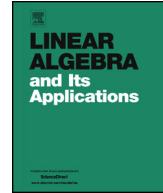




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## The nullity theorem for principal pivot transform



Robert Brijder

Hasselt University and Transnational University of Limburg, Belgium

### ARTICLE INFO

#### Article history:

Received 24 July 2013

Accepted 5 September 2013

Available online 4 October 2013

Submitted by R. Brualdi

#### MSC:

05C50

15A03

15A09

#### Keywords:

Nullity theorem

Principal pivot transform

Schur complement

Local complementation

Nullity polynomial

### ABSTRACT

We generalize the nullity theorem of Gustafson (1984) [8] from matrix inversion to principal pivot transform. Several special cases of the obtained result are known in the literature, such as a result concerning local complementation on graphs. As an application, we show that a particular matrix polynomial, the so-called nullity polynomial, is invariant under principal pivot transform.

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## 1. Introduction

Motivated by the well-known linear complementarity problem, Tucker [13] defined a matrix operation to study combinatorial equivalence of matrices. A slight modification of this matrix operation (some signs are different in the definition) became known as *principal pivot transform*. Principal pivot transform partially (component-wise) inverts a given matrix along a given set of indices, and it is applied in various settings such as mathematical programming and numerical analysis, see [12] for an overview.

The nullity theorem [8], independently discovered in [7, Theorem 2], establishes a one-to-one correspondence between the submatrices of a nonsingular matrix and the submatrices of its inverse such that the nullities of the submatrices are retained by the correspondence. The power of the nullity theorem is well illustrated in [11]. The main result of this paper (Theorem 1) generalizes the nullity

E-mail address: [robert.brijder@uhasselt.be](mailto:robert.brijder@uhasselt.be).

theorem to principal pivot transform (for which matrix inversion is a special case). We show that several other special cases of this main result are known in the literature, including a result on graphs, and we show that a particular matrix polynomial, the so-called nullity polynomial, is invariant under principal pivot transform (Corollary 2).

**2. Notation and terminology**

For finite sets  $U$  and  $V$ , a  $U \times V$ -matrix  $A$  (over some field  $\mathbb{F}$ ) is a matrix where the rows are indexed by  $U$  and the columns are indexed by  $V$ , i.e.,  $A$  is formally a function  $U \times V \rightarrow \mathbb{F}$ . Hence, the order of the rows/columns is not fixed (i.e., interchanging rows or columns is mute). Note that, e.g., the rank  $r(A)$ , the nullity (i.e., dimension of the null space)  $n(A)$ , and the inverse  $A^{-1}$  are well defined for  $A$  (the latter, of course, only when  $A$  is square and nonsingular). We denote for  $i \in U$  and  $j \in V$ , the value of the  $(i, j)$ th entry of  $A$  by  $A[i, j]$ . For  $X \subseteq U$  and  $Y \subseteq V$ , the submatrix of  $A$  induced by  $(X, Y)$ , denoted by  $A[X, Y]$ , is the restriction of  $A$  to  $X \times Y$ .

Similarly, a vector indexed by  $V$  is formally a function  $V \rightarrow \mathbb{F}$ , and we denote the element of  $v$  corresponding to  $i \in V$  by  $v[i]$ . As usual, the family of vectors indexed by  $V$  is denoted by  $\mathbb{F}^V$ . For  $Y \subseteq V$ , we let  $\iota_{Y,V}$  be the usual injection  $\mathbb{F}^Y \rightarrow \mathbb{F}^V$  by padding zeros. More precisely,  $(\iota_{Y,V}(w))[x] = w[x]$  if  $x \in Y$  and  $(\iota_{Y,V}(w))[x] = 0$  if  $x \in V \setminus Y$ . Similarly, we let  $\pi_{Y,V}$  be the usual projection of  $\mathbb{F}^V \rightarrow \mathbb{F}^Y$  by disregarding the entries with indices in  $V \setminus Y$ . If  $V$  is clear from the context, we simply write  $\iota_Y$  and  $\pi_Y$  for  $\iota_{Y,V}$  and  $\pi_{Y,V}$ , respectively.

**3. Principal pivot transform**

In this section we recall principal pivot transform, which is an operation for square matrices, see [12] for an overview.

Let  $A$  be a  $V \times V$ -matrix (over an arbitrary field  $\mathbb{F}$ ) and let  $X \subseteq V$  be such that the corresponding principal submatrix  $A[X, X]$  is nonsingular. The *principal pivot transform* (PPT for short) of  $A$  on  $X$ , denoted by  $A * X$ , is defined as follows:

$$A * X = \begin{matrix} & X & & V \setminus X \\ & & & \\ & & & \\ & & & \end{matrix} \begin{pmatrix} A[X, X]^{-1} & -A[X, X]^{-1}A[X, V \setminus X] \\ A[V \setminus X, X]A[X, X]^{-1} & A[V \setminus X, V \setminus X] - A[V \setminus X, X]A[X, X]^{-1}A[X, V \setminus X] \end{pmatrix}. \tag{1}$$

Matrix  $A * X[V \setminus X, V \setminus X]$  is called the *Schur complement* of  $A[X, X]$  in  $A$  [14].

Principal pivot transform can be considered a partial inverse, as  $A$  and  $A * X$  are related as follows, where the vectors  $x_1$  and  $y_1$  correspond to the elements of  $X$ :

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{if and only if} \quad A * X \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}. \tag{2}$$

Eq. (2) characterizes PPT, see [12, Theorem 3.1]. Note that if  $A$  is nonsingular, then  $A * V = A^{-1}$ . Also note that by Eq. (2) PPT is an involution (operation of order 2), and more generally, if  $(A * X) * Y$  is defined, then it is equal to  $A * (X \Delta Y)$ , where  $\Delta$  denotes symmetric difference.

We denote by  $A \# X$  the matrix obtained from  $A$  by replacing every row of  $A$  with index  $x \in V \setminus X$  by  $i_x^T$  where  $i_x$  is the vector having value 1 at index  $x$  and 0 elsewhere. Note that  $A \# X \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ x_2 \end{pmatrix}$  with  $x_1, x_2$ , and  $y_1$  from Eq. (2). From this it follows that if  $A[X, X]$  is nonsingular, then  $(A \# X)^{-1} = (A * X) \# X$ .

**4. Nullity theorem for principal pivot transform**

The following theorem is used in [6] to generalize the recursive relation for interlace polynomials from graphs [2,1] to arbitrary square matrices over arbitrary fields.

**Proposition 1.** (See [6].) Let  $A$  be a  $V \times V$ -matrix (over some field) and let  $Z \subseteq V$  be such that  $A[Z, Z]$  is nonsingular. Then, for all  $X \subseteq V$ ,  $n(A * Z[X, X]) = n(A[X \Delta Z, X \Delta Z])$ .

We now recall the nullity theorem.

**Proposition 2** (The nullity theorem). (See [8].) Let  $A$  be a nonsingular  $V \times V$ -matrix (over some field). Then, for all  $X, Y \subseteq V$ ,  $n(A^{-1}[X, Y]) = n(A[V \setminus Y, V \setminus X])$ .

**Remark 1.** We remark that the nullity theorem is in the literature often formulated in an unnecessary cumbersome way, since rows and columns of matrices are conventionally ordered and moreover not explicitly indexed. Instead, the concise description of the nullity theorem in Proposition 2 shows its nature as an elementary result.

Since  $A * V = A^{-1}$ , it is natural to ask whether Propositions 1 and 2 may be put under a common umbrella. We now show that this is indeed the case. Moreover, we relate the null spaces of the submatrices of  $A$  and  $A * Z$ .

**Theorem 1** (Nullity theorem for PPT). Let  $A$  be a  $V \times V$ -matrix (over some field) and let  $Z \subseteq V$  be such that  $A[Z, Z]$  is nonsingular. Then, for all  $X, Y \subseteq V$ ,

$$\ker(A * Z[X, Y]) = \ker(A[X \Delta R, Y \Delta R] \cdot (A\sharp Z)^{-1}[Y \Delta R, Y]) \quad \text{and} \tag{3}$$

$$n(A * Z[X, Y]) = n(A[X \Delta R, Y \Delta R]), \tag{4}$$

where  $R = Z \setminus (X \Delta Y)$ .

**Proof.** Let  $v \in \ker(A[X, Y])$ . Then  $A[X, Y]v = 0$ , and so  $\pi_X(A\iota_Y(v)) = 0$ .

If  $Ax = y$ , then for all  $i \in V$ ,

$$((A\sharp Z)x)[i] = \begin{cases} y[i] & \text{if } i \in Z, \\ x[i] & \text{otherwise,} \end{cases} \tag{5}$$

and, by Eq. (2),

$$((A * Z)(A\sharp Z)x)[i] = \begin{cases} x[i] & \text{if } i \in Z, \\ y[i] & \text{otherwise.} \end{cases} \tag{6}$$

By Eq. (5),  $((A\sharp Z)\iota_Y(v))[i] = 0$  if  $i \in (X \cap Z) \cup ((V \setminus Y) \setminus Z) = V \setminus (Y \Delta R)$ . Thus  $(A * Z \cdot A\sharp Z)\iota_Y(v) = (A * Z[V, Y \Delta R] \cdot A\sharp Z[Y \Delta R, Y])v$ .

By Eq. (6),  $((A * Z \cdot A\sharp Z)\iota_Y(v))[i] = 0$  if  $i \in (X \setminus Z) \cup ((V \setminus Y) \cap Z) = X \Delta R$ . Thus  $(A * Z[X \Delta R, Y \Delta R] \cdot A\sharp Z[Y \Delta R, Y])v = \pi_{X \Delta R}(A * Z \cdot A\sharp Z)\iota_Y(v) = 0$ . Consequently,  $v \in \ker(A * Z[X \Delta R, Y \Delta R] \cdot A\sharp Z[Y \Delta R, Y])$ . Hence

$$\ker(A[X, Y]) \subseteq \ker(A * Z[X \Delta R, Y \Delta R] \cdot A\sharp Z[Y \Delta R, Y]). \tag{7}$$

We show that if  $v_1, v_2 \in \ker(A[X, Y])$  are distinct, then  $A\sharp Z[Y \Delta R, Y]v_1 \neq A\sharp Z[Y \Delta R, Y]v_2$ . Assume to the contrary that  $A\sharp Z[Y \Delta R, Y]v_1 = A\sharp Z[Y \Delta R, Y]v_2$ . Recall that  $((A\sharp Z)\iota_Y(v))[i] = 0$  if  $i \in V \setminus (Y \Delta R)$ . Hence,  $A\sharp Z\iota_Y(v_1) = A\sharp Z\iota_Y(v_2)$ , and thus  $\iota_Y(v_1) = \iota_Y(v_2)$  as  $A\sharp Z$  is nonsingular. Consequently,  $v_1 = v_2$ —a contradiction. We thus obtain, by (7),  $n(A * Z[X \Delta R, Y \Delta R]) \geq n(A[X, Y])$ .

We apply now (7) to  $A := A * Z$ ,  $X := X \Delta R$ , and  $Y := Y \Delta R$ . We obtain  $\ker(A * Z[X \Delta R, Y \Delta R]) \subseteq \ker((A * Z) * Z[(X \Delta R) \Delta R', (Y \Delta R) \Delta R'] \cdot A * Z\sharp Z[(Y \Delta R) \Delta R', Y \Delta R])$  where  $R' = Z \setminus ((X \Delta R) \Delta (Y \Delta R)) = Z \setminus (X \Delta Y) = R$ . Hence we have

$$\ker(A * Z[X \Delta R, Y \Delta R]) \subseteq \ker(A[X, Y] \cdot (A\sharp Z)^{-1}[Y, Y \Delta R]) \tag{8}$$

as  $(A\sharp Z)^{-1} = A * Z\sharp Z$ . We again obtain that if  $v_1, v_2 \in \ker(A * Z[X \Delta R, Y \Delta R])$  are distinct, then  $(A\sharp Z)^{-1}[Y, Y \Delta R]v_1 \neq (A\sharp Z)^{-1}[Y, Y \Delta R]v_2$ . Thus, by (8),  $n(A[X, Y]) \geq n(A * Z[X \Delta R, Y \Delta R])$ .

Consequently,  $n(A[X, Y]) = n(A * Z[X \Delta R, Y \Delta R])$  and the inclusions of (7) and (8) are equalities. By change of variables  $A := A * Z$ , we have (3) and (4).  $\square$

Note that Propositions 1 and 2 are Eq. (4) of Theorem 1 for the cases  $X = Y$  and  $Z = V$ , respectively. The case  $Y = V \setminus X$  is also of particular interest, and so we explicitly state it here.

**Corollary 1.** *Let  $A$  be a  $V \times V$ -matrix (over some field) and let  $Z \subseteq V$  be such that  $A[Z, Z]$  is nonsingular. Then, for all  $X \subseteq V$ ,  $n(A * Z[X, V \setminus X]) = n(A[X, V \setminus X])$ .*

Equivalently, we have  $r(A * Z[X, V \setminus X]) = r(A[X, V \setminus X])$ .

Special cases of Corollary 1 have been considered in the literature. Oum [10, Corollary 4.14] shows, through the use of Lagrangian chain-groups, that Corollary 1 holds for the case where  $A$  is skew-symmetric or symmetric. In the next section we show that a result on graphs of Bouchet [3] can also be seen as a special case of Corollary 1.

The polynomial  $q(A) = \sum_{X \subseteq V} y^{n(A[X, X])}$  is a straightforward generalization of the interlace polynomial [2,1] and is shown to be invariant under PPT [6]. Due to Theorem 1, we may now define another polynomial that is invariant under PPT. Let us define the (extended) nullity polynomial for a  $V \times V$ -matrix  $A$  by

$$p(A) = \sum_{X, Y \subseteq V} y^{n(A[X, Y])}.$$

**Lemma 1.** *For each  $Z \subseteq V$ , the function  $f_Z : 2^V \times 2^V \rightarrow 2^V \times 2^V$  defined by  $f_Z(X, Y) = (X \Delta R_{X, Y, Z}, Y \Delta R_{X, Y, Z})$  with  $R_{X, Y, Z} = Z \setminus (X \Delta Y)$  is a one-to-one correspondence.*

**Proof.** Since the domain and codomain of  $f_Z$  are finite and equal, it suffices to show that  $f_Z$  is injective. Assume that  $f_Z(X, Y) = f_Z(X', Y')$ . Then  $X \Delta R_{X, Y, Z} = X' \Delta R_{X', Y', Z}$  and  $Y \Delta R_{X, Y, Z} = Y' \Delta R_{X', Y', Z}$ . Thus  $X \Delta Y = (X \Delta R_{X, Y, Z}) \Delta (Y \Delta R_{X, Y, Z}) = (X' \Delta R_{X', Y', Z}) \Delta (Y' \Delta R_{X', Y', Z}) = X' \Delta Y'$ . Therefore  $R_{X, Y, Z} = R_{X', Y', Z}$ , and so  $X = X'$  and  $Y = Y'$ .  $\square$

By Lemma 1 and Theorem 1, for each  $i \in \{0, \dots, |V|\}$ , the number of submatrices of  $A$  of nullity  $i$  is invariant under PPT. Hence we have the following.

**Corollary 2.** *Let  $A$  be a  $V \times V$ -matrix, and  $Z \subseteq V$  such that  $A[Z, Z]$  is nonsingular. Then  $p(A) = p(A * Z)$ .*

### 5. Graphs

Let  $G = (V, E)$  be a simple graph, i.e., without loops or parallel edges. We write  $V(G) = V$  and  $E(G) = E$ . The neighborhood of  $v \in V$  in  $G$ , denoted by  $N_G(v)$ , is  $\{w \in V \mid \{v, w\} \in E(G)\}$ . The local complement of  $G$  at  $v \in V$ , denoted by  $G^v$ , is obtained from  $G$  by replacing the subgraph of  $G$  induced by  $N_G(v)$  by its complementary subgraph. Hence, if  $u, w \in N_G(v)$  are distinct, then  $\{u, w\} \in E(G)$  if and only if  $\{u, w\} \notin E(G^v)$ . Graphs  $G$  and  $G'$  are said to be locally equivalent if there is a (possibly empty) sequence of local complementations such that  $G'$  is obtained from  $G$ . Since local complementation is an involution, local equivalence induces an equivalence relation.

The adjacency matrix  $A(G)$  of  $G$  is the  $V(G) \times V(G)$ -matrix over  $GF(2)$  where for all  $u, v \in V$ ,  $A(G)[u, v] = 1$  if and only if  $\{u, v\} \in E(G)$  (note that the diagonal entries are 0). The following result is from Bouchet [3] (see also [4, Section 3]), and is rediscovered in [9, Proposition 2.6].

**Proposition 3.** (See [3].) *Let  $G$  and  $G'$  be simple graphs that are locally equivalent. Then, for all  $X \subseteq V$ ,  $r(A(G')[X, V(G) \setminus X]) = r(A(G)[X, V(G) \setminus X])$ .*

We remark that the function which, for a simple graph  $G$ , assigns every  $X \subseteq V(G)$  to the value  $r(A(G)[X, V(G) \setminus X])$ , is called the connectivity function in [3] and the cut-rank in [9].

Let, for  $X \subseteq V$ ,  $I_X$  be the  $V(G) \times V(G)$ -matrix over  $GF(2)$  where for all  $u, v \in V(G)$ ,  $I_X[u, v] = 1$  if and only if  $u = v \in X$ . By Eq. (1) it is easy to see that  $A(G^v) = ((A(G) + I_{\{v\}}) * \{v\}) + I_{N_G(v) \cup \{v\}}$  for all  $v \in V(G)$  (see also, e.g., [5]). Proposition 3 follows now readily from Corollary 1.

**Proof of Proposition 3.** It suffices to consider the case  $G' = G^v$  with  $v \in V(G)$  as the general case follows by iteration. We have  $A(G^v)[X, V(G) \setminus X] = (((A(G) + I_{\{v\}}) * \{v\}) + I_{N_G(v) \cup \{v\}})[X, V(G) \setminus X] = ((A(G) + I_{\{v\}}) * \{v\})[X, V(G) \setminus X]$ . By Corollary 1,  $r(((A(G) + I_{\{v\}}) * \{v\})[X, V(G) \setminus X]) = r((A(G) + I_{\{v\}})[X, V(G) \setminus X]) = r(A(G)[X, V(G) \setminus X])$ .  $\square$

## Acknowledgement

We thank Hendrik Jan Hoogeboom for careful reading and useful comments on the paper.

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