold $\mathcal{W}^u(\widetilde{Q}^-)$ of the hyperbolic saddle equilibrium at $\widetilde{Q}^-$ is tracked in forward 'time,' and is matched, for ε positive and small, to the strong stable manifold $\mathcal{W}^s(\widetilde{Q}^+)$ of the origin $\widetilde{Q}^+$. (Here, we recall that the heteroclinic connection between $\widetilde{Q}^-$ and $\widetilde{Q}^+$ that is realized in the intersection of these manifolds reduces to the singular orbit Γ in the limit as $\varepsilon \to 0$.) Since both $\mathcal{W}^u(\widetilde{Q}^-)$ and $\mathcal{W}^s(\widetilde{Q}^+)$ correspond, in fact, to families of manifolds that are parametrized by $\tilde{c}$, and extended to the complex domain, the matching is accomplished using a Gevrey-1 version of the Implicit Function Theorem along a curve $\tilde{c} = \tilde{c}(\varepsilon)$, which uniquely determines $c_{\mathrm{crit}}(m) = m^{-1}\tilde{c}(m^{-1})$; see also [7, Proposition 4.2].

The required analysis is performed in the framework of the blown-up vector field induced by (11), *i.e.*, of the two coordinate charts K_2 and K_1 introduced in Section 2, and draws heavily on methods from complex analysis. In the process, we substantially refine the asymptotic estimates (in $\tilde{c}$ and ε) for $\mathcal{W}^u(\widetilde{Q}^-)$ and $\mathcal{W}^s(\widetilde{Q}^+)$ that were derived in [7, Section 4]: in Section 3.1, we show that the manifold $\mathcal{W}^u(\widetilde{Q}^-)$ is analytic when restricted to the inner region, but that it loses analyticity in its transition through the intermediate region; correspondingly, the manifold $\mathcal{W}^s(\widetilde{Q}^+)$ is merely $\mathcal{C}^\infty$-smooth, as discussed in Section 3.2. In particular, these refined estimates then translate into the Gevrey-1 asymptotics of c_{crit} postulated in (6), for m sufficiently large. Finally, the bound in (7) is obtained by performing a ‘truncation to the least term,’ as explained in Section 3.3.

3.1. Asymptotics of $\mathcal{W}^u(\widetilde{Q}^-)$

In this subsection, we discuss the asymptotics of the unstable manifold $\mathcal{W}^u(\widetilde{Q}^-)$ of $\widetilde{Q}^-$: we first consider the corresponding manifold $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ in the rescaling chart K_2, *i.e.*, in the inner region; then, we describe the transition through the intermediate region, which is studied in the phase-directional chart K_1. Finally, reverting to the original $(U, \widetilde{V}, \varepsilon)$-variables, we extend the resulting asymptotics to the outer region (away from the blow-up locus).

3.1.1. The inner region (chart K_2)

We recall that $\mathcal{W}^u(\widetilde{Q}^-)$ corresponds to the unstable manifold $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ of the hyperbolic saddle point $\widetilde{Q}_2^-$, after transformation to (w_2, z_2, r_2)-coordinates; cf. Section 2.3. Since that manifold is analytic for w_2 in any compact subset of $[0, \infty)$, it can be represented as a regular perturbation of the singular orbit Γ_2^- or, equivalently, as the graph of some function ζ that is analytic in w, as well as in the parameters $\tilde{c}$ and r:

$$z = \zeta(w, \tilde{c}, r) = \sum_{n=0}^{\infty} z_n(w, \tilde{c}) r^n, \quad \text{with } \zeta(0, \tilde{c}, r) = 0. \tag{19}$$

(Here and in the remainder of this section, we omit the subscript 2 for convenience of notation.)

For $r = 0$ in (19), we have $\zeta(w, \tilde{c}, 0) = z_0(w, \tilde{c}) \equiv z_0(w)$, where we recall the expression for z_0 from (16). It is evident that $|z_0(w)| \leqslant 2$. To obtain corresponding bounds on the higher-order coefficient functions $z_n(w, \tilde{c})$, for $n = 1, 2, 3, \ldots$, we first rewrite (14) with w as the independent variable:

$$\frac{\partial z}{\partial w}(w, \tilde{c}, r) = \frac{f(w, r)}{z - r\tilde{c}w}, \tag{20}$$

where the function $f(w, r)$ is defined as in (15).

Lemma 1. *The function f can be written as $f(w, r) = 2w \exp[-w\psi(rw)]$, where*

$$\psi(x) = -\frac{\ln(1 - x)}{x}$$

is analytic at $x = 0$, with $\psi(0) = 1$ and radius of convergence 1. Furthermore, ψ satisfies $|\psi(x)| \geqslant \ln 2$, as well as $\Re\psi(x) \geqslant \ln 2$ and $\Im\psi(x) \leqslant \frac{\pi}{2}$, for all x with $|x| < 1$.

Proof. All statements are immediately obvious, except for the bounds on $|\psi(x)|$, $\Re\psi(x)$, and $\Im\psi(x)$; the latter can be obtained by observing that the function $\frac{1}{\psi}$ is well defined and continuous on the closed unit ball $\overline{B}(0, 1)$, with $(\frac{1}{\psi})(1) := 0$. Applying the maximum principle to $\frac{1}{\psi}$, we find that $\min_{\overline{B}(0,1)} |\psi(x)| = \min_{\{|x|=1,\, x\neq 1\}} |\psi(x)|$. For $|x| = 1$, we write $x = \mathrm{e}^{i\phi}$, and we consider the function

$$\begin{aligned} g(\phi) &= \left|\ln(1 - \cos\phi - i\sin\phi)\right|^2 \\ &= \left|\ln\sqrt{2 - 2\cos\phi} - i\arctan\frac{\sin\phi}{1 - \cos\phi}\right|^2 = \left(\ln\sqrt{2 - 2\cos\phi}\,\right)^2 + \left(\arctan\frac{\sin\phi}{1 - \cos\phi}\right)^2. \end{aligned}$$

Next, we prove that $g(\phi)$ assumes its minimum at $\phi = \pi$, which will imply that $|\psi(x)|$ is minimal at $x = -1$. Defining $y = \frac{\sin\phi}{1-\cos\phi}$, we observe that $\cos\phi = \frac{y^2-1}{y^2+1}$ then, as well as that $\phi = \pi$ yields $y = 0$. Substituting into g, we obtain a simplified function $\tilde{g}$, as follows:

$$\tilde{g}(y) = \left(\ln\sqrt{\frac{4}{y^2 + 1}}\right)^2 + (\arctan y)^2, \quad \text{with } y \in (-\infty, \infty).$$

Correspondingly, we now show that $\tilde{g}$ is minimal at $y=0$. By symmetry, we may restrict to studying $\tilde{g}$ for $y \in [0,\infty)$: since the right-hand side in $\tilde{g}'(y)(y^2+1) = -2\ln 2y + y\ln(y^2+1) + 2\arctan y$ vanishes at $y=0$, and since its derivative $2 - 2\ln 2 + \ln(y^2+1)$ is strictly positive, it follows that $\tilde{g}'(y) > 0$ for all $y>0$, which proves the bound on $|\psi(x)|$. The bounds on $\Re\psi(x)$ and $\Im\psi(x)$ can be obtained in a similar fashion. □

Remark 3. Lemma 1 implies, in particular, that $|\arg\psi(x)|$ is uniformly bounded away from $\frac{\pi}{2}$.

By Lemma 1, the function f is analytic at $r=0$, for w in any compact subset of $[0,\infty)$; hence, the series in (19) is uniformly convergent on any such subset. However, as $w \to \infty$, the convergence becomes weaker; specifically, we claim that the series only satisfies Gevrey-1 growth properties at $w=\infty$, *i.e.*, that there exist positive constants A and B such that

$$\bigl|z_n(w,\tilde{c})\bigr| \leqslant AB^{n+1}(n+1)! \quad \text{for all } w \in [0,\infty) \text{ and } n = 0,1,2,\ldots, \tag{21}$$

where $\tilde{c}$ is close to its singular value $\tilde{c}_0 = 2$ [7, Lemma 4.1]. To prove the bound on $|z_n|$ in (21), we first estimate the corresponding remainder terms in the series expansion in (19): we write

$$\zeta^{[n]}(w,\tilde{c},r) := \left(\zeta(w,\tilde{c},r) - \sum_{k=0}^{n-1} z_k(w,\tilde{c})r^k\right) r^{-n}, \tag{22}$$

where we note that $z_n(w,\tilde{c}) = \zeta^{[n]}(w,\tilde{c},0)$. (Here and in the following, we will consider the sum from 0 to $n-1$ to be empty when $n=0$.) Applying the Residue Theorem, we find

$$\zeta^{[n]}(w,\tilde{c},r) = \frac{1}{2\pi i}\oint \frac{\zeta(w,\tilde{c},s)}{s^n(s-r)}\,ds, \tag{23}$$

where the integration is performed along a complex contour encircling both $s=0$ and $s=r$, with a counterclockwise orientation. Next, we show that the function ζ is bounded on a sufficiently large complex domain.

Proposition 2. *For $\theta > 0$ sufficiently small and $\eta \in (0,1)$, there exist constants $M>0$, $C_0>0$, and $R>0$ such that $z = \zeta(w,\tilde{c},r)$ extends analytically to the domain defined by*

$$|w| \geqslant M, \qquad |\tilde{c}-2| \leqslant C_0, \qquad |r| \leqslant R, \qquad |rw| \leqslant \eta, \quad \text{and} \quad \arg w \in [-\theta,\theta].$$

Furthermore, there exist constants $K_0>0$ and $\beta>0$ such that, on that domain,

$$\bigl|\zeta(w,\tilde{c},r)\bigr| \leqslant K_0 \quad \text{and} \quad \left|\frac{\partial\zeta}{\partial w}(w,\tilde{c},r)\right| \leqslant K_0|w|\mathrm{e}^{-\beta|w|}.$$

The constants M, R, and β can be chosen independently of η; finally, the constant β satisfies $\beta = \ln 2 + \mathcal{O}(\theta)$, where θ can be taken as small as required.

Proof. Choosing $M>0$ sufficiently large, we may assume that $z_0(M)$ is as close to its asymptotic limit 2 as desired. Since $\zeta(w,\tilde{c},r) \sim z_0(w)$, to lowest order, we find

$$\bigl|\zeta\bigl(M\mathrm{e}^{i\phi},\tilde{c},r\bigr) - 2\bigr| \leqslant \frac{2-|\tilde{c}|\eta}{4} \tag{24}$$

for $\phi \in [-\theta,\theta]$ and $|r| \leqslant R$, with R sufficiently small. By bounding $|\frac{\partial z}{\partial w}|$, we now prove that the restriction of Eq. (20) to the domain that is defined by $|z-2| \leqslant \frac{2-|\tilde{c}|\eta}{2}$ has a solution on that restricted domain. To that end, we observe that, under the above condition on z,

$$|z - \tilde{c}rw| \geqslant 2 - |z-2| - |\tilde{c}rw| \geqslant \frac{2-|\tilde{c}|\eta}{2}. \tag{25}$$

In order to bound the function $f(w,r)$, we estimate the argument $w\psi(rw)$ of the exponential in the definition of f; see Lemma 1:

$$\Re\bigl(w\psi(rw)\bigr) = |w|\bigl|\psi(rw)\bigr|\cos\arg\bigl(w\psi(rw)\bigr) = |w|\frac{\cos\arg(w\psi(rw))}{\cos\arg\psi(rw)}\Re\psi(rw).$$

Recalling that $|\arg\psi(rw)|$ is bounded away from $\frac{\pi}{2}$, cf. Remark 3, as well as that $|\arg w| = \mathcal{O}(\theta)$, we may apply the identity $\frac{\cos(x+y)}{\cos y} = \cos x - \sin x \tan y = 1 + \mathcal{O}(x)$ to obtain $\frac{\cos\arg(w\psi(rw))}{\cos\arg\psi(rw)} = 1 + \mathcal{O}(\theta)$. Since, moreover, $\Re\psi(rw) \geqslant \ln 2$, we have $\Re(w\psi(rw)) \geqslant \beta_\theta|w|$, where β_θ is some constant that tends towards $\ln 2$ as $\theta \to 0$. Hence, it follows that $|f(w,r)| = |2w\mathrm{e}^{-w\psi(rw)}| \leqslant 2|w|\mathrm{e}^{-\beta|w|}$, with $\beta = \ln 2 + \mathcal{O}(\theta)$, as claimed.

Next, fixing w in the restricted domain defined above, and letting ϕ be its complex argument, we write

$$\zeta(w,\tilde{c},r) = \zeta\big(Me^{i\phi},\tilde{c},r\big) + \int_{Me^{i\phi}}^{w} \frac{f(\omega,r)}{\zeta(\omega,\tilde{c},r)-\tilde{c}r}\,d\omega,$$

where the integration is performed along a complex line segment (with fixed argument). Making use of (24) and (25), we obtain

$$\begin{aligned}\big|\zeta(w,\tilde{c},r)-2\big| &\leqslant \big|\zeta(M,\tilde{c},r)-2\big| + \int_{M}^{|w|}\big|f(\omega,r)\big|\frac{2}{2-|\tilde{c}|\eta}\,|d\omega| \\ &\leqslant \frac{2-|\tilde{c}|\eta}{4} + \frac{4}{2-|\tilde{c}|\eta}\int_{M}^{|w|}\omega e^{-\beta\omega}\,d\omega < \frac{2-|\tilde{c}|\eta}{2},\end{aligned}$$

which is satisfied as long as

$$\int_{M}^{\infty}\omega e^{-\beta\omega}\,d\omega < \frac{(2-|\tilde{c}|\eta)^2}{16}. \tag{26}$$

As $M\to\infty$, the left-hand side in (26) tends to 0, which implies that we can integrate the solution of (20) inside the region $\{|z-2|\leqslant\frac{2-|\tilde{c}|\eta}{2}\}$ as far as we like (in w). Clearly, we have $|\zeta(w,\tilde{c},r)|\leqslant 2+\frac{2-|\tilde{c}|\eta}{2}$ inside that region; given (20) and the above estimate for f, we also find

$$\left|\frac{\partial\zeta}{\partial w}(w,\tilde{c},r)\right| \leqslant \frac{|f(w,r)|}{|\zeta(w,\tilde{c},r)-\tilde{c}rw|} \leqslant 2|w|e^{-\beta|w|}\frac{2}{2-|\tilde{c}|\eta}.$$

Finally, taking K_0 to be the maximum of $2+\frac{2-|\tilde{c}|\eta}{2}$ and $\frac{4}{2-|\tilde{c}|\eta}$, we obtain the desired result. □

Remark 4. The requirement that $\beta\gtrsim\ln 2$ in the statement of Proposition 2 is a consequence of the bound on $\Re\psi$ obtained in Lemma 1; in particular, that restriction will imply that the coefficient functions $z_n(w,\tilde{c})$ are Gevrey-1 of type $(\ln 2)^{-1}$ and not of type 1, as one may have expected intuitively.

Next, we make use of the estimates obtained in Proposition 2 to bound $\zeta^{[n]}$ – or, equivalently, the coefficient functions z_n – in the limit as $w\to\infty$ in chart K_2; since, by (13), that limit corresponds to taking $\varepsilon_1=0$ in the phase-directional chart K_1, the resulting large-w asymptotics of $\mathcal{W}^u(\tilde{Q}^-)$ will allow us to define a corresponding manifold there.

Proposition 3. *For $\theta>0$ sufficiently small and $\eta\in(0,1)$, there exist constants $M>0$, $C_0>0$, and $R>0$ such that the function $z=\zeta(w,\tilde{c},r)$ is analytic on the domain defined by*

$$|w|\geqslant M,\qquad |\tilde{c}-2|\leqslant C_0,\qquad |r|\leqslant R,\qquad |rw|\leqslant\eta,\quad \text{and}\quad \arg w\in[-\theta,\theta].$$

Moreover, on that domain, the coefficient functions $\{z_n\}_{n=0}^{\infty}$ in (19) satisfy

$$\big|z_n(w,\tilde{c})\big| \leqslant \big|\zeta^{[n]}(w,\tilde{c},r)\big| \leqslant AB^{n+1}(n+1)!, \tag{27}$$

for some (positive) constants A and B. Finally, there exists a sequence of functions $\{z_n^\infty(\tilde{c})\}_{n=0}^{\infty}$ with

$$\big|z_n(w,\tilde{c})-z_n^\infty(\tilde{c})\big| \leqslant A\int_{|w|}^{\infty}\omega^{n+1}e^{-\beta\omega}\,d\omega \leqslant AB^{n+1}(n+1)!, \tag{28}$$

for all w with $|w|\geqslant M$ and $\arg w\in[-\theta,\theta]$. At the expense of decreasing θ and R and increasing A, the constant B can be chosen as close to $(\ln 2)^{-1}$ as desired.

Proof. Let $\mu\in(\eta,1)$ be chosen arbitrarily; then, we may assume that the estimates derived in Proposition 2 are valid for $|rw|\leqslant\mu$. Writing $\varphi:=\frac{\partial\zeta}{\partial w}$ for the derivative of ζ with respect to w, we have

$$\varphi^{[n]}(w,\tilde{c},r) = \frac{1}{2\pi i}\oint\frac{\varphi(w,\tilde{c},s)}{s^n(s-r)}\,ds,$$

where the integration is performed along a contour with $|s| = \frac{\mu}{|w|}$. Considering values of r inside the disc that is defined by $|r| \leqslant \frac{\eta}{|w|}$, we find

$$\left|\varphi^{[n]}(w, \tilde{c}, r)\right| \leqslant \frac{1}{2\pi} K_0 |w| \mathrm{e}^{-\beta|w|} \left(\frac{|w|}{\mu}\right)^n \left(\frac{|w|}{\mu - \eta}\right) 2\pi \frac{\mu}{|w|} = K_0 \frac{\mu^{1-n}}{\mu - \eta} |w|^{n+1} \mathrm{e}^{-\beta|w|}.$$

For w in the domain specified in the formulation of the proposition, we write $\phi = \arg w$ and $\zeta^{[n]}(w, \tilde{c}, r) = \zeta^{[n]}(M\mathrm{e}^{i\phi}, \tilde{c}, r) + \int_{M\mathrm{e}^{i\phi}}^{w} \varphi^{[n]}(s, \tilde{c}, r)\, ds$. Making the substitution $x = \beta|w|$, we obtain

$$\begin{aligned}\left|\zeta^{[n]}(w, \tilde{c}, r)\right| &\leqslant \left|\zeta^{[n]}\left(M\mathrm{e}^{i\phi}, \tilde{c}, r\right)\right| + \frac{K_0 \mu^{1-n}}{\mu - \eta} \beta^{-n-2} \int_{\beta M}^{\beta|w|} x^{n+1} \mathrm{e}^{-x}\, dx \\ &\leqslant \left|\zeta^{[n]}(M, \tilde{c}, r)\right| + \frac{K_0 \mu^{1-n}}{\mu - \eta} \beta^{-n-2} \int_0^\infty x^{n+1} \mathrm{e}^{-x}\, dx = \left|\zeta^{[n]}(M, \tilde{c}, r)\right| + \frac{K_0 \mu^{1-n}}{\mu - \eta} \beta^{-n-2} (n+1)!.\end{aligned}$$

Since the function $\zeta(M, \tilde{c}, r)$ is analytic, $\zeta^{[n]}(M, \tilde{c}, r)$ can be bounded by a (convergent) geometric series in terms of n; hence, there certainly exists $A > 0$ such that $|\zeta^{[n]}(w, \tilde{c}, r)| \leqslant AB^{n+1}(n+1)!$, with $B = (\beta\mu)^{-1}$, which proves (27). Since, moreover, μ can be chosen arbitrarily close to 1, we can take B as close to $(\ln 2)^{-1}$ as desired.

Finally, we define $\varphi_n(w, \tilde{c}) = \varphi^{[n]}(w, \tilde{c}, 0)$ to be the n-th order Taylor coefficient of φ at $r = 0$; then, $\frac{\partial z_n}{\partial w} = \varphi_n$. As φ_n is exponentially decreasing at $w = \infty$, by the above, it follows that the limit

$$z_n^\infty(\tilde{c}) := \lim_{w \to \infty,\, \arg w \in [-\theta, \theta]} z_n(w, \tilde{c})$$

is well defined and that it satisfies $|z_n^\infty| \leqslant AB^{n+1}(n+1)!$; furthermore, we have

$$\left|z_n(w, \tilde{c}) - z_n^\infty(\tilde{c})\right| = \left|\int_w^\infty \varphi_n(\omega, \tilde{c})\, d\omega\right| \leqslant \int_{|w|}^\infty \frac{K_0 \mu^{1-n}}{\mu - \eta} \omega^{n+1} \mathrm{e}^{-\beta\omega}\, d\omega.$$

Simplifying this estimate further by replacing the lower limit of integration by 0, we obtain (28), which completes the proof. □

We remark that the bound on the error incurred when approximating z_n by z_n^∞ in (28) is too pessimistic in the large-w regime, as $\int_{|w|}^\infty \omega^{n+1} \mathrm{e}^{-\beta\omega}\, d\omega = \mathcal{O}(|w|^{n+1} \mathrm{e}^{-\beta|w|})$ for $|w| \to \infty$; however, (28) will turn out to be optimal for $|w|$ small, cf. the proof of Proposition 4 below.

Remark 5. Here and in the following, A will denote a generic constant whose value will remain unspecified, whereas $B = (\beta\mu)^{-1}$ will always be defined as in the statement of Proposition 3.

3.1.2. The intermediate region (chart K_1)

Given the asymptotics of the unstable manifold $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ of $\widetilde{Q}_2^-$ in the inner region, we now translate the corresponding estimates into the intermediate region, which is studied in the phase-directional chart K_1. Recalling that $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ can be represented as the graph of a function ζ_2, with $z_2 = \zeta_2(w_2, \tilde{c}, r_2)$, and applying the coordinate change $\kappa_{21} : K_2 \to K_1$ defined in (13), we find

$$z_1 = \zeta_1\left(\varepsilon_1^{-1}, \tilde{c}, r_1\varepsilon_1\right) =: \tilde{\zeta}_1(r_1, \tilde{c}, \varepsilon_1) \tag{29}$$

for the corresponding manifold $\mathcal{W}_1^u(\widetilde{Q}_1^-) := \kappa_{21}(\mathcal{W}_2^u(\widetilde{Q}_2^-))$ in K_1. (Here, $\tilde{\zeta}_1$ is some new, appropriately defined function of r_1, $\tilde{c}$, and ε_1.) The following result is an immediate consequence of Proposition 3, in combination with (13):

Corollary 1. *For $|\varepsilon| \geqslant 0$ sufficiently small, the manifold $\mathcal{W}_1^u(\widetilde{Q}_1^-)$ is described by a function*

$$z_1 = \tilde{\zeta}_1(r_1, \tilde{c}, \varepsilon_1)$$

that is analytic on the domain defined by

$$|\varepsilon_1| \leqslant M^{-1}, \qquad |\tilde{c} - 2| \leqslant C_0, \qquad |r_1\varepsilon_1| \leqslant R, \qquad |r_1| \leqslant \eta, \quad \textit{and} \quad \arg\varepsilon_1 \in [-\theta, \theta].$$

Corollary 1 implies, in particular, that the manifold $\mathcal{W}_2^u(\widetilde{Q}_2^-)$ extends to a neighborhood of the equilibrium point $P_1 = (0, 2, 0) \in \ell_1$ in chart K_1; recall Section 2.3.

Remark 6. When applying the blow-up technique in the framework of geometric singular perturbation theory, one typically constructs invariant manifolds in compact regions of the rescaling chart K_2; see, *e.g.*, [7,18] and the references therein. Subsequently, the domain of definition of these manifolds has to be extended by transformation to a phase-directional chart, such as is given by K_1. In our case, however, Proposition 3 already shows the existence of an invariant manifold in a domain that is 'larger than compact' in the w_2-direction, since we allow for $|w_2| \leqslant \frac{\eta}{r_2}$; recall Proposition 2. Consequently, the extension of that manifold to chart K_1 is a straightforward corollary, *i.e.*, we do not need to invoke Proposition 1.

Our next result bounds the error incurred when approximating the manifold $\mathcal{W}_1^u(\widetilde{Q}_1^-)$ – or, rather, the function $\tilde{\zeta}_1$ defined in (29) – by its formal power series expansion with coefficients $\{z_n^\infty\}$:

Proposition 4. *For $|\varepsilon_1| \leqslant M^{-1}$ and $\arg \varepsilon_1 \in [-\theta, \theta]$, there holds*

$$\left|\tilde{\zeta}_1(\eta, \tilde{c}, \varepsilon_1) - \sum_{k=0}^{n-1} z_k^\infty(\tilde{c})(\eta\varepsilon_1)^k\right| \leqslant AB^{n+1}(n+1)!|\eta\varepsilon_1|^n, \tag{30}$$

where $n = 1, 2, 3, \ldots,$ A is some (positive) constant, and M, θ, and B are defined as in Proposition 3.

Proof. Recalling the definition of $\tilde{\zeta}_1$, we estimate the left-hand side in (30) as

$$\begin{aligned}\left|\tilde{\zeta}_1(\eta, \tilde{c}, \varepsilon_1) - \sum_{k=0}^{n-1} z_k^\infty(\tilde{c})(\eta\varepsilon_1)^k\right| &= \left|\zeta_1\bigl(\varepsilon_1^{-1}, \tilde{c}, \eta\varepsilon_1\bigr) - \sum_{k=0}^{n-1} z_k^\infty(\tilde{c})(\eta\varepsilon_1)^k\right| \\ &\leqslant \left|\zeta_1\bigl(\varepsilon_1^{-1}, \tilde{c}, \eta\varepsilon_1\bigr) - \sum_{k=0}^{n-1} z_k\bigl(\varepsilon_1^{-1}, \tilde{c}\bigr)(\eta\varepsilon_1)^k\right| + \sum_{k=0}^{n-1}\bigl|z_k\bigl(\varepsilon_1^{-1}, \tilde{c}\bigr) - z_k^\infty(\tilde{c})\bigr||\eta\varepsilon_1|^k. \end{aligned}\tag{31}$$

By (22), the first term on the right-hand side in (31) corresponds to

$$\zeta_2(w_2, \tilde{c}, r_2) - \sum_{k=0}^{n-1} z_k(w_2, \tilde{c})r_2^k = \zeta_2^{[n]}(w_2, \tilde{c}, r_2)r_2^n$$

evaluated at $(w_2, r_2) = (\varepsilon_1^{-1}, \eta\varepsilon_1)$, which is bounded by $AB^{n+1}(n+1)!|\eta\varepsilon_1|^n$; recall (27).

Next, considering the second term in (31), we find

$$\begin{aligned}\sum_{k=0}^{n-1}\bigl|z_k\bigl(\varepsilon_1^{-1}, \tilde{c}\bigr) - z_k^\infty(\tilde{c})\bigr||\eta\varepsilon_1|^k &\leqslant \sum_{k=0}^{n-1} AB(k+1)!|B\eta\varepsilon_1|^k = AB(n+1)!\sum_{k=0}^{n-1}\frac{(k+1)!}{(n+1)!}|B\eta\varepsilon_1|^k \\ &\leqslant AB(n+1)!\left(\frac{1}{n(n+1)}\sum_{k=0}^{n-2}|B\eta\varepsilon_1|^k + \frac{1}{n+1}|B\eta\varepsilon_1|^{n-1}\right),\end{aligned}$$

where we have additionally made use of (28). Keeping $|B\eta\varepsilon_1| \leqslant 1$, we conclude that the terms inside the brackets are bounded by $\frac{2}{n+1}$, which proves (30). □

3.1.3. The outer region

Having described the transition of $\mathcal{W}^u(\widetilde{Q}^-)$ through the intermediate region, it remains to determine the resulting asymptotics in the outer region: 'blowing down' the graph $z_1 = \tilde{\zeta}_1(r_1, \tilde{c}, \varepsilon_1)$, *i.e.*, reverting to (W, Z, ε)-variables, we find that $\mathcal{W}^u(\widetilde{Q}^-)$ can be represented as $Z = \tilde{\zeta}(W, \tilde{c}, \varepsilon W^{-1})$. Evaluating that function at $W = (r_1 =)\eta$, for $\eta \in (0,1)$, we obtain the intersection of the blown-down manifold $\mathcal{W}^u(\widetilde{Q}^-)$ with the hyperplane defined by $\{W = \eta\}$. The following result is an immediate consequence of Proposition 4:

Proposition 5. *For any $\eta \in (0,1)$, the intersection of $\mathcal{W}^u(\widetilde{Q}^-)$ with $\{W = \eta\}$ is described by a function*

$$Z = \gamma(\tilde{c}, \varepsilon) := \tilde{\zeta}\bigl(\eta, \tilde{c}, \varepsilon\eta^{-1}\bigr)$$

that satisfies

$$\left|\gamma(\tilde{c}, \varepsilon) - \sum_{k=0}^{n-1} z_k^\infty(\tilde{c})\varepsilon^k\right| \leqslant AB^{n+1}(n+1)!|\varepsilon|^n$$

for $n = 1, 2, 3, \ldots$. In other words, γ is analytic in a sector of the complex plane, with $\arg\varepsilon \in [-\theta, \theta]$, and is Gevrey-1 asymptotic to the formal power series

$$Z \sim \sum_{n=0}^{\infty} z_n^{\infty}(\tilde{c})\varepsilon^n,$$

where the coefficients $\{z_n^{\infty}\}_{n=0}^{\infty}$ are defined as in the proof of Proposition 3.

Setting $\eta = \rho$, with ρ as in the definition of Σ_1^{out} in (18), we have $Z^{\mathrm{out}} := Z|_{\{W=\rho\}} = \tilde{\zeta}(\rho, \tilde{c}, \varepsilon\rho^{-1})$ for the intersection of $\mathcal{W}^u(\widetilde{Q}^-)$ with the section Σ^{out} that is obtained from Σ_1^{out} after blow-down. In particular, it follows that $\widetilde{V}^{\mathrm{out}} = \widetilde{V}|_{\{U=1-\rho\}} = -Z^{\mathrm{out}} + \tilde{c} =: \gamma_-^{\mathrm{out}}(\tilde{c}, \varepsilon)$ in the original $(U, \widetilde{V}, \varepsilon)$-variables.

3.2. Asymptotics of $\mathcal{W}^s(\widetilde{Q}^+)$

Finally, we derive the asymptotics of the strong stable manifold $\mathcal{W}^s(\widetilde{Q}^+)$ of the origin $\widetilde{Q}^+$, which is defined in the outer region, *i.e.*, in $(U, \widetilde{V})$-space, with $\tilde{c}$ and ε as parameters. Clearly, that manifold can be written as the graph of a function υ, with $\widetilde{V} = \upsilon(U, \tilde{c}, \varepsilon)$, whose series expansion about $\varepsilon = 0$ is identically zero (to all orders in ε). While υ is certainly not analytic at $\varepsilon = 0$, as its Taylor series expansion does not converge to the nonzero function $\widetilde{V}$, we can nevertheless show that $\widetilde{V}$ is $\mathcal{C}^{\infty}$-smooth (in ε) in a complex sector containing the positive real axis, with the origin as its vertex:

Proposition 6. *For $|\varepsilon| \geqslant 0$ sufficiently small, the manifold $\mathcal{W}^s(\widetilde{Q}^+)$ is described by a function*

$$\widetilde{V} = \upsilon(U, \tilde{c}, \varepsilon)$$

that is analytic (and, in fact, exponentially small) in ε, with $\arg\varepsilon \in [-\theta, \theta]$. Moreover, υ is $\mathcal{C}^{\infty}$-smooth at $\varepsilon = 0$.

Proof. The proof is based on a standard fixed point argument: we consider the set of continuous functions $v(U, \tilde{c}, \varepsilon)$ that are defined on $\mathcal{V} := [0, U_0] \times \overline{B}(2, C_0) \times \overline{\Omega}$, where $\overline{B}(2, C_0)$ is the closed complex ball around $\tilde{c}_0 = 2$ with radius C_0 and where $\overline{\Omega}$ denotes the topological closure of $\Omega = \{\varepsilon \in \mathbb{C} \mid 0 < |\varepsilon| < \varepsilon_0, \arg\varepsilon \in [-\theta, \theta]\}$. (Here, the positive constants C_0, U_0, θ, and ε_0 will be taken as small as required.) We denote by $\mathcal{E}$ the subset of the set of these functions that are furthermore analytic with respect to $(\tilde{c}, \varepsilon)$ on $(0, U_0) \times B(2, C_0) \times \Omega$, uniformly in U. Finally, we define the norm $\|v\| := \sup_{(U,\tilde{c},\varepsilon)\in\mathcal{V}} |U^{-1}v(U, \tilde{c}, \varepsilon)|$ on $\mathcal{E}$, and we let $\mathcal{E}_R$ be the subset of $\mathcal{E}$ with $\|v\| \leqslant R$. Then, it is easy to see that $(\mathcal{E}_R, \|\cdot\|)$ is a Banach space.

Given the equations in (9), it follows that the manifold $\widetilde{V} = \widetilde{V}(U, \tilde{c}, \varepsilon)$ can be interpreted as a fixed point of the functional $\tau : \mathcal{E}_R \to \mathcal{E}_R$, with

$$v \mapsto \tau(v) := \int_0^U \frac{-\frac{2}{\varepsilon^2}u^{\frac{1}{\varepsilon}}(1-u)}{v(u) - \tilde{c}u}\,du.$$

Since τ is a contraction mapping with Lipschitz constant less than 1, for R sufficiently small, there exists a fixed point υ for τ in $\mathcal{E}_R$. Hence, the corresponding manifold $\mathcal{W}^s(\widetilde{Q}^+)$ is analytic with respect to ε in a complex sector with vertex at the origin.

It remains to show that the function υ is exponentially small (in ε) in that sector: we have

$$|\upsilon| = \bigl|\tau(\upsilon)\bigr| \leqslant \frac{2}{|\varepsilon|^2(\tilde{c} - R)} \int_0^U \bigl|u^{\frac{1}{\varepsilon}-1}\bigr|\,du,$$

where the integrand is bounded from above by $|U^{\frac{1}{\varepsilon}-1}|$. Thus, we find that $|\upsilon| \leqslant \frac{C}{|\varepsilon|^2}|U_0^{\frac{1}{\varepsilon}}|$ for some constant $C > 0$, which is exponentially small with respect to ε (uniformly in U and $\tilde{c}$). Since the derivatives of exponentially small functions which are defined on a complex sector are also exponentially small, it is evident that υ has a $\mathcal{C}^{\infty}$-smooth extension down to $\varepsilon = 0$, which completes the proof. □

Remark 7. We note that the smoothness of $\widetilde{V}$ does not follow from the standard Stable Manifold Theorem: since the vector field in (9) is only finitely smooth for $(U, \widetilde{V}, \varepsilon)$ in any *a priori* given neighborhood of the origin, that theorem merely implies that $\mathcal{W}^s(\widetilde{Q}^+)$ is $\mathcal{C}^k$-smooth, non-uniformly in ε.

Finally, the intersection of $\mathcal{W}^s(\widetilde{Q}^+)$ with the section Σ^{out}, which we denote by γ_+^{out}, is found by evaluating the function υ at $U = 1 - \rho$: $\widetilde{V}^{\mathrm{out}} = \widetilde{V}|_{\{U=1-\rho\}} = \upsilon(1 - \rho, \tilde{c}, \varepsilon) =: \gamma_+^{\mathrm{out}}(\tilde{c}, \varepsilon)$. In particular, by Proposition 6, γ_+^{out} is exponentially small in $\varepsilon \in [0, \varepsilon_0]$, uniformly in $\tilde{c}$.

Remark 8. The loss of analyticity of $\mathcal{W}^s(\widetilde{Q}^+)$ can also be understood in terms of the asymptotics of $\mathcal{W}^u(\widetilde{Q}^-)$, as discussed in Section 3.2: while the corresponding expansions are uniformly convergent in compact subsets in the inner region, that uniformity is lost as one approaches infinity in chart K_2.

3.3. End of proof of Theorem 1

In this subsection, we conclude the proof of Theorem 1: matching the two manifolds $\mathcal{W}^u(\widetilde{Q}^-)$ and $\mathcal{W}^s(\widetilde{Q}^+)$ in the section Σ^{out}, we obtain a curve $\tilde{c}(\varepsilon)$ which determines the critical wave speed c_{crit}; then, we discuss the question of the optimal truncation point in the corresponding series expansion.

3.3.1. Matching $\mathcal{W}^u(\widetilde{Q}^-)$ and $\mathcal{W}^s(\widetilde{Q}^+)$

It was already established in [7, Proposition 4.2] that the manifolds $\mathcal{W}^u(\widetilde{Q}^-)$ and $\mathcal{W}^s(\widetilde{Q}^+)$ agree for $(\tilde{c},\varepsilon)=(2,0)$ – as they both reduce to the singular orbit Γ then – and that their intersection is transverse as $\tilde{c}$ is varied: defining $\mathcal{D}(\tilde{c},\varepsilon):=\gamma_-^{\text{out}}(\tilde{c},\varepsilon)-\gamma_+^{\text{out}}(\tilde{c},\varepsilon)$, one has $\mathcal{D}(2,0)=0$ as well as $\frac{\partial\mathcal{D}}{\partial\tilde{c}}(2,0)=-1\neq 0$. Moreover, the manifolds found in that intersection are Gevrey-1, in the sense that their series expansions with respect to ε exhibit factorial growth properties, as specified for instance in (21). Finally, the difference between the values of the functions γ_-^{out} and γ_+^{out} and their truncated expansions satisfies those same growth properties; recall Propositions 5 and 6.

Hence, by the Implicit Function Theorem, the two manifolds must coincide along a curve $\tilde{c}=\tilde{c}(\varepsilon)$. From a Gevrey version of that theorem, which can *e.g.* be found in [17], it then immediately follows that the function $\tilde{c}(\varepsilon)$ has a Gevrey-1 series expansion with respect to ε and that

$$\left|\tilde{c}(\varepsilon)-\sum_{k=0}^{n-1}\tilde{c}_k\varepsilon^k\right|\leqslant AB^{n+1}(n+1)!\varepsilon^n, \tag{32}$$

for some positive constants A and B. (Specifically, B can be chosen as close to $(\ln 2)^{-1}$ as desired, as stated in the proof of Proposition 3.) The first part of the statement of Theorem 1 is then obtained by defining $c_{\text{crit}}(m)=m^{-1}\tilde{c}(m^{-1})$, as in [7], and by noting that c_{crit} has an expansion in m^{-1} of the form in (5), with coefficients $c_k=\tilde{c}_{k-1}$ for $k=1,2,3,\ldots.$

Remark 9. While the analysis in Sections 3.1 and 3.2 allows for complex values of $\tilde{c}$ and ε, as w_2 and r_2 – or, equivalently, ε_1 – are assumed to vary in complex sectors containing the positive real axis (with vertex at the origin), it follows from [7, Theorem 1.1] that the critical wave speed c_{crit} must be a real function of the real parameter m^{-1}, in agreement with physical intuition.

3.3.2. The optimal truncation point

Finally, the estimate in (32) leads to the truncation to the least term stated in the second part of Theorem 1: given any fixed m, there exists an optimal truncation point to which the series expansion in (5) has to be summed so that it is closest to the actual value of the function $c_{\text{crit}}(m)$. That point is calculated by determining

$$\min_{n\geqslant 1}R_n,\quad \text{where } R_n:=AB^{n+1}\frac{(n+1)!}{m^n}$$

denotes the error of the truncation of the expansion for $\tilde{c}$ after the n-th term. Comparing R_n with R_{n-1}, one finds $R_n-R_{n-1}=AB^n\frac{n!}{m^n}(B(n+1)-m)$, which implies that R_n is minimal when $n+1\approx\frac{m}{B}$. Setting $n+1=[\frac{m}{B}]$, where the square brackets again denote the integer nearest to $\frac{m}{B}$, we have $\frac{m}{B}-\frac{1}{2}\leqslant n+1\leqslant\frac{m}{B}+\frac{1}{2}$ and, hence,

$$AB^{n+1}\frac{(n+1)!}{m^n}\leqslant AB\left(\frac{m}{B}\right)^{\frac{3}{2}}\left(\frac{B}{m}\right)^{\frac{m}{B}}\Gamma\left(\frac{m}{B}+\frac{3}{2}\right).$$

(Here, $\Gamma(\cdot)$ denotes the Gamma function, and we have used the identity $\Gamma(x+1)=x\Gamma(x)$, which is certainly valid for $x\in\mathbb{R}^+$.) Finally, making use of the fact that $x^{-x}\Gamma(x+\frac{3}{2})\mathrm{e}^x=\sqrt{2\pi}x(1+\mathcal{O}(x^{-1}))$ when $x=\frac{m}{B}$ is large, we find

$$AB\left(\frac{m}{B}\right)^{\frac{3}{2}}\left(\frac{B}{m}\right)^{\frac{m}{B}}\Gamma\left(\frac{m}{B}+\frac{3}{2}\right)\leqslant A\sqrt{2\pi}m\left(\frac{m}{B}\right)^{\frac{3}{2}}\mathrm{e}^{-\frac{m}{B}}\left(1+\mathcal{O}\left(Bm^{-1}\right)\right) \tag{33}$$

for $m\geqslant m_0$, with $m_0>0$ sufficiently large. Substituting (33) into (32) and recalling that $c=m^{-1}\tilde{c}$, we obtain (7), which completes the proof of Theorem 1.

Remark 10. As $B\gtrsim(\ln 2)^{-1}$, the above discussion suggests that $n\approx\ln 2m-1\approx 0.69315m-1$ (or, rather, the corresponding nearest integer $[n]$) would provide an appropriate truncation point for evaluating the critical wave speed $c_{\text{crit}}(m)$.

4. Derivation of Table 1

In this section, we outline our numerical algorithm for the evaluation of the coefficients c_k in the series expansion for the critical wave speed c_{crit} in (5) with fixed, but arbitrary, precision, for any $k \geqslant 1$. The first coefficient ($c_1 = 2$) was determined previously in [7,23,27], while the value of the second coefficient ($c_2 \approx -0.31191$) was calculated numerically in [23,27] and verified analytically in [7]. However, due to the inherent singular character of the problem, higher-order coefficients in the expansion are increasingly hard to obtain. Here, we calculate c_k explicitly up to $k = 10$, as given in Table 1. Our approach is based on rigorous interval arithmetic, which allows us to bound the numerical error that invariably accumulates in the evaluation of these coefficients, even for relatively small k: as the required accuracy far exceeds what is provided by standard double precision arithmetic, we used the GNU Arbitrary Precision Arithmetic library [12] for C, in conjunction with the MPFI (Multiple Precision Floating-point Interval) library [25], to obtain reliable intervals which must contain the coefficients c_k. (In our case, the accuracy was chosen sufficiently high to guarantee that the diameters of these intervals – and, hence, the resulting error maxima – will be less than 10^{-10}.) Finally, usage of the GNU MPFR (Multiple Precision Floating-point Rounding) library [11] ensured that all relevant intervals are correctly rounded outwards.

Conceptually, the argument is again based on the geometric framework introduced in Section 2, in that the phase space of the vector field in (9) is decomposed into the inner, intermediate, and outer regions identified there. However, while the asymptotics of $\mathcal{W}^u(\widetilde{Q}^-)$ was analyzed separately in the coordinate charts corresponding to these regions, recall Section 3.1, it suffices to restrict to the rescaling chart K_2 here: as observed already in [7, Remark 9], the regularity of the transition through the intermediate region implies that the limit as $w_2 \to \infty$ in K_2 is well defined; see also Remark 2.

To approximate the manifold $\mathcal{W}^u(\widetilde{Q}^-)$ in that transition, we need to determine the coefficient functions $z_n(w, \tilde{c})$ in the expansion for $z = \zeta(w, \tilde{c}, r)$, cf. (19), where we have again omitted the subscript 2 for convenience of notation. Making use of the fact that this expansion represents a formal invariant manifold for the vector field in (20), substituting and collecting like powers of r, and recalling the expression for z_0 from (16), we obtain a recursive sequence of differential equations for the functions z_n when $n = 1, 2, 3, \ldots$:

$$\sum_{k=0}^{n} z_k z'_{n-k} - \tilde{c} w z'_{n-1} = f_n(w). \tag{34}$$

(Here, f_n is the Taylor coefficient of order n of $f(w, r)$ about $r = 0$, and the prime now denotes differentiation with respect to w.) Elementary properties of $z_n(w, \tilde{c})$, such as the fact that z_n is a polynomial of degree n in $\tilde{c}$ (with smooth, w-dependent coefficients), as well as that $z_n(0, \tilde{c}) = 0$, are easily derived from the above recursion.

Next, we integrate (34) between 0 and w to find the recursive formula

$$z_0 z_n + \frac{1}{2}\sum_{k=1}^{n-1} z_k z_{n-k} - \tilde{c} w z_{n-1} + \tilde{c}\int_0^w z_{n-1}(\omega, \tilde{c})\, d\omega = F_n(w), \tag{35}$$

where $F_n(w) = \int_0^w f_n(\omega)\, d\omega$ denotes the antiderivative of f_n. While the recursion in (35) theoretically allows us to evaluate the coefficient functions z_n, the resulting integrals are nested for $n \geqslant 2$, which implies that, in practice, these functions have to be approximated.

That approximation can be accomplished as follows: for $0 < w_0 < w_1 < \infty$, we divide the (positive) real w-axis into the three intervals $[0, w_0]$, $[w_0, w_1]$, and $[w_1, \infty)$, which correspond to the inner, intermediate, and outer regions, respectively. Since the dynamics in the inner region is highly regular, the functions $z_n(w, \tilde{c})$ can be approximated by polynomials in $(w, \tilde{c})$ when $w \in [0, w_0]$. (Similarly, the polynomial approximations for z_0, $\frac{1}{z_0}$, and $F_n(w)$ that are required for the recursion in (35) are provided by the corresponding (univariate) Taylor series expansions, truncated at sufficiently high order.) For $w \in [w_1, \infty)$, *i.e.*, in the outer region, we approximate z_0 by a bivariate polynomial in (w, e^{-w}), as $z_0 = 2w\sqrt{1-x}|_{x=(1+w)\mathrm{e}^{-w}}$; since the functions F_n that occur in (35) – as well as the corresponding integrals of these functions – are of that same form, the resulting approximation for z_n is a polynomial in $(w, \mathrm{e}^{-w}, \tilde{c})$. Finally, an approximation for z_n in the intermediate region is obtained by partitioning the interval $[w_0, w_1]$ uniformly into subintervals on which the functions z_0, $\frac{1}{z_0}$, and F_n can be replaced by their Chebyshev interpolants.

It then remains to evaluate the coefficients $\tilde{c}_k$ in the corresponding series expansion for $\tilde{c}(r)$: the discussion in Section 3 implies that $\tilde{c}$ formally solves the relation

$$\sum_{n=0}^{\infty} z_n^{\infty}(\tilde{c}) r^n \equiv \tilde{c} \quad \text{or, equivalently,} \quad \sum_{n=0}^{\infty} z_n^{\infty}\left(\sum_{k=0}^{\infty} \tilde{c}_k r^k\right) r^n \equiv \sum_{k=0}^{\infty} \tilde{c}_k r^k, \tag{36}$$

which is obtained from (19) for $w \to \infty$. (Here, we have taken into account that $\widetilde{V} \to 0$ to all orders in ε and, hence, that $Z \to \tilde{c}$, by Section 2.2.) Since z_n is approximated by a polynomial in $(w, \mathrm{e}^{-w}, \tilde{c})$ on $[w_1, \infty)$, it is elementary to extract the asymptotic coefficients z_n^{∞}; thus, for $n = 0, \ldots, 4$, one finds

$$z_0^{\infty}(\tilde{c}) = 2.0,$$

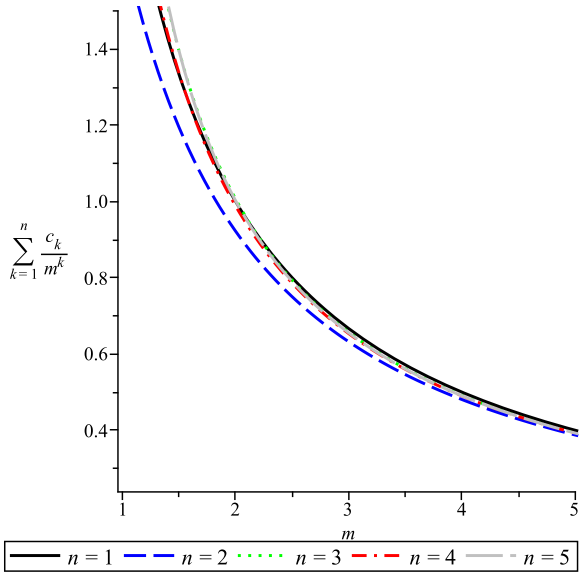


Fig. 4. The partial sums $\sum_{k=1}^{n} \frac{c_k}{m^k}$ for $n = 1, \dots, 5$ and $m \in [1, 5]$.

$$z_1^\infty(\tilde{c}) = -3.0 + 1.34405\tilde{c},$$
$$z_2^\infty(\tilde{c}) = 4.750 - 3.74689\tilde{c} + 0.95982\tilde{c}^2,$$
$$z_3^\infty(\tilde{c}) = -7.8750 + 10.68013\tilde{c} - 5.86670\tilde{c}^2 + 1.10091\tilde{c}^3,$$
$$z_4^\infty(\tilde{c}) = 13.54688 - 30.87285\tilde{c} + 30.21994\tilde{c}^2 - 13.21187\tilde{c}^3 + 2.11427\tilde{c}^4.$$

(In particular, one confirms that the function $Z \sim \sum_{n=0}^{\infty} z_n^\infty(\tilde{c})r^n$ has a divergent series expansion in r; cf. Proposition 5.) Substituting into (36), solving implicitly for $\tilde{c}_k$, and recalling that $c_k = \tilde{c}_{k-1}$, with $k = 1, 2, 3, \dots$, one obtains Table 1, as claimed.

5. Discussion

Since the proof of Theorem 1 is based on singular perturbation techniques, with $\varepsilon = m^{-1}$ as the (small) perturbation parameter, our analysis of the Gevrey properties of the critical wave speed c_{crit} for (1) is merely valid in the asymptotic limit where the polynomial degree m in $f_m(u) = 2u^m(1-u)$ tends to infinity. Correspondingly, the expansion in (5) is only guaranteed to approximate c_{crit} well for potentially very large m. Still, it follows that the smaller m is, the fewer terms in the series expansion for c_{crit} need to be considered: unlike in the theory of convergent power series, one does not automatically obtain a better approximation by including more terms in the truncation. In our case, the dependence of the optimal truncation point on the value of m is a reflection of the fact that the expansion in (5) is divergent and, specifically, Gevrey-1: the number of terms that need to be retained will (at least asymptotically) be linear in m.

The calculation of additional coefficients in (5) may seem irrelevant once it has been shown that the expansion is divergent. However, as is well known [5], divergent Gevrey-type series can display seemingly convergent behavior to very high order: while the observation that $|c_{k+2}| < |c_k|$ for $k = 1, \dots, 8$, in combination with the alternating signs of these coefficients, seems to indicate convergence, we fully expect the series expansion in (5) to diverge, and the coefficients c_k to exhibit factorial growth for k sufficiently large, as stated in (6). In that sense, our study confirms the widely known fact that the evaluation of only a few terms in a series expansion (as was done here) is no reliable indicator of its convergence properties.

Furthermore, this seeming convergence also explains why the first few truncations of (5) almost coincide – or, equivalently, why the correction that is provided by higher-order terms in the series seems negligible – for small values of m, in particular on the scale applied in [27, Fig. 3(a)]; see our Fig. 4 for comparison. Nevertheless, our analysis shows that a truncation to the least term will, in fact, be optimal; recall Section 3.3. Correspondingly, low-order truncations of (5) will approximate c_{crit} well even for relatively large m: on numerical grounds, one expects the one-term truncation $c_{\text{crit}}(m) \sim \frac{2}{m}$ to be optimal for $m \in [2, m_1)$, with $m_1 \approx 4$, in agreement with previous heuristic observations on the quality of that approximation [7,27]; similarly, it was conjectured in [7] that the two-term truncation will be optimal on some finite interval $[m_1, m_2)$, and so on. While we rigorously confirm this conjecture here, the determination of the interval endpoints m_k would require knowledge of the optimal Gevrey type – and, hence, of the (Borel) summability – of the formal series expansion in (5).

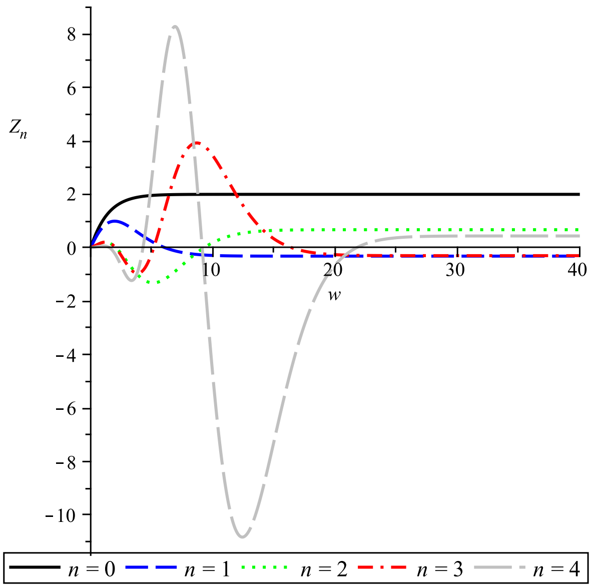


Fig. 5. The coefficient functions $Z_n(w)$ for $n = 0, \ldots, 4$ and $w \in [0, 40]$.

The convergence properties of (5) are closely related to the large-w asymptotics of the coefficient functions $z_n(w, \tilde{c})$ in the expansion for $\mathcal{W}^u(\widetilde{Q}^-)$, *i.e.*, for the function $\zeta(w, \tilde{c}, r)$: replacing $\tilde{c}$ with its formal series expansion in r and substituting into the definition of ζ in (19), we denote the w-dependent coefficients in the resulting composite expansion by $Z_n(w)$. (Here, we again suppress the subscript 2 for convenience of notation.) For illustration, we have plotted the first five of these functions in Fig. 5. Throughout, one observes large-amplitude oscillatory behavior that plateaus for some finite value of w before Z_n levels off to its asymptotic (constant) limit z_n^∞ as $w \to \infty$, in accordance with the estimates obtained in Section 3. Since these oscillations grow approximately like $(n+1)!$, cf. again Fig. 5, and since the corresponding peaks shift towards infinity (in w) as n increases, it follows that there can exist no neighborhood about infinity where all coefficient functions z_n are bounded. (Similarly, the terms $f_n(w)$ in the recursion in (34) which defines these functions are not uniformly analytic on $[0, \infty)$.) While it is surprising that the evaluation of the coefficients c_k via the implicit relation in (36) then yields a seemingly convergent series expansion for c_{crit} (to the order considered here), the underlying cause is unclear to us.

Finally, the discussion in Section 4 implies that the asymptotic coefficients $z_n^\infty(\tilde{c})$ which are obtained in the limit as $w \to \infty$ also grow approximately like $(n+1)!$, as predicted in Section 3; in particular, both z_n and z_n^∞ satisfy Gevrey-1 growth estimates, as specified in (27) and (28), respectively. Correspondingly, the expansion for $\zeta(w, \tilde{c}, r)$ in (19) is only Gevrey-1 with respect to r, with coefficient functions $z_n(w, \tilde{c})$ that are $\mathcal{C}^\infty$-smooth in w and analytic in $\tilde{c}$; recall Proposition 3. The analysis of the transition of $\mathcal{W}^u(\widetilde{Q}^-)$ through the intermediate region thus represents the cornerstone of our argument, as was also the case in [7].

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