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Characterizations of Nuclearity in Fréchet Spaces

L. EGGHE

Limburgs Universitair Centrum, B-3610 Diepenbeek, Belgium

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We extend the result of A. Bellow (*Proc. Nat. Acad. Sci. USA* **73**, No. 6 (1976), 1798–1799) on the characterization of finite-dimensional Banach spaces, to a characterization of nuclearity for Fréchet spaces. Those spaces are nuclear iff every Pettis-bounded and Pettis-uniformly integrable amart is mean convergent. Several other characterizations are given.

INTRODUCTION, TERMINOLOGY, AND NOTATIONS

Let (Ω, Σ, μ) be a probability space, and X a Fréchet space. A function $f: \Omega \to X$ is called μ-integrable if there is a sequence $(f_n)_{n=1}^{\infty}$ of step functions from Ω into X such that

(i) $\lim_{n\to\infty} f_n(\omega) = f(\omega) = \mu$-a.e.;

(ii) $\lim_{n\to\infty} \int_\Omega p(f_n - f)\, d\mu = 0$, for every continuous seminorm p on X.

It makes sense to define $\int_A f\, d\mu = \lim_{n\to\infty} \int_A f_n\, d\mu$. We define $L_X{}^1(\mu)$ as the space of classes $[f]$, where f is μ-integrable and $g \in [f]$ iff $g = f$, μ-a.e. We put a "mean" topology on $L_X{}^1(\mu)$ by means of

$$q(f) = \int_\Omega p(f)\, d\mu,$$

where p is an arbitrary continuous seminorm on X. So $L_X{}^1(\mu)$ becomes a Fréchet space. Another topology on $L_X{}^1(\mu)$ is used, the Pettis topology: Let U be an arbitrary zero neighborhood, and f in $L_X{}^1(\mu)$. Put

$$P_U(f) = \sup_{x' \in U^\circ} \int_\Omega |x'(f)|\, d\mu$$

where U° denotes the polar of U, w.r.t. the duality $\langle X, X' \rangle$. We call a sequence

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$(f_n)_{n=1}^{\infty}$ in $L_X^1(\mu)$, Pettis uniformly integrable if for every zero neighborhood U we have

$$\lim_{\mu(E)\to 0} \sup_{x' \in U^\circ} \int_E |x'(f_n)|\, d\mu = 0$$

uniformly in n.

Let $(\Sigma_n)_{n=1}^{\infty}$ be an increasing sequence of sub-σ-algebras of Σ. A sequence $(f_n, \Sigma_n)_{n=1}^{\infty}$ is called adapted to (Σ_n) in each f_n is Σ_n-measurable. A stopping time τ is a function of Ω into $N \cup \{\infty\}$ such that $\{\tau = n\} \in \Sigma_n$ for every n. We denote T the set of all bounded stopping times on Ω, directed in the natural way. Let $(f_n, \Sigma_n)_{n=1}^{\infty}$ be an adapted sequence. We write f_τ for the function $f_{\tau(\omega)}(\omega)$. $(f_n, \Sigma_n)_{n=1}^{\infty}$ is called an X-valued amart if each f_n is μ-integrable and

$$\left(\int_\Omega f_\tau\, d\mu\right)_{\tau \in T}$$

converges in X. Further information on amarts in Banach spaces is found in [1, 5].

In [2], Bellow proved the following.

THEOREM A. *For a Banach space X the following assertions are equivalent*:

(1) *X is of finite dimension.*

(2) *Every X-valued amart $(f_n, \Sigma_n)_{n=1}^{\infty}$, such that* $\sup_{\tau \in T} \int \|f_\tau\|\, d\mu < \infty$, *converges to a limit strongly a.e.*

(3) *Every X-valued amart $(f_n, \Sigma_n)_{n=1}^{\infty}$, such that $\|f_n(\omega)\| \leqslant 1$ for every n in N and ω in Ω, converges to a limit strongly a.e..*

We may also replace "strongly a.e." by "in the mean," in (3). The key tool in Bellow's proof is the lemma of Dvoretzky and Rogers. This is not very useful in more general spaces because of the norm-inequality in this lemma. A good theorem to use in the general setting is the theorem of Dvoretzky and Rogers on unconditional convergent and absolutely convergent series. This theorem has an extension, to characterize nuclearity in Fréchet spaces (see [10, 4.2.5]). It is this theorem we will use for extending Theorem A to Fréchet spaces.

1. THE THEOREM

THEOREM B. *Let X be a Fréchet space. The following assertions are equivalent.*

(i) *X is nuclear.*

(ii) *Every Pettis-bounded and Pettis-uniformly integrable amart $(f_n, \Sigma_n)_{n=1}^{\infty}$, in L_X^1 is L_X^1-convergent (i.e.: mean convergent).*

(iii) *For every Pettis-bounded and Pettis-uniformly integrable amart* $(f_n, \Sigma_n)_{n=1}^{\infty}$ *in* $L_X{}^1$, *there is a martingale* $(g_n, \Sigma_n)_{n=1}^{\infty}$ *which is* $L_X{}^1$*-convergent, such that* $f_n = g_n + h_n$, *with* $h_n \to 0$ *in* $L_X{}^1$*-sense.*

(iv) *On* $L_X{}^1$ *the Pettis topology is the same as the mean topology.*

(v) *Every Pettis-convergent amart is* $L_X{}^1$*-convergent.*

Proof. (ii) $\Rightarrow$ (i). Suppose that X is not nuclear. By [10, 4.2.5], there is a sequence $(x_n)_{n=1}^{\infty}$ in X which is summable and not absolutely summable. Hence, there is a continuous seminorm p on X such that for every n in N, there is an $m > n$ such that $\sum_{k=n+1}^{m} p(x_k) > 1$.

We build inductively a new sequence $(y_n)_{n=1}^{\infty}$.

(1) Let $n_1 \in \mathbb{N}$ be the smallest natural number such that

$$\alpha_1 = \sum_{k=1}^{n_1} p(x_k) > 1.$$

Hence $\sum_{k=1}^{n_1} p(y_k) = 1$, with $y_k = x_k/\alpha_1$ $(k = 1, \ldots, n_1)$. Call $J^{(1)} = \{1, \ldots, n_1\}$. Let J_1 be the smallest set of consecutive natural numbers, starting with $n_1 + 1 = \max J^{(1)} + 1$ such that

$$\alpha_1 = \sum_{k \in J_1} p(x_k) > p(y_1).$$

Hence $\sum_{k \in J_1} p(y_k) = p(y_1)$, with $y_k = (p(y_1)/\alpha_1) \cdot x_k$ $(k \in J_1)$.

(2) Let J_{n_1} be a set consisting of consecutive natural numbers, starting with $\max J_{n_1 - 1} + 1$, such that

$$\alpha_{n_1} = \sum_{k \in J_{n_1}} p(x_k) > p(y_{n_1}).$$

Hence $\sum_{k \in J_{n_1}} p(y_k) = p(y_{n_1})$, with $y_k = (p(y_{n_1})/\alpha_{n_1}) \cdot x_k$ $(k \in J_{n_1})$. Call

$$J^{(2)} = \bigcup_{i_1=1}^{n_1} J_{i_1} = \bigcup_{i_1 \in J^{(1)}} J_{i_1}.$$

We can do the same with $J^{(2)}$ that we did with $J^{(1)}$: first with J_1, then with J_2, and so on until J_{n_1}. This gives respectively:

$$\{J_{1,i_2} \,\|\, i_2 \in J_1\}, \{J_{2,i_2} \,\|\, i_2 \in J_2\}, \ldots, \{J_{n_1,i_2} \,\|\, i_2 \in J_{n_1}\}.$$

We denote

$$J^{(3)} = \bigcup_{i_1 \in J^{(1)}} \bigcup_{i_2 \in J_{i_1}} J_{i_1,i_2},$$

and so on. The inductive step is clear. To write it down explicitly would be confusing, because of the intricate indices appearing! It is now trivial that $\{J^{(n)} \| n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$. Since

$$\sum_{k \in J^{(n)}} p(y_k) = 1$$

for every $n \in \mathbb{N}$, we have that

$$\sum_{n=1}^{\infty} p(y_n) = \infty.$$

Furthermore, since $p(y_n) \leqslant p(x_n)$, for every $n \in \mathbb{N}$, and by [10, pp. 23–26]: $(y_n)_{n=1}^{\infty}$ is summable. We again call this sequence $(x_n)_{n=1}^{\infty}$. So our sequence $(x_n)_{n=1}^{\infty}$ satisfies

$$\sum_{i_0 \in J^{(1)}} p(x_{i_0}) = 1 \qquad (\alpha_1)$$

$$\sum_{i_1 \in J_1} p(x_{i_1}) = p(x_1) \qquad (\beta_1)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\sum_{i_1 \in J_{n_1}} p(x_{i_1}) = p(x_{n_1}) \qquad (\beta_{n_1})$$

$$\sum_{i_2 \in J_{1,1}} p(x_{i_2}) = p(x_{n_1+1}) \qquad (\gamma_1)$$

and so on.

The formulas (α_1); (β_1),..., $(\beta_{n'})$; (γ_1),... indicate the way to divide $[0, 1)$ into intervals of the same form. As a matter of fact,

$$\pi_1 = \{A_1, ..., A_{n_1}\},$$

where $A_1 = [0, p(x_1))$, $A_2 = [p(x_1), p(x_1) + p(x_2))$,..., $\pi_2 = \{A_k \| k \in J^{(2)}\} \geqslant \pi_1$, where $A_{n_1+1} = [0, p(x_{n_1+1}))$, $A_{n_1+2} = [p(x_{n_1+1}), p(x_{n_1+1}) + p(x_{n_1+2}))$,... In general,

$$\pi_n = \{A_k \| k \in J^{(n)}\}.$$

We have of course: $\pi_n \geqslant \pi_m$ iff $n \geqslant m$. Define

$$f_n = \sum_{k \in J^{(n)}} \frac{x_k}{p(x_k)} \chi_{A_k}$$

for every n in $\mathbb{N}$. Denote $\Sigma_n = \sigma(\pi_n)$ for every n in $\mathbb{N}$. So we have constructed the sequence $(f_n, \Sigma_n)_{n=1}^{\infty}$. It is an amart, which converges in the Pettis topology to zero. We even can prove that it converges to zero in Pettis-sense, for stopping times:

So, if T denotes the directed set of stopping times, corresponding to $(\Sigma_i)_{i=1}^{\infty}$, then

$$\lim_{\tau\in T} \sup_{x'\in U^\circ} \int_0^1 |x'(f_\tau)|\, d\mu = 0$$

for every zero neighborhood U. Indeed, for $\tau \in T$, with $n_\tau = \min\{\tau(\omega) \,\|\, \omega \in [0, 1)\}$ and $n_\tau' = \max\{\tau(\omega) \,\|\, \omega \in [0, 1)\}$, we have

$$\sup_{x'\in U^\circ} \int |x'(f_\tau)|\, d\mu = \sup_{x'\in U^\circ} \sum_{k=n_\tau}^{n_\tau'} \int_{\{\tau=k\}} |x'(f_k)|\, d\mu$$

$$= \sup_{x'\in U^\circ} \sum_{k=n_\tau}^{n_\tau'} \sum_{j\in D_k} \frac{|x'(x_j)|}{p(x_j)} \mu(A_j), \qquad (*)$$

where D_k is the set of indices k for which

$$\{\tau = k\} = \bigcup_{j\in D_k} A_j$$

(note that $\{A_j \,\|\, j \in D_k\} \subset \pi_k$ since $\{\tau = k\} \in \Sigma_k\}$.

Now $\mu(A_j) = p(x_j)$, by construction. Furthermore it is trivial by construction that $k \neq k' \Rightarrow D_k \cap D_{k'} = \varnothing$. So $\sum_{k=n_\tau}^{n_\tau'} \sum_{j\in D_k}$ is a sum where every index j appears just once. Furthermore the lowest j in this sum can be as high as we wish, by taking τ high enough (in T, $\leqslant$,). So by [10, p. 25], and since $\mathbb{N} \subset T$, we have that (*) goes to 0 for τ going through T. So (f_n) is a Pettis-convergent amart. Note that in the Banach space case, the sequence (f_n) is *uniformly bounded.*

Now (f_n) is not L_X^1-convergent. If it were convergent, its limit would certainly be zero, by the above convergencies. But

$$\int_0^1 p(f_n) = \sum_{k\in J^{(n)}} p(x_k) = 1$$

for every n in $\mathbb{N}$. This ends the main part of the proof.

The proof also shows the implications (iv) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i) (although this long proof is not needed for the implication (iv) $\Rightarrow$ (i) as we see a bit further on).

Remark. Since $(f_n, \Sigma_n)_{n=1}^{\infty}$ is also p-bounded we have given a new proof of Theorem A.

(i) $\Rightarrow$ (iv) Let f be a step function

$$f = \sum_{i=1}^{n} a_i \chi_{A_i}$$

and let (Ω, Σ, μ) be a measure space. We can suppose the A_i disjoint. From [10, 4.1.5], we have for every continuous seminorm p on X a zero neighborhood V and a Radon measure ν on the (weak*-compact) polar V°, such that, for every ω in Ω:

$$\begin{aligned} p(f(\omega)) &\leqslant \int_{V^\circ} \left| \left\langle \sum_{i=1}^{n} a_i \chi_{A_i}(\omega), x' \right\rangle \right| d\nu(x') \\ &= \sum_{i=1}^{n} \left[\int_{V^\circ} |\langle a_i , x' \rangle| \, d\nu(x') \right] \chi_{A_i}(\omega) \end{aligned}$$

So

$$\begin{aligned} \int_{\Omega} p(f(\omega))\, d\mu(\omega) &\leqslant \sum_{i=1}^{n} \left[\int_{V^\circ} |\langle a_i , x' \rangle| \, d\nu(x') \right] \mu(A_i) \\ &= \int_{V^\circ} \left[\sum_{i=1}^{n} |\langle a_i , x' \rangle| \, \mu(A_i) \right] d\nu(x') \\ &\leqslant \sup_{x' \in V^\circ} \left[\sum_{i=1}^{n} |\langle a_i , x' \rangle| \, \mu(A_i) \right] \nu(V^\circ) \\ &= \sup_{x' \in V^\circ} \left[\int_{\Omega} |\langle x', f(\omega) \rangle| \, d\mu(\omega) \right] \cdot \nu(V^\circ). \end{aligned}$$

Since $\nu(V^\circ) < \infty$, we have proved the assertion for step functions. If $f \in L_X{}^1(\mu)$, then there is a sequence of step functions $(f_n)_{n=1}^{\infty}$, mean convergent to f.

Then, with $q(\cdot) = \int_\Omega p(\cdot)\, d\mu$ and $P = \sup_{x' \in V^\circ} \int_\Omega |\langle x', \cdot \rangle| \, d\mu$:

$$\begin{aligned} q(f) &= \lim_n q(f_n) \\ &\leqslant \nu(V^\circ) \lim_n P(f_n) \\ &= \nu(V^\circ)\, P(f). \end{aligned}$$

(i) $\Rightarrow$ (ii) Since X is nuclear and Fréchet, it is (RNP) (see [7]). Hence every $L_X{}^1$-bounded and uniformly integrable martingale is $L_X{}^1$-convergent (this is well known in Banach spaces [3, 4], and the extension to Fréchet spaces is immediately seen).

So (by (i) $\Rightarrow$ (iv)), every Pettis-bounded and Pettis-uniformly integrable martingale is L_X^1-convergent. Let $(f_n, \Sigma_n)_{n=1}^{\infty}$ be an arbitrary Pettis-bounded and Pettis-uniformly integrable amart. Hence. ((i) $\Rightarrow$ (iv)) it is L_X^1-bounded and uniformly integrable. The Riesz decomposition theorem [6] also applies in case X is Fréchet and (RNP). So $f_n = g_n + h_n$, where (g_n, Σ_n) is a martingale, and where h_n is Pettis-convergent to 0; so also L_X^1-convergent to 0, and hence uniformly integrable. Thus (g_n) is L_X^1-bounded and uniformly integrable, and consequently L_X^1-convergent. Thus also (f_n).

We proved at the same time (i) $\Rightarrow$ (iii). Since (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (v) are obvious, the theorem is completely proved. ▮

Remark. Our theorem seems to be new even if X is a Banach space. It thus gives further equivalent formulations of finite dimensionality.

Further remarks. (1) We can prove in a simple way: Let X be a Fréchet space. Then (i) is equivalent to (vi) The Pettis-bounded subsets of $L_X^1(\mu)$ are mean bounded.

Proof. (i) $\Rightarrow$ (iv) See the theorem. (iv) $\Rightarrow$ (vi) Tiivial. (vi) $\Rightarrow$ (i) Suppose X not nuclear; we have by [10] a summable, not absolutely summable sequence $(x_n)_{n=1}^{\infty}$ in X. Take the dyadic division of order n of $[0, 1)$, for every n in N, and call

$$A_i^n = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right), \qquad 1 \leqslant i \leqslant 2^n.$$

Put $f_n = \sum_{i=1}^{2^n} 2^n \cdot x_i \cdot \chi_{A_i^n}$. Then it is trivial that $(f_n)_{n=1}^{\infty}$ is a Pettis-bounded and not meanbounded amart $((f_n)_{n=1}^{\infty}$ cannot serve to prove theorem B because the sequence is not Pettis uniformly integrable).

(2) When working not in Fréchet spaces, but in sequentially complete dual metric spaces, we can (by [10, 4.2.5]) prove: (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). Furthermore (i) $\not\Rightarrow$ (ii) here, since in [8] we have given an example of a nuclear sequentially complete dual metric space without (RNP).

Note added in proof. More recently, we proved the following theorem (again using the Dvoretzky–Rogers theorem and the fundamental result in [1, p. 279]):

THEOREM. *Let X be a Fréchet space. The following assertions are equivalent*:

(i) *X is nuclear.*

(ii) *Every mean bounded amart (f_n, Σ_n) is of class (B): i.e.,*

$$\sup_{\tau \in T} \int_{\Omega} p(f_\tau)\, d\mu < +\infty$$

for every continuous seminorm p on X.

(iii) *For every mean bounded and uniformly integrable amart* (f_n, Σ_n) *(and uniformly bounded in case* X *is a Banach space), and for every continuous seminorm* p *on* X: $(p(f_n), \Sigma_n)$ *is an amart.*

This generalizes the result in [2], and gives a new proof for it.

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