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THE GREEN RINGS OF TAFT ALGEBRAS

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Abstract. We compute the Green ring of the Taft algebra $H_n(q)$, where n is a positive integer greater than 1 and q is an n-th root of unity. It turns out that the Green ring $r(H_n(q))$ of the Taft algebra $H_n(q)$ is a commutative ring generated by two elements subject to certain relations defined recursively. Concrete examples for $n = 2, 3, ..., 8$ are given.

Introduction

The tensor product of representations of a Hopf algebra is an important ingredient in the representation theory of Hopf algebras and quantum groups. In particular, the decomposition of the tensor product of indecomposable modules into a direct sum of indecomposables has received enormous attention. For modules over a group algebra this information is encoded in the structure of the Green ring (or the representation ring) for finite groups, [1–4, 9, 11]. For modules over a Hopf algebra or a quantum group there are results by Cibils on a quiver quantum group [7], by Witherspoon on the quantum double of a finite group [18], by Gunnlaugsdóttir on the half quantum groups (or Taft algebras) [10], and by Chin on the coordinate Hopf algebra of quantum SL(2) at a root of unity [8]. However, the Green rings of those Hopf algebras are either equal to the Grothendick rings (in the semisimple cases) or not yet computed because of the complexity.

In this paper, we compute the Green rings of Taft algebras. It turns out that the Green ring of a Taft algebra is much more complicated than its Grothendick ring. In Section 1, we recall some basic definitions and results and make some preparations for the rest of the paper. In Section 2, we recall the indecomposable modules over the Taft algebra $H_n(q)$ from [7, 10], using the terminology of matrix representations, where n is a positive integer $\geqslant 2$ and q is a primitive n-th root of unity in the ground field k. There are n^2 non-isomorphic finite dimensional indecomposable modules over $H_n(q)$, and all of them are uniserial. Moreover, for each $1 \leqslant l \leqslant n$, there are exactly n finite dimensional indecomposable $H_n(q)$-modules $M(l, r)$, $r \in \mathbb{Z}_n$, up to isomorphism. Every indecomposable projective $H_n(q)$-module is n-dimensional.

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In Section 3, we describe the Green ring of Taft algebra $H_n(q)$. We first recall the decomposition formula of the tensor product of two indecomposable modules over $H_n(q)$ from [7, 10]. From the decomposition formula, we know that the tensor product of any two $H_n(q)$-modules is commutative. Moreover, the tensor product of two indecomposable non-projective modules has a simple summand if and only if the two indecomposable modules have the same dimension. Finally, we describe the structure of the Green ring $r(H_n(q))$ of $H_n(q)$. We show that the Green ring $r(H_n(q))$ is generated by two elements subject to certain relations which can be defined recursively.

1. Preliminaries

Throughout, we work over a fixed field k. Unless otherwise stated, all algebras, Hopf algebras and modules are defined over k; all modules are left modules and finite dimensional; all maps are k-linear; dim, $\otimes$ and Hom stand for $\dim_k$, $\otimes_k$ and Hom_k, respectively. For the theory of Hopf algebras and quantum groups, we refer to [12, 14–16].

Let $0 \neq q \in k$. For any integer $n > 0$, set $(n)_q = 1 + q + \cdots + q^{n-1}$. Observe that $(n)_q = n$ when $q = 1$, and

$$(n)_q = \frac{q^n - 1}{q - 1}$$

when $q \neq 1$. Define the q-factorial of n by $(0)!_q = 1$ and $(n)!_q = (n)_q(n-1)_q \cdots (1)_q$ for $n > 0$. Note that $(n)!_q = n!$ when $q = 1$, and

$$(n)!_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{(q - 1)^n}$$

when $n > 0$ and $q \neq 1$. The q-binomial coefficients $\begin{pmatrix} n \\ i \end{pmatrix}_q$ are defined inductively as follows for $0 \leqslant i \leqslant n$:

$$\begin{pmatrix} n \\ 0 \end{pmatrix}_q = 1 = \begin{pmatrix} n \\ n \end{pmatrix}_q \qquad \text{for } n \geqslant 0,$$

$$\begin{pmatrix} n \\ i \end{pmatrix}_q = q^i \begin{pmatrix} n-1 \\ i \end{pmatrix}_q + \begin{pmatrix} n-1 \\ i-1 \end{pmatrix}_q \qquad \text{for } 0 < i < n.$$

It is well-known that $\begin{pmatrix} n \\ i \end{pmatrix}_q$ is a polynomial in q with integer coefficients and with value at $q = 1$ is equal to the usual binomial coefficient $\begin{pmatrix} n \\ i \end{pmatrix}$, and that

$$\begin{pmatrix} n \\ i \end{pmatrix}_q = \frac{(n)!_q}{(i)!_q(n-i)!_q}$$

when $(n-1)!_q \neq 0$ and $0 < i < n$ (see [12, page 74]).

Throughout this paper, we fix an integer $n \geqslant 2$ and assume that the field k contains an n-th primitive root q of unity. Then we have

$$\begin{pmatrix} n \\ i \end{pmatrix}_q \neq 0, \quad (i)!_q \neq 0, \quad \text{for } 0 < i < n,$$

and that n is not divisible by the characteristic of k, i.e. $\frac{1}{n} \in k$.

The Taft algebra $H_n(q)$ is generated by two elements g and h subject to the relations (see [17])

$$g^n = 1, \ h^n = 0, \ hg = qgh.$$

$H_n(q)$ is a Hopf algebra with coalgebra structure Δ and antipode S given by

$$\begin{array}{lll} \Delta(g) = g \otimes g, & \Delta(h) = 1 \otimes h + h \otimes g, & \varepsilon(g) = 1, \\ \varepsilon(h) = 0, & S(g) = g^{-1} = g^{n-1}, & S(h) = -q^{-1}g^{n-1}h. \end{array}$$

Note that $\dim H_n(q) = n^2$ and $\{g^i h^j | 0 \leqslant i, j \leqslant n-1\}$ forms a k-basis for $H_n(q)$. When $n = 2$, $H_2(q)$ is exactly Sweedler's 4-dimensional Hopf algebra.

Let H be a Hopf algebra. The representation ring $r(H)$ and $R(H)$ can be defined as follows: $r(H)$ is the abelian group generated by the isomorphism classes $[V]$ of finite dimensional H-modules V modulo the relations $[M \oplus V] = [M] + [V]$. The multiplication of $r(H)$ is given by the tensor product of H-modules, that is, $[M][V] = [M \otimes V]$. Then $r(H)$ is an associative ring. $R(H)$ is an associative k-algebra defined by $k \otimes_{\mathbb{Z}} r(H)$. Note that $r(H)$ is a free abelian group with a $\mathbb{Z}$-basis $\{[V] | V \in \text{ind}(H)\}$, where $\text{ind}(H)$ denotes the category of finite dimensional indecomposable H-modules.

2. Representations of $H_n(q)$

For a module M over a finite dimensional algebra A, let $\text{rl}(M)$ denote the Loewy length (=radical length=socle length) of M, and let $\text{l}(M)$ denote the length of M. Let $P(M)$ denote the projective cover of M, and let $I(M)$ denote the injective hull of M.

Cibils constructed an nd-dimensional Hopf algebra $kZ_n(q)/I_d$ in [7], where q is an n-th root of unity in k with order d. He classified the indecomposable modules over $kZ_n(q)/I_d$ and gave the decomposition of the tensor products of two arbitrary indecomposable modules there. When q is a primitive n-th root of unity, $kZ_n(q)/I_n$ is isomorphic to $H_n(q)$ (see [7]). Therefore, from [7], one can get the classification of indecomposable modules and the decomposition of the tensor product of two indecomposable modules over $H_n(q)$. For completeness, we will describe the indecomposable modules over $H_n(q)$ in this section, using the terminology of matrix representation.

Let $G(H_n(q))$ denote the group of group-like elements in $H_n(q)$. Then $G(H_n(q)) = \{1, g, \cdots, g^{n-1}\}$ is a cyclic group of order n generated by g. The group algebra $kG(H_n(q))$ is a Hopf subalgebra of $H_n(q)$. There is a Hopf algebra epimorphism $\pi : H_n(q) \to kG(H_n(q))$ defined by $\pi(g) = g$ and $\pi(h) = 0$. Since k contains an n-th primitive root of unity, the group algebra $kG(H_n(q))$ is semisimple. It follows that $\text{Ker}\pi = \langle h \rangle \supseteq J(H_n(q))$, the Jacobson radical of $H_n(q)$. On the other hand, since $H_n(q)h = hH_n(q)$ and $h^n = 0$, $J(H_n(q)) \supseteq (h) = H_n(q)h$, the ideal of $H_n(q)$ generated by h. Hence $\text{Ker}\pi = (h) = J(H_n(q))$. Thus, an $H_n(q)$-module M is semisimple if and only if $h \cdot M = 0$, and moreover M is simple if and only if $h \cdot M = 0$ and M is simple as a module over the Hopf subalgebra $kG(H_n(q))$. Therefore, we have the following lemma.

Lemma 2.1. *There are n non-isomorphic simple $H_n(q)$-modules S_i, and each S_i is 1-dimensional and determined by*

$$g \cdot v = q^i v,\ h \cdot v = 0,\ v \in S_i,$$

where $i \in \mathbb{Z}_n := \mathbb{Z}/(n)$.

Note that $J(H_n(q))^m = H_n(q)h^m$ for all $m \geqslant 1$. Hence $J(H_n(q))^{n-1} \neq 0$, but $J(H_n(q))^n = 0$. This means that the Loewy length of $H_n(q)$ is n. Since every simple $H_n(q)$-module is 1-dimensional, $\mathrm{l}(M) = \dim(M)$ for any $H_n(q)$-module M.

Now let M be any $H_n(q)$-module. Since $J(H_n(q))^s = H_n(q)h^s = h^s H_n(q)$, we have $\mathrm{rad}^s(M) = h^s \cdot M$ for all $s \geqslant 1$.

Lemma 2.2. *Let $1 \leqslant l \leqslant n$ and $i \in \mathbb{Z}$. Then there is an algebra map $\rho_{l,i} : H_n(q) \to M_l(k)$ given by*

$$\rho_{l,i}(g) = \begin{pmatrix} q^i & & & & \\ & q^{i-1} & & & \\ & & q^{i-2} & & \\ & & & \ddots & \\ & & & & q^{i-l+1} \end{pmatrix},\ \rho_{l,i}(h) = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

Let $M(l,i)$ denote the corresponding left $H_n(q)$-module.

Proof. It follows from a straightforward verification. □

There is a k-basis $\{v_1, v_2, \cdots, v_l\}$ of $M(l,i)$ such that $g \cdot v_j = q^{i-j+1} v_j$ for all $1 \leqslant j \leqslant l$ and

$$h \cdot v_j = \begin{cases} v_{j+1}, & 1 \leqslant j \leqslant l-1, \\ 0, & j = l. \end{cases}$$

Hence we have $v_j = h^{j-1} \cdot v_1$ for all $2 \leqslant j \leqslant l$. Such a basis is called a *standard basis* of $M(l,i)$. For any integer i, we will often regard i as its image under the canonical projection $\mathbb{Z} \to \mathbb{Z}_n := \mathbb{Z}/(n)$. We have the following lemma.

Lemma 2.3. *For any $1 \leqslant l \leqslant n$ and $i \in \mathbb{Z}$, let $M(l,i)$ be the $H_n(q)$-module defined as in Lemma 2.2. Then:*

(1) $\mathrm{soc}(M(l,i)) = kv_l \cong S_{i-l+1}$ *and* $M(l,i)/\mathrm{rad}(M(l,i)) \cong S_i$.
(2) $M(l,i)$ *is indecomposable and uniserial.*
(3) *If $1 \leqslant l' \leqslant n$ and $i' \in \mathbb{Z}$, then $M(l,i) \cong M(l',i')$ if and only if $l' = l$ and $i' = i$ in $\mathbb{Z}_n$.*

Proof. (1) Since $J(H_n(q)) = (h) = hH_n(q) = H_n(q)h$, $\mathrm{soc}(M(l,i)) = \{v \in M(l,i) | h \cdot v = 0\} = kv_l$ and $\mathrm{rad}(M(l,i)) = h \cdot M(l,i) = \mathrm{span}\{v_2, \cdots, v_l\}$. It follows that $\mathrm{soc}(M(l,i)) \cong S_{i-l+1}$ and $M(l,i)/\mathrm{rad}(M(l,i)) \cong S_i$.

(2) By (1), $\mathrm{soc}(M(l,i))$ is simple, and hence $M(l,i)$ is indecomposable. Since $h^{l-1} \cdot M(l,i) \neq 0$ and $h^l \cdot M(l,i) = 0$, $\mathrm{rl}(M(l,i)) = l$. Hence $\mathrm{l}(M(l,i)) = \mathrm{rl}(M(l,i))$, and so $M(l,i)$ is uniserial.

(3) Obvious. □

As a consequence, we obtain the following:

Corollary 2.4. *Let $1 \leqslant l \leqslant n$ and $i \in \mathbb{Z}_n$. Then:*

(1) *$M(l,i)$ is simple if and only if $l = 1$. In this case, $M(1,i) \cong S_i$.*
(2) *$M(l,i)$ is projective (injective) if and only if $l = n$.*
(3) *$M(n,i) \cong P(S_i) \cong I(S_{i+1})$.*

Proof. (1): Follows from Lemma 2.3(1).

(2) and (3): Note that any finite dimensional Hopf algebra is a Frobenius algebra and hence is a self-injective algebra. If $l = n$, then it follows from [5, Lemma 3.5] that $M(n,i)$ is projective and injective.

For any $0 \leqslant i \leqslant n-1$, let $e_i = \frac{1}{n}\sum\limits_{j=0}^{n-1} q^{-ij}g^j$. Then $\{e_0, e_1, \cdots, e_{n-1}\}$ is a set of orthogonal idempotents such that $\sum\limits_{i=0}^{n-1} e_i = 1$. We also have $ge_i = q^i e_i$ and $h^{n-1}e_i \neq 0$. Therefore, $H_n(q)e_i = \mathrm{span}\{e_i, he_i, \cdots, h^{n-1}e_i\} \cong M(n,i)$. Thus, we have a decomposition of the regular module $H_n(q)$ as follows:

$$H_n(q) = \bigoplus_{i=0}^{n-1} H_n(q)e_i \cong \bigoplus_{i=0}^{n-1} M(n,i).$$

Hence $M(n,i) \cong P(S_i)$, and $M(n,0)$, $M(n,1)$, $\cdots$, $M(n,n-1)$ are all non-isomorphic indecomposable projective $H_n(q)$-modules. So (2) and (3) follow from Lemma 2.3. □

Since the indecomposable projective $H_n(q)$-modules are uniserial, any indecomposable $H_n(q)$-module is uniserial and is isomorphic to a quotient of an indecomposable projective module. Thus, we have the following theorem (see [7, page 467]).

Theorem 2.5. *Up to isomorphism, there are n^2 indecomposable finite dimensional $H_n(q)$-modules as follows:*

$$\{M(l,i) | 1 \leqslant l \leqslant n, 0 \leqslant i \leqslant n-1\}.$$

3. The Green ring of Taft algebra $H_n(q)$

We already know that there are n^2 non-isomorphic indecomposable modules over $H_n(q)$. They are

$$\{M(l,r) | 1 \leqslant l \leqslant n, r \in \mathbb{Z}_n\}.$$

The following lemma follows from a straightforward verification.

Lemma 3.1. *Let $1 \leqslant l \leqslant n$ and $r, r' \in \mathbb{Z}_n$. Then*

$$M(l,r) \otimes S_{r'} \cong S_{r'} \otimes M(l,r) \cong M(l, r+r')$$

as $H_n(q)$-modules. In particular, $S_r \otimes S_{r'} \cong S_{r+r'}$ and $M(l,r) \cong S_r \otimes M(l,0) \cong M(l,0) \otimes S_r$.

Cibils and Gunnlaugsdóttir derived the decomposition formulas of the tensor product of two indecomposable modules over $kZ_n(q)/I_n$ and the half-quantum group u_q^+ in [7] and [10], respectively. From [7, Theorem 4.1] or [10, Theorem 3.1], one gets the following Propositions 3.2, 3.3 and 3.4.

Proposition 3.2. *Let $2 \leqslant l \leqslant n$ and $r, r' \in \mathbb{Z}_n$. Then we have the $H_n(q)$-module isomorphisms*

$$M(l,r) \otimes M(n,r') \cong M(n,r') \otimes M(l,r) \cong \bigoplus_{i=1}^{l} M(n, r+r'+i-l).$$

Proposition 3.3. *Let $1 \leqslant l, l' < n$ and $r, r' \in \mathbb{Z}_n$. If $l + l' \leqslant n$, then*

$$M(l,r) \otimes M(l',r') \cong \bigoplus_{i=1}^{l_0} M(|l-l'|-1+2i, r+r'+i-l_0),$$

where $l_0 = \min\{l, l'\}$.

Proposition 3.4. *Let $1 \leqslant l, l' < n$ and $r, r' \in \mathbb{Z}_n$. If $l + l' > n$, then*

$$M(l,r) \otimes M(l',r') \cong (\bigoplus_{i=1}^{n-l_1} M(|l-l'|-1+2i, r+r'+i-l_0)) \oplus (\bigoplus_{i=1}^{l+l'-n} M(n, r+r'+1-i)),$$

where $l_0 = \min\{l, l'\}$ and $l_1 = \max\{l, l'\}$.

Following Propositions 3.3 and 3.4, we obtain the following:

Corollary 3.5. *Let $1 \leqslant l, l' \leqslant n-1$ and $r, r' \in \mathbb{Z}_n$. Then there is a simple summand in $M(l,r) \otimes M(l',r')$ if and only if $l = l'$.*

The following property of $M(l,r)$ can be derived from Lemma 3.1 and Propositions 3.2, 3.3 and 3.4.

Corollary 3.6. *Let $1 \leqslant l, l' \leqslant n$ and $r, r' \in \mathbb{Z}_n$. Then*

$$M(l,r) \otimes M(l',r') \cong M(l',r') \otimes M(l,r).$$

From Theorem 2.5 and Corollary 3.6, one can deduce the following known result (see [7, page 467]).

Corollary 3.7. *For any $H_n(q)$-modules M and N, there is an $H_n(q)$-module isomorphism*

$$M \otimes N \cong N \otimes M.$$

In the sequel, we let $a = [S_{-1}]$ and $x = [M(2,0)]$ in the Green ring $r(H_n(q))$ of $H_n(q)$. From Corollary 3.7, we know that $r(H_n(q))$ is a commutative ring.

Lemma 3.8.

(1) *$a^n = 1$ and $[M(l,r)] = a^{n-r}[M(l,0)]$ for all $2 \leqslant l \leqslant n$ and $r \in \mathbb{Z}_n$.*
(2) *If $n > 2$, then $[M(l+1,0)] = x[M(l,0)] - a[M(l-1,0)]$ for all $2 \leqslant l \leqslant n-1$.*
(3) *$x[M(n,0)] = (a+1)[M(n,0)]$.*
(4) *$r(H_n(q))$ is generated by a and x as a ring.*

Proof. (1): Follows from Lemma 3.1 since $[S_0]$ is the identity of the ring $r(H_n(q))$.

(2): If $n > 2$ and $2 \leqslant l \leqslant n-1$, then by Propositions 3.3 and 3.4 and Lemma 3.1, we have

$$\begin{aligned} M(2,0) \otimes M(l,0) &\cong M(l-1,-1) \oplus M(l+1,0) \\ &\cong S_{-1} \otimes M(l-1,0) \oplus M(l+1,0). \end{aligned}$$

It follows that $[M(l+1,0)] = x[M(l,0)] - a[M(l-1,0)]$.

(3): By Proposition 3.2 and Lemma 3.1, we have

$$\begin{aligned} M(2,0) \otimes M(n,0) &\cong M(n,-1) \oplus M(n,0) \\ &\cong S_{-1} \otimes M(n,0) \oplus M(n,0) \\ &\cong (S_{-1} \oplus S_0) \otimes M(n,0). \end{aligned}$$

It follows that $x[M(n,0)] = (a+1)[M(n,0)]$.

(4): Follows from (1), (2) and (3). □

Corollary 3.9. *Let $u_1, u_2, \cdots$ be a series of elements of the ring $r(H_n(q))$ defined recursively by $u_1 = 1$, $u_2 = x$ and*

$$u_l = xu_{l-1} - au_{l-2}, \ l \geqslant 3.$$

Then $[M(l,0)] = u_l$ for all $1 \leqslant l \leqslant n$ and $(x-a-1)u_n = 0$.

Proof. Follows from Lemma 3.8. □

Let $\mathbb{Z}[y,z]$ be the polynomial algebra over $\mathbb{Z}$ in two variables y and z. We define a *generalized Fibonacci polynomial* $f_n(y,z) \in \mathbb{Z}[y,z]$, $n \geqslant 1$, recursively as follows:

$$f_1(y,z) = 1, \ f_2(y,z) = z, \text{ and } f_n(y,z) = zf_{n-1}(y,z) - yf_{n-2}(y,z), \ n \geqslant 3.$$

Let I be the ideal of $\mathbb{Z}[y,z]$ generated by polynomials $y^n - 1$ and $(z-y-1)f_n(y,z)$.

With the above notation, we have the following main result.

Theorem 3.10. *The Green ring $r(H_n(q))$ of $H_n(q)$ is isomorphic to the quotient ring $\mathbb{Z}[y,z]/I$.*

Proof. By Lemma 3.8(4), $r(H_n(q))$ is generated, as a ring, by a and x. Hence there is a unique ring epimorphism ϕ from $\mathbb{Z}[y,z]$ to $r(H_n(q))$ such that $\phi(y) = a$ and $\phi(z) = x$. Since $a^n = 1$ by Lemma 3.8(1), $\phi(y^n - 1) = 0$. Let $\{u_i\}_{i \geqslant 1}$ be the series of elements of $r(H_n(q))$ given in Corollary 3.9. It is easy to see that $\phi(f_1(y,z)) = u_1$ and $\phi(f_2(y,z)) = u_2$. Now let $i \geqslant 3$ and assume that $\phi(f_{i-2}(y,z)) = u_{i-2}$ and $\phi(f_{i-1}(y,z)) = u_{i-1}$. Then

$$\begin{aligned} \phi(f_i(y,z)) &= \phi(zf_{i-1}(y,z) - yf_{i-2}(y,z)) \\ &= \phi(z)\phi(f_{i-1}(y,z)) - \phi(y)\phi(f_{i-2}(y,z)) \\ &= xu_{i-1} - au_{i-2} = u_i. \end{aligned}$$

Thus $\phi(f_i(y,z)) = u_i$ for all $i \geqslant 1$. In particular, we have $\phi(f_n(y,z)) = u_n$, and hence $\phi((z-y-1)f_n(y,z)) = (x-a-1)u_n = 0$ by Corollary 3.9. It follows that $\phi(I) = 0$ and that ϕ induces a ring epimorphism $\overline{\phi} : \mathbb{Z}[y,z]/I \to r(H_n(q))$ such that $\overline{\phi}(\overline{v}) = \phi(v)$ for all $v \in \mathbb{Z}[y,z]$, where $\overline{v}$ denotes the image of v under the natural epimorphism $\mathbb{Z}[y,z] \to \mathbb{Z}[y,z]/I$.

Let A be the subring of $r(H_n(q))$ generated by a. $A = \mathbb{Z}\langle a\rangle$ is the group ring of the cyclic group $\langle a\rangle$ over $\mathbb{Z}$. By Corollary 3.9 we have $u_1 = 1 \in A$ and $u_2 = x \in Ax \subset A + Ax$. By induction on i one can show that $u_i \in A + Ax + \cdots + Ax^{i-1}$ for all $i \geqslant 1$. Hence $u_i \in A + Ax + \cdots + Ax^{n-1}$ for all $1 \leqslant i \leqslant n$. Thus for all $1 \leqslant i \leqslant n$ and $r \in \mathbb{Z}_n$, by Lemma 3.8(1) we have $[M(i,r)] = a^{n-r}[M(i,0)] = a^{n-r}u_i \in A + Ax + \cdots + Ax^{n-1}$. It follows that $r(H_n(q)) = A + Ax + \cdots + Ax^{n-1}$. Since A is a free $\mathbb{Z}$-module with a $\mathbb{Z}$-basis $\{a^i | 0 \leqslant i \leqslant n-1\}$, $r(H_n(q))$ is generated by elements $a^i x^j, 0 \leqslant i,j \leqslant n-1$, as a $\mathbb{Z}$-module. Since $r(H_n(q))$ is a free $\mathbb{Z}$-module of rank n^2, $\{a^i x^j | 0 \leqslant i,j \leqslant n-1\}$ forms a $\mathbb{Z}$-basis for $r(H_n(q))$. Hence one can define a $\mathbb{Z}$-module homomorphism:

$$\psi : r(H_n(q)) \to \mathbb{Z}[y,z]/I, \quad a^i x^j \mapsto \overline{y^i z^j} = \overline{y}^i \overline{z}^j, \quad 0 \leqslant i,j \leqslant n-1.$$

Obviously, $\mathbb{Z}[y,z]/I$ is generated by elements $\overline{y^i z^j}, 0 \leqslant i,j \leqslant n-1$, as a $\mathbb{Z}$-module. Now we have

$$\psi\overline{\phi}(\overline{y^i z^j}) = \psi\phi(y^i z^j) = \psi(a^i x^j) = \overline{y^i z^j}$$

for all $0 \leqslant i,j \leqslant n-1$. Hence $\psi\overline{\phi} = \mathrm{id}$, and so $\overline{\phi}$ is injective. Thus, $\overline{\phi}$ is a ring isomorphism. □

The coefficients of the generalized Fibonacci polynomial $f_n(y,z)$ can be computed. They are quite similar to those of *the standard generalized Fibonacci polynomial* defined by

$$F_1(y,z) = 1, \; F_2(y,z) = z, \text{ and } F_n(y,z) = zF_{n-1}(y,z) + yF_{n-2}(y,z), \; n \geqslant 3.$$

For completeness, we compute $f_n(y,z)$ in the following lemma, which might be found elsewhere.

Lemma 3.11. *Let $\mathbb{Z}[y,z]$ be the polynomial algebra over $\mathbb{Z}$ in two variables y and z. Then for any $n \geqslant 1$, we have*

$$f_n(y,z) = \sum_{i=0}^{[(n-1)/2]} (-1)^i \left[\begin{array}{c} n-1-i \\ i \end{array} \right] y^i z^{n-1-2i}. \tag{1}$$

Proof. We prove it by induction on n. It is easy to check that equation (1) holds for $1 \leqslant n \leqslant 4$. Now let $n > 4$ and assume that the equation holds for smaller positive integers. If $n = 2m + 1$ is odd, then we have

$$\begin{aligned}
f_n(y,z) &= zf_{2m}(y,z) - yf_{2m-1}(y,z) \\
&= \sum_{i=0}^{m-1} (-1)^i \begin{bmatrix} 2m-1-i \\ i \end{bmatrix} y^i z^{2m-2i} \\
&\quad - \sum_{i=0}^{m-1} (-1)^i \begin{bmatrix} 2m-2-i \\ i \end{bmatrix} y^{i+1} z^{2m-2-2i} \\
&= \sum_{i=0}^{m-1} (-1)^i \begin{bmatrix} 2m-1-i \\ i \end{bmatrix} y^i z^{2m-2i} \\
&\quad + \sum_{i=1}^{m} (-1)^i \begin{bmatrix} 2m-1-i \\ i-1 \end{bmatrix} y^i z^{2m-2i} \\
&= z^{2m} + \sum_{i=1}^{m-1} (-1)^i (\begin{bmatrix} 2m-1-i \\ i \end{bmatrix} + \begin{bmatrix} 2m-1-i \\ i-1 \end{bmatrix}) y^i z^{2m-2i} \\
&\qquad + (-1)^m y^m \\
&= \sum_{i=0}^{m} (-1)^i \begin{bmatrix} 2m-i \\ i \end{bmatrix} y^i z^{2m-2i} \\
&= \sum_{i=0}^{[(n-1)/2]} (-1)^i \begin{bmatrix} n-1-i \\ i \end{bmatrix} y^i z^{n-1-2i}.
\end{aligned}$$

If $n = 2(m + 1)$ is even, then we have

$$\begin{aligned}
f_n &= zf_{2m+1}(y,z) - yf_{2m}(y,z) \\
&= \sum_{i=0}^{m} (-1)^i \begin{bmatrix} 2m-i \\ i \end{bmatrix} y^i z^{2m+1-2i} \\
&\quad - \sum_{i=0}^{m-1} (-1)^i \begin{bmatrix} 2m-1-i \\ i \end{bmatrix} y^{i+1} z^{2m-1-2i} \\
&= \sum_{i=0}^{m} (-1)^i \begin{bmatrix} 2m-i \\ i \end{bmatrix} y^i z^{2m+1-2i} \\
&\quad + \sum_{i=1}^{m} (-1)^i \begin{bmatrix} 2m-i \\ i-1 \end{bmatrix} y^i z^{2m+1-2i} \\
&= z^{2m+1} + \sum_{i=1}^{m} (-1)^i (\begin{bmatrix} 2m-i \\ i \end{bmatrix} + \begin{bmatrix} 2m-i \\ i-1 \end{bmatrix}) y^i z^{2m+1-2i} \\
&= z^{2m+1} + \sum_{i=1}^{m} (-1)^i \begin{bmatrix} 2m+1-i \\ i \end{bmatrix} y^i z^{2m+1-2i} \\
&= \sum_{i=0}^{m} (-1)^i \begin{bmatrix} 2m+1-i \\ i \end{bmatrix} y^i z^{2m+1-2i} \\
&= \sum_{i=0}^{[(n-1)/2]} (-1)^i \begin{bmatrix} n-1-i \\ i \end{bmatrix} y^i z^{n-1-2i}.
\end{aligned}$$

Thus the proof is completed. □

Now we can easily derive the Green rings $r(H_n(q))$ for $n = 2, 3, ..., 8$.

Corollary 3.12. *When $n = 2$, $r(H_2(q)) \cong \mathbb{Z}[y, z]/(y^2 - 1, (z - y - 1)z)$.*
When $n = 3$, $r(H_3(q)) \cong \mathbb{Z}[y, z]/(y^3 - 1, (z - y - 1)(z^2 - y))$.
When $n = 4$, $r(H_4(q)) \cong \mathbb{Z}[y, z]/(y^4 - 1, (z - y - 1)(z^3 - 2yz))$.
When $n = 5$, $r(H_5(q)) \cong \mathbb{Z}[y, z]/(y^5 - 1, (z - y - 1)(z^4 - 3yz^2 + y^2))$.
When $n = 6$, $r(H_6(q)) \cong \mathbb{Z}[y, z]/(y^6 - 1, (z - y - 1)(z^5 - 4yz^3 + 3y^2z))$.
When $n = 7$, $r(H_7(q)) \cong \mathbb{Z}[y, z]/(y^7 - 1, (z - y - 1)(z^6 - 5yz^4 + 6y^2z^2 - y^3))$.
When $n = 8$, $r(H_8(q)) \cong \mathbb{Z}[y, z]/(y^8 - 1, (z - y - 1)(z^7 - 6yz^5 + 10y^2z^3 - 4y^3z))$.

Remark 3.13.
(1) One can easily see that the Grothendick ring of $H_n(q)$ is the group ring $k\mathbb{Z}_n$ generated by the simple module $M(1, 0)$. From the above examples, we see that the Green ring is much more complicated than the Grothendick ring.
(2) The Green rings of generalized Taft algebras and the Green rings of monomial Hopf algebras [6] can be computed in a similar way. However, the computations of the Green ring of the small quantum group or the Green ring of the quantum double of a Taft algebra seem to be much more complicated, as they are not finitely generated [5, 13].
(3) Since the module category of a quasitriangular Hopf algebra H is braided monoidal, the Green ring of H is commutative. The Taft algebra $H_n(q)$ is not quasitriangular in the case $n > 2$ (not even almost cocommutative; see [7]), but its Green ring is commutative. This leads to the following question: can we characterize the class of Hopf algebras whose Green ring is commutative?

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