Matter-dependent production functions for economic modelling of firms

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Abstract

Governments design various policies to support the transition to a biobased economy. The design of these policies is complex and predicting their effectiveness is a major challenge. Quantitative research in environmental economics can provide valuable assistance for such policy design by scenario analysis and predicting models. This type of research requires also microeconomic modelling of material- or energy- intensive firms. These types of firms can be modelled by integrating material and energy flows as a productive input in the production function of the firm. However, this integration is not self-evident. Literature indicates several limitations in this respect.

This paper derives a new production function for material or energy intensive firms in forecasting microeconomic models. This EMod function is based on standard requirements for economic production functions, but also on additional physical restrictions. The algebraic form is non-linear, and presents interesting characteristics concerning the balance between capital, labour and energy inputs. The function has been applied in three types of models to test its feasibility. Its behaviour is coherent with the principles that led to the derivation and it differs markedly from modelling results with a standard Cobb Douglas function.

Keywords: Production function, energy economics, agent-based modelling

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1 Introduction

Industry is confronted with systemic environmental problems such as dependency on scarce fossil resources, toxicity of end products and waste accumulation. Companies react by examining new processes for the production of energy and materials based on renewable organic matter. Industry heavily invests in new production chains, and numerous innovations contribute to the growing applications of renewable feedstock for the production of energy, heat and materials. Many research projects investigate novel pathways to valorise organic matter as high quality chemical compounds, food additives or traditional fodder replacements. Governments actively want to support this transition to a biobased economy, as these solutions could potentially create several jobs, and eliminate many systemic environmental problems related to the use of fossil resources. It is however uncertain how this support should be organised most effectively. There are also various sustainability problems related to the use of renewable feedstock.

Detailed economic research can help to provide more precise advice for governments. It can investigate the effectiveness of policies that support this transition or effects on regional sustainability. Complex models can help to shed light on the interactions between material and energy flows on the one hand and economic dynamics during this transition on the other. Quantitative environmental economics offer many instruments in this respect. Some of these environmental economic models address a specific challenge: the direct integration of economic parameters such as labour and capital together with physical parameters such as energy flows, emissions, raw materials and waste streams. These models are suitable given the intertwined relations between physical flows and economic value creation. A particular group of methods is based on Agent-based models (ABM). ABM is increasingly applied in various economic fields. It is a type of computer based models that represents economic actors as autonomous, social and learning actors. These actors move within a framework that defines legal structures, availability of inputs and demands from different markets. Each company is modelled with specific production functions, growth decisions and a historical background. In principle, ABMs are the equivalent in economic sciences of the petri dish experiments for biological sciences. Because ABM can leave behind the principle of economic equilibrium, it is applicable for the study of the co-evolution of different markets or the transitions in industrial sectors.

This research investigates the industrial sectors that consist of highly material and energy-intensive economic actors. These are the actors that will most be affected by the transition and should adapt their strategies accordingly. Our focus lies on companies that are or will be using important streams of renewable organic resources. These firms transform one set of material and energetic flows into another more valuable set. Any kind of firm that produces bulk goods falls into this category. For instance, a paper factory produces a flux of paper and cardboard from fibre pulp, water, additives and large energy flows. Steel factories, chemical plants, biofuel production units all are applicable. It includes also more specialised industrial niches, such as waste water treatment installations, flour mills or various types of electricity plants. Industrialised forms of agriculture also apply. Pig farms for instance are according to this view installations that absorb piglets, energy, water and pig fodder to produce live pigs and manure. Biorefineries, large farms or food transforming factories are equally possible.

Industrial applications for whom this approach is less applicable are the firms providing services for a major part of their turnover, or firms where the value of the produced goods is largely determined by the information content and less by the material content. This is the case for instance for pharmaceutical companies, computer manufacturers, book publishers or high-end clothing production. But even considering this restriction, the target group of firms still covers a relatively wide field. The targeted firms have two common characteristics. First of all, they are highly dependent on material and energy feedstock as production factors next to the standard production factors capital and labour. The efficiency of material and energy transformation is crucial for their economic competitiveness. And secondly, the targeted firms are always installations that can transform matter and energy with a finite capacity. Within an economic model, both the characteristics need to be reflected. They should be integrated in the production function that is used to model the firm. However, this integration is not self-evident.

There is a long scientific tradition in the integration of physical resources as an input for economic production functions. The use of production functions in economics dates from the 19th century [9]. Theoretical forms have allowed diverse and multiple inputs since a long time [8]. New applications of production functions emerged with the interest in environmental economics. Here multiple inputs and outputs were considered. These additional inputs and outputs included emissions, waste, primary resources, or energy. These are physical and material inputs, unlike labour and capital. The energy crises provoked a distinct and growing line of research integrating energy and matter as a separate production input, next to labour and capital [2, 3, 16]. For instance, KLEM-models consider four inputs and differentiate between energy and material inputs [2, 16]. Energy contains here all fossil fuels, nuclear heat used for electricity generation. Material inputs are minerals and primary resources for a particular sector. These quantities are aggregated in monetary terms, by multiplying them with their respective prices.

Recently there is a renewed interest in the development of economic models taking explicitly matter and energy into account. In a more fundamental way, Van den Bergh [22] outlines precise definitions for substitutability between energy and capital, and develops general forms for production functions integrating both. In practise, very few projects employ these integrations in economic evolutionary or agent-based models. An important exception is the work of Safarzynska [17, 19, 18] where the model is based on a CES-function and includes the technical electricity generation efficiency as a part of the agent's productivity parameters.

There are however indications in scientific literature that standard economic production functions cannot integrate physical input resources unconditionally. Several authors indicate that physical resources need to respect fundamental physical restrictions as well as economic restrictions. For a general case, Coelli et al [4] show that efficiency models with production functions do not respect the material balance condition, being a fundamental physical law. They use this finding to define an optimal efficiency that is coherent with the material balance condition as well. Unfortunately, their approach requires that both inputs and outputs contain factors that are accounted for in material units, such as weight or MJ. If the output is defined as one single economic output in monetary terms, then the restrictions imposed by Coelli et al [4] are no longer applicable.

In the context of macro-economic analysis, Kuemmel et al. detail several physical constraints for matter and energy that lead to modifications of the economic production function [10, 14]. First, they indicate the decreasing marginal productivity of capital when the relative labour or energy input tend to zero. It is argued that capital cannot be productive by itself, it needs labour or energy inputs to enable production. Secondly, in a tendency towards full automation of the economy, when labour input falls to zero, there is a linear relation between the needed capital and the needed energy input. This relation indicates the final energy efficiency of the fully automated capital stock. And finally, the overall elasticity of scale is set to one. Growth or efficiency increase are related to creativity and economic shocks in time, whereas the production function represents the constant economic productivity between two shocks [12]. Based on these restrictions, Kuemmel et al. derive a new production function, the Linex function, linear in energy and exponential in labour and capital inputs. The importance of this distinction was illustrated in macro-economic analysis, where results indicate a large influence of the growth rate of energy inputs on the growth rate of national economies [1]. Other production functions did not yield the same results. Recently, Stresing et al.[21] showed that the dependence of national production on energy inputs is much larger when considering the specific character of the energy input than with a standard Cobb-Douglas production function and a common input elasticity of about 5%.

A final constraint on the effect of energy inputs is the concept of maximum transformation capacity as defined above. Ayres et al. also describe this limitation of capacity utilisation when interpreting the results obtained with the Linex function [11]. In our case, we regard production capacity at the firm level, for the particular type of firms targeted in this research. The maximum transformation capacity is solely defined by the actual capital and labour input. Modifying the capital or labour input can increase the production capacity. For instance, a biodiesel plant can be designed to produce 300.000 barrels of biodiesel a year. At its optimal production regime, the firm produces only 220.000, because higher production would call for input materials to be transported over a longer distance, causing no marginal profit. However, if the biodiesel prices rise, the firm can consider to increase production. In practise this requires investments in the installation, enabling a higher production rate with the same staff, or it leads to a larger labour input to keep the installation operational during a longer period of the year. It is very common in practise that an increase of production always requires a simultaneous investment of capital or labour together with the additional input of physical resources, matter and energy. Only rarely the production level can be raised significantly by simply increasing the input of physical resources. The actual capital and labour input of the firm determine in practise its maximal capacity by which the physical input resources can be transformed into final products. This restriction is not reflected in standard economic production functions.

In order to build a coherent model for material- and energy-intensive firms, another algebraic form is necessary that responds to these restrictions. Many of these restrictions are mainly related to macroeconomic assessments. This makes for instance the Linex function is not directly suitable for ABM-models. Moreover, Saunders [20] also indicated that the Linex production function is not concave in labour, thereby not respecting a standard economic requirement for production functions, and this can lead to unwanted deviation in forecasting models.

This paper proposes a new production function that underpins the behaviour of an economic actor within an ABM. This new function is derived based on two aspects. The first aspect is the characteristic dependency on material input factors, and thus the integration of material and energy input. The second aspect is the set of physical restrictions on the use of material and energy as outlined above.

This paper is structured as follows. In section 2 we formalize the conditions and expectations of the function. An algebraic form is derived and the first ideas on its behaviour are illustrated. In section 3 we test the new algebraic form. The function is applied in three consecutive steps. It is first used in a data regression. Then the results are used for a analysis of the optimal input cost shares. Finally, the function is used in a predictive model, where the modelled firms are subjected to price changes in the market. The response to this market shock is calculated. In order to provide a reference point, the results are compared with parallel results using a standard Cobb-Douglas function. Section 4 concludes.

2 Derivation and properties of the proposed algebraic form

The algebraic function is derived for three productive inputs : capital, labour and energy, defined according to table 1. The last category gathers both material and energy inputs. These two factors can be aggregated using multiple physical units. For instance in Cumulative Energy (CE) [7] or exergy [6]. This gives an advantage that compared to the earlier KLEM-models, the inputs of material and energy no longer need to be separated, they can be aggregated in common terms respecting their physical value. We will therefore derive the algebraic forms for only one "energy and matter" input x.

An algebraic form of the production function can be derived based on the theoretical and functional restrictions it has to respect. Standard economic theory requires at least the following characteristics [5, 8]:

- Nonnegativity :

$$y(k,l,x) \in \Re^{+} \tag{1}$$

- Weak essentiality :

$$\text{If} \quad y(k,l,x) > 0 \quad \text{then} \quad k \neq 0 \vee l \neq 0 \vee x \neq 0 \tag{2}$$

- Non decreasing in inputs : For instance for capital:

$$\text{If} \quad k_1 \geq k_2 \quad \text{then} \quad y(k_1,l,x) \geq y(k_2,l,x) \tag{3}$$

  The rules for non-decreasing output in labour and energy are analogous.

- Concavity, the law of decreasing marginal input productivities, for instance for capital :

$$\frac{\partial^2 y}{\partial^2 k} < 0 \tag{4}$$

  The rules for concavity in labour and energy are similar.

- Additional to these restrictions, a production function with multiple inputs is required to be twice cross-differentiable for each input.

Table 1: Nomenclature

Description	Symbol	Unit	Description	Symbol
Production output	y	[Units]	Output price	p
Capital input	k	[EUR]	Capital unit price	w_k
Labour input	l	[FTE]	Labour unit price	w_l
Material and energy input	x	[MJ]	Material and energy unit price	w_x

These are the most common constraints imposed to production functions by standard economic theory. These constraints can be relaxed under certain circumstances, but we intend to respect these in a first phase.

The particular nature of the material and energy input adds two more restrictions. One of the restrictions defined by Kuemmel et al [14, 13] remains valid in this approach. They describe the diminishing productivity of capital. If the energy input is negligible compared to the capital input, the marginal productivity of capital tends to zero. If E_k is the capital elasticity of the function then :

$$\lim_{x \to 0} E_k = 0 \tag{5}$$

The second restriction is imposed by our interpretation of the company as a matter and energy transforming unit. This leads to the concept of a maximal transformation capacity $c(k, l)$ as a function of k and l. The firm is not obliged to operate at its maximum capacity, the produced output can be lower than $c(k, l)$. With increasing energy and matter input x, the total production does not increase indefinitely, but it tends asymptotically to the maximum capacity $c(k, l)$. This practical restriction is not represented when the production function is modelled by a Cobb Douglas function [2]

In this approximation, the output can increase indefinitely when the material input rises, without additional input of labour nor capital. Surely, when fitting this production function to real data, this is not a problem. The relationship between the different inputs is implicit in the data itself. But for modelling of functions as autonomous agents, there are no relations between the inputs defined. So extrapolation from a realistic situation can produce very unrealistic outcomes. Within the model, several additional constraints can provide links between the inputs to avoid this deviation, but this leads in practise to overly restricted behaviour as the inputs become directly dependent one on another. A more appropriate algebraic form for the production function has to take account for the maximum transformation capacity of the firm directly. There are many algebraic forms that approach a maximum limit asymptotically for increasing x. The proposed EMod form also follows the restrictions outlined above :

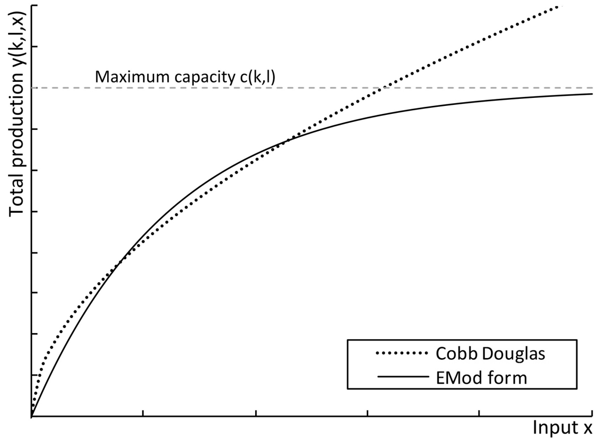
$$y(k, l, x) \quad = \quad c(k, l)(1 - e^{-m(k,l)x^{\gamma} + g(k,l)}) \tag{6}$$

In this functional form, $c(k, l)$ is the maximum transformation capacity of the firm, and $m(k, l)$, $c(k, l)$ and $g(k, l)$ are functions of capital and labour independent of x. The exponent of the energy factor is $\gamma > 0$. The form is compared to a standard Cobb-Douglas function in figure 1.

The behaviour of the function is determined by the three subfunctions, which are still to be defined. To control the behaviour, we evaluate two features of production functions as indicators.

[2]The general form of the Cobb-Douglas function used in this paper is : $y(k, l, x) = c_0 k^{\rho} l^{\sigma} x^{\gamma}$.

Figure 1: Contrary to the Cobb-Douglas form, the proposed EMod form tends asymptotically to a finite maximum production.



- Scale elasticity :

$$S = \sum_{Inputs} \frac{d\ln(y)}{d\ln(i)} \tag{7}$$

The Linex-function is built with a constant scale elasticity of 1. This can be considered too rigid for particular technologies. The scale elasticity should not be restricted to 1, and in the best case it should be constant over the entire domain.

- Elasticity of substitution between capital and labour : The elasticity of substitution between capital and labour is defined as :

$$ES_{k,l} = \frac{d\ln(\frac{k}{l})}{d\ln(\frac{MPL}{MPK})} \tag{8}$$

Imposing the form 6 is a very strong restriction on the behaviour of the function. These forms impose an input hierarchy where the effect of x is governed by k and l. It is therefore all the more important to uphold some flexibility with regards to the inputs k and l. Especially the substitutability between the two main inputs k and l needs to be preserved. In the best case, the solution should respect a constant elasticity of substitution between k and l over the entire range of inputs irrespective of x. If this restriction needs to be relaxed, it should be controlled that the input elasticity of substitution remains not constant but at least computable for all values of x.

Based on functional form 6, we assume $g(k,l) = 0$ for reasons of simplicity, so the input elasticities can be calculated as :

$$
\begin{aligned}
E_k &= \frac{k}{y}\frac{\partial y}{\partial k} = \frac{k}{c(k,l)}\frac{\partial c(k,l)}{\partial k} + kx^\gamma \frac{e^{-m(k,l)x^\gamma}}{1-e^{-m(k,l)x^\gamma}}\frac{\partial m(k,l)}{\partial k} &(9)\\
E_l &= \frac{l}{y}\frac{\partial y}{\partial l} = \frac{l}{c(k,l)}\frac{\partial c(k,l)}{\partial l} + lx^\gamma \frac{e^{-m(k,l)x^\gamma}}{1-e^{-m(k,l)x^\gamma}}\frac{\partial m(k,l)}{\partial l} &(10)\\
E_x &= \frac{x}{y}\frac{\partial y}{\partial x} = \gamma x^\gamma m(k,l)\frac{e^{-m(k,l)x^\gamma}}{1-e^{-m(k,l)x^\gamma}} &(11)
\end{aligned}
$$

Combining 9 with restriction 5 leads to the first relation between $c(k,l)$ and $m(k,l)$:

$$\frac{k}{c(k,l)}\frac{\partial c(k,l)}{\partial k} + \frac{k}{m(k,l)}\frac{\partial m(k,l)}{\partial k} = 0 \tag{12}$$

If the production function is required to exhibit constant elasticity to scale, then the scale effect in the exponential part of the function should be cancelled out. This leads to a second relation for $m(k,l)$:

$$m(\lambda k, \lambda l) = \frac{1}{\lambda^\gamma} m(k,l) \tag{13}$$

The clearest functions for $c(k,l)$ and $m(k,l)$ that respect both 12 and 13 are :

$$
\begin{aligned}
m(k,l) &= \frac{m_0}{k^\rho l^{\gamma-\rho}} &(14)\\
c(k,l) &= c_0 k^\rho l^{\gamma-\rho} &(15)\\
y(k,l,x) &= c_0 k^\rho l^{\gamma-\rho}(1-e^{-\frac{m_0 x^\gamma}{k^\rho l^{\gamma-\rho}}}) &(16)
\end{aligned}
$$

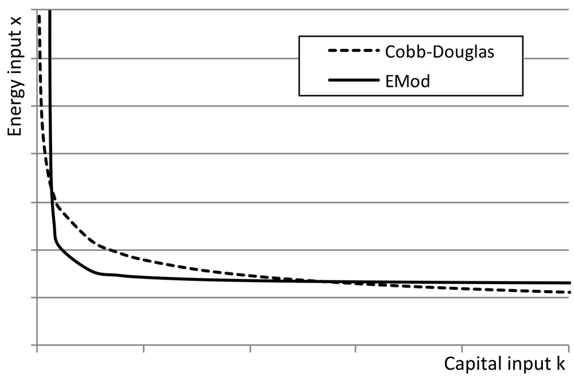
Here c_0, m_0, ρ and γ are constants. The overall input elasticity of scale in this case is equal to γ, and as such solely determined by the exponential of the energy inputs. For many applications this is too restrictive.

If the requirement for a constant elasticity to scale is relaxed, we can allow one more degree of freedom. This allows independent power factors for k and l, so the functions can be defined as:

$$
\begin{aligned}
m(k,l) &= \frac{m_0}{k^\rho l^\sigma} &(17)\\
c(k,l) &= c_0 k^\rho l^\sigma &(18)\\
y(k,l,x) &= c_0 k^\rho l^\sigma(1-e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}) &(19)
\end{aligned}
$$

Equation 19 describes the EMod function. This function is fixed by the choice of the five constants c_0, m_0, ρ, σ and γ. In this solution the elasticity of scale is no longer constant over the range of inputs, but it has the freedom to vary over a large range. At the same time, the elasticity of substitution between k and l remains constant and equal to 1, independent of k, l or x. A derivation of this is presented in annex A. A first indication of the characteristics of equation 19 is illustrated in figure 2. This figure compares two isoproduction lines in the

Figure 2: The isoproduction line of the resulting function (EMod) is much steeper than the isoproduction line of a Cobb-Douglas function



capital-energy field, while labour is constant. Contrary to the Cobb-Douglas isoproduction line, equation 19 leads to a very steep behaviour with rising x or k. The obtained isoproduction line resembles more closely a Leontief isoproduction line.

The corresponding input elasticities of equation 19 are :

$$E_k = \frac{k}{y}\frac{\partial y}{\partial k} = \rho + \frac{kx^\gamma \rho(-m_0)}{k^{\rho+1} l^\sigma} \frac{e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}}{1 - e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}} \tag{20}$$

$$= \rho(1 - \frac{m_0 x^\gamma}{k^\rho l^\sigma} \frac{e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}}{1 - e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}}) \tag{21}$$

$$E_l = \frac{l}{y}\frac{\partial y}{\partial l} = \sigma + \frac{lx^\gamma \sigma(-m_0)}{k^\rho l^{\sigma+1}} \frac{e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}}{1 - e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}} \tag{22}$$

$$= \sigma(1 - \frac{m_0 x^\gamma}{k^\rho l^\sigma} \frac{e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}}{1 - e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}}) \tag{23}$$

$$E_x = \frac{x}{y}\frac{\partial y}{\partial x} = \frac{\gamma m_0 x^\gamma}{k^\rho l^\sigma} \frac{e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}}{1 - e^{-\frac{m_0 x^\gamma}{k^\rho l^\sigma}}} \tag{24}$$

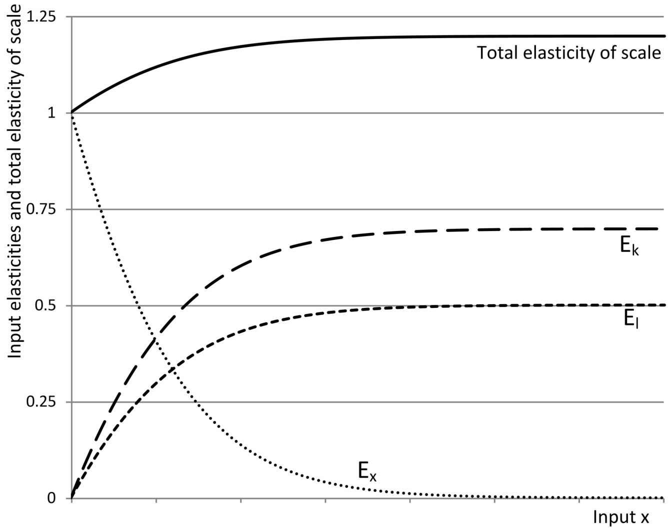
This leads to an unexpected relation between the input elasticities :

$$E_k = \rho(1 - \frac{E_x}{\gamma}) \tag{25}$$

$$E_l = \sigma(1 - \frac{E_x}{\gamma}) \tag{26}$$

This interdependence of the input elasticities is illustrated in figure 3. This figure illustrates the separate input elasticities of scale for $\rho = 0.7$, $\sigma = 0.5$ and $\gamma = 1$. When the energy input x is small or negligible, the input elasticities of labour and capital tend to zero. This is a logical consequence for the capital input elasticity corresponding to restriction 5. The behaviour of the labour input elasticity follows because of the substitutability between labour and capital. In

Figure 3: The variation of the three input elasticities and the total elasticity of scale in function of energy input x



this range of small energy inputs, the production function behaves as an exponential function of x only. This clarifies also why the total elasticity of the firm tends to γ when the energy input is very small, independent of any of the constants. At the other end of the spectrum, when x is very large, the energy input elasticity is negligible, and production can only be changed by modifying k or l. The total elasticity of scale in this case is constant and equals $\rho + \sigma$. At this end of the spectrum, the production function behaves as a Cobb-Douglas function in k and l. It is the transition between the two extremes that is of interest. In the range where k, l and x are balanced, the use of this function should more appropriate than a Cobb-Douglas or a standard exponential function.

3 Applications

3.1 Using the function in a data regression

The function of equation 19 is fitted to a dataset of 645 observations of diary farms between 1995 and 2001 assembled by Meul et al [15]. This dataset combines information on outputs, capital and labour inputs, and material and energy inputs. The material and energy inputs are in this case aggregated by measurements in Cumulative Energy. This approach is including also the indirect energy and solar irradiation, being the energy needed for the creation of the cow fodder and other materials. The origin of the data and more information on the particular aggregation procedure can be found in [15]. Some general indicators for this dataset are given in table 2.

The function parameters are estimated using an Ordinary Least Squares (OLS) estimation technique. A regression using a standard Cobb-Douglas function is provided as a benchmark. The regression results are illustrated in table 3. The results cannot be compared directly with each other. ρ, σ and γ are the input elasticities for capital, labour and energy respectively in the case of a Cobb-Douglas. But these constants do not have the same significance for the EMod function. A first approximation of the goodness of fit can be provided by the R^2 indicators, and it is quite similar for both regressions.

The constants ρ and σ add up to 1.6 in the case of the EMod function. This implies that the elasticity of scale varies between 1.15 and 1.6. To judge the usefulness of the current EMod form, it is important to know if the average elasticity is comparable to the one found by the Cobb-Douglas regression. If the average elasticity turns out to be low, then the regression with the EMod form converged to a suboptimal point where the influence of k and l was neglected, and all variation is imputed to x. Whereas if the average elasticity tends to 1.6, then the opposite has happened, and the regression converged to that part of the spectrum that only regards k and l as meaningful inputs. The Cobb-Douglas regression shows a constant elasticity of scale of 1.21. The input elasticities have been calculated for each data point in the set. The results of these calculations are given in table 4. This shows that the average scale elasticity using the EMod form is estimated at 1.20. This is very close to the estimation of the

Table 2: Data characteristics for 645 observations of Diary farms

	Observations	Mean	St.Dev.	Min.	Max	Unit
Output : Earnings	645	150 292	68 765	20 445	622 791	EUR
Capital	645	400 913	188 608	60 623	1 142 400	EUR
Labour	645	1.48	0.34	0.35	3.50	FTE
Energy inputs	645	1 175	491	253	3 579	GJ

Table 3: Results of data regression

EMod production function		Cobb-Douglas production function	
c_0	295	c_0	47.6
m_0	0.20		
ρ	0.513	ρ	0.214
σ	0.792	σ	0.260
γ	1.155	γ	0.735
R^2	0.965	R^2	0.966
$Adj.R^2$	0.965	$Adj.R^2$	0.966
Residual SS	2.73e10	Residual SS	5.98e11

Table 4: Distribution of local input elasticities with the EMod regression

Variable	Observations	Mean	St.Dev.	Min.	Max	Cobb-Douglas
E_k	645	0.163	0.051	0.048	0.355	0.214
E_l	645	0.251	0.079	0.074	0.547	0.260
E_x	645	0.790	0.115	0.358	1.047	0.735
Total elasticity of scale	645	1.203	0.015	1.169	1.260	1.209

Cobb-Douglas function. But more importantly, it shows that the EMod form converged to a point where all three inputs are required for maximal estimation accuracy, and that the data points are fitted to the transition range of the curve.

This indicates that the particular algebraic form can be fitted to actual data with satisfactory results. However, the results are not substantially different from regressions with a standard Cobb-Douglas function. This first estimation does not show very similar results but no added value when using this functional form for data regression.

3.2 Optimal input cost shares

The second purpose of the new algebraic form is to deliver a function with a distinctly different behaviour than standard economic production functions when used for predictive modelling. This new behaviour should reflect the limitation of a maximum transformation capacity as outlined above.

This section further elaborates on the results of the regressions in table 3. These results give two functions fitted on the same dataset. We now want to compare the optimal input cost shares for these two functions. A range of outputs is defined from 6.400 to 400.000. For each output, the profit-maximising input shares are determined. To that effect, market data are introduced with prices for inputs and outputs. These are the same for both functions.

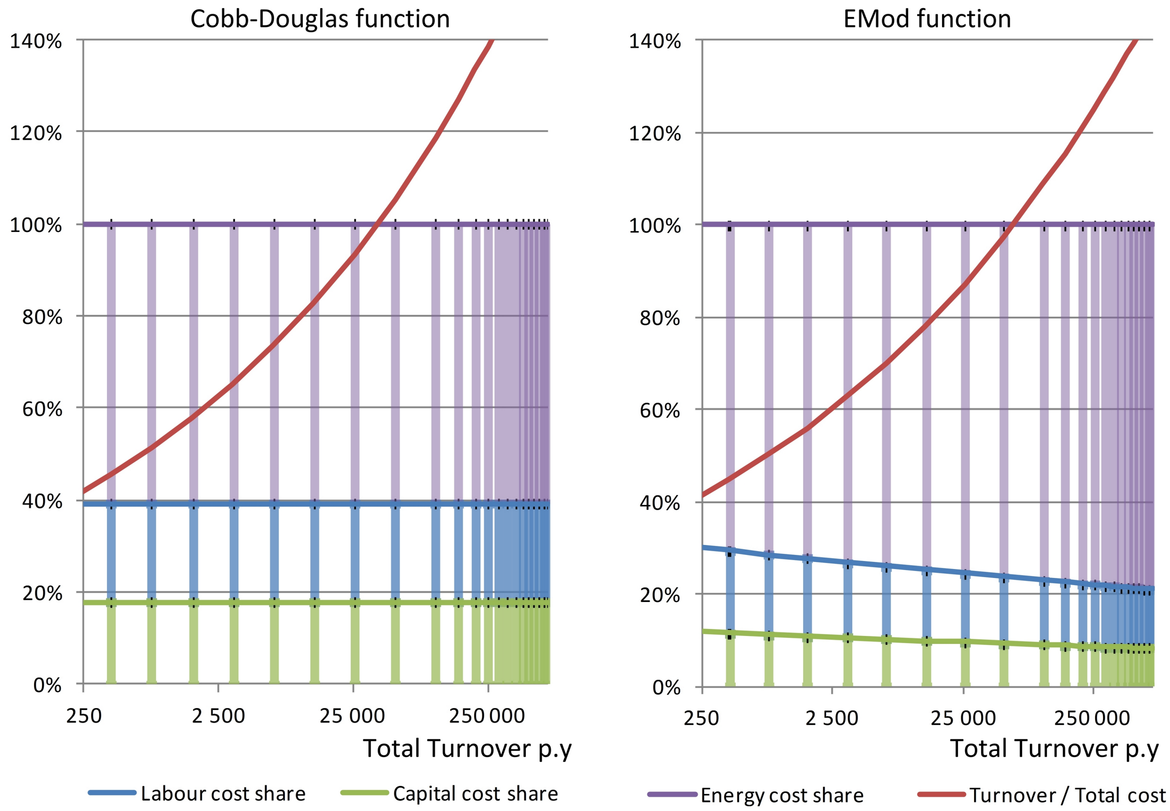
The equations for the optimal cost shares are derived in annex B profit-maximising point differs between a Cobb-Douglas approximation (CD) and an EMod approximation (EMod). The CD cost shares are constant, regardless of the produced output. In this case, the input cost shares are 18%, 21% and 61% for capital, labour and energy respectively. The EMod cost shares are varying with a tendency for an increasing energy cost share with increasing output. With the same market prices, the profits obtained in CD are steadily higher than with EMod. These results are illustrated in table 5. This could be a consequence of the fact that the EMod function is an inherently constrained function and is not capable of unlimited exponential growth.

Table 5: Profit-maximisation for a fixed output y leads to varying input cost shares

Total production	**Cobb Douglas approximation**	**EMod Approximation**			
Output y	Maximum profit	Maximum profit	Capital	Labour	Energy
6 400	-2 309	-2 751	10 %	16 %	74 %
12 800	-2 648	-3 581	10 %	15 %	75 %
25 600	-1 801	-3 739	10 %	15 %	76 %
51 200	2 598	-1 375	9 %	14 %	76 %
102 400	16 193	8 137	9 %	14 %	77 %
150 000	31 800	19 957	9 %	14 %	77 %
200 000	50 060	34 256	9 %	14 %	78 %
250 000	69 681	49 930	9 %	13 %	78 %
300 000	90 344	66 660	9 %	13 %	78 %
350 000	111 846	84 243	9 %	13 %	78 %
400 000	134 047	102 538	8 %	13 %	78 %

The results are also illustrated in figure 4. The large variation in input cost shares shows a large difference with the approach of the Cobb-Douglas function.

Figure 4: The EMod function leads to variable optimal cost shares and lower profit margins than a Cobb-Douglas function



3.3 Optimal input cost shares with restricted cash-flow after a market change

In a next step the market prices are altered and consequently also the market equilibria. Each firm, determined in the first step, adjusts its cost shares. Profits are again maximised. But there is a single restriction that the total cash flow of the company cannot increase. So the total of the costs incurred for capital, labour and energy should remain the same. The adaptations show clearly the differences in behaviour between the EMod function and the Cobb-Douglas function. Annex B details the equations used to find the optimised cost shares before and after the market change.

The market price of energy is reduced with 25%. Logically, production should increase for all firms in both functions. It is also expected to see profits rise, as the inputs are less expensive, but the output price remains unchanged. The first point of interest is to see the changes in inputs and input cost shares and to compare the differences between the two functions. The results are presented in table 6.

Table 6: The total Cash Flow being constant, the market responses of firms modelled with Cobb Douglas or with the EMod function are remarkably different.

	Cobb Douglas approximation			**EMod approximation**					
Initial turnover [EUR]	New Turnover [EUR]	New profit [EUR]	Profit increase [EUR]	New Turnover [EUR]	Change in cost share			New profit [EUR]	Profit increase [EUR]
					Capital	Labour	Energy		
6 400	7 908	-802	1 508	7 373	6.7%	10.4%	-17.1%	-1 778	973
12 800	15 815	367	3 015	15 170	5.9%	9.1%	-15.1%	-1 211	2 370
25 600	31 631	4 230	6 031	31 009	5.2%	8.1%	-13.3%	1 670	5 409
51 200	63 262	14 660	12 062	63 082	4.6%	7.2%	-11.8%	10 507	11 882
102 400	126 523	40 316	24 123	127 870	4.1%	6.4%	-10.5%	33 607	25 470
150 000	185 337	67 137	35 337	188 475	3.9%	6.0%	-9.9%	58 432	38 475
200 000	247 116	97 177	47 116	252 361	3.7%	5.7%	-9.4%	86 617	52 361
250 000	308 895	128 577	58 895	316 404	3.6%	5.5%	-9.1%	116 334	66 404
300 000	370 674	161 018	70 674	380 574	3.5%	5.4%	-8.8%	147 234	80 574
350 000	432 453	194 299	82 453	444 833	3.4%	5.2%	-8.6%	179 076	94 833
400 000	494 232	228 279	94 232	509 170	3.3%	5.1%	-8.5%	211 708	109 170

The cost shares of the Cobb Douglas function do not change. The price decrease of energy is met with an increase in energy intake. The overall energy cost share remains the same. The increased energy intake leads to a larger production and to larger profits. This is however not the behaviour that was intended for the type of company under investigation. Only rarely the production can increase by increasing the energy input. A simultaneous investment in capital or labour is needed in practise, so the estimation of the firm's response with a Cobb Douglas function is not likely.

The cost shares of the EMod function on the other hand do change simultaneously. Every increase in energy intake is accompanied by a smaller investment in capital and in labour. The cost shares of labour and capital rise significantly to the detriment of the energy cost share. This is a more realistic reaction. This is also illustrated in figure 5.

The firm adapts its inputs and it also increases its profits. This is illustrated in figure 6. The Cobb-Douglas model shows a consistent profit increase that follows the advantages of scale exhibited by the function. The EMod model shows smaller profit increases at small capacities of production. At regular production capacities, the overall profit increase due to adaptation after the market shock is relatively similar for both models.

Figure 5: Whereas for the Cobb-Douglas model only the energy intage increases, both the capital and the energy intake increase for the EMod model.

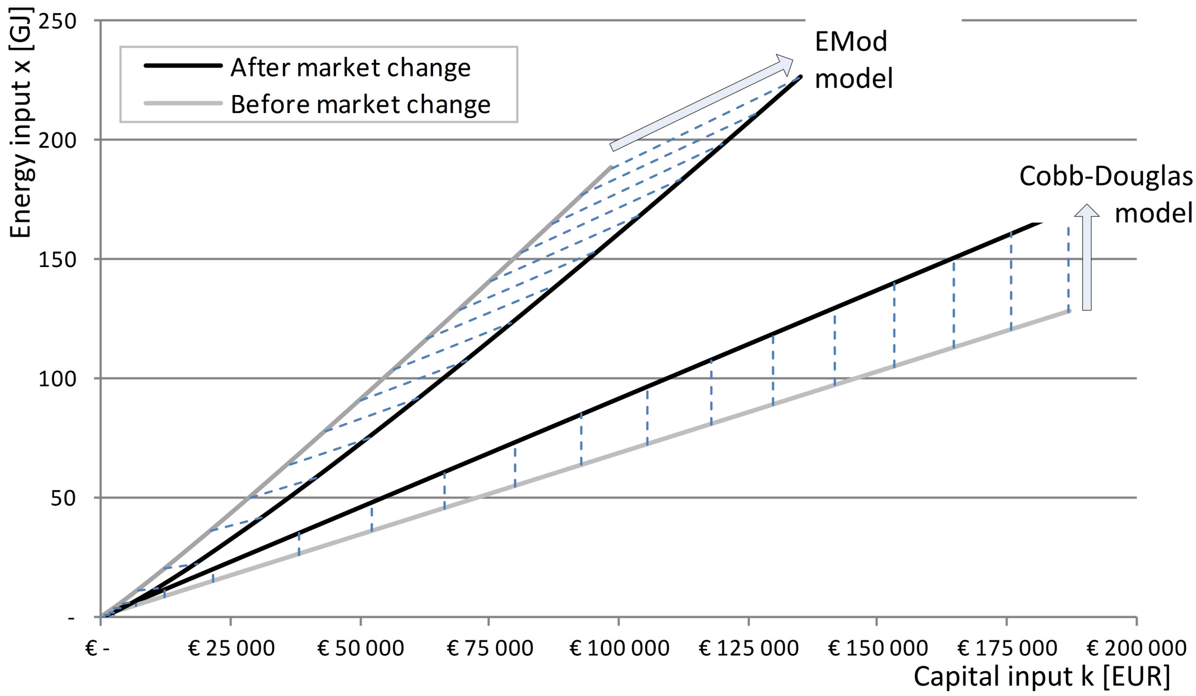
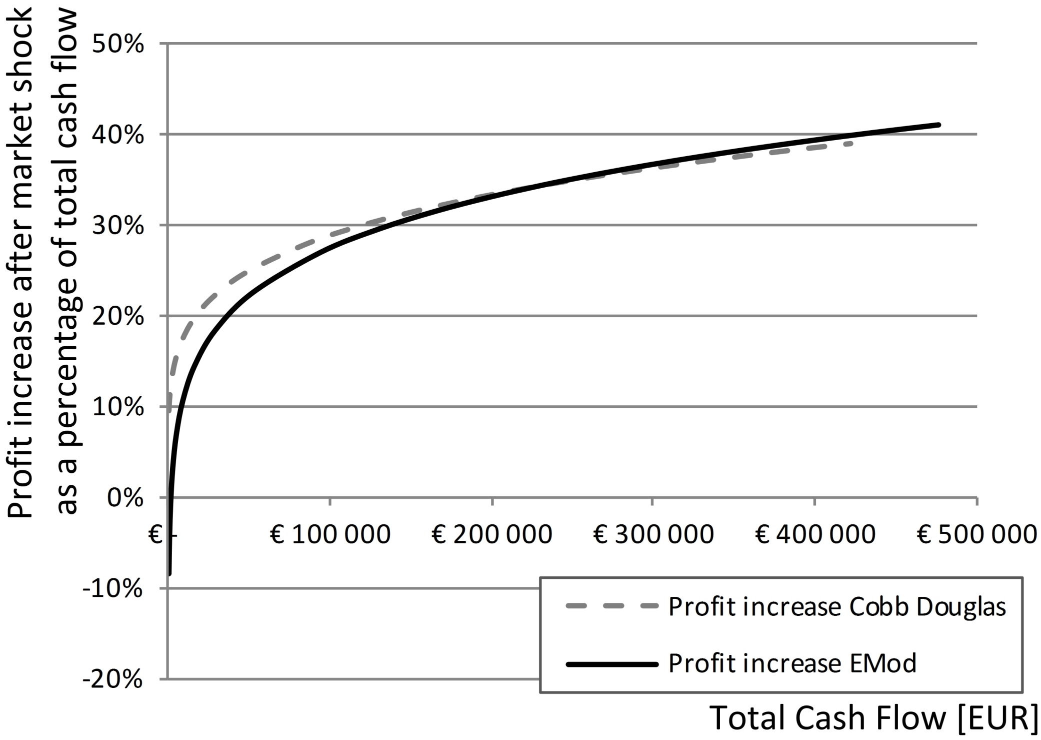


Figure 6: The EMod model and the Cobb-Douglas model show similar profit increases over the range of production.



4 Conclusions

The microeconomic modelling of material- and energy-intensive firms can be done with standard economic production functions, that integrate one or several material and energy input factors. However, this approach does encounter some principle obstacles, because standard production functions do not always reflect precisely the physical limitations of material and energy use. This is not an issue when the model is set up to fit functions to existing data. The physical restrictions of material and energy use will be present in the data implicitly. However, it might be a problem for predicting or extrapolating models.

In this paper a new production function was derived exactly for this purpose. This EMod function is based on standard requirements for economic production functions, but also on additional constraints related to the physical restrictions on the transformation of matter and energy. The resulting algebraic form is convenient, because it can be defined by only five parameters. It allows modelling of different technologies with varying scale advantages and ensures a constant elasticity of substitution between capital and labour. However the form is non-linear which might oblige the use of complicated econometric methods.

The EMod function has been applied in order to test its utility, first for data regression. The results are very similar to those of a standard Cobb-Douglas regression. But similarity as such is not a sufficient argument to adopt this EMod function. The non-linearity makes data regression certainly more complex. However, when using this function in a optimisation model or a partial equilibrium model, the results are very different and show the added value of this approach. We compared the behaviour of the EMod function under influence of changing external market factors. The main differences in behaviour are the changes in optimal input cost shares. These cost shares no longer are constant, and they tend to favour larger energy cost shares. Furthermore, when a price modification changes the market equilibrium, the EMod function presents a more restricted behaviour. Investments in additional energy or material input also necessitate related investments in capital and labour. This is coherent with the principles of microeconomic behaviour that led to the derivation of the EMod function. The overall predictions of profit increase is the same whether the EMod or the Cobb-Douglas function is applied. But the underlying dynamics and substitution between inputs to obtain this profit increase are very different between the two models.

More diversified applications of this EMod function can learn more about the advantages and the limitations to its use. Secondly, it is foreseen in future research to compare the behaviour also with other production functions such as the Leontief or Translog function.

Several models of environmental economics simulate economic agents that require a large input of physical resources, such as agricultural flows, energy flows or minerals. These models can opt for the use of the EMod form, as it is designed to simulate much closer the physical behaviour of the installations. At this preliminary stage, it is still required to accompany this with a parallel implementation of standard economic equations. This ensures a correct interpretation of the obtained results.

A Appendix A : Elasticity of substitution between k and l for the EMod form

The elasticity of substitution between capital and labour is defined as :

$$ES_{k,l} = \frac{d\ln(\frac{k}{l})}{d\ln(\frac{MPL}{MPK})} \tag{27}$$

The marginal products of capital and labour (MPK and MPL respectively) are:

$$\begin{aligned}
MPK = \frac{\partial y}{\partial k} &= c_0\rho k^{\rho-1}l^{\sigma}(1-e^{-\frac{m_0x^{\gamma}}{k^{\rho}l^{\sigma}}}) + c_0k^{\rho}l^{\sigma}\frac{m_0x^{\gamma}}{k^{\rho}l^{\sigma}}\frac{\rho}{k}e^{-\frac{m_0x^{\gamma}}{k^{\rho}l^{\sigma}}} &(28)\\
&= \frac{c_0\rho}{k}(k^{\rho}l^{\sigma}(1-e^{-\frac{m_0x^{\gamma}}{k^{\rho}l^{\sigma}}}) + m_0x^{\gamma}e^{-\frac{m_0x^{\gamma}}{k^{\rho}l^{\sigma}}}) &(29)\\
MPL = \frac{\partial y}{\partial l} &= \frac{c_0\sigma}{l}(k^{\rho}l^{\sigma}(1-e^{-\frac{m_0x^{\gamma}}{k^{\rho}l^{\sigma}}}) + m_0x^{\gamma}e^{-\frac{m_0x^{\gamma}}{k^{\rho}l^{\sigma}}}) &(30)
\end{aligned}$$

This indicates that the ratio between the two marginal products is :

$$\begin{aligned}
\frac{MPL}{MPK} &= \frac{\sigma k}{\rho l} &(31)\\
\ln(\frac{MPL}{MPK}) &= \ln\frac{\sigma}{\rho} + \ln\frac{k}{l} &(32)
\end{aligned}$$

Equation 32 shows a linear relationship, so that $ES_{k,l} = 1$.

B Appendix B : Optimal cost shares for profit-maximising firms

The optimal input cost shares of the firm depend on the prevailing market prices. Competitive markets are assumed and the individual firms have no market power. The output market price is p. The input prices for capital, labour and energy are w_k, w_l and w_x respectively. This allows to define the profit π of the firm as:

$$\pi = py - w_kk - w_ll - w_xx \tag{33}$$

In the first step, the total production y of the firm is fixed. So the optimal cost shares can be found with the first order differentiation which leads to :

$$\begin{aligned}
\frac{w_kk}{py} &= \frac{k}{y}\frac{\partial y}{\partial k} = E_k &(34)\\
\frac{w_ll}{py} &= \frac{l}{y}\frac{\partial y}{\partial l} = E_l &(35)\\
\frac{w_xx}{py} &= \frac{x}{y}\frac{\partial y}{\partial x} = E_x &(36)
\end{aligned}$$

For the Cobb Douglas function, the input elasticities are the coefficients of the functions, and are thus constant. The former equations allow thus to determine the optimal input cost shares directly.

For the EMod function, the solution has to be found iteratively. The input elasticities allow the following simplifications :

$$\frac{w_x x}{py} = E_x \tag{37}$$

$$\frac{w_k k}{py} = \rho(1 - \frac{E_x}{\gamma}) \tag{38}$$

$$\frac{w_l l}{py} = \sigma(1 - \frac{E_x}{\gamma}) \tag{39}$$

In the first phase, the output y is constant and fixed on beforehand. First a estimation for x is fixed, $\breve{x}$. Then the related quantities for k and l are determined, which lead to an output estimation $\breve{y}$. $\breve{y}$ is compared to y and $\breve{x}$ is adopted accordingly with the following relations :

$$w_k \breve{k} = \rho py(1 - \frac{w_x \breve{x}}{\gamma py}) \tag{40}$$

$$w_l \breve{l} = \sigma py(1 - \frac{w_x \breve{x}}{\gamma py}) \tag{41}$$

$$p\breve{y} = c_0 \breve{k}^\rho \breve{l}^\sigma (1 - e^{-\frac{m_0 \breve{x}^\gamma}{\breve{k}^\rho \breve{l}^\sigma}}) \tag{42}$$

In the second phase, y is no longer constant, but is also adapted influenced by the modified market price for energy $\dot{w}_x$. So the production moves from the original equilibrium $y(k, l, x)$ to the new one $\dot{y}(\dot{k}, \dot{l}, \dot{x})$. One can follow the same procedure as above to determine the new equilibrium, by adding the constraint that the total cash flow CF remains the same.

$$CF = w_k k + w_l l + w_x x = w_k \dot{k} + w_l \dot{l} + \dot{w}_x \dot{x} \tag{43}$$

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