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NON-COMMUTATIVE $\mathbb{P}^1$-BUNDLES OVER COMMUTATIVE SCHEMES

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Abstract. In this paper we develop the theory of non-commutative $\mathbb{P}^1$-bundles over commutative (smooth) schemes. Such non-commutative $\mathbb{P}^1$-bundles occur in the theory of D-modules but our definition is more general. We can show that every non-commutative deformation of a Hirzebruch surface is given by a non-commutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ in our sense.

Contents

1. Introduction

In this paper we develop the theory of non-commutative $\mathbb{P}^1$-bundles over commutative (smooth) schemes. Such non-commutative $\mathbb{P}^1$-bundles occur in the theory of D-modules (see [5]) but our definition is more general. The extra generality is needed to cover basic examples in non-commutative algebraic geometry [30]. As an indication that our definition is the "right one", we present a proof that every non-commutative deformation of a Hirzebruch surface is given by a non-commutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (see below).

Let us explain our definition. Assume that X is a scheme of finite type over a field k. Following [30] and later [23, 24], we define a $\operatorname{shbimod}(X-X)$ as the category of coherent $\mathcal{O}_{X\times X}$ modules whose support is finite over X on the left and right. We call the elements of $\operatorname{shbimod}(X-X)$ "sheaf-bimodules" to distinguish them from the somewhat more general bimodules which were introduced in [31]. The category of coherent sheaves on X may be identified with the objects in $\operatorname{shbimod}(X-X)$ supported on the diagonal.

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Convolution makes shbimod$(X-X)$ into a monoidal category so we may define a "$\mathbb{Z}$-graded sheaf-algebra" on X to be a graded algebra object in shbimod$(X-X)$. If $\mathcal{A}$ is a graded sheaf-algebra, then we may define a category $\mathrm{Gr}(\mathcal{A})$ of graded $\mathcal{A}$-modules. Following [1], we define $\mathrm{QGr}(\mathcal{A})$ as $\mathrm{Gr}(\mathcal{A})$ divided by the modules which are direct limits of right bounded ones.

A first approximative approach to non-commutative $\mathbb{P}^1$-bundles on X, advocated in [23, 24, 30], is to consider abelian categories of the form $\mathrm{QGr}(\mathcal{A})$, where $\mathcal{A}$ is a graded sheaf-algebra on X which resembles the symmetric algebra of a locally free sheaf of rank two on X.

In order to explain this definition, we need a notion of locally free sheaf in shbimod$(X-X)$. We say that $\mathcal{E} \in$ shbimod$(X-X)$ is locally free (of rank n) if $\mathrm{pr}_{1*}\mathcal{E}$ and $\mathrm{pr}_{2*}\mathcal{E}$ are locally free (of rank n). If $\mathcal{E} \in$ shbimod$(X-X)$, then we may define the tensor algebra $T_X\mathcal{E}$ in the obvious way. If $\mathcal{E}$ is locally free of rank two, then in [23, 24, 30] a non-commutative symmetric algebra of rank two associated to $\mathcal{E}$ is defined as a graded sheaf-algebra of the form $T_X\mathcal{E}/(\mathcal{Q})$ where $\mathcal{Q} \subset \mathcal{E}\otimes\mathcal{E}$ is $\mathcal{Q}$ is locally free of rank one. While this is a reasonable definition, there are some problems with it.

- It is not so easy to find suitable $\mathcal{Q}$ inside $\mathcal{E}\otimes\mathcal{E}$ (see the complicated computations in [30]).
- The dependence of $\mathrm{QGr}(T_X\mathcal{E}/(Q))$ on $\mathcal{Q}$ has not been made clear.

In this paper we solve these problems by showing that $\mathcal{Q}$ is actually superfluous (!) if X is smooth. In other words the theory can be set up in a manner which does not depend on an additional choice of $\mathcal{Q}$.

We need the concept of a sheaf-$\mathbb{Z}$-algebra on X. This is a sheaf-algebra version of a usual $\mathbb{Z}$-algebra [7, 27]. Thus a sheaf-$\mathbb{Z}$-algebra on X is defined by giving for $i,j \in \mathbb{Z}$ an object $\mathcal{A}_{ij}$ in shbimod$(X-X)$ together with "multiplication maps" $\mathcal{A}_{ij}\otimes\mathcal{A}_{jk}\to\mathcal{A}_{ik}$ and "identity maps" $\mathcal{O}_X\to\mathcal{A}_{ii}$ satisfying the usual axioms. As in the graded case, we may define abelian categories $\mathrm{Gr}(\mathcal{A})$ and $\mathrm{QGr}(\mathcal{A})$.

Let $\mathcal{E}$ be locally free of rank n. Then it is easy to show that $-\otimes_{\mathcal{O}_X}\mathcal{E}$ has a right adjoint $-\otimes_{\mathcal{O}_X}\mathcal{E}^*$, where $\mathcal{E}^* \in$ shbimod$(X-X)$ is also locally free of rank n (this depends on X being smooth). Repeating this construction, we may define $\mathcal{E}^{*2}=\mathcal{E}^{**}$ by requiring that $-\otimes_{\mathcal{O}_X}\mathcal{E}^{**}$ is the right adjoint of $-\otimes_{\mathcal{O}_X}\mathcal{E}^*$. By induction we define $\mathcal{E}^{*0}=\mathcal{E}$, $\mathcal{E}^{*m+1}=(\mathcal{E}^{*m})^*$ for $m\ge 0$, and by considering left adjoints we may define $\mathcal{E}^{*m}$ for $m<0$.

Standard properties of adjoint functors yield a bimodule inclusion $i_m : \mathcal{O}_X \hookrightarrow \mathcal{E}^{*m}\otimes\mathcal{E}^{*(m+1)}$.

We now define $\mathbb{S}(\mathcal{E})$ as the $\mathbb{Z}$-algebra which satisfies

(a) $\mathbb{S}(\mathcal{E})_{mm}=\mathcal{O}_X$;
(b) $\mathbb{S}(\mathcal{E})_{m,m+1}=\mathcal{E}^{*m}$;
(c) $\mathbb{S}(\mathcal{E})$ is freely generated by the $\mathbb{S}(\mathcal{E})_{m,m+1}$, subject to the relations given by the images of i_m.

Definition 1.1. The non-commutative $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ on X associated to the locally free sheaf bimodule of rank two $\mathcal{E}$ on X is the category $\mathrm{QGr}(\mathbb{S}(\mathcal{E}))$.

It is easy to see that if $\mathcal{E}$ is an ordinary commutative vector bundle of rank two on X, then $\mathrm{Gr}(S_X(\mathcal{E}))\cong\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. Thus the notion of a non-commutative $\mathbb{P}^1$-bundle is a generalization of the commutative one. This is no longer true in higher rank, but even then the algebra $\mathbb{S}(\mathcal{E})$ could be interesting in its own right.

We will show (see §4.2) that if $\mathcal{E} \in \mathrm{shbimod}(X - X)$ is locally free of rank n and $\mathcal{Q} \subset \mathcal{E} \otimes \mathcal{E}$ is of rank one and satisfies a suitable non-degeneracy condition, then $\mathrm{Gr}(T_X\mathcal{E}/(\mathcal{Q}) = \mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. This shows that the current definition of $\mathbb{P}^1$-bundles is indeed a generalization of the earlier one.

Let us now give a more detailed description of the content of this paper. Our first main result is the following.

Theorem 1.2. *If $\mathcal{E}$ is locally free of rank two, then $\mathbb{S}(\mathcal{E})$ is a noetherian sheaf-$\mathbb{Z}$-algebra in the sense that* $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ *is a locally noetherian Grothendieck category.*

To prove this, we follow a standard approach (see [3]) which consists in defining a suitable quotient $\mathcal{D}$ of $\mathcal{A} = \mathbb{S}(\mathcal{E})$ through the functor of point-modules. The sheaf-$\mathbb{Z}$-algebra $\mathcal{D}$ will be noetherian by construction, and we will show that there is an invertible ideal $\mathcal{J} \subset \mathcal{A}_{\geq 2}$ such that $\mathcal{D} = \mathcal{A}/\mathcal{J}$. Then we may conclude by invoking a suitable variant of the Hilbert basis theorem.

Point-modules over sheaf-($\mathbb{Z}$)-algebras have been defined in Adam Nyman's Ph.D. thesis [20], and he has shown that the corresponding functor is representable (under suitable hypotheses). In particular it follows from his results that the point functor of $\mathbb{S}(\mathcal{E})$ is representable by $\mathbb{P}_{X\times X}(\mathcal{E})$. We reproduce the proof of this fact, since we need the exact nature of the bijections involved.

Our second main result is the following.

Theorem 1.3. *Assume that Z is a Hirzebruch surface. Then every deformation of Z is a non-commutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.*

For a precise definition of the notion of deformation, we refer to §7.2 (which is based on [29]). The proof of Theorem 1.3 is based on the observation that on Z there are canonical exceptional line bundles which may be lifted to any deformation. Imitating some standard constructions in commutative algebraic geometry using the resulting objects yields the desired result.

After this paper was put on `arXiv` the theory of non-commutative $\mathbb{P}^1$-bundles has been further developed. In [21, 22] it was proved that they are Ext-finite and satisfy a classical form of Serre duality. These papers use Theorem 6.1.2 below. In return the current proof of Theorem 1.3 uses some results from [21, 22].

In [16] it was shown that non-commutative $\mathbb{P}^1$-bundles share a number of geometric properties with their commutative counterparts. These results are stated in the language of non-commutative algebraic geometry (where Grothendieck categories play the role of spaces, see, e.g., [26, 31]). In this setting one may define a structure map $f : \mathbb{P}(\mathcal{E}) \to X$ and Izuru Mori shows that the fibers do not intersect. He also defines a certain "quasi-section" for f and computes its self-intersection. In [17] Izuru Mori computes the derived category of non-commutative $\mathbb{P}^1$-bundles.

In [8] the authors attack the reverse question. They generalize a standard characterization of ruled surfaces [12] to the non-commutative case. Due to some new non-commutative phenomena that have to be dealt with, they do not yet obtain a full analogue but nonetheless non-commutative $\mathbb{P}^1$-bundles appear as a basic example. Along the way the authors prove that non-commutative $\mathbb{P}^1$-bundles satisfy the Bondal-Kapranov strengthening of Serre duality [6] and are "strongly noetherian" (which is important for the construction of Hilbert schemes in this generality [2]).

2. Notation and conventions

Unless otherwise specified, all schemes below will be of finite type over a field k.

3. Sheaf-bimodules

3.1. **Generalities.** In the current and the next section we recapitulate the definition of sheaf-bimodules from [30] and we give additional properties. Since we will need to work with certain families of objects it will be convenient to develop the material over a base-scheme S. In the applications we will assume $S = \operatorname{Spec} k$.

Below S is a scheme and $\alpha : X \to S$, $\beta : Y \to S$, $\gamma : Z \to S$ will be S-schemes. An S-central coherent $(X-Y)$-sheaf-bimodule $\mathcal{E}$ is by definition a coherent $\mathcal{O}_{X\times_S Y}$-module such that the support of $\mathcal{E}$ is finite over both X and Y. We denote the corresponding abelian category by $\operatorname{shbimod}_S(X-Y)$. More generally, an S-central $(X-Y)$-sheaf-bimodule will be a quasi-coherent sheaf on $X \times_S Y$, which is a filtered direct limit of objects in $\operatorname{shbimod}_S(X-Y)$. We denote the corresponding category by $\operatorname{ShBimod}_S(X-Y)$. An object $\mathcal{E}$ in $\operatorname{ShBimod}_S(X-Y)$ defines a right exact functor $-\otimes_{\mathcal{O}_X} \mathcal{E} : \operatorname{Qch}(X) \to \operatorname{Qch}(Y)$ commuting with direct sums via $\operatorname{pr}_{2*}(\operatorname{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times_S Y}} \mathcal{E})$. If $\mathcal{F}$ is an object in $\operatorname{ShBimod}_S(Y-Z)$, then the tensor product $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}$ is defined as $\operatorname{pr}_{13*}(\operatorname{pr}_{12}^* \mathcal{E} \otimes_{\mathcal{O}_{X\times Y\times Z}} \operatorname{pr}_{23}^* \mathcal{F})$. It is easy to show that this definition yields all the expected properties (see [30]).

Now assume that we have finite S-maps $u : W \to X$ and $v : W \to Y$. If $\mathcal{U}$ is a quasi-coherent $\mathcal{O}_W$-module, then we denote the $(X-Y)$-bimodule $(u,v)_*\mathcal{U}$ by ${}_u\mathcal{U}_v$. Any bimodule $\mathcal{E}$ can be presented in this form since we may take W to be the scheme-theoretic support of $\mathcal{E}$. From the definition it is easy to check that $-\otimes {}_u\mathcal{U}_v = v_*(u^*(-) \otimes_{\mathcal{O}_W} \mathcal{U})$.

It is useful to know that the functor $-\otimes_{\mathcal{O}_X} \mathcal{E}$ actually determines $\mathcal{E}$. Let us define $\operatorname{Bimod}(X-Y)$ as the category of right exact functors $\operatorname{Qch}(X) \to \operatorname{Qch}(Y)$ commuting with direct sums (this is equivalent to the definition in [31]). Then we have a functor

$$F : \operatorname{ShBimod}_S(X-Y) \to \operatorname{Bimod}(X-Y),$$

which sends $\mathcal{E}$ to the functor $-\otimes_{\mathcal{O}_X} \mathcal{E}$. We have the following result.

Lemma 3.1.1. *The functor F is fully faithful.*

Proof. We have to show how to reconstruct $\mathcal{E}$ from the functor $-\otimes_{\mathcal{O}_X} \mathcal{E}$.

Choose an affine open covering $X = \bigcup_i U_i$, and let $u_i : U_i \to X$, $u_{ij} : U_i \cap U_j \to X$ be the inclusion maps.

Assume that $H : \operatorname{Qch}(X) \to \operatorname{Qch}(Y)$ is a right exact functor commuting with direct sums. Then $H(u_{i*}\mathcal{O}_{U_i})$ will be a quasi-coherent sheaf on Y with an $\mathcal{O}_X(U_i)$ structure. There is a corresponding quasi-coherent sheaf H_i on $U_i \times_S Y$.

In a similar way we find quasi-coherent sheaves H_{ij} on $(U_i \cap U_j) \times_S Y$ together with maps $H_i \mid U_i \cap U_j \to H_{ij}$. We define $\mathcal{F} = \ker(\bigoplus_i u_{i*}H_i \to \bigoplus_{i\neq j} u_{ij*}H_{ij})$. It is easy to see that if $H = -\otimes_{\mathcal{O}_X} \mathcal{E}$, then $\mathcal{F} = \mathcal{E}$. □

It would be interesting to give a more precise characterization of the essential image of the functor F. One useful observation is that if $\mathcal{E} \in \operatorname{ShBimod}_S(X-Y)$, then $-\otimes_{\mathcal{O}_X} \mathcal{E}$ preserves exactness of short exact sequence of vector bundles. This leads to the following example.

Example 3.1.2. Let $S = \operatorname{Spec} k$, $X = \mathbb{P}^1$, and let $H : \operatorname{Qch}(X) \to \operatorname{Qch}(X)$ be the functor given by $H(\mathcal{M}) = \mathcal{O}_{\mathbb{P}^1} \otimes_k H^1(X, \mathcal{M})$. Then H does not preserve exactness of

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1)^2 \to \mathcal{O}_{\mathbb{P}^1} \to 0,$$

and hence it is not in the essential image of F.

If we compute $\mathcal{F}$ as in the proof of Lemma 3.1.1, then we find $\mathcal{F} = 0$ which gives another reason why H is not in the essential image of F.

A partial result in this context has been obtained by Nyman in [19].

Definition 3.1.3. An object $\mathcal{E}$ in $\operatorname{shbimod}_S(X-Y)$ is locally free on the left (right) (of rank n) if $\operatorname{pr}_{1*}\mathcal{E}$ ($\operatorname{pr}_{2*}\mathcal{E}$) is locally free on X (Y) (of rank n).

The following lemma shows that tensor products of locally free bimodules behave as they should.

Lemma 3.1.4. *Assume that $\mathcal{E} \in \operatorname{shbimod}_S(X-Y)$ and $\mathcal{F} \in \operatorname{shbimod}_S(Y-Z)$ are locally free on the left. Then $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}$ is also locally free on the left. Furthermore if $\mathcal{E}$ and $\mathcal{F}$ have constant rank on the left, then so does $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}$, and the left rank of $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}$ is the product of the left ranks of $\mathcal{E}$ and $\mathcal{F}$.*

Proof. As above we may assume $\mathcal{E} = {}_u\mathcal{U}_v$, $\mathcal{F} = {}_p\mathcal{V}_q$, where $\mathcal{U}$ is a coherent W-module for finite maps $u : W \to X$, $v : W \to Y$. Then $\operatorname{pr}_{1*}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) = u_*(\mathcal{U} \otimes_{\mathcal{O}_W} v^* p_* \mathcal{V})$. Here $\mathcal{V}' = v^* p_* \mathcal{V}$ is a locally free $\mathcal{O}_W$-module. Thus we have to show that if $u : W \to X$ is a finite map and $\mathcal{U}, \mathcal{V}'$ are coherent $\mathcal{O}_W$-modules such that $\mathcal{V}'$ is locally free and $u_*\mathcal{U}$ is locally free, then $u_*(\mathcal{U} \otimes_{\mathcal{O}_W} \mathcal{V}')$ is locally free. Since the question is local on X, we may reduce to the case that X is affine. Then W is affine as well and hence $\mathcal{V}'$ is a direct summand of a free $\mathcal{O}_W$-module. So we reduce to the case $\mathcal{V}' = \mathcal{O}_W$ which is obvious.

It is sufficient to prove the assertion on the rank for all pullbacks $\operatorname{Spec} l \to S$ for l algebraically closed. Hence we may assume that $S = \operatorname{Spec} l$ with k algebraically closed.

Now let m, n be, respectively, the left rank of $\mathcal{E}$ and $\mathcal{F}$. We have to show that $\operatorname{length}(\mathcal{O}_x \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) = mn$ for all closed points $x \in X$. Since $\mathcal{O}_x \otimes_{\mathcal{O}_X} \mathcal{E}$ is an extension of m objects of the form $\mathcal{O}_{y_i}$ for some $y_i \in Y$, this is clear. □

In the sequel we will use the following lemma to show that certain sheaves are locally free.

Lemma 3.1.5. *Assume that $\psi : R \to S$ is a local ring homomorphism between noetherian commutative local rings with maximal ideals m, n. Let $u : M \to N$ be a morphism between finitely generated S modules where N is in addition flat over R. Assume that $u \otimes_R R/m$ is injective with S/mS-free cokernel. Then u is also injective with S-free cokernel.*

Proof. Let C be the cokernel of u. By hypotheses C/mC is free over S/mS. Choose an isomorphism $(S/mS)^k \to C/mC$ and lift this to a map $\theta : S^k \to C$. Let T be its cokernel. Tensoring with R/m yields $T/mT = 0$. Since $\psi(m) \subset n$, we obtain $T = 0$ by Nakayama's lemma. Now factor θ through a map $\theta' : S^k \to N$, and let K be the pullback of θ' and u. Thus we have an exact sequence,

$$0 \to K \to M \oplus S^k \xrightarrow{(u,\theta')} N \to 0.$$

Since N is flat over R, this sequence remains exact if we tensor with R/mR. Since $(S/mS)^k$ is isomorphic to $\operatorname{coker} u \otimes_R R/m$, we deduce that $K/mK = 0$. By Nakayama's lemma we obtain $K = 0$. This clearly implies what we want. □

If α is smooth, then we will say that α is *equidimensional* if the fibers of α are equidimensional and if furthermore they all have the same dimension. We will say that α is of *relative dimension* n if it is equidimensional and if all fibers have dimension n.

The following result will be very convenient:

Proposition 3.1.6. *Assume that* α, β *are smooth and equidimensional of the same relative dimension. Then* $\mathcal{E} \in \operatorname{shbimod}(X - Y)$ *is locally free on the left if and only if it is locally free on the right.*

Proof. Assume that $\mathcal{E}$ is locally free on the left. We will show that it is also locally free on the right. First consider the case that $S = \operatorname{Spec} k$. Then X and Y are regular of the same dimension. As above we may assume that $\mathcal{E} = {}_\delta\mathcal{U}_\epsilon$ for finite maps $\delta : W \to X$, $\epsilon : W \to Y$. We then have the following chain of implications:

$$\begin{aligned} \delta_*\mathcal{U} \text{ is locally free} &\Rightarrow \delta_*\mathcal{U} \text{ is maximal Cohen-Macaulay} \\ &\Rightarrow \mathcal{U} \text{ is maximal Cohen-Macaulay on } W \\ &\Rightarrow \epsilon_*\mathcal{U} \text{ is maximal Cohen-Macaulay} \\ &\Rightarrow \epsilon_*\mathcal{U} \text{ is locally free.} \end{aligned}$$

The last implication follows from the fact that Y is regular.

Now consider the case where S is general. From the hypotheses that $\delta_*\mathcal{U}$ is locally free over X, we obtain that $\mathcal{U}$ is flat over S and hence $\epsilon_*\mathcal{U}$ is also flat over S (since ϵ is finite).

Thus $\epsilon_*\mathcal{U}$ is flat over S. Since ϵ is finite, the formation of $\epsilon_*\mathcal{U}$ commutes with base change. By the above discussion we know that for every $s \in S$ we have that $\epsilon_*(\mathcal{U}_s)$ is locally free over Y_s. Then Lemma 3.1.5 with $M = 0$ shows that $\epsilon_*\mathcal{U}$ itself is locally free. □

Below we assume that $\alpha : X \to S$, $\beta : Y \to S$, $\gamma : Z \to S$ are smooth and equidimensional of the same relative dimension.

Now assume that $\mathcal{E}$ is an object in $\operatorname{shbimod}_S(X - Y)$ which is locally free on the left (and hence on the right). We will define/construct the right and left duals $\mathcal{E}^*$, ${}^*\mathcal{E}$ to $\mathcal{E}$. For brevity we restrict the discussion below to the right dual. Everything has obvious analogues for the left dual.

We want $\mathcal{E}^* \in \operatorname{shbimod}_S(Y - X)$, and in addition we should have

$$\operatorname{Hom}_Y(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{B}) = \operatorname{Hom}_X(\mathcal{A}, \mathcal{B} \otimes_{\mathcal{O}_Y} \mathcal{E}^*).$$

According to Lemma 3.1.1 this property defines $\mathcal{E}^*$ up to unique isomorphism, if it exists.

We now describe $- \otimes_{\mathcal{O}_Y} \mathcal{E}^*$ more precisely. With the same notation as before, we assume $\mathcal{E} = {}_u\mathcal{U}_v$ where $\mathcal{U} \in \operatorname{coh}(W)$. Let us denote with $v^!$ the right adjoint to v_*. Then it is easy to verify that one has

$$\mathcal{E}^* = {}_v\, \mathcal{H}om_W(\mathcal{U}, v^! \mathcal{O}_Y)_u$$

from which in particular we deduce

$$\operatorname{pr}_{1*}(\mathcal{E}^*) \cong \operatorname{pr}_{2*}(\mathcal{E})^*. \tag{3.1}$$

Thus the left structure of $\mathcal{E}^*$ is given by the dual of the right structure of $\mathcal{E}$.

Let $Rv^!$ be the right derived functor to $v^!$ (note that this is somewhat at variance with the usual definitions). Then it is clear that we also have

$$\mathcal{E}^* = {}_v\, \mathcal{H}om_W(\mathcal{U}, Rv^! \mathcal{O}_Y)_u = {}_v\, \mathrm{R}\mathcal{H}om_W(\mathcal{U}, Rv^! \mathcal{O})_u. \tag{3.2}$$

Furthermore if $\omega_{X/S}$ denotes the relative dualizing complex, then we have $Rv^!(\mathcal{O}_Y) = \omega_{W/S} \otimes_{\mathcal{O}_W} v^*\omega_{Y/S}^{-1}$ from which we deduce

$$\mathcal{E}^* = \omega_{Y/S}^{-1} \otimes_{\mathcal{O}_Y} {}_v(\mathcal{U}^D)_u, \tag{3.3}$$

where $(-)^D$ denotes the Cohen-Macaulay dual. By symmetry we have a similar formula

$$^*\mathcal{E} = {}_v(\mathcal{U}^D)_u \otimes_{\mathcal{O}_X} \omega_{X/S}^{-1}, \tag{3.4}$$

where $^*\mathcal{E}$ is defined as $\mathcal{E}^*$ but using left adjoints.

Lemma 3.1.7. *We have* $\mathcal{E}^{**} = \omega_{X/S}^{-1} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_Y} \omega_{Y/S}$.

Proof. The author learned this beautiful formula from notes by Kontsevich [13] where it is shown that it holds more generally in the setting of derived categories. In our current setting it follows trivially from (3.3).

Corollary 3.1.8. *The left rank of $\mathcal{E}$ equals the right rank of $\mathcal{E}^*$ and vice versa.*

Proof. According to (3.1), the left structure of $\mathcal{E}^*$ is given by the ordinary vector bundle dual of the right structure of $\mathcal{E}$. Thus the right rank of $\mathcal{E}$ equals the left rank of $\mathcal{E}^*$. In the same way we find that the right rank of $\mathcal{E}^*$ equals the left rank of $\mathcal{E}^{**}$. Now from Lemma 3.1.7 we easily obtain that the left rank of $\mathcal{E}^{**}$ equals the left rank of $\mathcal{E}$, which finishes the proof. □

The following lemma will be used many times.

Lemma 3.1.9. *The formation of $(-)^*$ is compatible with base change for locally free coherent sheaf-bimodules.*

Proof. If $\mathcal{E}$ is a locally free coherent sheaf-bimodule on X and we have a base extension $T \to S$, then using the formula (3.3) we see that there is at least a map of sheaf-bimodules $(\mathcal{E}^*)_T \to (\mathcal{E}_T)^*$. Then by looking at the left or right structure, we see that this map is an isomorphism. □

Using standard properties of adjoint functors together with Lemma 3.1.1 we obtain canonical maps in $\mathrm{ShBimod}_S(X - Y)$,

$$i : \mathcal{O}_X \to \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{E}^*,$$
$$j : \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_Y.$$

In the sequel we will need some properties of these maps.

Proposition 3.1.10. (1) *i is injective and its cokernel is locally free.*
(2) *j is surjective (and hence its kernel is trivially locally free).*

Proof. We only consider (1) since (2) is similar. With a similar method as the one that was used in the proof of Proposition 3.1.6, it suffices to prove this in the case that $S = \operatorname{Spec} k$. If we restrict to this case, then it is sufficient to prove that for all closed points $x \in X$ the map

$$\mathcal{O}_x \to \mathcal{O}_x \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{E}^*$$

is non-zero. Now this map is obtained by adjointness from the identity map

$$\mathcal{O}_x \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_x \otimes_{\mathcal{O}_X} \mathcal{E}.$$

Since this map is obviously non-zero, we are done. □

Below it will be convenient to have a slight generalization of the relationship that exists between members of a pair $(\mathcal{E},\mathcal{E}^*)$. Therefore, we make the following definition.

Definition 3.1.11. $\mathcal{Q}\in \operatorname{shbimod}_S(X-Z)$ is invertible if there exists $\mathcal{Q}^{-1}\in \operatorname{shbimod}_S(Z-X)$ together with isomorphisms $\mathcal{Q}\otimes_{\mathcal{O}_Z}\mathcal{Q}^{-1}\cong\mathcal{O}_X$ and $\mathcal{Q}^{-1}\otimes_{\mathcal{O}_X}\mathcal{Q}\cong \mathcal{O}_Z$.

Using the results in [4] or [1], one obtains that $\mathcal{Q}\in\operatorname{shbimod}_S(X-Z)$ is invertible if and only if $\mathcal{Q}\cong {}_{\mathrm{id}_X}(\mathcal{L})_\beta$, where $\mathcal{L}\in\operatorname{Pic}(X)$ and β is an S isomorphism between X and Z.

Definition 3.1.12. Let $\mathcal{E},\mathcal{F}$ be locally free objects, respectively, in $\operatorname{shbimod}(X-Y)$ and $\operatorname{shbimod}(Y-Z)$. Assume that $\mathcal{Q}$ is an invertible object in $\operatorname{shbimod}(X-Z)$, and assume furthermore that $\mathcal{Q}$ is contained in $\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{F}$. We say that $\mathcal{Q}$ is non-degenerate if the composition

$$\mathcal{E}^*\otimes_{\mathcal{O}_X}\mathcal{Q}\to\mathcal{E}^*\otimes_{\mathcal{O}_X}\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{F}\to\mathcal{F}$$

is an isomorphism.

Clearly. if $\mathcal{Q}$ is non-degenerate in $\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{F}$, then we have

$$\mathcal{E}^*\cong\mathcal{F}\otimes_{\mathcal{O}_Z}\mathcal{Q}^{-1}. \tag{3.5}$$

3.2. **Sheaf-algebras and sheaf-$\mathbb{Z}$-algebras.** In this section the notation will be as in the previous section. It is clear that $\operatorname{ShBimod}_S(X-X)$ is a monoidal category, so we can routinely define algebras and I-algebras in this category (see [7] for the definition of ordinary $\mathbb{Z}$-algebras. If we replace the indexing set $\mathbb{Z}$ by an arbitrary set I, then we obtain the notion of an I-algebra). We will call these (S-central) sheaf-algebras and (S-central) sheaf-I-algebras. For example a sheaf-algebra on X is an object $\mathcal{A}$ in $\operatorname{ShBimod}_S(X-X)$ together with a multiplication map $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{A}\to\mathcal{A}$ and a unit map $\mathcal{O}_X\to\mathcal{A}$ having the usual properties. If $\mathcal{A}$ is a sheaf-algebra on X, then we define $\operatorname{Mod}(\mathcal{A})$ as the category consisting of objects in $\operatorname{Qch}(X)$ together with a multiplication map $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{A}\to\mathcal{M}$, again satisfying the usual properties. In the same way we may define $\operatorname{ShBimod}(\mathcal{A}-\mathcal{A})$. This and similar notions will be used routinely in the sequel. We leave the obvious definitions to the reader.

The previous paragraph makes clear what we mean by a sheaf-I-algebra on X. However in the sequel we will use this notion in somewhat greater generality. So we will discuss this next.

Assume that Ξ is a family of S schemes $\alpha_i : X_i\to S$ indexed by $i\in I$. A sheaf-I-algebra on Ξ is defined by giving for $i,j\in I$ an object $\mathcal{A}_{ij}$ in $\operatorname{ShBimod}_S(X_i-X_j)$ together with "multiplication maps" $\mathcal{A}_{ij}\otimes_{\mathcal{O}_{X_j}}\mathcal{A}_{jk}\to\mathcal{A}_{ik}$ and an "identity map" $\mathcal{O}_{X_i}\to\mathcal{A}_{ii}$ satisfying the usual axioms.

If $\mathcal{A}$ is a sheaf-Ξ-algebra, then an $\mathcal{A}$-module is a formal direct sum $\bigoplus_{i\in I}\mathcal{M}_i$, where $\mathcal{M}_i\in\operatorname{Qch}(X_i)$ together with multiplication maps $\mathcal{M}_i\otimes_{\mathcal{O}_{X_i}}\mathcal{A}_{ij}\to\mathcal{M}_j$, again satisfying the usual axioms. We denote the category of $\mathcal{A}$-modules by $\operatorname{Gr}(\mathcal{A})$. It is easy to see that $\operatorname{Gr}(\mathcal{A})$ is a Grothendieck category.

Unless otherwise specified, we will now assume that $I=\mathbb{Z}$ even though some (but not all) notions below make sense more generally. We will say that $\mathcal{A}$ is noetherian if $\operatorname{Gr}(\mathcal{A})$ is a locally noetherian abelian category. In the case that $\mathcal{A}$ is noetherian, we borrow a number of definitions from [1]. Let $M\in\operatorname{Gr}(\mathcal{A})$. We say that M is *left*, resp. *right, bounded* if $M_i=0$ for $i\ll 0$, resp. $i\gg 0$. We say that

M is *bounded* if M is both left and right bounded. We say M is *torsion* if it is a direct limit of right bounded objects. We denote the corresponding category by $\operatorname{Tors}(\mathcal{A})$. Following [1] we also put $\operatorname{QGr}(\mathcal{A}) = \operatorname{Gr}(\mathcal{A})/\operatorname{Tors}(\mathcal{A})$. Furthermore, we define the following functors: $\tau : \operatorname{Gr}(\mathcal{A}) \to \operatorname{Tors}(\mathcal{A})$ is the torsion functor associated to $\operatorname{Tors}(\mathcal{A})$; $\pi : \operatorname{Gr}(\mathcal{A}) \to \operatorname{QGr}(\mathcal{A})$ is the quotient functor; $\omega : \operatorname{QGr}(A) \to \operatorname{Gr}(A)$ is the right adjoint to π; and finally $\widetilde{(-)} = \omega\pi$.

In these notes we will use the convention that if Xyz is an abelian category, then xyz denotes the full subcategory of Xyz whose objects are given by the noetherian objects. Following this convention we introduce $\operatorname{qgr}(\mathcal{A})$ and $\operatorname{tors}(\mathcal{A})$. Note that if $M \in \operatorname{tors}(\mathcal{A})$, then M is right bounded, just as in the ordinary graded case. It is also easy to see that $\operatorname{qgr}(\mathcal{A})$ is equal to $\operatorname{gr}(\mathcal{A})/\operatorname{tors}(\mathcal{A})$. We put $\mathcal{A}_{\geq l} = \bigoplus_{j-i\geq l} \mathcal{A}_{ij}$ and similarly $\mathcal{A}_{\leq l} = \bigoplus_{j-i\leq l} A_{ij}$. $\mathcal{A}_{\geq 0}$ and $\mathcal{A}_{\leq 0}$ are both sheaf-$\mathbb{Z}$-subalgebras of $\mathcal{A}$ and $\mathcal{A}_{\geq l}$ and $\mathcal{A}_{\leq l}$ are sheaf-bimodules over $\mathcal{A}_{\geq 0}$ and $\mathcal{A}_{\leq 0}$, respectively.

We say that $\mathcal{A}$ is positive if $\mathcal{A} = \mathcal{A}_{\geq 0}$.

Lemma 3.2.1 ([18]). *$\mathcal{A}$ is noetherian if and only if $\mathcal{A}_{\geq 0}$ and $\mathcal{A}_{\leq 0}$ are noetherian.*

We will use the following generalization of the Hilbert basis-theorem.

Lemma 3.2.2. *Assume that $\mathcal{A}$ is positive, and let $I \subset \mathcal{A}_{\geq 1}$ be an invertible ideal in $\mathcal{A}$ (that is an invertible object in* $\operatorname{ShBimod}(\mathcal{A} - \mathcal{A})$ *which is contained in $\mathcal{A}$). If $\mathcal{A}/I$ is noetherian, then so is $\mathcal{A}$.*

$\mathcal{A}$ is said to be strongly graded if the canonical map $\mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{A}_{jk} \to Ascr_{ik}$ is surjective for all i, j, k. We have [18]

Lemma 3.2.3. *If $\mathcal{A}$ is strongly graded, then the restriction functor* $\operatorname{Gr}(\mathcal{A}) \to \operatorname{Mod}(\mathcal{A}_{ii}) : M \mapsto M_i$ *is an equivalence of categories for all i.*

An interesting fact about sheaf-$\mathbb{Z}$-algebras is that they admit a useful form of twisting. Let $\mathcal{A}$ be a sheaf-$\mathbb{Z}$-algebra over Ξ, and let $\Xi' = (X'_i)_{i\in\mathbb{Z}}$ be another family of S-schemes. Let $\mathcal{T}_i$ be invertible objects in $\operatorname{ShBimod}_S(X_i - X'_i)$. Define the sheaf-$\mathbb{Z}$-algebra $\mathcal{B}$ via

$$\mathcal{B}_{ij} = \mathcal{T}_i^{-1} \otimes_{\mathcal{O}_{X_i}} \mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{T}_j.$$

It is easy to see that the functor

$$\oplus_i \mathcal{M}_i \mapsto \oplus_i \mathcal{M}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{T}_i$$

defines an equivalence $\operatorname{Gr}(\mathcal{A}) \cong \operatorname{Gr}(\mathcal{B})$.

3.3. **Ampleness.** If $\Xi = (\alpha_i : X_i \to S)_{i\in\mathbb{Z}}$ and $\Omega = (\beta_i : Y_i \to S)_{i\in\mathbb{Z}}$ are collections of S-schemes, then a map $\gamma : \Omega \to \Xi$ is a collection of maps $(\gamma_i : Y_i \to X_i)_{i\in\mathbb{Z}}$ such that $\alpha_i\gamma_i = \beta_i$. Now assume that the following condition holds for γ:

(C) Let $i, j \in \mathbb{Z}$ be arbitrary, and let Z be an arbitrary closed subset of $Y_i \times_S Y_j$ which is finite over both factors. Then the image of Z in $X_i \times_S X_j$ is also finite over both factors.

Example 3.3.1. Here is an example of why this condition is not vacuous even if $Y_i \to X_i$ is proper. Let $S = \operatorname{Spec} k$, and let $(E, +)$ be an elliptic curve over k. Assume $Y_i = Y_j = E \times E$ and $X_i = X_j = E$ where γ_i is the projection on the first factor in $E \times E$. Let $Z \subset (E \times E) \times (E \times E)$ be the graph of the automorphism $E \times E \to E \times E : (x, y) \mapsto (x + y, y)$. Then the projection of Z on $E \times E$ is $E \times E$ and hence is not finite over both factors.

If $\mathcal{B}$ is a sheaf-$\mathbb{Z}$-algebra on Ω and γ satisfies (C), then we may define sheaf-$\mathbb{Z}$-algebra $\gamma_*(\mathcal{B})$ on Ξ by

$$\gamma_*(\mathcal{B})_{ij} = (\gamma_i, \gamma_j)_*(\mathcal{B}_{ij}).$$

There is a canonical functor $\gamma_* : \mathrm{Gr}(\mathcal{B}) \to \mathrm{Gr}(\gamma_*\mathcal{B}) : \oplus_i \mathcal{M}_i \mapsto \oplus_i \gamma_{i,*}\mathcal{M}_i$. This functor factors through a functor $\bar{\gamma}_* : \mathrm{QGr}(\mathcal{B}) \to \mathrm{QGr}(\gamma_*\mathcal{B})$. In the sequel we will study the properties of this functor in some special cases.

Let us now assume that $\mathcal{B}$ is a positive sheaf-$\mathbb{Z}$-algebra on Ω such that all $\mathcal{B}_{ij}$ are coherent. Assume furthermore that all γ_i are proper. Examining [4, 30] leads to the following notion.

Definition 3.3.2. $\mathcal{B}$ is ample for γ if the following conditions hold:

(1) $\mathcal{B}$ is noetherian.
(2) For every $i \in \mathbb{Z}$ and $\mathcal{M} \in \mathrm{coh}(Y_i)$, we have that $\mathcal{M} \otimes_{\mathcal{O}_{Y_i}} \mathcal{B}_{ij}$ is relatively generated by global sections for the map γ_j for $j \gg 0$.
(3) For every $i \in \mathbb{Z}$, $k > 0$, and $\mathcal{M} \in \mathrm{coh}(Y_i)$, we have that $R^k\gamma_{j*}(\mathcal{M} \otimes_{\mathcal{O}_{Y_i}} \mathcal{B}_{ij}) = 0$ for $j \gg 0$.

Generalizing [1, 4, 30], we then obtain:

Theorem 3.3.3. *Assume that condition (C) holds and that all γ_i are proper. Assume furthermore that $\mathcal{B}$ is ample for γ. Then $\bar{\gamma}_*$ is an equivalence of categories. In addition $\gamma_*(\mathcal{B})$ is noetherian and the functor γ_* preserves noetherian objects.*

3.4. **Point-modules.** Point-modules over sheaf-($\mathbb{Z}$)-algebras have been introduced by Adam Nyman in his Ph.D. thesis [20]. We reproduce his definition below.

We first introduce another notion of local freeness. If $\alpha : X \to S$ is an S-scheme and $P \in \mathrm{coh}(X)$, then we say that P is coherent over S if the support of P is finite over S.

We say that P is locally free (of rank n) over S if P is coherent over S and α_*P is locally free (of rank n). If P is locally free of rank one over S, then it is of the form ζ_*Q for a unique section $\zeta : S \to X$ of α and Q a line bundle on S. Using a slight abuse of notation, we write P^{-1} for $\zeta_*(Q^{-1})$. If $\alpha : X \to S$ and $\beta : Y \to S$ are S-schemes and if $P_1 \in \mathrm{coh}(X)$, $P_2 \in \mathrm{coh}(Y)$ are locally free of rank one over S, then so is

$$P_1 \boxtimes_S P_2 = \mathrm{pr}_1^*(P_1) \otimes_{\mathcal{O}_{X\times_S Y}} \mathrm{pr}_2^*(P_2).$$

Note that if $P_1 = \zeta_{1*}(Q_1)$ and $P_2 = \zeta_{2*}(Q_2)$, then

$$P_1 \boxtimes_S P_2 = (\zeta_1, \zeta_2)_*(Q_1 \otimes_{\mathcal{O}_S} Q_2).$$

We will need the following result.

Lemma 3.4.1. *Assume that $\alpha : X \to S$ and $\beta : Y \to S$ are S-schemes, and let $\mathcal{E} \in \mathrm{ShBimod}_S(X - Y)$. Let $P_0 \in \mathrm{coh}(X)$, $P_1 \in \mathrm{coh}(Y)$ be locally free of rank one over S. Then we have canonical isomorphisms:*

$$\mathrm{Hom}_{\mathcal{O}_Y}(P_0 \otimes_{\mathcal{O}_X} \mathcal{E}, P_1) \cong \mathrm{Hom}_{\mathcal{O}_{X\times_S Y}}(\mathcal{E}, P_0^{-1} \boxtimes_S P_1). \tag{3.6}$$

Furthermore, under this isomorphism, epimorphisms correspond to each other.

Proof. This is a direct computation. Let $P_0 = \zeta_{0*}(Q_0)$, $P_1 = \zeta_{1*}(Q_1)$ where $\zeta_1 : S \to X$, $\zeta_2 : S \to Y$ are sections of α and β, respectively. We have

$$P_0 \otimes_{\mathcal{O}_X} \mathcal{E} = \mathrm{pr}_{2*}(\mathrm{pr}_1^* \zeta_{0*}Q_0 \otimes_{\mathcal{O}_{X\times_S Y}} \mathcal{E}).$$

Thus we have

$$\operatorname{Hom}_{\mathcal{O}_Y}(P_0 \otimes_{\mathcal{O}_X} \mathcal{E}, P_1) = \operatorname{Hom}_{\mathcal{O}_S}(\zeta_1^* \operatorname{pr}_{2*}(\operatorname{pr}_1^* \zeta_{0*} Q_0 \otimes_{\mathcal{O}_{X\times_S Y}} \mathcal{E}), Q_1).$$

If we look at the pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{(\operatorname{id}_X, \zeta_1\alpha)} & X \times_S Y \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\operatorname{pr}_2} \\ S & \xrightarrow{\zeta_1} & Y, \end{array}$$

then we find

$$\begin{aligned} \zeta_1^* \operatorname{pr}_{2*}(\operatorname{pr}_1^* \zeta_{0*} Q_0 \otimes_{\mathcal{O}_{X\times_S Y}} \mathcal{E}) &= \alpha_*(\operatorname{id}_X, \zeta_1\alpha)^*(\operatorname{pr}_1^* \zeta_{0*} Q_0 \otimes_{\mathcal{O}_{X\times_S Y}} \mathcal{E}) \\ &= \alpha_*(\zeta_{0*} Q_0 \otimes_{\mathcal{O}_X} (\operatorname{id}_X, \zeta_1\alpha)^*\mathcal{E}) \\ &= \alpha_*\zeta_{0*}(Q_0 \otimes_{\mathcal{O}_S} \zeta_0^*(\operatorname{id}_X, \zeta_1\alpha)^*\mathcal{E}) \\ &= Q_0 \otimes_{\mathcal{O}_S} (\zeta_0, \zeta_1)^*(\mathcal{E}). \end{aligned}$$

We now compute

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_S}(Q_0 \otimes_{\mathcal{O}_S} (\zeta_0, \zeta_1)^*\mathcal{E}, Q_1) &= \operatorname{Hom}_{\mathcal{O}_S}((\zeta_0, \zeta_1)^*\mathcal{E}, Q_0^{-1} \otimes_{\mathcal{O}_S} Q_1) \\ &= \operatorname{Hom}_{\mathcal{O}_{X\times_S Y}}(\mathcal{E}, (\zeta_0, \zeta_1)_*(Q_0^{-1} \otimes_{\mathcal{O}_S} Q_1)) \\ &= \operatorname{Hom}_{\mathcal{O}_{X\times_S Y}}(\mathcal{E}, P_0^{-1} \boxtimes_S P_1). \end{aligned}$$

To prove the claim about preservation of epimorphisms one simply checks that epimorphisms are preserved in each individual step. □

Now assume that $\mathcal{A}$ is a positively graded sheaf-$\mathbb{Z}$-algebra on Ξ. Just as in the case of ordinary algebras, one may define a concept of point-modules in $\operatorname{Gr}(\mathcal{A})$.

Definition 3.4.2. An m-shifted point-module over $\mathcal{A}$ is an $\mathcal{A}$-module P generated in degree m such that for $n \geq m$ we have that P_n is locally free of rank one over S. A 0-shifted point module will simply be called a point-module. An extended point-module over $\mathcal{A}$ is an $\mathcal{A}$-module P such that for all m, $P_{\geq m}$ is an m-shifted point-module.

To study point-modules it will be convenient to introduce the notion of a truncated point-module. Let $[m:n]=\{m, m+1, \ldots, n\}$, and let $\mathcal{A}_{[m:n]} = \bigoplus_{m\leq i,j\leq n} \mathcal{A}_{ij}$. Clearly, $\mathcal{A}_{[m:n]}$ is an $[m:n]$-algebra. There are obvious restriction functors $\operatorname{Gr}(\mathcal{A}) \to \operatorname{Gr}(\mathcal{A}_{[m:n]})$ and $\operatorname{Gr}(\mathcal{A}_{[m:n]}) \to \operatorname{Gr}(\mathcal{A}_{[m':n']})$ when $m' \geq m, n' \leq n$.

We define an $[m:n]$-truncated $\mathcal{A}$ point-module P as an $\mathcal{A}_{[m:n]}$-module generated in degree m such that for $n \geq i \geq m$ we have that P_i is locally free of rank one.

It is natural to declare two (truncated, extended, shifted) point modules P, Q to be equivalent if there exists a line bundle $\mathcal{L}$ on S such that $Q_n = \alpha_n^*\mathcal{L} \otimes_{\mathcal{O}_{X_n}} P_n$.

The main feature of (extended) point-modules is that they define certain sheaf-$\mathbb{Z}$-algebras which may be used to study $\mathcal{A}$. Let P be an extended point-module over $\mathcal{A}$. Thus for every i we have that P_i is locally free of rank one over S and hence $P_i = \zeta_{i*}(Q_i)$ where ζ_i is a section of α_i and $Q_i \in \operatorname{Pic}(S)$.

We define $\mathcal{B}_{ij}(P) = Q_i^{-1} \otimes_S Q_j$. Thus $\mathcal{B}(P) = \bigoplus_{ij} \mathcal{B}_{ij}(P)$ is a strongly graded sheaf-$\mathbb{Z}$-algebra on S. Let $\Omega = (S)_{i\in\mathbb{Z}}$ be the trivial constant system of S-schemes, and let $\zeta : \Omega \to \Xi$ be defined by $(\zeta_i)_i$. Then the right $\mathcal{A}$-module structure of P

yields, through Lemma 3.4.1, a surjective map $\mathcal{A}_{m,n} \to \zeta_* \mathcal{B}_{m,n}(P)$, and a straightforward verification shows that this map is compatible with multiplication. Hence we obtain a surjective map of sheaf-$\mathbb{Z}$-algebras $\mathcal{A} \to \zeta_* \mathcal{B}(P)$.

In the sequel we will need families of the concepts that were introduced above. If $\theta : W \to S$ is an S-scheme, then we can consider the base extended algebra $\mathcal{A}_W$ which is just $\bigoplus_{m,n} (\theta, \theta)^* (\mathcal{A}_{m,n})$ where we have denoted the base extension of θ to a map $X_{n,W} \to X_n$ also by θ. We define a family of point-modules over $\mathcal{A}$ parametrized by W to be a point-module on $\mathcal{A}_W$. Families of extended and truncated point-modules are defined in a similar way.

Assume that P is a family of extended point-modules parametrized by W. Then $\mathcal{B}(P)$ is a W-central sheaf-$\mathbb{Z}$-algebra on W. As above we have $P_i = \zeta_{i*}(Q_i)$ where $Q_i \in \operatorname{Pic}(W)$ and ζ_i is a section of $X_i \to X_{i,W}$. We may write ζ_i as (μ_i, id_W) with μ_i a map $W \to X_i$.

Lemma 3.4.3. *The image of (μ_i, μ_j) lies inside the support of $\mathcal{A}_{ij}$.*

Proof. By the definition of a point-module, we have a surjective map

$$P_i \otimes_{\mathcal{O}_{X_{i,W}}} \mathcal{A}_{W,ij} \to P_j,$$

which according to Lemma 3.4.1 corresponds to a surjective map

$$(\theta, \theta)^* (\mathcal{A}_{ij}) \to P_i^{-1} \boxtimes_W P_j.$$

Thus the image of (ζ_i, ζ_j) lies inside $(\theta, \theta)^{-1}(\operatorname{Supp} \mathcal{A}_{ij})$. It follows that the image of $(\mu_i, \mu_j) = (\theta \circ \zeta_i, \theta \circ \zeta_j)$ lies inside $\operatorname{Supp} \mathcal{A}_{ij}$. This proves what we want. □

Corollary 3.4.4. *Assume that $\theta : W \to S$ is proper and that all $\mathcal{A}_{ij}$ are coherent. Then the μ_i are proper. Let $\Omega = (W)_{i \in \mathbb{Z}}$ be the constant system associated to W, and let $\mu : \Omega \to \Xi$ be given by $(\mu_i)_i$. Then μ satisfies (C), and the map $\mathcal{A}_W \to \mathcal{B}(P)$ by adjointness gives rise to a map $\mathcal{A} \to \mu_* \mathcal{B}(P)$.*

Proof. The map μ_i is the composition $W \xrightarrow{\zeta_i} X_{i,W} \xrightarrow{\theta} X_i$. The first map is a section and so it is a closed immersion. In particular it is proper. The second map is also proper since it is the base extension of a proper map. Thus μ_i is also proper.

Now we can verify (C). Since (μ_i, μ_j) is proper, it is sufficient to verify that the image of (μ_i, μ_j) is finite on the left and right. This is clear since by the previous lemma this image is contained in the support of $\mathcal{A}_{ij}$ and $\mathcal{A}_{ij}$ was coherent by hypotheses. □

Equivalences among families of point-modules are defined in the same way as for ordinary point-modules (see above). For use in the sequel we introduce the following (somewhat ad hoc) notation.

$\mathrm{Points}_{m,n}(W)$	equivalence classes of $[m:n]$-truncated point-modules parametrized by W.
$\mathrm{Points}_m(W)$	equivalence classes of m-shifted point-modules parametrized by W.
$\mathrm{Points}(W)$	equivalence classes of extended point-modules parametrized by W.

4. Non-commutative symmetric algebras

4.1. **Generalities.** We will consider the following particular case of a sheaf-$\mathbb{Z}$-algebra. Let $\alpha : X \to S$, $\beta : Y \to S$ be smooth equidimensional maps of the same relative dimension, and let $\mathcal{E} \in \text{shbimod}_S(X - Y)$ be locally free.

Define

$$X_n = \begin{cases} X & \text{if } n \text{ is even,} \\ Y & \text{if } n \text{ is odd.} \end{cases} \tag{4.1}$$

In a similar way we define

$$\alpha_n = \begin{cases} \alpha & \text{if } n \text{ is even,} \\ \beta & \text{if } n \text{ is odd.} \end{cases} \tag{4.2}$$

We define $\mathcal{E}^{*n}$ as in the introduction, i.e.,

$$\mathcal{E}^{*n} = \begin{cases} \mathcal{E}^{\overbrace{*\cdots *}^{n}} & \text{if } n > 0, \\ \mathcal{E} & \text{if } n = 0, \\ {}^{\overbrace{*\cdots *}^{-n}}\mathcal{E} & \text{if } n < 0. \end{cases}$$

We then define $\mathbb{S}(\mathcal{E})$ as the sheaf-$\mathbb{Z}$-algebra generated by the $\mathcal{E}^{*n}$ subject to the relations $i(\mathcal{O}_{X_n})$. More precisely

$$\mathcal{A}_{mn} = \begin{cases} 0 & \text{if } n < m, \\ \mathcal{O}_{X_n} & \text{if } n = m, \\ \mathcal{E}^{*m} & \text{if } n = m+1, \\ \mathcal{E}^{*m} \otimes \cdots \otimes \mathcal{E}^{*n-1} / & \\ \quad (i(\mathcal{O}_{X_m}) \otimes \mathcal{E}^{*m+2} \otimes \cdots \otimes \mathcal{E}^{*n-1} & \\ \quad + \cdots + \mathcal{E}^{*m} \otimes \cdots \otimes \mathcal{E}^{*n-3} \otimes i(\mathcal{O}_{X_{n-2}})) & \text{if } n \geq m+2. \end{cases}$$

We say that $\mathbb{S}(\mathcal{E})$ is a non-commutative symmetric algebra in *standard form.*

In the sequel it will sometimes be convenient to define more general symmetric algebras. We will do so now, and then we will show that these more general symmetric algebras are equivalent to those in standard form.

Let $\alpha_n : X_n \to S$ be arbitrary smooth equidimensional maps of the same relative dimension. Assume that $(\mathcal{E}_n)_n$, $(\mathcal{Q}_n)_n$ are, respectively, a series of locally free objects in $\text{shbimod}(X_n - X_{n+1})$ and invertible objects in $\text{shbimod}(X_n - X_{n+2})$ which are non-degenerate subobjects of $\mathcal{E}_n \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{E}_{n+1}$. We then define $\mathcal{A}$ to be the $(X_n)_n$-sheaf-$\mathbb{Z}$-algebra generated by the $\mathcal{E}_n$ subject to the relations $\mathcal{Q}_n$. Thus $\mathcal{A}_{nn} = \mathcal{O}_{X_n}$, $\mathcal{A}_{n,n+1} = \mathcal{E}_n$ and $\mathcal{A}_{n,n+2} = \mathcal{E}_n \otimes \mathcal{E}_{n+1}/\mathcal{Q}_n$, etc. We will call an algebra of the form $\mathcal{A}$ a non-commutative symmetric algebra. We expect a non-commutative symmetric algebra to have good homological properties but this has only been proved in the rank two case (see below).

Now let $X = X_0$, $\alpha = \alpha_0$, $Y = X_1$, $\beta = \alpha_1$, and define X'_n, α'_n in the same way as X_n, α_n in (4.1), (4.2). Thus

$$X'_n = \begin{cases} X_0 = X & \text{if } n \text{ is even,} \\ X_1 = Y & \text{if } n \text{ is odd} \end{cases}$$

and

$$\alpha'_n = \begin{cases} \alpha_0 = \alpha & \text{if } n \text{ is even,} \\ \alpha_1 = \beta & \text{if } n \text{ is odd.} \end{cases}$$

Using (3.5) we find

$$\begin{aligned} \mathcal{E}_1 &= \mathcal{E}_0^* \otimes_{\mathcal{O}_X} \mathcal{Q}_0, \\ \mathcal{E}_2 &= \mathcal{Q}_0^{-1} \otimes_{\mathcal{O}_X} \mathcal{E}_0^{**} \otimes_{\mathcal{O}_Y} \mathcal{Q}_1, \\ \mathcal{E}_3 &= \mathcal{Q}_1^{-1} \otimes_{\mathcal{O}_Y} \mathcal{E}_0^{***} \otimes_{\mathcal{O}_X} \mathcal{Q}_0 \otimes \mathcal{Q}_2. \end{aligned}$$

Continuing, we find that for $n \in \mathbb{Z}$ there exist invertible $\mathcal{Q}'_n \in \operatorname{shbimod}(X'_n - X_n)$ such that

$$\mathcal{E}_n = \mathcal{Q}_n'^{-1} \otimes_{\mathcal{O}_{X'_n}} \mathcal{E}_0^{*n} \otimes_{\mathcal{O}_{X'_{n+1}}} \mathcal{Q}'_{n+1} \tag{4.3}$$

and

$$\mathcal{Q}_n = \mathcal{Q}_n'^{-1} \otimes_{\mathcal{O}_{X'_n}} \mathcal{Q}'_{n+2}.$$

The inclusion

$$\mathcal{Q}_n \hookrightarrow \mathcal{E}_n \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{E}_{n+1}$$

becomes an inclusion

$$\mathcal{Q}_n'^{-1} \otimes_{\mathcal{O}_{X'_n}} \mathcal{Q}'_{n+2} \hookrightarrow \mathcal{Q}_n'^{-1} \otimes_{\mathcal{O}_{X'_n}} \mathcal{E}_0^{*n} \otimes_{\mathcal{O}_{X'_{n+1}}} \mathcal{E}_0^{*(n+1)} \otimes_{\mathcal{O}_{X'_{n+2}}} \mathcal{Q}'_{n+2},$$

and it is easy to see that this inclusion is derived from the canonical inclusion $i_n : \mathcal{O}_{X'_n} \to \mathcal{E}_0^{*n} \otimes_{\mathcal{O}_{X'_{n+1}}} \mathcal{E}^{*(n+1)}$.

Thus we have shown that every non-commutative symmetric algebra is obtained from one in standard form by twisting (see §3.2).

We will say that $\mathcal{A}$ is a non-commutative symmetric algebra of rank r if $\mathcal{E}_0$ has rank r on both sides. From Corollary 3.1.8 together with (4.3) we then obtain that all $\mathcal{E}_n$ have rank r on both sides.

4.2. Relation with the definition from [30, 24, 23].

Let X be a scheme, and let $\mathcal{E} \subset \operatorname{shbimod}_S(X - X)$ be locally free. Let $\mathcal{Q} \in \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ be a non-degenerate invertible subobject, and let $\mathcal{H} = T_X(\mathcal{E})/(\mathcal{Q})$. The following lemma makes the connection between $\mathcal{H}$ and $\mathbb{S}(\mathcal{E})$.

Lemma 4.2.1. *We have* $\operatorname{Gr}(\mathcal{H}) \cong \operatorname{Gr}(\mathbb{S}(\mathcal{E}))$.

Proof. If $\mathcal{A}$ is a sheaf-$\mathbb{Z}$-graded algebra on X, then we define the $\mathbb{Z}$-graded sheaf-algebra $\check{\mathcal{A}}$ by

$$\check{\mathcal{A}}_{ij} = \mathcal{A}_{j-i}. \tag{4.4}$$

It is clear that we have $\operatorname{Gr}(\check{\mathcal{A}}) = \operatorname{Gr}(\mathcal{A})$. Furthermore it is also clear that $\check{\mathcal{A}}$ is a non-commutative symmetric algebra with $\mathcal{E}_i = \mathcal{E}$ and $\mathcal{Q}_i = \mathcal{Q}$ for all i. Since such a non-commutative symmetric algebra is obtained by twisting from $\mathbb{S}(\mathcal{E})$, we are done. □

4.3. **Point-modules over non-commutative symmetric algebras of rank two.** We let the notation be as in the previous sections but we assume in addition that $\mathcal{A}$ has rank two. We start with the following result.

Proposition 4.3.1. *Assume that $P_{[m:m+1]}$ is an $[m : m+1]$-truncated point-module over $\mathcal{A}$. Then there exist unique (up to isomorphism) $[m-1 : m+1]$ and $[m : m+2]$-truncated point modules $P_{[m-1:m+1]}$ and $P_{[m:m+2]}$ whose restriction is equal to $P_{[m:m+1]}$.*

Proof. Both claims are similar, so we only consider the second one. Since we may shift $\mathcal{A}$, we may without loss of generality assume that $m = 0$. In that case P is described by a triple (P_0, P_1, ϕ) where $P_0 \in \mathrm{coh}(X_0)$, $P_1 \in \mathrm{coh}(X_1)$ are locally free of rank one over S and $\phi : P_0 \otimes_{\mathcal{O}_X} \mathcal{E}_0 \to P_1$ is a surjective map. We have to extend this triple to a quintuple $(P_0, P_1, P_2, \phi, \psi)$ where $P_2 \in \mathrm{coh}(X_2)$ is also locally free of rank one over S, and $\psi : P_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{E}_1 \to P_2$ is another surjective map. The entries in such a quintuple are not arbitrary since the relation $\mathcal{Q}_0$ has to be satisfied. To clarify this restriction, we note that point-modules and truncated point-modules are preserved under twisting (see §3.2). Hence we may without loss of generality assume that $\mathcal{A}$ is in standard form, i.e., $\mathcal{A} = \mathbb{S}(\mathcal{E})$ for some sheaf-bimodule $\mathcal{E}$ which is locally free of rank two on both sides.

In order for $(P_0, P_1, P_2, \phi, \psi)$ to define an object in a $\mathrm{Gr}(\mathcal{A}_{[0:2]})$ module we need that the composition $P_0 \to P_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{E} \otimes_{\mathcal{O}_{X_1}} \mathcal{E}^* \xrightarrow{\phi\otimes\mathcal{E}^*} P_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{E}^* \xrightarrow{\psi} P_2$ is equal to zero since this composition represents the action of $\mathcal{Q}_0$. From Lemma 4.3.2 below it follows that this composition may be described in the following alternative way:

$$P_0 \xrightarrow{\phi^*} P_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{E}^* \xrightarrow{\psi} P_2, \tag{4.5}$$

where ϕ^* is obtained from ϕ by adjointness. Thus the pair (ψ, P_2) is a quotient of $\operatorname{coker} \phi^*$. If we now show that $\operatorname{coker} \phi^*$ is itself locally free of rank one, then we are done. This last fact follows from Lemma 4.3.4 below. □

Lemma 4.3.2. *Assume that (L, R) is a pair of adjoint functors and assume that we have objects A, B, together with a map $\phi : LA \to B$. Then the composition $A \to RLA \xrightarrow{R\phi} RB$ is equal to $\phi^* : A \to RB$.*

Proof. This is standard. □

Lemma 4.3.3. *Let $\mathcal{E} \in \mathrm{shbimod}_S(X - Y)$ be locally free on the left, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which is locally free over S. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$ is also locally free over S. If $\mathcal{E}$ has constant rank m on the left and similarly if the S-rank of $\mathcal{F}$ is constant and equal to n, then the S-rank of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$ is constant as well and equal to mn.*

Proof. This is a direct consequence of Lemma 3.1.4 if we view $\mathcal{F}$ as an $(S - X)$-bimodule. □

Lemma 4.3.4. *Let $\alpha : X \to S$, $\beta : Y \to S$ be smooth equidimensional maps of the same relative dimension. Let $\mathcal{E} \in \mathrm{shbimod}_S(X - Y)$ be locally free of rank two on both sides. Assume that we have objects $P_0 \in \mathrm{coh}(X)$, $P_1 \in \mathrm{coh}(Y)$ which are locally free of rank one over S, together with a surjective map $\phi : P_0 \otimes_{\mathcal{O}_X} \mathcal{E} \to P_1$. Then the adjoint map $\phi^* : P_0 \to P_1 \otimes_{\mathcal{O}_Y} \mathcal{E}^*$ is injective and has a cokernel which is locally free of rank one over S.*

Proof. Using Lemma 3.1.5 it suffices to prove this in the case that $S = \operatorname{Spec} k$. But then it is sufficient to show that ϕ^* is not zero (as $P_1 \otimes_{\mathcal{O}_Y} \mathcal{E}^*$ has rank two by Lemma 4.3.3). Since ϕ is not zero, this is clear. ☐

Using the bijections exhibited in Proposition 4.3.1 together with the fact that the relations in $\mathcal{A}$ have degree two we now easily obtain:

Theorem 4.3.5. *The sets of extended point-modules, m-shifted point-modules, and $[m:n]$-truncated point-modules for $n \geq m+1$ over $\mathcal{A}$ are all in bijection. These bijections are given by the appropriate restriction functors.*

Corollary 4.3.6. *The functors* $\operatorname{Points}_{\mathcal{A}}$, $\operatorname{Points}_{m,\mathcal{A}}$, *and* $\operatorname{Points}_{m,n,\mathcal{A}}$ *(for $n \geq m+1$) are all naturally equivalent.*

It follows from the proof of Proposition 4.3.1 (see (4.5)) that if P is an extended point-module over $\mathcal{A}$, then there are exact sequences on X_{j+2}

$$0 \to P_j \otimes_{\mathcal{O}_{X_j}} \mathcal{Q}_j \to P_{j+1} \otimes_{\mathcal{O}_{X_{j+1}}} \mathcal{E}_{j+1} \to P_{j+2} \to 0. \tag{4.6}$$

In fact this was only shown if $\mathcal{A}$ is in standard form, but the general case follows by twisting. Now write P_j in the usual form $\zeta_{j*}(Q_j)$ where ζ_j is a section of α_j and $Q_j \in \operatorname{Pic}(S)$. Then applying α_{j+2*} to (4.6) we obtain an exact sequence on S,

$$0 \to Q_j \otimes_{\mathcal{O}_S} \zeta_j^* \operatorname{pr}_{1*}(\mathcal{Q}_j) \to Q_{j+1} \otimes_{\mathcal{O}_S} \zeta_{j+1}^* \operatorname{pr}_{1*}(\mathcal{E}_{j+1}) \to Q_{j+2} \to 0.$$

Put $\mathcal{B} = \mathcal{B}(P)$. Tensoring the previous exact sequence on the left with Q_i^{-1} yields an exact sequence

$$0 \to \mathcal{B}_{ij} \otimes_{\mathcal{O}_S} \zeta_j^* \operatorname{pr}_{1*}(\mathcal{Q}_j) \to \mathcal{B}_{ij+1} \otimes_{\mathcal{O}_S} \zeta_{j+1}^* \operatorname{pr}_{1*}(\mathcal{E}_{j+1}) \to \mathcal{B}_{ij+2} \to 0. \tag{4.7}$$

By dualizing (4.6), tensoring on the left with $\mathcal{Q}_j$, applying a suitable variant of (3.5), applying α_{j*}, tensoring with Q_k and finally changing indices we obtain the following analogous exact sequence

$$0 \to \zeta_{i+2}^* \operatorname{pr}_{2*}(\mathcal{Q}_i) \otimes_{\mathcal{O}_S} \mathcal{B}_{i+2j} \to \zeta_{i+1}^* \operatorname{pr}_{2*}(\mathcal{E}_i) \otimes_{\mathcal{O}_S} \mathcal{B}_{i+1j} \to \mathcal{B}_{ij} \to 0. \tag{4.8}$$

4.4. **Projective bundles associated to quasi-coherent sheaves.** If Z is a scheme and $\mathcal{U}$ is a coherent sheaf on Z, then we define $\mathbb{P}_Z(\mathcal{U}) = \underline{\operatorname{Proj}}\, S_Z\mathcal{U}$, where $S_Z\mathcal{U} = \bigoplus_n S_Z^n\mathcal{U}$ denotes the symmetric algebra of $\mathcal{U}$. On $E = \mathbb{P}_Z(\mathcal{U})$ there is a canonical line bundle denoted by $\mathcal{O}(1)$ or $\mathcal{O}_E(1)$ which corresponds to $(S_Z\mathcal{U})(1)$.

If W is an arbitrary scheme and χ is a W-point of $\mathbb{P}_Z(\mathcal{U})$, then χ defines a pair $(\chi', \mathcal{L})$ where χ' is the composition $W \xrightarrow{\chi} \mathbb{P}_Z(\mathcal{U}) \to Z$ and $\mathcal{L} \in \operatorname{Pic}(W)$ is given by $\chi^*(\mathcal{O}(1))$. Clearly, $\mathcal{L}$ is a quotient of $\chi'^*(\mathcal{U})$. It is standard that conversely every pair $(\chi', \mathcal{L})$ where χ' is a map $W \to Z$ and $\mathcal{L} \in \operatorname{Pic}(W)$ is a quotient of $\chi'^*(\mathcal{U})$ corresponds to a unique $\chi : W \to \mathbb{P}_Z(\mathcal{U})$.

We will use the following result in the following sections.

Lemma 4.4.1. *Let $x \in Z$ and let $m_x \subset \mathcal{O}_{Z,x}$ be the maximal ideal. Then the scheme-theoretic closed fiber of x in $\mathbb{P}_Z(\mathcal{U})$ is equal to $\mathbb{P}_{k(x)}(\mathcal{U}_x/m_x\mathcal{U}_x)$. In particular it is equal to some $\mathbb{P}^n_{k(x)}$.*

Here is a somewhat more specialized result.

Proposition 4.4.2. *Assume that $\beta : Z \to X$ is a map of schemes, and assume that $\mathcal{E} \in \operatorname{coh}(Z)$ is coherent over X. Then the obvious map $o : \mathbb{P}_Z(\mathcal{E}) \to \mathbb{P}_X(\beta_*\mathcal{E})$ is a closed immersion. If X is a smooth connected curve over k and $\mathcal{E}$ is locally free of rank two over X, then o is either surjective or else its image is a divisor.*

Proof. All claims are local on X so we may and will assume that $X = \operatorname{Spec} R$ is affine. In addition we may replace Z by the scheme-theoretic support of $\mathcal{E}$, i.e., we may assume that β is finite. It follows that Z is also affine, say $Z = \operatorname{Spec} T$. Therefore $\mathcal{E}$ is obtained from a finitely generated T module E and $\mathbb{P}_Z(\mathcal{E}) = \operatorname{Proj} S_T(E)$, $\mathbb{P}_X(\mathcal{E}) = \operatorname{Proj} S_R(E)$. The map o is obtained from the obvious map $S_R(E) \to S_T(E)$.

To prove that o is a closed immersion, we simply remark that $S_T(E) \to S_R(E)$ is surjective in degree ≥ 1.

Now we make the additional hypotheses on our data, i.e., X is a smooth connected curve over k and $\mathcal{E}$ is locally free of rank two over X. To prove our claim, we may now make the additional simplifying assumption that $X = \operatorname{Spec} R$ where R is a discrete valuation ring.

The fact that E is Cohen-Macaulay implies that T has no embedded components. So T is free of rank one or two over R and R embeds in T.

If T is free of rank one, then $T = R$ and hence o is an isomorphism. So assume that T has rank two. Thus $T = R[z]$ where z satisfies a monic quadratic equation over R.

We now have to show that the kernel K of $S_R(E) \to S_T(E)$ is generated by one element. Let $E = Rx+Ry$. Then K is generated by $(z\cdot x)x - x(z\cdot x)$, $(z\cdot y)x - y(z\cdot x)$ and $(z\cdot y)y - y(z\cdot y)$. Write $z\cdot x = ax + by$, $z \cdot y = cx + dy$ with $a, b, c, d \in R$. Then

$$\begin{aligned}
(z\cdot x)x - x(z\cdot x) &= byx - bxy = 0,\\
(z\cdot y)x - y(z\cdot x) &= cxx + dyx - ayx - byy = cx^2 + (d-a)xy - by^2,\\
(z\cdot y)y - y(z\cdot y) &= dxy - dyx = 0.
\end{aligned}$$

Thus K is indeed generated by a single quadratic element. □

Remark 4.4.3. The preceding result is false if X is not a curve.

Consider the following example: $Z = X \times Y$ with $X = Y = A^2$, $\Delta \subset X \times Y$ is the diagonal and Γ is the graph of $(x, y) \mapsto (-x, -y)$. Let $\mathcal{E} = \mathcal{O}_\Delta \oplus \mathcal{O}_\Gamma$. Counting dimensions of fibers, we see that $\mathbb{P}_{X\times Y}(\mathcal{E})$ has dimension 2.

Clearly, $\mathbb{P}_{X\times Y}(\mathcal{E})$ contains two closed subsets given, respectively, by $\mathbb{P}_{X\times Y}(\mathcal{O}_\Delta) = \Delta$ and $\mathbb{P}_{X\times Y}(\mathcal{O}_\Gamma) = \Gamma$ which must be irreducible components since they also have dimension 2. Furthermore, outside the point $(o, o) \in X \times Y$ the map $\Delta \coprod \Gamma \to \mathbb{P}_{X\times Y}(\mathcal{E})$ is an isomorphism. However, the fiber F of (o, o) in $\mathbb{P}_{X\times Y}(\mathcal{E})$ is $\mathbb{P}^1$ whereas $\Delta \coprod \Gamma$ gives us at most two points.

Thus F must be contained in an additional irreducible component. If this irreducible component is not F itself, then it must contain some points of $\mathbb{P}_{X\times Y}(\mathcal{E})$ not above (o, o). But then F must be equal to Δ or Γ, which is a contradiction. It follows that $\mathbb{P}_{X\times Y}(\mathcal{E})$ is not equidimensional and in particular it cannot be a divisor in $\mathbb{P}_X(\operatorname{pr}_1 \mathcal{E})$.

The problem with this example is that the support $\Delta \cup \Gamma$ of $\mathcal{E}$ is not Cohen-Macaulay.

4.5. **Representability of the point functor.** The following result has been proved by Adam Nyman [20]. We reproduce the proof since we need the exact nature of the isomorphisms involved.

Theorem 4.5.1. *The functor* $\operatorname{Points}_{\mathcal{A}}$ *is representable by* $\mathbb{P}_{X\times_S Y}(\mathcal{E}_0)$.

Proof. In view of the above discussion, it is clearly sufficient to prove this for $\mathrm{Points}_{0,1,\mathcal{A}}$. We will start by giving an alternative description of $\mathrm{Points}_{0,1,\mathcal{A}}(S)$. Without loss of generality we may assume that $\mathcal{A} = \mathbb{S}(\mathcal{E})$.

An object in $\mathrm{Points}_{0,1,\mathcal{A}}(S)$ has a unique representative of the form (P_0, P_1, ϕ) where $\alpha_{0,*}(\mathcal{P}_0) = \mathcal{O}_S$ and $\phi : P_0 \otimes_{\mathcal{O}_X} \mathcal{E} \to P_1$ is an epimorphism. There exist sections ζ_0, ζ_1 of α, β and an element Q_1 of $\mathrm{Pic}(S)$ such that $P_0 = \zeta_{0,*}(\mathcal{O}_S)$ and $P_1 = \zeta_{1,*}(Q_1)$.

According to Lemma 3.4.1, ϕ corresponds to an epimorphism $\phi' : \mathcal{E} \to P_0^{-1} \boxtimes_S P_1$ and furthermore $P_0 \boxtimes_S P_1 = (\zeta_0, \zeta_1)_*(Q_0)$. Since $(\zeta_0, \zeta_1)_*(Q_0)$ contains all information to reconstruct ζ_0, ζ_1, and Q_0, we conclude that $\mathrm{Points}_{\mathcal{A},0,1}(S)$ is in one-to-one correspondence with the set of quotients of $\mathcal{E}$ on $X \times_S Y$ which are of rank one over S.

If we apply this discussion before the statement of the theorem with $Z = X \times_S Y$, $\mathcal{U} = \mathcal{E}$, $W = S$, then we find

$$\mathrm{Points}_{\mathcal{A},0,1}(S) = \mathrm{Hom}_{\mathrm{Sch}}(S, \mathbb{P}_{X\times_S Y}(\mathcal{E})).$$

Since this bijection is obviously compatible with base extension, we find that the functor $\mathrm{Points}_{\mathcal{A},0,1}$ is represented by $\mathbb{P}_{X\times_S Y}(\mathcal{E})$. This finishes the proof. □

5. Properties of the universal point algebra

From now on we assume that our base scheme S is $\mathrm{Spec}\, k$, and therefore we will omit S from the notation. Otherwise the notation will be as in the previous section.

5.1. **A vanishing result.**

Theorem 5.1.1. *Let $s : E \to \bar{E}$ be a projective map of relative dimension one. Let $\mathcal{L} \in \mathrm{coh}(E)$, and assume that the restriction to every fiber of $\mathcal{L}$ is generated by global sections and has vanishing higher cohomology. Then $R^i s_* \mathcal{L} = 0$ for $i > 0$ and the canonical map $s^* s_* \mathcal{L} \to \mathcal{L}$ is surjective.*

Proof. This is not an immediate consequence of semi-continuity since we are not assuming that $\mathcal{L}$ is flat over $\bar{E}$.

We use the theorem on formal functions. For $y \in \bar{E}$ let $E_n = E \times_{\bar{E}} \mathrm{Spec}\, \mathcal{O}_{\bar{E},y}/m_y^n$ where m_y is the maximal ideal corresponding to y. In addition let $\mathcal{L}_n$ be the restriction of $\mathcal{L}$ to E_n. Then one has [10, Thm. III.11.1]

$$(R^i s_* \mathcal{L})\hat{}_y = \operatorname*{proj\,lim}_n H^i(E_n, \mathcal{L}_n).$$

Thus in order to show that $R^i s_*(\mathcal{L}) = 0$ for $i > 0$ it is sufficient to show that

$$(H1_n) \qquad H^i(E_n, \mathcal{L}_n) = 0$$

for all y and all n.

Similarly, it is easy to see that for $s^* s_* \mathcal{L} \to \mathcal{L}$ to be surjective it is sufficient that the condition

$$(H2_n) \qquad \Gamma(E_n, \mathcal{L}_n) \otimes_k \mathcal{O}_{E_n} \to \mathcal{L}_n \text{is surjective}$$

holds for all y and all n.

Our proof will be by induction on n. It follows from the hypotheses that $(H1_1)$ and $(H2_1)$ are satisfied.

Assume now that $(H1_n)$ and $(H2_n)$ are satisfied. We have an exact sequence

$$m_y^n/m_y^{n+1} \otimes_k \mathcal{L}_1 \to \mathcal{L}_{n+1} \to \mathcal{L}_n \to 0.$$

Thus $\mathcal{F} = \ker(\mathcal{L}_{n+1} \to \mathcal{L}_n)$ is the quotient of a sheaf with vanishing higher cohomology, and since we are in dimension 1 it follows that $\mathcal{F}$ itself has vanishing higher cohomology.

Thus it follows that

$$0 \to H^0(E_1, \mathcal{F}) \to H^0(E_{n+1}, \mathcal{L}_{n+1}) \to H^0(E_n, \mathcal{L}_n) \to 0$$

is exact, and furthermore the induction hypotheses imply that $H^i(E_{n+1}, \mathcal{L}_{n+1}) = 0$ for $i > 0$. So this proves $(H2_{n+1})$.

In order to prove $(H1_{n+1})$ we use the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{L}_{n+1} & \longrightarrow & \mathcal{L}_n & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^0(E_1,\mathcal{F})\otimes_k \mathcal{O}_E & \longrightarrow & H^0(X,\mathcal{L}_{n+1})\otimes_k \mathcal{O}_E & \longrightarrow & H^0(X,\mathcal{L}_n)\otimes_k \mathcal{O}_E & \longrightarrow & 0 \end{array}$$

Since the outermost vertical maps are surjective, the same holds for the middle one. This proves $(H1_{n+1})$. □

5.2. **The case of non-commutative symmetric algebras.** In this section the notation is as before. In particular $\mathcal{A}$ is a non-commutative symmetric algebra of rank two over $\Xi = (X_i)_{i\in\mathbb{Z}}$ (see §4). As usual we put $\mathcal{E}_i = \mathcal{A}_{i,i+1}$. By definition $\mathcal{E}_i$ has rank two on both sides.

Put $E^j = \mathbb{P}_{X_j \times X_{j+1}}(\mathcal{E}_j)$. Since E^j represents $\mathrm{Points}_{\mathcal{A}}$, there is a universal extended point P^j over $\mathcal{A}_{E^j}$. We now let $\mathcal{B}^j = \mathcal{B}(P^j)$ be the associated sheaf-$\mathbb{Z}$-algebras, and we aim to study these in more detail. As above let $\zeta_i^j : E^j \to X_{i,E^j}$ be the support of P_i^j. We may write ζ_i^j as a pair $(\mu_i^j, \mathrm{id}_{E^j})$ where μ_i^j is a map from E^j to X_i. Again as above we write $P_i^j = \zeta_{i,*}^j(Q_i^j)$ for $Q_i^j \in \mathrm{Pic}(E^j)$. We will also write $\alpha_i^j : X_{i,E^j} \to E^j$ for the map obtained by base extension from $\alpha_i : X_i \to \operatorname{Spec} k$.

Our first observation is that since the E^j all represent the same functor, there must exist isomorphisms $\theta^j : E^{j+1} \to E^j$ and objects $L^j \in \mathrm{Pic}(E^j)$ such that

$$P_i^{j+1} = \alpha_i^{j+1*}(L^{j+1}) \otimes_{\mathcal{O}_{X_{i,E^{j+1}}}} \theta^{j*}(P_i^j).$$

This may be rewritten as $\mu_i^{j+1} = \mu_i^j \theta^j$ and $Q_i^{j+1} = L^{j+1} \otimes_{\mathcal{O}_{E^{j+1}}} \theta^{j*} Q_i^j$ from which we deduce

$$\mathcal{B}_{mn}^{j+1} = \theta^{j*}(\mathcal{B}_{mn}^j).$$

In the sequel we will define $\theta^{jl} : E^j \to E^l$ as the composition $\theta^l \theta^{l+1} \cdots \theta^{j-1}$ if $j \geq l$ and by a similar formula if $j < l$. Thus we find

$$\mu_i^j = \mu_i^l \theta^{jl}$$

and

$$\mathcal{B}_{mn}^j = \theta^{jl*} \mathcal{B}_{mn}^l.$$

From the proof that E^m represents $\mathrm{Points}_{\mathcal{A}}$ it follows that $\mathcal{B}_{m,m+1}^m = \mathcal{O}_{E^m}(1)$ and (μ_m^m, μ_{m+1}^m) is the projection map $E^m = \mathbb{P}_{X_m \times X_{m+1}}(\mathcal{E}_m) \to X_m \times X_{m+1}$. This allows us to describe $\mathcal{B}_{mn}^i$ in terms of the $\mathcal{O}_{E^j}(1)$ and the isomorphisms θ^{pq}.

Let $E^j \xrightarrow{s_i^j} \bar{E}_i^j \xrightarrow{\bar{\mu}_i^j} X_i$ be the Stein factorization of μ_i^j. To understand these factorizations, let us first consider μ_j^j and μ_{j+1}^j which together represent the canonical map $E^j \to X_j \times X_{j+1}$. As an intermediate step consider the Stein factorization

$E^j \to G^j \to X_j \times X_{j+1}$ of this last map. By construction [10, Cor. III.11.5] G^j is finite over the scheme theoretic image Z^j of E^j in $X_j \times X_{j+1}$. Since Z^j is finite over both X_j and X_{j+1} we obtain from the construction of $\bar{E}^j$ [10, Cor. III.11.5] that $E^j \to G^j \to X_j$ and $E^j \to G^j \to X_{j+1}$ are the Stein factorizations of $\mu_j^j : E^j \to X_j$ and $\mu_{j+1}^j : E^j \to X_{j+1}$, respectively. In particular we obtain $\bar{E}_j^j = \bar{E}_{j+1}^j$ and $s_j^j = s_{j+1}^j$. Now using the fact that Stein factorizations are (obviously) compatible with isomorphisms, we obtain from this by applying suitable θ^{pq} that $\bar{E}_j^p = \bar{E}_{j+1}^p$ and $s_j^p = s_{j+1}^p$ for all p. Thus $\bar{E}_j^p$ and s_j^p are independent of j and we may write $\bar{E}_j^p = \bar{E}^p$, $s_j^p = s^p$. Thus the result of this discussion is that we have commutative diagrams:

$$\begin{array}{ccc} E^p & \xrightarrow{\theta^{pq}} & E^q \\ {\scriptstyle s^p}\downarrow & & \downarrow{\scriptstyle s^q} \\ \bar{E}^p & \xrightarrow{\bar{\theta}^{pq}} & \bar{E}^q \\ {\scriptstyle \bar{\mu}_i^p}\downarrow & & \downarrow{\scriptstyle \bar{\mu}_i^q} \\ X_i & = & X_i. \end{array} \tag{5.1}$$

Now we investigate the scheme-theoretic closed fibers of s^j.

By Lemma 4.4.1 the scheme-theoretic fibers of $E^j \to Z^j$ are either points or $\mathbb{P}^1$'s and hence in particular they are connected. The fibers of $E^j \to \bar{E}^j$ are also connected by the properties of the Stein factorization. Hence it follows that the map $\bar{E}^j \to Z^j$ is set theoretically a bijection. In particular s^j and $E^j \to Z^j$ have the same closed fibers. We conclude that the fibers if s^j are either points or $\mathbb{P}^1$'s.

Now let Ω be the constant system of schemes $(E^0)_{i\in\mathbb{Z}}$, and let $\mu^0 = (\mu_i^0)_i$. From Corollary 3.4.4 it follows that μ^0 satisfies condition (C). We can now prove the following technical result which will be used below in the proof that a non-commutative symmetric algebra is noetherian (see §6.3 below).

Theorem 5.2.1. *$\mathcal{B}_{\geq 0}^0$ is ample for μ^0.*

Proof. Since $\mathcal{B}^0$ is strongly graded and $\mathcal{B}_{00}^0 = \mathcal{O}_{E^0}$, it is clear that $\mathcal{B}^0$ and hence $\mathcal{B}_{\geq 0}^0$ is noetherian. So we need only verify the conditions 2. and 3. from Definition 3.3.2. Let $\mathcal{M} \in \mathrm{coh}(E^0)$.

We compute

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{ij}^0 &= \mathcal{M} \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{i,i+1}^0 \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{i+1,i+2}^0 \otimes_{\mathcal{O}_{E^0}} \cdots \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{j-1,j}^0 \\ &= \mathcal{M} \otimes_{\mathcal{O}_{E^0}} \theta_*^{i0}(\mathcal{B}_{i,i+1}^i) \otimes_{\mathcal{O}_{E^0}} \cdots \otimes_{\mathcal{O}_{E^0}} \theta_*^{j-1,0}(\mathcal{B}_{j-1,j}^{j-1}) \\ &= \mathcal{M} \otimes_{\mathcal{O}_{E^0}} \theta_*^{i0}(\mathcal{O}_{E^i}(1)) \otimes_{\mathcal{O}_{E^0}} \cdots \otimes_{\mathcal{O}_{E^0}} \theta_*^{j-1,0}(\mathcal{O}_{E^{j-1}}(1)). \end{aligned}$$

Since

$$R\mu_{j*}^0(\mathcal{M} \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{ij}^0) = \bar{\mu}_{j*}^0 R s_*^0(\mathcal{M} \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{ij}^0)$$

and $\bar{\mu}_j^0$ is finite, it is sufficient to prove the analogues of 2. and 3. in Definition 3.3.2 for $R^i s_*^0(\mathcal{M} \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{ij}^0)$. According to Theorem 5.1.1, we have to show that $\mathcal{M} \otimes_{\mathcal{O}_{E^0}} \mathcal{B}_{ij}^0$ when restricted to the $\mathbb{P}^1$ fibers of s^0 becomes eventually generated by global sections. This follows from the fact that according to (5.1) the $\mathbb{P}^1$-fibers are preserved under the θ's and the fact that $\mathcal{O}_{E^m}(1)$ when restricted to a $\mathbb{P}^1$-fiber of s^m is equal to $\mathcal{O}_{\mathbb{P}^1}(1)$. □

6. On the structure of non-commutative symmetric algebras of rank two

In this section the notation is the same as in the previous ones.

6.1. **Ranks and exact sequences.** Let $e_i \in \Gamma(X_n, \mathcal{A}_{nn}) = \Gamma(X_n, \mathcal{O}_{X_n})$ be the section corresponding to 1. The structure of the relations in $\mathcal{A}$ implies that there is an exact sequence of $(\mathcal{O}_{X_m} - \mathcal{A})$-sheaf-bimodules given by

$$\mathcal{Q}_m \otimes_{\mathcal{O}_{X_{m+2}}} e_{m+2}\mathcal{A} \to \mathcal{E}_m \otimes_{\mathcal{O}_{X_{m+1}}} e_{m+1}\mathcal{A} \to e_m\mathcal{A} \to 0. \tag{6.1}$$

We will show below that this exact sequence is exact on the left.

The following proposition is proved in the same way as Proposition 4.3.1 and Theorem 4.3.5.

Proposition 6.1.1. *Assume that $Q_{[0:n]}$ is an object in $\mathrm{Gr}(\mathcal{A}_{[0:n]})$ with the following properties:*

(1) $(Q_{[0:n]})_i \neq 0$ *for all* $i \in \{0, \ldots, n\}$.
(2) $Q_{[0:n]}$ *is generated in degree zero.*
(3) $(Q_{[0:n]})_0$ *and* $(Q_{[0:n]})_1$ *have finite length and*

$$\dim H^0(Q_{[0:n]})_0 = \dim H^0(Q_{[0:n]})_1 = 1.$$

Then $Q_{[0:n]}$ is a $[0:n]$-truncated point-module. Similarly if Q is an object in $\mathrm{Gr}(\mathcal{A})$ satisfying suitable analogues of (1)–(3), then Q is a point-module.

From the fact that a point-module is uniquely determined by its restriction to $\mathcal{A}_{[0:1]}$, one obtains that if k is algebraically closed, then for every $x \in X$ there is at least one point-module P such that $P_0 = \mathcal{O}_x$.

Now we will consider line-modules. For x a rational point in X_m we define $L_{m,x} = \mathcal{O}_x \otimes_{\mathcal{O}_{X_m}} e_m\mathcal{A}$. For simplicity we write L_x for $L_{0,x}$.

If P is a point-module, then we have

$$\mathrm{Hom}_{\mathcal{A}}(L_x, P) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_x, P) = \begin{cases} k & \text{if } P_0 = \mathcal{O}_x, \\ 0 & \text{otherwise.} \end{cases}$$

Thus it follows that if k is algebraically closed, then every L_x maps onto at least one point-module. In the same way one sees that $L_{m,x}$ maps to an m-shifted point-module.

Let $L_x \to P$ be a surjective map to a point-module, and let K be its kernel. Since $\mathrm{length}(L_x)_1 = 2$ and $\mathrm{length}\, P_1 = 1$, we deduce that $K_1 \cong \mathcal{O}_y$ for some $y \in X$. Thus there is a non-zero map $L_{1,y} \to K_1$. Since $\mathrm{coker}(L_{1,y} \to L_x)$ has the same truncation to $\mathcal{A}_{[0:1]}$ as P, it follows from Proposition 6.1.1 that we have an exact sequence

$$L_{1,y} \to L_x \to P \to 0. \tag{6.2}$$

We will call this a standard exact sequence. A similar standard exact sequence exists for $L_{m,x}$:

$$L_{m+1,y} \to L_{m,x} \to P \to 0, \tag{6.3}$$

where P is now an m-shifted point-module.

We can now prove the following result.

Theorem 6.1.2. *We have*

(1) $\mathcal{A}_{m,n}$ *is locally free of rank* $n - m + 1$ *on both sides.*

(2) *The exact sequences* (6.1) *and* (6.3) *are exact on the left.*

Proof. Without loss of generality we may assume that k is algebraically closed. As far as (1) is concerned, we will only consider the left structure of $\mathcal{A}$. The statement about the right structure follows by symmetry.

Assume that we have shown that $\mathcal{A}_{m,n}$ is locally free on the left of rank $n-m+1$ for $n-m\le t$. We tensor (6.1) on the left with $\mathcal{O}_x$. Since $\operatorname{length}(\mathcal{O}_x\otimes_{\mathcal{O}_{X_m}}\mathcal{Q}_m)=1$ and $\operatorname{length}(\mathcal{O}_x\otimes_{\mathcal{O}_{X_m}}\mathcal{E}_m)=2$, we obtain that $\mathcal{O}_x\otimes_{\mathcal{O}_{X_m}}\mathcal{Q}_m=\mathcal{O}_{x'}$ and $\mathcal{O}_x\otimes_{\mathcal{O}_{X_m}}\mathcal{E}_m$ is an extension of $\mathcal{O}_{x''}$ and $\mathcal{O}_{x'''}$ for some $x',x'',x'''\in X$.

This yields

$$\begin{aligned}&\operatorname{length}(L_{m,x})_{m+t+1}\\ &\qquad\ge \operatorname{length}(L_{m+1,x''})_{m+t+1}+\operatorname{length}(L_{m+1,x'''})_{m+t+1}\\ &\qquad\quad-\operatorname{length}(L_{m+2,x'})_{m+t+1}=t+2.\end{aligned}$$

On the other hand we have from (6.3)

$$\begin{aligned}\operatorname{length}(L_{m,x})_{m+t+1}&\le 1+\operatorname{length}(L_{m+1,y})_{m+t+1}\\ &=t+2.\end{aligned}$$

Combining these two inequalities yields $\operatorname{length}(L_{m,x})_{m+t+1}=t+2$ for all m,x. Since $(L_{m,x})_{m+t+1}=\mathcal{O}_x\otimes_{\mathcal{O}_X}\mathcal{A}_{m,m+t+1}$ this yields that $\mathcal{A}_{m,m+t+1}$ is locally free of rank $t+2$ on the left.

By induction we obtain the corresponding statement for all m,n. From this we easily obtain that (6.1) and (6.3) are exact on the left. $\square$

6.2. **Two different types.** We need the following notation. Let $X=X'\bigcup X''$ and $Y=Y'\bigcup Y''$ be disjoint unions of schemes, and let $p',p'':X',X''\to X$, $q',q'':Y',Y''\to Y$ be the inclusion maps. Assume $\mathcal{M}'\in\operatorname{ShBimod}(X'-Y')$, $\mathcal{M}''\in\operatorname{ShBimod}(X''-Y'')$. Then we define $\mathcal{M}'\boxplus\mathcal{M}''$ as $(p',q')_*(\mathcal{M}')\oplus(p'',q'')_*(\mathcal{M}'')$. We use a similar construction for sheaf-$\mathbb{Z}$-algebras. We leave the obvious definitions to the reader.

We will now analyze the $\mathcal{F}\in\operatorname{shbimod}(X-Y)$ which are locally free of rank two on both sides. As usual we assume that X, Y are smooth of the same dimension and equidimensional.

Let Z be the scheme theoretic support of $\mathcal{F}$. Since $\mathcal{F}$ is Cohen-Macaulay, all components of Z have the same dimension and there are no embedded components.

Assume that Z has an irreducible component Z' on which the restriction of $\mathcal{F}$ has rank two (generically). Z' lies over connected components X' and Y' of X and Y. Let X'' and Y'' be the union of the other connected components of X and Y. Counting ranks we see that there can be no other irreducible components of Z lying above X' and Y' and hence $\mathcal{F}=\mathcal{F}'\boxplus\mathcal{F}''$ where $\mathcal{F}'\in\operatorname{shbimod}(X'-Y')$ and $\mathcal{F}''\in\operatorname{shbimod}(X''-Y'')$.

Let us return to $\mathcal{F}'$. Since Z' is integral and has degree one over X' and Y' and since X' and Y' are furthermore integrally closed we obtain that Z is the graph of an isomorphism $\sigma:X\to Y$ and $\mathcal{F}$ is a vector bundle of rank two on Z'.

It is clear that $\mathbb{S}(\mathcal{F})=\mathbb{S}(\mathcal{F}')\boxplus\mathbb{S}(\mathcal{F}'')$. A similar decomposition then holds for every non-commutative symmetric algebra by twisting. Furthermore we leave it to the reader to check that $\operatorname{Gr}(\mathbb{S}(\mathcal{F}'))$ is equivalent to $\operatorname{Gr}(S_{Z'}(\mathcal{F}'))$ and hence corresponds to a commutative $\mathbb{P}^1$-bundle.

To formalize this let us make the following definition.

Definition 6.2.1. Let $\mathcal{A}$ be a non-commutative symmetric algebra of rank two, and let $\mathcal{E} = \mathcal{A}_{01}$. We say that $\mathcal{A}$ is of Type I if $\mathcal{E}$ is a rank two bundle over the graph of an automorphism and we say that $\mathcal{A}$ is of Type II if the restrictions of $\mathcal{E}$ to the irreducible components of its support all have rank one generically.

Thus we have obtained the following result.

Proposition 6.2.2. *Let $\mathcal{A}$ be a non-commutative symmetric algebra of rank two. Then $\mathcal{A} = \mathcal{A}' \boxplus \mathcal{A}''$ where $\mathcal{A}'$ is of Type I and $\mathcal{A}''$ is of type II.* $\mathrm{Gr}(\mathcal{A}')$ *is equivalent to the category of graded modules over the symmetric algebra of a rank two vector bundle over a smooth scheme.*

Example 6.2.3. The most basic example of a Type II symmetric algebra is obtained by embedding a smooth elliptic curve C as a divisor of degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and letting $\mathcal{E} = {}_u\mathcal{L}_v$ where $\mathcal{L}$ is a line bundle on C and $(u, v) : C \to \mathbb{P}^1 \times \mathbb{P}^1$ denotes the embedding. Such non-commutative symmetric algebras appeared naturally in [30] and provided one of the motivations for writing the current paper.

6.3. **Non-commutative symmetric algebras of rank two are noetherian.** Since to prove $\mathcal{A}$ is noetherian we may treat the cases of Type I and Type II individually, and since the Type I case is easy, we assume throughout that $\mathcal{A}$ is of Type II.

From Theorems 5.2.1 and 3.3.3 we obtain that $\mu_*\mathcal{B}^0{}_{\geq 0}$ is noetherian. Furthermore by construction there is a map $\mathcal{A} \to \mu^0_*\mathcal{B}^0{}_{\geq 0}$. We would like to use this map in order to analyze $\mathcal{A}$. However the analysis is complicated by the fact that E^0 may have components of different dimensions if $\dim X_n > 1$ (see Remark 4.4.3).

Therefore we will use the following trick. We will let F^j be the union of all components in E^j which are of maximal dimension, and we let $t^j : F^j \to E^j$ be the inclusion map. It is clear that θ^{jl} restricts to a map $F^j \to F^l$ which we will also denote by θ^{jl}.

Let $\mathcal{C}_{mn} = t^*(\mathcal{B}^0{}_{mn})$. Then $\mathcal{C} = \bigoplus_{m \leq n} \mathcal{C}_{mn}$ is a $\mathbb{Z}$-algebra on F^0.

Put $\lambda^j_i = \mu^j_i t^i$ and $\lambda = (\lambda^0_i)_i$. From the fact that $\mathcal{B}_{\geq 0}$ is ample for μ (Theorem 5.2.1) we easily obtain that $\mathcal{C}$ is ample for λ. We will now analyze the map $\mathcal{A} \to \lambda_*\mathcal{C}$.

Step 1. The map $\mathcal{A}_{ii} \to (\lambda_*\mathcal{C})_{ii}$ is monic. If we denote its cokernel by $\mathcal{S}_{ii}$, then $\mathcal{S}_{ii}$ is locally free of rank one on both sides.

To see this, we will show that

$$\mathrm{pr}_{1*}(\mathcal{A}_{ii}) \to \mathrm{pr}_{1*}(\lambda_*(\mathcal{C})_{ii}) \tag{6.4}$$

is monic and its cokernel is locally free of rank one. The corresponding statement for the right structure is similar.

We have $\mathcal{O}_{X_i} = \mathrm{pr}_{1*}(\mathcal{A}_{ii})$ and $\mathrm{pr}_{1*}(\mathcal{C}_{ii}) = \mathrm{pr}_{1*}(\lambda^0_i, \lambda^0_i)_*(\mathcal{O}_{F^0}) = \lambda^0_{i*}(\mathcal{O}_{F^0}) = \lambda^0_{i*}\theta^{i0}_*(\mathcal{O}_{F^i}) = \lambda^i_{i*}(\mathcal{O}_{F^i})$. So we need to show that $\mathcal{O}_{X_i} \to \lambda^i_{i*}(\mathcal{O}_{F^i})$ is monic and that its cokernel is locally free of rank one.

Put $B = \mathbb{P}_{X_i}(\mathrm{pr}_{1*}(\mathcal{E}_i))$, and let $\mathcal{O}_B(n) = \mathcal{O}_{\mathbb{P}_{X_i}(\mathrm{pr}_{1*}(\mathcal{E}_i))}(n)$. Denote the projection map $B \to X_i$ by p. By Proposition 4.4.2 the map $E^i \to B$ is a closed immersion. So the composition $F^i \to E^i \to B$ is a closed immersion as well. We denote this composition by v. Now since $\mathcal{A}$ is of Type II it easy to see that $\dim F^i = \dim X_i$. Hence F is a divisor in B. Generically $\mathcal{E}_0$ will be invertible over its support and hence generically F will have degree two over X_i. Since according to [10, II. Ex. 7.9] one has $\mathrm{Pic}(B) = \mathrm{Pic}(X_i) \times \mathbb{Z}^x$ where x is the number of connected components of X and the factor $\mathbb{Z}^x$ corresponds to the degrees over the generic fibers, it follows that $\mathcal{O}_B(-F^i) = \mathcal{L} \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_B(-2)$ where $\mathcal{L} \in \operatorname{Pic}(X_i)$.

We now apply Rp_* to the exact sequence

$$0 \to \mathcal{O}_B(-F^i) \to \mathcal{O}_B \to v_* \mathcal{O}_{F^i} \to 0. \tag{6.5}$$

Using the known properties of the map $p : B \to X_i$ [10, Ex. III.8.4] we extract from the long exact sequence for Rp_* a short exact sequence

$$0 \to \mathcal{O}_{X_i} \to \lambda^i_{i*}(\mathcal{O}_{F^i}) \to \wedge^2(\operatorname{pr}_{1*} \mathcal{E}_i)^* \otimes_{\mathcal{O}_{X_i}} \mathcal{L} \to 0. \tag{6.6}$$

This proves what we want.

We obtain in addition that $R^h \lambda^i_{i*}(\mathcal{O}_{F^i}) = 0$ for $h > 0$. This may be rephrased as the next step.

Step 2. $R^h(\lambda^0_i, \lambda^0_i)_*(\mathcal{C}_{ii}) = 0$ for $h > 0$.

Step 3. The map $\mathcal{A}_{i,i+1} \to (\lambda_* \mathcal{C})_{i,i+1}$ is an isomorphism.

Arguing as in Step 1, we reduce the problem to showing that the canonical map $\operatorname{pr}_{1*}(\mathcal{E}_i) \to \lambda^i_{i*}(\mathcal{O}_{F^i}(1))$ is an isomorphism.

Tensoring (6.5) by $\mathcal{O}_B(1)$ and applying Rp_*, we obtain what we want and in addition we obtain $R^h \lambda^i_{i*}(\mathcal{O}_{F^i}(1)) = 0$ for $h > 0$. This then yields the next step.

Step 4. $R^h(\lambda^0_i, \lambda^0_{i+1})_*(\mathcal{C}_{i,i+1}) = 0$ for $h > 0$. Indeed the image of $(\lambda^0_i, \lambda^0_{i+1})$ is finite over X_i. Thus it is sufficient to prove $\operatorname{pr}_{1*} R^h(\lambda^0_i, \lambda^0_{i+1})_*(\mathcal{C}_{i,i+1}) = 0$. By the Leray spectral sequence this then follows from $R^h \lambda^0_{i,*}(\mathcal{C}_{i,i+1}) = 0$, which is a restatement of $R^h \lambda^i_{i*}(\mathcal{O}_{F^i}(1)) = 0$.

Step 5. Now we translate the exact sequence (4.7) to our current situation. It becomes

$$0 \to \mathcal{C}_{ij} \otimes_{\mathcal{O}_{F^0}} \lambda^{0*}_j \operatorname{pr}_{1*}(\mathcal{Q}_j) \to \mathcal{C}_{ij+1} \otimes_{\mathcal{O}_{F^0}} \lambda^{0*}_{j+1} \operatorname{pr}_{1*}(\mathcal{E}_{j+1}) \to \mathcal{C}_{ij+2} \to 0.$$

Using Steps 2 and 4, one obtains by induction that the following sequence is exact:

$$\begin{aligned} 0 \to (\lambda^0_i, \lambda^0_j)_*(\mathcal{C}_{ij}) \otimes_{\mathcal{O}_{X_j}} \mathcal{Q}_j \\ \to (\lambda^0_i, \lambda^0_{j+1})_*(\mathcal{C}_{ij+1}) \otimes_{\mathcal{O}_{X_{j+1}}} \mathcal{E}_{j+1} \to (\lambda^0_i, \lambda^0_{j+2})_* \mathcal{C}_{ij+2} \to 0 \end{aligned}$$

and furthermore that $R^h(\lambda^0_i, \lambda^0_j)_*(\mathcal{C}_{i,j}) = 0$ for $h > 0$.

Step 6. The map $\mathcal{A}_{ii+2} \to (\lambda_* \mathcal{C})_{ii+2}$ is an epimorphism. If we denote its kernel by $\mathcal{T}_{ii+2}$, then $\mathcal{T}_{ii+2} = \mathcal{S}_{ii} \otimes_{\mathcal{O}_{X_i}} \mathcal{Q}_i$. In particular $\mathcal{T}_{ii+2}$ is locally free of rank one on both sides.

To prove these statements we consider the following commutative diagram with exact rows.

(6.7)

$$\begin{array}{ccccccccc} 0 & \to & (\lambda^0_i, \lambda^0_i)_*(\mathcal{C}_{ii}) \otimes_{\mathcal{O}_{X_i}} \mathcal{Q}_i & \to & (\lambda^0_i, \lambda^0_{i+1})_*(\mathcal{C}_{ii+1}) \otimes_{\mathcal{O}_{X_{i+1}}} \mathcal{E}_{i+1} & \to & (\lambda^0_i, \lambda^0_{i+2})_* \mathcal{C}_{ii+2} & \to & 0 \\ & & \uparrow & & \cong\uparrow & & \uparrow & & \\ 0 & \to & \mathcal{A}_{ii} \otimes_{\mathcal{O}_{X_i}} \mathcal{Q}_i & \to & \mathcal{A}_{ii+1} \otimes_{\mathcal{O}_{X_{i+1}}} \mathcal{E}_{i+1} & \to & \mathcal{A}_{ii+2} & \to & 0 \end{array}$$

(The second row is the dual version of (6.1).) Applying the snake lemma to (6.7) together with Step 1 yields what we want.

Step 7. Assume $j \geq i-1$. Then the complex

$$0 \to \mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{T}_{jj+2} \to \mathcal{A}_{ij+2} \to (\lambda_i^0, \lambda_{j+2}^0)_*(\mathcal{C}_{ij+2}) \to 0$$

is exact.

We prove this by induction on j. The cases $j = i-1, i$ were covered by the previous steps. Assume now $j \geq i+1$. We consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & (\lambda_i^0,\lambda_j^0)_*(\mathcal{C}_{ij}) \otimes_{\mathcal{O}_{X_j}} \mathcal{Q}_j & \to & (\lambda_i^0,\lambda_{j+1}^0)_*(\mathcal{C}_{ij+1}) \otimes_{\mathcal{O}_{X_{j+1}}} \mathcal{E}_{j+1} & \to & (\lambda_i^0,\lambda_{j+2}^0)_*\mathcal{C}_{ij+2} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{Q}_j & \to & \mathcal{A}_{ij+1} \otimes_{\mathcal{O}_{X_{j+1}}} \mathcal{E}_{j+1} & \to & \mathcal{A}_{ij+2} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{A}_{ij-2} \otimes_{\mathcal{O}_{X_{j-2}}} \mathcal{T}_{j-2j} \otimes_{\mathcal{O}_{X_j}} \mathcal{Q}_j & \to & \mathcal{A}_{ij-1} \otimes_{\mathcal{O}_{X_{j-1}}} \mathcal{T}_{j-1,j+1} \otimes_{\mathcal{O}_{X_{j+1}}} \mathcal{E}_{j+1} & \to & \mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{T}_{j,j+2} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By induction we may assume that the first two columns are exact. Hence so is the third column.

Step 8. The canonical maps $\mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{T}_{jj+2} \to \mathcal{A}_{ij+2}$ and $\mathcal{T}_{ii+2} \otimes_{X_{i+2}} \mathcal{A}_{i+2,j+2} \to \mathcal{A}_{ij+2}$ are monomorphisms, and furthermore they define an isomorphism $\mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{T}_{jj+2} \cong \mathcal{T}_{ii+2} \otimes_{X_{i+2}} \mathcal{A}_{i+2,j+2}$.

To see this, note that by the previous step we already know that the first map is a monomorphism. A similar proof involving (4.8) shows that the second map is also a monomorphism.

Since by definition $\mathcal{T}_{ii+2}$ goes to zero under the map $\mathcal{A} \to \lambda_*\mathcal{C}$, we also have that $\mathcal{T}_{ii+2} \otimes_{X_{i+2}} \mathcal{A}_{i+2,j+2}$ goes to zero. Thus the image of $\mathcal{T}_{ii+2} \otimes_{X_{i+2}} \mathcal{A}_{i+2,j+2}$ in $\mathcal{A}_{i,j+2}$ lies in the image of $\mathcal{A}_{ij} \otimes_{\mathcal{O}_{X_j}} \mathcal{T}_{jj+2}$. By symmetry the opposite inclusion will also hold, and hence we are done.

Step 9. $\mathcal{A}$ is noetherian.

By the previous steps we have an invertible ideal $\mathcal{J} \subset \mathcal{A}_{\geq 2}$ given by $\mathcal{J}_{ij} = \mathcal{A}_{ij-2} \otimes_{\mathcal{O}_{X_{i-2}}} \mathcal{T}_{j-2j} = \mathcal{T}_{ii+2} \otimes_{\mathcal{O}_{X_i}} \mathcal{A}_{i+2j}$ in $\mathcal{A}$ such that $(\mathcal{A}/\mathcal{J})_{\geq 1} = \mathcal{D}$ where $\mathcal{D}_{\geq 1} = \mathcal{C}_{\geq 1}$ and $\mathcal{D}_{ii} = \mathcal{O}_{X_i}$.

From the fact that $\mathcal{C}$ is noetherian and the fact that all $\mathcal{C}_{ij}$ are coherent, we easily obtain that $\mathcal{D}$ is noetherian. We may now conclude by invoking Lemma 3.2.2.

7. Non-commutative deformations of Hirzebruch surfaces

7.1. **Strongly ample sequences.** Let $\mathcal{E}$ be a noetherian abelian category. For us a sequence $(O(n))_{n\in\mathbb{Z}}$ of objects in $\mathcal{E}$ is *strongly ample* if the following conditions hold

(A1) For all $\mathcal{M} \in \mathcal{E}$ and for all n there is an epimorphism $\bigoplus_{i=1}^t O(-n_i) \to \mathcal{M}$ with $n_i \geq n$.

(A2) For all $\mathcal{M} \in \mathcal{E}$ and for all $i > 0$ one has $\mathrm{Ext}^i_{\mathcal{E}}(O(-n), \mathcal{M}) = 0$ for $n \gg 0$.

A strongly ample sequence $(O(n))_{n\in\mathbb{Z}}$ in $\mathcal{E}$ is ample in the sense of [25]. Hence using the methods of [1] or [25], one obtains $\mathcal{E} \cong \operatorname{qgr}(A)$ if $\mathcal{E}$ is Hom-finite, where A is the noetherian $\mathbb{Z}$-algebra $\bigoplus_{ij} \operatorname{Hom}_{\mathcal{E}}(O(-j), O(-i))$.

It would be interesting to know if a non-commutative $\mathbb{P}^1$-bundle always has an ample sequence. The next lemma is very weak but it is sufficient for us below.

Lemma 7.1.1. *Let $\mathcal{A}$ be a non-commutative symmetric algebra over $(X_n)_n$ (see §4) with all X_n being equal to a smooth projective scheme X. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Assume that $\mathcal{A}_{i,i+1}$ is generated by global sections on the right for all i and that for each m we have that $\mathcal{O}_X(-m) \otimes_{\mathcal{O}_X} \mathcal{A}_{mn} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ has vanishing cohomology for $n \gg 0$.*

Then $\operatorname{qgr}(\mathcal{A})$ has a strongly ample sequence given by

$$O(n) = \pi(\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} e_{-n}\mathcal{A}).$$

Proof. We have maps of $\operatorname{gr}(\mathcal{A})$-objects induced by the multiplication in $\mathcal{A}$

$$\mathcal{A}_{i,i+1} \otimes_{\mathcal{O}_X} e_{i+1}\mathcal{A} \to e_i\mathcal{A},$$

which are surjective in degree $\geq i+1$. Since $\mathcal{A}_{i,i+1}$ is generated by global sections on the right, these may be turned into maps

$$(e_{i+1}\mathcal{A})^{t_i} \to e_i\mathcal{A} \tag{7.1}$$

for certain t_i which are still surjective in degree $\geq i+1$.

Let $\mathcal{M} = \pi M$ with $M \in \operatorname{gr}(\mathcal{A})$ noetherian. Then there is some N such that $M_{\geq N}$ is generated in degree one. Hence there is some N', which we will take $\geq N$, such that there is an epimorphism

$$(\mathcal{O}_X(-N') \otimes_{\mathcal{O}_X} e_N\mathcal{A})^s \to M_{\geq N},$$

which, using the the maps given in (7.1), may be turned into epimorphisms

$$(\mathcal{O}_X(-N') \otimes_{\mathcal{O}_X} e_{N'}\mathcal{A})^s \to M_{\geq N'}.$$

This implies condition (A1). We now compute

$$\begin{aligned}\operatorname{RHom}_{\operatorname{QGr}(\mathcal{A})}(O(-n), \pi M) &= \operatorname{RHom}_X(\mathcal{O}_X(-n), R\omega(\pi M)_n)\\ &= R\Gamma(X, R\omega(\pi M)_n(n)).\end{aligned}$$

According to [21, Cor. 3.3+proof] and [21, Lemma 3.4] the map $M \to R\omega(\pi M)$ is an isomorphism in high degree. Hence for $n \gg 0$:

$$\operatorname{RHom}_{\operatorname{QGr}(\mathcal{A})}(O(-n), \pi M) = R\Gamma(X, M_n(n)).$$

Thus $\operatorname{Hom}_{\operatorname{QGr}(\mathcal{A})}(O(-n), -)$ has finite cohomological dimension. To prove (A2), we may then assume that $\mathcal{M} = O(-m) = \pi(\mathcal{O}_X(-m) \otimes_{\mathcal{O}_X} e_m\mathcal{A})$ for m large. Since in that case

$$M_n(n) = \mathcal{O}_X(-m) \otimes_{\mathcal{O}_X} \mathcal{A}_{mn} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n),$$

we are done. □

7.2. **Deformations of abelian categories.** For the convenience of the reader we will repeat the main statements from [29]. We first recall briefly some notions from [14]. Throughout R will be a commutative noetherian ring and $\mathrm{mod}(R)$ is its category of finitely generated modules.

Let $\mathcal{C}$ be an R-linear abelian category. Then we have bifunctors $-\otimes_R - : \mathcal{C}\times \mathrm{mod}(R) \to \mathcal{C}$, $\mathrm{Hom}_R(-,-) : \mathrm{mod}(R)\times\mathcal{C}\to\mathcal{C}$ defined in the usual way. These functors may be derived in their $\mathrm{mod}(R)$-argument to yield bi-delta-functors $\mathrm{Tor}_i^R(-,-)$, $\mathrm{Ext}^i_R(-,-)$. An object $M\in\mathcal{C}$ is *R-flat* if $M\otimes_R -$ is an exact functor, or equivalently if $\mathrm{Tor}_i^R(M,-)=0$ for $i>0$.

By definition (see [14, §3]) $\mathcal{C}$ is *R-flat* if Tor_i^R or equivalently Ext^i_R is effaceable in its $\mathcal{C}$-argument for $i>0$. This implies that Tor_i^R and Ext^i_R are universal ∂-functors in both arguments.

If $f: R\to S$ is a morphism of commutative noetherian rings such that S/R is finitely generated and $\mathcal{C}$ is an R-linear abelian category, then $\mathcal{C}_S$ denotes the (abelian) category of objects in $\mathcal{C}$ equipped with an S-action. If f is surjective, then $\mathcal{C}_S$ identifies with the full subcategory of $\mathcal{C}$ given by the objects annihilated by $\ker f$. The inclusion functor $\mathcal{C}_S\to\mathcal{C}$ has right and left adjoints given by $\mathrm{Hom}_R(S,-)$ and $-\otimes_R S$, respectively.

Now assume that J is an ideal in R, and let $\widehat{R}$ be the J-adic completion of R. Recall that an abelian category $\mathcal{D}$ is said to be *noetherian* if it is essentially small and all objects are noetherian. Let $\mathcal{D}$ be an R-linear noetherian category, and let $\mathrm{Pro}(\mathcal{D})$ be its category of pro-objects. We define $\widehat{\mathcal{D}}$ as the full subcategory of $\mathrm{Pro}(\mathcal{D})$ consisting of objects M such that $M/MJ^n\in\mathcal{D}$ for all n and such that in addition the canonical map $M\to \operatorname{proj\,lim}_n M/MJ^n$ is an isomorphism. The category $\widehat{\mathcal{D}}$ is $\widehat{R}$-linear. The following is basically a reformulation of Jouanolou's results [11].

Proposition 7.2.1 (see [29, Prop. 2.2.5])**.** *$\hat{\mathcal{D}}$ is a noetherian abelian subcategory of* $\mathrm{Pro}(\mathcal{D})$.

There is an exact functor

$$\Phi:\mathcal{D}\to\widehat{\mathcal{D}}: M\mapsto \operatorname*{proj\,lim}_n M/MJ^n, \tag{7.2}$$

and we say that $\mathcal{D}$ is complete if Φ is an equivalence of categories. In addition we say that $\mathcal{D}$ is *formally flat* if $\mathcal{D}_{R/J^n}$ is R/J^n-flat for all n.

Definition 7.2.2. Assume that $\mathcal{C}$ is an R/J-linear noetherian flat abelian category. Then an R-deformation of $\mathcal{C}$ is a formally flat complete R-linear abelian category $\mathcal{D}$ together with an equivalence $\mathcal{D}_{R/J}\cong\mathcal{C}$.

In general, to simplify the notation, we will pretend that the equivalence $\mathcal{D}_{R/J}\cong\mathcal{C}$ is just the identify.

Thus below we consider the case that $\mathcal{D}$ is complete and formally flat and $\mathcal{C}=\mathcal{D}_{R/J}$. The following definition turns out to be natural.

Definition 7.2.3 (see [29, 1.1])**.** Assume that $\mathcal{E}$ is a formally flat noetherian R-linear abelian category. Let $\mathcal{E}_t$ be the full subcategory of $\mathcal{E}$ consisting of objects annihilated by a power of J. Let $M,N\in\mathcal{E}$. Then the *completed* Ext*-groups* between M, N are defined as

$$`\,\mathrm{Ext}^i_{\widehat{\mathcal{E}}}(M,N)=\mathrm{Ext}^i_{\mathrm{Pro}(\mathcal{E}_t)}(M,N).$$

An R-linear category $\mathcal{E}$ is said to be Ext-*finite* if $\operatorname{Ext}^i_{\mathcal{E}}(M,N)$ is a finitely generated R-module for all i and all objects M, $N \in \mathcal{E}$. Assuming Ext-finiteness, the completed Ext-groups become computable.

Proposition 7.2.4 ([29, Prop. 2.5.3]). *Assume that $\mathcal{E}$ is a formally flat noetherian R-linear abelian category and that $\mathcal{E}_{R/J}$ is* Ext-*finite. Then* $`\operatorname{Ext}^i_{\widehat{\mathcal{E}}}(M,N) \in \operatorname{mod}(\hat{R})$ *for $M, N \in \widehat{\mathcal{E}}$ and furthermore*

$$`\operatorname{Ext}^i_{\widehat{\mathcal{E}}}(M,N) = \operatorname{proj}\lim_k \operatorname{inj}\lim_l \operatorname{Ext}^i_{\mathcal{E}_{R/J^l}}(M/MJ^l, N/NJ^k).$$

If M is in addition R-flat, then

$$`\operatorname{Ext}^i_{\widehat{\mathcal{E}}}(M,N) = \operatorname{proj}\lim_k \operatorname{Ext}^i_{\mathcal{E}_{R/J^k}}(M/MJ^k, N/NJ^k).$$

The results below allow one to lift properties from $\mathcal{C}$ to $\mathcal{D}$. They follow easily from the corresponding infinitesimal results ([15, Thm. A], [14, Prop. 6.13], [29]).

Proposition 7.2.5. *Let $M \in \mathcal{C}$ be a flat object such that*

$$\operatorname{Ext}^i_{\mathcal{C}}(M, M \otimes_{R/J} J^n/J^{n+1}) = 0$$

for $i = 1, 2$ and $n \geq 1$. Then there exists a unique R-flat object (up to non-unique isomorphism) $\overline{M} \in \mathcal{D}$ such that $\overline{M}/\overline{M}J \cong M$.

Proposition 7.2.6. *Let $\overline{M}, \overline{N} \in \mathcal{D}$ be flat objects and put $\overline{M}/\overline{M}J = M$, $\overline{N}/\overline{N}J = N$. Assume that for all X in* $\operatorname{mod}(R/J)$ *we have* $\operatorname{Ext}^i_{\mathcal{C}}(M, N \otimes_{R/J} X) = 0$ *for a certain $i > 0$. Then we have* $`\operatorname{Ext}^i_{\mathcal{D}}(\overline{M}, \overline{N} \otimes_R X) = 0$ *for all* $X \in \operatorname{mod}(R)$.

Proposition 7.2.7. *Let $\overline{M}, \overline{N} \in \mathcal{D}$ be flat objects and put $\overline{M}/\overline{M}J = M$, $\overline{N}/\overline{N}J = N$. Assume that for all X in* $\operatorname{mod}(R/J)$ *we have* $\operatorname{Ext}^1_{\mathcal{C}}(M, N \otimes_{R/J} X) = 0$. *Then* $\operatorname{Hom}_{\mathcal{D}}(\overline{M}, \overline{N})$ *is R-flat and furthermore for all X in* $\operatorname{mod}(R)$ *we have*

$$\operatorname{Hom}_{\mathcal{D}}(\overline{M}, \overline{N} \otimes_R X) = \operatorname{Hom}_{\mathcal{D}}(\overline{M}, \overline{N}) \otimes_R X.$$

Let us also mention Nakayama's lemma [29].

Lemma 7.2.8. *Let $M \in \mathcal{D}$ be such that $MJ = 0$. Then $M = 0$.*

The following result is a version of "Grothendieck's existence theorem".

Proposition 7.2.9 (see [29, Prop. 4.1]). *Assume that R is complete, and let $\mathcal{E}$ be an* Ext-*finite R-linear noetherian category with a strongly ample sequence $(O(n))_n$. Then $\mathcal{E}$ is complete, and furthermore if $\mathcal{E}$ is flat, then we have for $M, N \in \mathcal{E}$,*

$$\operatorname{Ext}^i_{\mathcal{E}}(M,N) = `\operatorname{Ext}^i_{\mathcal{E}}(M,N). \tag{7.3}$$

The following result shows that the property of being strongly ample lifts well.

Theorem 7.2.10 (see [29, Thm. 4.2]). *Assume that R is complete and that $\mathcal{C}$ is* Ext-*finite, and let $O(n)_n$ be a sequence of R-flat objects in $\mathcal{D}$ such that $(O(n)/O(n)J)_n$ strongly ample. Then*

(1) *$O(n)_n$ is strongly ample in $\mathcal{D}$;*
(2) *$\mathcal{D}$ is flat (instead of just formally flat);*
(3) *$\mathcal{D}$ is* Ext-*finite as R-linear category.*

7.3. **Deformations of Hirzebruch surfaces.** Below (R, m) is a complete commutative local noetherian ring with residue field $k = R/m$. Everything will now either be over k or over R. Although in the main part of this paper we have set up the theory over a base scheme of finite type over a k, it is not difficult to see that the results remain valid over $\operatorname{Spec} R$. We will use this without further comment. When we say that something is "compatible with base change", we mean compatible with the passage from R to k. We usually abbreviate $- \otimes_R k$ by $(-)_k$. We also use a subscript k to indicate that something is defined over k.

We let X_k be the Hirzebruch surface $\mathbb{P}(\mathcal{E}_k)$ with $\mathcal{E}_k = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(h)$, $h \geq 0$, and we let $\mathcal{D}$ be an R-deformation of $\mathcal{C} = \operatorname{coh}(X_k)$ in the sense of §7.2. The rest of this section will be devoted to proving the following result.

Theorem 7.3.1. *There exists a sheaf-bimodule $\mathcal{E}$ over $\mathbb{P}^1_R$ such that $\mathcal{D}$ is equivalent to* $\operatorname{qgr}(\mathbb{S}(\mathcal{E}))$.

Let $t : X_k \to \mathbb{P}^1_k$ be the projection map. Then we have standard line bundles $\mathcal{O}_k(m, n) = t^*\mathcal{O}_{\mathbb{P}^1_k}(m) \otimes_{\mathcal{O}_{X_k}} \mathcal{O}_{X_k/\mathbb{P}^1}(n)$ on X_k. From the formula

$$R\Gamma(X_k, \mathcal{O}_{X_k}(m, n)) = R\Gamma(\mathbb{P}^1_k, \mathcal{O}_{X_k}(m) \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} Rt_*\mathcal{O}_{X_k/\mathbb{P}^1_k}(n)),$$

we deduce that in particular $H^i(X_k, \mathcal{O}_{X_k}(m, n)) = 0$ for $i > 0$ and $m, n \geq 0$.

Since the $\mathcal{O}_k(m, n)$ are exceptional in $\mathcal{C}$ they lift to objects $O(m, n)$ in $\mathcal{D}$ using Proposition 7.2.5. Furthermore from the ampleness criterion in [10, Cor. V.2.18] together with Theorem 7.2.10(1), it follows that $(O(n, n))_n$ is a strongly ample sequence in $\mathcal{D}$. By item (3) of the same theorem we obtain that $\mathcal{D}$ is Ext-finite.

We now define some R-linear $\mathbb{Z}$-algebras

$$C_n = \bigoplus_{j \geq i} \operatorname{Hom}(O(-j, -n), O(-i, -n))$$

as well as $(C_m - C_n)$-bimodules for $n \geq m$,

$$A_{mn} = \bigoplus_{j \geq i} \operatorname{Hom}(O(-j, -n), O(-i, -m)).$$

From Proposition 7.2.7 it follows that C_n and A_{mn} are R-flat and compatible with base change. Hence

$$C_{n,k} = \bigoplus_{j \geq i} \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(j - i)), \tag{7.4}$$

$$A_{mn,k} = \bigoplus_{j \geq i} \operatorname{Hom}(S^{n-m}\mathcal{E}_k(j - i)). \tag{7.5}$$

We can now look for some properties of $C_{n,k}$ that lift to C_n (see [28, §8.3] for a more elaborate example of how this is done).

(P1)

$$\operatorname{rk} C_{n,ij} = \begin{cases} j - i + 1 & \text{if } j \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

(P2) Define $V_{n,i} = C_{n,i,i+1}$. Then C_n is generated by the $(V_{n,i})_i$.

(P3) Put $K_{n,i} = \ker(V_{n,i} \otimes V_{n,i+1} \to C_{n,i,i+2})$. Then the relations between the $V_{n,i}$ in C_n are generated by the $K_{n,i}$.

(P4) Rank counting reveals that $\operatorname{rk} K_{n,i} = 1$. The R-module $K_{n,i}$ is generated by a non-degenerate tensor $r_{n,i}$ in $V_{n,i} \otimes_R V_{n,i+1}$.

Using these properties, it is now easy to describe C_n. After choosing suitable bases x_i, y_i in $V_{n,i}$, we may assume that $r_i = y_i x_{i+1} - x_i y_{i+1}$. Thus all C_n are in fact isomorphic to $\check{S}$ (see (4.4)) where S is the graded algebra $R[x, y]$. In particular $\operatorname{qgr}(C_n) \cong \operatorname{coh}(\mathbb{P}^1_R)$ for all n.

It also follows that after suitable reindexing A_{mn} becomes in a natural way a bigraded $S \otimes_R S$-module which we denote by A'_{mn}. We think of A'_{mn} as an S-S-bimodule with independent left and right grading. The required reindexing is given by

$$A'_{mn;ij} = A_{mn;-i,j}.$$

Here x, y act as x_{i-1}, y_{i-1} on the left and as x_j, y_j on the right.

The following diagram is commutative:

$$\begin{array}{ccc} \operatorname{gr}(C_m) & \xrightarrow{-\otimes_{C_m} A_{mn}} & \operatorname{gr}(C_n) \\ \Vert & & \Vert \\ \operatorname{gr}(S) & \xrightarrow{(-\otimes_S A'_{mn})_{0,-}} & \operatorname{gr}(S). \end{array}$$

Here by $(-)_{0,-}$ we mean taking the part of degree zero for the left grading.

Let $\mathcal{A}_{mn}$ be the quasi-coherent $\mathcal{O}_{\mathbb{P}^1_R} \boxtimes \mathcal{O}_{\mathbb{P}^1_R}$-module associated to $A'_{mn;ij}$.

Lemma 7.3.2. *$\mathcal{A}_{mn}$ is locally free on the left and right of rank $n - m + 1$. In addition*

$$\mathcal{A}_{mn,k} \cong \delta_* S^{n-m} \mathcal{E}_k,$$

where $\delta : \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ is the diagonal embedding.

Proof. We first observe that $\mathcal{A}_{mn}$ is in fact coherent. To this end it is sufficient to show that the diagonal submodule $\bigoplus_i A'_{mn;ii}$ is a finitely generated $\bigoplus_i S_i \otimes_R S_i$-module. This may be verified after tensoring with k.

From (7.5) one obtains

$$\begin{aligned} A'_{mn} \otimes_R k &= \bigoplus_{j+i \geq 0} \Gamma(X_k, O_k(j+i, n-m)) \\ &= \bigoplus_{j+i \geq 0} \Gamma(\mathbb{P}^1_k, S^{n-m} \mathcal{E}_k(j+i)). \end{aligned} \tag{7.6}$$

Thus

$$\bigoplus_i A'_{mn;ii} \otimes_R k = \bigoplus_{i \geq 0} \Gamma(\mathbb{P}^1_k, S^{n-m} \mathcal{E}_k(2i)). \tag{7.7}$$

The right-hand side of (7.7) is the graded-$\bigoplus_i S_{i,k} \otimes_k S_{i,k}$-module associated to the coherent $\mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^1_k}$-module $\delta_* S^{n-m} \mathcal{E}_k$ (for the ample line bundle given by $\mathcal{O}_k(1,1)$). Hence this graded module is finitely generated.

From the computation in the previous paragraph we also learn that $\mathcal{A}_{m,n} \otimes_R k$ is indeed given by the sheaf $S^{n-m} \mathcal{E}_k$ supported on the diagonal.

We claim that the support of $\mathcal{A}_{mn}$ is finite over both factors of $\mathbb{P}^1_R \times \mathbb{P}^1_R$. Again it is clearly sufficient to check this over k but then it follows from the explicit form of $\mathcal{A}_{m,n} \otimes_R k$ given above.

As indicated above A_{mn} is flat over R. Hence the same is true for $\mathcal{A}_{mn}$. Since $\mathcal{A}_{mn} \otimes_R k$ is locally free over both factors it follows from Lemma 3.1.5 that $\mathcal{A}_{m,n}$ is locally free on the left and on the right. By tensoring with k we deduce that the left and right rank of $\mathcal{A}_{m,n}$ are equal to $n - m + 1$. □

Lemma 7.3.3. *The functor* $-\otimes_{C_m} A_{mn}$ *sends* $\operatorname{gr}(C_m)$ *to* $\operatorname{gr}(C_n)$.

Proof. It is sufficient to prove that for every i we have that $e_i A_{mn}$ lies in $\operatorname{gr}(C_n)$. Since $e_i A_{mn}$ is a finitely generated R-module in every degree, we may prove this after specialization.

We compute

$$e_i A_{mn,k} = \bigoplus_{j\geq i} \Gamma(\mathbb{P}^1_k, S^{n-m}\mathcal{E}_k(j-i)).$$

Thus $e_i A_{mn,k}$ is up to finite length modules the graded S-module associated to the coherent $\mathbb{P}^1$-module $S^{n-m}\mathcal{E}_k(-i)$. Hence it is finitely generated. $\square$

Lemma 7.3.4. *There is a commutative diagram*

$$\begin{array}{ccc} \operatorname{gr}(C_m) & \xrightarrow{-\otimes_{C_m} A_{mn}} & \operatorname{gr}(C_n) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \operatorname{coh}(\mathbb{P}^1_R) & \xrightarrow[-\otimes_{\mathbb{P}^1_R}\mathcal{A}_{mn}]{} & \operatorname{coh}(\mathbb{P}^1_R). \end{array} \tag{7.8}$$

Proof. We first have to construct a natural transformation

$$\begin{array}{ccc} \operatorname{gr}(C_m) & \xrightarrow{-\otimes_{C_m} A_{mn}} & \operatorname{gr}(C_n) \\ {\scriptstyle\pi}\downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle\pi} \\ \operatorname{coh}(\mathbb{P}^1_R) & \xrightarrow[-\otimes_{\mathbb{P}^1_R}\mathcal{A}_{mn}]{} & \operatorname{coh}(\mathbb{P}^1_R). \end{array}$$

Taking into account the equivalences $\operatorname{gr}(C_m) = \operatorname{gr}(S)$, this diagram may be rewritten as

$$\begin{array}{ccc} \operatorname{gr}(S) & \xrightarrow{(-\otimes_S A'_{mn})_{0,-}} & \operatorname{gr}(S) \\ {\scriptstyle\pi}\downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle\pi} \\ \operatorname{coh}(\mathbb{P}^1_R) & \xrightarrow[\pi M\mapsto\pi([\omega_1\pi_1(M\otimes_S A'_{mn})]_{0,-})]{} & \operatorname{coh}(\mathbb{P}^1_R). \end{array} \tag{7.9}$$

Here ω_1 is ω applied to the left grading and similarly for π. The natural transformation is now obtained by functoriality from the canonical map

$$M\otimes_S A'_{mn} \to \pi_1\omega_1(M\otimes_S A'_{mn}).$$

We claim this natural transformation is an isomorphism. Both branches of the diagram (7.8) represent right exact functors so it is sufficient to consider the value on the projective generators $S_k(i)$ of $\operatorname{gr}(S_k)$. This verification may be done after specialization.

We find

$$\begin{aligned}S_k(i)\otimes_{S_k}A'_{mn,k}&=\bigoplus_{pq}A'_{mn;p+i,q,k}\\&=\bigoplus_{p+q+i\geq 0}\Gamma(\mathbb{P}^1_k,S^{n-m}\mathcal{E}_k(q+p+i)),\end{aligned}$$

where we have used (7.6). An easy verification shows that

$$\pi_1\omega_1(S_k(i)\otimes_{S_k}A'_{mn,k})=\bigoplus_{p,q}\Gamma(\mathbb{P}^1_k,S^{n-m}\mathcal{E}_k(q+p+i))$$

from which we deduce that

$$(\pi_1\omega_1(S_k(i)\otimes_{S_k}A'_{mn,k})/(S_k(i)\otimes_{S_k}A'_{mn,k}))_{l,-}$$

is finite dimensional for any l. This implies that the natural transformation in (7.9) is in fact a natural isomorphism. □

The natural morphism $A_{mn}\otimes_{C_n}A_{nt}\to A_{mt}$ induces via diagram (7.8) a natural transformation of functors

$$-\otimes_{\mathbb{P}^1_R}(\mathcal{A}_{mn}\otimes_{\mathbb{P}^1_R}\mathcal{A}_{nt})\to -\otimes_{\mathbb{P}^1_R}\mathcal{A}_{mt}.$$

Using Lemma 3.1.1, one obtains from this a morphism of bimodules

$$\mathcal{A}_{mn}\otimes_{\mathbb{P}^1_R}\mathcal{A}_{nt}\to\mathcal{A}_{mt}. \tag{7.10}$$

Using a similar argument, one shows that this morphism of bimodules satisfies the associativity axiom and hence produces a sheaf-$\mathbb{Z}$-algebra on $\mathbb{P}^1_R$ given by

$$\mathcal{A}=\bigoplus_{n\geq m}\mathcal{A}_{mn}.$$

From the fact that $A_{mm}=C_m$, one easily obtains $\mathcal{A}_{mm}=\mathcal{O}_{\mathbb{P}^1_R}$. Making explicit the proof of Lemma 3.1.1 one obtains that over k (7.10) is given by the canonical maps

$$S^{n-m}\mathcal{E}_k\otimes_{\mathbb{P}^1_k}S^{t-n}\mathcal{E}_k\to S^{t-m}\mathcal{E}_k.$$

Therefore by a suitable version of Nakayama's lemma we deduce that (7.10) is an epimorphism and hence $\mathcal{A}$ is generated by $\mathcal{E}_n\overset{\text{def}}{=}\mathcal{A}_{n,n+1}$.

Let Q_n be the kernel of $\mathcal{A}_{n,n+1}\otimes_{\mathbb{P}^1_R}\mathcal{A}_{n+1,n+2}\to\mathcal{A}_{n,n+2}$. We claim that this kernel is non-degenerate in $\mathcal{A}_{n,n+1}\otimes_{\mathbb{P}^1_R}\mathcal{A}_{n+1,n+2}$.

In Lemma 3.1.9 we have shown that the dualizing of bimodules is compatible with base change. From this it easily follows that it is sufficient to check the non-degenerateness of $\mathcal{Q}_n$ over k where it is obvious.

Now let $\mathcal{A}'$ be the $\mathbb{Z}$-algebra generated by the $\mathcal{E}_{n,n+1}$ subject to the relations given by the $\mathcal{Q}_n$. By construction there is a surjective map $\mathcal{A}'\to\mathcal{A}$. Since $\mathcal{A}'$ and $\mathcal{A}$ are locally free in each degree and have the same rank, it follows that this surjective map must actually be an isomorphism.

So summarizing we have shown the following:

Lemma 7.3.5. *$\mathcal{A}$ is a non-commutative symmetric algebra over $\mathbb{P}^1_R$.*

By §4.1 it follows that $\mathrm{gr}(\mathcal{A})\cong\mathrm{gr}(\mathbb{S}(\mathcal{E}))$ with $\mathcal{E}=\mathcal{A}_{01}$, and this equivalence preserves right bounded modules. Hence to finish the proof of Theorem 7.3.1 it is sufficient to show that $\mathrm{qgr}(\mathcal{A})\cong\mathcal{D}$.

Put

$$C = \bigoplus_{\substack{(i,m),(j,n)\\ i\le j\\ m\le n}} \mathrm{Hom}(O(-j,-n), O(-i,-m)).$$

Then C is a $\mathbb{Z}^2$-algebra, and we have an exact functor

$$\Sigma : \mathrm{Gr}(C) \to \mathrm{Gr}(\mathcal{A}),$$

which is defined as follows. Let $M \in \mathrm{Gr}(C)$. Then $M_n \overset{\mathrm{def}}{=} M_{-,n}$ is a right C_n-module. Furthermore the right action of C on M induces maps

$$M_m \otimes_{C_m} A_{mn} \to M_n. \tag{7.11}$$

Put $\mathcal{M}_n = \pi(M_n) \in \mathrm{Qch}(\mathbb{P}^1_R)$. Thanks to Lemma 7.3.4 the maps (7.11) become maps

$$\mathcal{M}_m \otimes_{\mathbb{P}^1_R} \mathcal{A}_{mn} \to \mathcal{M}_n,$$

and one checks that $\Sigma\mathcal{M} \overset{\mathrm{def}}{=} \bigoplus_n \mathcal{M}_n$ defines an object in $\mathrm{Gr}(\mathcal{A})$. Put $\sigma M = \pi\Sigma M \in \mathrm{QGr}(\mathcal{A})$ (where here π is the quotient functor $\mathrm{Gr}(\mathcal{A}) \to \mathrm{QGr}(\mathcal{A})$).

We claim that Σ sends finitely generated objects in $\mathrm{Gr}(C)$ to objects in $\mathrm{gr}(\mathcal{A})$. It suffices to prove this for the projective generators $e_{im}C$.

We have for $n \ge m$,

$$e_{im}C_{-,n} = \bigoplus_{j\ge i} \mathrm{Hom}_{\mathcal{D}}(O(-j,-n), O(-i,-m)).$$

Hence we have to prove that the right-hand side is a finitely generated C_n-module. Since the summands $\mathrm{Hom}_{\mathcal{D}}(O(-j,-n), O(-i,-m))$ are all finitely generated R-modules, we may do this after specialization. We get

$$e_{im}C_{-,n} \otimes_R k = \bigoplus_{j\ge i} \Gamma(\mathbb{P}^1_k, S^{n-m}\mathcal{E}_k(j-i)),$$

which is indeed finitely generated. For reference below we note that from this computation we also get

$$\pi(e_{im}C_{-,n} \otimes_R k) = S^{n-m}\mathcal{E}_k(-i)$$

(where here π is the quotient functor $\mathrm{Gr}(C_{n,k}) \to \mathrm{QGr}(C_{n,k}) \cong \mathrm{Qch}(\mathbb{P}^1_k)$), and thus

$$\Sigma(e_{im}C \otimes_R k) = \bigoplus_{n\ge m} S^{n-m}\mathcal{E}_k(-i)$$

so that finally we get

$$\sigma(e_{im}C \otimes_R k) = O_k(-i,-m). \tag{7.12}$$

Since

$$\mathrm{Hom}_C(e_{jn}C, e_{im}C) = e_{im}Ce_{jn} = C_{(i,m)(j,n)} = \mathrm{Hom}_{\mathcal{D}}(O(-j,-n), O(-i,-m)),$$

functoriality yields a morphism of R-modules (for $j \ge i$, $n \ge m$)

$$\mathrm{Hom}_{\mathcal{D}}(O(-j,-n), O(-i,-m)) \to \mathrm{Hom}_{\mathrm{qgr}(\mathcal{A})}(\sigma(e_{jn}C), \sigma(e_{im}C)).$$

The left-hand side is R-flat and commutes with base change as indicated above. We claim that this is true for the right-hand side as well.

Lemma 7.3.6. $\mathrm{qgr}(\mathcal{A})$ *is a deformation of* $\mathrm{qgr}(\mathcal{A})_k = \mathrm{qgr}(\mathcal{A}_k) = \mathrm{qgr}(S\mathcal{E}_k) = \mathrm{coh}(\mathbb{P}^1_k)$.

Proof. According to [21], $\mathrm{qgr}(\mathcal{A})$ is Ext-finite. Therefore, according to Proposition 7.2.9 it is sufficient to prove that $\mathrm{qgr}(\mathcal{A})$ has a strongly ample sequence. To this end we verify the conditions for Lemma 7.1.1. It is standard that these conditions lift from k to R and hence we may check them over k. Over k they follow from the explicit description of $\mathcal{A}_{mn,k}$ given in Lemma 7.3.2. □

Lemma 7.3.7. *The R-module*

$$\mathrm{Hom}_{\mathrm{qgr}(\mathcal{A})}(\sigma(e_{jn}C), \sigma(e_{im}C))$$

for $i \leq j$ and $m \leq n$ is flat and compatible with base change. Furthermore the canonical map

$$\mathrm{Hom}_{\mathcal{D}}(O(-j,-n), O(-i,-m)) \to \mathrm{Hom}_{\mathrm{qgr}(\mathcal{A})}(\sigma(e_{jn}C), \sigma(e_{im}C)) \tag{7.13}$$

constructed above is an isomorphism.

Proof. We first discuss the first statement. Given Lemma 7.3.6 it is sufficient to check that $\sigma(e_{jn}C) \otimes_R k$ satisfies the conditions of Proposition 7.2.7. It is easy to see that $\sigma(e_{jn}C)$ is compatible with base change and is R-flat. One may then invoke the explicit description of $\sigma(e_{jn}C \otimes_R k)$ given in (7.12).

To prove the last statement we note that this is true over k by (7.12). We may then invoke Nakayama's lemma for R (given that everything is compatible with base change as we have shown above). □

Proof of Theorem 7.3.1. Given our preparatory work, it is sufficient to prove $\mathcal{D} \cong \mathrm{qgr}(\mathcal{A})$. By Theorem 7.2.10 we obtain that $(O(n,n))_n$ is an ample sequence in $\mathcal{D}$. Given (7.13) and the $\mathbb{Z}$-algebra version of the Artin-Zhang theorem [1] it is sufficient to prove that $(\sigma(e_{-n,-n}C))_n$ forms a strongly ample sequence in $\mathrm{qgr}(\mathcal{A})$. Using Lemma 7.3.6 together with Theorem 7.2.10 this may be checked over k. Then we invoke again the explicit description of $\sigma(e_{-n,-n}C \otimes_R k)$ given in (7.12). □

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