

The Green rings of the generalized Taft Hopf algebras

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ABSTRACT. In this paper, we investigate the Green ring $r(H_{n,d})$ of the generalized Taft algebra $H_{n,d}$, extending the results of Chen, Van Oystaeyen and Zhang (to appear in Proc. of AMS). We shall determine all nilpotent elements of the Green ring $r(H_{n,d})$. It turns out that each nilpotent element in $r(H_{n,d})$ can be written as a sum of indecomposable projective representations. The Jacobson radical $J(r(H_{n,d}))$ of $r(H_{n,d})$ is generated by one element, and its rank is $n - n/d$. Moreover, we will present all the finite dimensional indecomposable representations over the complexified Green ring $R(H_{n,d})$ of $H_{n,d}$. Our analysis is based on the decomposition of the tensor product of indecomposable representations and the observation of the solutions for the system of equations associated to the generating relations of the Green ring $r(H_{n,d})$.

1. Introduction

Let \mathcal{C} be the category of finite dimensional representations of a Hopf algebra H over a field K . In the study of the monoidal structure of \mathcal{C} one has to consider the decomposition of the tensor product of representations in \mathcal{C} , in particular, the tensor product of two indecomposable representations in \mathcal{C} . However, in general, very little is known about how a tensor product of two indecomposable representations decomposes into a direct sum of indecomposable representations. One method of addressing this problem is to consider the tensor product as the multiplication of the Green ring (or the representation ring) $r(H)$, and to study the ring properties of the Green ring. In general, it is relatively easy to explore the ring structure of the complexified Green ring $R(H)$ as we shall see in this paper.

A lot of work have been done in this direction. Firstly, Green [15, 16], Benson and Carlson, etc., considered the semi-simplicity of the representation ring $r(KG)$ for modular representations of a finite group G (see [4]). One of the interesting results they obtained is that KG is of finite representation type if and only if there are no nilpotent elements in $r(KG)$. In general, it is difficult to determine all nilpotent elements of $r(KG)$ if KG is of infinite representation type (see [1, 18, 28]). For the Green rings of Hopf algebras, if H is a finite dimensional semi-simple Hopf algebra, then the Green ring $r(H)$ is equal to the Grothendieck ring and is semi-simple (see, e.g. [23, 32]). If H is the enveloping algebra of a complex semi-simple Lie algebra, the Green ring has been studied by Cartan and Weyl (see

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[14]). Here we would like to mention the recent work by Sergeev and Veselov for basic classical Lie superalgebras (see [26]). Darpö and Herschend have presented a general description of the Green ring of the polynomial Hopf algebra $K[x]$ in [11] in case the ground field K is perfect. For representation rings of quantum algebras, we refer to the work by Domokos and Lenagan (see [12]) and the work by Chin (see [10]).

Recently, Cibils in [8] determined all the graded Hopf algebras on a cycle path coalgebra (which are just equal to the generalized Taft algebras (see [27, 30])), and considered the decomposition of two indecomposable representations (see also [17]). Moreover, Cibils also computed the Green ring of the Sweedler 4-dimensional Hopf algebra by generators and relations and asked how to compute the Green ring of $K\mathbb{Z}_n(q)/I_d$. In [31], Wakui computed the Green rings of all non semisimple Hopf algebras of dimension 8, one of which is $K\mathbb{Z}_4(q)/I_2$, over an algebraically closed field K of characteristic 0 by generators and relations. More recently, Chen, Van Oystaeyen and Zhang (see [7]) has explicitly described the Green ring $r(H_n)$ of the Taft Hopf algebra H_n by two generators and two relations.

The aforementioned works motivate us to investigate the structure of the Green ring $r(H_{n,d})$ of the generalized Taft Hopf algebra $H_{n,d}$, which in turn could be helpful to understand the structure and the classification of non-semisimple monoidal categories. It is well-known that the pointed Hopf algebra $H_{n,d}$ is neither commutative nor cocommutative, and is even not quasitriangular in general. However, it is of finite representation type, and monomial, that is, all the relations are given by paths. Thus the study of the Green ring of $H_{n,d}$ can help us to understand more the tensor categories of finite representation type.

The main aim of this paper is to compute the generators and generating relations of the Green ring $r(H_{n,d})$, to determine explicitly all nilpotent elements of $r(H_{n,d})$, and to give all finite dimensional indecomposable representations of the complexified Green ring $R(H_{n,d})$. The paper is organized as follows. In Section 2, we recall the definitions and the basic representation theory of the generalized Taft algebra $H_{n,d}$. There are nd non-isomorphic finite dimensional indecomposable modules over $H_{n,d}$. For each $1 \leq l \leq d$, there are exactly n indecomposable $H_{n,d}$ -modules with dimension l . Every indecomposable projective $H_{n,d}$ -module is d -dimensional. In section 3, we prove that the Green ring $r(H_{n,d})$ of $H_{n,d}$ is generated by two elements subject to two relations, one of which is associated with a generalized Fibonacci polynomial (see Theorem 3.2).

In order to describe all nilpotent elements and the structure of the Green rings $r(H_{n,d})$ and $R(H_{n,d})$, we compute, in Section 4, explicitly the roots of the generalized Fibonacci polynomials appeared in the generating relations of the Green rings $r(H_{n,d})$. It turns out that the generalized Fibonacci polynomial $F_n(a, x)$ has $n - 1$ distinct complex roots (see Proposition 4.2 for detail).

In section 5, we analyze the distinct solutions of the system of equations associated to the aforementioned generalized Fibonacci polynomial. By using a one to one correspondence between the set of solutions for the system of equations and the set of the isomorphism classes of irreducible modules over $R(H_{n,d})$, we obtain $nd - n + \frac{n}{d}$ irreducible modules of dimension 1 over $R(H_{n,d})$. This gives us a description of all nilpotent elements of $r(H_{n,d})$, and concludes that the Jacobson radical $J(r(H_{n,d}))$ is a principal ideal (i.e. generated by one element). Moreover, we will determine all finite dimensional reducible and indecomposable modules over

the Green ring $H_{n,d}$. It is interesting that the number of indecomposable representations of the Green ring $R(H_{n,d})$ and that of $H_{n,d}$ are both equal to nd . In the final section, we apply the same technique used on $r(H_{n,d})$ to describe all nilpotent elements and finite dimensional indecomposable representations of the projective class ring of $H_{n,d}$. Cibils studied the projective class rings in [9] for basic and split Hopf algebras.

Throughout, we work over a fixed algebraically closed field K of characteristic 0. Unless otherwise stated, all algebras, Hopf algebras and modules are defined over K ; all modules are left modules and finite dimensional; all maps are K -linear; \dim and \otimes stand for \dim_K and \otimes_K , respectively. For the theory of Hopf algebras and quantum groups, we refer to [21, 24, 25, 29].

2. Representation theory of the generalized Taft algebra $H_{n,d}$

In the sequel, we fix two integers $n, d \geq 2$ such that $d \mid n$. Assume that q is a primitive d -th root of unity. In [27] (see also [2]) Radford considered the following Hopf algebra $H_{n,d} = H_{n,d}(q)$ generated by two elements g and h subject to the relations:

$$g^n = 1, \quad h^d = 0, \quad hg = qgh.$$

The algebra $H_{n,d}$ is a Hopf algebra with comultiplication Δ , counit ε , and antipode S given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(h) &= 1 \otimes h + h \otimes g, & \varepsilon(g) &= 1, \\ \varepsilon(h) &= 0, & S(g) &= g^{-1} = g^{n-1}, & S(h) &= -q^{-1}g^{n-1}h. \end{aligned}$$

Note that the dimension of $H_{n,d}$ is dn , and the set $\{g^i h^j \mid 0 \leq i \leq n-1, 0 \leq j \leq d-1\}$ forms a PBW basis for $H_{n,d}$.

In case $d = n$, then $H_n = H_{n,n}$ is the n^2 -dimensional Taft (Hopf) algebra (see [30]). For this reason, $H_{n,d}$ is called a generalized Taft algebra in [6, 20]. The Hopf algebra $H_{n,d}$ can be also approached by quiver and relations, that is, $H_{n,d}$ is isomorphic to the quiver quantum group $KZ_n(q)/I_d$ constructed by Cibils in [8].

In order to compute the Green ring of $H_{n,d}$, we need to recall the classification of finite dimensional indecomposable representations of $H_{n,d}$. Thanks to Theorem 4.3 in [6], we have the following structures of $H_{n,d}$ as algebras. Throughout the paper, we let $m = n/d$. Then

LEMMA 2.1. *Let H_d be the Taft algebra of dimension d^2 . Then*

$$H_{n,d} \cong \overbrace{H_d \times H_d \times \cdots \times H_d}^{m \text{ copies}}. \quad \square$$

Recall that an algebra A is called Nakayama if each indecomposable projective left and right module has a unique composition series. Since $H_{n,d}$ is a Nakayama algebra, its representation theory is not difficult to describe (see [3, 8, 20]). In [8] Cibils classified the indecomposable modules over $kZ_n(q)/I_d$, and gave the decomposition formulas of the tensor product of two arbitrary indecomposable modules. However, using the forms of the presentations in [8], it is hard to work out explicitly the generators and the generating relations for the Green rings of $H_{n,d}$. For this reason, we will rewrite in the following all indecomposable modules over $H_{n,d}$, and reformulate the decomposition formulas of the tensor product of two indecomposable modules.

Let $\mathbb{Z}_n := \mathbb{Z}/(n)$. For $1 \leq l \leq d$ and $i \in \mathbb{Z}_n$, denote by $M(l, i)$ the k -vector space with a K -basis $\{v_0, v_1, \dots, v_{l-1}\}$. Let ω be a primitive n -th root of unity so that $\omega^m = q$. Define an action of $H_{n,d}$ on $M(l, i)$ as follows:
 $g \cdot v_j = \omega^i q^{-j} v_j$ for all $0 \leq j \leq l - 1$ and

$$h \cdot v_j = \begin{cases} v_{j+1}, & 0 \leq j \leq l - 2, \\ 0, & j = l - 1. \end{cases}$$

Then $M(l, i)$ is an indecomposable module and the set $\{M(l, i) \mid i \in \mathbb{Z}_n, 1 \leq l \leq d\}$ forms a complete list of non-isomorphic indecomposable $H_{n,d}$ -modules. Moreover, $M(l, i)$ is simple if and only if $l = 1$; and $M(l, i)$ is projective indecomposable if and only if $l = d$ (for the proof, see [7, 8, 20]).

LEMMA 2.2. *We have the following isomorphisms:*

$$\begin{aligned} M(1, i) \otimes M(l, r) &\cong M(l, r) \otimes M(1, i) \cong M(l, r + i), \\ M(l, r) &\cong M(1, r) \otimes M(l, 0) \cong M(l, 0) \otimes M(1, r) \end{aligned}$$

for all $1 \leq l \leq d$ and $i, r \in \mathbb{Z}_n$. □

To find the decomposition of the tensor product $M(l, i) \otimes M(s, j)$, it is enough to find the decomposition of the tensor product $M(l, 0) \otimes M(s, 0)$. The following proposition tell us that it suffices to give an explicit evaluation of the tensor product of any indecomposable representation with the two-dimensional indecomposable representation $M(2, 0)$.

PROPOSITION 2.3. *We have the following isomorphisms:*

- (1) $M(2, 0) \otimes M(l, 0) \cong M(l + 1, 0) \oplus M(l - 1, -m)$ for all $2 \leq l \leq d - 1$ and $d > 2$.
- (2) $M(2, 0) \otimes M(d, 0) \cong M(d, 0) \oplus M(d, -m)$.

PROOF. (1) For $d > 2$ and $2 \leq l \leq d - 1$, let $\{v_0, v_1\}$ and $\{w_0, w_1, \dots, w_{l-1}\}$ be the standard basis of indecomposable representations $M(2, 0)$ and $M(l, 0)$ respectively. Consider the submodule V of $M(2, 0) \otimes M(l, 0)$ generated by $v_0 \otimes w_0$. From the action $g \cdot h^i \cdot (v_0 \otimes w_0) = q^{-i} h^i \cdot (v_0 \otimes w_0)$, it follows that $V = \langle h^i \cdot (v_0 \otimes w_0) \mid i \in \mathbb{N} \rangle$. By induction, we obtain

$$h^i \cdot (v_0 \otimes w_0) = v_0 \otimes w_i + (1 + q^{-1} + \dots + q^{-(i-1)})v_1 \otimes w_{i-1}$$

for $0 \leq i \leq l - 1$, and

$$h^l \cdot (v_0 \otimes w_0) = (1 + q^{-1} + \dots + q^{-(l-1)})v_1 \otimes w_{l-1} \neq 0$$

since q is a primitive d -th root of unity and $0 \leq l \leq d - 1$. It follows that $h^{l+1} \cdot (v_0 \otimes w_0) = 0$. By the construction of $M(l + 1, 0)$, this implies that $M(2, 0) \otimes M(l, 0)$ contains an indecomposable submodule V isomorphic to $M(l + 1, 0)$. On the other hand, if we set

$$e_i = h^i \cdot (v_0 \otimes w_1 - q^{-(l-1)}(1 + q + \dots + q^{l-2})v_1 \otimes w_0)$$

for $i \in \mathbb{N}$, then by induction again we have

$$e_i = v_0 \otimes w_{i+1} - q^{-(l-1)}(1 + q + \dots + q^{l-i-2})v_1 \otimes w_i$$

for $0 \leq i \leq l - 2$ and, in particular, we have

$$e_{l-2} = v_0 \otimes w_{l-1} - q^{-(l-1)}v_1 \otimes w_{l-2} \neq 0 \text{ and } e_{l-1} = 0.$$

Notice that $g \cdot e_i = q^{-i-1}e_i = \omega^{-m}q^{-i}e_i$. Thus, $M(2, 0) \otimes M(l, 0)$ contains an indecomposable submodule M generated by e_0 and isomorphic to $M(l - 1, -m)$. Since $M(l + 1, 0)$ has a unique composition series

$$0 \rightarrow M(1, -lm) \rightarrow \dots \rightarrow M(l - 1, -2m) \rightarrow M(l, -m) \rightarrow M(l + 1, 0),$$

we have $M(l - 1, -m) \cap M(l + 1, 0) = \{0\}$. Comparing the dimensions we obtain $M(2, 0) \otimes M(l, 0) \cong M(l + 1, 0) \oplus M(1, -m)$.

(2) As in the proof of (1), we let $\{v_0, v_1\}$ and $\{w_0, w_1, \dots, w_{d-1}\}$ be the standard basis of the indecomposable representations $M(2, 0)$ and $M(d, 0)$ respectively. Consider the submodule V of $M(2, 0) \otimes M(d, 0)$ generated by $v_0 \otimes w_0$. Since $g \cdot h^i \cdot (v_0 \otimes w_0) = q^{-i}h^i \cdot (v_0 \otimes w_0)$, we have

$V = \langle h^i \cdot (v_0 \otimes w_0) \mid i \in \mathbb{Z} \rangle$. Now by induction, we get

$$h^i \cdot (v_0 \otimes w_0) = v_0 \otimes w_i + (1 + q^{-1} + \dots + q^{-(i-1)})v_1 \otimes w_{i-1}$$

for $0 \leq i \leq d - 1$, and

$$h^d \cdot (v_0 \otimes w_0) = (1 + q^{-1} + \dots + q^{-(d-1)})v_1 \otimes w_{d-1} = 0.$$

These imply that $M(2, 0) \otimes M(d, 0)$ contains an indecomposable submodule V isomorphic to $M(d, 0)$.

On the other hand, if we let N be the submodule of $M(2, 0) \otimes M(d, 0)$ generated by $v_0 \otimes w_1$, then by induction, we obtain the following:

$$h^i \cdot (v_0 \otimes w_1) = v_0 \otimes w_{i+1} + (q^{-1} + \dots + q^{-i})v_1 \otimes w_i$$

for $0 \leq i \leq d - 2$ and

$$h^{d-1} \cdot (v_0 \otimes w_1) = (q^{-1} + \dots + q^{-(d-1)})v_1 \otimes w_{d-1} \neq 0.$$

Thus we get $h^d \cdot (v_0 \otimes w_1) = 0$. Note that $g \cdot h^i \cdot (v_0 \otimes w_1) = q^{-i-1}v_0 \otimes w_1 = \omega^{-m}q^{-i}v_0 \otimes w_1$. It follows that $M(2, 0) \otimes M(d, 0)$ contains an indecomposable submodule N isomorphic to $M(d, -m)$. Comparing the dimensions we obtain that $M(2, 0) \otimes M(d, 0) \cong M(d, 0) \oplus M(d, -m)$. \square

In order to describe the Green ring $r(H_{n,d})$ of $H_{n,d}$, we also need the decomposition of the tensor product $M(d, 0) \otimes M(d, 0)$.

PROPOSITION 2.4. *We have the following decomposition:*

$$M(d, 0) \otimes M(d, 0) \cong M(d, 0) \oplus M(d, -m) \oplus \dots \oplus M(d, -(d - 1)m).$$

PROOF. Let $\{v_0, v_1, \dots, v_{d-1}\}$ be the standard basis of the indecomposable representation $M(d, 0)$. Similar to the proof of the Proposition 2.3, for $0 \leq i \leq d - 1$ we get that $\langle v_0 \otimes v_i \rangle$, the submodule of $M(d, 0) \otimes M(d, 0)$ generated by element $v_0 \otimes v_i$, is isomorphic to $M(d, -im)$. Now comparing the dimensions, we have an isomorphism:

$$M(d, 0) \otimes M(d, 0) \cong M(d, 0) \oplus M(d, -m) \oplus \dots \oplus M(d, -(d - 1)m). \quad \square$$

3. Generators and relations for the Green ring $r(H_{n,d})$

The aim of this section is to describe explicitly the generators and the generating relations for the Green ring $r(H_{n,d})$ of the generalized Taft algebra $H_{n,d}$.

Let H be a Hopf algebra over a field K . Recall that the Green ring or the representation ring $r(H)$ of H can be defined as follows. As a group $r(H)$ is the free abelian group generated by the isomorphism classes $[V]$ of finite dimensional H -modules V modulo the relations $[M \oplus V] = [M] + [V]$. The multiplication of $r(H)$ is

given by the tensor product of H -modules, that is, $[M][V] = [M \otimes V]$. Then $r(H)$ is an associative ring with identity given by $[K_\varepsilon]$, where K_ε is the trivial 1-dimensional H -module. Note that $r(H)$ is a free abelian group with a \mathbb{Z} -basis $\{[V] \mid V \in \text{ind}(H)\}$, where $\text{ind}(H)$ denotes the set of finite dimensional indecomposable H -modules. The complexified Green ring $R(H)$, which we shall call the Green algebra for short, is an associative \mathbb{C} -algebra defined by $\mathbb{C} \otimes_{\mathbb{Z}} r(H)$.

Now we return to the case $H = H_{n,d}$. We begin with the following lemma, which follows from Lemma 2.2, Proposition 2.3 and 2.4.

LEMMA 3.1. *The following equations hold in $r(H_{n,d})$.*

- (1) $[M(1, -1)]^n = 1$ and $[M(l, r)] = [M(1, -1)]^{n-r}[M(l, 0)]$ for all $1 \leq l \leq d$ and $r \in \mathbb{Z}_n$.
- (2) $[M(l + 1, 0)] = [M(2, 0)][M(l, 0)] - [M(1, -1)]^m[M(l - 1, 0)]$ for all $2 \leq l \leq d - 1$ and $d > 2$.
- (3) $[M(2, 0)][M(d, 0)] = (1 + [M(1, -1)]^m)[M(d, 0)]$.
- (4) $[M(d, 0)][M(d, 0)] = \sum_{i=0}^{d-1} [M(d, -im)]$. □

As a consequence, we have the following corollary.

COROLLARY 3.2. *The Green ring $r(H_{n,d})$ is a commutative ring generated by $[M(1, -1)]$ and $[M(2, 0)]$.* □

Therefore, $r(H_{n,d})$ can be identified with a quotient of the polynomial ring $\mathbb{Z}[y, z]$. In the sequel, we shall determine the relations of the two generators. Let $F_s(y, z)$ be the generalized Fibonacci polynomials defined by

$$F_{s+2}(y, z) = zF_{s+1}(y, z) - yF_s(y, z)$$

for $s \geq 1$, while $F_0(y, z) = 0$, $F_1(y, z) = 1$, $F_2(y, z) = z$. These generalized Fibonacci polynomials were found appearing in the generating relations of the Green rings of the Taft algebras H_n (see [7]). The general form of the polynomials is as follows:

LEMMA 3.3. [7, Lemma 3.11] *For $s \geq 2$ we have*

$$F_s(y, z) = \sum_{i=0}^{\lfloor (s-1)/2 \rfloor} (-1)^i \binom{s-1-i}{i} y^i z^{s-1-2i},$$

where $\lfloor \frac{s-1}{2} \rfloor$ denotes the biggest integer which is not bigger than $\frac{s-1}{2}$. □

Now we are ready to present the Green ring $r(H_{n,d})$.

THEOREM 3.4. *The Green ring $r(H_{n,d})$ of $H_{n,d}$ is isomorphic to the quotient ring of the polynomial ring $\mathbb{Z}[y, z]$ modulo the ideal I generated by the following elements*

$$y^n - 1, (z - y^m - 1)F_d(y^m, z).$$

Moreover, under the isomorphism the polynomial $F_s(y^m, z)$ corresponds to $[M(s, 0)]$ for $2 \leq s \leq d$.

PROOF. It follows from Lemma 3.1 that

$$([M(2, 0)] - (1 + [M(1, -1)]^m))[M(d, 0)] = 0, [M(1, -1)]^n = 1$$

and

$$[M(d, 0)] = F_d([M(1, -1)]^m, [M(2, 0)]).$$

These lead to a natural surjective \mathbb{Z} -algebra homomorphism

$$\Phi : \mathbb{Z}[y, z]/I \rightarrow r(H_{n,d}), \Phi(y) = [M(1, -1)], \Phi(z) = [M(2, 0)]$$

from $\mathbb{Z}[y, z]/I$ to $r(H_{n,d})$. Observe that as a free \mathbb{Z} -module, $\mathbb{Z}[y, z]/I$ has a \mathbb{Z} -basis $\{y^i z^j \mid 0 \leq i \leq n - 1, 0 \leq j \leq d\}$. This means that $\mathbb{Z}[y, z]/I$ and $r(H_{n,d})$ both have the same rank nd as free \mathbb{Z} -modules. It follows that Φ must be an isomorphism. The other assertions now follow from the fact that the following relations hold in $r(H_{n,d})$

$$[M(l + 1, 0)] = [M(2, 0)][M(l, 0)] - [M(1, -1)]^m [M(l - 1, 0)]$$

for all $2 \leq l \leq d - 1$ and $d > 2$. □

One can easily see that the Grothendieck ring of $H_{n,d}$ is the group ring $\mathbb{Z}\mathbb{Z}_n$ generated by $[M(1, -1)]$. From the above theorem, we see that the Green ring of $H_{n,d}$ is much more complicated than the Grothendieck ring of $H_{n,d}$. Another interesting factor is that the Green ring of $H_{n,d}$ is commutative although $H_{n,d}$ is not quasitriangular. Thus the Green rings of the generalized Taft algebras $H_{n,d}$ (here n, d can be chosen) provide more examples that a non-quasitriangular Hopf algebra possesses a commutative Green ring. However, we do have non-commutative Green rings. For example, the Green ring of the small quantum group is not commutative when $p \geq 3$ (see [22]).

Now let A be the subalgebra of $r(H_{n,d})$ generated by $[M(1, -m)]$ and $[M(2, 0)]$. Then by Lemma 3.1 we have

$$r(H_{n,d}) = A \bigoplus A[M(1, -1)] \bigoplus \cdots \bigoplus A[M(1, -1)]^{(m-1)}.$$

By Theorem 3.10 in [7] and Theorem 3.4, it is easy to see that A is isomorphic to $r(H_d)$, the Green ring of the Taft algebra H_d . Now the following theorem is straightforward.

THEOREM 3.5. *As a \mathbb{Z} -algebra, the Green ring $r(H_{n,d})$ is isomorphic to $r(H_d)[x]/(x^m - a)$, where $a = [M(1, -m)]$. □*

REMARK 3.6. *For a finite-dimensional Hopf algebra H over a field K , the antipode S induces an anti-ring endomorphism $*$ of the Green ring $r(H)$, see [23]. In the case where H is the generalized Taft algebra $H_{n,d}$, it is easy to see that the anti-ring endomorphism $*$ is given by $[M(1, -1)]^* = [M(1, -1)]^{-1}$ and $[M(2, 0)]^* = [M(1, -1)]^{-m} [M(2, 0)]$. It follows that the anti-isomorphism $*$ is an involution whereas S is not. This is because S^2 is inner and $V^{**} \cong V$ for any finite dimensional H -module, see [23].*

4. Roots of the generalized Fibonacci polynomials

In order to determine all nilpotent elements and the structure of the Green rings $r(H_{n,d})$ and $R(H_{n,d})$, we need to compute explicitly all roots of the generalized Fibonacci polynomial $F_d(y, z)$ associated to the generating relations of the Green ring $r(H_{n,d})$. We shall set up in this section a general framework, which is of interest in its own.

Let $\mathbb{C}[x]$ be the polynomial algebra over the complex number field \mathbb{C} with variable x . Fix $0 \neq a \in \mathbb{C}$, the generalized Fibonacci polynomials $F_s(a, x)$ are defined recursively:

$$F_0(a, x) = 0, F_1(a, x) = 1, F_{s+2}(a, x) = xF_{s+1}(a, x) - aF_s(a, x)$$

for $s \geq 0$. $F_s(a, x)$ has the following combinatorial formula as shown in Section 3:

$$F_s(a, x) = \sum_{i=0}^{\lfloor (s-1)/2 \rfloor} (-1)^i \binom{s-1-i}{i} a^i x^{s-1-2i}$$

for $s \geq 1$. From these combinatorial expressions, it is easy to see that $F_s(a, x)$ is a polynomial in $\mathbb{C}[x]$ with degree $s - 1$.

REMARK 4.1. *The classical Fibonacci polynomials $F_s(x)$ are defined by the linear recurrence*

$$F_{s+2}(x) = xF_{s+1}(x) + F_s(x)$$

for $s \geq 0$ and $F_0(x) = 0, F_1(x) = 1$, that is, $F_s(x) = F_s(-1, x)$. These polynomials are of great importance in the study of many subjects such as algebra, geometry, and number theory. Obviously, these polynomials have a close relation with the famous Fibonacci numbers. The Fibonacci polynomials are essentially Chebychev polynomials, of which the roots can be computed. Thus $F_s(x)$ has $s - 1$ distinct roots given by (see, e.g., [19]) :

$$x_j = 2i \cos \frac{\pi j}{s}, \quad 1 \leq j \leq s - 1. \quad \square$$

The following proposition give us the roots of generalized Fibonacci polynomial $F_s(a, x)$.

PROPOSITION 4.2. *Let $s \geq 2$. Then the generalized Fibonacci polynomial $F_s(a, x)$ has $s - 1$ distinct complex roots given by*

$$x_j = 2\sqrt{a} \cos \frac{j\pi}{s}, \quad 1 \leq j \leq s - 1.$$

PROOF. Let $x = by$ and $b = -i\sqrt{a} \in \mathbb{C}$. Then

$$F_s(a, x) = b^{s-1} F_s(-1, y) = b^{s-1} F_s(y).$$

Note that the roots of $F_s(y)$ are $y_j = 2i \cos \frac{j\pi}{s}$, $1 \leq j \leq s - 1$. It follows that the roots of $F_s(a, x)$ are

$$x_j = 2\sqrt{a} \cos \frac{j\pi}{s}, \quad 1 \leq j \leq s - 1. \quad \square$$

We need the following corollary in Section 5.

COROLLARY 4.3. *For every $1 \leq j \leq s - 1$, there exists a $2s$ -th root $\eta_j \neq 1$ of unity, such that*

$$x_j = 2\sqrt{a} \cos \frac{\pi j}{s} = \sqrt{a}(\eta_j + \eta_j^{-1}).$$

PROOF. For each $j, 1 \leq j \leq s - 1$, let $\eta_j = \cos \frac{j\pi}{s} + i \sin \frac{j\pi}{s} = e^{\frac{j\pi i}{s}}$. Then $\eta_j \neq 1$ is a $2s$ -th root of unity and

$$2\sqrt{a} \cos \frac{\pi j}{s} = \sqrt{a}(\eta_j + \eta_j^{-1}). \quad \square$$

To end this section, we solve the system of equations determined by the generating relations of the Green ring $r(H_{n,d})$. For $0 \leq k \leq n - 1$, let $\omega_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ be an n -th root of unity. By Proposition 4.2, all distinct roots of the generalized Fibonacci polynomial $F_d(\omega_k^m, x)$ are given by

$$\sigma_{k,j} = 2\sqrt{\omega_k^m} \cos \frac{\pi j}{d}, \quad 1 \leq j \leq d - 1.$$

We have the following lemma.

LEMMA 4.4. *The following system of equations*

$$(1) \quad \begin{cases} y^n = 1, \\ (z - y^m - 1)F_d(y^m, z) = 0 \end{cases}$$

has $(d - 1)n + m$ distinct solutions in \mathbb{C} , where $m = n/d$, and the solutions are given by

$$(2) \quad \begin{aligned} \mathfrak{T} = & \{(\omega_k, 2) \mid 0 \leq k \leq n - 1, d \mid k\} \\ & \cup \{(\omega_k, \sigma_{k,j}) \mid 0 \leq k \leq n - 1, 1 \leq j \leq d - 1\}. \end{aligned}$$

PROOF. By Proposition 4.2, it is not difficult to see that the solutions of the system (1) are as follows:

$$\{(\omega_k, 1 + \omega_k^m), (\omega_k, \sigma_{k,j}) \mid 0 \leq k \leq n - 1, 1 \leq j \leq d - 1\}.$$

Thanks to Corollary 4.3, the roots of the generalized Fibonacci polynomial $F_d(\omega_k^m, x)$ can be written as

$$\sigma_{k,j} = 2\sqrt{\omega_k^m} \cos \frac{\pi j}{d} = \sqrt{\omega_k^m}(\eta_j + \eta_j^{-1}),$$

where $\eta_j = \cos \frac{j\pi}{d} + i \sin \frac{j\pi}{d}$ is a $2d$ -th root of unity, and $1 \leq j \leq d - 1$. Now we divide k , $0 \leq k \leq n - 1$, in two cases.

Case 1: $d \mid k$. In this case, for $1 \leq j \leq d - 1$ we have $\sigma_{k,j} \neq 2$. Thus, the following set gives md distinct roots of the system (1):

$$\{(\omega_k, 2), (\omega_k, \sigma_{k,j}) \mid 0 \leq k \leq n - 1, d \mid k, 1 \leq j \leq d - 1\}.$$

Case 2: $d \nmid k$. In this case, $\eta_k = \sqrt{\omega_k^m} \neq 1$. So we have

$$\sigma_{k,k} = \sqrt{\omega_k^m}(\eta_k + \eta_k^{-1}) = \sqrt{\omega_k^m}(\sqrt{\omega_k^m} + \sqrt{\omega_k^{-m}}) = 1 + \omega_k^m.$$

Since the generalized Fibonacci polynomial $F_d(\omega_k^m, z)$ possesses $d - 1$ distinct roots $\sigma_{k,j}$ for each k , this implies that for each $\omega_k^m \neq 1$, the system (1) has $d - 1$ distinct roots $\{(\omega_k, \sigma_{k,j}) \mid 1 \leq j \leq d - 1\}$.

To summarize, we have in total

$$md + (n - m)(d - 1) = nd - n + m$$

distinct solutions for the system (1) of equations. □

5. Nilpotent elements in $r(H_{n,d})$ and representation theory of $R(H_{n,d})$

In this section, we determine all nilpotent elements in the Green ring $r(H_{n,d})$ and classify the finite dimensional indecomposable representations over the Green algebra $R(H_{n,d})$. We first list the irreducible representations of the Green algebra $R(H_{n,d})$.

THEOREM 5.1. *Let $d \geq 2$. Then the Green algebra $R(H_{n,d})$ has exactly $nd - n + m$ irreducible modules and each irreducible module is 1-dimensional.*

PROOF. By Theorem 3.4, we know that the Green algebra $R(H_{n,d})$ of $H_{n,d}$ is commutative and isomorphic to $\mathbb{C}[y, z]/I$, where the ideal I is generated by the following elements

$$y^n - 1, (z - y^m - 1)F_d(y^m, z).$$

Since \mathbb{C} is an algebraically closed field of characteristic 0, the dimension of each irreducible module is 1. By Lemma 4.4, we know that the system (1) has $nd - n + m$

distinct solutions given by \mathfrak{T} in (2). For each solution $(\lambda, \mu) \in \mathfrak{T}$, one can define an irreducible $R(H_{n,d})$ -module $\mathbb{C}_{\lambda,\mu}$ on the vector space \mathbb{C} by $y \cdot 1 = \lambda, z \cdot 1 = \mu$.

It is clear that $(\lambda, \mu) \mapsto \mathbb{C}_{\lambda,\mu}$ gives a one to one correspondence between the set of solutions for the system (1) and the set of the isomorphism classes of irreducible modules over $R(H_{n,d})$. □

THEOREM 5.2. *Let $d \geq 2$. The set of nilpotent elements in $r(H_{n,d})$ is equal to*

$$\langle [M(d, i)] - [M(d, j)] \mid i \equiv j \pmod{m} \rangle,$$

that is, the Jacobson radical $J(r(H_{n,d}))$ of $r(H_{n,d})$ has a \mathbb{Z} -basis

$$\{[M(d, im + j)] - [M(d, (i - 1)m + j)] \mid 1 \leq i \leq d - 1, 0 \leq j \leq m - 1\}.$$

PROOF. By Proposition 2.4 we have the $H_{n,d}$ -modules decomposition

$$M(d, 0) \otimes M(d, 0) \cong \bigoplus_{i=0}^{d-1} M(d, -im).$$

Thus, for $i, j \in \mathbb{Z}_n$, we get

$$\begin{aligned} M(d, i) \otimes M(d, j) &\cong M(1, i) \otimes M(1, j) \otimes M(d, 0) \otimes M(d, 0) \\ &\cong M(1, i + j) \otimes \bigoplus_{s=0}^{d-1} M(d, -sm) \\ &\cong \bigoplus_{s=0}^{d-1} M(d, i + j - sm). \end{aligned}$$

This implies that if $i \equiv j \pmod{m}$, then we have

$$M(d, i) \otimes M(d, i) \cong M(d, j) \otimes M(d, j) \cong \bigoplus_{s=0}^{d-1} M(d, i + j - sm) \cong M(d, i) \otimes M(d, j).$$

It follows that in the Green ring $r(H_{n,d})$

$$[M(d, i)]^2 = [M(d, j)]^2 = [M(d, i)][M(d, j)] = [M(d, j)][M(d, i)]$$

for $i \equiv j \pmod{m}$. Therefore, if $i \equiv j \pmod{m}$, we have that

$$([M(d, i)] - [M(d, j)])^2 = 0,$$

i.e., $0 \neq [M(d, i)] - [M(d, j)]$ is an nilpotent element in $r(H_{n,d})$. Since \mathbb{Z} is a principal ideal integral domain and hence each submodule of free \mathbb{Z} -module is free. Note that the set

$$\{[M(d, im + j)] - [M(d, (i - 1)m + j)] \mid 1 \leq i \leq d - 1, 0 \leq j \leq m - 1\} \subset J(r(H_{n,d}))$$

is independent over \mathbb{Z} . Hence the rank r of $J(r(H_{n,d}))$ as \mathbb{Z} -module is large or equal to $(d - 1)m = n - m$. On the other hand, it is easy to see from Theorem 5.1 that the dimension $\dim_{\mathbb{C}}(J(R(H_n))) = n - m$. But the fact that $\dim_{\mathbb{C}}(J(R(H_n))) \geq r$ is clear, and hence $r = n - m$. Thus, the set

$$\{[M(d, im + j)] - [M(d, (i - 1)m + j)] \mid 1 \leq i \leq d - 1, 0 \leq j \leq m - 1\}$$

forms a \mathbb{Z} -basis of $J(r(H_{n,d}))$. □

As a consequence, we obtain the following:

COROLLARY 5.3. *The Jacobson radical $J(r(H_{n,d}))$ is a principal ideal generated by the element $([M(1, m)] - 1)[M(d, 0)]$.*

PROOF. For $1 \leq i \leq d - 1, 0 \leq j \leq m - 1$, by Lemma 3.1, we have

$$\begin{aligned} & [M(d, im + j)] - [M(d, (i - 1)m + j)] \\ &= ([M(1, im + j)] - [M(1, (i - 1)m + j)])[M(d, 0)] \\ &= [M(1, (i - 1)m + j)]([M(1, m) - 1])[M(d, 0)]. \end{aligned}$$

Note that $[M(1, (i - 1)m + j)]$ is invertible in $r(H_{n,d})$. Hence, the assertion follows from Theorem 5.2. \square

In the sequel, we shall classify the indecomposable but non-irreducible finite dimensional modules over the Green algebra $R(H_{n,d})$. For each $k, 0 \leq k \leq n - 1$, such that $d \nmid k$, write ω_k for $\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$, an n -th root of unity, and let $V(k)$ be a 2-dimensional \mathbb{C} -vector space with a basis $\{v_1, v_2\}$. Define an action of the Green algebra $R(H_{n,d})$ on $V(k)$ as follows:

$$y \cdot v_i = \omega_k v_i, z \cdot v_1 = (1 + \omega_k^m)v_1, z \cdot v_2 = v_1 + (1 + \omega_k^m)v_2.$$

Then we have the following.

LEMMA 5.4. *Let $0 \leq k, t \leq n - 1, d \nmid k, d \nmid t$. Then $V(k)$ is an indecomposable and reducible module of $R(H_{n,d})$. Moreover, $V(k) \cong V(t)$ if and only if $k = t$.*

PROOF. $V(k)$ is indecomposable because $1 + \omega_k^m$ is a double root of the equation

$$(z - \omega_k^m - 1)F_d(\omega_k^m, z) = 0$$

by Lemma 4.4 and the action of z corresponds to the following Jordan block

$$\begin{pmatrix} 1 + \omega_k^m & 1 \\ 0 & 1 + \omega_k^m \end{pmatrix}.$$

Observe that $\mathbb{C}v_1$ is a submodule of $V(k)$. Therefore, $V(k)$ is reducible which is direct from Theorem 5.1. The rest is straightforward. \square

THEOREM 5.5. *Let V be a finite dimensional indecomposable and reducible module of the Green algebra $R(H_{n,d})$. Then there exists $k, 0 \leq k \leq n - 1$, and $d \nmid k$, such that $V \cong V(k)$.*

PROOF. Assume that V is a finite dimensional indecomposable module with dimension $s \geq 2$. Since the actions of y and z on V are commutative and $y^n = 1$, there exist a basis $\{v_1, v_2, \dots, v_s\}$ of V such that with respect to this basis, the matrices corresponding to the actions of the generators y and z are respectively $Y = \omega_k E_s$, where $\omega_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ is an n -th root of unity for some $0 \leq k \leq n - 1$ and E_s is the $s \times s$ identity matrix, and the Jordan block:

$$Z = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}_{s \times s}$$

for some $\lambda \in \mathbb{C}$. Notice that the matrix Z must satisfy the following matrix equation

$$(Z - (\omega_k^m + 1)E_s) \prod_{j=1}^{d-1} (Z - \sigma_{k,j}E_s) = 0,$$

where $\{\sigma_{k,j} \mid 1 \leq j \leq d - 1\}$ is the set of roots of the generalized Fibonacci polynomial $F_d(\omega_k^m, z)$, see Proposition 4.2.

Since for each j the matrix $Z - \sigma_{k,j}E_s$ is a Jordan block, $Z - \sigma_{k,j}E_s$ is invertible if $\lambda \neq \sigma_{k,j}$. Moreover, $\sigma_{k,j}$ for $1 \leq j \leq d - 1$ are distinct and $\sigma_{k,k} = 1 + \omega_k^m$ for $d \nmid k$. It follows that $\omega_k^m \neq 1, s = 2$ and $\lambda = 1 + \omega_k^m$. Therefore, we have that $V \cong V(k)$ for some $0 \leq k \leq n - 1$ such that $d \nmid k$. \square

Now we can describe the blocks of the Green algebra $R(H_{n,d})$.

COROLLARY 5.6. (1) *The set $\{\mathbb{C}_{\lambda,\mu}, V(k) \mid (\lambda, \mu) \in \mathfrak{T}, 0 \leq k \leq n - 1, d \nmid k\}$ forms a complete list of finite dimensional indecomposable representations of $R(H_{n,d})$ with cardinal number nd . Moreover, $\mathbb{C}_{\lambda,\mu}$ is projective if and only if $\mu \neq 1 + \lambda^m$ and $V(k)$ is projective for each k .*

(2) *There are $nd - 2(n - m)$ blocks of dimension 1 and $n - m$ blocks of dimension 2.* \square

6. Comparing the Green ring and the projective class ring of $H_{n,d}$

In this section, we compare the structure of the Green ring $r(H_{n,d})$ and the structure of the projective class ring $p(H_{n,d})$. Recall that the projective class ring $p(H_{n,d})$ of $H_{n,d}$ is the subalgebra of $r(H_{n,d})$ generated by the projective and semisimple representations of $H_{n,d}$ (see [9] for details). It is easy to see that $p(H_{n,d})$ has a \mathbb{Z} -basis $\{[M(1, i)], [M(d, i)] \mid 0 \leq i \leq n - 1\}$. In [9] Cibils determined the structure of the complexified projective class algebra $P(H_d)$ of the Taft algebra H_d , which is isomorphic to $\mathbb{C}^2 \times \mathbb{C}[\epsilon]^{d-1}$, where $\mathbb{C}[\epsilon]$ is the algebra of dual numbers $\mathbb{C}[x]/(x^2)$. Applying the same technique used on $r(H_{n,d})$ and $R(H_{n,d})$, we can easily obtain the following result.

PROPOSITION 6.1. *Let $p(H_{n,d})$ be the projective class ring of $H_{n,d}$ and assume $d \geq 2$. Then as a \mathbb{Z} -algebra, $p(H_{n,d})$ is generated by $[M(1, -1)]$ and $[M(d, 0)]$, and is isomorphic to $\mathbb{Z}[y, z]/I$ with I generated by the following elements:*

$$y^n - 1, z^2 - (1 + y^m + y^{2m} + \dots + y^{(d-1)m})z.$$

PROOF. Follows from Lemma 3.1. \square

Since the nilpotent elements of $r(H_{n,d})$ stem from projective indecomposable modules, we have that the Jacobson radical of $r(H_{n,d})$ is equal to the Jacobson radical of $p(H_{n,d})$.

COROLLARY 6.2. *The set of nilpotent elements of $p(H_{n,d})$ is equal to the set of nilpotent elements of $r(H_{n,d})$.* \square

The following proposition summarizes the irreducible representations of the complexified projective class algebra $P(H_{n,d}) = \mathbb{C} \otimes_{\mathbb{Z}} p(H_{n,d})$.

For each $k, 0 \leq k \leq n - 1$, let $\omega_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ be an n -th root of unity. Let $\mathbb{C}_k = \mathbb{C}$ for each k , and $\mathbb{C}_{k,d} = \mathbb{C}$ for $d \mid k$, as vector spaces. Define the actions of $P(H_{n,d})$ on \mathbb{C}_k and $\mathbb{C}_{k,d}$ respectively as follows:

$$y \cdot 1 = \omega_k, z \cdot 1 = 0$$

and

$$y \cdot 1 = \omega_k, z \cdot 1 = d$$

Then we have the following.

PROPOSITION 6.3. (1) *There are $n + m$ non-isomorphic irreducible 1-dimensional $P(H_{n,d})$ -representations*

$$\{\mathbb{C}_k \mid 0 \leq k \leq n - 1\} \cup \{\mathbb{C}_{k,d} \mid 0 \leq k \leq n - 1, d \mid k\}$$

(2) *There are $n - m$ non-isomorphic reducible indecomposable 2-dimensional $P(H_{n,d})$ -representations V_k , where $0 \leq k \leq n - 1, d \nmid k$, and V_k has a basis $\{v_1, v_2\}$ with the module structure given by*

$$y \cdot v_1 = \omega_k v_1, y \cdot v_2 = \omega_k v_2, z \cdot v_1 = 0, z \cdot v_2 = v_1.$$

Moreover, the set $\{\mathbb{C}_k, \mathbb{C}_{k,d}, V_i\}$ forms a complete list of non-isomorphic indecomposable $P(H_{n,d})$ -representations. \square

We finish the paper with a remark on the stable Green ring of $H_{n,d}$. The stable Green ring was introduced in the study of Green rings for the modular representation theory of finite groups. It is a quotient of the Green ring modulo all projective representations (see[4, 13]). In our situation, it is easy to see from Theorem 3.4 that the stable Green ring $St(H_{n,d})$ of $H_{n,d}$ is generated by $[M(1, -1)]$ and $[M(2, 0)]$ and is isomorphic to $\mathbb{Z}[y, z]/J$, where the ideal J is generated by the following elements:

$$y^n - 1, F_d(y^m, z).$$

By Theorem 5.2, the stable Green ring $St(H_{n,d})$ has no nilpotent elements, and hence it is semiprimitive.

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References

- [1] L. Archer, On certain quotients of the Green rings of dihedral 2-groups, *J. Pure & Appl. Algebra* 212 (2008), 1888-1897. MR2414693 (2009c:20006)
- [2] N. Andruskiewitsch and H. J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3 , *J. Algebra* 209(1998), 658-691. MR1659895 (99k:16075)
- [3] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, in: Cambridge Stud. Adv. Math, vol. 36, Cambridge Univ. Press, Cambridge, 1995. MR1314422 (96c:16015)
- [4] D. J. Benson, Representation rings of finite groups, Representations of algebras (Durham, 1985), 181-C199, London Math. Soc. Lecture Note Ser., 116, Cambridge Univ. Press, Cambridge, 1986. MR897325 (88i:16037)
- [5] H. X. Chen, Finite-dimensional representations of a quantum double, *J. Algebra* 251 (2002), 751-789. MR1919152 (2003f:16060)
- [6] X. W. Chen, H. L. Huang, Y. Ye and P. Zhang, Monomial Hopf algebras. *J. Algebra* 275 (2004), 212-232. MR2047446 (2005e:16060)
- [7] H. X. Chen, F. V. Oystaeyen and Y. H. Zhang, The Green rings of Taft algebras, arXiv:1111.1837v2 [math.RT], to appear in Proc. of AMS.

- [8] C. Cibils, A quiver quantum groups, *Commun. Math. Phys.* 157 (1993), 459-477. MR1243707 (94m:16046)
- [9] C. Cibils, The projective class ring of basic and split Hopf algebras, *K-theory*. 17(1999), 385-393. MR1706125 (2000h:16014)
- [10] W. Chin, Special biserial coalgebras and representations of quantum $SL(2)$, arXiv:math/0609370v5 [math.QA]. MR2872433
- [11] E. Darpö and M. Herschend, On the representation ring of the polynomial algebra over perfect field, *Math. Z.*, 265(2011),605-615. MR2644312 (2011d:13023)
- [12] M. Domokos and T. H. Lenagan, Representation rings of quantum groups. *J. Algebra* 282 (2004), 103-128. MR2095574 (2005g:20080)
- [13] K. Erdmann, E. Green, N. Snashall, R. Taillefer, Representation theory of the Drinfeld double of a family of Hopf algebras, *J. Pure Applied Algebra* 204(2)(2006), 413-454. MR2184820 (2006i:16056)
- [14] W. Fulton and J. Harris, Representation theory, *Grad. Texts Math*, 129, Springer-Verlag, New York, 1991. MR1153249 (93a:20069)
- [15] J. A. Green, The modular representation algebra of a finite group, *Ill. J. Math.* 6(4) (1962), 607-619. MR0141709 (25:5106)
- [16] J. A. Green, A transfer theorem for modular representations, *J. Algebra.* 1(1964), 71-84. MR0162843 (29:147)
- [17] E. Gunnlaugsdóttir, Monoidal structure of the category of u_q^+ -modules, *Linear Algebra and its Applications* 365 (2003), 183-199. MR1987337 (2004g:17011)
- [18] H. Heldner, Nilpotent elements of order 3 in the Green ring of the dihedral 2-group, *Arch. Math(Basel)*, 63(1994), 193-196. MR1287246 (95f:20017)
- [19] V. E. Hoggart and J. M. Bicknell, Roots of Fibonacci polynomials, *Fibonacci Quarterly*, 11(3)(1973), 271-274. MR0323700 (48:2056)
- [20] H. L. Huang, H. X. Chen and P. Zhang, Generalized Taft algebras, *Algebra Coll.* 11(3)(2004),313-320. MR2081190 (2005e:16025)
- [21] C. Kassel, Quantum groups, Springer-Verlag, New York, 1995. MR1321145 (96e:17041)
- [22] H. Kondo and Y. Saito, Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to \mathfrak{sl}_2 , *J. Alg.* 330(2011), 103-129. MR2774620 (2012e:17037)
- [23] M. Lorenz, Representations of finite-dimensional Hopf algebras, *J. Alg.*188(1997), 476-505. MR1435369 (98i:16039)
- [24] S. Majid, Foundations of quantum group theory, Cambridge Univ. Press, Cambridge, 1995. MR1381692 (97g:17016)
- [25] S. Montgomery, Hopf Algebras and their actions on rings, *CBMS Series in Math.*, Vol. 82, Am. Math. Soc., Providence, 1993. MR1243637 (94i:16019)
- [26] A. N. Sergeev and A. P. Veselov, Grothendieck rings of basic classical Lie superalgebras, *Ann. Math.*, 173(2011), 663-703. MR2776360 (2012a:17011)
- [27] D. E. Radford, On the coradical of a finite-dimensional Hopf algebra, *Proc. Amer. Math. Soc.*, 53(1975), 9-15. MR0396652 (53:514)
- [28] I. Reiner, Nilpotent elements in rings of integral representations, *Proc. Amer. Math. Soc.*, 17(1966),270-274. MR0188306 (32:5745)
- [29] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969. MR0252485 (40:5705)
- [30] E. J. Taft, The order of the antipode of a finite-dimensional Hopf algebra, *Proc. Nat. Acad. Sci. USA* 68 (1971), 2631-2633. MR0286868 (44:4075)
- [31] M. Wakui, Various Structures Associated to the Representation Categories of Eight-Dimensional Nonsemisimple Hopf Algebras, *Algebra and representation theory*, 7(2004),491-515. MR2104670 (2005k:16089)
- [32] S. J. Witherspoon, The representation ring of the quantum double of a finite group, *J. Algebra* 179 (1996), 305-329. MR1367852 (96m:20015)

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