The Gelfand-Kirillov conjecture for semi-direct products of Lie algebras

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Abstract. Let g be an n-dimensional Lie algebra over a field k of characteristic zero and let W be a g-module such that $\dim W \geq n$. Sufficient conditions are given in order for the semi-direct product $g \oplus W$ to satisfy the Gelfand-Kirillov conjecture. This implies that this conjecture holds for an important class of Frobenius Lie algebras. Special attention is devoted to the case where $g = sl(2, k)$.

1. Introduction

Let L be a finite dimensional Lie algebra over a field k of characteristic zero, with basis $\{y_1, \ldots, y_s\}$. Let $U(L)$ be its enveloping algebra and let $D(L)$ be the quotient division ring of $U(L)$ with center $Z(D(L))$. Let $R(L)$ be the quotient field of the symmetric algebra $S(L)$. Denote by $i(L)$ the index of L, for which the following formula holds [D, 1.14.13].

$$i(L) = \dim L - \mathrm{rank}_{R(L)}([y_i, y_j])$$

and which coincides with the transcendence degree of $Z(D(L))$ over k if L is algebraic [RV, 4.6], [O1, p. 72].
In 1966, Gelfand and Kirillov formulated the following conjecture and settled it for nilpotent Lie algebras, $gl(n)$ and $sl(n)$ [GK1].

The GK-conjecture: Let L be algebraic and let k be algebraically closed, then $D(L)$ is isomorphic to a Weyl skew field $D_n(F)$ over a purely transcendental extension F of k.

Later on, they established this also for L semi-simple, but only for an extension

of the center $Z(D(L))$ [GK2].
In 1973, three separate proofs were given for the solvable case by Borho [BGR], Joseph [J] and McConnell [M]. Concerning the mixed case, Nghiem treated in [N] the semi-direct products $sl(n)$, $sp(2n)$ and $so(n)$ with their standard representation and proved the GK-conjecture for these.
In 1996, Alev, Van den Bergh and the author presented a family of counterexamples in [AOV1], focusing on semi-direct products of the Lie algebra of a nonspecial group with a representation admitting a trivial generic stabilizer. For example, the 9-dimensional semi-direct product of $sl(2)$ with two copies of the adjoint representation (the nonspecial group to be considered here is $PSL(2)$). This happens to be the smallest counterexample since the GK-conjecture holds in lower dimensions [AOV2].

In this paper we continue to consider semi-direct products. The following is our main result:

Theorem 1.1. Let g be an n-dimensional Lie algebra over a field k of characteristic zero and let W be a g-module such that $\dim W \geq n$. Put $K = R(W)$, the quotient field of the symmetric algebra $S(W)$, and let K^g be the subfield of invariants under the action of g. Consider the semi-direct product $L = g \oplus W$. We assume that:

(i) $g(f) = 0$ for some $f \in W^*$ (where $g(f)$ is the stabilizer of f, consisting of all $x \in g$ such that $f(xw) = 0$ for all $w \in W$)

(ii) K is a purely transcendental extension of K^g

(iii) $tr \deg_{K^g}(K) = n$

Then $D(L)$ is isomorphic to the Weyl skew field $D_n(F)$, where $F = Z(D(L)) = K^g$. If in addition the extension F/k is also rational then L satisfies the GK-conjecture.

Note that g is not assumed to be algebraic and also that k need not be algebraically closed. However, if g is algebraic, then so is L and in that case (iii) is an immediate consequence of (i) (Remark 2.3).
Theorem 1.1, which generalizes Corollary 2.3(1) of [AOV1], has some interesting consequences. In particular, it implies that an important class of Frobenius Lie algebras satisfy the GK-conjecture. Its proof is straightforward and provides a method for the explicit computation of the Weyl generators of $D(L)$. In section 3 this procedure

is applied to the 8-dimensional semi-direct product of $sl(2)$ with $W_2 \bigoplus W_1$ (where W_n is the $(n+1)$-dimensional irreducible $sl(2)$-module). Finally, section 4 is devoted to the semi-direct product of $sl(2)$ with W_n for $n \geq 5$ (k algebraically closed). This satisfies the GK-conjecture if and only if n is odd. In particular, $sl(2) \bigoplus W_6$ is a 10-dimensional counterexample to GK.

2. Proof of the main theorem and its consequences

Let g, W, L etc... be as above. In particular $L = g \bigoplus W$ is the semi-direct product of g with W in which $[x, w] = xw$, $x \in g$, $w \in W$ and in which W is an abelian ideal. Let $\{x_1, \ldots, x_n\}$ be a basis of g and let $\{e_1, \ldots, e_m\}$ be a basis of W. We now recall the following [O3, p. 708]:

Proposition 2.1 The following are equivalent:

1. $g(f) = 0$ for some $f \in W^*$
2. $\mathrm{rank}_K([x_i, e_j]) = n$
3. $i(L) = \dim W - \dim g$
4. W is a commutative polarization of L
5. K is a maximal subfield of $D(L)$

Moreover, if these conditions are satisfied then W is a faithful g-module and $Z(D(L)) = K^g$.

Remark 2.2 If k is algebraically closed, g a simple Lie algebra, acting irreducibly on W then the conditions of the proposition are satisfied if and only if $\dim g < \dim W$ [AVE]. See also [R, p. 196].

Proof of Theorem 1.1 Put $F = K^g = Z(D(L))$.
By assumption, we can find $q_1, \ldots, q_n \in K$, algebraically independent over F, such that $K = F(q_1, \ldots, q_n)$.
Next, we claim that the matrix

$$A = ([x_i, q_j]) \in K^{n \times n}$$

is invertible.
Since $e_j \in K$ there exists $f_j \in F(X_1, \ldots, X_n)$ such that $e_j = f_j(q_1, \ldots, q_n)$.
Then,

$$[x_i, e_j] = \sum_{s=1}^{n} [x_i, q_s] \frac{\partial f_j}{\partial q_s}$$

for all $i : 1, \ldots, n$ and $j : 1, \ldots, m$.
Therefore, we have the following equality of matrices:

$$([x_i, e_j]) = \underbrace{([x_i, q_s])}_{A} \left(\frac{\partial f_j}{\partial q_s} \right)$$

On the left hand side we have an $n \times m$ matrix of rank n. Consequently, $A \in K^{n \times n}$ is also of rank n, establishing our claim. Let $B = (b_{js}) \in K^{n \times n}$ be the inverse of A and put

$$p_j = \sum_{s=1}^{n} b_{js} x_s \quad j : 1, \ldots, n$$

Note that: $p_j \in g_K = K \otimes_k g$.
It follows that

$$\sum_{j=1}^{n} [x_i, q_j] p_j = \sum_{j,s} [x_i, q_j] b_{js} x_s = x_i$$

In other words, $p_1, \ldots, p_n \in g_K$ are the unique solutions of the following system of equations

$$\sum_{j=1}^{n} [x_i, q_j] p_j = x_i \quad i : 1, \ldots, n \quad (*)$$

This implies that (since K is commutative)

$$\sum_{j=1}^{n} [x, q_j] p_j = x \quad \text{for all} \quad x \in g_K \quad (**)$$

Next, we want to verify that

$$p_1, \ldots, p_n, q_1, \ldots, q_n$$

form a set of Weyl generators of $D(L)$ over F.
For all $i, j : 1, \ldots, n$ we have:

1) $[q_i, q_j] = 0$ since K is commutative.

2) $[p_i, q_j] = \delta_{ij}$
First we observe that for all $j, t : 1, \dots, n$:

$$[p_j, q_t] = [\sum_s b_{js} x_s, q_t] = \sum_s b_{js}[x_s, q_t] \in K$$

Using $(*)$ and the fact that K is commutative we get for all $i, t : 1, \dots, n$:

$$\sum_{j=1}^{n} [x_i, q_j][p_j, q_t] = \left[\sum_{j=1}^{n} [x_i, q_j] p_j, q_t\right] = [x_i, q_t]$$

Therefore we have the following equality of matrices $\in K^{n \times n}$:

$$\underbrace{([x_i, q_j])}_{A}([p_j, q_t]) = \underbrace{([x_i, q_t])}_{A}$$

Consequently, $[p_j, q_t] = \delta_{jt}$ as A is invertible.

3) $[p_i, p_j] = 0$
Using (2) we see that for all $s : 1, \dots, n$:
$[[p_i, p_j], q_s] = [[p_i, q_s], p_j] + [p_i, [p_j, q_s]] = 0.$
Hence, $[p_i, p_j] \in C(K) = K$, K being a maximal subfield of $D(L)$ by Proposition 2.1.
Next,

$$[x_i, p_s] = \left[\sum_{j=1}^{n} [x_i, q_j] p_j, p_s\right]$$

$$= \sum_{j=1}^{n} [[x_i, q_j], p_s]\, p_j + \sum_{j=1}^{n} [x_i, q_j][p_j, p_s]$$

$$= \sum_{j=1}^{n} [[x_i, p_s], q_j]\, p_j + \sum_{j=1}^{n} \underbrace{[x_i, [q_j, p_s]]}_{0}\, p_j + \sum_{j=1}^{n} [x_i, q_j][p_j, p_s]$$

$$= [x_i, p_s] + \sum_{j=1}^{n} [x_i, q_j][p_j, p_s]$$

(using $(**)$ since $[x_i, p_s] \in g_K$).
Hence, $\sum_{j=1}^{n} [x_i, q_j][p_j, p_s] = 0$ which forces $[p_j, p_s] = 0$ for all $j, s : 1, \dots, n$ as $A = ([x_i, q_j])$ is an invertible matrix.

4. $p_1, \dots, p_n, q_1, \dots, q_n$ generate $D(L)$ as a skew field over F.
From $q_1, \dots, q_n$ and F we obtain all of K as $K = F(q_1, \dots, q_n)$.
In particular, also $W \subset K$.
On the other hand, we also obtain the basis $x_1, \dots, x_n$ of g since $x_i = \sum_{j=1}^{n} [x_i, q_j] p_j$ and $[x_i, q_j] \in K$.

We may now conclude that $D(L) = D_n(F)$.

Remark 2.3 Suppose g is algebraic. Then the same holds for L and in that case (iii) is an immediate consequence of (i).

Proof. First, we show that L is algebraic. $\mathrm{ad}_L W$ is algebraic, since it is an abelian Lie subalgebra of $\mathrm{End} L$, consisting of nilpotent endomorphisms [C, p. 303]. Hence, $\mathrm{ad}_L L = \mathrm{ad}_L g + \mathrm{ad}_L W$ is also algebraic, being the sum of two algebraic Lie subalgebras of $\mathrm{End} L$ [C, p. 175]. Consequently, L is algebraic [C, p. 336].
Therefore,

$$\begin{aligned} tr\deg_k(K^g) &= tr\deg_k Z(D(L)) = i(L) \\ &= \dim W - \dim g = m - n \text{ (by Proposition 2.1)} \end{aligned}$$

From $k \subset K^g \subset K$ we see that

$$\begin{aligned} tr\deg_{K^g}(K) &= tr\deg_k(K) - tr\deg_k(K^g) \\ &= m - (m - n) = n \end{aligned}$$

Corollary 2.4 Assume $L = g \bigoplus W$ satisfies all the conditions of Theorem 1.1. In particular,

$$F = k(c_1, \dots, c_t) \quad \text{and} \quad K = F(q_1, \dots, q_n)$$

are purely transcendental extensions of k and F respectively (where $F = K^g = Z(D(L))$). Let $d \in DerL$ be an outer derivation of L such that

(i) $d(g) \subset g$

(ii) $d(c_i) = \alpha_i c_i$ for some $\alpha_i \in \mathbb{Q}$, $i : 1, \dots, t$

(iii) $d(q_j) = \lambda_j q_j$ for some $\lambda_j \in k$, $j : 1, \dots, n$

Then the semi-direct product $L \bigoplus kd$ satisfies the GK-conjecture.

Proof. We construct $p_1, \ldots, p_n \in D(L)$ as in the proof of Theorem 1.1. Then we know that $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ form a system of Weyl generators of $D(L)$ over F, a rational extension of k. In view of Lemma 4 of [AOV2] it suffices to show that $d(p_j) = -\lambda_j p_j$ for all $j : 1, \ldots, n$.

For this purpose, we let d act on both sides of formule $(*)$

$$x_i = \sum_{j=1}^{n} [x_i, q_j] p_j \quad i : 1, \ldots, n$$

We obtain:

$$\begin{aligned} d(x_i) &= \sum_{j=1}^{n} d([x_i, q_j]) p_j + \sum_{j=1}^{n} [x_i, q_j] d(p_j) \\ &= \sum_{j=1}^{n} [d(x_i), q_j] p_j + \sum_{j=1}^{n} [x_i, d(q_j)] p_j + \sum_{j=1}^{n} [x_i, q_j] d(p_j) \\ &= d(x_i) + \sum_{j=1}^{n} [x_i, \lambda_j q_j] p_j + \sum_{j=1}^{n} [x_i, q_j] d(p_j) \end{aligned}$$

(using $(**)$ since $d(x_i) \in g$)
Consequently $\sum_{j=1}^{n} [x_i, q_j](\lambda_j p_j + d(p_j)) = 0$ for all $i : 1, \ldots, n$. This implies that $\lambda_j p_j + d(p_j) = 0$ for all $j : 1, \ldots, n$ as the matrix $A = ([x_i, q_j]) \in K^{n \times n}$ is invertible.

Proposition 2.5 Let g be an n-dimensional Lie algebra over k and W an n-dimensional g-module such that $g(f) = 0$ for some $f \in W^*$.

(i) Then the semi-direct product $L = g \bigoplus W$ is Frobenius and satisfies the GK-conjecture. In fact, $D(L) \cong D_n(k)$.

(ii) We may assume that $g \subset \mathrm{End} W$. Let $T \subset \mathrm{End} W$ be an abelian Lie subalgebra consisting of diagonalizable endomorphisms of W such that $[T, g] \subset g$ and $T \cap g = \{0\}$. Then the semi-direct product $L_1 = (T \bigoplus g) \bigoplus W$ also satisfies the GK-conjecture.

Proof. (i) L is Frobenius since $i(L) = 0$ by Proposition 2.1. In particular, $K^g = Z(D(L)) = k$ [O2], [O4, p. 283].

Therefore, the conditions of Theorem 1.1 are trivially satisfied.
(ii) Let $\{t_1, \ldots, t_r\}$ be a basis of T. We can find a basis $\{q_1, \ldots, q_n\}$ of W such that $[t_i, q_j] = t_i(q_j) = \lambda_{ij} q_j$ for some $\lambda_{ij} \in k$ for all i, j.
We can construct $p_1, \ldots, p_n \in D(L)$ as in the proof of Theorem 1.1 such that $p_1, \ldots, p_n, q_1, \ldots, q_n$ is a system of Weyl generators of $D(L)$ over $k = Z(D(L))$.
Then each derivation $d_i = \mathrm{ad}_L t_i$ is a derivation of L which satisfies the conditions of Corollary 2.4, which implies that

$$[t_i, p_j] = d_i(p_j) = -\lambda_{ij} p_j \quad \text{for all } i, j$$

and so $[t_i, p_j q_j] = 0$.
First, we introduce for each $i : 1, \ldots, r$

$$u_i = \sum_{j=1}^{n} \lambda_{ij} p_j q_j \in D(L) \backslash \{0\}$$

Clearly, $[t_s, u_i] = 0$ for all s, i.
Note that

$$\begin{aligned} [u_i, p_s] &= [\sum_j \lambda_{ij} p_j q_j, p_s] = \sum_j \lambda_{ij} p_j [q_j, p_s] \\ &= -\lambda_{is} p_s \end{aligned}$$

Similarly, $[u_i, q_s] = \lambda_{is} q_s$.
In particular, $[u_i, p_s q_s] = 0$ for all s and hence also $[u_i, u_j] = 0$ for all i, j.
Next, we put

$$z_i = t_i - u_i \in D(L_1) \backslash \{0\}, \quad i : 1, \ldots, r$$

Then we observe that for all i, j:

$$[z_i, p_j] = [t_i - u_i, p_j] = [t_i, p_j] - [u_i, p_j] = 0$$

similarly, $[z_i, q_j] = 0$ and also $[z_i, z_j] = 0$. We now proceed step by step.

Step 1: $t = 1$
Because $z_1 = t_1 - u_1$ commutes with all the p's and q's we may conclude, as in the proof (case 2) of lemma 4 of [AOV2], that $p_1, \ldots, p_n, q_1, \ldots, q_n$ form a system of Weyl generators of $D(L_1)$ over $k(z_1)$.

Step 2 ($t = 2$) follows from step 1 using the same argument, and so on.
In the end, $p_1, \ldots, p_n, q_1, \ldots, q_n$ form a system of Weyl generators of $D(L_1)$ over $k(z_1, \ldots, z_r)$, a purely transcendental extension of k. In particular, $Z(D(L_1)) = k(z_1, \ldots, z_r)$.

Examples (all of type (i) of the proposition)

1. Let g be an n-dimensional Frobenius Lie algebra and let W be its adjoint representation. Then the Takiff Lie algebra $L = g \bigoplus W$ is Frobenius and $D(L) \cong D_n(k)$.

2. Let g be an n-dimensional reductive Lie algebra over k, k algebraically closed, and let W^* be a prehomogeneous g-module (i.e. W^* has an open orbit) such that $\dim W = n$. Then $L = g \bigoplus W$ is Frobenius [EO, p. 143] and $D(L) \cong D_n(k)$.

3. Let A be an n-dimensional (associative) Frobenius algebra with a unit. A becomes a Lie algebra g for the Lie bracket $[a, b] = ab - ba$ and $W = A$ becomes a g-module by left multiplication. Then $L = g \bigoplus W$ is a Frobenius Lie algebra [EO, p. 144] and $D(L) \cong D_n(k)$.

3. The explicit verification of $L = sl(2, k) \oplus W_2 \oplus W_1$

We want to demonstrate the method of Theorem 1.1 (and its proof) for this Lie algebra (which is $L_{8,2}$ of [AOV2, p. 567]) in order to obtain the Weyl generators of $D(L)$.
So, here $g = sl(2, k)$ (which is algebraic), with standard basis h, x, y and $W = W_2 \bigoplus W_1$, with standard basis $e_0, e_1, e_2;\ e_3, e_4$. The Lie brackets of these form the following matrix M:

	h	x	y	e_0	e_1	e_2	e_3	e_4
h	0	$2x$	$-2y$	$2e_0$	0	$-2e_2$	e_3	$-e_4$
x	$-2x$	0	h	0	$2e_0$	e_1	0	e_3
y	$2y$	$-h$	0	e_1	$2e_2$	0	e_4	0
e_0	$-2e_0$	0	$-e_1$	0	0	0	0	0
e_1	0	$-2e_0$	$-2e_2$	0	0	0	0	0
e_2	$2e_2$	$-e_1$	0	0	0	0	0	0
e_3	$-e_3$	0	$-e_4$	0	0	0	0	0
e_4	e_4	$-e_3$	0	0	0	0	0	0

Put $K = R(W) = k(e_0, e_1, e_2, e_3, e_4)$. First, we verify the conditions of Theorem 1.1.

(i) We notice that the 3×5 submatrix of M in the top right corner has rank 3 over K (since $\det \begin{pmatrix} -2e_2 & e_3 & -e_4 \\ e_1 & 0 & e_3 \\ 0 & e_4 & 0 \end{pmatrix} = 2e_2e_3e_4 - e_1e_4^2 \neq 0$). This implies that $\mathfrak{g}(f) = 0$ for some $f \in W^*$ by Proposition 2.1. In particular, $Z(D(L)) = K^{\mathfrak{g}}$.

(ii) Since L is algebraic we know that:

$$tr \deg_k(Z(D(L))) \;=\; i(L) = \dim W - \dim \mathfrak{g} = 2$$

Put

$$c_1 = e_1^2 - 4e_0e_2 \quad \text{and} \quad c_2 = e_0e_4^2 - e_1e_3e_4 + e_2e_3^2$$

We verify that $c_1, c_2 \in Z(D(L))$

$$\begin{aligned} [x, c_1] &= 2[x, e_1]e_1 - 4e_0[x, e_2] \\ &= 4e_0e_1 - 4e_0e_1 = 0 \\ [y, c_1] &= 2[y, e_1]e_1 - 4[y, e_0]e_2 \\ &= 4e_1e_2 - 4e_1e_2 = 0 \\ [x, c_2] &= e_0[x, e_4^2] - [x, e_1]e_3e_4 - e_1e_3[x, e_4] + [x, e_2]e_3^2 \\ &= 2e_0e_3e_4 - 2e_0e_3e_4 - e_1e_3^2 + e_1e_3^2 = 0 \\ [y, c_2] &= [y, e_0]e_4^2 - [y, e_1]e_3e_4 - e_1[y, e_3]e_4 + e_2[y, e_3^2] \\ &= e_1e_4^2 - 2e_2e_3e_4 - e_1e_4^2 + 2e_2e_3e_4 = 0 \end{aligned}$$

Put $F = k(c_1, c_2) \subset Z(D(L))$ and note that $tr \deg_k(F) = 2 = tr \deg(Z(D(L)))$. So, $Z(D(L))$ is algebraic over F.
We now consider the following elements of K:

$$q_1 = e_3 \quad q_2 = e_0e_3^{-1}e_4 \quad q_3 = 2q_2 - e_1$$

Then,

$$4e_0c_2 = (q_1q_3)^2 - c_1q_1^2 \qquad (***)$$

Indeed,

$$\begin{aligned} (q_1q_3)^2 - c_1q_1^2 &= (2q_1q_2 - q_1e_1)^2 - c_1q_1^2 \\ &= (2e_0e_4 - e_1e_3)^2 - (e_1^2 - 4e_0e_2)e_3^2 \\ &= 4e_0^2e_4^2 - 4e_0e_1e_3e_4 + e_1^2e_3^2 - e_1^2e_3^2 + 4e_0e_2e_3^2 \\ &= 4e_0(e_0e_4^2 - e_1e_3e_4 + e_2e_3^2) = 4e_0c_2 \end{aligned}$$

Next, $K = k(c_1, c_2, q_1, q_2, q_3)$.
Indeed, using $(***)$ we obtain e_0 from c_1, c_2, q_1, q_3. From q_2, q_3 we obtain e_1. From e_0, e_1 and $c_1 = e_1^2 - 4e_0e_2$ we get e_2. Finally, from e_0, $q_1 = e_3$ and $q_2 = e_0e_3^{-1}e_4$ we obtain e_4.
Clearly, $K = F(q_1, q_2, q_3)$, a purely transcendental extension of degree 3 over $F = k(c_1, c_2)$. The subfield $Z(D(L)) \subset K$ is algebraic over F. Hence, $Z(D(L)) = F$. By Theorem 1.1 we may conclude that $D(L) \cong D_3(F)$ where $F = k(c_1, c_2)$, a rational extension of k. So, L satisfies the GK-conjecture.
In order to construct $p_1, p_2, p_3 \in D(L)$ we need to calculate the following Lie brackets:

$$
\begin{aligned}
[h, q_1] &= [h, e_3] = e_3 \quad [h, q_2] = [h, e_0e_3^{-1}e_4] = 0 \\
[h, q_3] &= [h, 2q_2 - e_1] = 2[h, q_2] - [h, e_1] = 0 \\
[x, q_1] &= [x, e_3] = 0 \quad [x, q_2] = [x, e_0e_3^{-1}e_4] = e_0e_3^{-1}e_3 = e_0 \\
[x, q_3] &= [x, 2q_2 - e_1] = 2[x, q_2] - [x, e_1] = 2e_0 - 2e_0 = 0 \\
[y, q_1] &= [y, e_3] = e_4 \\
[y, q_2] &= [y, e_0e_3^{-1}e_4] = [y, e_0]e_3^{-1}e_4 - e_0e_3^{-2}[y, e_3]e_4 \\
&= e_1e_3^{-1}e_4 - e_0e_3^{-2}e_4^2 = e_3^{-2}(e_1e_3e_4 - e_0e_4^2) \\
&= e_3^{-2}(e_2e_3^2 - c_2) = e_2 - e_3^{-2}c_2 \\
[y, q_3] &= [y, 2q_2 - e_1] = 2[y, q_2] - [y, e_1] \\
&= 2e_2 - 2e_3^{-2}c_2 - 2e_2 = -2e_3^{-2}c_2
\end{aligned}
$$

Finally, p_1, p_2, p_3 are the solutions of the following equations:

$$
\begin{aligned}
h &= [h, q_1]p_1 + [h, q_2]p_2 + [h, q_3]p_3 = e_3p_1 \\
x &= [x, q_1]p_1 + [x, q_2]p_2 + [x, q_3]p_3 = e_0p_2 \\
y &= [y, q_1]p_1 + [y, q_2]p_2 + [y, q_3]p_3 \\
&= e_4p_1 + e_3^{-2}(e_2e_3^2 - c_2)p_2 - 2e_3^{-2}c_2p_3
\end{aligned}
$$

So, $p_1 = e_3^{-1}h \quad p_2 = e_0^{-1}x$ and

$$
\begin{aligned}
p_3 &= \frac{1}{2}e_3^2c_2^{-1}[e_4p_1 + e_3^{-2}(e_2e_3^2 - c_2)p_2 - y] \\
&= \frac{1}{2}c_2^{-1}[e_3e_4h + e_0^{-1}(e_2e_3^2 - c_2)x - e_3^2y]
\end{aligned}
$$

4. The semi-direct product $L = sl(2,k) \oplus W_n$

In this section we assume k to be algebraically closed. Let G be a connected semi-simple algebraic group over k with Lie algebra g and let W be a finite dimensional G-representation. Then W is also a g-representation. We recall that G is said to be special if any principal homogeneous G-space is locally trivial for the Zariski topology. For example, $GL(n,k)$, $SL(n,k)$ and $Sp(2n,k)$ are special [CS, p. 18]. We now recall the main result of [AOV1], due to M. Van den Bergh:

Theorem 4.1 Assume that the generic stabilizer (in G) of W is trivial and consider the semi-direct product $L = g \bigoplus W$. Then the following are equivalent:
(1) $D(L)$ is a Weyl skew field over some field.
(2) G is special.

We now focus our attention to the case where $G = SL(2,k)$. As before, we put $K = R(W)$, which we regard as the field of rational functions on W. Clearly, $K^{SL(2,k)} = K^{sl(2,k)}$. This field of invariants has the following interesting property, due to F.A. Bogomolov and P.I. Katsylo [B], [BK], [K1], [K2].

Theorem 4.2 $K^{SL(2,k)}$ is a purely transcendental extension of k.

Next, we replace W by W_n, the $(n+1)$-dimensional irreducible representation of $SL(2,k)$, usually represented by the space of binary forms of degree n.
For $n \leq 4$ we have already verified in [AOV2] that the Lie algebra $L = sl(2,k) \bigoplus W_n$ satisfies the GK-conjecture by providing the Weyl generators explicitly. See also the appendix (due to H. Kraft) of [AOV1] for $n = 3, 4$.
For larger n we now have the following simple criterion:

Proposition 4.3 Suppose $n \geq 5$. Then, $L = sl(2,k) \bigoplus W_n$ satisfies the GK-conjecture if and only if n is odd.

Proof Case 1: n is odd.
In this case we know that the generic stabilizer of W_n is trivial [P]. Since $SL(2,k)$ is special, $D(L)$ is a Weyl skew field over some field extension F of k by Theorem 4.1.
Clearly, $F = Z(D(L)) = K^{sl(2,k)}$ by Proposition 2.1 and Remark 2.2. But the

latter is rational over k by Theorem 4.2. Therefore, the "if"-part is settled.
Case 2: n is even.
In this situation the generic stabilizer of W_n is precisely $Z = \{I, -I\}$, the center of $SL(2,k)$ [P]. Hence, W_n can also be regarded as a representation space for the group $PSL(2,k) = SL(2,k)/Z$, with trivial generic stabilizer. Since $PSL(2,k)$ is not special we may conclude, using Theorem 4.1, that $D(L)$ cannot be a Weyl skew field over some field. In particular, L does not satisfy the GK-conjecture.

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