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STRONG CONVERGENCE OF PRAMARTS IN BANACH SPACES

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1. Introduction. Let E be a Banach space and $(X_n, \mathfrak{F}_n)$ be an adapted sequence on the probability space $(\Omega, \mathfrak{F}, P)$. We denote by T the set of all bounded stopping times with respect to $(\mathfrak{F}_n)$. $(X_n, \mathfrak{F}_n)$ is called a pramart if

$$(\|E^{\mathfrak{F}_\sigma}X_\tau - X_\sigma\|)_{\substack{\sigma \leqq \tau \\ \sigma, \tau \in T}}$$

converges to zero in probability, uniformly in $\tau \geqq \sigma$. The notion of pramart was introduced in [**6**]. A good property is the optional sampling property (see Theorem 2.4 in [**6**]). Furthermore the class of pramarts intersects the class of amarts, and every amart is a pramart if and only if $\dim E < \infty$ ([**2**], see also [**4**]). Pramarts behave indeed quite differently than amarts. Although the class of pramarts is large, they have good convergence properties as is seen in the next two results of Millet–Sucheston, [**6**], [**7**].

THEOREM 1.1. *Let $(X_n, \mathfrak{F}_n)$ be a real-valued pramart of class* (d), *i.e.*,

$$\liminf E(X_n^+) + \liminf E(X_n^-) < \infty.$$

Then (X_n) converges a.s.

THEOREM 1.2. *Let E have* (RNP) *and let $(X_n, \mathfrak{F}_n)$ be an L_E^1-bounded pramart. Suppose* (a) *or* (b) *is satisfied, where*

(a) *(X_n) is uniformly integrable.*
(b) *(X_n) is of class* (B) (*i.e.*, $\sup_{\tau \in T} \int_\Omega \|X_\tau\| < \infty$).

Then (X_n) converges strongly a.s.

(Uniform integrability is meant in the sense defined in [**5**].) This leaves the general problem (L. Sucheston):

Problem. Do L_E^1-bounded pramarts in a (RNP) Banach space converge strongly a.s.?

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In this paper we show that, even if the problem is solved affirmatively, it is not the most general class of pramarts that do converge a.s. in a (RNP) space. Indeed here we will show that a different class fulfills the convergence requirement:

THEOREM 1.3. *Let E have* (RNP). *Let $(X_n, \mathfrak{F}_n)$ be a general pramart (not necessarily $L_E{}^1$-bounded). Assume that there exists a subsequence (X_{nk}) which is uniformly integrable. Then (X_n) converges strongly a.s.*

2. Proof of theorem 1.3.

LEMMA 2.1. (Theorem 4.1 in[**6**]). *Let $(\mathfrak{F}_n)_{n=1}^{\infty}$ be an increasing sequence of sub-σ-algebras of the σ-algebra $\mathfrak{F}$. Let $f(\sigma, \tau)$ be a family of $\mathfrak{F}_\sigma$-measurable, E-valued random variables, defined for $\sigma, \tau \in T$, $\sigma \leqq \tau$ (E: Banach space). Assume for every $n \in \mathbf{N}$,*

$$1_{\{\sigma=n\}} f(\sigma, \tau) = 1_{\{\sigma=n\}} f(n, \tau).$$

If $f(\sigma, \tau)$ converges in probability to f_∞, then $f(\sigma, \tau)$ converges strongly a.s. to f_∞.

LEMMA 2.2. *Let E be an arbitrary Banach space, and $(X_n, \mathfrak{F}_n)$ an adapted sequence. $(X_n, \mathfrak{F}_n)$ is a pramart if and only if*

$$\lim_{\sigma\in T} \sup_{\substack{\tau\geqq\sigma \\ \tau\in T}} \|X_\sigma - E^{\mathfrak{F}_\sigma} X_\tau\| = 0 \quad \textit{in probability.}$$

Proof. Using the definition of a pramart and Lemma 2.1 with

$$f(\sigma, \tau) = X_\sigma - E^{\mathfrak{F}_\sigma} X_\tau,$$

and $f_\infty = 0$, we see that if $(X_n, \mathfrak{F}_n)$ is a pramart, we have that

$$(X_\sigma - E^{\mathfrak{F}_\sigma} X_\tau)_{\sigma\in T} \text{ converges a.s. to } 0,$$

uniformly in $\tau \geqq \sigma$. So:

$$\lim_{\sigma\in T} \sup_{\substack{\tau\geqq\sigma \\ \tau\in T}} \|X_\sigma - E^{\mathfrak{F}_\sigma} X_\tau\| = 0, \text{ a.s.}$$

and hence in probability.

We need another lemma:

LEMMA 2.3. *Let E be any Banach space and $(X_n, \mathfrak{F}_n)$ be any pramart. If there is a subsequence $(X_{n_k})_{k=1}^{\infty}$ which is Cesaro-mean convergent, then (X_n) itself converges strongly a.s.*

Proof. Fix any increasing sequence $(\tau_n)_{n=1}^{\infty}$ in T. Let us call $Y \in L_E{}^1$ the Cesaro-mean limit of X_{n_k}, and write

$$U_k = \frac{1}{k} \sum_{i=1}^{k} X_{ni}.$$

We have, for every $\omega \in \Omega$ and $m, n, k \in \mathbf{N}$:

$$\begin{aligned}
&\|X_{\tau_m}(\omega) - X_{\tau_n}(\omega)\| \\
&\leqq \|X_{\tau_m}(\omega) - E^{\mathfrak{F}_{\tau_m}}U_k(\omega)\| + \|E^{\mathfrak{F}_{\tau_m}}U_k(\omega) - E^{\mathfrak{F}_{\tau_m}}Y(\omega)\| \\
&+ \|E^{\mathfrak{F}_{\tau_m}}Y(\omega) - E^{\mathfrak{F}_{\tau_n}}Y(\omega)\| + \|E^{\mathfrak{F}_{\tau_n}}Y(\omega) - E^{\mathfrak{F}_{\tau_n}}U_k(\omega)\| \\
&+ \|E^{\mathfrak{F}_{\tau_n}}U_k(\omega) - X_{\tau_n}(\omega)\|.
\end{aligned}$$

Now:

$$\|X_{\tau_m} - E^{\mathfrak{F}_{\tau_m}}U_k\| \leqq \frac{1}{k}\sum_{i=1}^{k}\|X_{\tau_m} - E^{\mathfrak{F}_{\tau_m}}X_{n_i}\| = \frac{1}{k}\sum\nolimits_1^{(m)} + \frac{1}{k}\sum\nolimits_2^{(m)},$$

where $\sum_1^{(m)}$ is summation over these indices i such that $n_i \ngeqq \tau_m$. Since $(n_i)_{i=1}^{\infty}$ is cofinal, we have only a fixed finite number of n_i such that $n_i \ngeqq \tau_m$. $\sum_2^{(m)}$ is summation over the rest. So:

$$\frac{1}{k}\sum\nolimits_2^{(m)} \leqq \sup_{n_i \geqq \tau_m}\|X_{\tau_m} - E^{\mathfrak{F}_{\tau_m}}X_{n_i}\|.$$

Fix $\epsilon > 0$. $(E^{\mathfrak{F}_{\tau_m}}Y, \mathfrak{F}_{\tau_m})_{m=1}^{\infty}$ is trivially a mean-convergent martingale (to Y). So it converges in probability. Choose m_0 such that $m, n \geqq m_0$ implies

$$P\left(\left\{\left\|E^{\mathfrak{F}_{\tau_m}}Y - E^{\mathfrak{F}_{\tau_n}}Y\right\| > \frac{\epsilon}{5}\right\}\right) \leqq \frac{\epsilon}{5}.$$

By Lemma 2.2, choose m_1 such that $m \geqq m_1$ implies

$$P\left(\left\{\sup_{n_i \geqq \tau_m}\|X_{\tau_m} - E^{\mathfrak{F}_{\tau_m}}X_{n_i}\| > \frac{\epsilon}{10}\right\}\right) \leqq \frac{\epsilon}{10}.$$

For every fixed m in $\mathbf{N}$ we have that $(E^{\mathfrak{F}_{\tau_m}}U_k)_{k=1}^{\infty}$ converges to $E^{\mathfrak{F}_{\tau_m}}Y$ in the mean (since $E^{\mathfrak{F}_{\tau_m}}(\,\cdot\,)$ is an $L_E{}^1$-contraction), and hence in probability. Fix $m, n \geqq \max(m_0, m_1)$. Choose one k such that

(i) $\displaystyle \frac{1}{k}\sum\nolimits_1^{(m)} < \frac{\epsilon}{10}$.

(ii) $\displaystyle \frac{1}{k}\sum\nolimits_1^{(n)} < \frac{\epsilon}{10}$.

(iii) $\displaystyle P\left(\left\{\|E^{\mathfrak{F}_{\tau_m}}U_k(\omega) - E^{\mathfrak{F}_{\tau_m}}Y(\omega)\| > \frac{\epsilon}{5}\right\}\right) \leqq \frac{\epsilon}{5}$.

(iv) $\displaystyle P\left(\left\{\|E^{\mathfrak{F}_{\tau_n}}U_k(\omega) - E^{\mathfrak{F}_{\tau_m}}Y(\omega)\| > \frac{\epsilon}{5}\right\}\right) \leqq \frac{\epsilon}{5}$.

We now easily see that if $m, n \geqq \max(m_0, m_1)$:

$$P(\{\|X_{\tau_m}(\omega) - X_{\tau_n}(\omega)\| > \epsilon\}) < \epsilon.$$

Since convergence in probability is determined by a complete metric, we see that $(X_\tau)_{\tau \in T}$ converges in probability. By Lemma 2.1 (applied to $f(\sigma, \tau) = X_\tau$) $(X_n)_{n=1}^{\infty}$ converges strongly a.s., which finishes the proof.

THEOREM 2.4. *Let the Banach space E have* (RNP). *Let $(X_n, \mathfrak{F}_n)$ be a pramart for which there is a subsequence $(X_{n_k})_{n=1}^{\infty}$ which is uniformly integrable. Then $(X_n)_{n=1}^{\infty}$ itself converges strongly a.s.*

Proof. Let $(X_{n_k})_{k=1}^{\infty}$ be the uniformly integrable subsequence. Since $(X_{n_k}, \mathfrak{F}_{n_k})$ is obviously a pramart, it follows from Theorem 1.2 that $(X_{n_k})_{k=1}^{\infty}$ converges strongly a.s. Since (X_{n_k}) is uniformly integrable, it converges in $L_E{}^1$-sense. Hence Lemma 2.3 finishes the proof.

Remark 2.5. In Lemma 2.3 as well as in Theorem 2.4, we may change $(X_{n_k})_{k=1}^{\infty}$ into $(X_{\sigma_k})_{k=1}^{\infty}$, where $(\sigma_k)_{k=1}^{\infty}$ is an arbitrary cofinal increasing sequence of stopping times. This follows from the proof of Lemma 2.3, and, for Theorem 2.4, from the optional sampling property of parmarts, only applied cofinally [**6**].

We wish to indicate that Theorem 2.4 is a typical pramart result, in the following sense: Take $E = \mathbf{R}$, and let $(X_n, \mathfrak{F}_n)$ be an amart, which has a subsequence which is L^1-bounded. As remarked to me by G. A. Edgar, it follows from the Riesz-decomposition [**3**], that (X_n) itself is L^1-bounded. So the refinement of supposing only some boundedness of a subsequence instead of the whole sequence does not make much sense for amarts. That it does for pramarts is seen in the next two examples.

Example 2.6. With respect to constant σ-algebras, a pramart is just an a.s. convergent sequence. It is now easily seen that a sequence which is not $L_E{}^1$-bounded may admit a uniformly integrable subsequence.

Example 2.7. (Lemma 9.1 in [**6**]). Let $\Omega = [0, 1)$, (γ_n) be a strictly decreasing sequence in $[0, 1)$, with $\lim \gamma_n = 0$. No matter what vectors $x_n \in E$ we take, $(X_n, \mathfrak{F}_n)$ is a pramart, with

$$X_n = x_n 1_{[\gamma_{n+1}, \gamma_n)}$$

$$\mathfrak{F}_n = \sigma(X_1, \ldots, X_n).$$

Now it is trivial to choose (x_n) in such a way that (X_n) has a uniformly integrable subsequence, without (X_n) being $L_E{}^1$-bounded.

3. A result in Banach-Saks spaces.

Definition 3.1. A Banach space E is said to have the *Banach–Saks-Property* (BSP), if every bounded sequence (x_n) in E has a Cesaro convergent subsequence.

A non-trivial equivalent formulation is found in [**1**]:

THEOREM 3.2. *E has* (BSP) *if and only if every bounded sequence* (X_n) *in E has a subsequence, such that every subsequence of it converges Cesaro.*

Now using a diagonalisation procedure, together with Theorem 3.2, the same technique of proof as in Lemma 2.3 shows:

PROPOSITION 3.3. *Let E have* (BSP). *Then every $L_E{}^1$-bounded finitely generated (i.e.,: every $\mathfrak{F}_n$ is finite) pramart converges a.s.*

This proposition may have some relevance, when trying to construct a counterexample to the general problem, posed in the first section.

Note added in proof. In a forthcoming paper of A. Bellow and L. Egghe it is noted that in Lemma 2.1 (Theorem 4.1 in [**6**]) we need an additional requirement on $f(\sigma, \tau)$: a localization in the second variable too: If $A \in \mathscr{F}_\sigma$ and $\tau', \tau'' \in T, \tau', \tau'' \geqq \sigma$, such that $\tau'(\omega) = \tau''(\omega)$ on A, then

$$\chi_A f(\sigma, \tau') = \chi_A f(\sigma, \tau'').$$

Since we only apply Lemma 2.1 for $f(\sigma, \tau) = X_\sigma - E^{F_\sigma} X_\tau$ and for $f(\sigma, \tau) = X_\tau(\sigma, \tau \in T, \sigma \leqq \tau)$, we see that the additional requirement is also satisfied.

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