

The evaluation of geometric queries: constraint databases and quantifier elimination

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Abstract

We model the algorithmic task of geometric elimination (e.g., quantifier elimination in the elementary field theories of real and complex numbers) by means of certain constraint database queries, called *geometric queries*. As a particular case of such a geometric elimination task, we consider *sample point queries*. We show exponential lower complexity bounds for evaluating geometric queries in the general and in the particular case of sample point queries. Although this paper is of theoretical nature, its aim is to explore the possibilities and (complexity-)limits of computer implemented query evaluation algorithms for *Constraint Databases*, based on the principles of the most advanced geometric elimination procedures and their implementations, like, e.g., the software package "Kronecker" (see [11]). *This paper is based on [14] and is only a draft specially prepared for CESSI 2006, representing work in progress of the authors. It is not aimed for publication in the present form.*

1 Introduction and summary

The framework of *constraint databases* was introduced in 1990 by Kanelakis, Kuper and Revesz [17] as a generalization of the relational database model. Here, a database consists of a finite number of generalized relations, rather than classical relations. When we consider constraint databases over the real numbers, a generalized relation is finitely represented by a Boolean

combination of polynomial equalities and inequalities over the reals. These so called *constraint formulas* finitely represent possibly infinite sets in some real space \mathbf{R}^n . Therefore, the constraint database model provides an elegant and powerful model for applications that deal with infinite sets of points in some finite dimensional real space, and is hence well-suited for modelling, e.g., *spatial databases*. For example, the spatial relation consisting of the set of points on the northern hemisphere together with the points on the equator of the unit sphere in the three-dimensional space \mathbf{R}^3 can be represented by the constraint formula $x^2 + y^2 + z^2 = 1 \wedge z \geq 0$.

The constraint model has been extensively studied by now and various logic-based query languages have been considered [18]. First-order logic over the reals, $\text{FO}(+, \times, =, <, 0, 1)$, augmented with relation names to address (generalized) relations in the input database, is the standard query language for constraint databases. As an example we may consider, for an input database, which contains just one ternary relation represented by S , the query expressed by the $\text{FO}(+, \times, =, <, 0, 1, S)$ -sentence

$$(\exists r)(\forall x)(\forall y)(\forall z)(S(x, y, z) \rightarrow x^2 + y^2 + z^2 < r^2).$$

This query expresses that the three-dimensional spatial relation S is bounded.

The standard way to evaluate this query on a particular database, e.g., the hemisphere above, consists in replacing the subexpression $S(x, y, z)$ in the query expression by the formula $x^2 + y^2 + z^2 = 1 \wedge z \geq 0$ and next to eliminate from the resulting $\text{FO}(+, \times, =, <, 0, 1)$ -formula the quantifiers that were introduced by the query expression. In our example, this adds up to eliminating the quantifiers from

$$(\exists r)(\forall x)(\forall y)(\forall z)((x^2 + y^2 + z^2 = 1 \wedge z \geq 0) \rightarrow x^2 + y^2 + z^2 < r^2),$$

which would result in the value *true*.

In this paper, we do not consider the traditional application domains of constraint databases, such as spatial databases, but focus on a completely different domain, namely *geometric elimination theory*. Hereto, we extend the constraint database model in the sense that we allow databases also to contain functions, rather than only relations. In this paper, a typical (input) database schema will be of the form

$$(R_1, \dots, R_r; F_1, \dots, F_s)$$

with relation names R_i ($i = 1, \dots, r$) and function names F_j ($j = 1, \dots, s$). The relation names are interpreted, following the context, by algebraic or

semi-algebraic sets and the function names by polynomial, or exceptionally, rational functions defined over the complex or real numbers.

The reason to include function symbols is two-fold. Firstly, functions appear naturally as byproducts of quantifier-elimination procedures and it is therefore suitable to consider them as in- or outputs of such algorithms. So, it is natural to include them if we want to model appropriately the new application domain of geometric elimination theory. Typical examples of such functions are the determinant and the resultants of systems of n homogeneous equations in n unknowns in the linear and non-linear case, respectively.

The second reason is based on a complexity argument and explains why it does not suffice to represent a k -ary function just by a $(k + 1)$ -ary relation that stores the graph of the function. When we extend the constraint database model with functions and likewise extend the first-order query language including function symbols for the representation of input functions, we are sometimes able to write queries more economically with respect to the number of quantifiers. This leads in turn to more efficient evaluation of these queries. As an example consider a input schema containing a unary function symbol F . The first-order query formula

$$y = F(F(x))$$

of $\text{FO}(+, \times, =, <, 0, 1, F)$ defines, for each interpretation of the symbol F by a unary real-valued function f , all tuples (x, y) of real numbers satisfying $y = f^2(x)$. On the other hand, if we model the function symbol F by means of the graph of f , i.e., using a database schema containing a binary relation symbol R , then the relation $y = f^2(x)$ becomes first-order expressible by the $\text{FO}(+, \times, =, <, 0, 1, R)$ -query

$$(\exists z)(R(x, z) \wedge R(z, y)),$$

which contains a quantifier. Observe that the evaluation of this query cannot be done directly, it requires the elimination of this quantifier.

First-order logic over the real or complex numbers extended with relation and function names, used to address input relations and functions, allows to define output relations in the traditional way. On the other hand, the creation of output functions requires an extension of first-order logic by special terms in order to *specify* these functions. To illustrate this, we consider the following

example. Let be given the $\text{FO}(+, \times, =, <, 0, 1, F_{11}, \dots, F_{nn}, F)$ -formula

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n F_{i1}(u_1, \dots, u_m) x_1 + \cdots + F_{in}(u_1, \dots, u_m) x_n = 0 \right) \quad (\dagger)$$

$$\wedge \bigvee_{i=1}^n x_i \neq 0 \leftrightarrow F(u_1, \dots, u_m) = 0,$$

in which the the input schema is given by the function names F_{ij} ($i, j = 1, \dots, n$), representing polynomial input functions $f_{ij}(u_1, \dots, u_m)$ and where the output schema is given by the term F representing a polynomial function $f(u_1, \dots, u_m)$. The formula (\dagger) may be interpreted as a specification of the function $f(u_1, \dots, u_m)$ by the requirement that the condition $F(u_1, \dots, u_m) = 0$ in (\dagger) reflects that the linear homogeneous equation system given by the matrix $\Phi(u_1, \dots, u_m) = (f_{ij}(u_1, \dots, u_m))_{1 \leq i, j \leq n}$ has a non-zero solution. An example function that satisfies this specification is the determinant of the matrix Φ . Another example of a function that satisfies the specification (\dagger) is the square of the determinant. In Section ??, we will discuss generalizations of this example to non-linear systems of homogeneous equations and it turns out that possible interpretations of the output functions are resultants.

We remark that the variables u_1, \dots, u_m and x_1, \dots, x_n play a different rôle in the expression (\dagger) . Therefore, we shall use the following terminology: we call u_1, \dots, u_m *parameters* and x_1, \dots, x_n *variables*. The idea behind this distinction is the following. Suppose we are given a concrete database instance $(f_{11}(u_1, \dots, u_m), \dots, f_{nn}(u_1, \dots, u_m))$ of the the database schema $(F_{11}(u_1, \dots, u_m), \dots, F_{nn}(u_1, \dots, u_m))$.¹ This instance gives rise to a new database containing a single $(m + n)$ -ary relation described by the expression

$$\bigwedge_{i=1}^n \sum_{j=1}^m f_{ij}(u_1, \dots, u_m) x_j = 0.$$

This relation can be viewed as an algebraic family, which is parameterized by u_1, \dots, u_m , of database instances consisting of a single n -ary relation in the variables x_1, \dots, x_n .

¹Or better?: Suppose we are given a concrete algebraic family of database instances $(f_{11}(u_1, \dots, u_m), \dots, f_{nn}(u_1, \dots, u_m))$, parameterized by u_1, \dots, u_m , over the the database schema (F_{11}, \dots, F_{nn}) . This way we avoid using parameterized schemas, which are little bit strange. See also next page.

The expression (†) is exemplary for the way we shall specify output functions in this paper. Thus, output functions will typically take parameters as arguments.

We also remark that specifications like (†) are no longer first-order formulas, but rather higher-order expressions. On the other hand, we may view these expressions as queries in our new constraint database model (extended with function symbols). Indeed, a (constraint database) *query* is usually defined as a partial computable function that transforms a given * constraint database instance (over a previously fixed input schema) into a new constraint database instance (over an also previously fixed output schema). Observe that the example described by the expression (†) fits in this setup. Indeed, for a given instance of the input symbols F_{ij} ($i, j = 1, \dots, n$) any instantiation of the function symbol F , satisfying (†), can be considered as an output of the corresponding query.

In the sequel, any algorithm that produces an output satisfying a specification like (†) will be considered as a query evaluation algorithm.

Once the constraint database model is extended with functions *, the constraint database formalism may be used to describe the algorithmic task of geometric elimination. Hereto, we introduce the notion of *geometric query*, which is based on the distinction between parameters and variables, considered before. The idea is to apply queries, which are specified as above, to algebraically parameterized families of input databases. Roughly speaking, a geometric query is a transformation of algebraic families of constraint databases that is parameterization independent. In this sense, the query which transforms an given input instance $(f_{11}(u_1, \dots, u_m), \dots, f_{nn}(u_1, \dots, u_m))$ of the the previous database schema $(F_{11}(u_1, \dots, u_m), \dots, F_{nn}(u_1, \dots, u_m))$ into the determinant of the matrix $\Phi = (f_{ij}(u_1, \dots, u_m))_{1 \leq i, j \leq n}$, satisfies the specification (†) and is geometric.

A central contribution of this paper is the conclusion that in the largest possible sense, any known or thinkable geometric elimination procedure can be modeled as the evaluation of a suitable geometric query. This allows us now to use *descriptive* specifications in order to formulate basic tasks of geometric elimination theory. Note that the descriptive specifications are much more flexible and problem adaptive than the traditional operative ones as, e.g., the task of the computation of a certain determinant, resultant or Chow form. In order to make the statement of this conclusion precise, we have to concretize the data structures which will be used to represent the databases which occur as inputs and outputs of geometric query evaluation

algorithms, i.e., of elimination procedures.

In view of progress made in [] concerning the complexity of geometric elimination algorithms, we shall describe the relations and functions of a given database instance by the complexity model of essentially division-free boolean-arithmetic and arithmetic circuits, respectively (see Section ?? and [] for details on this model). By the way, we observe that the classical representation of database instances by quantifier-free $\text{FO}(+, \times, =, <, 0, 1)$ -formulas (or $\text{FO}(+, \times, =, 0, 1)$ -formulas) is contained in this complexity model.

With this conceptual tool at hand, we shall be able to certify the intrinsic non-polynomial character of geometric elimination (Theorem ??). However, this complexity result does not exclude that particular elimination tasks, as, e.g., the evaluation of certain resultants, would be solvable in polynomial time. Motivated by practical applications, we shall therefore consider the particular elimination task of evaluating *sample point queries*.

The classical data model for constraint databases [18] does not support data exploration and local visualization. Indeed, a quantifier-free formula in disjunctive normal form, describing the output of a query, allows the answering of, for instance, the membership question, but it does not allow an easy² exhibition of the output, by, e.g., the production of *sample points*, or, for low dimensions, a visualization of the output. To increase the tangibility of the output, we suggest considering a new type of query that produces sample points. Furthermore, it could be desirable to support an exploration of the neighborhood of a given sample point. Both aspects, namely finding a sample point and exploration of its neighborhood, encounter a simple expression in case of rationally parameterizable algebraic or semi-algebraic varieties. Therefore, we shall extend the concept of sample point query to queries that return rationally parameterized families of polynomial functions as output. Such queries will be called *extended sample point queries*. In Section ??, we shall prove that extended sample point queries, associated to first-order formulas containing a *fixed* number of quantifier alternations, cannot be evaluated in polynomial sequential time by so-called “branching-parsimonious algorithms”. This lower bound result suggests that further research on the complexity of query evaluation in constraint database theory should be directed towards the identification of database and query classes that have a strongly improved complexity behavior. As a pattern for the de-

²Remark: By *definable choice* [23] you can produce sample points in FO. The correctness of this statement is in the easy.

velopment of such a theory, we suggest a new type of elimination algorithms which are based on the notion of system degree and use non-conventional data structures (see [2, 3, 4, 8, 10, 11, 13, 15, 16, 20, 21]).

This paper introduces a number of new concepts for constraint database theory that sometimes require certain notions from algebraic complexity theory, algebraic geometry and commutative algebra. These notions can be found in standard textbooks, such as [6] (algebraic complexity theory), [1] (commutative algebra) and [22] (algebraic geometry). The reader only interested in database issues may read this paper while skipping these technical details (and in particular the rather involved proof of Theorems ?? and ?? below).

The remainder of this paper is organized as follows.

2 Preliminaries on the constraint database model

We define the notions of constraint database schema and instance. In Section ??, we will discuss representations of database instances. The definitions of constraint database schema and instance are generalizations of the traditional definitions of constraint databases [18] that also allow polynomial functions to be included in a database.

We assume the existence of an infinite set of relation names and function names.

Definition 1 An *constraint database schema* is a finite sequence $(R_1, \dots, R_r; F_1, \dots, F_s)$ of relation names R_i ($i = 1, \dots, r$) and function names F_j ($j = 1, \dots, s$), where r and s are integers. To each relation and function name R_i and F_j , natural numbers $ar(R_i)$ and $ar(F_j)$ are associated, called the *arity* of R_i and F_j , respectively (we remark that constants are modelled by function names of arity zero). \square

Let $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ be a constraint database schema. Further on, we shall be interested in expressing queries in first-order logic over the real numbers extended with the relation and function names appearing in \mathcal{S} . We shall write $\text{FO}(+, \times, =, <, 0, 1, \mathcal{S})$ for $\text{FO}(+, \times, =, <, 0, 1, R_1, \dots, R_r; F_1, \dots, F_s)$ and $\text{FO}(+, \times, =, 0, 1, \mathcal{S})$ for $\text{FO}(+, \times, =, 0, 1, R_1, \dots, R_r; F_1, \dots, F_s)$ for these

first-order languages with relation symbols R_1, \dots, R_r and function symbols F_1, \dots, F_s .

We denote by \mathbf{R} and \mathbf{C} the sets of the real and complex numbers.

Definition 2 Let $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ be a constraint database schema. An *constraint database instance* over \mathcal{S} is a finite sequence $(A_1, \dots, A_r; f_1, \dots, f_s)$ such that A_i is a semi-algebraic subset of $\mathbf{R}^{ar(R_i)}$ or a constructible subset³ of $\mathbf{C}^{ar(R_i)}$ ($i = 1, \dots, r$) and such that f_j is a polynomial or rational function from $\mathbf{R}^{ar(F_j)}$ to \mathbf{R} or from $\mathbf{C}^{ar(F_j)}$ to \mathbf{C} ($j = 1, \dots, s$). \square

Below, we will refer to an constraint *database schema* and *instance* simply as database schema and database instance.

Data objects such as the semi-algebraic and constructible sets and polynomial functions can be modeled in various ways. In the sequel, we shall use the term *data model* to refer to a conceptual model that is used to describe data objects. Quantifier-free first-order formulas over the reals represent an example of a data model which describes semi-algebraic sets. We use the term *data structure* to refer to the actual structure that implements the data models. For example, quantifier-free formulas may be given in disjunctive normal form and the polynomials appearing in them may be given in dense or sparse representation.

It is important to remark that there are different data structures for constraint database instances and that for each particular structure there are different representations of particular instances. For example, relation and function instances can not only be finitely described by means of (sometimes unique) first-order formulas over the real numbers but also by, e.g., (*never* unique) essentially division-free *arithmetic-boolean circuits* and *arithmetic circuits*, respectively (see Section ?? and ??). For the purpose of this section, it is enough to assume some fixed data structure. When we speak about a *representation* of a database instance, it will be with respect to this fixed data structure. The reader should think that representations of database instances in this data structure are typically *not* unique.

Next, we define the notion of query.

Definition 3 Let an input schema $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ and an output schema $\tilde{\mathcal{S}} = (\tilde{R}_1, \dots, \tilde{R}_p; \tilde{F}_1, \dots, \tilde{F}_q)$ be given.

³Maybe Chevally's QE for algebraically closed fields should be mentioned. The constraint database audience is probably not familiar with that.

A query over the input schema \mathcal{S} and output schema $\tilde{\mathcal{S}}$ is a partial mapping Q that maps database instances over \mathcal{S} to database instances over $\tilde{\mathcal{S}}$. This mapping Q can also be interpreted as a series of partial mappings $(Q_{\tilde{R}_1}, \dots, Q_{\tilde{R}_p}; Q_{\tilde{F}_1}, \dots, Q_{\tilde{F}_q})$, where $Q_{\tilde{R}_i}$ ($i = 1, \dots, p$) maps database instances over \mathcal{S} to semi-algebraic or constructible subsets of $\mathbf{R}^{ar(\tilde{R}_i)}$ or $\mathbf{C}^{ar(\tilde{R}_i)}$ and where $Q_{\tilde{F}_j}$ ($j = 1, \dots, q$) maps database instances over \mathcal{S} to a polynomial or rational function from $\mathbf{R}^{ar(\tilde{F}_j)}$ to \mathbf{R} or from $\mathbf{C}^{ar(\tilde{F}_j)}$ to \mathbf{C} . We shall always suppose that queries are induced by partial mappings which map representations of input instances to representations of output instances. Hence, semantically equivalent representations of input instances (interpreted as semi-algebraic or constructible sets or polynomial or rational functions) are mapped to semantically equivalent representations of output instances. These mappings of representations of input instances or output instances always will be given implicitly by the context and we shall not specify them further. The reader may assume that these mappings are computable (in some suitable sense), but we will not rely on this fact in this paper. \square

We use the terminology *relational query* and *functional query* in case of $q = 0$ and $p = 0$, respectively.

Below, we will simply refer to these mappings as *queries* whenever the input and output schemas are clear.

We will be especially interested in relational and functional queries that are expressible in extensions of first-order logic over the real numbers, $\text{FO}(+, \times, =, <, 0, 1)$, or complex number, $\text{FO}(+, \times, =, 0, 1)$. Indeed, given an input schema $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ and an output schema $\tilde{\mathcal{S}} = (\tilde{R}_1, \dots, \tilde{R}_p; \tilde{F}_1, \dots, \tilde{F}_q)$, we can consider $\text{FO}(+, \times, =, <, 0, 1, \mathcal{S}, \tilde{\mathcal{S}})$ and $\text{FO}(+, \times, =, 0, 1, \mathcal{S}, \tilde{\mathcal{S}})$ as logics to express relations between the input and output databases or as formalisms to specify relations and functions. In particular, any formula φ with k free variables in $\text{FO}(+, \times, =, <, 0, 1, \mathcal{S})$ or $\text{FO}(+, \times, =, 0, 1, \mathcal{S})$, when evaluated on a database instance over $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ defines a k -ary output relation, when we interpret variables to range over the real or complex numbers.

The formula φ may be considered as a relational query that corresponds to an output schema $\tilde{\mathcal{S}}$ which consist of a single k -ary relation symbol.

3 Geometric elimination algorithms modeled as geometric queries

In this section, we argue by means of a number of examples of elimination problems that geometric elimination algorithms can be modeled as geometric queries (to be defined further in this section) that satisfy some precise restrictions. We also discuss the constraint formalism as a specification language.

3.1 Elimination algorithms for non-parametric elimination problems

As a first example, let us consider the family (for varying polynomial function g) of elimination problems

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n x_i^2 - x_i = 0 \wedge g(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right), \quad (\dagger_a)$$

in which u_1, \dots, u_m are considered parameters. Elimination problems of the kind (\dagger_a) are typically solved by algorithms that produce as intermediate results suitable matrices $M_{x_1}, \dots, M_{x_n} \in \mathbf{Q}^{N \times N}$, which only depend on the system $\bigwedge_{i=1}^n x_i^2 - x_i = 0$ (in case of the elimination problem (\dagger_a) , we have $N = 2^n$). These matrices M_{x_1}, \dots, M_{x_n} are defined in such a way that the parameters u_1, \dots, u_m satisfy the first-order formula (\dagger_a) if and only if they satisfy the quantifier-free formula

$$\det g(u_1, \dots, u_m, M_{x_1}, \dots, M_{x_n}) = 0. \quad (\dagger_b)$$

By the way, let us observe that the formula (\dagger_b) represents a declarative specification of the matrices M_{x_1}, \dots, M_{x_n} .

If the system $\bigwedge_{i=1}^n x_i^2 - x_i = 0$ is changed by one that defines another zero-dimensional variety, in which polynomials of degree at most d appear, then N will also change and in fact be bounded by d^n . More generally, formulas of the form

$$(\exists x_1) \cdots (\exists x_n) (\varphi(x_1, \dots, x_n) \wedge g(u_1, \dots, u_m, x_1, \dots, x_n) = 0), \quad (\dagger)$$

where $\varphi(x_1, \dots, x_n)$ defines a zero-dimensional \mathbf{Q} -definable subvariety of \mathbf{C}^n , and g is a polynomial function, form a well-known class of elimination problems. Algorithms which solve these elimination problems produce typically

matrices M_{x_1}, \dots, M_{x_n} as output. These matrices depend in size and content only on the subformula $\varphi(x_1, \dots, x_n)$ and they provide an easy way to obtain an elimination formula, namely $\det g(u_1, \dots, u_m, M_{x_1}, \dots, M_{x_n}) = 0$, for arbitrary g . These algorithms can therefore be seen as a *pre-processing* of the formula $\varphi(x_1, \dots, x_n)$ into the matrices M_{x_1}, \dots, M_{x_n} , aimed to facilitate the expression of the solution $\det g(u_1, \dots, u_m, M_{x_1}, \dots, M_{x_n}) = 0$ of the solution of the elimination problem (†).

If we want to model this class of elimination problems and their elimination polynomials using the language of constraint databases, there are several ways of doing this. Here, we start with one simple formulation and then we explain how the formalism of constraint databases can be flexibly adapted to deal with more general situations.

For instance, we may produce the elimination problem (†_a) by applying the query expressed by the following FO(+, ×, =, 0, 1, F_1, \dots, F_n, G)-formula

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n F_i(x_1, \dots, x_n) = 0 \wedge G(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right) \quad (\dagger_c)$$

over the input schema $\mathcal{S} = (F_1, \dots, F_n, G)$, to the database instance (f_1, \dots, f_n, g) given by $f_i(x_1, \dots, x_n) := x_i^2 - x_i$ ($i = 1, \dots, n$) and some polynomial $g(u_1, \dots, u_m, x_1, \dots, x_n)$.

Similarly, the more general elimination problem (†) may be produced by the query that is expressed by the FO(+, ×, =, 0, 1, R, G)-formula

$$(\exists x_1) \cdots (\exists x_n) (R(x_1, \dots, x_n) \wedge G(u_1, \dots, u_m, x_1, \dots, x_n) = 0)$$

over the input schema $\mathcal{S}' = (R, G)$, to the database instance (A, g) , where A is supposed to be the zero-dimensional \mathbf{Q} -definable subvariety of \mathbf{C}^n which is given by the FO(+, ×, =, 0, 1)-formula $\varphi(x_1, \dots, x_n)$ and where $g(u_1, \dots, u_m, x_1, \dots, x_n)$ is some polynomial. In both cases, the output schema is $\tilde{\mathcal{S}} = \{\Delta\}$, where Δ is a function symbol, with $ar(\Delta) = m$. The database instance of $\tilde{\mathcal{S}}$ which represents the output of the query, is in both cases, the polynomial $\det g(u_1, \dots, u_m, M_{x_1}, \dots, M_{x_n})$.

Another way to produce the elimination problem (†) is the following: we suppose now that $g(u_1, \dots, u_m, x_1, \dots, x_n)$ is a fixed polynomial and we consider the query Q_g expressed by the FO(+, ×, =, 0, 1, R)-formula

$$(\exists x_1) \cdots (\exists x_n) (R(x_1, \dots, x_n) \wedge g(u_1, \dots, u_m, x_1, \dots, x_n) = 0)$$

over the input schema $\mathcal{T} = (R)$.

We restrict our attention to input instances given by $\text{FO}(+, \times, =, 0, 1)$ -formulas $\varphi(x_1, \dots, x_n)$ that define zero-dimensional varieties of fixed (strictly positive) cardinality N . Consequently, the output matrices M_{x_1}, \dots, M_{x_n} have also fixed dimension N . In this way, we may view the output $(M_{x_1}, \dots, M_{x_n})$ of the underlying elimination algorithm as an instance of a schema $\tilde{\mathcal{T}} = (\tilde{F}_1, \dots, \tilde{F}_\ell)$, with $\text{ar}(\tilde{F}_j) = 0$ ($j = 1, \dots, \ell$), where $\ell = n \cdot N^2$. Here, the function names \tilde{F}_j ($j = 1, \dots, \ell$) are designed to describe the constant entries of the matrices M_{x_1}, \dots, M_{x_n} (below, we will slightly abuse the notation by regarding $(M_{x_1}, \dots, M_{x_n})$ as an instance of the schema $\tilde{\mathcal{T}} = (\tilde{F}_1, \dots, \tilde{F}_\ell)$). Observe that this defines a query Q with input schema \mathcal{T} and output schema $\tilde{\mathcal{T}}$ which maps the given \mathbf{Q} -definable input instance $A \subset \mathbf{C}^n$ of cardinality N to $Q(A) := (M_{x_1}, \dots, M_{x_n})$.

On the other hand, elimination problems like (\dagger_b) , are obtained by applying the queries \tilde{Q}_g , expressed by quantifier-free $\text{FO}(+, \times, =, 0, 1, \tilde{F}_1, \dots, \tilde{F}_\ell)$ -formulas

$$\det g(u_1, \dots, u_m, \tilde{F}_1, \dots, \tilde{F}_\ell) = 0 \quad (\dagger_d)$$

over the input schema $\tilde{\mathcal{S}} = (\tilde{F}_1, \dots, \tilde{F}_\ell)$ to the database instance containing the $\ell = n \cdot N^2$ entries of the matrices M_{x_1}, \dots, M_{x_n} as constants.

Hence, the original input database A becomes pre-processed into a new databases consisting of the n -tuple of matrices $(M_{x_1}, \dots, M_{x_n})$ in such a way that, for arbitrary g , the query Q_g applied to the input database A and the query \tilde{Q}_g applied to $(M_{x_1}, \dots, M_{x_n})$ describe the same sets. For any input instance A , we have therefore

$$Q_g(A) = \tilde{Q}_g(Q(A)).$$

We observe that the pre-processing performed by the underlying elimination algorithm can be seen as the computation of a view on the original database instance A , that allows the replacement of the query Q_g , which is defined using quantifiers, by the query \tilde{Q}_g , which can be expressed without quantifiers.

Below, we shall pay particular attention to the following variant of the family of elimination problems (\dagger_a) , namely

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n x_i^2 - x_i = 0 \wedge y = g(u_1, \dots, u_m, x_1, \dots, x_n) \right). \quad (*_a)$$

Here, u_1, \dots, u_m are considered as parameters, y as free variable and $g(u_1, \dots, u_m, x_1, \dots, x_n)$ is a suitable polynomial function. Let $M_{x_1}, \dots, M_{x_n} \in$

$\mathbf{Q}^{2^n \times 2^n}$ be the matrices introduced before. Then all previous comments remain valid, mutatis mutandis, if one replaces the determinant in the quantifier-free formula (\dagger_b) by the characteristic polynomial (in the free variable y), of the matrix

$$g(u_1, \dots, u_m, M_{x_1}, \dots, M_{x_n}).$$

3.2 Elimination algorithms for parametric elimination problems

In this section, we discuss three variations of the previous examples of elimination problem.

3.2.1 First variation

We are going to consider the following variation of the example (\dagger) of Section 3.1, where the elimination problem takes the form

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n f_i(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \wedge g(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right). \quad (\dagger')$$

Here u_1, \dots, u_m are parameters, possibly subject to some first-order definable restriction (the parameters instances that satisfy this restriction are called *admissible* for the query (\dagger')) and f_1, \dots, f_n, g are polynomials in $u_1, \dots, u_m, x_1, \dots, x_n$.

For fixed polynomial $g(u_1, \dots, u_m, x_1, \dots, x_n)$ the elimination problem (\dagger') can be produced by applying the query Q'_g , expressed by the FO(+, \times , =, 0, 1, F_1, \dots, F_n)-formula

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n F_i(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \wedge g(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right) \quad (\dagger'_c)$$

over the input schema $\mathcal{S} = (F_1, \dots, F_n)$, where now $ar(F_i) = m + n$ ($i = 1, \dots, n$), to some suitable database instance (f_1, \dots, f_n) .

For the sake of consistency and conciseness of exposition, we shall require that the input database should satisfy the following two *flatness conditions*:

- the polynomials f_1, \dots, f_n form a regular sequence in $\mathbf{Q}[u_1, \dots, u_m, x_1, \dots, x_n]$;

- for $V := \{f_1 = 0, \dots, f_n = 0\} \subset \mathbf{C}^m \times \mathbf{C}^n$, the morphism of affine varieties $\pi : V \rightarrow \mathbf{C}^m$, induced by the canonical projection $\mathbf{C}^m \times \mathbf{C}^n \rightarrow \mathbf{C}^m$, is finite.

The elimination problem (\dagger') is typically solved by algorithms that produce matrices with *rational* entries $M_{x_1}(u_1, \dots, u_m), \dots, M_{x_n}(u_1, \dots, u_m) \in \mathbf{Q}(u_1, \dots, u_m)^{N \times N}$.

We observe, that the flatness conditions satisfied by the input database (f_1, \dots, f_n) imply that the characteristic polynomial (and in particular the determinant) of the matrix $g(u_1, \dots, u_m, x_1, \dots, x_n)$ becomes a *polynomial* expression in u_1, \dots, u_m .

Therefore, if we wish to describe, in a similar way as in Section 3.1, such an elimination algorithm as a query mapping input database instances (f_1, \dots, f_n) to output database instances $(M_{x_1}(u_1, \dots, u_m), \dots, M_{x_n}(u_1, \dots, u_m))$, the constraint database model should be extended and allow an output schema that is not apriori fixed, e.g., using dynamic arrays of function names in this case.

Since our actual constraint database model is limited to fixed schemas, we are obliged to require that the input database should satisfy a third flatness condition, namely that the typical fiber of the finite morphism of affine varieties $\pi : V \rightarrow \mathbf{C}^m$ should be of cardinality N , where N is a previously fixed, strictly positive integer.

Then we may fix an output schema $\tilde{\mathcal{S}} = (\tilde{F}_1, \dots, \tilde{F}_\ell)$, as before, where $\ell = n \cdot N^2$ and $ar(\tilde{F}_j) = m$ ($j = 1, \dots, \ell$). An instance of $\tilde{\mathcal{S}}$ is now given by ℓ *rational* functions belonging to $\mathbf{Q}(u_1, \dots, u_m)$.

We call a parameter point $(\alpha_1, \dots, \alpha_m) \in \mathbf{C}^m$ admissible if the (rational) entries of $M_{x_1}(u_1, \dots, u_m), \dots, M_{x_n}(u_1, \dots, u_m)$ are well-defined in $\alpha_1, \dots, \alpha_m$. Let us finally remark that the entries of the output matrices $(M_{x_1}(u_1, \dots, u_m), \dots, M_{x_n}(u_1, \dots, u_m))$ are not arbitrary rational functions in u_1, \dots, u_m . Indeed, for every input instance (f_1, \dots, f_n) over \mathcal{S} , and every two admissible parameter instances $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_m)$ for which $f_i(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n)$ and $f_i(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n)$ ($i = 1, \dots, n$) are the same polynomials (in x_1, \dots, x_n), also $M_{x_j}(\alpha_1, \dots, \alpha_m)$ and $M_{x_j}(\alpha'_1, \dots, \alpha'_m)$ will be equal. This remark will turn out to be crucial for the lower bound results that follow in the next section and motivate us to introduce the notion of “geometric query” at the end of this section.

3.2.2 Second variation

Let f_1, \dots, f_n be polynomials in the indeterminates $u_1, \dots, u_m, x_1, \dots, x_n$ and assume that f_1, \dots, f_n are homogeneous of degrees d_1, \dots, d_n with respect to x_1, \dots, x_n . The elimination problem

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n f_i(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \wedge \bigvee_{i=1}^n x_i \neq 0 \right)$$

can be obtained by applying the query, expressed by the $\text{FO}(+, \times, =, 0, 1, F_1, \dots, F_n)$ -formula

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n F_i(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \wedge \bigvee_{i=1}^n x_i \neq 0 \right)$$

over the input schema $\mathcal{S} = \{F_1, \dots, F_n\}$ with $\text{ar}(F_i) = m + n$ ($i = 1, \dots, n$) to the database instance (f_1, \dots, f_n) . Then the output of an elimination algorithm could be the resultant of the polynomials f_1, \dots, f_n with respect to x_1, \dots, x_n , which we denote by $\text{Res}_{x_1, \dots, x_n}^{d_1, \dots, d_n}(f_1, \dots, f_n)$. Remark that parametric systems of n homogeneous linear equations in the unknowns x_1, \dots, x_n represent a particular case of this situation. In this case the resultant becomes the determinant.

We observe now that for every input instance (f_1, \dots, f_n) over $\mathcal{S} = (F_1, \dots, F_n)$ and every two parameter instances $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_m)$ for which $f_i(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n)$ and $f_i(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n)$ ($i = 1, \dots, n$) are the same polynomials, also $\text{Res}_{x_1, \dots, x_n}^{d_1, \dots, d_n}(f_1(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n), \dots, f_n(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n))$ and $\text{Res}_{x_1, \dots, x_n}^{d_1, \dots, d_n}(f_1(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n), \dots, f_n(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n))$ are the same polynomial.

3.2.3 Third variation

As a last variation on of the example of Section 3.1, let us reconsider the family (for varying function g) of elimination problems

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n x_i^2 - x_i = 0 \wedge g(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right), \quad (\dagger_a)$$

where the part $\bigwedge_{i=1}^n x_i^2 - x_i = 0$ is considered as fixed. This example of a family of zero-dimensional elimination problems may seem trivial, but in fact,

no polynomial-time algorithm is known to solve it. It turns out that this example already illustrates the intrinsic difficulty of geometric elimination. For instance, elimination algorithms which use simultaneous Newton iteration to all zeroes of a zero-dimensional equation system lead to the consideration of this kind of problem, see, e.g., [1].

The complexity of the most efficient known elimination algorithms for problem (\dagger_a) depends exponentially on n , and linearly on the circuit complexity of g .

The elimination problems (\dagger_a) can be obtained by applying the query expressed by the $\text{FO}(+, \times, =, 0, 1, G)$ -formula

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n x_i^2 - x_i = 0 \wedge G(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right) \quad (\dagger'_c)$$

over the input schema $\mathcal{S} = (G)$, with $\text{ar}(G) = m + n$, to the database instance (g) consisting of the polynomial function g . The matrices $M_{x_1}, \dots, M_{x_n} \in \mathbf{Q}^{2^n \times 2^n}$, introduced in Section 3.1, are well-defined in this case and the elimination algorithm under consideration produces as output a polynomial of the form $\det g(u_1, \dots, u_m, M_{x_1}, \dots, M_{x_n})$. So, the elimination algorithm can be modeled by a query with input schema $\mathcal{S} = (G)$ and output schema $\tilde{\mathcal{S}} = (\tilde{G})$, where $\text{ar}(\tilde{G}) = m$.

Finally, we remark for any two admissible parameter instances $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_m)$ for which $g(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n)$ and $g(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n)$ ($i = 1, \dots, n$) are the same polynomials (in x_1, \dots, x_n), also $\det g(\alpha_1, \dots, \alpha_m, M_{x_1}, \dots, M_{x_n})$ and $\det g(\alpha'_1, \dots, \alpha'_m, M_{x_1}, \dots, M_{x_n})$ are the same values.

3.3 Definition of geometric queries

The observations made earlier in Section 3 motivate the notion of geometric query, which we shall introduce below. First, we shall formalize a distinction between parameters and variables.

3.3.1 Variables versus parameters

From now on, we shall distinguish between two types of indeterminates which we refer to as *variables* on the one hand and as *parameters* on the other hand.

The difference will be reflected in the notation: we use u_1, u_2, \dots to indicate *parameters* and x_1, x_2, \dots to indicate *variables*. In this section, we shall

limit our attention to this purely syntactic distinction between variables and parameters and postpone the discussion of their (difficult) semantic meaning to Section 4.

In particular, when dealing with geometric elimination, we shall always work with an input database schema $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ where all relation and function symbols are of arity $m + n$ and which are such that the first m indeterminates refer to parameters and the last n to variables. Our notation reflects this circumstance by $R_i(u_1, \dots, u_m; x_1, \dots, x_n)$ ($i = 1, \dots, r$) and $F_j(u_1, \dots, u_m; x_1, \dots, x_n)$ ($j = 1, \dots, s$).

Let A_i be instances of R_i and f_j be instances of F_j . We shall always assume that there are $\text{FO}(+, \times, =, <, 0, 1)$ -formulas describing $A_i(u_1, \dots, u_m; x_1, \dots, x_n)$ and polynomials (in the variables x_1, \dots, x_n) over the function field $\mathbf{Q}(u_1, \dots, u_m)$, describing $f_j(u_1, \dots, u_m; x_1, \dots, x_n)$ respectively. For an admissible instance $(\alpha_1, \dots, \alpha_m)$ of the parameters u_1, \dots, u_m , we shall always require that $f_j(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n)$ is a well-defined polynomial in x_1, \dots, x_n .

Definition 4 Let $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ be a database schema and let $(A_1, \dots, A_r; f_1, \dots, f_s)$ be a database instance over \mathcal{S} .

We call two admissible instances $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_m)$ of the parameters u_1, \dots, u_m *equivalent* in the given database $(A_1, \dots, A_r; f_1, \dots, f_s)$ (or simply $(A_1, \dots, A_r; f_1, \dots, f_s)$ -*equivalent*) if the following conditions are satisfied:

- $\{(x_1, \dots, x_n) \in \mathbf{R}^n \mid A_i(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n)\} = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid A_i(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n)\}$ for all $i = 1, \dots, r$; and
- $f_j(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n)$ and $f_j(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n)$ are the same polynomials (in x_1, \dots, x_n) for all $j = 1, \dots, s$. □

3.3.2 Definition of geometric queries

Definition 5 Let $\mathcal{S} = (R_1, \dots, R_r; F_1, \dots, F_s)$ be an input database schema, in which all relations and functions are of arity $m + n$ as discussed above, and let $\tilde{\mathcal{S}} = (\tilde{R}_1, \dots, \tilde{R}_p; \tilde{F}_1, \dots, \tilde{F}_q)$ be an output schema. For the sake of simplicity of exposition, we assume that the relations and functions appearing in instances over \mathcal{S} and $\tilde{\mathcal{S}}$ depend on the same parameters u_1, \dots, u_m .

A query $Q = (Q_{\tilde{R}_1}, \dots, Q_{\tilde{R}_p}; Q_{\tilde{F}_1}, \dots, Q_{\tilde{F}_q})$ over the input schema \mathcal{S} and output schema $\tilde{\mathcal{S}}$ is called a *geometric query* if for any database instance

$(A_1, \dots, A_r; f_1, \dots, f_s)$ and any two admissible $(A_1, \dots, A_r; f_1, \dots, f_s)$ -equivalent instances $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_m)$ for the parameters u_1, \dots, u_m , the parameter instances $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_m)$ are also admissible and equivalent in the database

$$Q(A_1, \dots, A_r; f_1, \dots, f_s) = (Q_{\tilde{R}_1}(A_1, \dots, A_r; f_1, \dots, f_s), \dots, Q_{\tilde{R}_p}(A_1, \dots, A_r; f_1, \dots, f_s); Q_{\tilde{F}_1}(A_1, \dots, A_r; f_1, \dots, f_s), \dots, Q_{\tilde{F}_q}(A_1, \dots, A_r; f_1, \dots, f_s)).$$

□

In the case of arbitrary first-order expressible queries, it may occur that the parameters appearing in the relations and functions in the output schema are different from those appearing in the relations and functions of the input schema. Nevertheless, this fact does not really restrict the applicability of our notion of geometric query, since the formula that defines the given first-order query may easily be rewritten (unnested) into an equivalent one in which the input and output schema depend on the same parameters. Since we will not rely on this more general situation, we shall not go into more details.

It is clear that our notion of geometric query captures the reality of existing elimination algorithms when the query-formula only uses symbols belonging to the given database schema, as, e.g., (\dagger_c) in Section 3.1. If the query-formula contains both subexpressions without database symbols and subexpressions with database symbols, as, e.g., (\dagger'_c) in Section 3.2.1 and (\dagger''_c) in Section 3.2.3, then the notion of geometric query still has an intuitive sense for elimination algorithms. In the general case, the query-formula must possibly be rewritten, before the formal definition of geometric query may be meaningfully applied.

We end this section with some trivial examples of geometric queries. The transformation that maps the function $u_1 u_2 x_1$ to the function x_1 is an example of a geometric query, as is the identity and the transformation that maps the functions $u_1 x_1, u_2 x_1$ to $u_1 u_2 x_1$.

The transformation that maps the function $u_1 u_2 x_1$ to the function u_1 is an example of a non-geometric query. Taking $(\alpha_1, \alpha_2) := (1, 0)$ and $(\alpha'_1, \alpha'_2) = (0, 1)$ gives two equal functions for what concerns $u_1 u_2 x_1$, but not for what concerns u_1 . □

3.4 Specification of queries

3.4.1 First-order specification

First-order logic over the real or complex numbers extended with relation and function names, used to address input relations and functions, allows to define output relations in the traditional way.

As remarked in the introduction, the creation of output functions requires an extension of first-order logic by special terms in order to *specify* these functions. The $\text{FO}(+, \times, =, <, 0, 1, F_{11}, \dots, F_{nn}, F)$ -formula (\dagger) in the Introduction is a first example of such a specification. Both the determinant and the square of the determinant of any database instance of the schema (F_{11}, \dots, F_{nn}) satisfy this specification.

A less trivial example is given by the specification formula

$$\begin{aligned}
 & \left(\bigwedge_{i=1}^n (\forall z)(\forall x_1) \cdots (\forall x_n) (F_i(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right. \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. \rightarrow F_i(u_1, \dots, u_m, zx_1, \dots, zx_n) = 0 \right) \\
 & \wedge (\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n F_i(u_1, \dots, u_m, x_1, \dots, x_n) = 0 \right. \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. \wedge \bigvee_{i=1}^n x_i \neq 0 \right) \leftrightarrow \tilde{F}(u_1, \dots, u_m) = 0
 \end{aligned} \tag{\dagger}$$

over the input schema $\mathcal{S} = (F_1, \dots, F_n)$, with $ar(F_i) = m + n$ ($i = 1, \dots, n$) and output schema $\tilde{\mathcal{S}} = (\tilde{F})$ with $ar(\tilde{F}) = m$.

This formula can be interpreted as a descriptive specification of a (typically geometric) query which maps instances (f_1, \dots, f_n) over $\mathcal{S} = (F_1, \dots, F_n)$ to instances (\tilde{f}) of the output schema $\tilde{\mathcal{S}} = (\tilde{F})$.

The reader should be aware that this formula, contrary to the formula (\dagger) of the Introduction, does not apriori restrict the degree of the input polynomials with respect to the x_1, \dots, x_n .

However, fixing an input (f_1, \dots, f_n) that satisfies the left-hand side of the equivalence in the specification (\dagger), the polynomials f_i satisfy for suitable non-negative integers d_i the condition $f_i(u_1, \dots, u_m, zx_1, \dots, zx_n) = z^{d_i} f_i(u_1, \dots, u_m, x_1, \dots, x_n)$ ($i = 1, \dots, n$). As we have already observed in Section 3.2.1, for each occurrence of degrees (d_1, \dots, d_n) the notion of resultant is well defined and denoted by $\text{Res}_{x_1, \dots, x_n}^{d_1, \dots, d_n}(f_1, \dots, f_n)$. Hence if we interpret the

function symbol \tilde{F} by $\text{Res}_{x_1, \dots, x_n}^{d_1, \dots, d_n}(f_1, \dots, f_n)$, we obtain the output of a query which satisfies the given specification.

The traditional informal specification of the resultant is operative, but as illustrated here, the formalism of constraint databases allows the declarative specification of objects such as resultants and determinants.

3.4.2 Higher-order specification

Now, we consider an example of a specification of an elimination task that cannot be formulated by means of a first-order formula. We use a specification in English, based on database schemas as before.

As, we have already seen in Section 3.1 and 3.2.3, for varying polynomial function $g(u_1, \dots, u_m, x_1, \dots, x_n)$, the family of elimination problems expressed by

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n x_i^2 - x_i = 0 \wedge y = g(u_1, \dots, u_m, x_1, \dots, x_n) \right), \quad (*_a)$$

can be interpreted as the application of the query expressed by the FO(+, ×, =, <, 0, 1, G)-formula

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n x_i^2 - x_i = 0 \wedge y = G(u_1, \dots, u_m, x_1, \dots, x_n) \right) \quad (\dagger''_c)$$

over the input schema $\mathcal{S} = (G)$ to the database instance (g) containing the polynomial function g .

The canonical elimination polynomial of $(*_a)$ is

$$p(y, u_1, \dots, u_m) := \prod_{(\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n} (y - g(u_1, \dots, u_m, \varepsilon_1, \dots, \varepsilon_n)). \quad (*_e)$$

The elimination algorithms, described in the geometric elimination literature, aim at obtaining a representation of the polynomial p , such that for any given values of the parameters u_1, \dots, u_m , p can be evaluated as a polynomial function in y , e.g., by means of a division-free arithmetic circuit. More specifically, the aim is to find polynomials $\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m)$ and a polynomial

$$q(t_1, \dots, t_\ell, y) = \sum_{0 \leq j \leq 2^n} g_j(t_1, \dots, t_\ell) y^j,$$

belonging to $\mathbf{Q}[t_1, \dots, t_\ell]$ and $\mathbf{Q}[t_1, \dots, t_\ell, y]$ respectively, such that the identity

$$\begin{aligned} p(y, u_1, \dots, u_m) &= q(\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m), y) \\ &= \sum_{j=0}^{2^n} q_j(\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m)) y^j \end{aligned}$$

is satisfied.

The idea behind this elimination strategy is that of *partial evaluation*. This means that the evaluation of p with respect to y is postponed and that $p(y, u_1, \dots, u_m)$ is written, for some fixed values of u_1, \dots, u_m , as a polynomial function q , in some pre-processed parameter dependent values $\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m)$ such that, for the evaluation of p in the input y , no branchings are needed anymore.

In the language of constraint databases, this pre-processing of $g(u_1, \dots, u_m, x_1, \dots, x_n)$ into the functions $\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m)$, can be modeled as a query with input schema $\mathcal{S} = (G)$ and output schema $\tilde{\mathcal{S}} = (\tilde{\Omega}_1, \dots, \tilde{\Omega}_\ell)$, with $ar(\tilde{\Omega}_j) = m$ ($j = 1, \dots, \ell$). We remark that, as soon as u_1, \dots, u_m and ℓ are fixed, also the database schema $\tilde{\mathcal{S}}$ becomes fixed.

In the geometric elimination literature, algorithms are (implicitly or explicitly) required to be *robust* (see [7] for the definition of the notion of robustness). In the formalism of constraint databases, robustness can be specified by requiring that the query, that transforms $g(u_1, \dots, u_m, x_1, \dots, x_n)$ into $\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m)$, is a *geometric query*. This means that for any two parameter instances $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_m)$ for which $g(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n)$ and $g(\alpha'_1, \dots, \alpha'_m, x_1, \dots, x_n)$ determine the same polynomials in x_1, \dots, x_n , also $\omega_1(\alpha_1, \dots, \alpha_m) = \omega_1(\alpha'_1, \dots, \alpha'_m), \dots, \omega_\ell(\alpha_1, \dots, \alpha_m) = \omega_\ell(\alpha'_1, \dots, \alpha'_m)$ holds. Therefore we may conclude that the query that transforms the input database instance (g) into the output database instance (p) is also a geometric query. On the other hand, we observe that $\omega(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m)$ may be interpreted as parameters of a fixed division-free circuit with input y , which is represented by the polynomial q and which is independent of the input g . This circuit evaluates the polynomial $p(u_1, \dots, u_m, y)$.

The task of transforming the elimination problem ($*_a$), given by the polynomial $g(u_1, \dots, u_m, x_1, \dots, x_n)$, into a collection of intermediate polynomials $\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m)$ that satisfy a number of restrictions, can

therefore be specified using the constraint database formalism, as we have just illustrated.

We conclude this section by mentioning that we may ask how long the vector $(\omega_1(u_1, \dots, u_m), \dots, \omega_\ell(u_1, \dots, u_m))$ has to be in order to be able to express $p(u_1, \dots, u_m, y)$ by a fixed division-free arithmetic circuit with input y , which is independent of the polynomial g . As the main result of this paper, it turns out that *the notion of geometric query implies, that the number ℓ , necessarily has to be exponentially big in the number of quantified variables, i.e., $\ell \geq 2^n$.*

4 Representation of databases and algorithmic model for query evaluation

4.1 Representation of databases

The function symbols appearing in a database schema will be interpreted by polynomial functions that are well-defined in any commutative ring. Typically, the data structure to implement these functions are division-free arithmetic circuits that allow the evaluation of the function in an arbitrary commutative ring.

The relation symbols appearing in a database schema will be interpreted by relations that may be evaluated in an arbitrary commutative ring. The relations are implemented by arithmetic boolean circuits.

Section 2

These representations may be considered as implementations of a certain datatype which allows in case of function symbols the evaluation of the function in an arbitrary commutative ring and in case of relation symbols the evaluation of membership

This paper uses notions from algebraic geometry and commutative algebra. These notions can be found in standard textbooks, such as [22] and we refer to Appendix A for an overview of them.

4.2 Algorithmic models and complexity measures.

4.2.1 Data structures: essentially division-free arithmetic circuits

The algorithmic problems considered in this paper will depend on *continuous parameters* and therefore the corresponding input data structures have to contain entries for these parameters. We call them *problem* or *input parameters*. Once such a parametric problem is given, the specialization of the parameters representing input objects are called (admissible) *problem* or *input instances* and these may in principle be algebraically dependent. An algorithm solving the given problem operates on the corresponding input data structure and produces for each admissible input instance an *output instance* which belongs to a previously chosen output data structure.

The procedures or algorithms considered in this paper operate with essentially division-free arithmetic circuits as basic data structures for the representation of inputs and outputs. An *essentially division-free arithmetic circuit* (or straight-line program) is an algorithmic device that can be represented by a labeled directed acyclic graph (*dag*) as follows: the circuit depends on certain input nodes, labeled by indeterminates over the field of the rationals \mathbf{Q} . These indeterminates are thought to be subdivided in two disjoint sets, representing the *parameters* and the *variables* of the given circuit. For the sake of definiteness, let U_1, \dots, U_r be the parameters and Y_1, \dots, Y_t the variables of the circuit. Let $K := \mathbf{Q}(U_1, \dots, U_r)$ be the field of quotients of polynomials in $\mathbf{Q}[U_1, \dots, U_r]$. We call K the *parameter field* of the circuit. The circuit nodes of indegree zero which are not inputs are labeled by elements of \mathbf{Q} , which are called the *scalars* of the circuit (here “indegree” means the number of incoming edges of the corresponding node). Internal nodes are labeled by arithmetic operations (addition, subtraction, multiplication and division). We require that the internal nodes of the circuit represent polynomials in the variables Y_1, \dots, Y_t . We call these polynomials the intermediate results of the given circuit. The coefficients of these polynomials belong to the parameter field K . In order to achieve this requirement, we allow in an essentially division-free circuit only divisions which involve elements of K . Thus essentially division-free circuits do not contain divisions involving intermediate results which depend on the variables Y_1, \dots, Y_t . A circuit which contains only divisions by non-zero elements of \mathbf{Q} is called *totally division-free*.

Finally we suppose that the given circuit contains one or more nodes

which are labeled as output nodes. The results of these nodes are called *outputs* of the circuit. Output nodes may occur labeled additionally by sign marks of the form “= 0” or “ $\neq 0$ ” or may remain unlabeled. Thus the given circuit represents by means of the output nodes which are labeled by sign marks a system of parametric polynomial equations and inequations. This system determines in its turn for each admissible parameter instance a locally closed set (i.e., an embedded affine variety) with respect to the Zariski topology of the affine space \mathbf{A}^t of variable instances. The output nodes of the given circuit which remain unlabeled by sign marks represent a parametric polynomial application (in fact a morphism of algebraic varieties) which maps for each admissible parameter instance the corresponding locally closed set into a suitable affine space. We shall interpret the system of polynomial equations and inequations represented by the circuit as a *parametric family of systems* in the variables of the circuit. The corresponding varieties constitute a *parametric family of varieties*. The same point of view is applied to the morphism determined by the unlabeled output nodes of the circuit. We shall consider this morphism as a *parametric family of morphisms*.

4.2.2 Complexity models

To a given essentially division-free arithmetic circuit we may associate different complexity measures and models. In this paper we shall be mainly concerned with *sequential computing time*, measured by the *size* of the circuit. Occasionally we will also refer to *parallel time*, measured by the *depth* of the circuit. In our main complexity model is the *total* one, where we take into account *all* arithmetic operations (additions, subtractions, multiplications and possibly occurring divisions) at *unit costs*. For purely technical reasons we shall also consider two *non-scalar* complexity models, one over the ground field \mathbf{Q} and the other one over the parameter field K . In the non-scalar complexity model over K we count only the *essential* multiplications (i.e. multiplications between intermediate results which actually involve variables and not only parameters). This means that K -linear operations (i.e. additions and multiplications by arbitrary elements of K) are *cost free*. Similarly, \mathbf{Q} -linear operations are not counted in the non-scalar model over \mathbf{Q} .

Let $\theta_1, \dots, \theta_m$ be the elements of the parameter field K computed by the given circuit. Since this circuit is essentially division-free we conclude that its outputs belong to $\mathbf{Q}[\theta_1, \dots, \theta_m][Y_1, \dots, Y_t]$. Let L be the non-scalar size (over K) of the given circuit and suppose that the circuit contains q output

nodes. Then the circuit may be rearranged (without affecting its non-scalar complexity nor its outputs) in such a way that the condition

$$m = L^2 + (2t - 1)L + q(L + t + 1) \quad (1)$$

is satisfied. In the sequel we shall always assume that we have already performed this rearrangement. Let $Y := (Y_1, \dots, Y_t)$, $\theta := (\theta_1, \dots, \theta_m)$ and let $f_1, \dots, f_q \in \mathbf{Q}[\theta][Y]$ be the outputs of the given circuit. Let Z_1, \dots, Z_m be new indeterminates and write $Z := (Z_1, \dots, Z_m)$. Then there exist polynomials $F_1, \dots, F_q \in \mathbf{Q}[Z, Y]$ such that $f_1 = F_1(\theta, Y), \dots, f_q = F_q(\theta, Y)$ holds. Let us write $f := (f_1, \dots, f_q)$ and $F := (F_1, \dots, F_q)$. Consider the object class

$$\mathcal{O} := \{F(\zeta, Y) : \zeta \in \mathbf{A}^m\}$$

which we think represented by the data structure $\mathcal{D} := \mathbf{A}^m$ by means of the obvious encoding which maps each code $\zeta \in \mathcal{D}$ to the object $F(\zeta, Y) \in \mathbf{C}[Y]^q$.

For the moment, let us consider as input data structure the Zariski open subset $\mathcal{U} \subseteq \mathbf{A}^r$ where the rational map $\theta = (\theta_1, \dots, \theta_m)$ is defined. Then the given essentially division-free arithmetic circuit represents an algorithm which computes for each input code $u \in \mathcal{U}$ an output code $\theta(u)$ representing the output object $f(u, Y) = F(\theta(u), Y)$. This algorithm is in the above sense essentially division-free. From identity (1) we deduce that the size m of the data structure \mathcal{D} is closely related to the non-scalar size L of the given circuit. In particular we have the estimate

$$\sqrt{m} - (t + q) \leq L. \quad (2)$$

Later we shall meet specific situations where we are able to deduce from a previous (mathematical) knowledge of the mathematical object $f = (f_1, \dots, f_q)$ a lower bound for the size of the output data structure of *any* essentially division-free algorithm which computes for an arbitrary input code $u \in \mathcal{U}$ the object $f(u, Y)$. Of course, in such situations we obtain by means of (2) a lower bound for the non-scalar size (over K) of *any* essentially division-free arithmetic circuit which solves the same task. In particular we obtain lower bounds for the total size and for the non-scalar size over \mathbf{Q} of all such arithmetic circuits.

4.2.3 Elimination problems

Given an essentially division-free arithmetic circuit as input, an *elimination problem* consists in the task of finding an essentially division-free output

circuit which describes the Zariski closure of the image of the morphism determined by the input circuit. The output circuit and the corresponding algebraic variety are also called a *solution* of the given elimination problem. We say that a given parameter point fixes an *instance* of the elimination problem under consideration. In this sense a problem instance is described by an input and an output (or solution) instance.

In this paper we restrict our attention to input circuits which are totally division-free and contain only output nodes labelled by “=0” and unlabelled output nodes. Mostly our output circuits will also be totally division-free and will contain only one output node, labelled by the mark “=0”. This output node will always represent a canonical elimination polynomial associated to the elimination problem under consideration (see Section 4.3 for more details).

In case that our output circuit contains divisions (depending only on parameters but not on variables), we require to be able to perform these divisions for any problem instance. In order to make this requirement sound, we admit in our algorithmic model certain limit processes in the spirit of de l’Hôpital’s rule (below we shall modelise these limit processes algebraically, in terms of places and valuations). The restriction we impose on the possible divisions in an output circuit represents a *first* fundamental geometric *uniformity requirement* for our algorithmic model.

4.2.4 Parametric elimination procedures

An algorithm which solves a given elimination problem may be considered as a (geometric) elimination procedure. However this simple minded notion is too restrictive for our purpose of showing lower complexity bounds for elimination problems. It is thinkable that there exists for every individual elimination problem an efficient ad hoc algorithm, but that there is no universal way to find and to represent all these ad hoc procedures. Therefore, a *geometric elimination procedure* in the sense of this paper will satisfy certain uniformity and universality requirements which we are going to explain now.

We modelise our elimination procedures by families of *arithmetic networks* (also called arithmetic-boolean circuits) which solve entire classes of elimination problems of *arbitrary* input size. In this sense we shall require the *universality* of our geometric elimination procedures. Moreover, we require that our elimination procedures should be essentially division-free.

In a universal geometric elimination procedure, branchings and divisions

by intermediate results (that involve only parameters, but not variables) cannot be avoided. From our elimination procedures we shall require to be *parsimonious* with respect to *branchings* (and divisions). In particular we shall require that our elimination procedures do not introduce branchings and divisions for the solution of a given elimination problem when traditional algorithms do not demand this (an example of such a situation is given by the flat families of elimination problems we are going to consider below in Section 4.3). This restriction represents a *second* fundamental *uniformity requirement* for our algorithmic model.

We call a universal elimination procedure *parametric* if it satisfies our first and second uniformity requirement, i.e., if the procedure does not contain branchings which otherwise could be avoided and if all possibly occurring divisions can be performed on all problem instances, in the way we have explained before. In this paper we shall only consider parametric elimination procedures.

We call a parametric elimination procedure *geometrically robust* if it produces for any input instance an output circuit which depends only on the mathematical objects “input equation system” and “input morphism” but not on their circuit representation. We shall apply this notion only to elimination problems given by (geometrically or scheme-theoretically) flat families of algebraic varieties. This means informally that a parametric elimination procedure is geometrically robust if it produces for flat families of problem instances “continuous” solutions.

Of course, our notion of geometric robustness depends on the (geometric or scheme-theoretical) context, i.e. it is not the same for schemes or varieties. In Section 4.3 we shall explain our idea of geometric robustness in the typical situation of flat families of algebraic varieties given by reduced complete intersections.

Traditionally, the size of a system of polynomial equations (and inequations) is measured in purely extrinsic, syntactic terms (e.g. number of parameters and variables, degree of the input polynomials, size and depth of the input circuit etc). However, there exists a new generation of symbolic and numeric algorithms which take also into account intrinsic, semantic (e.g. geometric or arithmetic) invariants of the input equation system in order to measure the complexity of elimination procedure under consideration more accurately.

In this paper we shall turn back to the traditional point of view. In [7] it was shown that, under certain universality and uniformity restrictions,

no parametric elimination procedure which includes efficient computation of Zariski closures and of generically squarefree parametric greatest common divisors for circuit represented algebraic families of polynomials, is able to solve an arbitrary elimination problem in polynomial (sequential) time, if time is measured in terms of circuit size and input length is measured in syntactical terms only.

4.3 Flat families of elimination problems

4.3.1 Definitions and preliminaries

Let, as before, let $U_1, \dots, U_r, X_1, \dots, X_n, Y$ be indeterminates over \mathbf{Q} . In the sequel we shall consider X_1, \dots, X_n and Y as variables and U_1, \dots, U_r as parameters. Let G_1, \dots, G_n and F be polynomials belonging to the \mathbf{Q} -algebra $\mathbf{Q}[U_1, \dots, U_r, X_1, \dots, X_n]$. Suppose that the polynomials G_1, \dots, G_n form a regular sequence in $\mathbf{Q}[U_1, \dots, U_r, X_1, \dots, X_n]$ defining thus an equidimensional subvariety $V := \{G_1 = 0, \dots, G_n = 0\}$ of the $(r + n)$ -dimensional affine space $\mathbf{A}^r \times \mathbf{A}^n$ over the field \mathbf{C} . The algebraic variety V has dimension r . Let δ be the (geometric) degree of V (this degree does not take into account multiplicities or components at infinity). Suppose furthermore that the morphism of affine varieties $\pi : V \rightarrow \mathbf{A}^r$, induced by the canonical projection of $\mathbf{A}^r \times \mathbf{A}^n$ onto \mathbf{A}^r , is finite and generically unramified (this implies that π is flat and that the ideal generated by G_1, \dots, G_n in $\mathbf{Q}[U_1, \dots, U_r, X_1, \dots, X_n]$ is radical). Let $\tilde{\pi} : V \rightarrow \mathbf{A}^{r+1}$ be the morphism defined by $\tilde{\pi}(z) := (\pi(z), F(z))$ for any point z of the variety V . The image of $\tilde{\pi}$ is a hypersurface of \mathbf{A}^{r+1} whose minimal equation is a polynomial of $k[U_1, \dots, U_r, Y]$ which we denote by P . Let us write $\deg P$ for the total degree of the polynomial P and $\deg_Y P$ for its partial degree in the variable Y . Observe that P is monic in Y and that $\deg P \leq \delta \deg F$ holds. Furthermore, for a Zariski dense set of points u of \mathbf{A}^r , we have that $\deg_Y P$ is the cardinality of the image of the restriction of F to the finite set $\pi^{-1}(u)$. The polynomial $P(U_1, \dots, U_r, F)$ vanishes on the variety V .

Let us consider an arbitrary point $u = (u_1, \dots, u_r)$ of \mathbf{A}^r . For arbitrary polynomials $A \in \mathbf{Q}[U_1, \dots, U_r, X_1, \dots, X_n]$ and $B \in \mathbf{Q}[U_1, \dots, U_r, Y]$ we denote by $A^{(u)}$ and $B^{(u)}$ the polynomials $A(u_1, \dots, u_r, X_1, \dots, X_n)$ and $B(u_1, \dots, u_r, Y)$ which belong to $\mathbf{Q}(u_1, \dots, u_r)[X_1, \dots, X_n]$ and $\mathbf{Q}(u_1, \dots, u_r)[Y]$ respectively. Similarly we denote for an arbitrary polynomial $C \in \mathbf{Q}[U_1, \dots, U_r]$ by $C^{(u)}$ the value $C(u_1, \dots, u_r)$ which belongs to

the field $\mathbf{Q}(u_1, \dots, u_r)$. The polynomials $G_1^{(u)}, \dots, G_n^{(u)}$ define a zero dimensional subvariety $V^{(u)} := \{G_1^{(u)} = 0, \dots, G_n^{(u)} = 0\} = \pi^{-1}(u)$ of the affine space \mathbf{A}^n . The degree (cardinality) of $V^{(u)}$ is bounded by δ . Denote by $\tilde{\pi}^{(u)} : V^{(u)} \rightarrow \mathbf{A}^1$ the morphisms induced by the polynomial $F^{(u)}$ on the variety $V^{(u)}$. Observe that the polynomial $P^{(u)}$ vanishes on the (finite) image of the morphism $\tilde{\pi}^{(u)}$. Observe also that the polynomial $P^{(u)}$ is not necessarily the minimal equation of the image of $\tilde{\pi}^{(u)}$.

We call the equation system $G_1 = 0, \dots, G_n = 0$ and the polynomial F a *flat family of elimination problems depending on the parameters* U_1, \dots, U_r and we call P the associated *elimination polynomial*. An element $u \in \mathbf{A}^r$ is considered as a *parameter point* which determines a *particular problem instance* (see Section 4.2). The equation system $G_1 = 0, \dots, G_n = 0$ together with the polynomial F is called the *general instance* of the given flat family of elimination problems and the elimination polynomial P is called the *general solution* of this flat family.

The *problem instance* determined by the parameter point $u \in \mathbf{A}^r$ is given by the equations $G_1^{(u)} = 0, \dots, G_n^{(u)} = 0$ and the polynomial $F^{(u)}$. The polynomial $P^{(u)}$ is called a *solution* of this particular problem instance. We call two parameter points $u, u' \in \mathbf{A}^r$ *equivalent* (in symbols: $u \sim u'$) if $G_1^{(u)} = G_1^{(u')}, \dots, G_n^{(u)} = G_n^{(u')}$ and $F^{(u)} = F^{(u')}$ holds. Observe that $u \sim u'$ implies $P^{(u)} = P^{(u')}$. We call polynomials $A \in \mathbf{Q}[U_1, \dots, U_r, X_1, \dots, X_n]$, $B \in \mathbf{Q}[U_1, \dots, U_r, Y]$ and $C \in \mathbf{Q}[U_1, \dots, U_r]$ *invariant* (with respect to \sim) if for any two parameter points u, u' of \mathbf{A}^r with $u \sim u'$ the respective identities $A^{(u)} = A^{(u')}$, $B^{(u)} = B^{(u')}$ and $C^{(u)} = C^{(u')}$ hold.

4.3.2 Arithmetic circuits

An arithmetic circuit in $\mathbf{Q}[U_1, \dots, U_r, Y]$ with scalars in $\mathbf{Q}[U_1, \dots, U_r]$ is a totally division-free arithmetic circuit in $\mathbf{Q}[U_1, \dots, U_r, Y]$, say β , modelised in the following way: β is given by a directed acyclic graph whose internal nodes are labelled as before by arithmetic operations. There is only one input node of β , labelled by the variable Y . The other nodes of indegree zero the circuit β may contain, are labelled by arbitrary elements of $\mathbf{Q}[U_1, \dots, U_r]$. These elements are considered as the *scalars of* β . We call such an arithmetic circuit β *invariant* (with respect to the equivalence relation \sim) if all its scalars are invariant polynomials of $\mathbf{Q}[U_1, \dots, U_r]$. Considering instead of Y the variables X_1, \dots, X_n as inputs, one may analogously define the notion of an arithmetic circuit in $\mathbf{Q}[U_1, \dots, U_r, X_1, \dots, X_n]$ with scalars in $\mathbf{Q}[U_1, \dots, U_r]$

and the meaning of its invariance. However, typically we shall limit ourselves to circuits in $\mathbf{Q}[U_1, \dots, U_r, Y]$ with scalars in $\mathbf{Q}[U_1, \dots, U_r]$.

4.3.3 Geometrically robust parametric elimination problems

We are now ready to characterise in the given situation what we mean by a *geometrically robust parametric elimination procedure*. Suppose that the polynomials G_1, \dots, G_n and F are given by a totally division-free arithmetic circuit β in $\mathbf{Q}[U_1, \dots, U_r, X_1, \dots, X_n]$. A geometrically robust parametric elimination procedure accepts the circuit β as input and produces as output an *invariant* circuit Γ in $\mathbf{Q}[U_1, \dots, U_r, Y]$ with scalars in $\mathbf{Q}[U_1, \dots, U_r]$, such that Γ represents the polynomial P . Observe that in our definition of geometric robustness we did not require that β is an invariant circuit because this would be too restrictive for the modelling of concrete situations in computational elimination theory.

The invariance property required for the output circuit Γ means the following: let $u = (u_1, \dots, u_r)$ be a parameter point of \mathbf{A}^r and let $\Gamma^{(u)}$ be the arithmetic circuit in $\mathbf{Q}(u_1, \dots, u_r)[Y]$ obtained from the circuit Γ evaluating in the point u the elements of $\mathbf{Q}[U_1, \dots, U_r]$ which occur as scalars of Γ . Then the invariance of Γ means that the circuit $\Gamma^{(u)}$ depends only on the particular *problem instance* determined by the parameter point u but not on u itself. Said otherwise, a geometrically robust elimination procedure produces the solution of a particular problem instance in a way which is independent of the possibly different representations of the given problem instance.

By definition, a geometrically robust parametric elimination procedure produces always the *general* solution of the flat family of elimination problems under consideration. This means that for flat families, geometrically robust parametric elimination procedures do not introduce branchings in the output circuits. It turns out that the following meta-statement becomes true: *within the standard philosophy of commutative algebra, none of the known (exponential time) parametric elimination procedures can be improved to a polynomial time algorithm*. For this purpose it is important to remark that the known *parametric* elimination procedures (which are without exception based on linear algebra as well as on comprehensive Gröbner basis techniques) are all geometrically robust.

The invariance property of these procedures is easily verified in the situation of a flat family of elimination problems. One has only to observe that all known elimination procedures accept the input polynomials G_1, \dots, G_n and

F in their dense or sparse *coefficient representation* or as *evaluation black box* with respect to the variables X_1, \dots, X_n .

Finally let us observe that robust elimination procedures can be specified as geometric queries.

5 A lower complexity bound for evaluating geometric queries

As main result of this paper, we obtain that *for geometric FO - queries the size of the output data schema may become necessarily exponentially big in the number of quantified variables occurring in the query.*

The proof of this fact goes along the lines of Theorem below.

6 Sample point queries

6.1 Sample point queries and generalized sample point queries

6.1.1 The example of rationally parameterized families of polynomial functions.

A particular instance of interest is the case that the semi-algebraic set A is contained in \mathbf{R}^{m+n+1} and represents a rational family of polynomial functions from \mathbf{R}^n to \mathbf{R} . To be more precise, let $\pi : \mathbf{R}^{m+n+1} \rightarrow \mathbf{R}^m$ be the canonical projection of any point of \mathbf{R}^{m+n+1} on its first m coordinates. Suppose that A is non-empty and that for any $u = (u_1, \dots, u_m) \in \pi(A)$ the semi-algebraic set $(\{u\} \times \mathbf{R}^{n+1}) \cap A$ is the graph of an n -variate polynomial $f_u \in \mathbf{R}[x_1, \dots, x_n]$. It is a natural extension of our previously introduced sample-point query to ask for a procedure which enables us for each $u \in \pi(A)$ and each $x \in \mathbf{R}^n$ to compute the value of $f_u(x)$. The output of such a procedure may be a purely existential prenex first-order formula in the free variables u_1, \dots, u_m and x_1, \dots, x_n which represents for each $u \in \pi(A)$ a division-free arithmetic circuit which evaluates the polynomial f_u (observe that there exists a uniform degree bound for all these polynomials). One easily verifies that all our requirements on the semi-algebraic set A , except that of the polynomial character of the function represented by the graph $(\{u\} \times \mathbf{R}^{n+1}) \cap A$, are first-

order definable over the reals. Nevertheless, over the complex numbers, when A is a constructible (i.e., a first-order definable subset of \mathbf{C}^{m+n+1}), all these requirements are first-order expressible. This leads us to a new type of computable queries which return on input a semi-algebraic or constructible set A as above and an element $u \in \pi(A)$, a first-order formula which represents a division-free arithmetic circuit evaluating the polynomial f_u . Uniformity of query evaluation with respect to u is expressed by the requirement that the terms contained in this formula have to depend *rationally* on u .

In the following, we shall refer to this type of queries as *extended sample point queries*. We shall refer to u_1, \dots, u_m as the *parameters* and to x_1, \dots, x_n as the *variables* of the query.

Suppose now that A is a constructible subset of \mathbf{C}^{m+n+1} with irreducible Zariski-closure. Let V be the Zariski-closure of $\pi(A)$. Then V is an irreducible affine subvariety of \mathbf{C}^m . We denote the function field of V by K . It is not difficult to see that for generically chosen parameter points $u \in \pi(A)$, the extended sample point query associated to A can be realized by a greatest common divisor computation in the polynomial ring $K[x_1, \dots, x_n]$.

6.1.2 Variables versus parameters.

The previous example is motivated by spatial data that come from physical observation and are only known with uncertainty. Another motivation comes from parametric optimization. Optimization problems described in can also be studied in parametric form, i.e., in the case where the linear inequalities and the target function contain coefficients that depend on a time parameter [5] and [14]. In this case, an optimum is not an arbitrary set of sample points but an analytic (or at least continuous) function which depends on a time parameter.

We are now going to explain why we distinguished between the parameters u_1, \dots, u_m and the variables x_1, \dots, x_n in our discussion of rationally parameterized families of polynomial functions. In the example above, let $\Phi(u_1, \dots, u_m; x_1, \dots, x_m, y)$ be a quantifier-free formula which defines the semi-algebraic or constructible set A . Let us suppose that Φ contains a subformula $\Psi(u_1, \dots, u_m)$ which expresses an internal algebraic dependency between the parameters u_1, \dots, u_m . With respect to the variables x_1, \dots, x_n there is no such subformula contained in Φ . For the sake of simplicity, we shall suppose that Ψ defines the set $\pi(A)$ and that there exists a formula $\Omega(u_1, \dots, u_m; x_1,$

$\dots, x_n, y)$ such that $\Phi(u_1, \dots, u_m; x_1, \dots, x_m, y)$ can be written as

$$\Psi(u_1, \dots, u_m) \wedge \Omega(u_1, \dots, u_m; x_1, \dots, x_m, y).$$

Below we shall meet natural examples of parameterized algebraic families of polynomial functions where the formula Ψ becomes of uncontrolled size and is of few interest, whereas the formula Ω becomes the relevant part of the output information of a suitable elimination algorithm. This situation occurs for instance when the points u of \mathbf{R}^m satisfying the formula Ψ are given in parametric form (i.e., when they are image points of some polynomial or rational map coming from some affine source space). In this case, we are only interested in the subformula Ω , since points $u \in \mathbf{R}^m$ satisfying Ψ can easily be produced in sufficient quantity. In subsequent queries, x_1, \dots, x_n may appear as bounded variables, whereas the parameters u_1, \dots, u_m are not supposed to be subject to quantification. The example of the u_1, \dots, u_m expressing uncertainty in physical spatial data illustrates this. These different rôles motivate us to distinguish between u_1, \dots, u_m and x_1, \dots, x_n and to call them parameters and variables, respectively.

6.1.3 The branching-parsimonious algorithmic model.

In the model, that we are going to use in the sequel, parameters and variables receive a different treatment. (Free) variables may be specialized into arbitrary real (or complex) values, whereas the specialization of parameters may be subject to certain restrictions. In the above example of a rationally parameterized family of polynomial functions, the n -tuple of variables (x_1, \dots, x_n) may be specialized into any point of the affine space \mathbf{R}^n (or \mathbf{C}^n), whereas the m -tuple of parameters (u_1, \dots, u_m) may only be specialized into points satisfying $\Psi(u_1, \dots, u_m)$, i.e., into points belonging to $\pi(A)$. Once the n -tuple of variables (x_1, \dots, x_n) is specialized into a point of the corresponding affine space, this point cannot be modified anymore. However, we allow infinitesimal modifications of a given specialization of the m -tuple of parameters (u_1, \dots, u_m) within the domain of definition determined by the formula Ψ . In the branching-parsimonious model, we require that an arithmetic boolean circuit which represents the semi-algebraic or constructible set A does not contain divisions which involve the variables x_1, \dots, x_n . Similarly, for a given point $u \in \pi(A)$, we require that the arithmetic circuit representing the polynomial f_u is division-free. However, divisions by algebraic expressions in the parameters u_1, \dots, u_m are sometimes unavoidable (e.g., in the case of

parametric greatest common divisor computations; see [7, 9]). Therefore, we allow certain limited divisions by algebraic expressions which depend only on the parameters u_1, \dots, u_m . More precisely, we allow that the arithmetic boolean circuits representing the set A or the output of the corresponding extended sample point query computes certain, but not arbitrary, rational functions in the parameters u_1, \dots, u_m , called *scalars* of the circuit. However, we do not allow the division of a positive-degree polynomial in the variables x_1, \dots, x_n by a non-constant scalar. In the above sense, we require for our branching-parsimonious algorithmic model that arithmetic boolean circuits are *essentially division-free with respect to variables* (see [7, 9] for a precise definition).

6.1.4 Branching-free output representations of extended sample point queries.

Since we allow certain infinitesimal modifications of the parameters u_1, \dots, u_m within their domain of definition, we sometimes may replace divisions (and corresponding branchings) by limit processes in the spirit of L'Hôpital's rule. It is possible to mimic algebraically this kind of limit process by places (see [19] for the notion of place and [7, 9] for motivations of this idea).

Branchings corresponding to divisions can trivially be avoided by restricting input data. Therefore a meaningful notion of branching-parsimonious (or branching-free) algorithm requires the consideration of Zariski-closures of input data sets. This may partially explain the rather technical assumptions and tools in the following ad hoc definition of a branching-free representation of the output of an extended sample point query.

Suppose now that in the example above A is a constructible subset of \mathbf{C}^{m+n+1} with irreducible Zariski-closure B . Let V be the Zariski-closure of $\pi(A)$ in \mathbf{C}^m . Then V is an irreducible affine variety whose function field we denote by K . Moreover, the irreducible affine variety B is birationally equivalent to $V \times \mathbf{C}^n$. Suppose furthermore that $\pi(B) = V$ holds and that B represents a rationally parameterized family of polynomial functions which extends the family represented by A . Then we say that the extended sample point query associated with A admits a *branching-free output representation* if there exists an essentially division-free, single-output arithmetic circuit β with inputs x_1, \dots, x_n and scalars $\theta_1, \dots, \theta_s \in K$ satisfying the following conditions:

- (i) for any point $u \in \pi(A)$ where the rational functions $\theta_1, \dots, \theta_s$ are

defined, the division-free arithmetic circuit, obtained from β by specializing the scalars $\theta_1, \dots, \theta_s$ into the complex values $\theta_1(u), \dots, \theta_s(u)$, evaluates the polynomial f_u ;

- (ii) for any point $u \in V$ and any place $\varphi : K \rightarrow \mathbf{C} \cup \{\infty\}$ whose valuation ring extends the local ring of the affine variety V at the point u , the values $\varphi(\theta_1), \dots, \varphi(\theta_s)$ are finite and uniquely determined by u (therefore we shall write $\theta_1(u) := \varphi(\theta_1), \dots, \theta_s(u) := \varphi(\theta_s)$).

Let an arithmetic circuit β be given as above. Then we call β a branching-free representation of the output of the extended sample point query associated to A .

Observe that the output of the circuit β represents a polynomial belonging to $K[x_1, \dots, x_n]$ whose coefficients satisfy condition (ii). Moreover, the arithmetic circuit β constitutes a division-free representation of the extended sample point query associated to the Zariski-closure B of A . Finally, let us remark that for any $u \in V$, $x \in \mathbf{C}^n$ and $y \in \mathbf{C}$, the point (u, x, y) belongs to B if and only if the circuit β_u , obtained from β by replacing the scalars $\theta_1, \dots, \theta_s$ by the complex numbers $\theta_1(u), \dots, \theta_s(u)$, computes on input x the output y .

We require that a *branching-parsimonious query evaluation algorithm* produces a branching-free output representation of the given extended sample point query if the query admits such a representation.

Let us also observe that extended sample point queries appear in a natural way if we apply the constraint database concept to data processing in the context of approximation theory and functional analysis.

6.2 A lower complexity bound for evaluating sample point queries

In this section, we restrict our attention to constraint databases defined in the language $\text{FO}(+, \times, 0, 1, =)$ over the complex numbers. We shall consider two ternary relational predicates, namely $S(v_1, v_2, w)$ and $P(v_1, v_2, w)$. Our query language will therefore be $\text{FO}(+, \times, 0, 1, =, S, P)$. Let L, n be given natural numbers and let $r := (L + n + 1)^2$. For any polynomial $f \in \mathbf{C}[x_1, \dots, x_n]$, we denote by $L(f)$ the minimal non-scalar size of all division-free arithmetic circuits with inputs x_1, \dots, x_n and scalars from \mathbf{C} which evaluate the poly-

nomial f . Let

$$W_{L,n} := \{f \in \mathbf{C}[x_1, \dots, x_n] \mid L(f) \leq L\}.$$

One sees easily that all polynomials contained in $W_{L,n}$ have degree at most 2^L and that $W_{L,n}$ forms a \mathbf{Q} -definable object class which has a \mathbf{Q} -definable holomorphic encoding by the continuous data structure \mathbf{C}^r . Observe that Zariski-closure $\overline{W_{L,n}}$ of $W_{L,n}$ is a \mathbf{Q} -definable, absolutely irreducible algebraic variety consisting of the polynomials of $\mathbf{C}[x_1, \dots, x_n]$ which have approximate complexity at most L . Moreover, the affine variety $\overline{W_{L,n}}$ forms a cone in its ambient space (i.e., for any $\lambda \in \mathbf{C}$ we have $\lambda \overline{W_{L,n}} \subseteq \overline{W_{L,n}}$). For details on complexity and data structure models we refer to [6, 7].

Let z_1, \dots, z_r and y be new variables. Choose now a directed acyclic graph $\mathcal{D}_{L,n}$ representing a generic, division-free arithmetic circuit with input nodes x_1, \dots, x_n , output node y and scalar nodes z_1, \dots, z_r such that any polynomial of $W_{L,n}$ may be evaluated by the division-free arithmetic circuit obtained from $\mathcal{D}_{L,n}$ by a suitable specialization of the parameters z_1, \dots, z_r into complex values. Without loss of generality, we may assume that the number of internal nodes of $\mathcal{D}_{L,n}$ is of order $O((L+n)^2)$. Translating the structure of the directed acyclic graph $\mathcal{D}_{L,n}$ into first-order logic one infers easily a formula $\Psi_{L,n}(S, P, z_1, \dots, z_r, x_1, \dots, x_n, y)$ in the free variables $z_1, \dots, z_r, x_1, \dots, x_n, y$ of the query language $\text{FO}(+, \times, 0, 1, =, S, P)$ such that $\Psi_{L,n}(S, P)$ satisfies the following conditions.

- (i) $\Psi_{L,n}(S, P)$ is prenex, purely existential and of length $O((L+n)^2)$;
- (ii) interpreting the predicates S and P in $\Psi_{L,n}$ as the graphs of the addition and the multiplication of complex numbers, and specializing the variables z_1, \dots, z_r into the complex numbers ζ_1, \dots, ζ_r , the formula $\Psi_{L,n}(S, P, \zeta_1, \dots, \zeta_r, x_1, \dots, x_n, y)$ describes the graph of the polynomial of $\mathbf{C}[x_1, \dots, x_n]$ computed by the arithmetic circuit, obtained from $\mathcal{D}_{L,n}$ by specializing the scalars z_1, \dots, z_r into ζ_1, \dots, ζ_r .

Let $m := 4(L+n)^2 + 2$. From [7], Corollary 3 (see also [9, Lemma 4]) we deduce that there exists an identification sequence $\gamma_1, \dots, \gamma_m \in \mathbf{Q}^n$ for the object class $\overline{W_{L,n}}$. Let $\Delta_{L,n}(S, P)$ be a closed $\text{FO}(+, \times, 0, 1, =, S, P)$ -formula saying that S and P are the graphs of two binary operations which map \mathbf{C}^2 into \mathbf{C} , that $\gamma_1, \dots, \gamma_n$ is an identification sequence for the object class of applications from \mathbf{C}^n to \mathbf{C} , defined by the $\text{FO}(+, \times, 0, 1, =, S, P)$ -formula

$\Psi_{L,n}(S, P)$ and that this object class is not empty. Without loss of generality, we may assume that $\Delta_{L,n}(S, P)$ has length $O((L+n)^2)$ and is prenex with a fixed number of quantifier alternations (which is independent of L and n).

We consider now the FO(+, \times , 0, 1, =, S, P)-formulas $\Phi_{L,n}(S, P, u_1, \dots, u_r, x_1, \dots, x_n, y)$ defined by

$$(\exists z_1) \cdots (\exists z_r) (\Psi_{L,n}(S, P, z_1, \dots, z_r, x_1, \dots, x_n, y) \wedge \bigwedge_{1 \leq k \leq m} \Psi_{L,n}(S, P, z_1, \dots, z_r, \gamma_k, u_k)) \wedge \Delta_{L,n}(S, P)$$

and $\Omega_{L,n}(S, P, u_1, \dots, u_r)$ defined by

$$(\exists z_1) \cdots (\exists z_r) (\bigwedge_{1 \leq k \leq m} \Psi_{L,n}(S, P, z_1, \dots, z_r, \gamma_k, u_k)) \wedge \Delta_{L,n}(S, P).$$

Without loss of generality, we may assume that $\Phi_{L,n}(S, P)$ and $\Omega_{L,n}(S, P)$ are prenex formulas of length $O((L+n)^2)$ having a *fixed* number of quantifier alternations and containing the free variables $u_1, \dots, u_m; x_1, \dots, x_n, y$ and u_1, \dots, u_m , respectively. Let $\pi : \mathbf{C}^{m+n+1} \rightarrow \mathbf{C}^m$ be the canonical projection which maps each point of \mathbf{C}^{m+n+1} on its first m coordinates and let D be a constraint database over the schema (S, P) over the complex numbers. Suppose that D satisfies the formula $\Delta_{L,n}(S, P)$. With respect to the database D , the formula $\Phi_{L,n}(S, P, u_1, \dots, u_r, x_1, \dots, x_n, y)$ defines a non-empty constructible subset $A_{L,n}(D)$ of \mathbf{C}^{m+n+1} and the formula $\Omega_{L,n}(S, P, u_1, \dots, u_r)$ defines the set $\pi(A_{L,n}(D))$. Moreover, for any $u \in \pi(A_{L,n}(D))$, the formula $\Phi_{L,n}(S, P, u, x_1, \dots, x_n, y)$ describes the graph of a n -variate polynomial map. Therefore, it makes sense to consider, for any natural numbers n and L , the generalized sample point query associated to the formula $\Phi_{L,n}(S, P, u_1, \dots, u_r, x_1, \dots, x_n, y)$. Suppose now that there is given a *branching-parsimonious* procedure \mathcal{P} which evaluates this family of extended sample point queries. We are now going to analyze the complexity behaviour of \mathcal{P} for this query on the particular input database D , where S and P are interpreted as the graphs of the sum and the product of complex numbers.

We are now able to state and to prove the main complexity result of this paper.

Theorem 1 *Let notations and assumptions be as before. Then the branching-parsimonious procedure \mathcal{P} requires sequential time $2^{\Omega(n)}$ in order*

to evaluate on input the database D the extended sample point query associated to the size $O(n^2)$ first-order formula $\Phi_{n,n}(S, P)$. In particular, extended sample point queries associated to first-order formulas with a fixed number of quantifier alternations cannot be evaluated by branching-parsimonious procedures in polynomial time.

Proof. The arguments we are now going to use follow the general lines of the proofs of [9, Theorem 5] and [7, Theorem 4].

For the moment let us fix the integer parameters L and n .

Observe that the closed formula $\Delta_{L,n}(S, P)$ is valid on the database D . Therefore the constructible set $A_{L,n} := A_{L,n}(D)$ is nonempty.

Let $B_{L,n}$ and $V_{L,n}$ be the Zariski-closures of $A_{L,n}$ and $\pi(A_{L,n})$ in \mathbf{C}^{m+n+1} and \mathbf{C}^m , respectively.

Let $\lambda_{L,n} := \overline{W}_{L,n} \times \mathbf{C}^n \rightarrow \mathbf{C}^{m+n+1}$ and $\mu_{L,n} := \overline{W}_{L,n} \rightarrow \mathbf{C}^m$ be the morphisms of \mathbf{Q} -definable affine varieties defined for $f \in \overline{W}_{L,n}$ and $x \in \mathbf{C}^n$ by $\lambda_{L,n}(f, x) := (f(\gamma_1), \dots, f(\gamma_m), x, f(x))$ and $\mu_{L,n}(f) := (f(\gamma_1), \dots, f(\gamma_m))$. From the syntactic form of $\Phi_{L,n}(S, P)$ and $\Omega_{L,n}(S, P)$ one infers immediately that

$$\lambda_{L,n}(W_{L,n} \times \mathbf{C}^n) = A_{L,n}$$

and

$$\mu_{L,n}(W_{L,n}) = \pi(A_{L,n})$$

holds.

Therefore $B_{L,n}$ and $V_{L,n}$ are \mathbf{Q} -definable absolutely irreducible affine varieties and we may consider $\lambda_{L,n}$ and $\mu_{L,n}$ as dominant morphism mapping $\overline{W}_{L,n} \times \mathbf{C}^n$ into $B_{L,n}$ and $\overline{W}_{L,n}$ into $V_{L,n}$. Observe that $\overline{W}_{L,n}$ and $V_{L,n}$ form closed cones in their respective ambient spaces. Since $\gamma_1, \dots, \gamma_m$ were chosen as an identification sequence for the object class $\overline{W}_{L,n}$, we may conclude that $\mu_{L,n} : \overline{W}_{L,n} \rightarrow V_{L,n}$ is an injective dominant morphism of closed affine cones, which is homogeneous of degree one.

Therefore $\mu_{L,n}$ is a finite, bijective and birational morphism of affine varieties (see, e.g., [22, I.5.3 Theorem 8 and proof of Theorem 7], [9, Lemma 4] or [7, Lemma 5]).

Let $\tilde{\pi} : \mathbf{C}^{m+n+1} \rightarrow \mathbf{C}^{m+n}$ be the canonical projection which maps each point of \mathbf{C}^{m+n+1} on its first $m+n$ coordinates. Then $\tilde{\pi} \circ \lambda_{L,n} : \overline{W}_{L,n} \times \mathbf{C}^n \rightarrow V_{L,n} \times \mathbf{C}^n$ is a finite bijective and birational morphism of affine varieties and therefore $\lambda_{L,n} : \overline{W}_{L,n} \times \mathbf{C}^n \rightarrow B_{L,n}$ has the same property. This implies

$\pi(B_{L,n}) = V_{L,n}$ and that $B_{L,n}$ represents a rationally parameterized family of polynomial functions which extends the family represented by $A_{L,n}$.

Let $K_{L,n}$ the function field over \mathbf{C} of the absolutely irreducible variety $V_{L,n}$ and let $R_{L,n}$ be the \mathbf{C} -algebra of all rational functions θ of $K_{L,n}$ such that for any point $u \in V_{L,n}$ and any place $\varphi : K_{L,n} \rightarrow \mathbf{C} \cup \{\infty\}$ whose valuation ring extends the local ring of the affine variety $V_{L,n}$ at the point u , the value $\varphi(\theta)$ is finite and uniquely determined by u . Thus, for $u \in V_{L,n}$ and $\varphi : K_{L,n} \rightarrow \mathbf{C} \cup \{\infty\}$ as above, we may associate to any polynomial $f := \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in R_{L,n}[x_1, \dots, x_n]$ the polynomial $f(u, x_1, \dots, x_n) := \sum a_{i_1 \dots i_n}(u) x_1^{i_1} \dots x_n^{i_n} := \sum \varphi(a_{i_1 \dots i_n}) x_1^{i_1} \dots x_n^{i_n}$, which belongs to $\mathbf{C}[x_1, \dots, x_n]$.

Since $\mu_{L,n} : \overline{W}_{L,n} \rightarrow V_{L,n}$ is a finite, bijective and birational morphism of affine varieties, we conclude that $R_{L,n}$ contains the coordinate ring of the affine variety $\overline{W}_{L,n}$ (see e.g. [19]). This implies that there exists a polynomial $f_{L,n} \in R_{L,n}[x_1, \dots, x_n] \subset K[x_1, \dots, x_n]$ with the following property: for any $u \in V_{L,n}$, $x \in \mathbf{C}^n$ and $y \in \mathbf{C}$, the point (u, x, y) belongs to $B_{L,n}$ if and only if $f_{L,n}(u, x) = y$ holds.

A branching-free output representation of the extended sample point query associated to the constructible set $A_{L,n}$ can now easily be realized by any arithmetic circuit which first computes all monomial terms of the polynomial $f_{L,n}$ and finally sums them up.

Therefore, the given branching-parsimonious query evaluation procedure \mathcal{P} produces on input consisting of the database D and the formula $\Phi_{L,n}(S, P)$ a branching-free representation of the extended sample point query associated to $A_{L,n}$. This branching-free representation is realized by an essentially division-free single-output arithmetic circuit $\beta_{L,n}$ with inputs x_1, \dots, x_n and scalars $\theta_1^{(L,n)}, \dots, \theta_{s_{L,n}}^{(L,n)}$ belonging to $R_{L,n}$ such that $\beta_{L,n}$ computes at its output the polynomial $f_{L,n} \in R_{L,n}[x_1, \dots, x_n]$.

Let t and ℓ_1, \dots, ℓ_n be new variables and let us now consider the polynomial $g_n := t \prod_{1 \leq i \leq n} (\ell_i + x_i)$ defining the constructible object class

$$\Gamma_n := \left\{ \tau \prod_{1 \leq i \leq n} (\lambda_i + x_i) \mid \tau, \lambda_1, \dots, \lambda_n \in \mathbf{C} \right\}$$

of n -variate complex polynomial functions.

Observe that each element of Γ_n has nonscalar sequential time complexity at most n .

Therefore the Zariski-closure $\overline{\Gamma}_n$ of the object class Γ_n is contained in $\overline{W}_{n,n}$.

Observe that $\bar{\Gamma}_n$ is an absolutely irreducible, \mathbf{Q} -definable affine variety. Since $\mu_{n,n} : \bar{W}_{n,n} \rightarrow V_{n,n}$ is a finite morphism of irreducible affine varieties, we conclude that $C_n := \mu_{n,n}(\bar{\Gamma}_n)$ is an absolutely irreducible, \mathbf{Q} -definable closed affine subvariety of $V_{n,n}$.

Let E_n and L_n be the coordinate ring and the rational function field over \mathbf{C} of the absolutely irreducible affine variety C_n .

Observe that we may identify E_n with $\mathbf{C}[g_n(t, \ell_1, \dots, \ell_n, \gamma_1), \dots, g_n(t, \ell_1, \dots, \ell_n, \gamma_m)]$ and L_n with $\mathbf{C}(g_n(t, \ell_1, \dots, \ell_n, \gamma_1), \dots, g_n(t, \ell_1, \dots, \ell_n, \gamma_m))$. Therefore we may consider E_n as a \mathbf{C} -subdomain of the polynomial ring $\mathbf{C}[t, \ell_1, \dots, \ell_n]$ and L_n as a \mathbf{C} -subfield of $\mathbf{C}(t, \ell_1, \dots, \ell_n)$.

The rational functions $\theta_1^{(n)}, \dots, \theta_{s_{n,n}}^{(n)}$ of the affine variety $V_{n,n}$ may be not defined on the subvariety C_n . Nevertheless, since they belong to the \mathbf{C} -algebra $R_{n,n}$, one verifies easily that there exist rational functions $\sigma_1^{(n)}, \dots, \sigma_{s_{n,n}}^{(n)}$ of the affine variety C_n satisfying the following condition: for any point $u \in C_n$ and any place $\psi : L_n \rightarrow \mathbf{C} \cup \{\infty\}$ whose evaluation ring extends the local ring of C_n at the point u , the values of ψ at $\sigma_1^{(n)}, \dots, \sigma_{s_{n,n}}^{(n)}$ are given by $\psi(\sigma_1^{(n)}) = \theta_1^{(n,n)}(u), \dots, \psi(\sigma_{s_{n,n}}^{(n)}) = \theta_{s_{n,n}}^{(n,n)}(u)$ and therefore finite and uniquely determined by u .

In particular, the rational functions $\sigma_1^{(n)}, \dots, \sigma_{s_{n,n}}^{(n)} \in L_n$ are integral over the \mathbf{C} -algebra E_n and hence contained in the polynomial ring $\mathbf{C}[t, \ell_1, \dots, \ell_n]$ (see, e.g., [19]). Therefore we may consider $\sigma_1^{(n)}, \dots, \sigma_{s_{n,n}}^{(n)}$ as polynomials in the variables t and ℓ_1, \dots, ℓ_n (i.e. as elements of $\mathbf{C}[t, \ell_1, \dots, \ell_n]$).

From $g_n = t \prod_{1 \leq i \leq n} (\ell_i + x_i)$, we infer the identities

$$g_n(0, \ell_1, \dots, \ell_n, \gamma_1) = 0, \dots, g_n(0, \ell_1, \dots, \ell_n, \gamma_m) = 0.$$

Since the polynomials $\sigma_1^{(n)}, \dots, \sigma_{s_{n,n}}^{(n)}$ depend integrally from $g_n(t, \ell_1, \dots, \ell_n, \gamma_m), \dots, g_n(t, \ell_1, \dots, \ell_n, \gamma_m)$, we deduce now easily that the polynomials $\sigma_1^{(n)}(0, \ell_1, \dots, \ell_n), \dots, \sigma_{s_{n,n}}^{(n)}(0, \ell_1, \dots, \ell_n)$ do not depend on the variables ℓ_1, \dots, ℓ_n , i.e., they belong to \mathbf{C} .

Let now β_n be the division-free arithmetic circuit with scalars in $\mathbf{C}[t, \ell_1, \dots, \ell_n]$ obtained by replacing in the circuit $\beta_{n,n}$ the scalars $\theta_1^{(n,n)}, \dots, \theta_{s_{n,n}}^{(n,n)}$ by the polynomials $\sigma_1^{(n)}, \dots, \sigma_{s_{n,n}}^{(n)}$.

One verifies easily that the circuit $\tilde{\beta}_n$ computes the polynomial

$$g_n = t \prod_{1 \leq i \leq n} (\ell_i + x_i) = \sum_{\substack{\delta_1, \dots, \delta_n, \varepsilon_1, \dots, \varepsilon_n \in \{0,1\} \\ \delta_1 + \varepsilon_1 = 1, \dots, \delta_n + \varepsilon_n = 1}} t \ell_1^{\delta_1} \dots \ell_n^{\delta_n} x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}.$$

Let $v_1, \dots, v_{s_{n,n}}$ be new variables. From the directed acyclic graph structure of $\tilde{\beta}_n$ (or $\beta_{n,n}$) one deduces immediately that for each $(\delta_1, \dots, \delta_n) \in \{0, 1\}^n$ there exists a polynomial $Q_{(\delta_1, \dots, \delta_n)}^{(n)} \in \mathbf{Q}[v_1, \dots, v_{s_{n,n}}]$ satisfying the condition $Q_{(\delta_1, \dots, \delta_n)}^{(n)}(\sigma_1^{(n)}, \dots, \sigma_{s_{n,n}}^{(n)}) = t \ell_1^{\delta_1} \dots \ell_n^{\delta_n}$. Let $Q_n : \mathbf{C}^{s_{n,n}} \rightarrow \mathbf{C}^{2^n}$ the polynomial map defined by $Q_n := (Q_{(\delta_1, \dots, \delta_n)}^{(n)}; (\delta_1, \dots, \delta_n) \in \{0, 1\}^n)$.

Consider now an arbitrary integer $1 \leq \rho \leq 2^n$ and let $\lambda_{\rho,1} := \rho^{2^0}, \dots, \lambda_{\rho,n} := \rho^{2^{n-1}}$, $\lambda_\rho := (\lambda_{\rho,1}, \dots, \lambda_{\rho,n})$ and $\alpha_\rho^{(n)} : \mathbf{C} \rightarrow \mathbf{C}^{s_{n,n}}$ and $\beta_\rho^{(n)} : \mathbf{C} \rightarrow \mathbf{C}^{2^n}$ be the parameterized algebraic curves defined for $\tau \in \mathbf{C}$ by

$$\alpha_\rho^{(n)}(\tau) := (\sigma_1^{(n)}(\tau, \lambda_\rho), \dots, \sigma_{s_{n,n}}^{(n)}(\tau, \lambda_\rho))$$

and

$$\beta_\rho^{(n)}(\tau) := (\tau \lambda_{\rho,1}^{\delta_1} \dots \lambda_{\rho,n}^{\delta_n}; (\delta_1, \dots, \delta_n) \in \{0, 1\}^n) = (\tau \rho^j; 0 \leq j < 2^n).$$

Observe that the functional identity

$$\beta_\rho^{(n)} = Q_n \circ \alpha_\rho^{(n)} \tag{3}$$

is valid.

Since the polynomials $\sigma_1^{(n)}(0, \ell_1, \dots, \ell_n), \dots, \sigma_{s_{n,n}}^{(n)}(0, \ell_1, \dots, \ell_n)$ do not depend on the variables ℓ_1, \dots, ℓ_n , there exists a point $a_n \in \mathbf{C}^{s_{n,n}}$, independent on ρ , such that $\alpha_\rho^{(n)}(0) = a_n$ holds.

Let us denote the derivatives of $\alpha_\rho^{(n)}$, $\beta_\rho^{(n)}$ and Q_n by $\frac{d\alpha_\rho^{(n)}}{d\tau}$, $\frac{d\beta_\rho^{(n)}}{d\tau}$ and DQ_n . Furthermore let $\omega_\rho^{(n)} := \frac{d\alpha_\rho^{(n)}}{d\tau}(0) \in \mathbf{C}^{s_{n,n}}$, let $\eta_n : \mathbf{C}^{s_{n,n}} \rightarrow \mathbf{C}^{2^n}$ be the \mathbf{C} -linear map defined by $\eta_n := (DQ_n)(a_n)$ and observe that $\frac{d\beta_\rho^{(n)}}{d\tau}(0) = (\rho^j; 0 \leq j < 2^n)$ holds. Applying the chain rule to (3), we infer the following identities:

$$\begin{aligned} (\rho^j; 0 \leq j < 2^n) &= \frac{d\beta_\rho^{(n)}}{d\tau}(0) = (DQ_n)(\alpha_\rho^{(n)}(0))\left(\frac{d\alpha_\rho^{(n)}}{d\tau}(0)\right) \\ &= (DQ_n)(a_n)(\omega_\rho^{(n)}) = \eta_n(\omega_\rho^{(n)}). \end{aligned}$$

Since $(\rho^j)_{1 \leq \rho \leq 2^n, 0 \leq j < 2^n}$ is a nonsingular Vandermonde matrix, we conclude now that the image of the \mathbf{C} -linear map $\eta_n : \mathbf{C}^{s_{n,n}} \rightarrow \mathbf{C}^{2^n}$ contains 2^n linear independent points. Therefore η_n is surjective. This implies $s_{n,n} \geq 2^n$.

Therefore the arithmetic circuit $\beta_{n,n}$, which represents the output produced by the procedure \mathcal{P} on input consisting of the database D and the formula $\Phi_{n,n}$, contains at least 2^n scalars.

This implies that the nonscalar size of the circuit $\beta_{n,n}$ is at least $2^{\frac{n}{2}} - n - 1$. In conclusion, the procedure \mathcal{P} requires $2^{\Omega(n)}$ sequential time in order to produce the output $\beta_{n,n}$ on input consisting of the data base D and the size $O(n^2)$ formula $\Phi_{n,n}$. \square

The main outcome of Theorem 1 and its proof can be paraphrased as follows: *constraint database theory applied to quite natural computation tasks, as, e.g., branching-parsimonious interpolation of low complexity polynomials, leads necessarily to non-polynomial sequential time lower bounds.*

In view of the $P_{\mathbf{R}} \neq NP_{\mathbf{R}}$ conjecture in the algorithmic model of Blum–Shub–Smale over the real and complex numbers, it seems unlikely that this worst case complexity behavior can be improved substantially if we drop some or all of our previously introduced requirements on queries and their output representations. Nevertheless we wish to stress that these requirements constitute a fundamental technical ingredient for the argumentation in the proof of Theorem 1.

7 Conclusion and future research on the complexity of query evaluation

In this paper, we have emphasized the importance of *data structures* and their effect on the complexity of quantifier elimination.

However, the intrinsic inefficiency of quantifier-elimination procedures represents a bottle-neck for real-world implementations of constraint database systems. As we have argued, it is unlikely that constraint database systems that are based on general purpose quantifier-elimination algorithms will ever become efficient. Also, restriction to work with linear data, as in most existing constraint database systems [18, Part IV], will also not lead to more efficiency. A promising direction is the study of a concept like the *system degree*, that has shown to be a fruitful notion for the complexity analysis of quantifier elimination in elementary geometry and has been implemented in the software package (polynomial equation solver "Kronecker", see [11]). In the context of query evaluation in constraint databases, the notion of system degree is still unsatisfactory since it is determined both by the query formula and the quantifier-free formulas describing the input database relations. It is a task for future constraint database research to develop a well-adapted complexity invariant in the spirit of the system degree in elimination theory.

Another direction of research is the study of query evaluation for first-order languages that capture certain genericity classes. For example, the first-order logic FO(**between**) has point variables rather than being based on real numbers and it captures the fragment of first-order logic over the reals that expresses queries that are invariant under affine transformations of the ambient space [12]. Although a more efficient complexity of query evaluation in this language cannot be expected, it is interesting to know whether languages such as FO(**between**) have quantifier elimination themselves (after an augmentation with suitable predicates).

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Appendix A: Language and tools from algebraic geometry.

Let k be the field \mathbf{Q} , \mathbf{R} or \mathbf{C} . We assume k to be “effective” with respect to arithmetic operations as addition/subtraction, multiplication/division. Let \bar{k} be an algebraically closed field containing k (in the sequel we shall call such a field *an algebraic closure of k*).

We denote by \mathbf{N} the set of natural numbers and by \mathbf{Z}^+ the set of non-negative integers.

Fix $n \in \mathbf{Z}^+$ and let X_0, \dots, X_n be indeterminates over k . We denote by $\mathbf{A}^n := \mathbf{A}^n(\bar{k})$ the n -dimensional affine space and by $\mathbf{P}^n := \mathbf{P}^n(\bar{k})$ the n -dimensional projective space over \bar{k} . The spaces \mathbf{A}^n and \mathbf{P}^n are thought to be endowed with their respective Zariski topologies over k and with their respective sheaves of k -rational functions with values in \bar{k} . Thus the points of \mathbf{A}^n are elements (x_1, \dots, x_n) of \bar{k} and the points of \mathbf{P}^n are (non uniquely) represented by non-zero elements (x_0, \dots, x_n) of \bar{k}^{n+1} and denoted by $(x_0 : \dots : x_n)$. The indeterminates X_1, \dots, X_n are considered as the coordinate functions of the affine space \mathbf{A}^n . The coordinate ring (of polynomial functions) of \mathbf{A}^n is identified with the polynomial ring $k[X_1, \dots, X_n]$. Similarly we consider the (graded) polynomial ring $k[X_0, \dots, X_n]$ as the projective coordinate ring of \mathbf{P}^n . Consequently we represent rational functions of \mathbf{P}^n as quotients of homogeneous polynomials of equal degree belonging to $k[X_0, \dots, X_n]$. Let F_1, \dots, F_s be polynomials which belong to $k[X_1, \dots, X_n]$ or are homogeneous and belong to $k[X_0, \dots, X_n]$. We denote by $\{F_1 = 0, \dots, F_s = 0\}$ or $V(F_1, \dots, F_s)$ the algebraic set of common zeroes of the polynomials F_1, \dots, F_s in \mathbf{A}^n and \mathbf{P}^n respectively. We consider the set $V := \{F_1 = 0, \dots, F_s = 0\}$ as (Zariski-)closed (affine or projective) subvariety of its ambient space \mathbf{A}^n or \mathbf{P}^n and call V the affine or projective variety defined by the polynomials F_1, \dots, F_s . We think the variety V to be equipped with the induced Zariski topology and its sheaf of rational functions. The irreducible components of V are defined with respect to its Zariski topology over k . We call V irreducible if V contains a single irreducible component and equidimensional if all its irreducible components have the same dimension. The dimension $\dim V$ of the variety V is defined as the maximal dimension of all its irreducible components. If V is equidimensional we define its (geometric) degree as the number of points arising when we intersect V with $\dim V$ many generic (affine) linear hyperplanes of its ambient space

\mathbf{A}^n or \mathbf{P}^n . For an arbitrary closed variety V with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_t$ we define its degree as $\deg V := \deg \mathcal{C}_1 + \dots + \deg \mathcal{C}_t$. With this definition of degree the intersection of two closed subvarieties V and W of the same ambient space satisfies the Bézout inequality

$$\deg V \cap W \leq \deg V \deg W$$

We denote by $k[V]$ the affine or (graded) projective coordinate ring of the variety V . If V is irreducible we denote by $k(V)$ its field of rational functions. In case that V is a closed subvariety of the affine space \mathbf{A}^n we consider the elements of $k[V]$ as \bar{k} -valued functions mapping V into \bar{k} . The restrictions of the projections X_1, \dots, X_n to V generate the coordinate ring $k[V]$ over k and are called the coordinate functions of V . The data of n coordinate functions of V fixes an embedding of V into the affine space \mathbf{A}^n . Morphisms between affine and projective varieties are induced by polynomial maps between their ambient spaces which are supposed to be homogeneous if the source and target variety is projective.

Replacing the ground field k by its algebraic closure \bar{k} , we may apply all this terminology again. In this sense we shall speak about the Zariski topologies and coordinate rings over \bar{k} and sheaves of \bar{k} -rational functions. In this more general context varieties are defined by polynomials with coefficients in \bar{k} . If we want to stress that a particular variety V is defined by polynomials with coefficients in the ground field k , we shall say that V is *k-definable* or *k-constructible*. The same terminology is applied to any set determined by a (finite) boolean combination of *k-definable* closed subvarieties of \mathbf{A}^n or \mathbf{P}^n . By a *constructible* set we mean simply a \bar{k} -constructible one. Constructible and \bar{k} -constructible sets are always thought to be equipped with their corresponding Zariski topology. In case of $k := \mathbf{Q}$ and $\bar{k} := \mathbf{C}$ we shall sometimes also consider the *euclidean* (i.e. “*strong*”) topology of \mathbf{A}^n and \mathbf{P}^n and their constructible subsets.

The rest of our terminology and notation of algebraic geometry and commutative algebra is standard and can be found in [19], [22], and in [1].