CALABI-YAU COALGEBRAS

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Abstract. We present a method for constructing the minimal injective resolution of a simple comodule of a path coalgebra of quivers with relations. Dual to the Calabi-Yau condition of algebras, we introduce the concept of a Calabi-Yau coalgebra. Then we describe the Calabi-Yau coalgebras of lower global dimensions. An appendix is included for listing some properties of cohom functors.

Introduction

Calabi-Yau algebras and categories, because of their links to mathematical physics, algebraical geometry, representation theory, ... etc., were intensively studied in recent years (cf.[9, 2, 13, 1, 8]). Bocklandt proved that a graded Calabi-Yau algebra (defined by a finite quiver) of dimension 2 is a preprojective algebra, while a 3-dimensional graded Calabi-Yau algebra is determined by a superpotential (cf.[1]). Recently Berger and Taillefer in [2] defined a special class of nongraded Calabi-Yau algebras, namely the Poincaré-Birkhoff-Witt deformations of graded Calabi-Yau algebras of dimension 3 . But for general nongraded Calabi-Yau algebras, we don't have much information of them. In fact, it is not easy to study Calabi-Yau the property for general nongraded algebras. Nevertheless, a noetherian complete algebra shares many similar properties with a connected graded algebra (cf. [18, 7]). Our naive idea is to consider when a noetherian complete algebra is Calabi-Yau. It is well-known that a noetherian complete algebra with cofinite Jacobson radical is the dual algebra of an artinian coalgebra (cf. [12]). Since any coalgebra has the locally finite property, we could take this advantage to attack the problem by using coalgebras. However the Calabi-Yau property of coalgebras has not yet been studied. So in this paper we try to lay a foundation for Calabi-Yau coalgebras. In a subsequent paper, we will discuss the Calabi-Yau property of artinian coalgebras (cf. [11]).

The paper is organised as follows. In Section 1, we introduce the Calabi-Yau condition to coalgebras, and discuss some basic properties. In Section 2, we show that a Calabi-Yau coalgebra of dimensions 0 is exactly a cosemisimple coalgebra. In particular, a semiperfect coalgebra is Calabi-Yau if and only if it is cosemisimple. A Calabi-Yau coalgebra of dimension 1 is precisely a direct sum of (possibly infinite) copies of the coalgebra $\mathbf{k}[x]$. For Calabi-Yau coalgebras of higher dimensions, we will

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mainly focus us on the path coalgebras of quivers with relations. This is reasonable because any coalgebra is Morita-Takeuchi equivalent to a basic coalgebra (cf. [5]), which is a subcoalgebra of the path coalgebra of its Gaberial quiver (cf. [6]) if we work over an algebraically closed field. For a path coalgebra of a quiver (Q, Ω) with relations (cf. [16]), we give a construction of the first two steps of the minimal injective resolutions of a simple comodule through the arrows and the relations of Q. This enables us to analyze the necessity conditions on the quivers and the relations such that the path coalgebras are Calabi-Yau of dimensions 2 or 3. From there we obtain the results dual to the ones in [1].

Throughout $\mathbf{k}$ is an algebraically closed field of characteristic zero. All the algebras and coalgebras involved are over $\mathbf{k}$; unadorned $\otimes$ means $\otimes_{\mathbf{k}}$ and Hom means $\mathrm{Hom}_{\mathbf{k}}$.

1. Calabi-Yau condition for coalgebras

Let C be a coalgebra. Let $\mathcal{D}^b({}^C\mathcal{M})$ be the bounded derived category of C. Consider the full triangulated subcategory $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ consisting of complexes with finite dimensional cohomology. Recall that a $\mathbf{k}$-linear category $\mathcal{T}$ is *Hom-finite* if for any two objects X and Y in $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is a finite dimensional $\mathbf{k}$-vector space. In general, $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ is not a Hom-finite $\mathbf{k}$-linear category. Since any simple comodule is finite dimensional, we have that $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ is Hom-finite if and only if, for any simple comodules M and N, $\mathrm{Ext}^i_C(N, M)$ is finite dimensional for all $i \geq 0$. For example, $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ is Hom-finite if C is a (left) strictly quasi-finite coalgebra (cf. [10]). Recall that a Hom-finite triangulated category $\mathcal{T}$ is called a *Calabi-Yau category of dimension n* if there are natural isomorphisms $\mathrm{Hom}_{\mathcal{T}}(X, Y)^* \cong \mathrm{Hom}_{\mathcal{T}}(Y, X[n])$ for all $X, Y \in \mathcal{T}$, that is, the n-th shift functor is a Serre functor (cf. [1, Appendix]).

Definition 1.1. A coalgebra C is called a left *Calabi-Yau coalgebra of dimension n* (simply written as CY-n) if

(i) $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ is Hom-finite;

(ii) $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ is a Calabi-Yau category of dimension n.

Similarly, we can define right Calabi-Yau coalgebras. Note that for a semiperfect or an artinian coalgebra C, C is left Calabi-Yau if and only if it is right Calabi-Yau. But in general, we don't know whether or not left Calab-Yau and right Calabi-Yau are equivalent. In the following, a CY coalgebra always means a left CY coalgebra.

We now list some basic properties of a Calabi-Yau coalgebra:

Proposition 1.2. (i) *If C and D are Morita-Takeuchi equivalent, then C is CY-n if and only if D is CY-n;*

(ii) *If C is CY-n, then the global dimension* $\mathrm{gldim} C = n$*;*

(iii) *If $\{C_\lambda\}_{\lambda \in \Lambda}$ is a set of CY-n coalgebras, then $\oplus_{\lambda \in \Lambda} C_\lambda$ is a CY-n coalgebra.*

Proof. the statement (i) is trivial. The statement (ii) follows from the fact that the global dimension of C is equal to the supremum of the injective dimension of simples comodules (cf. [15]).

(iii) Let C be a coalgebra, and $e_1, e_2 \in C^*$ be a pair of central orthogonal idempotents. We have that for any objects $X, Y \in \mathcal{D}^+({}^C\mathcal{M})$, $\mathrm{Hom}_{\mathcal{D}^+({}^C\mathcal{M})}(Xe_1, Ye_2) = 0$. Now let $C = \oplus_{\lambda\in\Lambda} C_\lambda$, and let $e_\lambda \in C^*$ be the central idempotent whose restriction to C_λ is the counit ε_λ of C_λ and e_λ sends C_β to zero if $\beta \neq \lambda$. For $X \in \mathcal{D}^b_{fd}({}^C\mathcal{M})$, note that there are only finitely many idempotents $e_{\lambda_1}, \ldots, e_{\lambda_n}$ such that $Xe_{\lambda_i} \neq 0$ in $\mathcal{D}^b_{fd}({}^C\mathcal{M})$. Hence we have the following natural isomorphisms for any $X, Y \in \mathcal{D}^b_{fd}({}^C\mathcal{M})$,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^b_{fd}({}^C\mathcal{M})}(X,Y) &= \mathrm{Hom}_{\mathcal{D}^b_{fd}({}^C\mathcal{M})}(\oplus_{\lambda\in\Lambda} Xe_\lambda, \oplus_{\lambda\in\Lambda} Ye_\lambda)\\
&\cong \oplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{D}^b_{fd}({}^C\mathcal{M})}(Xe_\lambda, Ye_\lambda)\\
&\cong \oplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{D}^b_{fd}({}^C\mathcal{M})}(Ye_\lambda, Xe_\lambda[n])^*\\
&\cong \mathrm{Hom}_{\mathcal{D}^b_{fd}({}^C\mathcal{M})}(\oplus_{\lambda\in\Lambda} Ye_\lambda, \oplus_{\lambda\in\Lambda} Xe_\lambda[n])^*\\
&\cong \mathrm{Hom}_{\mathcal{D}^b_{fd}({}^C\mathcal{M})}(Y, X[n])^*.
\end{aligned}$$

Therefore (iii) holds. □

2. Calabi-Yau coalgebras of dimension 0 and 1

In this section, we will see that the structures of the Calabi-Yau coalgebras of dimensions 0 and 1 are quite simple. A CY-0 coalgebra is nothing but a cosemisimple coalgebra as we will see in the next proposition.

Proposition 2.1. *A coalgebra C is CY-0 if and only if C is cosemisimple.*

Proof. We only need to prove that a cosemisimple coalgebra is CY-0. By [4, Cor.3.6], C is a symmetric coalgebra. Hence, for any left C-comodule N, we have the left C^*-isomorphisms $N^* \cong \mathrm{Hom}_{C^*}(N, C) \cong \mathrm{Hom}_{C^*}(N, C^*)$ (cf. [4, Theorem 5.3]). Since C is cosemisimple, $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ is exactly the homotopy category of bounded complexes with finite dimensional cohomology, and any complex in $\mathcal{D}^b_{fd}({}^C\mathcal{M})$ is split. It is sufficient to show that the natural isomorphisms in Definition 1.1 hold for all finite dimensional comodules. Let N and M be any finite dimensional comodules. We have the following natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_C(M,N) &\cong \mathrm{Hom}_{C^*}(M,N)\\
&\cong \mathrm{Hom}_{C^*}(M,N^{**})\\
&\cong \mathrm{Hom}(M\otimes_{C^*} N^*, \mathbf{k})\\
&\cong \mathrm{Hom}(M\otimes_{C^*} \mathrm{Hom}_{C^*}(N,C^*), \mathbf{k})\\
&\cong \mathrm{Hom}_{C^*}(N,M)^*.
\end{aligned}$$

It follows that C is CY-0. □

Corollary 2.2. *Let C be a semiperfect coalgebra. Then C is CY if and only if C is cosemisimple.*

Proof. Since C is semiperfect, the injective envelop of a simple comodule is finite dimensional. Hence there are enough finite dimensional injective comodules. Let E be a finite dimensional injective comodule. If C is CY-n, then $\mathrm{Ext}^n_C(E,E)\cong \mathrm{Hom}_C(E,E)^*\neq 0$. This implies that $n=0$. Therefore C is cosemisimple. □

Remark 2.3. *From the proof of Corollary 2.2, we see that if C is a non-cosemisimple CY coalgebra, then any injective comodule should be of infinite dimension.*

It is well-known that a hereditary coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to a path coalgebra (cf. [3]), a CY-1 coalgebra over an algebraically closed field must be Morita-Takeuchi equivalent to a path coalgebra. Let us recall some notations about quivers and path coalgebras. Let Q be a quiver with the set of vertices Q_0 and the set of arrows Q_1. For an arrow $a\in Q_1$, we use $s(a)$ to denote the source of a and $t(a)$ to denote the target of a. A nontrivial path is a sequence of arrows $p=a_1a_2\cdots a_n$ with $s(a_{i+1})=t(a_i)$. We say that the length of p is n. Sometimes a vertex is called a path of length 0. For a path $p=a_1a_2\cdots a_n$, we define $s(p)=s(a_1)$ and $t(p)=t(a_n)$. Denote by $\mathbf{k}Q$ the path algebra of Q (in general, $\mathbf{k}Q$ has no unit). The the multiplication of two pathes p,q is defined as pq if $s(q)=t(p)$ and 0 otherwise. Let a be an arrow and $p=a_1a_2\cdots a_n$ be a path. We define $pa^{-1}=a_1\cdots a_{n-1}$ if $a=a_n$ and 0 otherwise. If $x=k_1p_1+\cdots+k_np_n$ is a linear combination of paths, then we define $xa^{-1}=k_1p_1a^{-1}+\cdots+k_np_na^{-1}$. Similarly, if q is a general path, we define $pq^{-1}=r$ if $p=rq$.

Following the notations of [14], we use CQ to denote the path coalgebra of Q. There is a nondegenerated bilinear map $\langle\ ,\ \rangle : CQ\times \mathbf{k}Q\longrightarrow \mathbf{k}$ defined by $\langle p,q\rangle=\delta_{p,q}$ (the Kronecker delta), where p,q are two pathes of the quiver Q. The bilinear form $\langle\ ,\ \rangle$ induces an injective map $\iota:\mathbf{k}Q\longrightarrow (CQ)^*$. Clearly, ι preserves the multiplications (cf. [14]). Since CQ is a $(CQ)^*$-bimodule, for $x\in CQ$ and $y\in \mathbf{k}Q$, the notion $\iota(y)x$ (or $x\iota(y)$) makes sense. We list two simple properties of the bilinear form $\langle\ ,\ \rangle$, which will be frequently used.

Lemma 2.4. (i) *Let a be an arrow, $x\in CQ$ and $y\in\mathbf{k}Q$. Then $\langle xa^{-1},y\rangle=\langle x,ya\rangle$.*

(ii) *Let $x\in CQ$ and $y\in\mathbf{k}Q$. Assume $y=k_1p_1+\cdots+k_np_n$ with $k_1k_2\cdots k_n\neq 0$ and $s(p_1)=\cdots=s(p_n)$. If $\iota(y)x=0$ then $\langle x,y\rangle=0$.*

Proof. (i) is obvious. The statement (ii) follows from (i) and the fact that $xp^{-1}=\iota(p)x$ for any path p. □

Let $i\in Q_0$ be a vertex, let $e_i\in(CQ)^*$ be the idempotent corresponding to i, and let S_i be the simple left CQ-comodule corresponding to i. For simplicity, we write C for CQ. The minimal injective resolution of S_i can be written as follows:

$$0\longrightarrow S_i\longrightarrow e_iC\xrightarrow{f}\bigoplus_{a\in Q_1,t(a)=i} e_{s(a)}C\to 0, \tag{1}$$

where f is defined by $f(x) = \sum_{t(a)=i} \iota(a)x$ in which the multiplication is the left C^*-module action, and we regard $\iota(a)x$ as an element in $e_{s(a)}C$. Here we need to point out that if two arrows a and b have the same source and the same target in the above sequence, we should distinguish $e_{s(a)}C$ from $e_{s(b)}C$ in the direct sum $\bigoplus_{a\in Q_1;t(a)=i} e_{s(a)}C$.

Proposition 2.5. *Let C be a coalgebra. Then C is CY-1 if and only if C is Morita-Takeuchi equivalent to a direct sum of copies of $\mathbf{k}[x]$, where $\mathbf{k}[x]$ is the path coalgebra of the quiver Q with one vertex and one arrow.*

Proof. Note that $\mathbf{k}[x]$ is a CY-1 coalgebra because the dual algebra of $\mathbf{k}[x]$ is the power series algebra which is noetherian and CY-1 (cf. [11]). Hence a direct sum of copies of $\mathbf{k}[x]$ is also CY-1.

Conversely, assume that C is CY-1. Without loss of generality, we can further assume that C is the path coalgebra of a quiver Q. Let $i, j \in Q_0$. By the CY property, $\dim \mathrm{Ext}^1_C(S_j, S_i) = \dim \mathrm{Ext}^0_C(S_i, S_j) = \delta_{i,j}$. From the minimal resolution (1) of S_i, we have $\dim \mathrm{Ext}^1_C(S_j, S_i) = \#\{a \in Q_1 | s(a) = j, t(a) = i\}$. It follows that for vertices i, j and $i \neq j$, there exist no arrows from i to j, and for each vertex i there is a unique arrow from i to itself. Therefore, C is a direct sum of copies of $\mathbf{k}[x]$. □

3. Minimal injective resolutions of the simple comodules

In order to investigate CY-2 and CY-3 coalgebras, we need more notations about path coalgebras of quivers with relations. Recall from [16] that a *quiver with relations* means a pair (Q, Ω) where Ω is a two-side ideal of $\mathbf{k}Q$ contained in $\mathbf{k}Q_{\geq 2}$. Note that the path algebra $\mathbf{k}Q$ is graded. If Ω is a graded ideal, then we say that the relations are *homogeneous*. The path coalgebra of (Q, Ω) is the following subcoalgebra of CQ:

$$C(Q, \Omega) = \{x \in CQ | \langle x, \Omega\rangle = 0\}.$$

It is well known that any coalgebra is Morita-Takeuchi equivalent to a basic coalgebra and a basic coalgebra over an algebraically closed field is isomorphic to a subcoalgebra of the path coalgebra of its Gabriel quiver (cf. [6]). However not every subcoalgebra C of a path coalgebra is of the form $C = C(Q, \Omega)$ (cf. [14]). We wish to understand when a path coalgebra of a quiver with relations is CY-2 or CY-3. To this end, we need to investigate the minimal injective resolutions of simple comodules. For convenience, we introduce some temporary notations. Let $p = a_1a_2\cdots a_n$ be a nontrivial path. We define $lead(p) = \{a_n\}$, the set of leading arrow. If p is a trivial path, then we let $lead(p) = \emptyset$. For $x \in \mathbf{k}Q$ and $x = k_1p_1 + \cdots + k_np_n$ with $k_1k_2\cdots k_n \neq 0$, we define $lead(x) = lead(p_1) \cup \cdots \cup lead(p_n)$. If S is a subset of $\mathbf{k}Q$, then we define $lead(S) = \bigcup_{x\in S} lead(x)$.

We choose a generating set of the relation ideal Ω in the following way. Denote by $\mathbf{k}Q_{\leq n}$ the subspace of linear combinations of paths of length less or equal to n. Then the path algebra $\mathbf{k}Q$ is a filtered algebra. The restriction of the filtration to Ω

results in a filtration on Ω. Set $\Omega(n) = \Omega \cap \mathbf{k}Q_{\leq n}$. Note that $\Omega(0) = \Omega(1) = 0$. For $i, j \in Q_0$, let $\mathbf{k}Q_{i,j}$ be the subspace of all the linear combinations of paths from i to j. Set $\Omega(n)_{i,j} = \Omega(n) \cap \mathbf{k}Q_{i,j}$. Then $\Omega(n) = \bigoplus_{i,j\in Q_0} \Omega(n)_{i,j}$. Choose a basis $B(2)_{i,j}$ of $\Omega(2)\cap\mathbf{k}Q_{i,j}$ for all $i, j \in Q_0$. Set $R(2) = \bigcup_{i,j\in Q_0} B(2)_{i,j}$. Then $R(2)$ is a basis of $\Omega(2)$. For $n \geq 2$, let $I(n)$ be the ideal of $\mathbf{k}Q$ generated by $\Omega(n)$. Set $I(n)_{i,j} = I(n) \cap \mathbf{k}Q_{i,j}$. Then $I(n) = \bigoplus_{i,j\in Q_0} I(n)_{i,j}$. For $i, j \in Q_0$, let $V(n+1)_{i,j}$ be a subspace of $\Omega(n+1)_{i,j}$ such that $\Omega(n+1)_{i,j} = (I(n)_{i,j} \cap \Omega(n+1)_{i,j}) \oplus V(n+1)_{i,j}$. Choose a basis $B(n+1)_{i,j}$ of $V(n+1)_{i,j}$. Set $R(n+1) = \bigcup_{i,j\in Q_0} B(n+1)_{i,j}$. Now let $R = \bigcup_{n\geq 2} R(n)$. The set R possesses the following properties:

(i) R generates the ideal Ω, and R is minimal, that is, any proper subset of R can not generates Ω;

(ii) each element of R is a combination of paths with common source and common target.

We call a subset R of Ω satisfying the properties (i) and (ii) *a minimal set of relations* of (Q, Ω).

Because of the property (ii) above, for an element $r \in R$ we may write $s(r)$ for the common source and $t(r)$ for the common target. Note that if Ω is a graded ideal then we may choose the minimal set R to be homogeneous, that is, each element of R is a combination of paths with the same length.

Let (Q, Ω) be a quiver with relations. We say that the relation ideal Ω is *locally finite* if there is a minimal set of relations R such that for every pair of vertices (i, j) the set $\{r \in R | s(r) = i, t(r) = j\}$ is a finite set or empty. Note that the locally finite property of Ω is independent of the choice of the minimal set of relations R.

Now we are ready to construct the minimal injective resolution of a simple comodule of certain path coalgebra. Let (Q, Ω) be a quiver with relations, and $C = C(Q, \Omega)$. Let R be a minimal set of relations. Assume that Ω is locally finite. Let S_n be the simple comodule corresponding to the vertex n. Let $R_n = \{r \in R | t(r) = n\}$. We construct a sequence:

$$0 \longrightarrow S_n \longrightarrow e_nC \xrightarrow{f} \bigoplus_{a\in Q_1, t(a)=n} e_{s(a)}C \quad \xrightarrow{g} \bigoplus_{r\in R_n} e_{s(r)}C, \tag{2}$$

where the first map is the embedding map, and f and g are constructed as follows. For $x \in e_nC$, $f(x) = \displaystyle\sum_{t(a)=n, a\in Q_1} \iota(a)x$, where $\iota(a)x$ is regarded as an element in $e_{s(a)}C$. Similar to the sequence (1), we should distinguish $e_{s(a)}C$ from $e_{s(b)}C$ if a and b have the same source and the same target. For $x \in e_{s(a)}C$ with $a \notin lead(R_n)$, define $g(x) = 0$. If $x \in e_{s(a)}C$ with $a \in lead(R_n)$, then we define $g(x) = \sum_{r\in R_n} \iota(ra^{-1})x$. Since Ω is locally finite, g is well defined. Here $\iota(ra^{-1})x$ is viewed as an element in $e_{s(r)}C$, and we also need to distinguish $e_{s(r)}C$ from $e_{s(r')}C$ in the direct sum $\bigoplus_{t(r)=n} e_{s(r)}C$ if r and r' share the same source (and the same target).

Theorem 3.1. *Let (Q, Ω) be a quiver with relations such that Ω is locally finite. Then the sequence (2) constructed above is exact.*

Proof. We first prove $gf = 0$. For $x \in e_nC$, we may write x as $x = \sum_{i=1}^{s} k_ip_i + \sum_{j=1}^{m} t_jq_j$, where p_i's are paths such that $lead(p_i) \nsubseteq lead(R_n)$ for all $i = 1, \ldots, s$ and q_j's are paths such that $lead(q_j) \subseteq lead(R_n)$ for all $j = 1, \ldots, m$. Let a_j be the leading arrow of q_j. We have

$$\begin{array}{rcl} gf(x) & = & gf(\sum_{j=1}^{m} t_jq_j) \\ & = & g(\sum_{j=1}^{m} t_j\iota(a_j)q_j) \\ & = & \sum_{j=1}^{m} t_j(\sum_{r\in R_n} \iota(ra_j^{-1})\iota(a_j)q_j) \\ & \overset{(I)}{=} & \sum_{j=1}^{m} t_j(\sum_{r\in R_n} \iota(r)q_j) \\ & = & \sum_{r\in R_n} \iota(r)x \\ & = & \sum_{r\in R_n} \sum_{(x)} x_{(1)}\langle x_{(2)}, r\rangle \\ & = & 0, \end{array}$$

where the last identity holds because C is a coalgebra of $\mathbf{k}Q$ and any element in C is orthogonal with Ω. For the identity (I) one may check it straightforward. We should point out that $\iota(ra_j^{-1})\iota(a_j)$ may not equal to $\iota(r)$.

Next we show the inclusion $\ker g \subseteq \mathrm{im} f$. Given an element $y \in \ker g$, then y is in one of the following two cases.

Case (1): if $y \in e_{s(a)}C$ with $a \notin lead(R_n)$, we let $x = ya$. In this case it is clear that $\iota(r)x = 0$ for all $r \in R$.

Case (2): if $y \in \bigoplus_{a\in lead(R_n)} e_{s(a)}C$, we can assume that $y = k_1\alpha_1 + \cdots + k_m\alpha_m$ with $k_1k_2\cdots k_m \neq 0$ and $\alpha_i \in e_{s(a_i)}C$ $(i = 1, \ldots, m)$. In this case we have

$$\begin{array}{rcl} g(y) & = & g(\sum_{i=1}^{m} k_i\alpha_i) \\ & = & \sum_{i=1}^{m} k_i(\sum_{r\in R_n} \iota(ra_i^{-1})\alpha_i) \\ & = & \sum_{i=1}^{m} k_i\left(\sum_{r\in R_n} \iota(ra_i^{-1})\iota(a_i)(\alpha_ia_i)\right) \\ & = & \sum_{i=1}^{m} k_i \sum_{r\in R_n} \iota(r)(\alpha_ia_i) \\ & = & \sum_{r\in R_n} \iota(r) \sum_{i=1}^{m}(k_i\alpha_ia_i), \end{array}$$

where the forth identity holds because α_ia_i is a linear combination of paths with the same leading arrow a_i. Let $x = \sum_{i=1}^{m} k_i\alpha_ia_i$. Recall that $\iota(r)x \in e_{s(r)}C$. Hence $g(y) = 0$ implies $\iota(r)x = 0$ for all $r \in R_n$. Of course, if $r \notin R_n$, we certainly have $\iota(r)x = 0$.

To show that in both cases $y \in \mathrm{im} f$, we only need to show that $x \in e_nC$ since $y = f(x)$. This is equivalent to proving that $\langle x, \Omega\rangle = 0$. Indeed, for any $\beta \in \Omega$, we write $\beta = \sum_{i=1}^{s} k_ip_ir_i + \sum_{j=1}^{m} t_jp'_jr'_jq_j$ where $r_i, r'_j \in R$ and q_j is a nontrivial path for every j. Let a_j be the leading arrow of q_j for $j = 1, \dots, m$. We have

$$\begin{aligned}\langle x, \beta\rangle &= \sum_{i=1}^{s} k_i \langle x, p_i r_i\rangle + \sum_{j=1}^{m} t_j \langle x, p'_j r'_j q_j\rangle \\ &= \sum_{i=1}^{s} k_i \langle \iota(r_i)x, p_i\rangle + \sum_{j=1}^{m} t_j \langle \iota(a_j)x, p'_j r'_j q_j a_j^{-1}\rangle \\ &= 0,\end{aligned}$$

where the last identity holds because $\iota(a_j)x \in C$ and $p'_j r'_j q_j a_j^{-1} \in \Omega$. This completes the proof. □

Remark 3.2. *There is a right version of the sequence (2). Let $C = C(Q, \Omega)$ be as in Theorem 3.1, and let S_m be the right simple comodule corresponding to the vertex m. The first two steps of the minimal injective resolution of S_m is:*

$$0 \longrightarrow S_m \longrightarrow Ce_m \xrightarrow{f} \bigoplus_{a\in Q_1, s(a)=n} Ce_{t(a)} \xrightarrow{g} \bigoplus_{r\in R, s(r)=m} Ce_{t(r)}, \tag{3}$$

where the maps f and g are defined as follows. Define $f(x) = \sum_{s(a)=m, a\in Q_1} x\iota(a)$, for $x \in Ce_m$, and for $x \in Ce_{t(a)}$, define $g(x) = \sum_{s(r)=m, r\in R} x\iota(a^{-1}r)$.

Since all the items (except the simple comodule S_n) in the sequence (2) are injective and the socle of each injective comodule is contained in the image of the map, the sequence (2) is the first two steps of the minimal injective resolution of S_n.

Corollary 3.3. *Let (Q, Ω) be a quiver with relations. If $C = C(Q, \Omega)$ is CY, then, for any pair of vertices (i, j), there are at most finitely many arrows from i to j.*

Proof. The sequence

$$0 \longrightarrow S_n \longrightarrow e_n C \xrightarrow{f} \bigoplus_{a\in Q_0, t(a)=n} e_{s(a)} C$$

as a part of the sequence (2) is always exact for any path coalgebra of a quiver with relations. Then the result follows from the hypothesis that the derived category of complexes with finite dimensional cohomology is Hom-finite. □

4. Calabi-Yau coalgebras of dimensions 2 and 3.

With the preparation of the preceding sections, we can now deduce some necessity conditions for a quiver with relations so that its path coalgebra is CY-2 or CY-3. The following theorems are dual to the corresponding results in [1].

Theorem 4.1. *Let (Q, Ω) be a quiver with locally finite relation ideal Ω. Let R be a minimal set of relations of (Q, Ω). Assume that $C = C(Q, \Omega)$ is CY-2. Then we have the following.*

(i) *For each vertex n, there is a unique element $r \in R$ such that $s(r) = t(r) = n$;*

(ii) *let* $W = \{ra^{-1} | a \in Q_1, r \in R\}$, *and let* $\overline{\Omega} = (W)$ *be the ideal generated by the elements in* W. *Then* $\overline{\Omega} = \mathbf{k}Q_{\geq 1}$;

(iii) *For any two vertices* n, m, $\#\{a \in Q | s(a) = n, t(a) = m\} = \#\{b \in Q_1 | s(b) = m, t(b) = n\}$; *for any vertex* n, *there are at most finitely many arrows starting from* n, *and at most finitely many arrows ending at* n.

(iv) *If* Ω *is a graded ideal of* $\mathbf{k}Q$, *then any element* r *in* R *is a linear combination of paths of length 2.*

Proof. Since C is of global dimension 2, the minimal injective resolution of S_n is:

$$0 \longrightarrow S_n \longrightarrow e_nC \xrightarrow{f} \bigoplus_{a \in Q_1, t(a)=n} e_{s(a)}C \xrightarrow{g} \bigoplus_{r \in R_n} e_{s(r)}C \longrightarrow 0.$$

Now by the CY property, we have $\dim \mathrm{Ext}^2_C(S_m, S_n) = \dim \mathrm{Hom}_C(S_n, S_m) = \delta_{n,m}$. Hence (i) follows.

(ii) It is not hard to see $\overline{\Omega} \supseteq \Omega$. Hence the coalgebra $\overline{C}$ defined by $\{x \in CQ | \langle x, \overline{\Omega} \rangle = 0\}$ is a subcoalgebra of C. View $\overline{C}$ as a left C-comodule. Following (i), the minimal injective resolution of S_n reads as follows:

$$0 \longrightarrow S_n \longrightarrow e_nC \xrightarrow{f} \bigoplus_{a \in Q_1, t(a)=n} e_{s(a)}C \xrightarrow{g} e_nC \longrightarrow 0.$$

Let M be a finite dimensional C-subcomodule of $\overline{C}$. Applying $\mathrm{Hom}_C(M, -)$ to the above injective resolution of S_n, we obtain that $\mathrm{Ext}^2_C(M, S_n)$ is the cokernel of $g_* = \mathrm{Hom}_C(M, g)$. Assume that r is the unique element in R such that $s(r) = t(r) = n$. We claim that $g_* = 0$. Indeed, let $h \in \mathrm{Hom}_C(M, \bigoplus_{a \in Q_1, t(a)=n} e_{s(a)}C)$. Since M is finite dimensional, the image of h lies in $\oplus_{i=1}^k e_{s(a_i)}C$ for finitely many arrows $a_1, \ldots, a_n$. We still use h to denote the induced morphism $h : M \longrightarrow \oplus_{i=1}^k e_{s(a_i)}C$. By Prop. A.4 in the appendix, we have the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(M, \oplus_{i=1}^k e_{s(a_i)}C) \arrow[r, "g_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_C(M, e_nC) \arrow[d, "\cong"] \\
h_C(\oplus_{i=1}^k e_{s(a_i)}C, M)^* \arrow[r, "h_C(g,M)^*"] \arrow[d, "\cong"'] & h_C(e_nC, M)^* \arrow[d, "\cong"] \\
\oplus_{i=1}^k (Me_{s(a_i)})^* \arrow[r, "\theta"] & (Me_n)^*,
\end{tikzcd}$$

where θ is the dual of the morphism: $Me_n \xrightarrow{\cdot\iota(ra_i^{-1})} \oplus_{i=1}^k Me_{s(a_i)}$. Recall that M is contained in $\overline{C}$, and so is Me_n. Hence $\langle Me_n, \overline{\Omega} \rangle = 0$, and $Me_n\iota(ra_i^{-1}) = 0$ for all i. Thus the map θ in the diagram is the zero map. It follows that the map g_* is the zero map as well. So the claim follows. Thus we have $\mathrm{Ext}^2_C(M, S_n) \cong \mathrm{Hom}_C(M, e_nC)$. By the CY property, $\mathrm{Ext}^2_C(M, S_n) \cong \mathrm{Hom}_C(S_n, M)^*$. Therefore

$$1 \geq \dim \mathrm{Hom}_C(S_n, M) = \dim \mathrm{Hom}(M, e_nC) = \dim h_C(e_nC, M) = \dim Me_n. \tag{4}$$

Suppose that there is a nonzero element $\overline{c} \in \overline{C}$ such that it is a combination of nontrivial paths, say, $\overline{c} = k_1p_1 + \cdots + k_mp_m$ with $k_1 \neq 0$. Let D be the subcoalgebra of $\overline{C}$ generated by $\overline{c}$. Then D can be viewed as a finite dimensional left subcomodule of $\overline{C}$. Since D must contain the vertex $s(p_1)$ and $\overline{c}$, we deduce that $\dim De_{s(p_1)}$ is at least 2. This contradicts with the fact (4). Hence $\overline{C}$ is exactly the subcoalgebra of CQ generated by the vertices of Q. Therefore $\overline{\Omega}$ must be the ideal of $\mathbf{k}Q$ generated by all the arrows of Q.

(iii) The first part follows from the fact $\dim \mathrm{Ext}^1_C(S_n, S_m) = \dim \mathrm{Ext}^1_C(S_m, S_n)$. The second part follows from (i) and (ii) since the arrows in $\overline{\Omega}$ starting from a vertex n are contained in the ideal generated by $\{ra^{-1}|a \in Q_1\}$, where r is the unique element r in R with $s(r) = n$.

(iv) If Ω is graded, then any element in R is homogeneous. For any element $r \in R$, we may assume $r = k_1p_1 + \cdots + k_mp_m$ with $k_1k_2\cdots k_m \neq 0$. By (i) and (ii), there are at least one path among $p_1, \dots, p_m$ of length 2. Otherwise $\overline{\Omega}$ could not be $\mathbf{k}Q_{\geq 1}$. This forces all the paths $p_1, \dots, p_m$ to be of length 2. □

Theorem 4.2. *Let (Q, Ω) be as in Theorem 4.1. Assume $C = C(Q, \Omega)$ is CY-3.*

(i) *Let R be a minimal relation set. For any vertices i, j, we have $\#\{a \in Q_1|s(a) = i, t(a) = j\} = \#\{r \in R|s(r) = j, t(r) = i\}$;*

(ii) *Assume further that the ideal Ω is graded, and for any vertices $i, j \in Q_0$ and any integer $n \geq 1$, there are only finitely many paths of length n starting from i and ending at j. Then we may choose a minimal relation set R such that every element of R is a combination of paths with a fixed length;*

(iii) *Under the assumptions of (ii), we may choose a minimal relation set R and a correspondence $\nu : Q_1 \to R$ such that $s(\nu(a)) = t(a), t(\nu(a)) = s(a)$ for all $a \in Q_1$, and for each arrow b with $t(b) = i$, $r_b = \sum\limits_{s(a)=i, a\in Q_1} k_b a r_a b^{-1}$, where $r_a = \nu(a)$ and $k_b \in \mathbf{k}$.*

Proof. (i) By the CY property, $\dim \mathrm{Ext}^1_C(S_j, S_i) = \dim \mathrm{Ext}^2_C(S_i, S_j)$. From the minimal injective resolutions of S_i and of S_j, one easily obtain $\#\{a \in Q_1|s(a) = i, t(a) = j\} = \dim \mathrm{Ext}^1_C(S_j, S_i)$ and $\#\{r \in R|s(r) = j, t(r) = i\} = \dim \mathrm{Ext}^2_C(S_i, S_j)$.

(ii) Since Ω is graded, we may choose a minimal relation set R such that every element of R is a combination of paths with the same length. Since the global dimension of C is 3, the minimal injective resolution of S_i is of the following form

$$0 \longrightarrow S_i \longrightarrow e_iC \xrightarrow{f} \bigoplus_{t(a)=i} e_{s(a)}C \xrightarrow{g} \bigoplus_{r\in R, t(r)=i} e_{s(r)}C \xrightarrow{\eta} e_iC \longrightarrow 0, \tag{5}$$

where f and g is the map formed in (2). We want to construct the map η explicitly. Since C is graded, all the maps in the above sequence are graded maps. The map p is determined by a sequence of maps $\eta_r : e_{s(r)}C \to e_iC$ for $r \in R$ and $t(r) = i$. Note that $\mathrm{Hom}_C(e_{s(r)}C, e_iC) \cong h_C(e_iC, e_{s(r)}C)^* \cong (e_{s(r)}Ce_i)^*$. Since η is a graded map, the finiteness assumption on Q implies that there is homogeneous element $\alpha_r \in e_{s(r)}Ce_i$ such that $\eta_r(x) = \iota(\alpha_r)x$ for all $x \in e_{s(r)}C$, where we view $e_{s(r)}Ce_i$ as a subset of $\mathbf{k}Q$. Let $\overline{\Omega}$ be the ideal of $\mathbf{k}Q$ generated by the set $R\bigcup\{\alpha_r|r \in R\}$, and let $\overline{C} = \{x|\langle x, \overline{\Omega}\rangle = 0\}$. Then $\overline{C}$ is a subcoalgebra of C. For any finite dimensional subcoalgebra D of $\overline{C}$, a similar argument to the one in the proof of Theorem 4.1 shows that $\mathrm{Ext}^3_C(D, S_i) \cong \mathrm{Hom}_C(D, e_iC)$. By the CY property we obtain

$$\dim De_i = \dim h_c(e_iC, D) = \dim \mathrm{Hom}_C(D, e_iC) = \dim \mathrm{Hom}_C(S_i, D) \leq 1$$

for all $i \in Q_0$, which implies that $\overline{C}$ must be the coradical of C. Hence $\overline{\Omega} = \mathbf{k}Q_{\geq 1}$. Moreover, for any vertices i, j, we have $\#\{\alpha_r|s(r) = j, t(r) = i\} = \#\{a \in Q_1|s(a) = i, t(a) = j\}$ by (i). Since $\Omega \subseteq \mathbf{k}Q_{\geq 2}$, we obtain that α_r is a linear combination of arrows starting from $t(r)$ and ending at $s(r)$ for all α_r, and that the elements in $\{\alpha_r|s(r) = j, t(r) = i\}$ are linearly independent. Since all the maps in the resolution (5) are graded the statement (ii) follows.

(iii) Following (ii), we may choose a minimal relation set R whose elements are linear combination of paths with fixed length. Also the map $p_r : e_{s(r)}C \to e_iC$ is determined by an element that is a linear combination of arrows. Once again, for any pair of vertices (i, j), the facts that $\#\{\alpha_r|s(r) = j, t(r) = i\} = \#\{a \in Q_1|s(a) = i, t(a) = j\}$ and that the elements in $\{\alpha_r|s(r) = j, t(r) = i\}$ are linearly independent enable us to combine the elements in $R_{j,i} := \{r \in R|s(r) = j, t(r) = i\}$ linearly to obtain a new set $R'_{j,i}$ such that $\eta_{r'} : e_{s(r')}C \to e_iC$ is defined by $\eta_{r'}(x) = \iota(a)x$ with a an arrow starting from j and ending at i for all $r' \in R'_{j,i}$. Moreover, we have established a correspondence $\nu_{i,j} : \{a \in Q_1|s(a) = i, t(a) = j\} \to R'_{j,i}$. Now let $R' = \bigcup_{i,j} R'_{i,j}$. We obtain a correspondence $\nu : Q_1 \to R'$ such that $\eta_{\nu(a)} : e_{s(\nu(a))}C \to e_iC$ is defined by $\eta_{\nu(a)}(x) = \iota(a)x$. For simplicity, we assume that R itself has the above properties, and write $r_a = \nu(a)$ for all $a \in Q_1$. Let $a \in Q_1$ with $t(a) = i$. For any $x \in e_{s(a)}C$, we have

$$0 = \eta g(x) = \eta\left(\sum_{s(d)=i} \iota(r_d a^{-1})x\right) = \sum_{s(d)=i} \iota(d)\iota(r_d a^{-1})x = \sum_{s(d)=i} \iota(d r_d a^{-1})x.$$

This implies $\langle e_{s(a)}C, \sum_{s(d)=i} dr_d a^{-1}\rangle = 0$. Since $\sum_{s(d)=i} dr_d a^{-1}$ is a linear combination of paths with the same target $s(a)$, we obtain $\langle C, \sum_{s(d)=i} dr_d a^{-1}\rangle = 0$. Hence $\sum_{s(d)=i} dr_d a^{-1} \in \Omega$ by the finiteness assumption on Q.

For $\xi \in \mathrm{Ext}^2_C(S_i, S_j)$ and $\gamma \in \mathrm{Ext}^1_C(S_j, S_i)$, we compute the Yoneda products $\xi * \gamma$ and $\gamma * \xi$. Assume there are n arrows from j to i labeled as $a_1, \ldots, a_n$, and m arrows starting from i labeled as $d_1, \ldots, d_m$. Note that $\dim \mathrm{Ext}^2_C(S_i, S_j) = n = \dim \mathrm{Ext}^1_C(S_j, S_i)$. As we have seen that for every u $(1 \leq u \leq n)$, $\sum_{v=1}^m d_v r_{d_v} a_u^{-1} \in \Omega$, we obtain

$$\sum_{v=1}^{m} d_v r_{d_v} a_u^{-1} = \sum_{w=1}^{n} l_{uw} r_{a_w}, \text{ where } l_{uw} \in \mathbf{k}. \tag{6}$$

We rewrite the minimal injective resolution of S_i as follows:

$$0 \longrightarrow S_i \longrightarrow e_iC \xrightarrow{f_i} \bigoplus_{u=1}^{n} e_{s(a_u)}C \quad \oplus \bigoplus_{t(a)=i,,a\neq a_u} e_{s(a)}C \quad \xrightarrow{g_i} \bigoplus_{v=1}^{m} e_{s(r_{d_v})}C \xrightarrow{\eta_i} e_iC \longrightarrow 0.$$

Since $\mathrm{Ext}^1_C(S_j, S_i) \cong \mathrm{Hom}_C(S_j, \bigoplus_{u=1}^n e_{s(a_u)}C)$, γ can be represented by a map $f_\gamma \in \mathrm{Hom}_C(S_j, \bigoplus_{u=1}^n e_{s(a_u)}C)$. similarly, ξ can be represented by a map $g_\xi \in \mathrm{Hom}_C(S_i, \bigoplus_{v=1}^m e_{s(r_{d_v})}C)$. Consider now the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S_j \arrow[r] \arrow[dr, "f_\gamma"] & e_jC \arrow[r, "f_j"] \arrow[d, "\theta"] & \bigoplus_{t(b)=j} e_{s(b)}C \arrow[r, "g_j"] \arrow[d, "\zeta"] & \bigoplus_{t(r)=j} e_{s(r)}C \arrow[d, "\varphi"] \\
S_i \arrow[r] & e_iC \arrow[r, "f_i"'] & \bigoplus_{u=1}^{n} e_{s(a_u)}C \quad \oplus \bigoplus_{t(a)=i,,a\neq a_u} e_{s(a)}C \arrow[r, "g_i"'] & \bigoplus_{v=1}^{m} e_{s(r_{d_v})}C \arrow[r, "\eta_i"'] & e_iC.
\end{tikzcd}$$

Now assume $f_\gamma : S_j \to \bigoplus_{u=1}^n e_{s(a_u)}C$ is defined by $x \mapsto (k_1x, \dots, k_nx)$ where k_ux is regarded as an element in $e_{s(e_u)}C$. Now we can construct the maps θ, ζ and φ as follows. For $x \in e_jC$, let $\theta(x) = (k_1x, \dots, k_nx)$; for $x \in e_{s(b)}C$, let $\zeta(x) = \sum_{u,v} k_u\iota(r_{d_v}a_u^{-1}b^{-1})x$. For the map φ, we notice that $\bigoplus\limits_{t(r)=j} e_{s(r)}C = \bigoplus\limits_{t(r)=j,s(r)\neq i} e_{s(r)}C \oplus \bigoplus\limits_{u=1}^{n} e_{s(r_{a_u})}C$. Now if $s(r) \neq i$ and $x \in e_{s(r)}C$, set $\varphi(x) = 0$, and if $x \in e_{s(r_{a_w})}C$, set $\varphi(x) = \sum_{u=1}^n k_ul_{uw}x$, where l_{uw}'s are the coefficients in the identity (6). Now it is straightforward to check that the diagram above is commutative. Suppose that $g_\xi \in \mathrm{Hom}_C(S_i, \bigoplus_{v=1}^m e_{s(r_{d_v})}C)$ is defined by $g_\xi(x) = (\overline{k}_1x, \dots, \overline{k}_mx)$ where we view $\overline{k}_vx$ as an element of $e_{s(r_{d_v})}C$. Then $\gamma*\xi \in \mathrm{Ext}^3_C(S_i, S_i)$ is represented by the map $\Psi := \varphi \circ g_\xi \in \mathrm{Hom}_C(S_i, e_iC)$. Further, if $x \in S_i$, we see that $\Psi(x) = \sum_{u,w=1}^n \overline{k}_wk_ul_{uw}x$.

Similarly, we can see that $\xi * \gamma$ is represented by a map $\psi \in \mathrm{Hom}_C(S_j, e_jC)$ with $\psi(y) = \sum_{u=1}^n \overline{k}_uk_uy$ for all $y \in e_jC$. Since C is CY-3, by [1, Appendix] there are trace map $\mathrm{Tr}_i : \mathrm{Ext}^3_C(S_i, S_i) \to \mathbf{k}$ for all i such that $\mathrm{Tr}_i(\gamma*\xi) = \mathrm{Tr}_j(\xi*\gamma)$. Since $\mathrm{Ext}^3_C(S_i, S_i)$ is of dimension 1 for all i, the trace maps $\mathrm{Tr}_i : \mathrm{Ext}^3_C(S_i, S_i) \to \mathbf{k}$ is represented by a scalar λ_i. Hence we have $\lambda_i \sum_{u,w=1}^n \overline{k}_wk_ul_{uw} = \lambda_j \sum_{u=1}^n \overline{k}_uk_u$ for arbitrary choices of $(k_1, \dots, k_n)$ and $(\overline{k}_1, \dots, \overline{k}_n)$. It follows that $l_{uw} = 0$ if $u \neq w$, and $l_{uu} = \frac{\lambda_j}{\lambda_i}$ for all $u = 1, \dots, n$. Hence the identity (6) is equivalent to $\sum_{v=1}^m d_vr_{d_v}a_u^{-1} = \frac{\lambda_j}{\lambda_i}r_{a_u}$. $\square$

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Appendix A.

In this appendix, we list some properties of the cohom functors of categories of comodules. These properties are probably well known. Since we could not find any reference, we give a complete account of proofs here.

Let C be an arbitrary coalgebra. If ${}^C M$ is a quasi-finite comodule, there is a cohom functor $h_C(M,-) : {}^C\mathcal{M} \to Vect_{\mathbf{k}}$. The cohom functor $h_C(M,-)$ is left adjoint to the tensor functor, that is; for any left C-comodule X and any vector space V, we have a natural isomorphism (cf. [17])

$$\Phi_{X,V} : \mathrm{Hom}(h_C(M,X),V) \longrightarrow \mathrm{Hom}_C(X, M\otimes V).$$

Let η be the unit of the adjoint pair $(h_C(M,-), M\otimes -)$.

Let D be another coalgebra. If X is a C-D-bicomodule, then $h_C(M,X)$ is a right D-comodule. Regard $h_C(M,X)$ as a left D^*-module. Then $\mathrm{Hom}(h_C(M,X),V)$ is a right D^*-module. Simultaneously, $\mathrm{Hom}_C(X, M\otimes V)$ is also a right D^*-module with the right D^*-action induced by the left D^*-action on X.

Lemma A.1. *The natural isomorphism $\Phi_{X,V}$ is right D^*-module isomorphism.*

Proof. We have to show $\Phi_{X,V}(f\cdot d^*) = \Phi_{X,V}(f)\cdot d^*$ for all $f \in \mathrm{Hom}(h_C(M,X),V)$ and $d^* \in D^*$. Note that $\Phi_{X,V}(f)$ is the composition $X \xrightarrow{\eta_X} M\otimes h_C(M,X) \xrightarrow{id\otimes f} M\otimes V$. We use ρ_X^D to denote the right D-comodule structure map of X, and use $\rho^D_{h_C(M,X)}$ to denote the right D-comodule structure map of $h_C(M,X)$. We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "\rho_X^D"'] & M\otimes h_C(M,X) \arrow[d, "id\otimes\rho^D_{h_C(M,X)}"'] & \\
X\otimes D \arrow[r, "\eta_X\otimes id"] \arrow[d, "id\otimes d^*"'] & M\otimes h_C(M,X)\otimes D \arrow[r, "id\otimes f\otimes d^*"] \arrow[d, "id\otimes id\otimes d^*"'] & M\otimes V \arrow[d, "="] \\
X \arrow[r, "\eta_X"] & M\otimes h_C(M,X) \arrow[r, "id\otimes f"] & M\otimes V.
\end{tikzcd}$$

So, we obtain

$$\begin{aligned}
(\Phi_{X,V}(f)\cdot d^*)(x) &= \Phi(f)\circ(id\otimes d^*)\circ\rho_X^D(x)\\
&= (id\otimes f)\circ(\eta_X)\circ(id\otimes d^*)\circ\rho_X^D(x)\\
&= (id\otimes f)\circ(id\otimes id\otimes d^*)\circ(id\otimes\rho^D_{h_C(M,X)})\circ\eta_X(x)\\
&= (id\otimes f)\circ(id\otimes f\cdot d^*)\circ\eta_X(x)\\
&= \Phi_{X,V}(f\cdot d^*)(x).
\end{aligned}$$

Hence $\Phi_{X,V}$ is a right D^*-module morphism. □

Lemma A.2. *Let ${}^C M$ and ${}^C N$ be quasi-finite comodules, and $f : M \to N$ be a comodule morphism. Let ${}^C X^D$ be a bicomodule. We have a commutative diagram of* *right D^*-module morphisms:*

$$\begin{tikzcd}
\mathrm{Hom}(h_C(M,X),V) \arrow[r, "\mathrm{Hom}(h_C(f,X),V)"] \arrow[d, "\Phi_{M,V}"'] & \mathrm{Hom}(h_C(N,X),V) \arrow[d, "\Phi_{N,V}"'] \\
\mathrm{Hom}_C(X,M\otimes V) \arrow[r, "\mathrm{Hom}_C(X,f\otimes V)"] & \mathrm{Hom}_C(X,N\otimes V).
\end{tikzcd}$$

Proof. The diagram follows from the following commutative diagrams in which the morphisms are natural ones:

$$\begin{tikzcd}
\mathrm{Hom}(\varinjlim_{\lambda}\mathrm{Hom}_C(X_\lambda,M)^*,V) \arrow[r] \arrow[d] & \varprojlim_{\lambda}\mathrm{Hom}(\mathrm{Hom}_C(X_\lambda,M)^*,V) \arrow[r] \arrow[d] & \varprojlim_{\lambda}\mathrm{Hom}_C(X_\lambda,M\otimes V) \arrow[d] \\
\mathrm{Hom}(\varinjlim_{\lambda}\mathrm{Hom}_C(X_\lambda,N)^*,V) \arrow[r] & \varprojlim_{\lambda}\mathrm{Hom}(\mathrm{Hom}_C(X_\lambda,N)^*,V) \arrow[r] & \varprojlim_{\lambda}\mathrm{Hom}_C(X_\lambda,N\otimes V),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(h_C(M,X),V) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}(\varinjlim_{\lambda}\mathrm{Hom}_C(X_\lambda,M)^*,V) \arrow[d] \\
\mathrm{Hom}(h_C(N,X),V) \arrow[r, "\cong"] & \mathrm{Hom}(\varinjlim_{\lambda}\mathrm{Hom}_C(X_\lambda,N)^*,V),
\end{tikzcd}$$

$$\begin{tikzcd}
\varprojlim_{\lambda}\mathrm{Hom}_C(X_\lambda,M\otimes V) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_C(X,M\otimes V) \arrow[d] \\
\varprojlim_{\lambda}\mathrm{Hom}_C(X_\lambda,N\otimes V) \arrow[r, "\cong"] & \mathrm{Hom}_C(X,N\otimes V).
\end{tikzcd}$$

In the above diagrams, the limits run through all the finite dimensional left C^*-subcomodule of X. □

Let $e \in C^*$ be an idempotent. Let ${}^C Y$ be a comodule. Then Ye is a left eCe-comodule. We have a left C-comodule morphism (cf. [5]):

(7) $$\theta_Y : Y \to eC\square_{eCe}Ye,\ y \mapsto \sum_{(y)} ey_{(-1)} \otimes y_{(0)}e,$$

where $\sum_{(y)} y_{(-1)} \otimes y_{(0)} = \rho(y)$. Since eC is a quasi-finite left C-comodule, we have a natural isomorphism

(8) $$\Psi : \mathrm{Hom}_C(Y, eC\square_{eCe}Ye) \longrightarrow \mathrm{Hom}_{eCe}(h_C(eC,Y),Ye).$$

Now let $\xi_Y = \Phi(\theta_Y) : h_C(eC,Y) \longrightarrow Ye$. If Y is a C-D-bicomodule, then θ_Y is a morphism of right D-comodules. One may check that $\xi_Y = \Phi(\theta_Y)$ is also a morphism of right D-comodules.

Lemma A.3. *The map ξ_Y defined above is an isomorphism. Moreover, we have a natural isomorphism*

$$\xi : h_C(eC,-) \to (-)e$$

of functors from the category of left C-comodules to the category of left eCe-comdules.

Proof. We already know from [5] that the functor $h_C(eC,-)$ is natural isomorphic to $(-)e$. We need to show that ξ is exactly the natural isomorphism between these two functors. From the proof of [5, Theorem 1.5], we know that there is a natural isomorphism

$$\Psi : \mathrm{Hom}_C(Y, eC\Box_{eCe}Z) \longrightarrow \mathrm{Hom}_{eCe}(Ye, Z),$$

for $Y \in {}^C\mathcal{M}$ and $Z \in {}^{eCe}\mathcal{M}$. Moreover, given $f \in \mathrm{Hom}_C(Y, eC\Box_{eCe}Z)$, we have $\Psi(f) = (e \otimes id) \circ f$. Now let $Z = Ye$. One sees that $\Psi(\theta_Y) = id_{Ye}$. So, $\theta_Y : Y \to eC\Box_{eCe}Ye$ is the unit map of the adjoint pair $((-)e, eC\Box_{eCe}-)$. Since $(h_C(eC,-), eC\Box_{eCe}-)$ is also an adjoint pair, the unit map θ_Y induces an isomorphism through the isomorphism (8). That is, $\xi_Y : h_C(eC, Y) \longrightarrow Ye$ is a natural isomorphism. $\square$

Proposition A.4. *Let X be a left C-comodule, and $e_1, e_2 \in C^*$ be idempotents. Given an element $c^* \in e_2C^*e_1$, we have a left C-comodule morphism $e_1C \xrightarrow{c^*\cdot} e_2C$ and a commutative diagram:*

$$\begin{tikzcd}
h_C(e_2C, X) \arrow[r, "h_C(c^*\cdot, X)"] \arrow[d, "\xi_X^2"'] & h_C(e_1C, X) \arrow[d, "\xi_X^1"] \\
Xe_2 \arrow[r, "\cdot c^*"] & Xe_1,
\end{tikzcd}$$

where ξ_X^1 and ξ_X^2 are natural isomorphisms formed in Lemma A.3 corresponding to idempotents e_1 and e_2 respectively.

Proof. To show the diagram to be commutative, it suffices to prove that the following compositions of morphisms coincide:

$$\varphi : M \xrightarrow{\eta_M^2} e_2C \otimes h_C(e_2C, X) \xrightarrow{id\otimes\xi_X^2} e_2C \otimes Xe_2 \xrightarrow{id\otimes\cdot c^*} e_2C \otimes Xe_1,$$

$$\phi : M \xrightarrow{\eta_M^2} e_2C \otimes h_C(e_2C, X) \xrightarrow{id\otimes h_C(c^*\cdot, X)} e_2C \otimes h_C(e_1C, X) \xrightarrow{id\otimes\xi_X^1} e_2C \otimes Xe_1,$$

where η_M^2 is the unit of the corresponding adjoint pair. By Lemma A.3, we have $\varphi = (id \otimes \cdot c^*) \circ \theta_X^2$, where θ_X^2 is formed in (7) corresponding to the idempotent e_2. In the commutative diagrams of Lemma A.2, if we set $N = e_1C$, $M = e_2C$ and $V = h_C(e_2C, X)$, then we obtain the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_M^1"] \arrow[d, "\eta_M^2"'] & e_1C \otimes h_C(e_1C, X) \arrow[d, "c^*\cdot\otimes id"] \\
e_2C \otimes h_C(e_2C, X) \arrow[r, "id\otimes h_C(c^*\cdot, X)"] & e_2C \otimes h_C(e_1C, X).
\end{tikzcd}$$

Hence

$$\begin{aligned}
\phi &= (id \otimes \xi_X^1) \circ (c^* \cdot \otimes id) \circ \eta_X^1 \\
&= (c^* \cdot \otimes id) \circ (id \otimes \xi_X^1) \circ \eta_X^1 \\
&= (c^* \cdot \otimes id) \circ \theta_X^1,
\end{aligned}$$

where θ_X^1 is formed in (7) corresponding to the idempotent e_1. Now for any element $x \in X$, we have

$$\begin{aligned}\varphi(x) &= \textstyle\sum_{(x)} e_2 x_{(-1)} \otimes x_{(0)} e_2 c^* \\ &= \textstyle\sum_{(x)} e_2 x_{(-1)} \otimes x_{(0)} e_2 c^* e_1 \\ &= \textstyle\sum_{(x)} x_{(-5)} e_2(x_{(-4)}) \otimes e_2(x_{(-3)}) c^*(x_{(-2)}) e_1(x_{(-1)}) x_{(0)} \\ &= \textstyle\sum_{(x)} x_{(-4)} e_2(x_{(-3)}) \otimes c^*(x_{(-2)}) e_1(x_{(-1)}) x_{(0)},\end{aligned}$$

and

$$\begin{aligned}\phi(x) &= \textstyle\sum_{(x)} c^* e_1 x_{(-1)} \otimes x_{(0)} e_1 \\ &= \textstyle\sum_{(x)} e_2 c^* e_1 x_{(-1)} \otimes x_{(0)} e_1 \\ &= \textstyle\sum_{(x)} x_{(-4)} e_2(x_{(-3)}) \otimes c^*(x_{(-2)}) e_1(x_{(-1)}) x_{(0)}.\end{aligned}$$

Hence $\varphi = \phi$ as desired. □

Remark A.5. *If X is a C-D-bicomodule, then the morphisms in the commutative diagram of last proposition are morphisms of right D-comodules.*

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