

PRIMARY BIRTH OF CANARD CYCLES IN SLOW-FAST CODIMENSION 3 ELLIPTIC BIFURCATIONS

RENATO HUZAK, PETER DE MAESSCHALCK AND FREDDY DUMORTIER

Hasselt University, Campus Diepenbeek, Agoralaan Gebouw D
3590 Diepenbeek, Belgium

(Communicated by Xingfu Zou)

ABSTRACT. In this paper we continue the study of “large” small-amplitude limit cycles in slow-fast codimension 3 elliptic bifurcations which is initiated in [8]. Our treatment is based on blow-up and good normal forms.

1. Introduction. This paper deals with the canard phenomenon and corresponding limit cycles of canard type, as they appear in slow-fast families of vector fields in the (x, y) -plane. The prototypical example where the canard phenomenon appears is the Van der Pol system, where the unique singular point of a specific slow-fast structured vector field undergoes a Hopf bifurcation upon variation of a control parameter. Letting ϵ denote the singular parameter separating the two time scales in the Van der Pol system, a weighted rescaling of (x, y, ϵ) exposes a rescaled system of differential equations in rescaled variables (X, Y) , where the slow-fast structure has disappeared and where a traditional Hopf bifurcation is observed. The periodic orbit(s) seen in the (X, Y) -space are considered small-amplitude limit cycles in the traditional (x, y) -plane, since they typically are contained in an $O(\epsilon)$ -neighbourhood of the Hopf point. Traditional techniques from dynamical systems and bifurcation theory can be used to deal with those small-amplitude limit cycles. Besides these cycles, the (x, y) -plane may also contain so-called slow-fast cycles of canard type, with properties typically associated to slow-fast systems. It is precisely at the interface between small-amplitude cycles and canard cycles that a delicate analysis is needed. Those families of cycles are unbounded in (X, Y) -space and yet close to the origin in (x, y) -coordinates. Pioneering work has been done in [12] (in a codimension 1 scenario) and in [2] for systems that are locally similar to the Van der Pol system in higher codimension.

In [2], the way the interface between small-amplitude cycles and canard cycles is examined is by blowing up the origin. The traditional rescaling from above is seen in one of the charts of the blow-up construction. In the blow-up construction, one can also examine other charts, the so-called phase directional rescaling charts. It is precisely those charts that become important in a study of the birth of canard cycles. We remind the reader that in most papers on slow-fast systems using blow-up, the phase directional rescaling charts are only studied minimally, just enabling the contributing authors to trace orbits passing through these charts and to focus

2000 *Mathematics Subject Classification.* Primary: 34E15, 37G15, 34E20, 34C23, 34C26.

Key words and phrases. Blow-up, center manifold, cyclicity, singular perturbations, normal form.

attention to the traditional rescaling charts. Indeed, only a few properties are needed of these phase directional rescaling charts to examine large amplitude canard cycles, the so-called detectable cycles, see also [10].

Here, we also study the birth of canard cycles, i.e. cycles at the interface between small-amplitude cycles and canard cycles, but in a more degenerate context: while the underlying bifurcation mechanism in the Van der Pol equation is a codimension 1 Hopf bifurcation, we now consider a codimension 3 singularity in a slow-fast context. The essential extra difficulty is that besides the singular parameter ϵ , we now have 3 extra parameters unfolding the codimension 3 singularity, and even without a slow-fast structure, a blow-up is needed to give a complete study. As can be expected, we present a study where two blow-up constructions are combined: one to unfold the codimension 3 singularity (a “primary blow-up”), and one to dissolve the slow-fast structure (a “secondary blow-up”). Cycles that are bounded in coordinates after the secondary blow-up could be called small-amplitude cycles. As they grow in size under influence of some perturbation parameter, they meet the boundary of the secondary blow-up and give birth to intermediary-sized cycles: those cycles are of size $O(1)$ in coordinates after the primary blow-up. This birth of canard cycles is largely similar to the one treated in [2]. In primary blow-up coordinates, the intermediary-sized cycles are already of canard type, but may continue growing until they meet the boundary of the primary blow-up and give birth to canard cycles of size $O(1)$ in original coordinates. This birth process actually differs more than a bit from the situation discussed in [2] and [12], and the study of limit cycles in this situation is the topic of this paper.

One of the main properties of the primary blow-up is that it is not ϵ that is included in the blow-up construction, since the primary blow-up does not involve dissolving the slow-fast structure; instead, the primary blow-up is involved with desingularizing a codimension 3 singularity. As a consequence, the family of vector fields being blown up has a slow-fast structure both before *and after* blow-up.

Instead of presenting a general technique for treating a birth of canard situation in case of a blow-up preserving the slow-fast structure, we choose to demonstrate the (quite intricate) techniques in a situation for which a lot of results already have been obtained:

- We study a slow-fast codimension 3 singularity that is the slow-fast variant of a well-studied codimension 3 singularity (see [5]).
- The desingularization of the singularity using a primary and secondary blow-up has been worked out before (see [8]). Both the detectable canard cycles ([3]) and the small-amplitude limit cycles ([8]) have been characterized before. On top of that, the birth problem associated to the secondary blow-up has been dealt with. (See [8].)

We claim that the results are general enough to be useful in other situations, and furthermore the results contribute to a complete understanding of the slow-fast codimension 3 singularities and nearby limit cycles.

In Section 2, we present a more detailed setting and introduce the setting using a precise system of equations in mind. In Section 3, we carry out the desingularization via blowing-ups. Section 4 contains precise statements of the results we aim to prove. Section 5 finally contains detailed proofs of the results.

The technique of blow-up is crucial in this paper, not only in proving the results, but already in stating the results. We therefore refer the interested reader to [13] for more informations about desingularizations of nilpotent singularities in families

of planar vector fields, and continue under the assumption that reader has sufficient background.

2. Slow-fast codimension 3 elliptic bifurcations. We deal with the slow-fast family of planar systems

$$X_{\bar{\epsilon},b,\lambda} : \begin{cases} \dot{x} = y \\ \dot{y} = -xy + \bar{\epsilon} \left(b_0 + b_1x + b_2x^2 - x^3 + x^4 \bar{H}(x, \lambda) + y^2 G(x, y, \lambda) \right), \end{cases} \quad (1)$$

where G and \bar{H} are smooth, $\bar{\epsilon} \geq 0$ is the singular parameter that is kept small, $b = (b_0, b_1, b_2)$ are regular perturbation parameters close to 0 and $\lambda \in \Lambda$, with Λ a compact subset of some euclidean space. The family $X_{\bar{\epsilon},b,\lambda}$ represents slow-fast codimension 3 elliptic bifurcations, in analogy with the terminology introduced in [5] and [8].

Let us recall that [5] is devoted to the study of generic bifurcations of three-parameter families of planar vector fields around singular points whose linear parts are nilpotent. The authors dealt with three categories: the saddle case, the elliptic case and the focus case. Dealing with slow-fast systems ($\bar{\epsilon} \sim 0$, $\bar{\epsilon} > 0$), in [8] we distinguish only between the elliptic case (1) and the saddle case, obtained by putting the + -sign in front of x^3 in (1). The focus case is possible in the family (1) if the parameter $\bar{\epsilon} > 0$ is sufficiently large, but then (1) is not of slow-fast type. For more details we refer to [8].

The family $X_{\bar{\epsilon},b,\lambda}$ contains for $\bar{\epsilon} = 0$ a curve of singular points given by $\{y = 0\}$. All points on the curve except for the origin are normally hyperbolic. Of course, the dynamics for $\bar{\epsilon} = 0$ can be studied by canceling the common factor y and seeing that orbits of $X_{0,b,\lambda}$ take the form $y = y(x)$, with $\frac{dy}{dx} = -x$. In other words, orbits lie on parabolas that intersect the curve of singular points transversally, except at the origin, where the parabola has a second-order contact with the curve (see Figure 1). The origin $(x, y) = (0, 0)$ is a so-called contact point, and we observe that it is of nilpotent type.

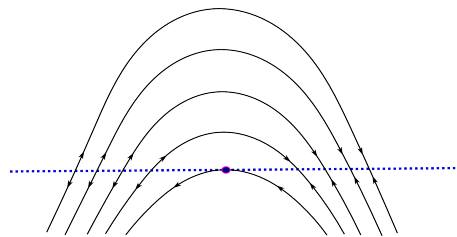


FIGURE 1. The dynamics of $X_{0,b,\lambda}$.

The $\bar{\epsilon}$ -perturbation may cause the curve $y = 0$ to perturb into some invariant curve with a dynamics on it, but with a speed that is $O(\bar{\epsilon})$. This way, so-called detectable canard limit cycles may appear: a fast movement along the top of a parabola above $\{y = 0\}$ is followed by a slow movement connecting the two ends of the parabola along $y = 0$.

As mentioned above the papers [8] and [3] are devoted to the study of the limit cycles that may appear in the family $X_{\bar{\epsilon},b,\lambda}$ perturbing from $X_{0,(0,0,0),\lambda}$. The paper [3] deals with systems (1) but emphasizes passage near the generic turning point

$(x, y) = (0, 0)$ in order to study the detectable canard limit cycles, whereas in the paper [8] the focus is on small amplitude limit cycles near $(x, y) = (0, 0)$.

Let us first explain what is our goal of the present paper. We reparametrize the b -parameters, by introducing weighted spherical parameters as used in [8]:

$$(b_0, b_1, b_2) = (r^3 \bar{B}_0, r^2 B_1, r B_2), \quad r \geq 0, \quad B = (\bar{B}_0, B_1, B_2) \in \mathbb{S}^2.$$

We obtain an $(\bar{\epsilon}, B, r, \lambda)$ -family of vector fields in \mathbb{R}^2 :

$$\begin{cases} \dot{x} = y \\ \dot{y} = -xy + \bar{\epsilon} \left(r^3 \bar{B}_0 + r^2 B_1 x + r B_2 x^2 - x^3 + x^4 \bar{H}(x, \lambda) + y^2 G(x, y, \lambda) \right). \end{cases} \quad (2)$$

For $r = 0$, system (2) has no limit cycles near $(x, y) = (0, 0)$. For $r \neq 0, r \sim 0$, system (2) has been studied in an $(\bar{\epsilon}, B, \lambda)$ -uniform neighborhood of the origin in (x, y) -space (see [2], at least in the Liénard setting, i.e. $G \equiv 0$, and [1]). This local study is valid in a small neighborhood of $(x, y) = (0, 0)$ and the domain on which the arguments of [2] and [1] can be applied shrinks to $(x, y) = (0, 0)$ when $r \rightarrow 0$.

The idea in [8] was to start the study of limit cycles of (2) in a neighborhood of the origin $(x, y) = (0, 0)$ that *does not shrink to the origin when $\bar{\epsilon} \rightarrow 0$ and $r \rightarrow 0$* . This means that the transition from small amplitude limit cycles to small (but detectable) canard limit cycles has to be considered. *The goal of the present paper is to study this “birth of canards” in the (x, y) – plane, i.e. so-called “large” small amplitude limit cycles.*

As mentioned above, in [8] slow-fast codimension 3 saddle bifurcations have been studied. In slow-fast codimension 3 saddle bifurcations, one finds no birth of canards because detectable canard limit cycles are not present (see [8] and [3]).

Now we want to provide few details on what is shown in [8] and how the above mentioned birth of canards comes into play. In [8], the best way we saw to study the small limit cycles problem of (2) was by applying the so-called “primary” blow-up:

$$(x, y, r) = (u\bar{x}, u^2\bar{y}, u\bar{r}), \quad (3)$$

with $\bar{x}^2 + \bar{y}^2 + \bar{r}^2 = 1$ and $\bar{r} \geq 0, u \geq 0, u \sim 0$. We also divided the system we got by u . The precise elaboration can be found in [8] or in Section 3. Roughly speaking this blow-up transforms the r -family of quite “degenerate” two-dimensional problems (2) into a less degenerate, but still $\bar{\epsilon}$ -singular, three-dimensional problem. Instead of having to work in a r -uniform neighborhood of the origin, we now have to consider a neighborhood inside $\{u \geq 0\}$ of the primary blow-up locus $\{(u, \bar{x}, \bar{y}, \bar{r}); u = 0, \bar{x}^2 + \bar{y}^2 + \bar{r}^2 = 1, \bar{r} \geq 0\}$.

As it is usual in working along a sphere it is preferable to work in different charts. He have the family chart “ $\bar{r} = 1$ ” and the phase-directional charts “ $\bar{x} = \pm 1, \bar{y} = \pm 1$ ”. The family chart is the traditional rescaling chart, and it amounts to making a standard rescaling of the phase variables (x, y) . The phase-directional charts link the family chart to the original phase plane and we use them to study dynamics of (2) near the “equator” $\{(u, \bar{x}, \bar{y}, \bar{r}); u = 0, \bar{r} = 0, \bar{x}^2 + \bar{y}^2 = 1\}$ of the primary blow-up locus.

In [8] we detected, depending on region in the B -space, all possible closed curves (so-called limit periodic sets) on the (primary) blow-up locus which can produce limit cycles of the blown-up vector field for $u\bar{r} > 0$ and $\bar{\epsilon} > 0$, and we saw that a birth of canards, *near a limit periodic set with large parts on the “equator” of the blow-up locus*, is possible only for those values of B which are in the slow-fast Hopf region: $\bar{B}_0 \sim 0, B_1 = -1$ and B_2 in an arbitrarily large compact interval. The

subject of this paper is a study of cyclicity of the limit periodic set near which the birth of canards occurs. In the slow-fast Hopf region, some of the results from [8] have been proved by using the cyclicity results that we will prove in this paper (see Theorem 2.4, Theorem 2.5 and Section 3.8 in [8]).

Let us recall that, instead of using coordinates on \mathbb{S}^2 ($B \in \mathbb{S}^2$), we used different charts of the sphere (jump chart, slow-fast Bogdanov-Takens chart, slow-fast Hopf chart, etc.) and we proved that, outside the slow-fast Hopf region (i.e. chart), system (2) has at most one hyperbolic limit cycle, in an $(\bar{\epsilon}, r, \lambda)$ -uniform neighborhood of $(x, y) = (0, 0)$. The size of this limit cycle goes to zero when $r \rightarrow 0$ because the essential parts of the study of this case have been done in the family chart “ $\bar{r} = 1$ ”; for more details we refer to [8].

Here we point out that the proofs of all these results from [8] were based on performing an extra blow-up at the origin $(\bar{x}, \bar{y}, \bar{\epsilon}) = (0, 0, 0)$ which depends on region in the B -space in which one looks. We called it “secondary” blow-up.

Being interested in the birth of canards, we consider (2) where B is in the slow-fast Hopf region. We introduce the following rescaling:

$$(\bar{\epsilon}, \bar{B}_0) = (\epsilon^2 E, \epsilon B_0), \quad \epsilon \geq 0, \quad \epsilon \sim 0, \quad (E, B_0) \in \mathbb{S}^1, \quad E \geq 0.$$

The calculations will be performed, as usual, in charts. When E is in any compact interval and $B_0 = \pm 1$, then the system (2) has no small-amplitude limit cycles (hence no birth of canards occurs); for details see [8]. When $E = 1$ and $B_0 \sim 0$, then (2) changes into

$$\begin{cases} \dot{x} = y \\ \dot{y} = -xy + \epsilon^2 \left(\epsilon r^3 B_0 - r^2 x + B_2 r x^2 - x^3 + x^4 \bar{H}(x, \lambda) + y^2 G(x, y, \lambda) \right), \end{cases} \quad (4)$$

where G and \bar{H} are smooth, $\epsilon > 0$ is the singular parameter that is kept small, $r > 0$ is a regular parameter that is kept small, B_0 is a regular parameter close to 0, B_2 is a regular parameter in an arbitrary compact subset of \mathbb{R} and $\lambda \in \Lambda$, with Λ a compact subset of some euclidean space. If we introduce a new variable $Y = y + \frac{1}{2}x^2$, then (4) changes into

$$X_{\epsilon, r, B_0, B_2, \lambda} : \begin{cases} \dot{x} = y - \frac{1}{2}x^2 \\ \dot{y} = \epsilon^2 \left(\epsilon r^3 B_0 - r^2 x + B_2 r x^2 - x^3 + x^4 \bar{H}(x, \lambda) \right. \\ \left. + (y - \frac{1}{2}x^2)^2 G(x, y - \frac{1}{2}x^2, \lambda) \right). \end{cases} \quad (5)$$

where we denote Y again by y . We prefer to work in the so-called Liénard plane.

We recall that when $G = 0$ and \bar{H} is polynomial, the given family (1) of vector fields is of (generalized) polynomial Liénard type of degree $(1, n)$, where $n = \deg \bar{\epsilon}(\dots)$. The 1 in $(1, n)$ comes from the degree of the polynomial in front of y in \dot{y} . Determining the maximum number of limit cycles of a Liénard type vector field of degree (m, n) is one of the major open problems in the field of planar dynamics (see [14]), and [8], [3], [9] and this paper contribute to the extensive research in this area, in the case $n \geq 4$ and $m = 1$. We point out that there is a strong link between results on slow-fast type Liénard equations, as treated in this paper, and general Liénard equations, see [6] and [7].

As mentioned in the introduction, in Section 3 we study system (5) in the family chart and in the phase-directional charts of the primary blow-up (3), and we detect a limit periodic set on the blow-up locus near which a birth of canards occurs (see

Figure 2). In [8], this limit periodic set was denoted by L_{00} . In this paper, we denote it by Γ . The fact that Γ is on the primary blow-up locus, with a part on the “equator” of the primary blow-up locus, explains the title “Primary birth of canard cycles ...” that we have chosen for this paper.

In Section 4 we define a difference map near Γ which enables us to introduce the notion of cyclicity of Γ . Then we state the results about the cyclicity of Γ , depending on the parameter B_2 . When $B_2 \neq 0$, then at most one limit cycle may appear near Γ . When $B_2 \sim 0$, then the cyclicity of Γ depends on the higher order terms in \bar{H} , in analogy with the results in [2]. If we suppose that $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$, then at most two limit cycles may come near Γ .

In Section 5, we give a detailed proof of all statements formulated in Section 4. We use similar methods as in [2] to study the difference map near Γ ; we exploit symmetries that are present both in the system (5) and in the blow-up construction (3), we use C^k -normal forms (see [11]) near semi-hyperbolic singularities P_0^+ and P_0^- on Γ (see Figure 2), etc.

3. Blow-up construction and detection of Γ . If we add the equation $\dot{r} = 0$ to (5), we obtain a $\tau = (\epsilon, B_0, B_2, \lambda)$ -family of vector fields in \mathbb{R}^3 :

$$X_\tau := X_{\epsilon,r,B_0,B_2,\lambda} + 0 \frac{\partial}{\partial r}.$$

As mentioned in the introduction, we consider the so-called primary blow-up map (defining a singular change of coordinates)

$$\Theta : \mathbb{R}^+ \times \mathbb{S}^2 \rightarrow \mathbb{R}^3 : (u, (\bar{x}, \bar{y}, \bar{r})) \mapsto (x, y, r) = (u\bar{x}, u^2\bar{y}, u\bar{r}), \quad \bar{r} \geq 0.$$

The blown-up vector field is defined as the pullback of the original vector field X_τ divided by u :

$$\bar{X}_\tau := \frac{1}{u} \Theta^* X_\tau. \tag{6}$$

In order to detect Γ for $\epsilon = u = B_0 = 0$, we have to study the blown-up vector field \bar{X}_τ near the blow-up locus $\{0\} \times \mathbb{S}_+^2$, in the family chart “ $\bar{r} = 1$ ” (see Section 3.1) and in the phase-directional chart “ $\bar{y} = 1$ ” (see Section 3.2). (We denote by \mathbb{S}_+^2 the half-sphere in $(\bar{x}, \bar{y}, \bar{r})$ -space with $\bar{r} \geq 0$.) The phase-directional charts “ $\bar{x} = \pm 1, \bar{y} = -1$ ” are not relevant when studying the limit periodic set Γ .

3.1. Family chart “ $\bar{r} = 1$ ”. We use the following family rescaling of (5):

$$(x, y, r) = (u\bar{x}, u^2\bar{y}, u), \tag{7}$$

with (\bar{x}, \bar{y}) in an arbitrarily large disk in \mathbb{R}^2 and $u = r \geq 0$. The blown-up field is a τ -family of 3-dimensional vector fields (after division by the positive factor r):

$$X_\tau^{(1)} : \begin{cases} \dot{\bar{x}} = \bar{y} - \frac{1}{2}\bar{x}^2 \\ \dot{\bar{y}} = \epsilon^2 \left(\epsilon B_0 - \bar{x} + B_2 \bar{x}^2 - \bar{x}^3 + r \bar{x}^4 \bar{H}(r\bar{x}, \lambda) \right. \\ \quad \left. + r \left(\bar{y} - \frac{1}{2}\bar{x}^2 \right)^2 G(r\bar{x}, r^2 \left(\bar{y} - \frac{1}{2}\bar{x}^2 \right), \lambda) \right) \\ \dot{r} = 0. \end{cases} \tag{8}$$

The vector field $X_\tau^{(1)}$ represents a singular perturbation problem with ϵ as singular parameter. If we treat $X_\tau^{(1)}$ as a (τ, r) -family of 2-dimensional vector fields, it can be easily seen that $X_\tau^{(1)}$ for $\epsilon = 0$ has a curve of singularities (a critical curve)

given by $\{\bar{y} - \frac{1}{2}\bar{x}^2 = 0\}$. The critical curve consists of partially hyperbolic singularities, except at the origin $(\bar{x}, \bar{y}) = (0, 0)$, where we deal with a nilpotent singularity (so-called turning point). We see that the curve is normally attracting when $\bar{x} > 0$ and normally repelling when $\bar{x} < 0$.

The ϵ -perturbation may cause the curve $\{\bar{y} - \frac{1}{2}\bar{x}^2 = 0\}$ to perturb to some invariant curve which follows the attracting part of the critical curve until it reaches the section $\{\bar{x} = 0\}$ and then follows the repelling part of the critical curve. To see that this connection between the attracting part and the repelling part of the critical curve is possible for $\bar{x} \sim 0$, we desingularize $X_\tau^{(1)}$ near the origin $(\bar{x}, \bar{y}, \epsilon) = (0, 0, 0)$ using the so-called secondary blow-up $(\bar{x}, \bar{y}, \epsilon) = (v\tilde{x}, v^2\tilde{y}, v\tilde{\epsilon})$ where $v \geq 0$, $(\tilde{x}, \tilde{y}, \tilde{\epsilon}) \in \mathbb{S}^2$ and $\tilde{\epsilon} \geq 0$.

In the family chart “ $\tilde{\epsilon} = 1$ ” one obtains, after dividing by v , the family of vector fields

$$\begin{cases} \dot{\tilde{x}} = \tilde{y} - \frac{1}{2}\tilde{x}^2 \\ \dot{\tilde{y}} = B_0 - \tilde{x} + O(v). \end{cases}$$

For $B_0 = v = 0$, dynamics of the above system are of center type with a regular orbit $\gamma = \{\tilde{y} = \frac{1}{2}\tilde{x}^2 - 1\}$ connecting the end point of the attracting part of the critical curve and the end point of the repelling part of the critical curve, and pointing from the attracting part to the repelling part. The end points are located on the “equator” of the secondary blow-up locus $\{v = 0\}$ and can be studied in phase-directional charts “ $\tilde{x} = \pm 1, \tilde{y} = \pm 1$ ” (see also [8]).

For $\bar{x} \neq 0$, we can consider the slow dynamics along $\{\bar{y} - \frac{1}{2}\bar{x}^2 = 0\}$:

$$\bar{x}' = -1 + B_2\bar{x} - \bar{x}^2 + r\bar{x}^3\bar{H}(r\bar{x}, \lambda). \tag{9}$$

When parameter B_2 is kept in any compact set $K \subset]-2, 2[$, then (9) is strictly negative in an arbitrary compact set in the \bar{x} -space by taking r small enough. For $(B_2, r) \sim (\pm 2, 0)$, a saddle-node singularity appears in (9), located near $\bar{x} = \pm 1$. When B_2 is kept in any compact set $K \subset \mathbb{R} \setminus [-2, 2]$, then one has two simple singularities in the slow dynamics, for $r \sim 0$.

Combining the case $\bar{x} \sim 0$ and the case $\bar{x} \neq 0$ we find that a passage from $\bar{x} = +\infty$ to $\bar{x} = -\infty$, along the critical curve, is possible when the slow dynamics has no simple singularities. As a consequence, the critical curve, for $r = 0$, will be a part of the limit periodic set Γ (see Figure 2).

3.2. Phase-directional chart “ $\bar{y} = 1$ ”. As we are interested in the points of intersection of the critical curve $\{\bar{y} - \frac{1}{2}\bar{x}^2 = 0\}$ with the “equator” of the primary blow-up locus, we consider the phase-directional chart “ $\bar{y} = 1$ ” where the blow-up map is

$$(x, y, r) = (U\bar{X}, U^2, UR), \quad U \geq 0, \tag{10}$$

where $U \sim 0$, $R \geq 0$ and (\bar{X}, R) is in an arbitrarily large disk in \mathbb{R}^2 . We obtain a blown-up field which, after dividing by U , can be written as

$$X_\tau^{(2)} : \begin{cases} \dot{\bar{X}} = 1 - \frac{1}{2}\bar{X}^2 - \frac{1}{2}\epsilon^2\bar{X}\Psi(\bar{X}, U, R, \tau) \\ \dot{U} = \frac{1}{2}\epsilon^2U\Psi(\bar{X}, U, R, \tau) \\ \dot{R} = -\frac{1}{2}\epsilon^2R\Psi(\bar{X}, U, R, \tau), \end{cases} \tag{11}$$

where $\Psi(\bar{X}, U, R, \tau) = \epsilon R^3 B_0 - R^2 \bar{X} + RB_2 \bar{X}^2 - \bar{X}^3 + U \bar{X}^4 \bar{H}(U \bar{X}, \lambda) + U(1 - \frac{1}{2} \bar{X}^2)^2 G(U \bar{X}, U^2(1 - \frac{1}{2} \bar{X}^2), \lambda)$.

On $\{U = 0, R = 0\}$ $X_\tau^{(2)}$ has singularities at $\bar{X} = \pm\sqrt{2} + O(\epsilon^2)$ which represent the above mentioned intersection points for $\epsilon = 0$. The eigenvalues of the linear part at $P_\epsilon^\pm = (\pm\sqrt{2} + O(\epsilon^2), 0, 0)$ are given by $(\mp\sqrt{2} + O(\epsilon^2), \mp\epsilon^2(\sqrt{2} + O(\epsilon^2)), \pm\epsilon^2(\sqrt{2} + O(\epsilon^2)))$. Hence, we find that P_ϵ^\pm are hyperbolic (resonant) saddles for $\epsilon > 0$ and semi-hyperbolic singularities for $\epsilon = 0$.

For $\epsilon > 0$, dynamics of $X_\tau^{(2)}$ near P_ϵ^\pm , restricted to $\{U = 0\}$ (the blow-up locus), are of saddle type. In P_ϵ^+ (resp. P_ϵ^-) we have the \bar{X} -axis as stable manifold (resp. unstable manifold). For $U = R = 0$ and $\epsilon \geq 0$, dynamics of $X_\tau^{(2)}$ points from P_ϵ^- to P_ϵ^+ . As a consequence, Γ will contain the part of the “equator” of the (primary) blow-up locus between P_0^- and P_0^+ .

3.3. Combining the family chart “ $\bar{r} = 1$ ” and the phase-directional chart “ $\bar{y} = 1$ ”. The primary blow-up locus $\{0\} \times \mathbb{S}_+^2$ can be considered as a 2-dimensional closed disc which we denote by \bar{D} . Let Γ denote the limit periodic set on \bar{D} defined as the union of the critical curve $\{\bar{y} - \frac{1}{2}\bar{x}^2 = 0\}$ and the regular arc A of $\partial\bar{D}$ between P_0^- and P_0^+ (see Figure 2). Hence the limit periodic set Γ represents a limiting situation, for $\epsilon = u = 0$.

As mentioned in the introduction, we are interested in an upper bound on the number of limit cycles of \bar{X}_τ that can bifurcate from Γ , for $\epsilon > 0$ and $u > 0$.

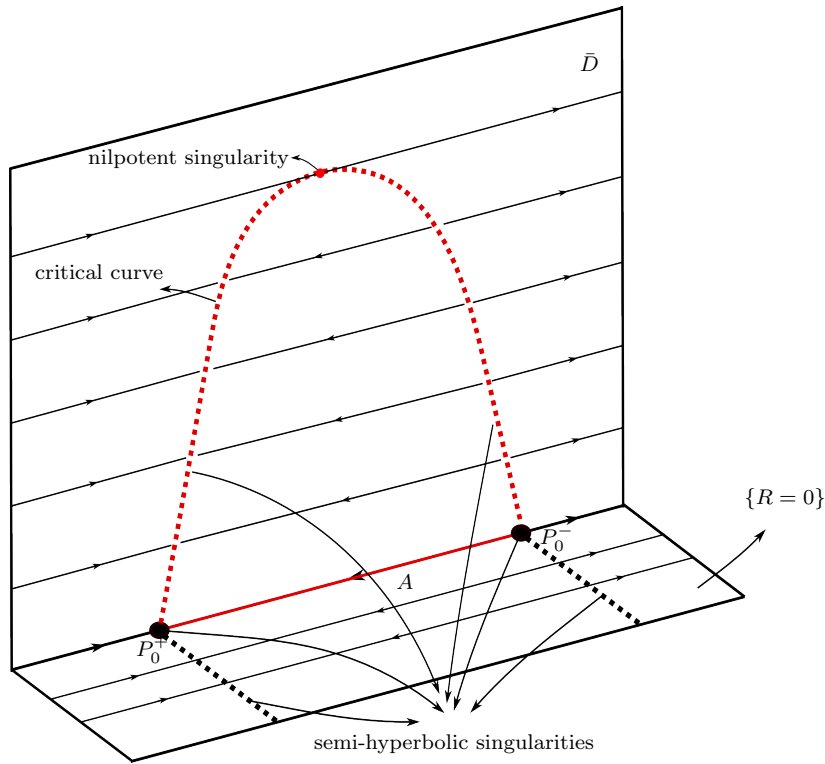


FIGURE 2. The dynamics of the vector field \bar{X}_τ on \bar{D} for $\epsilon = 0$, and the singular cycle $\Gamma = \{critical\ curve\} \cup A$.

4. Statement of the results. We suppose that $\tau \in [0, \epsilon_0] \times [-B_0^0, B_0^0] \times [-B_2^0, B_2^0] \times \Lambda$ where $\epsilon_0 > 0$ and $B_0^0 > 0$ are small and fixed, and $B_2^0 > 2$ is arbitrarily large and fixed. We denote by \mathcal{B} the compact set $[-B_0^0, B_0^0] \times [-B_2^0, B_2^0]$.

As it is usual in studying the cyclicity of a limit periodic set with two ‘‘corners’’ (e.g. Γ with P_0^- and P_0^+), it is preferable to link limit cycles to zeros of a difference map rather than to fixed points of a return map. We choose two sections $\Sigma_0^1 \subset \{\bar{X} = 0\}$ and $\Sigma_0^2 \subset \{\bar{x} = 0\}$. Section Σ_0^1 is expressed in the coordinates (\bar{X}, U, R) of $X_\tau^{(2)}$ and parametrized by $(U, R) \in [-U^0, U^0] \times [0, R^0]$, where $U^0 > 0$ and $R^0 > 0$ are small and fixed; section Σ_0^2 is expressed in the coordinates (\bar{x}, \bar{y}, r) of $X_\tau^{(1)}$ and parametrized by (\bar{y}, r) where $\bar{y} \sim 0$ and $r \sim 0$ (see Figure 3).

We are interested in examining orbits of \bar{X}_τ and $-\bar{X}_\tau$, defined in (6), that start at the section Σ_0^1 and meet the section Σ_0^2 in finite time. To be more precise, we denote by $o_{U,R,\tau}^+$ (resp. $o_{U,R,\tau}^-$) the forward orbit (resp. the backward orbit) of \bar{X}_τ starting at the point $(U, R) \in \Sigma_0^1$, for $\tau \in [0, \epsilon_0] \times \mathcal{B} \times \Lambda$. Now we can define the following set in (U, R, τ) -space:

$$\mathcal{D} = \{(U, R, \tau) : o_{U,R,\tau}^+ \text{ and } o_{U,R,\tau}^- \text{ reach } \Sigma_0^2 \text{ in finite time}\}.$$

Remark 1. Based on Section 3.1 and Section 3.2, orbit $o_{U,R,\tau}^+$ (resp. $o_{U,R,\tau}^-$) follows the arc A until it comes close to point P_0^+ (resp. P_0^-), then follows the attracting (resp. repelling) branch of the critical curve $\{\bar{y} - \frac{1}{2}\bar{x}^2 = 0\}$, for an appropriate parameter B_2 , and meets Σ_0^2 , close to the turning point $(\bar{x}, \bar{y}) = (0, 0)$. Since γ defined in Section 3.1 contains $(\tilde{x}, \tilde{y}) = (0, -1)$, $\bar{y} = \epsilon^2 \tilde{y}$ and \bar{x} -component of (8) is equal to \bar{y} for $\bar{x} = 0$, we have that orbit $o_{U,R,\tau}^\pm$ intersects Σ_0^2 transversally. Hence we may add ‘‘transversally’’ in definition of $\mathcal{D} \subset [-U^0, U^0] \times]0, R^0[\times]0, \epsilon_0[\times \mathcal{B} \times \Lambda$.

A first result deals with the smoothness of the transition map from Σ_0^1 to Σ_0^2 along the trajectories of \bar{X}_τ in positive and negative time.

Theorem 4.1. *There exists a small ball W around $(U, R) = (0, 0)$ such that for any degree $k \geq 1$ of smoothness there exist $0 < \epsilon_k \leq \epsilon_0$ so that the mappings*

$$\begin{aligned} \mathcal{H}_+ : \mathcal{D}_k &= \mathcal{D} \cap (W \times [0, \epsilon_k] \times \mathcal{B} \times \Lambda) \rightarrow \Sigma_0^2 : \\ (U, R, \tau) &\mapsto (\bar{y}, r) = (-\epsilon^2 h_+(U, R, \tau), UR) \end{aligned} \tag{12}$$

and

$$\begin{aligned} \mathcal{H}_- : \mathcal{D}_k &= \mathcal{D} \cap (W \times [0, \epsilon_k] \times \mathcal{B} \times \Lambda) \rightarrow \Sigma_0^2 : \\ (U, R, \tau) &\mapsto (\bar{y}, r) = (-\epsilon^2 h_-(U, R, \tau), UR) \end{aligned} \tag{13}$$

(defined by following respectively the orbits $o_{U,R,\tau}^+$ and $o_{U,R,\tau}^-$ until they reach Σ_0^2) are C^∞ and have C^k -extensions to $\overline{\mathcal{D}_k}$. Moreover, functions h_+ and h_- are strictly positive and C^∞ with C^k -extensions to $\overline{\mathcal{D}_k}$.

A proof of Theorem 4.1 is given in Section 5.1.

Remark 2. Whereas the functions \mathcal{H}_\pm are C^∞ on \mathcal{D}_k (\bar{X}_τ is C^∞), presence of hyperbolic saddles P_ϵ^\pm , $\epsilon > 0$, weakens the obtained smoothness: one has C^k -smoothness on $\overline{\mathcal{D}_k}$ for all k , in the sense that possibly $\epsilon_k \rightarrow 0$ as $k \rightarrow +\infty$ (see also [3]). The boundary of the domain \mathcal{D}_k of \mathcal{H}_\pm includes the sets $\{\epsilon = 0\}$ and $\{R = 0\}$, and might include other parts of the parameter space, where orbits $o_{U,R,\tau}^\pm$ get trapped at a saddle-node but nearby orbits can meet $\{\bar{x} = 0\}$. Let us recall that such a saddle-node may appear for $B_2 \sim \pm 2$, in the family chart ‘‘ $\bar{r} = 1$ ’’ (Section 3.1).

The functions h_+ and h_- in respectively (12) and (13) play a central role in the search for the limit cycles near Γ . Zeros of $\delta := h_+ - h_-$, for $U > 0$, $R > 0$ and $\epsilon > 0$, correspond to periodic orbits Hausdorff-close to Γ . To be more precise, for each fixed value of (r, τ) , $r > 0$, we can treat the function δ as 1-variable function defined on “segment” $I_r^+ := \{(U, R); (U, R, \tau) \in \mathcal{D}_k, UR = r, U \geq 0\}$, for some $k \geq 1$. We want to study the number of isolated zeros (counted with multiplicity) of the function δ on I_r^+ for each fixed value of (r, τ) where $r > 0$.

To study the isolated zeros of δ on I_r^+ , we will consider its Lie-derivative $\mathcal{L}_Y \delta = U \frac{\partial \delta}{\partial U} - R \frac{\partial \delta}{\partial R}$ along the vector field $Y = U \frac{\partial}{\partial U} - R \frac{\partial}{\partial R}$. The reason to introduce this Lie-derivative is that the equation $\{\mathcal{L}_Y \delta = 0\}$ can be reduced to a simpler form than the equation $\{\delta = 0\}$ which contains exponential terms (see Section 5.2).

As the vector field Y has no zeros on $\{UR = r\}$, $r > 0$, Rolle’s theorem will permit to find the maximum number of the zeros of δ from zeros of $\mathcal{L}_Y \delta$. This trick with a Lie-derivative along a vector field is used in [2].

We say that $(U_0, R_0, \tau_0) \in \mathcal{D}_k$, $U_0 > 0$, is a zero of multiplicity $n \geq 1$ of the function δ if $\delta(U_0, R_0, \tau_0) = (\mathcal{L}_Y \delta)(U_0, R_0, \tau_0) = \dots = (\mathcal{L}_Y^{n-1} \delta)(U_0, R_0, \tau_0) = 0$ and $(\mathcal{L}_Y^n \delta)(U_0, R_0, \tau_0) \neq 0$. We inductively define $\mathcal{L}_Y^m \delta$ as: $\mathcal{L}_Y^0 \delta = \delta$, $\mathcal{L}_Y^2 \delta = \mathcal{L}_Y(\mathcal{L}_Y \delta)$ and $\mathcal{L}_Y^m \delta = \mathcal{L}_Y(\mathcal{L}_Y^{m-1} \delta)$ for $m \geq 3$.

The notion of cyclicity of δ enables us to state main results in this paper.

Definition 4.2. a) $\delta(U, R, \tau)$ is said to have finite cyclicity at $B_2 = \bar{B}_2 \in [-B_2^0, B_2^0]$ if there exist $N \in \mathbb{N}_0$, $k \in \mathbb{N}_1$, $r_0 > 0$ sufficiently small, a small ball W around $(U, R) = (0, 0)$ and a small ball V around $(\epsilon, B_0, B_2) = (0, 0, \bar{B}_2)$ such that for each fixed value of $(r, \tau) \in]0, r_0] \times V \times \Lambda$ the number of isolated zeros (counting multiplicity) of $\delta(U, R, \tau)$ on $\{(U, R); (U, R, \tau) \in \mathcal{D}_k \cap (W \times V \times \Lambda), UR = r, U \geq 0\}$ is bounded by N (the number of isolated zeros of a function with an empty domain is 0).

b) the minimum of such N is called the cyclicity of δ at $B_2 = \bar{B}_2$. The cyclicity of Γ at $B_2 = \bar{B}_2$ is the cyclicity of δ at $B_2 = \bar{B}_2$.

Remark 3. Since we study the system (5) for $r > 0$, it is sufficient to define the cyclicity of δ for $U > 0$ and $R > 0$ ($r = UR$). As we can see in Theorem 4.1, the functions h_+ and h_- are defined not only for $U \geq 0$, but also for $U < 0$. Here we point out that symmetries defined in Section 5.2 include the variable U and play an important role in the study of the cyclicity of δ : in Section 5.2 we will use $h_-(U, R, \epsilon, B_0, B_2, \lambda) = h_+(-U, R, \epsilon, -B_0, -B_2, \lambda)$.

Our first main result states that for $B_2 \neq 0$ at most one (hyperbolic) limit cycle may perturb from Γ . The case $B_2 \neq 0$ is easy to treat because of the presence of the term $B_2 r x^2$ in (5) which makes sure that for $B_0 = 0$ the blown-up vector field \bar{X}_τ is far away from center behavior on the primary blow-up locus.

Theorem 4.3. a) If $-2 < \bar{B}_2 < 2$ and $\bar{B}_2 \neq 0$, then the cyclicity of Γ at $B_2 = \bar{B}_2$ is equal to 1. When $\bar{B}_2 > 0$, then we deal with a hyperbolic and attracting limit cycle. When $\bar{B}_2 < 0$, then we deal with a hyperbolic and repelling limit cycle.

b) If $\bar{B}_2 = \pm 2$, then the cyclicity of Γ at $B_2 = \bar{B}_2$ is equal to 1. When $\bar{B}_2 = 2$, then we deal with a hyperbolic and attracting limit cycle. When $\bar{B}_2 = -2$, then we deal with a hyperbolic and repelling limit cycle.

c) If $2 < |\bar{B}_2| \leq B_2^0$, then the cyclicity of Γ at $B_2 = \bar{B}_2$ is 0.

A proof of Theorem 4.3 is given in Section 5.3.

Remark 4. Let K denote an arbitrary compact subset of \mathbb{R} such that $K \subset]0, B_2^0]$. Theorem 4.3 implies that there exist $k \in \mathbb{N}_1, r_0 > 0$, a ball W around $(U, R) = (0, 0)$ and a ball V around $(\epsilon, B_0) = (0, 0)$ such that for each fixed value of $(r, \tau) \in]0, r_0] \times V \times K \times \Lambda$ the number of zeros (counting multiplicity) of $\delta(U, R, \tau)$ on $\{(U, R); (U, R, \tau) \in \mathcal{D}_k \cap (W \times V \times K \times \Lambda), UR = r, U \geq 0\}$ is bounded by 1. Hence, we have found that at most one hyperbolically attracting limit cycle can occur in an $(\epsilon, B_0, B_2, \lambda, r)$ -uniform neighborhood of Γ where $\epsilon > 0, \epsilon \sim 0, B_0 \sim 0, \lambda \in \Lambda, r > 0, r \sim 0$ and $B_2 \in K$. This result is used in the proof of Theorem 2.4. in [8].

When $B_2 \sim 0$, then one might have more than one limit cycle, as explained in [8], and it is due to the fact that one studies perturbations of a vector field of center type (symmetries introduced in Section 5.2 imply that \bar{X}_τ is of center type on the blow-up locus, for $B_0 = B_2 = 0$). In order to make this situation less degenerate, we assume that $\bar{H}(0, \lambda) \neq 0$ for each $\lambda \in \Lambda$, like in [8].

Theorem 4.4. *Suppose that $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. Then the cyclicity of Γ at $B_2 = 0$ is bounded by 2. In other words, no more than 2 limit cycles of \bar{X}_τ may bifurcate from Γ , for $B_2 \sim 0$.*

Remark 5. Theorem 4.4 implies that at most two limit cycles can be found in an $(\epsilon, B_0, B_2, \lambda, r)$ -uniform neighborhood of Γ where $\epsilon > 0, \epsilon \sim 0, B_0 \sim 0, B_2 \sim 0, \lambda \in \Lambda, r > 0$ and $r \sim 0$. This result is also used in [8].

A proof of Theorem 4.4 is given in Section 5.4.

5. Proofs of Theorem 4.1-Theorem 4.4. We start by proving Theorem 4.1. Besides proving Theorem 4.1, in Section 5.1 we get a nice structure of a forward transition map between Σ_0^1 and Σ_+^2 (see Figure 3). (For a precise definition of Σ_+^2 we refer to later sections.) That structure is given by (46) in Theorem 5.9, and in Section 5.2 it is used to obtain the exponential form (63) of $\mathcal{L}_Y \delta$.

We repeat once more that the exponential form of $\mathcal{L}_Y \delta$ will be used in the proof of Theorem 4.3 and in the proof of Theorem 4.4.

5.1. Proof of Theorem 4.1. We will provide an explicit proof for \mathcal{H}_+ , hence working in forward time; the treatment of \mathcal{H}_- is completely analogous.

We split \mathcal{H}_+ into three parts. Choose the sections Σ_0^1 and Σ_0^2 , as explained above. We also choose a section Σ_+^1 (resp. Σ_+^2) near P_ϵ^+ transverse to arc A (resp. transverse to the critical curve $\{\bar{y} = \frac{1}{2}\bar{x}^2\}$) (Figure 3). Σ_+^1 (resp. Σ_+^2) is parametrized by two regular parameters. We refer to Section 5.1.2 for precise definitions of Σ_+^1 and Σ_+^2 .

One defines (see Figure 3):

1. the regular transition map \mathcal{R}_+^τ near the arc A from Σ_0^1 to Σ_+^1 , defined by the flow of \bar{X}_τ ,
2. the Dulac map \mathcal{D}_+^τ describing the corner passage near P_ϵ^+ from Σ_+^1 to Σ_+^2 defined by the flow of \bar{X}_τ ,
3. the singular transition map \mathcal{S}_+^τ near the critical curve from Σ_+^2 to Σ_0^2 , defined by the flow of \bar{X}_τ ,
4. the transition maps $\mathcal{H}_\pm^\tau(U, R) := \mathcal{H}_\pm(U, R, \tau)$ near Γ from Σ_0^1 to Σ_0^2 defined by the flow of $\pm \bar{X}_\tau$. In particular $\mathcal{H}_+^\tau = \mathcal{S}_+^\tau \circ \mathcal{D}_+^\tau \circ \mathcal{R}_+^\tau$ for $(U, R, \tau) \in \mathcal{D}$.

Our goal is to study the transition maps \mathcal{R}_+^τ , \mathcal{D}_+^τ and \mathcal{S}_+^τ . We start with \mathcal{D}_+^τ . Choosing appropriate normalizing coordinates near P_0^+ will appear to be a helpful tool in simplifying the calculation of the transition map \mathcal{D}_+^τ .

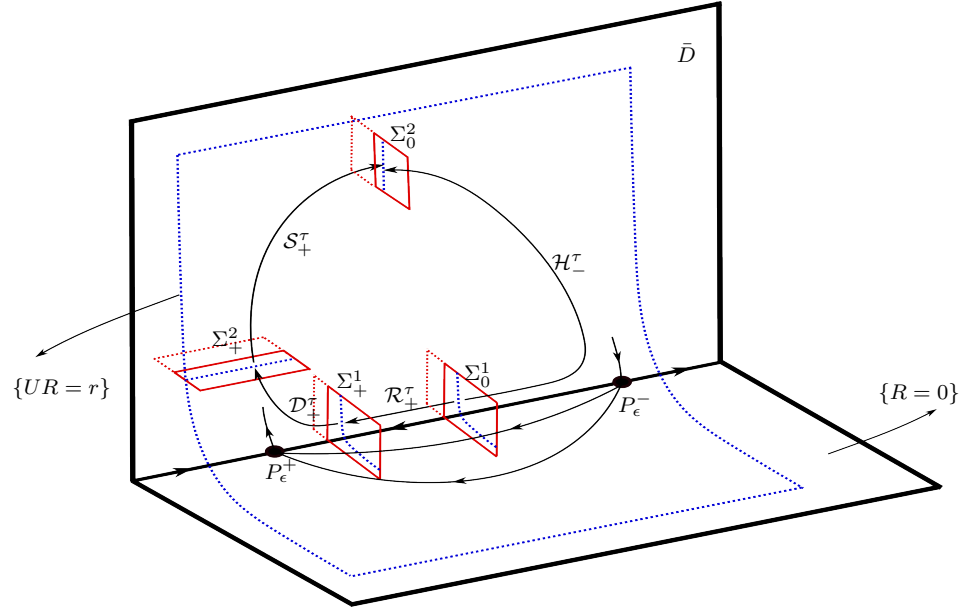


FIGURE 3. The maps \mathcal{R}_+^τ , \mathcal{D}_+^τ and \mathcal{S}_+^τ , for $\epsilon > 0$.

5.1.1. *Normalizing coordinates.* Since we want to study the corner passage near P_ϵ^+ , we will change $X_\tau^{(2)}$, defined in (11), near P_ϵ^+ by the equivalent family Y_τ defined as:

$$Y_\tau := -\frac{2}{\Psi(\bar{X}, U, R, \tau)} X_\tau^{(2)}. \tag{14}$$

As τ can be chosen in the compact set $[0, \epsilon_0] \times \mathcal{B} \times \Lambda$, we can suppose that Y_τ is defined in a fixed neighborhood of P_0^+ .

We now introduce the coordinate change

$$z = \bar{X} - \sqrt{2}. \tag{15}$$

In the coordinates (z, U, R) the vector field Y_τ can be written as

$$\tilde{Y}_\tau : \begin{cases} \dot{z} = \frac{2\sqrt{2}z + z^2}{\Psi(\sqrt{2} + z, U, R, \tau)} + \epsilon^2(\sqrt{2} + z) \\ \dot{U} = -\epsilon^2 U \\ \dot{R} = \epsilon^2 R. \end{cases} \tag{16}$$

Our goal is to normally linearize the vector field (16), i.e. to linearize the differential equation of the hyperbolic variable z in (16). We aim at getting a normal linearization of (16) that is as smooth as possible. The theorems that are presented in [11] provide normal linearizations that respect a lot of essential structure, combined with a maximal smoothness.

We consider first local invariant manifolds of (16) near the singularity $Q_\epsilon := P_\epsilon^+ - (\sqrt{2}, 0, 0)$. Section 3.2 implies that Q_ϵ is a hyperbolic saddle for $\epsilon > 0$ and a semi-hyperbolic singularity for $\epsilon = 0$. The well-known center manifold theorem implies now that for each $k \geq 1$ there exists a $C^k(B_0, B_2, \lambda)$ -family of local center manifolds of the extended family of vector fields $\tilde{Y}_\tau + 0 \frac{\partial}{\partial \epsilon}$ at $(z, U, R, \epsilon) = (0, 0, 0, 0)$, where $(B_0, B_2, \lambda) \in \mathcal{B} \times \Lambda$. In fact the (B_0, B_2, λ) -family of center manifolds of the extended family forms a τ -family of invariant manifolds of \tilde{Y}_τ at Q_ϵ .

It is known that in general one cannot expect the existence of a C^∞ center manifold of a C^∞ vector field. Since we deal with a specific C^∞ vector field, we utilize Theorem 1 in [11] to see that there exists a ball V around $(U, R) = (0, 0)$, $0 < \epsilon_1 \leq \epsilon_0$ and a C^1 -graph $z = \phi(U, R, \epsilon, B_0, B_2, \lambda)$ on $V \times [0, \epsilon_1] \times \mathcal{B} \times \Lambda$ that forms a τ -family of invariant manifolds of (16), with $\phi(0, 0, 0, B_0, B_2, \lambda) = 0$. Moreover, ϕ is C^∞ for $U \neq 0$ and there exists a decreasing sequence $(\epsilon_k)_{k \geq 1}$ for which ϕ is C^k on $V \times [0, \epsilon_k] \times \mathcal{B} \times \Lambda$.

The main benefit of Theorem 1 in [11] is hence the fact that a single center manifold of $\tilde{Y}_\tau + 0 \frac{\partial}{\partial \epsilon}$ is constructed that can be made as smooth as required, provided one takes ϵ small enough. It is clear that this result improves the center manifold theorem.

Remark 6. We fix the center manifold $z = \phi$ of $\tilde{Y}_\tau + 0 \frac{\partial}{\partial \epsilon}$ given in Theorem 1 in [11].

Now we have to straighten the center manifold $z = \phi$. After applying the coordinate change

$$Z = z - \phi \tag{17}$$

to (16), the z -component of (16) can be written as

$$\begin{cases} \dot{Z} = \frac{2\sqrt{2}(Z + \phi) + (Z + \phi)^2}{\Psi(\sqrt{2} + Z + \phi, U, R, \tau)} - \frac{2\sqrt{2}\phi + \phi^2}{\Psi(\sqrt{2} + \phi, U, R, \tau)} + \epsilon^2 Z. \end{cases} \tag{18}$$

Here we used the fact that the family of the invariant manifolds $z = \phi(U, R, \tau)$ of (16) is expressed by a solution to the partial differential equation

$$\frac{2\sqrt{2}\phi + \phi^2}{\Psi(\sqrt{2} + \phi, U, R, \tau)} + \epsilon^2(\sqrt{2} + \phi + U \frac{\partial \phi}{\partial U} - R \frac{\partial \phi}{\partial R}) = 0. \tag{19}$$

Remark 7. The equation (19) implies that $\phi = O(\epsilon^2)$.

Taking into account (18), one gets a family of vector fields

$$\begin{cases} \dot{Z} = -(A(U, R, \tau) + Z\kappa(U, R, Z, \tau))Z \\ \dot{U} = -\epsilon^2 U \\ \dot{R} = \epsilon^2 R, \end{cases} \tag{20}$$

The functions A and κ are C^∞ for $U \neq 0$ and C^k for $(U, R) \in V$, $|Z|$ small, $\epsilon \in [0, \epsilon_k]$, $(B_0, B_2) \in \mathcal{B}$ and $\lambda \in \Lambda$. Bearing in mind Remark 7, we find that

$$\begin{aligned} A(U, R, \tau) &= -\frac{\partial}{\partial Z} \left(\frac{2\sqrt{2}(Z + \phi) + (Z + \phi)^2}{\Psi(\sqrt{2} + Z + \phi, U, R, \tau)} \right) \Big|_{Z=0} - \epsilon^2 \\ &= \frac{-2\sqrt{2}}{-R^2\sqrt{2} + RB_2 2 - 2\sqrt{2} + U4\bar{H}(U\sqrt{2}, \lambda)} + O(\epsilon), \end{aligned} \tag{21}$$

where $O(\epsilon)$ is C^∞ for $U \neq 0$ and C^k for $(U, R) \in V$, $\epsilon \in [0, \epsilon_k]$, $(B_0, B_2) \in \mathcal{B}$ and $\lambda \in \Lambda$. Hence $A(0, 0, 0, B_0, B_2, \lambda) = 1$.

We are now in a position to use the following normal linearization theorem (see [11], Theorem 2):

Theorem 5.1. *Consider a family of vector fields (20), with the above-mentioned conditions. There exists a local C^1 -family of diffeomorphisms of the form*

$$(Z, U, R) \rightarrow (\tilde{z}, U, R), \tilde{z} = \Phi(Z, U, R, \tau),$$

with $\Phi(0, U, R, \tau) = 0$ and $\frac{\partial \Phi}{\partial Z}(0, U, R, \tau) = 1$, defined for $(Z, U, R) \in \tilde{V}$ near the origin and for $\epsilon \in [0, \epsilon_1]$ (up to shrinking ϵ_1 if necessary), $(B_0, B_2) \in \mathcal{B}$, $\lambda \in \Lambda$, conjugating the family (20) to

$$\begin{cases} \dot{\tilde{z}} = -A(U, R, \tau)\tilde{z} \\ \dot{U} = -\epsilon^2 U \\ \dot{R} = \epsilon^2 R. \end{cases} \tag{22}$$

Φ is C^∞ for $UR \neq 0$ and C^k on $\tilde{V} \times [0, \epsilon_k] \times \mathcal{B} \times \Lambda$, up to shrinking ϵ_k if necessary.

5.1.2. *Dulac map \mathcal{D}_+^τ near P_ϵ^+ .* In this section we express first the Dulac map \mathcal{D}_+^τ in normalizing coordinates, i.e. D_+^τ calculated from $\{\tilde{z} = 1\}$, parametrized by (U, R) , to $\{R = R_1\}$, parametrized by (\tilde{z}, U') , for the expression (22). We suppose that $R_1 > 0$ is small enough such that the section $\{R = R_1\}$ lies in the domain of the function A . The U' -component of D_+^τ is $\frac{UR}{R_1}$. Let us denote the \tilde{z} -component of D_+^τ by d_+^τ .

Proposition 5.2. *There exists a ball \tilde{W} around $(U, R) = (0, 0)$ such that the \tilde{z} -component $d_+^\tau(U, R) = d_+(U, R, \tau)$ of D_+^τ is C^k on $\Omega_k := (\tilde{W} \cap \{R > 0\}) \times]0, \epsilon_k] \times \mathcal{B} \times \Lambda$ and has a C^k -extension to the closure $\overline{\Omega}_k$, up to shrinking ϵ_k if necessary where $(\epsilon_k)_{k \geq 1}$ is introduced in Section 5.1.1. Moreover,*

$$\begin{aligned} d_+^\tau(U, R) &= \exp -\frac{1}{\epsilon^2} \int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' \\ &= \exp -\frac{1}{\epsilon^2} \left(\alpha_k(U, R, \tau) + \beta_k(U, R, \tau) \ln R \right), \end{aligned} \tag{23}$$

where α_k and β_k are C^k -functions on $\overline{\Omega}_k$ and $\beta_k(U, 0, \tau) = -A(0, 0, \tau)$.

Proof. We consider a τ -family of orbits γ_τ of (22) starting at $(1, U, R)$ where $U \sim 0$, $R \sim 0$ and $0 < R < R_1$, $\epsilon \sim 0$ and $\epsilon > 0$. The expression (22) implies that the (U, R) -component of γ_τ is expressed by

$$(U(t), R(t)) = (U \exp(-\epsilon^2 t), R \exp(\epsilon^2 t)).$$

The time to go from $\{\tilde{z} = 1\}$ to $\{R = R_1\}$ is given by

$$t(R) = \frac{1}{\epsilon^2} \ln \frac{R_1}{R}.$$

We now look at the first line of (22) where one substitutes $U(t), R(t)$. Integrating from $\{\tilde{z} = 1\}$ to $\{R = R_1\}$, one obtains

$$\ln \frac{d_+^\tau(U, R)}{1} = - \int_0^{t(R)} A(U \exp(-\epsilon^2 s), R \exp(\epsilon^2 s), \tau) ds.$$

Hence we have that

$$d_+^\tau(U, R) = \exp - \int_0^{\frac{1}{\epsilon^2} \ln \frac{R_1}{R}} A(U \exp(-\epsilon^2 s), R \exp(\epsilon^2 s), \tau) ds.$$

We make in the integral the change of variable: $R' = R \exp(\epsilon^2 s)$. This gives

$$d_+^\tau(U, R) = \exp - \frac{1}{\epsilon^2} \int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR'. \tag{24}$$

To end this let us use the change of variable $\bar{R} = \frac{R'}{R_1}$. Then (24) changes into

$$d_+^\tau(U, R) = \exp - \frac{1}{\epsilon^2} \int_{R/R_1}^1 \frac{A(\frac{UR}{\bar{R}R_1}, \bar{R}R_1, \tau)}{\bar{R}} d\bar{R}.$$

If we denote R/R_1 by V , then we have that

$$d_+^\tau(U, R) = \exp - \frac{1}{\epsilon^2} \int_V^1 \frac{f(V/\bar{R}, \bar{R}, U, \tau)}{\bar{R}} d\bar{R}, \tag{25}$$

where $f(V', \bar{R}, U, \tau) = A(UV', \bar{R}R_1, \tau)$. Let us recall that the domain of $A(U, R, \tau)$ only shrinks in the ϵ -direction when degree of smoothness k increases. If we use a Taylor formula at first order in $\bar{R} = 0$, then we have that $f(V', \bar{R}, U, \tau) = f(V', 0, U, \tau) + \bar{R}f_1(V', \bar{R}, U, \tau)$. As a consequence we have the following expression for the integral in (25):

$$\int_V^1 \frac{f(V/\bar{R}, 0, U, \tau)}{\bar{R}} d\bar{R} + \int_V^1 f_1(V/\bar{R}, \bar{R}, U, \tau) d\bar{R}, \tag{26}$$

Let us first study the first integral in (26). Using the change of variable $w = V/\bar{R}$ we have that

$$\int_V^1 \frac{f(V/\bar{R}, 0, U, \tau)}{\bar{R}} d\bar{R} = \int_V^1 \frac{f(w, 0, U, \tau)}{w} dw = -A(0, 0, \tau) \ln V + \bar{F}(V, U, \tau), \tag{27}$$

where \bar{F} is arbitrarily (finitely) smooth by taking small ϵ . To study the second integral in (26), we use the following lemma (see [3]):

Lemma 5.3. 1) Let $f(V, R)$ be a C^k -function on $[0, 1] \times [0, 1]$, $k \geq 2$, and let $b \geq 0$ be a fixed integer. Define

$$F(V) = \int_V^1 R^b f(V/R, R) dR, \quad V \in [0, 1].$$

Then $F(V)$ is of the form $F(V) = \alpha(V) + \beta(V)V \ln V$, for some C^{k-2} functions α and β , defined on $[0, 1]$, and $F(0) = \int_0^1 R^b f(0, R) dR$. Furthermore, $\beta = O(V^b)$, provided we have $k \geq b + 2$. 2) Should f depend smoothly on extra parameters up to some order, then so will the resulting functions α and β be smooth w.r.t. these parameters up to the same order.

Hence Lemma 5.3, with $b = 0$, implies that

$$\int_V^1 f_1(V/\bar{R}, \bar{R}, U, \tau) d\bar{R} = \tilde{\alpha}_k(V, U, \tau) + \tilde{\beta}_k(V, U, \tau)V \ln V, \tag{28}$$

where $\tilde{\alpha}_k, \tilde{\beta}_k$ are C^k functions provided f_1 is C^{k+2} for $k \geq 1$. Let us recall that f_1 is arbitrarily (finitely) smooth by taking ϵ small enough. Expressions (27) and (28),

together with the fact that $V = \frac{R}{R_1}$, imply that the integral in (25) can be written as

$$\begin{aligned} \tilde{\alpha}_k\left(\frac{R}{R_1}, U, \tau\right) + \bar{F}\left(\frac{R}{R_1}, U, \tau\right) + A(0, 0, \tau) \ln R_1 - \tilde{\beta}_k\left(\frac{R}{R_1}, U, \tau\right) \frac{R}{R_1} \ln R_1 \\ + \left(\tilde{\beta}_k\left(\frac{R}{R_1}, U, \tau\right) \frac{R}{R_1} - A(0, 0, \tau)\right) \ln R. \end{aligned} \tag{29}$$

If we denote the first line in (29) by α_k and the expression in front of $\ln R$ by β_k , then we obtain that

$$d_+^\tau(U, R) = \exp -\frac{1}{\epsilon^2} \left(\alpha_k(U, R, \tau) + \beta_k(U, R, \tau) \ln R \right), \tag{30}$$

where for each $k \geq 1$ α_k and β_k are C^k , by taking ϵ sufficiently small, and $\beta_k(U, 0, \tau) = -A(0, 0, \tau) = -1 + O(\epsilon)$.

It remains to study the smoothness of the transition map $d_+^\tau(U, R)$. We focus here to the fact that, by choosing ϵ small enough, the exponent $\frac{A(0, 0, \tau)}{\epsilon^2}$ can be chosen arbitrarily high, so that any degree of smoothness can locally be obtained. For the sake of completeness, let us prove it.

For $R > 0$ and $\epsilon > 0$, we have nothing to prove due to the smoothness of α_k and β_k .

Since $\beta_k(U, 0, \tau) < 0$, $d_+^\tau(U, R) \rightarrow 0$ when $R \rightarrow 0$ or $\epsilon \rightarrow 0$. Choose now any $k \geq 1$. In what follows, we show that, by choosing ϵ small enough, $\partial^I d_+^\tau(U, R) \rightarrow 0$ when $R \rightarrow 0$ where $I = (i_1, i_2, i_3, i_4, i_5, i_6)$, $1 \leq |I| \leq k$ and where $\partial^I = \partial^{|I|} / \partial U^{i_1} \partial R^{i_2} \partial \epsilon^{i_3} \partial B_0^{i_4} \partial B_2^{i_5} \partial \lambda^{i_6}$.

Straightforward calculations show that any derivative $\partial^I d_+^\tau(U, R)$, $1 \leq |I| \leq k$, is a finite linear combination of the following expressions:

$$\exp -\frac{1}{\epsilon^2} \left(\alpha_k + \beta_k \ln R \right) \prod_{j=1}^l \partial^{I_j} \left(-\frac{1}{\epsilon^2} \left(\alpha_k + \beta_k \ln R \right) \right), \tag{31}$$

where $|I_j| \geq 1$ and $\sum_{j=1}^l |I_j| = |I|$.

On account of (31) it is sufficient to show that by choosing ϵ small enough the expression

$$\frac{1}{\epsilon^m} \bar{\alpha}(U, R, \tau) |\ln R|^{n_1} \frac{1}{R^{n_2}} \exp -\frac{1}{\epsilon^2} \left(\alpha_k + \beta_k \ln R \right) \tag{32}$$

goes to 0 as $R \rightarrow 0$ where $\bar{\alpha}$ is continuous. We suppose that $m \in \mathbb{N}_1$, $n_1 \in \mathbb{N}_0$ and $n_2 \in \mathbb{N}_0$ are arbitrary and fixed.

$\bar{\alpha}$ in (32) is bounded and, with no loss of generality, we suppose that $\bar{\alpha} = 1$. The logarithm of (32) is

$$\begin{aligned} -m \ln \epsilon + n_1 \ln |\ln R| - n_2 \ln R - \frac{1}{\epsilon^2} \left(\alpha_k + \beta_k \ln R \right) \\ = \frac{1}{\epsilon^2} \left(-m \epsilon^2 \ln \epsilon + n_1 \epsilon^2 \ln |\ln R| - n_2 \epsilon^2 \ln R - \alpha_k - \beta_k \ln R \right) \end{aligned} \tag{33}$$

As $\ln |\ln R| \leq \ln \frac{1}{R}$ for $R \sim 0$, (33) is bounded above by

$$\frac{1}{\epsilon^2} \left(-m \epsilon^2 \ln \epsilon - \alpha_k + (-\beta_k - n_1 \epsilon^2 - n_2 \epsilon^2) \ln R \right). \tag{34}$$

Since $\beta_k \sim -1$, we have that $-\beta_k - n_1 \epsilon^2 - n_2 \epsilon^2 > 0$ by taking ϵ small enough. Hence (34) goes to $-\infty$ as $R \rightarrow 0$. It also goes to $-\infty$ as $\epsilon \rightarrow 0$, for R small. \square

Let us finally define sections Σ_+^1 and Σ_+^2 . We denote by

$$\varphi(\bar{X}, U, R) = \varphi_\tau(\bar{X}, U, R) = (\psi_\tau(\bar{X}, U, R), U, R) = (\tilde{Z}, U, R)$$

the τ -family of coordinate changes conjugating (Y_τ) , defined in (14), to (22) locally near P_ϵ^+ . φ is the succession of the mapping defined in (15), the mapping defined in (17), the mapping defined in Theorem 5.1 and the dilatation $(\tilde{z}, U, R) \rightarrow (-\tilde{z}/\tilde{z}_0, U, R) = (\tilde{Z}, U, R)$, for $\tilde{z}_0 > 0$ small enough such that for each $\epsilon \sim 0$, $(B_0, B_2) \in \mathcal{B}$ and $\lambda \in \Lambda$ section $\{(-\tilde{z}_0, U, R) \mid U \sim 0, R \sim 0, R \geq 0\}$ lies in the domain of the inverse of the diffeomorphism defined in Theorem 5.1. Let us remark that (22) is unchanged under the dilatation.

Let

$$\sigma_+^1 = \{(1, U, R) \mid U \sim 0, R \sim 0, R \geq 0\}$$

and

$$\sigma_+^2 = \{(\tilde{Z}, U', R_1) \mid \tilde{Z} \sim 0, \tilde{Z} \geq 0, U' \sim 0\}.$$

By taking $R_1 > 0$ sufficiently small and fixed one can suppose that these sections lie in the domain of φ^{-1} for each $\epsilon \sim 0$, $(B_0, B_2) \in \mathcal{B}$ and $\lambda \in \Lambda$. We now choose Σ_+^2 in the (\bar{X}, U, R) coordinates of Y_τ near P_ϵ^+ inside $\{R = R_1\}$ and transversally cutting the τ -family of invariant manifolds $\bar{X} = \sqrt{2} + \phi(U, R, \tau)$ of Y_τ (ϕ is fixed in Remark 6). Hence we can take $\Sigma_+^2 = \varphi^{-1}(\sigma_+^2)$. We parametrize Σ_+^2 by $\{\bar{X} \sim \sqrt{2}, U \sim 0\}$. We choose $\Sigma_+^1 = \varphi^{-1}(\sigma_+^1)$ transversally cutting the arc A . We parametrize Σ_+^1 by (U, R) . The sections Σ_+^1 and Σ_+^2 depend, in a C^k way, on τ bearing in mind that only ϵ decreases when the smoothness requirements increase.

The map \mathcal{D}_+^τ from Σ_+^1 to Σ_+^2 , defined by the flow of Y_τ (i.e. $X_\tau^{(2)}$), can be studied now, by Proposition 5.2, in the normalizing coordinates (\tilde{Z}, U, R) in the region $\{\tilde{Z} > 0, R > 0\}$, from σ_+^1 to σ_+^2 . It can be expressed by:

$$(U, R) \rightarrow (d_+(U, R, \tau), UR/R_1).$$

5.1.3. *Regular transition map \mathcal{R}_+^τ .* Let us recall that the map \mathcal{R}_+^τ , from a subset of Σ_0^1 to Σ_+^1 , is defined by the flow of $X_\tau^{(2)}$. The transition map \mathcal{R}_+^τ can be represented as going from a subset of Σ_0^1 to σ_+^1 , i.e. with values in the normalizing coordinates (\tilde{Z}, U, R) . One obtains map $R_+(U, R, \tau)$. Hence R_+ is expressed in (U, R) with values in (U, R) .

We split R_+ into two parts. Choose $\bar{X}_0 \in]0, \sqrt{2}[$, sufficiently close to $\sqrt{2}$, such that for each $\epsilon \sim 0$, $(B_0, B_2) \in \mathcal{B}$ and $\lambda \in \Lambda$ section $\{(\bar{X}_0, U, R) \mid U \sim 0, R \sim 0, R \geq 0\}$ lies in the domain of φ and such that \bar{X}_0 is strictly smaller than the \bar{X} -coordinate of $\varphi_\tau^{-1}(1, 0, 0)$. Let $T_+(U, R, \tau)$ represent the transition map from (a subset of) Σ_0^1 to $\{\bar{X} = \bar{X}_0\}$ along the trajectories of $X_\tau^{(2)}$.

Lemma 5.4. *There exists a ball \tilde{W}_1 around $(U, R) = (0, 0)$ and $0 < \tilde{\epsilon} \leq \epsilon_0$ such that T_+ is C^∞ on $\Omega^1 := (\tilde{W}_1 \cap \{R \geq 0\}) \times [0, \tilde{\epsilon}] \times \mathcal{B} \times \Lambda$ and*

$$T_+(U, R, \tau) = \left(U(1 + \epsilon^2 \bar{T}(U, R, \tau)), R(1 + \epsilon^2 \bar{T}(U, R, \tau))^{-1} \right), \tag{35}$$

where \bar{T} is C^∞ on Ω^1 .

Proof. Notice first that the parameter (B_0, B_2, λ) takes values in the compact set $\mathcal{B} \times \Lambda$. Hence the \bar{X} -component of $X_\tau^{(2)}$ is strictly positive for $\bar{X} \in [0, \bar{X}_0]$ and $(\epsilon, U, R) \sim (0, 0, 0)$. This means that the family (11) has no singularities between the sections Σ_0^1 and $\{\bar{X} = \bar{X}_0\}$, under the given conditions on the parameters.

In other words, the orbit of (11), starting at $(U, R) \in \Sigma_0^1$, $(U, R) \sim (0, 0)$, reaches $\{\bar{X} = \bar{X}_0\}$ in a finite time. Remark that this fact is clear for $\epsilon = 0$. The fundamental results on the existence, uniqueness and continuity with respect to parameters of solutions to initial value problems imply now that we deal with a C^∞ transition map $T_+ = (T_+^1, T_+^2)$ ($X_\tau^{(2)}$ is C^∞).

We know that T_+ preserves UR , i.e. $T_+^1(U, R, \tau)T_+^2(U, R, \tau) = UR$ for any τ . We also know that T_+ preserves $\{U = 0\}$ and $\{R = 0\}$. It means that there exist C^∞ -functions t_+^1 and t_+^2 , such that:

$$T_+^1(U, R, \tau) = Ut_+^1(U, R, \tau), \quad T_+^2(U, R, \tau) = Rt_+^2(U, R, \tau).$$

Hence we find that $t_+^1(0, 0, \tau) \neq 0$ and $t_+^2 = 1/t_+^1$, for any τ . We obtain that

$$T_+(U, R, \tau) = (Ut_+^1(U, R, \tau), R \frac{1}{t_+^1(U, R, \tau)}).$$

If $\epsilon = 0$, then $T_+(U, R, \tau) = (U, R)$. This means that there exist a C^∞ -function \bar{T} such that $t_+^1(U, R, \tau) = 1 + \epsilon^2 \bar{T}(U, R, \tau)$. □

Choose sections $\pi_\tau^\pm := \varphi(\{\bar{X}_0, U, R\} \mid U \sim 0, R \sim 0, R \geq 0\})$. We parametrize π_τ^\pm through φ by (U, R) . In order to finish the study of R_+ , we need to consider the regular transition map $F_+ : \pi_\tau^\pm \rightarrow \sigma_\pm^1$ along the trajectories of (22), expressed using the chosen parametrization on $\pi_\tau^\pm, \sigma_\pm^1$. Notice that φ leaves the (U, R) -component unchanged. This means that the regular map T_+ is expressed by (35) if we use for $\{\bar{X} = \bar{X}_0\}$ the normalizing coordinates (\bar{Z}, U, R) . Then we have

$$R_+(U, R, \tau) = F_+(T_+(U, R, \tau), \tau).$$

Lemma 5.5. *There exists a ball \widetilde{W}_2 around $(U, R) = (0, 0)$ such that F_+ is C^k on $\Omega_k^2 := (\widetilde{W}_2 \cap \{R \geq 0\}) \times [0, \epsilon_k] \times \mathcal{B} \times \Lambda$, up to shrinking ϵ_k if necessary, and*

$$F_+(U, R, \tau) = \left(U(1 + \epsilon^2 \bar{F}(U, R, \tau)), R(1 + \epsilon^2 \bar{F}(U, R, \tau))^{-1} \right),$$

where \bar{F} is C^k on Ω_k^2 .

Proof. The study of F_+ is analogous to the study of T_+ . Notice that the domain of A in (22) only shrinks in the ϵ -direction as the smoothness requirements increase, and that π_τ^\pm is a graph of a C^k -function which has the same smoothness property as A . □

Combining Lemma 5.4 and Lemma 5.5 we get:

Proposition 5.6. *There exists a ball \widetilde{W}_3 around $(U, R) = (0, 0)$ such that R_+ is C^k on $\Omega_k^3 := (\widetilde{W}_3 \cap \{R \geq 0\}) \times [0, \epsilon_k] \times \mathcal{B} \times \Lambda$, up to shrinking ϵ_k if necessary, and*

$$R_+(U, R, \tau) = \left(U(1 + \epsilon^2 \bar{R}(U, R, \tau)), R(1 + \epsilon^2 \bar{R}(U, R, \tau))^{-1} \right), \tag{36}$$

where \bar{R} is C^k on Ω_k^3 .

Proof. We have

$$\begin{aligned} R_+(U, R, \tau) &= F_+(T_+(U, R, \tau), \tau) \\ &= \left(U(1 + \epsilon^2 \bar{T})(1 + \epsilon^2 \bar{F}(T_+, \tau)), R(1 + \epsilon^2 \bar{T})^{-1}(1 + \epsilon^2 \bar{F}(T_+, \tau))^{-1} \right) \\ &= \left(U(1 + \epsilon^2 \bar{R}(U, R, \tau)), R(1 + \epsilon^2 \bar{R}(U, R, \tau))^{-1} \right). \end{aligned}$$

□

5.1.4. *Combining R_+ and D_+^τ .* In the following proposition we give the expression for the transition map

$$(D_+^\tau \circ R_+)(U, R, \tau)$$

going from (a subset of) Σ_0^1 to σ_+^2 , using the chosen parametrization on Σ_0^1, σ_+^2 .

Proposition 5.7. *There exists a ball \widetilde{W}_4 around $(U, R) = (0, 0)$ such that $D_+^\tau \circ R_+$ is well defined and C^k on $\Omega_k^4 := (\widetilde{W}_4 \cap \{R > 0\}) \times]0, \epsilon_k] \times \mathcal{B} \times \Lambda$ and has a C^k -extension to $\overline{\Omega_k^4}$, up to shrinking ϵ_k if necessary. Moreover,*

$$(D_+^\tau \circ R_+)(U, R, \tau) = \left(d_+^\tau(R_+(U, R, \tau)), \frac{UR}{R_1} \right),$$

where

$$d_+^\tau(R_+(U, R, \tau)) = \exp -\frac{1}{\epsilon^2} \left(\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' + \epsilon^2 l(U, R, \tau) \right), \tag{37}$$

with

$$l(U, R, \tau) = l_1^{(k)}(U, R, \tau) + l_2^{(k)}(U, R, \tau)R \ln R.$$

Functions $l_1^{(k)}, l_2^{(k)}$ are C^k on $\overline{\Omega_k^4}$. The function l does not depend on k .

Proof. The above-mentioned smoothness properties of $D_+^\tau \circ R_+$ follow directly from Proposition 5.2 and Proposition 5.6. From Section 5.1.2 and (36), it is clear that the U' -component of $D_+^\tau \circ R_+$ is

$$U(1 + \epsilon^2 \bar{R}(U, R, \tau)) \cdot R(1 + \epsilon^2 \bar{R}(U, R, \tau))^{-1} / R_1 = UR / R_1.$$

Using expressions (23) (we write $\alpha = \alpha_k$ and $\beta = \beta_k$) and (36) we obtain that

$$\begin{aligned} d_+^\tau(R_+(U, R, \tau)) &= \exp -\frac{1}{\epsilon^2} \left(\alpha(U(1 + \epsilon^2 \bar{R}), R(1 + \epsilon^2 \bar{R})^{-1}, \tau) \right. \\ &\quad \left. + \beta(U(1 + \epsilon^2 \bar{R}), R(1 + \epsilon^2 \bar{R})^{-1}, \tau) \ln R(1 + \epsilon^2 \bar{R})^{-1} \right) \\ &= \exp -\frac{1}{\epsilon^2} \left(\alpha(U, R, \tau) + \beta(U(1 + \epsilon^2 \bar{R}), R(1 + \epsilon^2 \bar{R})^{-1}, \tau) \ln R \right. \\ &\quad \left. + O(\epsilon^2 \bar{R}) - \beta(U(1 + \epsilon^2 \bar{R}), R(1 + \epsilon^2 \bar{R})^{-1}, \tau) \ln(1 + \epsilon^2 \bar{R}) \right). \end{aligned} \tag{38}$$

On account of Proposition 5.2, we have that

$$\beta(U, R, \tau) = -A(0, 0, \tau) + R\bar{\beta}(U, R, \tau). \tag{39}$$

Based on (39), we have

$$\begin{aligned} &\beta(U(1 + \epsilon^2 \bar{R}), R(1 + \epsilon^2 \bar{R})^{-1}, \tau) \\ &= -A(0, 0, \tau) + R(1 + \epsilon^2 \bar{R})^{-1} \bar{\beta}(U(1 + \epsilon^2 \bar{R}), R(1 + \epsilon^2 \bar{R})^{-1}, \tau) \\ &= -A(0, 0, \tau) + R(\bar{\beta}(U, R, \tau) + O(\epsilon^2 \bar{R})) \\ &= \beta(U, R, \tau) + RO(\epsilon^2 \bar{R}). \end{aligned} \tag{40}$$

Using (40) and the fact that $\ln(1 + \epsilon^2 \bar{R}(U, R, \tau)) = O(\epsilon^2 \bar{R})$, the expression (38) changes to

$$\begin{aligned} &d_+^\tau(R_+(U, R, \tau)) \\ &= \exp -\frac{1}{\epsilon^2} \left(\alpha(U, R, \tau) + \beta(U, R, \tau) \ln R + O(\epsilon^2 \bar{R}) + O(\epsilon^2 \bar{R})R \ln R \right) \\ &= \exp -\frac{1}{\epsilon^2} \left(\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' + O(\epsilon^2 \bar{R}) + O(\epsilon^2 \bar{R})R \ln R \right). \end{aligned}$$

This completes the proof. □

The section σ_+^2 is contained in a family directional chart and we can use for it better adapted coordinates. On section σ_+^2 we can consider coordinates (\bar{x}, r) , where (\bar{x}, \bar{y}, r) are the coordinates of $X_\tau^{(1)}$, defined in (8). The changes of coordinates

$$(\tilde{Z}, U') \rightarrow (\bar{x}, r)$$

are given by $r = U'R_1$ (φ leaves the (U, R) -component unchanged) and $\bar{x} = \bar{x}_{\tau,r}(\tilde{Z})$ (we have eliminated $U' = \frac{r}{R_1}$ in the expression of \bar{x}).

Lemma 5.8. *One has*

$$\bar{x}_{\tau,r}(\tilde{Z}) = \frac{1}{R_1} \left(\sqrt{2} + \phi\left(\frac{r}{R_1}, R_1, \tau\right) \right) - \frac{\tilde{z}_0}{R_1} \tilde{Z}(1 + O(\tilde{Z})), \tag{41}$$

where ϕ is fixed in Remark 6, $\tilde{z}_0 > 0$ is defined at the end of Section 5.1.2 and the domain of $O(\tilde{Z})$ only shrinks in the ϵ -direction as the smoothness requirements increase.

Proof. On account of blow-up formulas (7) and (10) we get the following relation between the coordinates \bar{x} and \bar{X} , on σ_+^2 :

$$\bar{x} = \frac{\bar{X}}{R_1}. \tag{42}$$

The mappings defined in (15) and (17), together with Theorem 5.1 and the dilatation defined in Section 5.1.2, imply the following relation between the coordinates \bar{X} and (\tilde{Z}, U') , on σ_+^2 :

$$\bar{X} = \sqrt{2} + \phi(U', R_1, \tau) - \tilde{z}_0 \tilde{Z}(1 + O(\tilde{Z})). \tag{43}$$

Based on Theorem 5.1, $O(\tilde{Z})$ in (43) has the required property stated in Lemma 5.8.

Putting together (42) and (43) we find the relation between \bar{x} and (\tilde{Z}, r) on σ_+^2 :

$$\bar{x} = \frac{1}{R_1} \left(\sqrt{2} + \phi\left(\frac{r}{R_1}, R_1, \tau\right) \right) - \frac{1}{R_1} \tilde{z}_0 \tilde{Z}(1 + O(\tilde{Z})). \tag{44}$$

□

Parameterizing section σ_+^2 by (\bar{x}, r) , the transition map from (a subset of) Σ_0^1 to σ_+^2 along the trajectories of \bar{X}_τ is equal to

$$\left(\bar{x}_{\tau,UR}(d_+^r(R_+(U, R, \tau))), UR \right). \tag{45}$$

We denote by $\bar{x}(U, R, \tau)$ the first component in (45). Combining Proposition 5.7 and Lemma 5.8 we obtain the final form for the transition map from (a subset of) Σ_0^1 to $\Sigma_+^2 = \varphi^{-1}(\sigma_+^2)$, where Σ_+^2 is parametrized by (\bar{x}, r) :

Theorem 5.9. *There exists a ball \widetilde{W}_5 around $(U, R) = (0, 0)$ such that $\bar{x}(U, R, \tau)$ is well defined and C^k on $\Omega_k^5 := (\widetilde{W}_5 \cap \{R > 0\}) \times]0, \epsilon_k] \times \mathcal{B} \times \Lambda$ and has a C^k -extension to $\overline{\Omega}_k^5$, up to shrinking ϵ_k if necessary. Moreover,*

$$\begin{aligned} \bar{x}(U, R, \tau) = & \frac{1}{R_1} \left(\sqrt{2} + \phi\left(\frac{UR}{R_1}, R_1, \tau\right) \right) \\ & - \exp - \frac{1}{\epsilon^2} \left(\int_R^{R_1} \frac{A\left(\frac{UR}{R'}, R', \tau\right)}{R'} dR' + \epsilon^2 L(U, R, \tau) \right), \end{aligned} \tag{46}$$

where

$$L(U, R, \tau) = L_1^{(k)}(U, R, \tau) + L_2^{(k)}(U, R, \tau)R \ln R.$$

Functions $L_1^{(k)}$ and $L_2^{(k)}$ are C^k on $\overline{\Omega_k^5}$. The function L does not depend on k .

Proof. The above-mentioned smoothness property of $\bar{x}(U, R, \tau)$ follows directly from Proposition 5.7 and Lemma 5.8. Taking into account (41) we get

$$\begin{aligned} \bar{x}(U, R, \tau) &= \frac{1}{R_1} \left(\sqrt{2} + \phi\left(\frac{UR}{R_1}, R_1, \tau\right) \right. \\ &\quad \left. - \frac{\tilde{z}_0}{R_1} d_+^\tau(R_+(U, R, \tau)) \left(1 + O(d_+^\tau(R_+(U, R, \tau))) \right) \right). \end{aligned} \tag{47}$$

Of course, we have that

$$1 + O(d_+^\tau(R_+(U, R, \tau))) = \exp -\frac{1}{\epsilon^2} \left(-\epsilon^2 \ln(1 + O(d_+^\tau(R_+(U, R, \tau)))) \right),$$

where $\ln(1 + O(d_+^\tau(R_+(U, R, \tau))))$ is arbitrarily (finitely) smooth by taking ϵ small enough. This follows directly from the fact that functions $O(\tilde{Z})$ in Lemma 5.8 and $d_+^\tau(R_+(U, R, \tau))$ are arbitrarily (finitely) smooth by taking ϵ small enough. If we use (37), then the expression (47) changes to

$$\begin{aligned} \bar{x}(U, R, \tau) &= \frac{1}{R_1} \left(\sqrt{2} + \phi\left(\frac{UR}{R_1}, R_1, \tau\right) \right. \\ &\quad \left. - \exp -\frac{1}{\epsilon^2} \left(\int_R^{R_1} \frac{A\left(\frac{UR}{R'}, R', \tau\right)}{R'} dR' + \epsilon^2 l(U, R, \tau) \right) \right. \\ &\quad \left. - \epsilon^2 \ln(1 + O(d_+^\tau(R_+(U, R, \tau)))) - \epsilon^2 \ln \frac{\tilde{z}_0}{R_1} \right). \end{aligned}$$

The rest of the proof is now trivial. □

5.1.5. *Transition map \mathcal{S}_+^τ and conclusion.* We consider the transition map \mathcal{S}_+^τ from Σ_+^2 to $\Sigma_0^2 \subset \{\bar{x} = 0\}$ along the trajectories of the vector field $X_\tau^{(1)}$ defined in (8). Remark that \mathcal{S}_+^τ can be treated entirely in the family chart “ $\bar{r} = 1$ ”. Section Σ_+^2 is parametrized by (\bar{x}, r) and section Σ_0^2 is parametrized by (\bar{y}, r) . One obtains map

$$\mathcal{S}_+^\tau(\bar{x}, r) = (s_+(\bar{x}, r, \tau), r).$$

As mentioned in Section 3.1, in the family chart we consider r as a regular parameter and observe that ϵ is a singular perturbation parameter. It is important to realize that the slow dynamics (9) are characterized by a regular flow box, with possible isolated saddle-node singularities for $(B_2, r) \sim (\pm 2, 0)$, located away from the origin $(\bar{x}, \bar{y}) = (0, 0)$. This case has already been treated in [4].

In order to be able to use the results in [4], we define a section

$$T = \{\tilde{x} = 0\},$$

along the secondary blow-up locus of the origin $(\bar{x}, \bar{y}) = (0, 0)$ (see Section 3.1). More precisely, T is defined in the family chart $\{\tilde{\epsilon} = 1\}$, and parametrized by the (secondary) blow-up coordinate \tilde{y} . We denote by $\tilde{y} = \hat{s}_+(\bar{x}, r, \tau)$ the transition map between Σ_+^2 and T . By following the orbits through the curve $\{\bar{x} = \bar{x}(U, R, \tau), \bar{y} = 1/R_1^2, r = UR\}$, $(U, R, \tau) \in \mathcal{D}, (\epsilon, U, R) \sim (0, 0, 0)$, in forward time until they reach T , we end up with a C^k -transition map $\tilde{y} = \hat{s}_+(\bar{x}(U, R, \tau), UR, \tau)$ with a C^k -extension to the closure of its domain. Degree of smoothness k can be chosen

arbitrarily high by taking ϵ small enough. We refer to Theorem 3.1 of [4] and Theorem 5.9 in this paper.

Since $\bar{y} = \epsilon^2 \tilde{y}$, we obtain $s_+ = \epsilon^2 \hat{s}_+$. For $(\epsilon, B_0) \sim (0, 0)$, \hat{s}_+ is strictly negative because $\gamma \cap T = (0, -1)$ where γ is defined in Section 3.1.

It is now clear that the transition map

$$\mathcal{H}_+(U, R, \tau) = (s_+(\bar{x}(U, R, \tau), UR, \tau), UR), (U, R, \tau) \in \mathcal{D}, (\epsilon, U, R) \sim (0, 0, 0),$$

has all the properties mentioned in Theorem 4.1 in this paper.

5.2. Difference map and Lie-derivative. In order to prove Theorems 4.3 and 4.4, we need to consider the difference map

$$\Delta(U, R, \tau) = \mathcal{H}_+(U, R, \tau) - \mathcal{H}_-(U, R, \tau), \tag{48}$$

where $(U, R, \tau) \in \mathcal{D}_k$ and $\mathcal{D}_k, \mathcal{H}_\pm$ are defined in (12) and (13).

The r -component of Δ is equal to 0. The \bar{y} -component of Δ can be written as $-\epsilon^2 \delta(U, R, \tau)$, where

$$\delta(U, R, \tau) = h_+(U, R, \tau) - h_-(U, R, \tau), \tag{49}$$

where $(U, R, \tau) \in \mathcal{D}_k$.

The expression of (5) is invariant under the symmetry:

$$S : (r, B_0, B_2) \rightarrow (-r, -B_0, -B_2). \tag{50}$$

From the family rescaling (7) and the symmetry S , defined in (50), it follows that the vector field $X_\tau^{(1)}$ is invariant under the symmetry:

$$S_F : (\bar{x}, \bar{y}, r, \epsilon, B_0, B_2, \lambda, t) \rightarrow (-\bar{x}, \bar{y}, -r, \epsilon, -B_0, -B_2, \lambda, -t). \tag{51}$$

The directional blow-up formula (10) and the symmetry S , defined in (50), imply that $X_\tau^{(2)}$ is invariant under the symmetry

$$S_P : (\bar{X}, U, R, \epsilon, B_0, B_2, \lambda, t) \rightarrow (-\bar{X}, -U, R, \epsilon, -B_0, -B_2, \lambda, -t). \tag{52}$$

By the invariance of $X_\tau^{(1)}$ (resp. $X_\tau^{(2)}$) under S_F (resp. S_P), defined in (51) (resp. (52)), one can write (49) as

$$\delta(U, R, \epsilon, B_0, B_2, \lambda) = h_+(U, R, \epsilon, B_0, B_2, \lambda) - h_+(-U, R, \epsilon, -B_0, -B_2, \lambda), \tag{53}$$

where $(U, R, \epsilon, B_0, B_2, \lambda) \in \mathcal{D}_k$.

To study isolated zeros of δ , we will consider its Lie-derivative $\mathcal{L}_Y \delta = U \frac{\partial \delta}{\partial U} - R \frac{\partial \delta}{\partial R}$ (see Section 4). It is clear that $\mathcal{L}_Y(UR) = 0$. We first apply the Lie-derivation to the expression $-\epsilon^2 h_+(U, R, \tau)$:

$$\begin{aligned} \mathcal{L}_Y(-\epsilon^2 h_+(U, R, \tau)) &= \mathcal{L}_Y(s_+(\bar{x}(U, R, \tau), UR, \tau)) \\ &= \frac{\partial s_+}{\partial \bar{x}}(\bar{x}(U, R, \tau), UR, \tau) \cdot \mathcal{L}_Y(\bar{x}(U, R, \tau)), \end{aligned} \tag{54}$$

where $(U, R, \tau) \in \mathcal{D}_k$.

Remark 8. From now on, we avoid mentioning the fact that degree of smoothness of $h_+(U, R, \tau)$, $\bar{x}(U, R, \tau)$, etc., depends on ϵ . We also do not specify domains of $h_+(U, R, \tau)$, $\bar{x}(U, R, \tau)$, etc., and we avoid pointing out that α_k, β_k , etc., depend on degree k of smoothness. We merely say that all these functions are C^k bearing in mind that any (finite) degree of smoothness can be obtained. Similarly, we say that $F(U, R, \tau)$ is C^k in $(U, R, R \ln R, \tau)$ if we can choose a sufficiently smooth function f such that $F(U, R, \tau) = f(U, R, R \ln R, \tau)$.

5.2.1. *Study of $\mathcal{L}_Y(\bar{x}(\mathbf{U}, \mathbf{R}, \tau))$.* On account of (23) and (46) we get

$$\begin{aligned} \mathcal{L}_Y(\bar{x}(U, R, \tau)) &= \frac{1}{\epsilon^2} \exp - \frac{1}{\epsilon^2} \left(\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' + \epsilon^2 L \right) \\ &\quad \cdot \mathcal{L}_Y(\alpha + \beta \ln R + \epsilon^2 L), \end{aligned} \tag{55}$$

where α and β are the C^k -functions defined in (23), and $L(U, R, \tau)$ is the C^k -function in $(U, R, R \ln R, \tau)$ introduced in Theorem 5.9.

We can write $P = \mathcal{L}_Y(\alpha + \beta \ln R + \epsilon^2 L)$. In order to study P , we need the following easy properties of the Lie-derivation:

Lemma 5.10. 1. $\mathcal{L}_Y(\ln R) = -1$.

2. If $\tilde{\nu}(U, R, \tau) = \nu(U, R, R \ln R, \tau)$ is C^k in $(U, R, R \ln R, \tau)$, then $\zeta(U, R, \tau) = \mathcal{L}_Y \tilde{\nu}(U, R, \tau)$ is C^k in $(U, R, R \ln R, \tau)$ and moreover $\zeta(0, 0, \tau) \equiv 0$ (one can also write: $\zeta = o(1)$).

Taking into account Lemma 5.10 and the fact that $\beta(U, 0, \tau) = -A(0, 0, \tau)$ (see Proposition 5.2), we obtain that

$$\begin{aligned} P(U, R, \tau) &= \mathcal{L}_Y(\alpha + \beta \ln R + \epsilon^2 L) \\ &= \mathcal{L}_Y \alpha + (\ln R) \mathcal{L}_Y \beta - \beta + \epsilon^2 \mathcal{L}_Y L \\ &= -\beta(U, R, \tau) + o(1), \end{aligned} \tag{56}$$

where $o(1)$ is C^k in $(U, R, R \ln R, \tau)$. Based on (56), we find that $P(0, 0, \tau) = A(0, 0, \tau) > 0$. As a consequence, the function $P(U, R, \tau)$ remains strictly positive for U, R small enough and its logarithm is C^k in $(U, R, R \ln R, \tau)$. Hence (55) changes to

$$(\mathcal{L}_Y \bar{x}(U, R, \tau))(U, R, \tau) = \frac{1}{\epsilon^2} \exp - \frac{1}{\epsilon^2} \left(\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' + \epsilon^2 \bar{L}(U, R, \tau) \right), \tag{57}$$

where \bar{L} is C^k in $(U, R, R \ln R, \tau)$.

5.2.2. *Study of $\frac{\partial s_+}{\partial \bar{x}}(\bar{x}(\mathbf{U}, \mathbf{R}, \tau), \mathbf{U}, \mathbf{R}, \tau)$.* The derivative $\frac{\partial s_+}{\partial \bar{x}}$ can be expressed in terms of an integral of divergence:

$$\frac{\partial s_+}{\partial \bar{x}}(\bar{x}, r, \tau) = \frac{-\epsilon^2 \pi(\bar{x}, r, \tau)}{s_+(\bar{x}, r, \tau)} \exp \int_{O_{\bar{x}, r, \tau}} \text{div} X_\tau^{(1)} dt, \tag{58}$$

where $O_{\bar{x}, r, \tau}$ is the orbit through $(\bar{x}, \frac{1}{R_1^2}, r) \in \Sigma_+^2$ in positive time until it hits Σ_0^2 and where

$$\begin{aligned} \pi(\bar{x}, r, \tau) &= \epsilon B_0 - \bar{x} + B_2 \bar{x}^2 - \bar{x}^3 + r \bar{x}^4 \bar{H}(r \bar{x}, \lambda) \\ &\quad + r \left(\frac{1}{R_1^2} - \frac{1}{2} \bar{x}^2 \right)^2 G(r \bar{x}, r^2 \left(\frac{1}{R_1^2} - \frac{1}{2} \bar{x}^2 \right), \lambda). \end{aligned} \tag{59}$$

This follows directly from the following well known result (see Proposition 2 of [10]):

Proposition 5.11. *Let f be a vector field on an open subset of \mathbb{R}^n . Let S_1 and S_2 be two open sections of \mathbb{R}^n , transverse to the flow of f . Assume $p \in S_1, q \in S_2$ and the orbit through p reaches q in finite time. Let $T : S_0 \subset S_1 \rightarrow S_2$ be the transition map defined in a neighborhood of p . If $\phi_i : U_i \rightarrow S_i$ are coordinates for S_i with $U_i \subset \mathbb{R}^{n-1}$, then*

$$\det(D(\phi_2^{-1} \circ T \circ \phi_1))(s_1) = \frac{\det(D\phi_1(s_1)|f(p))}{\det(D\phi_2(s_2)|f(q))} \exp \left\{ \int_{O(p,q)} \text{div} f dt \right\},$$

where $s_1 = \phi_1^{-1}(p)$, $s_2 = \phi_2^{-1}(q)$, and where $(D\phi_1(s_1)|f(p))$ is a matrix composed of the $n \times (n - 1)$ matrix $D\phi_1(s_1)$ and the column vector $f(p)$, and similarly for $(D\phi_2(s_2)|f(q))$. The integral is taken over the orbit $O(p, q)$ from p to q parametrized by t .

We denote the integral in (58) by $\mathcal{I}(\bar{x}, r, \tau)$.

From (58), we finally get:

$$\frac{\partial s_+}{\partial \bar{x}}(\bar{x}(U, R, \tau), UR, \tau) = \frac{\pi(\bar{x}(U, R, \tau), UR, \tau)}{h_+(U, R, \tau)} \exp \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau), \quad (60)$$

where we used $-\epsilon^2 h_+(U, R, \tau) = s_+(\bar{x}(U, R, \tau), UR, \tau)$.

Remark 9. 1. Taking into account Theorem 5.9 and (59) we see that function $\pi(\bar{x}(U, R, \tau), UR, \tau)$ in (60) is a strictly negative C^k -function by taking ϵ , U and R small enough. Let us recall that R_1 is sufficiently small and fixed such that Ψ in (11) is strictly negative for $\bar{X} \sim \sqrt{2}$, $U \sim 0$, $|R| \leq R_1$, $\epsilon \sim 0$, $(B_0, B_2) \in \mathcal{B}$ and $\lambda \in \lambda$.

2. On account of Theorem 4.1, we see that $h_+(U, R, \tau)$ is C^k and strictly positive.

Remark 9 implies that $-\frac{\pi(\bar{x}(U, R, \tau), UR, \tau)}{h_+(U, R, \tau)}$ is strictly positive and C^k , and its logarithm is a C^k -function. Hence, (60) can be written as

$$\frac{\partial s_+}{\partial \bar{x}}(\bar{x}(U, R, \tau), UR, \tau) = -\exp \frac{1}{\epsilon^2} (\epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) + \epsilon^2 \tilde{L}(U, R, \tau)), \quad (61)$$

where \tilde{L} is a C^k -function.

5.2.3. *Combining $\mathcal{L}_Y(\bar{\mathbf{x}}(\mathbf{U}, \mathbf{R}, \tau))$ and $\frac{\partial s_+}{\partial \bar{\mathbf{x}}}(\bar{\mathbf{x}}(\mathbf{U}, \mathbf{R}, \tau), \mathbf{UR}, \tau)$.* Bearing in mind (57) and (61), (54) can be written as

$$\begin{aligned} (\mathcal{L}_Y(-\epsilon^2 h_+))(U, R, \tau) &= -\frac{1}{\epsilon^2} \exp \frac{1}{\epsilon^2} \left(-\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' \right. \\ &\quad \left. + \epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) + \epsilon^2 L^*(U, R, \tau) \right), \end{aligned} \quad (62)$$

where L^* is C^k in $(U, R, R \ln R, \tau)$.

5.2.4. *Lie-derivative of δ .* Let us recall that $\tau = (\epsilon, B_0, B_2, \lambda)$. We denote by τ^* the parameter $(\epsilon, -B_0, -B_2, \lambda)$.

Lemma 5.12. *Suppose that $f(U, R, \tau)$ is C^k and write $g(U, R, \tau) = f(-U, R, \tau^*)$. Then*

$$(\mathcal{L}_Y g)(U, R, \tau) = (\mathcal{L}_Y f)(-U, R, \tau^*).$$

Proof. We have:

$$\begin{aligned} (\mathcal{L}_Y g)(U, R, \tau) &= U \frac{\partial g}{\partial U}(U, R, \tau) - R \frac{\partial g}{\partial R}(U, R, \tau) \\ &= (-U) \frac{\partial f}{\partial U}(-U, R, \tau^*) - R \frac{\partial f}{\partial R}(-U, R, \tau^*) \\ &= (\mathcal{L}_Y f)(-U, R, \tau^*). \end{aligned}$$

□

Using (53), (62) and Lemma 5.12, we find the Lie-derivative of δ :

$$\begin{aligned}
 (\mathcal{L}_Y \delta)(U, R, \tau) &= \frac{1}{\epsilon^4} \exp \frac{1}{\epsilon^2} \left(- \int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' \right. \\
 &\quad \left. + \epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) + \epsilon^2 L^*(U, R, \tau) \right) \\
 &\quad - \frac{1}{\epsilon^4} \exp \frac{1}{\epsilon^2} \left(- \int_R^{R_1} \frac{A(\frac{-UR}{R'}, R', \tau^*)}{R'} dR' \right. \\
 &\quad \left. + \epsilon^2 \mathcal{I}(\bar{x}(-U, R, \tau^*), -UR, \tau^*) + \epsilon^2 L^*(-U, R, \tau^*) \right). \tag{63}
 \end{aligned}$$

It is clear that for $\epsilon > 0$ and $R > 0$ the equation $\{\mathcal{L}_Y \delta = 0\}$ is equivalent to

$$\begin{aligned}
 & - \int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' + \int_R^{R_1} \frac{A(\frac{-UR}{R'}, R', \tau^*)}{R'} dR' \\
 & + \epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) - \epsilon^2 \mathcal{I}(\bar{x}(-U, R, \tau^*), -UR, \tau^*) + \epsilon^2 \bar{L}^*(U, R, \tau) = 0, \tag{64}
 \end{aligned}$$

where \bar{L}^* is a C^k -function in $(U, R, R \ln R, \tau)$ which is identically zero for $U = B_0 = B_2 = 0$. Our goal is to solve the equation (64) on segment $\{(U, R); UR = r, U \sim 0, U > 0, R \sim 0, R > 0\}$, for each possible (r, τ) such that $r \sim 0$ and $r > 0$.

We first simplify (64). It is clear that $\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR'$ goes to $+\infty$ as $R \rightarrow 0$. We aim at controlling the difference $\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' - \int_R^{R_1} \frac{A(\frac{-UR}{R'}, R', \tau^*)}{R'} dR'$ in (64).

Lemma 5.13. *One has that*

$$\begin{aligned}
 & \int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \epsilon, B_0, B_2, \lambda)}{R'} dR' \\
 & = (1 + \epsilon f_1(\epsilon)) \int_R^{R_1} \frac{A(\frac{UR}{R'}, R', 0, B_0, B_2, \lambda)}{R'} dR' + \epsilon f_2(U, R, \tau), \tag{65}
 \end{aligned}$$

where f_1 is a C^k -function, depending only on ϵ , and where f_2 is a C^k -function in $(U, R, R \ln R, \tau)$.

Proof. The expression (21) implies that $A(U, R, 0, B_0, B_2, \lambda) > 0$. If we keep the notation τ for $(\epsilon, B_0, B_2, \lambda)$ and denote $(0, B_0, B_2, \lambda)$ by τ_0 , then we obtain that

$$A(U, R, \tau) = A(U, R, \tau_0) \left(1 + \epsilon F_1(\tau) + \epsilon R F_2(U, R, \tau) + \epsilon U F_3(U, R, \tau) \right), \tag{66}$$

where F_1, F_2 and F_3 are C^k . If we suppose that $(U, R) = (0, 0)$ in (66), then we get

$$A(0, 0, \tau) = A(0, 0, \tau_0) \left(1 + \epsilon F_1(\tau) \right) > 0. \tag{67}$$

Taking into account (19) and the fact that the parameters B_0, B_2 and λ are accompanied by U or R in the expression Ψ , defined in (11), the family of invariant manifolds ϕ , fixed in Remark 6, does not depend on (B_0, B_2, λ) , for $(U, R) = (0, 0)$. As Ψ and ϕ appear in the expression (21), it follows that $A(0, 0, \tau)$ does not depend on (B_0, B_2, λ) . Formula (67) now implies that $F_1(\tau)$ does not depend on (B_0, B_2, λ) , hence $F_1(\tau) = f_1(\epsilon)$.

It is clear that we can write (66) as

$$A(U, R, \tau) = A(U, R, \tau_0) \left(1 + \epsilon f_1(\epsilon) \right) + \epsilon R F_2(U, R, \tau) + \epsilon U F_3(U, R, \tau), \tag{68}$$

for some new C^k -functions F_2 and F_3 . Using (68), we have that

$$\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' = (1 + \epsilon f_1(\epsilon)) \int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau_0)}{R'} dR' + \epsilon \int_R^{R_1} \frac{R' F_2(\frac{UR}{R'}, R', \tau)}{R'} dR' + \epsilon \int_R^{R_1} \frac{UR F_3(\frac{UR}{R'}, R', \tau)}{R'^2} dR'. \tag{69}$$

Take any C^k -function $F_4(U, R, \tau)$. In the proof of Proposition 5.2, we found that

$$\int_R^{R_1} \frac{F_4(\frac{UR}{R'}, R', \tau)}{R'} dR' = \alpha^*(U, R, \tau) + \beta^*(U, R, \tau) \ln R, \tag{70}$$

where α^* and β^* are C^k and where $\beta^*(U, 0, \tau) = -F_4(0, 0, \tau)$. If we choose $F_4(U, R, \tau) = RF_2(U, R, \tau)$, then we get $\beta^*(U, 0, \tau) = 0$. In other words, $\ln R$ in (70) is accompanied by R . Hence the second integral on the right hand side of (69) is a C^k -function in $(U, R, R \ln R, \tau)$. If we take $F_4(U, R, \tau) = UF_3(U, R, \tau)$ in (70), then we get $\beta^*(U, 0, \tau) = 0$. Hence the third integral on the right hand side of (69) is a C^k -function in $(U, R, R \ln R, \tau)$. \square

Using (21) and Lemma 5.13, we find that

$$-\int_R^{R_1} \frac{A(\frac{UR}{R'}, R', \tau)}{R'} dR' + \int_R^{R_1} \frac{A(\frac{-UR}{R'}, R', \tau^*)}{R'} dR' = (1 + \epsilon f_1(\epsilon)) C(U, R, B_2, \lambda) + \epsilon \bar{f}_2(U, R, \tau), \tag{71}$$

where \bar{f}_2 is a C^k -function in $(U, R, R \ln R, \tau)$ and identically zero for $U = B_0 = B_2 = 0$, and where

$$C(U, R, B_2, \lambda) = -8\sqrt{2} \int_R^{R_1} \frac{B_2 R' + \frac{UR}{R'} \left(\bar{H}(\frac{UR}{R'} \sqrt{2}, \lambda) + \bar{H}(\frac{-UR}{R'} \sqrt{2}, \lambda) \right)}{R' D(U, R, R', B_2, \lambda) D(-U, R, R', -B_2, \lambda)} dR', \tag{72}$$

with

$$D(U, R, R', B_2, \lambda) = -\sqrt{2}R'^2 + 2B_2R' - 2\sqrt{2} + 4\frac{UR}{R'} \bar{H}(\frac{UR}{R'} \sqrt{2}, \lambda).$$

Taking into account (71), (64) changes to

$$\epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) - \epsilon^2 \mathcal{I}(\bar{x}(-U, R, \tau^*), -UR, \tau^*) + (1 + \epsilon f_1(\epsilon)) C(U, R, B_2, \lambda) + \epsilon \tilde{L}^*(U, R, \tau) = 0, \tag{73}$$

where \tilde{L}^* is a C^k -function in $(U, R, R \ln R, \tau)$ and identically zero for $U = B_0 = B_2 = 0$.

It can be easily seen that C , defined in (72), is bounded for $R \sim 0$. We make this statement precise in the following lemma:

Lemma 5.14.

$$C(U, R, B_2, \lambda) = B_2 c_1(U, R, B_2, \lambda) + U c_2(U, R, B_2, \lambda), \tag{74}$$

where c_1 and c_2 are C^k -functions in $(U, R, R \ln R, B_2, \lambda)$ and where c_1 is strictly negative for $R \sim 0$ and $R \geq 0$.

Proof. Using (72), we can write C as

$$C(U, R, B_2, \lambda) = B_2 \int_R^{R_1} C_1\left(\frac{UR}{R'}, R', B_2, \lambda\right) dR' + \int_R^{R_1} \frac{UR}{R'^2} C_2\left(\frac{UR}{R'}, R', B_2, \lambda\right) dR', \tag{75}$$

where C_1 and C_2 are C^∞ -functions. Bearing in mind (70), we see that the first integral in (75) is a C^k -function in $(U, R, R \ln R, B_2, \lambda)$. We denote it by $c_1(U, R, B_2, \lambda)$. It is clear from (72) that c_1 is strictly negative.

The second integral in (75) we can write as

$$UR \int_R^{R_1} \frac{1}{R'^2} C_2\left(\frac{UR}{R'}, 0, B_2, \lambda\right) dR' + UR \int_R^{R_1} \frac{1}{R'} C_3\left(\frac{UR}{R'}, R', B_2, \lambda\right) dR', \tag{76}$$

where C_3 is a C^∞ -function. The second expression in (76) can, by (70), be written as $UC_4(U, R, B_2, \lambda)$ where C_4 is a C^k -function in $(U, R, R \ln R, B_2, \lambda)$.

If we use the change of variables: $\frac{R}{R'} = w$, then the first expression in (76) changes to

$$U \int_{\frac{R}{R_1}}^1 C_2(Uw, 0, B_2, \lambda) dw.$$

The above integral can be written as $UC_5(U, R, B_2, \lambda)$, for some C^∞ -function C_5 . We define now $c_2(U, R, B_2, \lambda) = C_4(U, R, B_2, \lambda) + C_5(U, R, B_2, \lambda)$. □

It remains to simplify the difference of divergence integrals in (73). The integral of divergence \mathcal{I} has been studied in detail in [10], if the slow dynamics (9) has no zeros, and in [4], if the slow dynamics (9) has isolated saddle-node singularities located away from the contact point $(\bar{x}, \bar{y}) = (0, 0)$.

As we mentioned above, a saddle-node singularity appears in (9) for $(B_2, r) \sim (\pm 2, 0)$, located near $\bar{x} = \pm 1$. Hence we may rely on [4]. When parameter B_2 is kept in any compact set $K \subset]-2, 2[$, then (9) is regular in an arbitrary compact set in the \bar{x} -space by taking r small enough. As a consequence, we are allowed to refer to [10]. When $2 < |B_2| \leq B_2^0$, then one has simple zeros in the slow dynamics and hence limit cycles near Γ are not possible.

We are now in a good position to prove Theorems 4.3 and 4.4.

5.3. Proof of Theorem 4.3. The symmetries S_F and S_P defined, respectively, by (51) and (52) imply that it suffices to prove Theorem 4.3 for B_2 strictly positive. Based on the discussion above, we distinguish three possibilities.

5.3.1. $B_2 \in K \subset]0, 2[$, K is any compact set. Our goal is to show that, by taking $U \sim 0, R \sim 0, R \geq 0, \epsilon \sim 0$ and $\epsilon \geq 0$, the left hand side of the equation (73) is strictly negative for any $B_0 \sim 0, B_2 \in K$ and $\lambda \in \Lambda$. This implies that the equation $\{\mathcal{L}_Y \delta = 0\}$ has no solutions along $\{UR = r\}$, under the given conditions on the parameters and variables, and for $(R, \epsilon) > (0, 0)$. Using the Rolle's Theorem one finds that the cyclicity of Γ at $B_2 \in K$ is bounded by one.

Consider first the expression

$$\epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) - \epsilon^2 \mathcal{I}(\bar{x}(-U, R, \tau^*), -UR, \tau^*).$$

We may use Theorem 5.9 and results in [10], for $B_2 \in -K \cup K$, to write the function $\epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau)$ as

$$I(B_2, UR, \lambda) + \varphi_1(U, R, \tau) + \varphi_2(U, R, \tau) \epsilon^2 \ln \epsilon, \tag{77}$$

where φ_1 and φ_2 are C^k -functions, including at $\epsilon = 0$ and $R = 0$, $\varphi_1 = O(\epsilon)$ and where $I(B_2, r, \lambda)$ represents the slow divergence integral, defined as:

$$I(B_2, r, \lambda) = \int_0^{\frac{\sqrt{2}}{R_1}} \frac{wdw}{-1 + B_2w - w^2 + rw^3\bar{H}(rw, \lambda)} < 0. \tag{78}$$

Using (77) we obtain that

$$\begin{aligned} &\epsilon^2\mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) - \epsilon^2\mathcal{I}(\bar{x}(-U, R, \tau^*), -UR, \tau^*) \\ &= I(B_2, UR, \lambda) - I(-B_2, -UR, \lambda) + \bar{\varphi}_1(U, R, \tau) + \bar{\varphi}_2(U, R, \tau)\epsilon^2 \ln \epsilon, \end{aligned} \tag{79}$$

where $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are C^k -functions, including at $\epsilon = 0$ and $R = 0$, and $\bar{\varphi}_1 = O(\epsilon)$.

Taking into account (79) and Lemma 5.14 the equation (73) can be written as

$$\begin{aligned} &I(B_2, UR, \lambda) - I(-B_2, -UR, \lambda) + C(U, R, B_2, \lambda) \\ &+ \epsilon\tilde{L}^*(U, R, \tau) + \bar{\varphi}_2(U, R, \tau)\epsilon^2 \ln \epsilon = 0, \end{aligned} \tag{80}$$

for some new C^k -function \tilde{L}^* in $(U, R, R \ln R, \tau)$.

Remark 10. When $B_2 \sim 0$, we also deal with a regular slow dynamics and we may hence use (80), where \tilde{L}^* and $\bar{\varphi}_2$ are identically zero by taking $U = B_0 = B_2 = 0$ (see Section 5.4).

Note that the left hand side of the equation (80) can be treated as a continuous (including $\epsilon = 0$) perturbation of $I(B_2, UR, \lambda) - I(-B_2, -UR, \lambda) + C(U, R, B_2, \lambda)$. Hence it suffices to show that the first line in (80) is strictly negative for $B_2 \in K$, $\lambda \in \Lambda$, $U \sim 0$, $R \sim 0$ and $R \geq 0$.

Using the change of variables $w = -w'$ in $I(-B_2, -UR, \lambda)$, it can be easily seen that

$$\begin{aligned} &I(B_2, UR, \lambda) - I(-B_2, -UR, \lambda) \\ &= \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{wdw}{-1 + B_2w - w^2 + URw^3\bar{H}(URw, \lambda)}. \end{aligned} \tag{81}$$

Consider now

$$\tilde{I}(\bar{x}, B_2, r, \lambda) = \int_{-\bar{x}}^{\bar{x}} \frac{wdw}{-1 + B_2w - w^2 + rw^3\bar{H}(rw, \lambda)}, \tag{82}$$

for $B_2 \in K$, $\lambda \in \Lambda$ and $r \sim 0$. From Lemma 3.1 in [8] we know that for any $\rho > 0$ small there exist $r_0 > 0$ and $\nu > 0$ sufficiently small such that $\tilde{I}(\bar{x}, B_2, r, \lambda) \leq -\nu$ for $\bar{x} \in [\rho, \frac{1}{\rho}]$, $B_2 \in K$, $r \in [-r_0, r_0]$ and $\lambda \in \Lambda$. If we take $\rho = \frac{R_1}{\sqrt{2}}$, we get that (81) is strictly negative for $B_2 \in K$, $\lambda \in \Lambda$, $U \sim 0$, $R \sim 0$ and $R \geq 0$.

To see that the first line in (80) is strictly negative for $B_2 \in K$, $\lambda \in \Lambda$, $U \sim 0$, $R \sim 0$ and $R \geq 0$, it suffices to observe that $C(U, R, B_2, \lambda)$ is, by Lemma 5.14, strictly negative for $B_2 \in K$, $\lambda \in \Lambda$, $U \sim 0$, $R \sim 0$ and $R \geq 0$. Hence the cyclicity of δ at $B_2 \in K$ is bounded by one.

To see that the cyclicity of Γ at $B_2 \in K$ is precisely one, we use Theorem 4.1. We claim that there exists a C^k -function $b_0(U, R, \epsilon, B_2, \lambda) \sim 0$, for $B_2 \in K$, $\lambda \in \Lambda$, $(U, R, \epsilon) \sim (0, 0, 0)$, $R \geq 0$ and $\epsilon \geq 0$, such that zeros of δ will occur for $B_0 = b_0(U, R, \epsilon, B_2, \lambda)$. In fact, since δ is C^k (see Theorem 4.1), $\delta(U, R, 0, 0, B_2, \lambda) = 0$ and $\frac{\partial \delta}{\partial B_0}(U, R, 0, 0, B_2, \lambda) \neq 0$ (B_0 is a breaking parameter), the implicit function theorem implies existence of unique C^k -function $b_0(U, R, \epsilon, B_2, \lambda)$ such that solution

of $\delta = 0$, for $\epsilon \sim 0$ and $B_0 \sim 0$, can only occur for $B_0 = b_0(U, R, \epsilon, B_2, \lambda)$. Hence the cyclicity of Γ at $B_2 \in K$ is at least one.

Since the left hand side of the equation (80) is strictly negative, we deal with a hyperbolically attracting limit cycle.

5.3.2. **B₂ ~ 2.** We invoke Lemma 5.14 and see that the function C in (73) is bounded. Of course we have that $\epsilon \tilde{L}^*$ introduced in (73) is bounded due to the fact that \tilde{L}^* is C^k in $(U, R, R \ln R, \tau)$. It remains to study $\epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau) - \epsilon^2 \mathcal{I}(\bar{x}(-U, R, \tau^*), -UR, \tau^*)$, i.e. the first line in (73).

Since the slow dynamics (9) is regular for $\bar{x} \leq 0$, the paper [4] shows us that the contribution $\epsilon^2 \mathcal{I}(\bar{x}(-U, R, \tau^*), -UR, \tau^*)$ is bounded. Using [4] once more we have that $\epsilon^2 \mathcal{I}(\bar{x}(U, R, \tau), UR, \tau)$ is dominated by the contribution of the saddle-node singularity and it tends to $-\infty$ as $(U, R, B_2, \epsilon) \rightarrow (0, 0, 2, 0)$.

Hence we find that the cyclicity of Γ at $B_2 = 2$ is bounded by one. Since the left hand side of (73) is always negative for (U, R, B_2, ϵ) near $(0, 0, 2, 0)$, we deal with a hyperbolically attracting limit cycle. We refer to Section 5.3.1 to see that the cyclicity of Γ at $B_2 = 2$ is one.

5.3.3. **2 < B₂ ≤ B₂⁰.** The cyclicity of Γ in this case is zero.

5.4. **Proof of Theorem 4.4.** We suppose that $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. Our goal is to solve equation (73) along the segment $\{(U, R) | UR = r, U \sim 0, U > 0, R \sim 0, R > 0\}$ for each fixed $(r, \epsilon, B_0, B_2) \sim (0, 0, 0, 0)$, $r > 0$, $\epsilon \geq 0$ and $\lambda \in \Lambda$. Remark 10 implies that the equation (73), for $B_2 \sim 0$, can be written as

$$I(B_2, UR, \lambda) - I(-B_2, -UR, \lambda) + C(U, R, B_2, \lambda) + \tilde{L}(U, R, \tau) = 0, \tag{83}$$

where $\tilde{L}(U, R, \tau)$ is $O(\epsilon)$, C^k in $(U, R, R \ln R, \tau, \epsilon^2 \ln \epsilon)$ and identically zero by taking $U = B_0 = B_2 = 0$. Hence, we can write

$$\tilde{L}(U, R, \tau) = UG_1(U, R, \tau) + B_0G_2(U, R, \tau) + B_2G_3(U, R, \tau), \tag{84}$$

where G_i , $i = 1, 2, 3$, is $O(\epsilon)$ and C^k in $(U, R, R \ln R, \tau, \epsilon^2 \ln \epsilon)$.

When $U = B_2 = 0$, then $I(B_2, UR, \lambda) - I(-B_2, -UR, \lambda)$ and $C(U, R, B_2, \lambda)$ are identically zero. Based on (81), we have that

$$\begin{aligned} I(B_2, UR, \lambda) - I(-B_2, -UR, \lambda) &= B_2 \left(- \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^2 dw}{(1+w^2)^2} + I_1(U, R, \tau) \right) \\ &+ UR \left(- \bar{H}(0, \lambda) \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^4 dw}{(1+w^2)^2} + I_2(U, R, \tau) \right), \end{aligned} \tag{85}$$

where I_1 and I_2 are $O(UR, B_2)$ and C^∞ . Using (72) we obtain

$$\begin{aligned} C(U, R, B_2, \lambda) &= B_2 \left(-4\sqrt{2} \int_R^{R_1} \frac{dR'}{(2+R'^2)^2} + \eta_1(U, R, \tau) \right) \\ &+ U \left(-8\sqrt{2}\bar{H}(0, \lambda) \int_R^{R_1} \frac{RdR'}{R'^2(2+R'^2)^2} + \eta_2(U, R, \tau) \right), \end{aligned} \tag{86}$$

where η_1 and η_2 are $O(U, B_2)$ and C^k in $(U, R, R \ln R, \tau)$ (see Lemma 5.14).

Taking into account (84), (85) and (86), (83) changes to:

$$\begin{aligned}
 U & \left(-\bar{H}(0, \lambda)R \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^4 dw}{(1+w^2)^2} - 8\sqrt{2}\bar{H}(0, \lambda) \int_R^{R_1} \frac{RdR'}{R'^2(2+R'^2)^2} + G_1(U, R, \tau) \right) \\
 & + B_2 \left(- \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^2 dw}{(1+w^2)^2} - 4\sqrt{2} \int_R^{R_1} \frac{dR'}{(2+R'^2)^2} + G_3(U, R, \tau) \right) \\
 & + B_0 G_2(U, R, \tau) = 0,
 \end{aligned} \tag{87}$$

for some new functions G_1, G_2, G_3 that are $O(U, B_2, \epsilon)$ and C^k in the variable $(U, R, R \ln R, \tau, \epsilon^2 \ln \epsilon)$.

Lemma 5.15. *We have that*

$$(i) \quad -R \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^4 dw}{(1+w^2)^2} - 8\sqrt{2} \int_R^{R_1} \frac{RdR'}{R'^2(2+R'^2)^2} = -2\sqrt{2} + f_1(R)$$

and

$$(ii) \quad - \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^2 dw}{(1+w^2)^2} - 4\sqrt{2} \int_R^{R_1} \frac{dR'}{(2+R'^2)^2} = -\frac{\pi}{2} + f_2(R),$$

where f_1 and f_2 are C^∞ -functions in R and identically zero when $R = 0$.

Proof. We get

$$\begin{aligned}
 & -R \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^4 dw}{(1+w^2)^2} - 8\sqrt{2} \int_R^{R_1} \frac{RdR'}{R'^2(2+R'^2)^2} \\
 & = -R \left(\int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^4 dw}{(1+w^2)^2} + 2 \int_{\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R}} \frac{w^4 dw}{(1+w^2)^2} \right) = -R \int_{-\frac{\sqrt{2}}{R}}^{\frac{\sqrt{2}}{R}} \frac{w^4 dw}{(1+w^2)^2} \\
 & = -\frac{\sqrt{2}(4+3R^2)}{(2+R^2)} + 3R \operatorname{Arcctg}\left(\frac{R}{\sqrt{2}}\right) = -2\sqrt{2} + O(R),
 \end{aligned}$$

where $O(R)$ is a C^∞ -function in R . In the first step we used the change of variable: $w = \frac{\sqrt{2}}{R'}$. Similarly, we obtain

$$\begin{aligned}
 & - \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^2 dw}{(1+w^2)^2} - 4\sqrt{2} \int_R^{R_1} \frac{dR'}{(2+R'^2)^2} \\
 & = - \int_{-\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R_1}} \frac{w^2 dw}{(1+w^2)^2} - 2 \int_{\frac{\sqrt{2}}{R_1}}^{\frac{\sqrt{2}}{R}} \frac{w^2 dw}{(1+w^2)^2} = - \int_{-\frac{\sqrt{2}}{R}}^{\frac{\sqrt{2}}{R}} \frac{w^2 dw}{(1+w^2)^2} \\
 & = \frac{\sqrt{2}R}{2+R^2} - \operatorname{Arcctg}\left(\frac{R}{\sqrt{2}}\right) = -\frac{\pi}{2} + O(R),
 \end{aligned}$$

where $O(R)$ is a C^∞ -function in R . □

With the help of Lemma 5.15 we infer that the equation (87) can be written as

$$\begin{aligned}
 U & \left(-\bar{H}(0, \lambda)2\sqrt{2} + G_1(U, R, \tau) \right) + B_0 G_2(U, R, \tau) \\
 & + B_2 \left(-\frac{\pi}{2} + G_3(U, R, \tau) \right) = 0,
 \end{aligned} \tag{88}$$

for some new functions G_1, G_2, G_3 that are $O(U, R, B_2, \epsilon)$ and C^k in the variable $(U, R, R \ln R, \tau, \epsilon^2 \ln \epsilon)$. In order to be able to apply an algorithm of derivation-division to the left hand side of (88), we have to get rid of the parameter B_0 in (88).

Again we invoke Theorem 4.1 and obtain that there exists a unique C^k -function $b_0(U, R, \epsilon, B_2, \lambda)$, for $(U, R, \epsilon, B_2) \sim (0, 0, 0, 0)$ and $\lambda \in \Lambda$, such that solutions of $\delta(U, R, \tau) = 0$, for $\epsilon \sim 0$ and $B_0 \sim 0$, can only occur for $B_0 = b_0(U, R, \epsilon, B_2, \lambda)$. Hence, it is sufficient to solve the following equation, with respect to variable R ($R > 0, R \sim 0$):

$$\delta\left(\frac{U_2 R_2}{R}, R, \epsilon, b_0(U_2, R_2, \epsilon, B_2, \lambda), B_2, \lambda\right) = 0, \tag{89}$$

for each fixed $(U_2, R_2) > (0, 0)$, $(U_2, R_2) \sim (0, 0)$, $\epsilon > 0, \epsilon \sim 0, B_2 \sim 0$ and $\lambda \in \Lambda$. We will prove that there exist $R_P > 0, U_P > 0, \epsilon_0 > 0, \tilde{B}_2 > 0$ sufficiently small such that for each $U_2 \in]0, U_P[, R_2 \in]0, R_P[, \epsilon \in]0, \epsilon_0], B_2 \in [-\tilde{B}_2, \tilde{B}_2]$ and $\lambda \in \Lambda$ the equation (89) has at most two solutions (counting with multiplicity) w.r.t. $R \in [\frac{U_2 R_2}{U_P}, R_P]$.

It can be easily seen that

$$-R \frac{\partial}{\partial R} \left(\delta\left(\frac{U_2 R_2}{R}, R, \tau\right) \right) = (\mathcal{L}_Y \delta) \left(\frac{U_2 R_2}{R}, R, \tau \right). \tag{90}$$

Taking into account that $\{\mathcal{L}_Y \delta = 0\}$ is equivalent, for $R > 0$ and $\epsilon > 0$, to the equation (88), (90) implies that $\left\{ \frac{\partial}{\partial R} \left(\delta\left(\frac{U_2 R_2}{R}, R, \epsilon, b_0(U_2, R_2, \epsilon, B_2, \lambda), B_2, \lambda\right) \right) = 0 \right\}$ is equivalent, for $R > 0$ and $\epsilon > 0$, to

$$\begin{aligned} \frac{U_2 R_2}{R} & \left(-\bar{H}(0, \lambda) 2\sqrt{2} + G_1\left(\frac{U_2 R_2}{R}, R, \epsilon, b_0(U_2, R_2, \epsilon, B_2, \lambda), B_2, \lambda\right) \right) \\ & + b_0(U_2, R_2, \epsilon, B_2, \lambda) G_2\left(\frac{U_2 R_2}{R}, R, \epsilon, b_0(U_2, R_2, \epsilon, B_2, \lambda), B_2, \lambda\right) \\ & + B_2 \left(-\frac{\pi}{2} + G_3\left(\frac{U_2 R_2}{R}, R, \epsilon, b_0(U_2, R_2, \epsilon, B_2, \lambda), B_2, \lambda\right) \right) = 0, \end{aligned} \tag{91}$$

where G_1, G_2, G_3 are identical to G_1, G_2, G_3 introduced in (88).

Since $\delta(0, R, \epsilon, 0, 0, \lambda) = 0$ and $\delta(U, 0, \epsilon, 0, 0, \lambda) = 0$, we find that

$$b_0(0, R, \epsilon, 0, \lambda) = 0 \text{ and } b_0(U, 0, \epsilon, 0, \lambda) = 0.$$

Hence we can write b_0 as:

$$b_0(U_2, R_2, \epsilon, B_2, \lambda) = U_2 R_2 \bar{b}_0(U_2, R_2, \epsilon, B_2, \lambda) + B_2 \tilde{b}_0(U_2, R_2, \epsilon, B_2, \lambda), \tag{92}$$

where \bar{b}_0 and \tilde{b}_0 are C^k -functions.

Denote by ξ parameter $(U_2, R_2, \epsilon, B_2, \lambda)$. Keeping in mind (92) the equation (91) changes to

$$\frac{U_2 R_2}{R} \left(-\bar{H}(0, \lambda) 2\sqrt{2} + \bar{G}_1(R, \xi) \right) + B_2 \left(-\frac{\pi}{2} + \bar{G}_2(R, \xi) \right) = 0, \tag{93}$$

where \bar{G}_1 and \bar{G}_2 are $O(\frac{U_2 R_2}{R}, R, B_2, \epsilon)$ and C^k in $(\frac{U_2 R_2}{R}, R, R \ln R, \xi, \epsilon^2 \ln \epsilon)$.

Let us now recall the notion of a strict Chebyshev system of C^k -functions of degree one as it was introduced in [2].

Definition 5.16. Let $\mathcal{F} = \{f_0, f_1\}$ be a sequence of C^k -functions defined on an interval $[a, b] \subset \mathbb{R}$. One says that \mathcal{F} is a strict Chebyshev system (in short, ST-system) on $[a, b]$ (of degree one) if one has that $f_0 \neq 0$ for all $z \in [a, b]$ and $(\frac{f_1}{f_0})'(z) \neq 0$ for all $z \in [a, b]$.

In the next proposition, we give an essential property of ST-systems (of degree one) that has been proven in [2].

Proposition 5.17. *Let $\mathcal{F} = \{f_0, f_1\}$ be a ST-system on $[a, b]$. Let $(\alpha_0, \alpha_1) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and let $f = \alpha_0 f_0 + \alpha_1 f_1$. Then the function f has at most one simple zero on $[a, b]$.*

Let us write

$$f_0(R, \xi) = -\frac{\pi}{2} + \bar{G}_2(R, \xi)$$

and

$$f_1(R, \xi) = R^{-1} \left(-\bar{H}(0, \lambda)2\sqrt{2} + \bar{G}_1(R, \xi) \right).$$

In what follows, we prove that there exist $R_P > 0$, $U_P > 0$, $\epsilon_0 > 0$, $\tilde{B}_2 > 0$ sufficiently small such that for each $U_2 \in]0, U_P[$, $R_2 \in]0, R_P[$, $\epsilon \in [0, \epsilon_0]$, $B_2 \in [-\tilde{B}_2, \tilde{B}_2]$ and $\lambda \in \Lambda$ the system $\{f_0(R, (U_2, R_2, \epsilon, B_2, \lambda)), f_1(R, (U_2, R_2, \epsilon, B_2, \lambda))\}$ is a ST-system of degree one in the variable $R \in [\frac{U_2 R_2}{U_P}, R_P]$.

It is clear that $f_0 < 0$ for $(\frac{U_2 R_2}{R}, R, B_2, \epsilon) \sim (0, 0, 0, 0)$. We have

$$\frac{f_1}{f_0} = R^{-1} \left(\vartheta_1 + \tilde{G}_1(R, \xi) \right) \tag{94}$$

where $\vartheta_1 = \frac{\bar{H}(0, \lambda)4\sqrt{2}}{\pi}$ and where \tilde{G}_1 is an $O(\frac{U_2 R_2}{R}, R, B_2, \epsilon)$ -function and C^k in $(\frac{U_2 R_2}{R}, R, R \ln R, \xi, \epsilon^2 \ln \epsilon)$. We can write

$$\tilde{G}_1(R, \xi) = \tilde{g}_1(\frac{U_2 R_2}{R}, R, R \ln R, \xi, \epsilon^2 \ln \epsilon),$$

where \tilde{g}_1 is a C^k -function. Applying $\frac{\partial}{\partial R}$ to (94) we get

$$\begin{aligned} R^{-1} \left(-\frac{U_2 R_2}{R^2} \partial_1 \tilde{g}_1 + \partial_2 \tilde{g}_1 + (1 + \ln R) \partial_3 \tilde{g}_1 \right) - R^{-2} (\vartheta_1 + \tilde{g}_1) \\ = R^{-2} \left(-\vartheta_1 - \frac{U_2 R_2}{R} \partial_1 \tilde{g}_1 + R \partial_2 \tilde{g}_1 + R(1 + \ln R) \partial_3 \tilde{g}_1 - \tilde{g}_1 \right), \end{aligned} \tag{95}$$

where \tilde{g}_1 and its derivatives are calculated in $(\frac{U_2 R_2}{R}, R, R \ln R, \xi, \epsilon^2 \ln \epsilon)$. The expression in (95) is non-zero by taking $(\frac{U_2 R_2}{R}, R, \epsilon, B_2) \sim (0, 0, 0, 0)$.

Using the Rolle’s Theorem one finds that the cyclicity of δ at $B_2 = 0$ is bounded by two. This completes the proof of Theorem 4.4.

REFERENCES

[1] P. De Maesschalck and F. Dumortier, *Slow-fast Bogdanov-Takens bifurcations*, *J. Differential Equations*, **250** (2011), 1000–1025.
 [2] F. Dumortier and R. Roussarie, *Birth of canard cycles*, *Discrete Contin. Dyn. Syst. Ser. S*, **2** (2009), 723–781.
 [3] P. De Maesschalck and F. Dumortier, *Singular perturbations and vanishing passage through a turning point*, *J. Differential Equations*, **248** (2010), 2294–2328.
 [4] P. De Maesschalck and F. Dumortier, *Canard cycles in the presence of slow dynamics with singularities*, *Proc. Roy. Soc. Edinburgh Sect. A*, **138** (2008), 265–299.
 [5] F. Dumortier, R. Roussarie, J. Sotomayor, and H. Zoladek, *Bifurcations of Planar Vector Fields*, *Lecture Notes in Mathematics*, **1480**, Springer-Verlag, Berlin, 1991.
 [6] Freddy Dumortier, *Compactification and desingularization of spaces of polynomial Liénard equations*, *J. Differential Equations*, **224** (2006), 296–313.
 [7] Robert Roussarie, *Putting a boundary to the space of Liénard equations*, *Discrete Contin. Dyn. Syst.*, **17** (2007), 441–448.
 [8] R. Huzak, P. De Maesschalck and F. Dumortier, *Limit cycles in slow-fast codimension 3 saddle and elliptic bifurcations*, *J. Differential Equations*, **255** (2013), 4012–4051.

- [9] Peter De Maesschalck and Freddy Dumortier, [Detectable canard cycles with singular slow dynamics of any order at the turning point](#), *Discrete Contin. Dyn. Syst.*, **29** (2011), 109–140.
- [10] P. De Maesschalck and F. Dumortier, [Time analysis and entry-exit relation near planar turning points](#), *J. Differential Equations*, **215** (2005), 225–267.
- [11] P. Bonckaert, P. De Maesschalck and F. Dumortier, [Well adapted normal linearization in singular perturbation problems](#), *J. Dynam. Differential Equations*, **23** (2011), 115–139.
- [12] M. Krupa and P. Szmolyan, [Relaxation oscillation and canard explosion](#), *J. Differential Equations*, **174** (2001), 312–368.
- [13] Daniel Panazzolo, [Desingularization of nilpotent singularities in families of planar vector fields](#), *Mem. Amer. Math. Soc.*, **158** (2002).
- [14] J. Llibre, [A survey on the limit cycles of the generalized polynomial Liénard differential equations](#), *Mathematical models in engineering, biology and medicine, AIP Conf. Proc.*, **1124** (2009), 224–233.

Received October 2013; revised April 2014.

E-mail address: renato.huzak@uhasselt.be

E-mail address: peter.demaesschalck@uhasselt.be

E-mail address: freddy.dumortier@uhasselt.be