Homological properties of a certain noncommutative Del Pezzo surface

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HOMOLOGICAL PROPERTIES OF A CERTAIN NONCOMMUTATIVE DEL PEZZO SURFACE

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Abstract. Recently, de Thanhoffer de Volcsey and Van den Bergh showed that Grothendieck groups of "noncommutative Del Pezzo surfaces" with an exceptional sequence of length 4 are isomorphic to one of three types, the third one not coming from a commutative Del Pezzo surface. In this paper, we adapt the theory of noncommutative $\mathbb{P}^1$-bundles as appearing in the work of Van den Bergh and Nyman to produce a sheaf $\mathbb{Z}$-algebra whose associated Proj has an exceptional sequence of length 4 for which the Gram matrix is of this third type. We show that this noncommutative scheme is noetherian and describe its local structure through the use of our generalized preprojective algebras ([3])

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1. Introduction and Overview

In the paper, [2], de Thanhoffer de Volcsey and Van den Bergh provide a numerical classification of possibly noncommutative Del Pezzo surfaces with an exceptional sequence of length 4. More precisely they consider a lattice Λ with a nondegenerate bilinear form $\langle -,-\rangle$ and consider the following sets of conditions

- there is an $s \in \mathrm{Aut}(\Lambda)$ such that $\langle x, sy\rangle = \langle y, x\rangle$ for $x, y \in \Lambda$
- $(s-1)$ is nilpotent
- $\mathrm{rk}(s-1) = 2$

- $\langle (s-1)x\,,(s-1)x\rangle < 0$ for $x \notin Ker(s-1)$

It is proved that the Grothendieck group $K(X)$, together with the Euler form, of a Del Pezzo surface X satisfies these conditions. The classification result they obtain is the following:

Theorem 1.1. *Let Λ satisfy the above conditions. Then Λ is isomorphic to $\mathbb{Z}^4$ where the matrix of the bilinear form is one of the following standard types:*

$$\begin{bmatrix}1&2&2&4\\0&1&0&2\\0&0&1&2\\0&0&0&1\end{bmatrix},\begin{bmatrix}1&2&3&5\\0&1&1&3\\0&0&1&2\\0&0&0&1\end{bmatrix}\quad and \quad\begin{bmatrix}1&2&1&5\\0&1&0&4\\0&0&1&2\\0&0&0&1\end{bmatrix}$$

The first two types correspond to the Grothendieck groups of the Del Pezzo surfaces $\mathbb{P}^1\times\mathbb{P}^1$ and $\mathbb{F}_1$ respectively. Moreover, it is an easy exercise to show that the third type in fact cannot correspond to a Del Pezzo surface. The goal of this paper is to construct a noncommutative analogue of a Del Pezzo surface equipped with an exceptional collection which forms a basis for the Euler form for which the Gram matrix is of the third type.

The entries of the matrix seem to suggest that we should find a 'noncommutative scheme' equipped with 2 'maps' to $\mathbb{P}^1$. We consider a construction which is an adaptation of Van den Bergh's theory of noncommutative $\mathbb{P}^1$-bundles. In [9], he considers a *symmetric sheaf $\mathbb{Z}$-algebras* $\mathbb{S}(\mathcal{E})$ constructed from a locally free bimodule $\mathcal{E}$ of rank $(2,2)$ and shows that $\mathbb{S}(\mathcal{E})$ can be regarded as the noncommutative analogue of a $\mathbb{P}^1$-bundle. In this paper we will apply the same construction where the bimodule $\mathcal{E}$ is of rank $(4,1)$ instead as suggested by the entries $(1,3)$ and $(2,4)$

In the first section, we recall the required background on (symmetric) sheaf $\mathbb{Z}$-algebras. For the benefit of the reader, we show how they relate to classical $\mathbb{P}^1$-bundles (Corollary 2.11).

In the next section, we describe their local behaviour. We prove that there exists a cover such that over each open subset, they can be regarded as a generalized preprojective algebra as introduced in the paper [3]. An immediate application of this result is that the category of graded $\mathcal{A}$-modules is a locally noetherian Grothendieck category (Theorems 3.1, 3.17).

In the final section, we introduce pullback functors $\Pi_n^*:\mathrm{QCoh}(X_n)\longrightarrow\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ and we adapt the results of [5] to our setting to prove a formula which computes the Ext groups of sheaves pulled back from X or Y:

Theorem. (See 4.1) Let $\mathcal{E}\in\mathrm{bimod}(X,Y)$ be locally free of rank (4,1). Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves on X_m respectively X_n for $m,n\in\mathbb{Z}$ such that $m\geq n-1$. Then

$$\mathrm{Ext}^i_{\mathrm{Proj}(\mathcal{A})}(\Pi_m^*\mathcal{F},\Pi_n^*\mathcal{G})\cong\mathrm{Ext}^i_{X_m}(\mathcal{F},\mathcal{G}\otimes\mathbb{S}(\mathcal{E})_{n,m})$$

for all $i\geq 0$.

This culminates in the construction of the desired noncommutative Del Pezzo surface:

Theorem. (See 4.4) Let $\mathcal{E}$ be the $\mathbb{P}^1$-bimodule ${}_f(\mathcal{O}_{\mathbb{P}^1})_{Id}$ and $\mathbb{S}(\mathcal{E})$ be the associated symmetric sheaf $\mathbb{Z}$-algebra. Then

$$\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}),\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)),\Pi_0^*(\mathcal{O}_{\mathbb{P}^1}),\Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))$$

is an exceptional sequence of graded $\mathbb{S}(\mathcal{E})$-modules for which the Gram matrix of the Euler form is given by

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Aknowledgements

Both authors wish to thank their supervior, Michel Van den Bergh for suggesting the construction and many helpful discussions along the way. They would also like to explicitly mention that most of section 4 is a straightforward adaptation of the beautiful techniques developed by Adam Nyman in [6, 7]

2. Symmetric Sheaf $\mathbb{Z}$-Algebras

2.1. **Definitions and construction.** Sheaf-bimodules were defined in [9] as follows:

Definition 2.1. Let $X \longrightarrow S$ and $Y \longrightarrow S$ be S-schemes. A coherent $X-Y$ bimodule $\mathcal{E}$ is a coherent $\mathcal{O}_{X\times_S Y}$-module such that the support of $\mathcal{E}$ is finite over X and Y. We denote the corresponding abelian category by $\mathrm{bimod}_S(X-Y)$. More generally an $X-Y$-bimodule is a quasi-coherent $\mathcal{O}_{X\times_S Y}$-module which is a filtered direct limit of objects in $\mathrm{bimod}_S(X-Y)$. The abelian category of $X-Y$-bimodules is denoted $\mathrm{BiMod}_S(X-Y)$. Finally, a bimodule $\mathcal{E}$ is called locally free if $\pi_{X*}(\mathcal{E})$ and $\pi_{Y*}(\mathcal{E})$ are locally free. If moreover $\pi_{X*}(\mathcal{E})$ and $\pi_{Y*}(\mathcal{E})$ have finite rank m and n respectively, then $\mathcal{E}$ is said to have rank (m,n).

The tensor product of $\mathcal{O}_{X\times_S Y\times_S Z}$-modules induces a tensor product $\mathrm{BiMod}_S(X-Y)\otimes \mathrm{BiMod}_S(Y-Z) \longrightarrow \mathrm{BiMod}_S(X-Z)$ given by

$$\mathcal{E}\otimes\mathcal{F} := \pi_{X\times Z*}\big(\pi^*_{X\times Y}(\mathcal{E})\otimes_{X\times Y\times Z}\pi^*_{Y\times Z}(\mathcal{F})\big)$$

Moreover for each $\mathcal{E}\in \mathrm{BiMod}_S(X-Y)$ there is a right exact functor :

$$-\otimes_X \mathcal{E} : \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(Y) : \mathcal{M}\otimes_Y \mathcal{E} := \pi_{Y*}\big(\pi^*_X(\mathcal{M})\otimes_{X\times Y}\mathcal{E}\big)$$

which is exact if and only if $\mathcal{E}$ is locally free. By [9, Lemma 3.1.1.] this functor determines $\mathcal{E}$ uniquely.

Definition 2.2. Let W be an S-scheme with finite S-maps $u : W \longrightarrow X$, $v : W \longrightarrow Y$. If $\mathcal{U}\in \mathrm{QCoh}(W)$, then we denote $(u,v)_*\mathcal{U}\in \mathrm{BiMod}_S(X-Y)$ as ${}_u\mathcal{U}_v$. One easily checks:

$$-\otimes {}_u\mathcal{U}_v = v_*(u^*(-)\otimes_W \mathcal{U})$$

Any bimodule isomorphic to one of the form ${}_u\mathcal{U}_u \cong {}_{Id}(u_*\mathcal{U})_{Id}$ is called *central.*

Definition 2.3. Let $(X_i \longrightarrow S)_{i\in\mathbb{Z}}$ be a collection of S-schemes.
A sheaf $\mathbb{Z}$-algebra $\mathcal{A}$, is a collection of $X_i - X_j$-bimodules $\mathcal{A}_{ij}$ together with maps $\mathcal{A}_{ij}\otimes\mathcal{A}_{jk} \longrightarrow \mathcal{A}_{ik}$ satisfying the usual associativity and unit properties (see [9]).
An $\mathcal{A}$-module is a sequence of X_i-modules $\mathcal{M}_i$ together with maps $\mathcal{M}_i\otimes\mathcal{A}_{ij} \longrightarrow \mathcal{M}_j$ again satifying the obvious axioms. The associated category is denoted $\mathrm{Gr}(\mathcal{A})$.
An $\mathcal{A}$-module is right bounded if $\mathcal{M}_i = 0$ for $i >> 0$. An $\mathcal{A}$-module is called torsion if it is a filtered colimit of right bounded modules. Let $\mathrm{Tors}(\mathcal{A})$ be the subcategory

of $\mathrm{Gr}(\mathcal{A})$ consisting of torsion modules. Then $\mathrm{Tors}(\mathcal{A})$ is Serre and the corresponding quotient category is denoted by $\mathrm{Proj}(\mathcal{A})$[1].
We have a projection functor $p : \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Proj}(\mathcal{A})$ with right adjoint ω (see [8]).

We shall need the following form of duality of bimodules:

Lemma 2.4. *Let $\mathcal{E} \in \mathrm{bimod}_S(X-Y)$ be locally free. Then there is a unique object $\mathcal{E}^* \in \mathrm{bimod}_S(Y-X)$ such that*

$$- \otimes_Y \mathcal{E}^* : \mathrm{QCoh}(Y) \longrightarrow \mathrm{QCoh}(X)$$

is the right adjoint of $- \otimes_X \mathcal{E}$, i.e for $\mathcal{M} \in \mathrm{QCoh}(X)$ and $\mathcal{N} \in \mathrm{QCoh}(Y)$:

$$\mathrm{Hom}_Y(\mathcal{M} \otimes \mathcal{E}, \mathcal{N}) \cong \mathrm{Hom}_X(\mathcal{M}, \mathcal{N} \otimes \mathcal{E}^*)$$

Proof. see [9, Section 3]. □

Remark 2.5. If $\mathcal{E} = {}_u\mathcal{U}_v$ then $\mathcal{E}^*$ is given by ${}_v\,\mathcal{H}om_W(\mathcal{U}, v^!\mathcal{O}_Y)_u$

The dual notion leads to a an object ${}^*\mathcal{E}$ such that

$$\mathrm{Hom}_X(\mathcal{N} \otimes {}^*\mathcal{E}, \mathcal{M}) \cong \mathrm{Hom}_Y(\mathcal{N}, \mathcal{M} \otimes \mathcal{E})$$

and Yoneda's lemma proves that $\mathcal{E} = {}^*(\mathcal{E}^*) = ({}^*\mathcal{E})^*$. Repeated application of duals leads to the following notation:

$$\mathcal{E}^{*n} = \begin{cases} \mathcal{E}^{\overbrace{*\cdots*}^{n}} & n \geq 0 \\ {}^{\overbrace{*\cdots*}^{-n}}\mathcal{E} & n < 0 \end{cases}$$

In this context it will be convenient to invoke the following notation:

$$X_n = X \text{ if } n \text{ is even and } Y \text{ if } n \text{ is odd} \tag{1}$$

There are unit and counit morphisms:

$$\begin{aligned} i_n &: \mathcal{O}_{X_n} \longrightarrow \mathcal{E}^{*n} \otimes \mathcal{E}^{*n+1} \\ j_n &: \mathcal{E}^{*n} \otimes \mathcal{E}^{*n-1} \longrightarrow \mathcal{O}_{X_n} \end{aligned} \tag{2}$$

Our next ingredient is that of a nondegenerate bimodule.

Definition 2.6. For $\mathcal{E} \in \mathrm{bimod}(X-Y)$ and $\mathcal{F} \in \mathrm{bimod}(Y-Z)$, a bimodule $\mathcal{Q} \subset \mathcal{E} \otimes_Y \mathcal{F}$ is *invertible* if there is a $\mathcal{Q}^{-1} \in \mathrm{bimod}(Z-X)$ such that $\mathcal{Q} \otimes \mathcal{Q}^{-1} \cong \mathcal{O}_X$ and $\mathcal{Q}^{-1} \otimes \mathcal{Q} \cong \mathcal{O}_Z$. If moreover the following canonical composition

$$\mathcal{E}^* \otimes_X \mathcal{Q} \longrightarrow \mathcal{E}^* \otimes_X \otimes \mathcal{E} \otimes_Y \mathcal{F} \longrightarrow \mathcal{F}$$

is in fact an isomorphism, then $\mathcal{Q}$ is said to be *nondegenerate*.

We can now state the definition of a symmetric sheaf $\mathbb{Z}$-algebra.

Definition 2.7. Let $(X_i \longrightarrow S)_{i\in\mathbb{Z}}$ be a sequence of S-schemes and let $\mathcal{E}_i$ be locally free $X_i - X_{i+1}$-bimodules. Then the *tensor sheaf $\mathbb{Z}$-algebra* $\mathbb{T}(\{\mathcal{E}_i\})$ is the sheaf $\mathbb{Z}$-algebra generated by the $\{\mathcal{E}_i\}$. More precisely

$$\mathbb{T}(\{\mathcal{E}_i\})_{m,n} = \begin{cases} 0 & n < m \\ {}_{Id}(\mathcal{O}_{X_m})_{Id} & n = m \\ \mathcal{E}_m \otimes \ldots \otimes \mathcal{E}_{n-1} & n > m \end{cases}$$

[1]The notation $\mathrm{QGr}(\mathcal{A})$ is standard as well.

If moreover for each i we are given a nondegenerate $X_i - X_{i+2}$-bimodule $\mathcal{Q}_i \subset \mathcal{E}_i \otimes \mathcal{E}_{i+1}$, then the *symmetric sheaf* $\mathbb{Z}$*-algebra* $\mathbb{S}(\{\mathcal{E}_i\}, \{\mathcal{Q}_i\})$ is the quotient of $\mathbb{T}(\{\mathcal{E}_i\})$ by the relations $(\mathcal{Q}_i)_i$. I.e. $\mathbb{S}(\{\mathcal{E}_i\}, \{\mathcal{Q}_i\})_{m,n}$ is defined as

$$\begin{cases} \mathbb{T}(\{\mathcal{E}_i\})_{m,n} & n \leq m+1 \\ \mathbb{T}(\{\mathcal{E}_i\})_{m,n}/[(\mathcal{Q}_m \otimes \ldots) + (\mathcal{E}_m \otimes Q_{m+1} \otimes \ldots) + \ldots + (\ldots \otimes Q_{n-2})] & n \geq m+2 \end{cases}$$

Given an X-Y- bimodule $\mathcal{E}$, the so-called *standard* tensor sheaf $\mathbb{Z}$-algebra $\mathbb{T}(\mathcal{E})$ and *standard* symmetric sheaf $\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ are constructed as above by taking X_n as in (1) and $\mathcal{E}_n, \mathcal{Q}_n$ as follows:

(3)
$$\begin{aligned} \mathcal{E}_n &= \mathcal{E}^{*n} \\ \mathcal{Q}_n &= i_n\left(\mathcal{O}_{X_n}\right) \subset \mathcal{E}^{*n} \otimes \mathcal{E}^{*n+1} \end{aligned}$$

There is a useful operation called twisting:

Theorem 2.8. *Let* $(X_i \longrightarrow S)_i$ *and* $(Y_i \longrightarrow S)_i$ *be* S*-schemes and* $\mathcal{A}$ *a sheaf* $\mathbb{Z}$*-algebra.*
Given a collection of invertible $X_i - Y_i$*-bimodules* $(\mathcal{T}_i)_i$*, one can construct a sheaf* $\mathbb{Z}$*-algebra* $\mathcal{B}$ *by*

$$\mathcal{B}_{ij} := \mathcal{T}_i^{-1} \otimes \mathcal{A}_{ij} \otimes \mathcal{T}_i$$

called the twist of $\mathcal{A}$ *by* $(\mathcal{T}_i)_i$*.*
There is an equivalence of categories given by the functor

$$\mathcal{T} : \mathrm{Gr}(\mathcal{A}) \cong \mathrm{Gr}(\mathcal{B}) : \mathcal{M}_i \longrightarrow \mathcal{M}_i \otimes \mathcal{T}_i$$

Finally, every sheaf $\mathbb{Z}$*-algebra can be obtained from a standard one by a twist.*

Proof. This is proven in section 4.1 of [9] □

The above theorem allows us to make the following definition:

Definition 2.9. A symmetric sheaf $\mathbb{Z}$-algebra is *commutative* if it is the twist of a standard symmetric sheaf $\mathbb{Z}$-algebra whose underlying bimodule is central.

2.2. **The Rank** $(2,2)$ **Case.** The following lemma (which was already announced but not proven in [9]) shows that commutative symmetric sheaf $\mathbb{Z}$-algebras of rank (2,2) are essentially commutative. We shall use the notation $\widehat{Sym_{X\times X}({}_{Id}\mathcal{V}_{Id})}$ to denote the sheaf-$\mathbb{Z}$-algebra whose (i,j)-component is $Sym_{X\times X}({}_{Id}\mathcal{V}_{Id})_{j-i}$ and considered its associated category of graded modules as in Definition 2.3.

Lemma 2.10. *Let* $\mathcal{V}$ *be a locally free* X*-module of rank 2. There is an equivalence of the form*

$$\mathrm{Gr}(\mathbb{S}({}_{Id}\mathcal{V}_{Id})) \xrightarrow{\mathcal{T}} \mathrm{Gr}\left(\widehat{\mathrm{Sym}_{X\times X}({}_{Id}\mathcal{V}_{Id})}\right) \xrightarrow{\cong} \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V}))$$

where $\mathcal{T}$ *is given by twisting through* $\left(\left(\bigwedge^2 \mathcal{V}\right)^{\lfloor \frac{i}{2} \rfloor}\right)_{i\in\mathbb{Z}}$.

Proof. The second equivalence in the composition follows from the fact

$$\mathcal{M}_i \otimes \mathrm{Sym}_{X\times X}({}_{Id}\mathcal{V}_{Id})_{j-i} = \mathcal{M}_i \otimes {}_{Id}\left(\mathrm{Sym}_X(\mathcal{V})_{j-i}\right)_{Id} = \mathcal{M}_i \otimes_X \mathrm{Sym}_X(\mathcal{V})_{j-i}$$

implying that both multiplications coincide. We thus only need to exhibit the first equivalence.
Let $\mathcal{E} = {}_{Id}\mathcal{V}_{Id}$. Using the explicit expression for the dual, we obtain

$$\mathcal{E}^* = {}_{Id}\mathcal{H}om(\mathcal{V}, Id^!\mathcal{O}_X)_{Id} = {}_{Id}(\mathcal{V}^*)_{Id}$$

In particular $\mathcal{E}^{*2n} = \mathcal{E} = {}_{Id}(\mathcal{V})_{Id}$ and $\mathcal{E}^{*2n+1} = \mathcal{E}^* = {}_{Id}(\mathcal{V}^*)_{Id}$ hold for all n.

Recall from [4, II ex.5.16.b] that the pairing $\mathcal{V} \otimes \mathcal{V} \longrightarrow \Lambda^2\mathcal{V}$ is perfect, implying there is an isomorphism

$$\mathcal{V}^* \otimes (\Lambda^2\mathcal{V}) \xrightarrow{\cong} \mathcal{V} \tag{4}$$

Let $(\mathcal{T}_i)_i = (\bigwedge^2 \mathcal{V})^{\lfloor \frac{i}{2} \rfloor}$. It follows that $\mathbb{T}(\mathcal{E})$ is obtained from a twist of the classical tensor $\mathbb{Z}$-algebra $\widehat{T_X(\mathcal{V})}$ and by Theorem 2.8 there is an equivalence

$$\mathrm{Gr}(\mathbb{T}(\mathcal{E})) \to \mathrm{Gr}(\widehat{\mathrm{T}_X(\mathcal{V})}) : \left(\mathcal{M}_i\right)_i \mapsto \left(\mathcal{M}_i \otimes (\Lambda^2\mathcal{V})^{\lfloor \frac{i}{2} \rfloor}\right)_i$$

specifically in each component:

$$\mathbb{T}(\mathcal{E})_{m,n} \cong {}_{Id}\left((\Lambda^2\mathcal{V})^{\lfloor \frac{m}{2} \rfloor} \otimes T_X(\mathcal{V})_{n-m} \otimes (\Lambda^2\mathcal{V})^{-\lfloor \frac{n}{2} \rfloor}\right)_{Id} \tag{5}$$

We now claim that the twisting in (5) induces a twisting

$$\mathbb{S}(\mathcal{E})_{m,n} \cong {}_{Id}\left((\Lambda^2\mathcal{V})^{\lfloor \frac{m}{2} \rfloor} \otimes \mathrm{Sym}_X(\mathcal{V})_{n-m} \otimes (\Lambda^2\mathcal{V})^{-\lfloor \frac{n}{2} \rfloor}\right)_{Id}$$

and hence an equivalence of categories:

$$\mathrm{Gr}(\mathbb{S}(\mathcal{E})) \to \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V})) : \left(\mathcal{M}_i\right)_i \mapsto \bigoplus_i \mathcal{M}_i \otimes (\Lambda^2\mathcal{V})^{\lfloor \frac{i}{2} \rfloor} \tag{6}$$

So we are left with proving the claim. For this we must understand what happens under (5) to the relations that define $\mathbb{S}(\mathcal{E})$ as a quotient of $\mathbb{T}(\mathcal{E})$.
As the relations are generated in degree 2 it suffices to consider $\mathbb{S}(\mathcal{E})_{m,m+2} \otimes {}_{Id}(\Lambda^2\mathcal{V})_{Id}$. This is the quotient of $\mathbb{T}(\mathcal{E})_{m,m+2} \otimes {}_{Id}(\Lambda^2\mathcal{V})_{Id} \cong {}_{Id}\left(T_X(\mathcal{V})_2\right)_{Id} = {}_{Id}\left(\mathcal{V} \otimes \mathcal{V}\right)_{Id}$ by the relation $i\left({}_{Id}(\mathcal{O}_X)_{Id}\right) \otimes {}_{Id}(\Lambda^2\mathcal{V})_{Id} \subset {}_{Id}\left(\mathcal{V} \otimes \mathcal{V}^* \otimes \Lambda^2\mathcal{V}\right)_{Id} \cong {}_{Id}\left(\mathcal{V} \otimes \mathcal{V}\right)_{Id}$. We have to check that this relation is exactly the one that defines $\mathrm{Sym}_X(\mathcal{V})$ as a quotient of $\mathrm{T}_X(\mathcal{V})$. The latter relation is defined locally, so it suffices to check on a trivializing open subset U for $\mathcal{V}$. If $\mathcal{V}|_U \cong \mathcal{O}_X|_U\, u \oplus \mathcal{O}_X|_U\, v$ then $i\left({}_{Id}(\mathcal{O}_X)_{Id}\right)$ is locally given by $u \otimes u^* + v \otimes v^*$. One checks that the isomorphism (4) maps $u^* \otimes (u \wedge v)$ to v and $v^* \otimes (u \wedge v)$ to $-u$, the induced relation in $\mathcal{V} \otimes \mathcal{V}$ is locally given by $u \otimes v - v \otimes u$, the defining relation of $\mathrm{Sym}_X(\mathcal{V})$. □

Corollary 2.11. *With the assumptions from the previous theorem we have an induced equivalence:*

$$\Phi : \mathrm{Proj}(\mathbb{S}({}_{Id}(\mathcal{V})_{Id})) \xrightarrow{\cong} \mathrm{Proj}(\mathrm{Sym}_X(\mathcal{V})) \xrightarrow{\cong} \mathrm{QCoh}(\mathbb{P}_X(\mathcal{V}))$$

Proof. The equivalence given in (6) obviously maps torsion modules onto torsion modules, hence it factors through $\mathrm{Proj}(\mathbb{S}({}_{Id}(\mathcal{V})_{Id}) \xrightarrow{\cong} \mathrm{Proj}(\mathrm{Sym}_X(\mathcal{V}))$.

The second equivalence is given by the following pair of functors

$$\operatorname{Proj}(\operatorname{Sym}_X(\mathcal{V})) \underset{p\circ\Gamma_* := p\left[\oplus_i \pi_*\big((-)(i)\big)\right]}{\overset{\widetilde{(-)}}{\rightleftarrows}} \operatorname{QCoh}(\mathbb{P}_X(\mathcal{V}))$$

Where π is the projection $\mathbb{P}_X(\mathcal{V}) \longrightarrow X$. □

2.3. **Truncation Functors and periodicity.** Any sheaf $\mathbb{Z}$-algebra is endowed with a sequence of *truncation* functors as follows: let $(X_i \longrightarrow S)_i$ be S-schemes and $\mathcal{A}$ a sheaf $\mathbb{Z}$-algebra. Then for each $n \in \mathbb{Z}$, consider the functor

$$\operatorname{Gr}(\mathcal{A}) \xrightarrow{(-)_n} \operatorname{QCoh}(X_n)$$

We shall need the following easy result on these functors:

Lemma 2.12. *Let $e_n\mathcal{A}$ be the right $\mathcal{A}$-module $(\mathcal{A}_{nm})_m$. There is an adjoint pair*

$$- \otimes e_n\mathcal{A} \;\dashv\; (-)_n$$

Proof. The proof of this is standard and left to the reader □

If $\mathcal{A}$ is a symmetric $\mathbb{Z}$-algebra in standard form, then there is a 2-periodic behaviour among these functors in the following way:

Proposition 2.13. *Let $\mathcal{A}$ be a symmetric sheaf $\mathbb{Z}$-algebra. Then there is an autoequivalence α on $\operatorname{Gr}(\mathcal{A})$ inducing a commuting diagram for each n*

$$\begin{array}{ccc} \operatorname{Gr}(\mathcal{A}) & \xrightarrow{(-)_n} & \operatorname{QCoh}(X_n) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\otimes\omega_{X_n/S}} \\ \operatorname{Gr}(\mathcal{A}) & \xrightarrow[(-)_{n+2}]{} & \operatorname{QCoh}(X_n) \end{array}$$

Proof. By Theorem 2.8 $\mathcal{A}$ is Morita equivalent to a symmetric sheaf $\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ in standard form with $\mathcal{E} \in \operatorname{bimod}(X - Y)$. Moreover by [9, 4.1.7] , we have

$$\mathcal{E}^{*2} \cong \omega_{X/S}^{-1} \otimes \mathcal{E} \otimes \omega_{Y/S}$$

Hence the twist by $(\omega_{X_i/S})_{i\in\mathbb{Z}}$ yields an equivalence

$$\mathcal{T} : \operatorname{Gr}(\mathbb{S}(\mathcal{E})) \xrightarrow{\cong} \operatorname{Gr}(\omega^{-1} \otimes \mathbb{S}(\mathcal{E}) \otimes \omega) \xrightarrow{\cong} Gr(\mathbb{S}(\mathcal{E}^{*2}))$$

Where we used the short-hand notation

$$\big(\omega^{-1} \otimes \mathbb{S}(\mathcal{E}) \otimes \omega\big)_{m,n} = \omega_{X_m/S}^{-1} \otimes \mathbb{S}(\mathcal{E})_{m,n} \otimes \omega_{X_n/S}$$

Next, the construction of a standard symmetric sheaf $\mathbb{Z}$-algebra implies that there is an equivalence $\Psi : \operatorname{Gr}(\mathbb{S}(\mathcal{E})(2)) \longrightarrow \operatorname{Gr}(\mathbb{S}(\mathcal{E}^{*2}))$. We now simply define

$$\alpha := (-2) \circ \Psi^{-1} \circ \mathcal{T} : \operatorname{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \operatorname{Gr}(\mathbb{S}(\mathcal{E}^{*2})) \longrightarrow \operatorname{Gr}(\mathbb{S}(\mathcal{E})(2)) \longrightarrow \operatorname{Gr}(\mathbb{S}(\mathcal{E}))$$

□

In the case of a sheaf $\mathbb{Z}$-algebra which is commutative (see 2.9) of rank (2,2), the 0^{th} truncation functor coincides with the pushforward functor in the following sense:

Theorem 2.14. *Let $\mathbb{S}({}_{Id}(\mathcal{V})_{Id})$ be a commutative symmetric sheaf $\mathbb{Z}$-algebra of rank (2,2) and let $\Phi : \mathrm{Proj}(\mathbb{S}({}_{Id}(\mathcal{V})_{Id})) \longrightarrow \mathrm{QCoh}(\mathbb{P}_X(\mathcal{V}))$ be the equivalence provided by Corollary 2.11. Then the following diagram commutes*

$$\begin{array}{ccccc} & & \mathrm{Gr}(\mathbb{S}({}_{Id}(\mathcal{V})_{Id})) & & \\ & \overset{\omega}{\nearrow} & & \overset{(-)_0}{\searrow} & \\ \mathrm{Proj}(\mathbb{S}({}_{Id}(\mathcal{V})_{Id})) & & & & \mathrm{QCoh}(X) \\ & \underset{\Phi}{\searrow} & & \underset{\pi_*}{\nearrow} & \\ & & \mathrm{QCoh}(\mathbb{P}_X(\mathcal{V})) & & \end{array}$$

Proof. Let $Z := \mathbb{P}_X(\mathcal{V})$ and $\mathcal{A} := \mathbb{S}({}_{Id}(\mathcal{V})_{Id})$. The explicit isomorphism we need to exhibit is

$$\pi_*\Big(\widetilde{\oplus_i(-)\otimes \mathcal{T}_i}\Big) \cong \big(\omega(-)\big)_0$$

Now by Lemma 2.12 and the definition of ω, the functor $\big(\omega(-)\big)_0$ is right adjoint to $p\big((-)\otimes e_0\mathcal{A}\big)$. Another formal computation using Corollary 2.11 shows that $\pi_*\Big(\widetilde{\oplus_i(-)\otimes \mathcal{T}_i}\Big)$ is right adjoint to $\mathcal{T}^{-1}\left[(p\circ\Gamma_*)\big(\pi^*(-)\big)\right] = p\left[\big(\pi_*\big(\pi^*(-)(i)\big)\otimes \mathcal{T}_i^{-1}\big)_i\right]$, which by the projection formula, simplifies to $p\left(\big((-)\otimes \pi_*\mathcal{O}_Z(i)\otimes \mathcal{T}_i^{-1}\big)_i\right)$. The unicity of adjoint functors thus reduces the claim to showing the isomorphism

$$\big((-)\otimes \pi_*\mathcal{O}_Z(i)\otimes \mathcal{T}_i\big)_i \cong (-)\otimes e_0\mathcal{A} \tag{7}$$

Since Z is a projective bundle over X, we have $\pi_*\big(\mathcal{O}_Z(i)\big) = \pi_*\big(\widetilde{\mathrm{Sym}_X(\mathcal{V})_i}\big)$ and since $\mathcal{V}$ has rank 2, by [4, Proposition II.7.11.a], $\pi_*\big(\widetilde{\mathrm{Sym}_X(\mathcal{V})_i}\big) = \mathrm{Sym}_X(\mathcal{V})_i$. Now, by the choice of $\mathcal{T}_i$, we have $\mathrm{Sym}_X(\mathcal{V})_i = \mathcal{A}_{0i}\otimes \mathcal{T}_i$. (7) thus becomes

$$\big((-)\otimes \pi_*\mathcal{O}_Z(i)\otimes \mathcal{T}_i^{-1}\big)_i = \big((-)\otimes \mathcal{A}_{0i}\otimes \mathcal{T}_i\otimes \mathcal{T}_i^{-1}\big)_i = \big((-)\otimes \mathcal{A}_{0i}\big)_i = (-)\otimes e_0\mathcal{A}$$

proving the claim. □

We also have 1-periodicity for the truncation functors in this case:

Proposition 2.15. *Let $\mathcal{V}$ be a locally free sheaf of rank 2 on X and $\mathbb{S}({}_{Id}\mathcal{V}_{Id})$ the associated symmetric sheaf $\mathbb{Z}$-algebra. Then there is an equivalence β and for each n, a line bundle $\mathcal{L}_n$ on X making the diagram*

$$\begin{array}{ccc} \mathrm{Gr}(\mathbb{S}({}_{Id}\mathcal{V}_{Id})) & \xrightarrow{(-)_n} & \mathrm{QCoh}(X) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle -\otimes\mathcal{L}_n} \\ \mathrm{Gr}(\mathbb{S}({}_{Id}\mathcal{V}_{Id})) & \xrightarrow[(-)_{n+1}]{} & \mathrm{QCoh}(X) \end{array}$$

commute.

Proof. By Lemma 2.10 there is a sequence of $X-X$-bimodules $\mathcal{T}_i$ such that the following is an equivalence of categories

$$\mathrm{Gr}(\mathbb{S}({}_{Id}\mathcal{V}_{Id})) \longrightarrow \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V})) : (\mathcal{M}_i)_i \mapsto \bigoplus_i \mathcal{M}_i\otimes\mathcal{T}_i$$

Let (-1) denote the inverse shift functor on $\mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V}))$, i.e. $(\mathcal{M}(-1))_i = \mathcal{M}_{i-1}$

and define β as being the autoequivalence making the diagram

$$\begin{array}{ccc} \mathrm{Gr}(\mathbb{S}({}_{Id}\mathcal{V}_{Id})) & \xrightarrow{\mathcal{T}} & \mathrm{Sym}_X(\mathcal{V}) \\ \beta\downarrow & & \downarrow(-1) \\ \mathrm{Gr}(\mathbb{S}({}_{Id}\mathcal{V}_{Id})) & \xrightarrow[\mathcal{T}]{} & \mathrm{Sym}_X(\mathcal{V}) \end{array}$$

commute. Since we clearly have $(-)_{n+1} \circ (-1) = (-)_n$, we get the required result by choosing the line bundle $\mathcal{L}_n := \mathcal{T}_n \otimes \mathcal{T}_{n+1}^{-1}$ with $\mathcal{T}_n$ as in the proof of Lemma 2.10. $\square$

Remark 2.16. the previous result of 1-periodicity clearly implies 2-periodicity after repeated application in the sence that

$$(-)_{n+2} \circ \beta^2 = \big(\mathcal{L}_{n+1} \otimes \mathcal{L}_n\big) \otimes (-)_n$$

hence one can wonder weather this coincides with Proposition 2.13. An explicit computation shows that this is not the case in general. Indeed, from the explicit form of $\mathcal{T}$ in Proposition 2.13 and β in Proposition 2.15, we obtain

$$\mathcal{L}_n = \big(\bigwedge^2 \mathcal{V}\big)^{\lfloor \frac{n}{2} \rfloor} \otimes \big(\bigwedge^2 \mathcal{V}\big)^{-\lfloor \frac{n+1}{2} \rfloor}$$

and $\mathcal{L}_{n+1} \otimes \mathcal{L}_n = \Big(\bigwedge^2(\mathcal{V})\Big)^{-1}$, which obviously does not coincide with $\omega_{X/S}$ in general.

3. Noetherianity of $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$

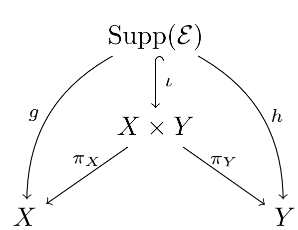
In this section we prove one of the main results of this paper:

Theorem 3.1. *Let X and Y be smooth varieties and $\mathcal{E} \in \mathrm{bimod}_S(X-Y)$ be locally free of rank (4,1). Then the category $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is locally noetherian.*

Throughout this section we will always assume that X, Y and $\mathcal{E} \in \mathrm{bimod}_S(X-Y)$ satisfy the conditions in the above theorem. The next lemma shows that under these assumptions, the bimodule $\mathcal{E}$ can written is a convenient form using a line bundle on Y and a finite map f.

Lemma 3.2. *Let X and Y be smooth varieties over an algebraically closed field k and let $\mathcal{E} \in \mathrm{bimod}_k(X,Y)$ be locally free of rank $(4,1)$. Then there is a line bundle $\mathcal{L}$ on Y and a finite surjective morphism $f : Y \longrightarrow X$ of degree 4 such that $\mathcal{E} \cong {}_f(\mathcal{L})_{Id}$.*

Proof. We consider the following diagram:



By definition 2.1: g and h are finite and in particular they are closed. As $g_*(\mathcal{E})$ and $h_*(\mathcal{E})$ are locally free, this immediately implies that g and h must be surjective. The construction of f relies on the fact that h is in fact an isomorphism. The main step in proving this is to show that $\mathrm{Supp}(\mathcal{E})$ is irreducible.

Let $V_1, \ldots, V_n$ be the irreducible components of $\mathrm{Supp}(\mathcal{E})$ ordered such that $\dim(V_i) \geq \dim(V_{i+1})$ and let $d := \dim(Y)$. The surjectivity of h implies that $\dim(V_1) \geq d$ and the finiteness implies $\dim(V_1) = d$. Let m be the largest integer such that $\dim(V_m) = d$. We first prove $m = 1$ and then conclude the proof by showing $m = n$ as well.
By way of contradiction assume $m > 1$. As h is finite, $\dim(h(V_i)) = \dim(V_i) = d$ and hence $h|_{V_i}$ is surjective for $i = 1, \ldots, m$. Then there is an open subset U of Y such that $h^{-1}(U) = \bigsqcup_{i=1}^{m} (h^{-1}(U) \cap V_i)$. This U is obtained by removing the images of the lower dimensional irreducible components and the intersections of $V_1, \ldots, V_m$. I.e.

$$Y \setminus U := \bigcup_{1 \leq i < j \leq m} h(V_i \cap V_j) \cup \bigcup_{i > m} h(V_i) \tag{8}$$

($Y \setminus U$ is closed as a finite union of closed subsets. Moreover all these closed sets have dimension strictly smaller than d hence U is non-empty and $h^{-1}(U) \cap V_i \neq \emptyset$ for $i = 1, \ldots, m$.) As $\mathcal{E}$ is coherent on $\mathrm{Supp}(\mathcal{E})$, it is locally free on $\mathrm{Supp}(\mathcal{E}) \setminus W$ where W is a finite union of closed subsets of dimension strictly smaller than d. Moreover $\mathcal{E}$ has constant rank on the components of $\mathrm{Supp}(\mathcal{E}) \setminus W$. Hence by reducing U even more, we may assume that

$$h^{-1}(U) = \bigsqcup_{i=1}^{m} (h^{-1}(U) \cap V_i) \tag{9}$$

with for each i: $\mathcal{E}|_{h^{-1}(U) \cap V_i}$ locally free of some constant rank r_i. Now for each i: $h_i : h^{-1}(U) \cap V_i \longrightarrow U$ is a finite, surjective morphism of varieties of some degree δ_i. Hence it is flat and by the above $h_{i,*}(\mathcal{E}|_{h^{-1}(U) \cap V_i})$ is a locally free sheaf of rank $r_i \cdot \delta_i > 0$. (9) implies that

$$(h_* \mathcal{E})|_U = \bigoplus_{i=1}^{m} h_{i,*}(\mathcal{E}|_{h^{-1}(U) \cap V_i}) \tag{10}$$

is a locally free sheaf of rank $\sum_{i=1}^{m} r_i \cdot \delta_i$. By assumption this rank is 1, which is only possible if $m = 1$ and $r_1 = \delta_1 = 1$.

Next we show $n = m = 1$. By way of contradiction assume $n > 1$. Then similar to (9) we define an open subset $U' \subset Y$ by

$$Y \setminus U' := \bigcup_{1 \leq i < j \leq n} h(V_i \cap V_j)$$

As for all $i < j : \dim(V_i \cap V_j) < \dim(V_2)$ we must have $h^{-1}(U') \cap V_2 \neq \emptyset$. On the other hand, as $\mathcal{E}|_{V_2}$ is coherent, $\mathcal{E}|_{h^{-1}(U') \cap V_2}$ is nonzero and hence $h_{2,*}\left(\mathcal{E}|_{h^{-1}(U') \cap V_2}\right)$ is a nonzero torsion sheaf on Y.

However as before we have

$$(h_*\mathcal{E})|_{U'} = \bigoplus h_{i,*}(\mathcal{E}|_{h^{-1}(U')\cap V_i})$$

leading to a contradiction as the left hand side is a line bundle and thus torsion free, whereas the right hand side has a nonzero torsion summand. Hence we have shown $m = n = 1$, such that $\operatorname{Supp}(\mathcal{E})$ is irreducible. In particular $h : \operatorname{Supp}(\mathcal{E}) \longrightarrow Y$ is a surjective, finite morphism of some degree $\delta > 0$ and $h_*(\mathcal{O}_{\operatorname{Supp}(\mathcal{E})})$ is a locally free $\mathcal{O}_Y$-module of rank δ. Locally on $h^{-1}(U)$ this sheaf has rank $\delta_1 = 1$, hence $\delta = 1$ and h is an isomorphism. One now easily checks that choosing $f := g \circ h^{-1}$ and $\mathcal{L} := h_*(\mathcal{E})$ gives an isomorphism of $X - Y$-bimodules $\mathcal{E} \cong {}_f(\mathcal{L})_{Id}$. □

Remark 3.3. Throughout the text we shall assume that $\mathcal{E}$ is given in the above form, i.e. $\mathcal{E} = {}_f(\mathcal{L})_{Id}$.

3.1. **Restricting to an open subset.** The first step in the proof of Theorem 3.1 is showing that there is an appropriate notion of restricting $\mathbb{S}(\mathcal{E})$ to an open subset of X and that the statement of Theorem 3.1 can be reduced to an open cover of X in this sense.
Throughout this section we will use the following notation:
If $U \subset X$ is an open subset, then we define $U^n \subset X_n$ as follows:

$$U^n = \begin{cases} U & \text{if } n \text{ is even} \\ f^{-1}(U) & \text{if } n \text{ is odd} \end{cases}$$

By construction U^n is an open subset of X_n and it is an affine open subset whenever U is because f is a finite morphism. For a bimodule $\mathcal{F} \in \operatorname{bimod}_S(X_n - X_{n+1})$, a sheaf $\mathbb{Z}$-algebra $\mathcal{A}$ and an $\mathcal{A}$-module $\mathcal{M}$ we will use the notation $|_U$ to denote the restriction to the corresponding open subset. I.e.

$$\begin{aligned} \mathcal{F}|_U &:= \mathcal{F}|_{U^n \times U^{n+1}} \\ (\mathcal{A}|_U)_{m,n} &:= (\mathcal{A}_{m,n})|_U = (\mathcal{A}_{m,n})|_{U^m \times U^n} \\ (\mathcal{M}|_U)_n &:= (\mathcal{M}_n)|_{U^n} \end{aligned}$$

To ensure that the restrictions of $\mathcal{A}$ to an open subset in turn has the structure of a sheaf $\mathbb{Z}$-algebra, we need the following technical condition:

Lemma 3.4. *Let $\mathcal{A}$ be a sheaf $\mathbb{Z}$-algebra and $U \subset X$ an open subset such that for each m, n:* $\operatorname{Supp}((\mathcal{A}_{m,n})|_{U^m \times X_n}) \subset U^m \times U^n$ *and* $\operatorname{Supp}((\mathcal{A}_{m,n})|_{X_m \times U^n}) \subset U^m \times U^n$. *Then*

i) $\mathcal{A}|_U$ *has an induced algebra structure.*
ii) Restriction of modules to U defines a functor $|_U : \operatorname{Gr}(\mathcal{A}) \to \operatorname{Gr}(\mathcal{A}|_U)$

Proof. i) We must show that for all $l, m, n \in \mathbb{Z}$ there are multiplication morphisms $\mathcal{A}_{l,m}|_U \otimes \mathcal{A}_{m,n}|_U \to \mathcal{A}_{l,n}|_U$ induced by the morphisms $\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n} \to \mathcal{A}_{l,n}$. The latter induces a morphism of $U^l - U^n$-bimodules:

$$(\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n})|_U \to \mathcal{A}_{l,n}|_U$$

Now the claim follows from the following chain of isomorphisms:

$$
\begin{aligned}
(\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n})|_U &= \left(\pi_{X_l,X_n*}\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m}) \otimes_{X_l\times X_m\times X_n} \pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\right)\right)|_{U^l\times U^n} \\
&= \pi_{U^l,U^n*}\left(\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m}) \otimes_{X_l\times X_m\times X_n} \pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\right)|_{U^l\times X_m\times U^n}\right) \\
&= \pi_{U^l,U^n*}\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m})|_{U^l\times X_m\times U^n} \otimes \pi^*_{X_m,X_n}(\mathcal{A}_{m,n})|_{U^l\times X_m\times U^n}\right) \\
&= \pi_{U^l,U^n*}\left(\pi^*_{U^l,X_m}(\mathcal{A}_{l,m}|_{U^l\times X_m}) \otimes_{U^l\times X_m\times U^n} \pi^*_{X_m,U^n}(\mathcal{A}_{m,n}|_{X_m\times U^n})\right) \\
&= \pi_{U^l,U^n*}\left(\pi^*_{U^l,U^m}(\mathcal{A}_{l,m}|_{U^l\times U^m}) \otimes_{U^l\times U^m\times U^n} \pi^*_{U^m,U^n}(\mathcal{A}_{m,n}|_{U^m\times U^n})\right) \\
&= \mathcal{A}_{l,m}|_U \otimes \mathcal{A}_{m,n}|_U
\end{aligned}
$$

Where π_{U^l,X_m} and π_{U^l,U^m} are the projections $\pi_{U^l,X_m} : U^l \times X_m \times U^n \to U^l \times X_m$ and $\pi_{U^l,U^m} : U^l \times U^m \times U^n \to U^l \times U^m$, with similar definitions for π_{X_m,U^n} and π_{U^m,U^n}.

The first equality is the definition of tensor product of bimodules

$$\mathrm{bimod}(X_l - X_m) \times \mathrm{bimod}(X_m - X_n) \to \mathrm{bimod}(X_l - X_n)$$

The second equality follows from the commutation of pushforward and restriction of sheaves. The third equality follows from the commutation of tensor product of sheaves and restriction. The fourth equality follows from the commutation of pullback and restriction of sheaves. The fifth equality follows the assumption of the lemma. The last equality is the definition of multiplication

$$\mathrm{bimod}(U^l - U^m) \times \mathrm{bimod}(U^m - U^n) \to \mathrm{bimod}(U^l - U^n)$$

ii) This essentially reduces to showing $(\mathcal{M}_i \otimes \mathcal{A}_{i,j})|_{U_j} = (\mathcal{M}|_U)_i \otimes (\mathcal{A}|_U)_{i,j}$ which is completely similar to i).

□

As an immediate corollary we have

Corollary 3.5. *For any $U \subset X$,*

i) $\mathbb{S}(\mathcal{E})|_U$ has an algebra structure induced by $\mathbb{S}(\mathcal{E})$
ii) There is a functor $|_U : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \to \mathrm{Gr}(\mathbb{S}(\mathcal{E})|_U)$
iii) There is an isomorphism of symmetric sheaf $\mathbb{Z}$-algebras: $\mathbb{S}(\mathcal{E})|_U \cong \mathbb{S}(\mathcal{E}|_U)$

Proof. i+ii) As $\mathcal{E}$ is given by ${}_f(\mathcal{L})_{Id}$, the conditions in Lemma 3.4 are obviously satisfied if $\mathcal{A} = \mathbb{S}(\mathcal{E})$. For iii) We first show that for all $n \in \mathbb{N}$ there is a natural isomorphism

$$\eta_{\mathcal{E}} : (\mathcal{E}^{*n})|_U = (\mathcal{E}|_U)^{*n} \tag{11}$$

By Remark 2.5 we see by induction that for each $i \geq 0$ there is a line bundle $\mathcal{L}_i$ such that

$$
\begin{aligned}
\mathcal{E}^{*2i} &= {}_f(\mathcal{L}_i)_{Id} \\
\mathcal{E}^{*2i+1} &= {}_{Id}(\mathcal{L}_i)_f
\end{aligned}
$$

where $\mathcal{L}_0 = \mathcal{L}$. The explicit form of the dual (11) shows that it suffices to exhibit isomorphisms

$${}_f\left(\mathcal{H}om_Y(\mathcal{L}_i, f^!\mathcal{O}_X)\right)_{Id_X}\Big|_U \cong {}_{f|_U}\left(\mathcal{H}om_{f^{-1}(U)}((\mathcal{L}_i)|_{f^{-1}(U)}, (f|_U)^!\mathcal{O}_U)\right)_{Id_U}$$

However as restriction to open affine subsets commutes with f_*, $\mathcal{H}om_Y$ and $f^!$, this isomorphism is immediate.
Note that (11) is valid for $n < 0$ as well since $(-)^{*(-n)}$ is the inverse of $(-)^{*n}$. Finally, the naturality of $\eta_{\mathcal{E}}$ immediately implies that the restricted unit morphisms $i_n|_U$ coincides with

$$_{Id}\left(\mathcal{O}_{U^n}\right)_{Id} \longrightarrow \left(\mathcal{E}|_U\right)^{*n} \otimes \left(\mathcal{E}|_U\right)^{*n+1}$$

Implying in particular that $\eta_{\mathcal{E}}$ induces an isomorphism

$$i_n({}_{Id}\left(\mathcal{O}_{U^n}\right)_{Id}) \cong i_n({}_{Id}\left(\mathcal{O}_{X^n}\right)_{Id})|_{U^n}$$

and we can extend $\eta_{\mathcal{E}}$ to an isomorphism

$$\mathbb{S}(\mathcal{E})|_U \cong \mathbb{S}(\mathcal{E}|_U)$$ □

Lemma 3.6. *Let $\bigcup_l U_l$ be a finite open cover for X. Moreover assume that $\mathcal{A}$ is a sheaf $\mathbb{Z}$-algebra such that the conditions in Lemma 3.4 are satisfied for all U_l, then*

$$\forall l : \mathcal{M}|_{U_l} \in \mathrm{Gr}(\mathcal{A}|_{U_l}) \text{ is noetherian } \Rightarrow \mathcal{M} \in \mathrm{Gr}(\mathcal{A}) \text{ is noetherian}$$

Proof. Suppose we are given an ascending chain of subobjects of $\mathcal{M}^n \subset \mathcal{M}$ in $\mathrm{Gr}(\mathcal{A})$ such that the restriction of this chain to any of the U_l stabilizes. As there are only finitely many U_l, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all l: $(\mathcal{M}^n)|_{U_l} = (\mathcal{M}^{n+1})|_{U_l}$. By a degree-wise application of the glueing axiom, the graded modules $\mathcal{M}^n$ and $\mathcal{M}^{n+1}$ must coincide. □

3.2. **covering by relative Frobenius pairs.** Lemma 3.6 shows that proving that a given set of generators is in fact a set of noetherian generators can be done locally. In this subsection we construct an open cover $X = \bigcup_l U_l$ for which the categories $\mathrm{Gr}(\mathbb{S}(\mathcal{E})|_{U_l})$ can explicitly described (see 3.13). For this cover, the sections satisfy a relative version of the Frobenius property as introduced in the paper [3], whose definition and properties we recall below:

Definition 3.7. We say that S/R is *relative Frobenius* of rank n if:

- S is a free R-module of rank n.
- $\mathrm{Hom}_R(S, R)$ is isomorphic to S as S-module.

Remark 3.8. It is clear that if R is a field, then S/R being relative Frobenius coincides with S being a finite dimensional Frobenius algebra in the classical sense.

We shall need the following notation: for a relatively Frobenius pair, let $M := {}_RS_S$. This $R-S$-bimodule can be considered a $R \oplus S$ bimodule by letting only the R-component act on the left and only the S-component on the right. Similarly, we let $N := {}_SS_R$ and consider it an $R \oplus S$-bimodule by letting only the component S act on the left and only the component R act on the right. We now define

$$T(R, S) := T_{R\oplus S}(M \oplus N)$$

Note that by construction, in degree 2, we have $M \otimes_{R\oplus S} M = N \otimes_{R\oplus S} N = 0$, hence

$$T(R, S)_2 = (M_{R\oplus S}N) \oplus (N \otimes_{R\oplus S} M) = ({}_RS \otimes_S S_R) \oplus ({}_SS \otimes_R S_R)$$

The algebra we will be concerned in will be a quotient of $T(R, S)$ as follows: let λ be a generator of $\mathrm{Hom}_R(S, R)$ as an S-module. The R-bilinear form $\langle a\, , b\rangle := \lambda(ab)$ is clearly nondegenerate and hence we can find dual R-bases $(e_i)_i$, $(f_j)_j$ satisfying

$$\lambda(e_i f_j) = \delta_{ij}$$

Definition 3.9. For a relative Frobenius pair S/R, the *generalized preprojective algebra* $\Pi_R(S)$ is given by

$$T(R,S)/(\text{rels})$$

where the relations are in degree 2 given by

$$1 \otimes 1 \in {}_R S \otimes_S S_R$$
$$\sum_i e_i \otimes f_i \in {}_S S \otimes_R S_S$$

Remark 3.10. If S is the ring $R^{\oplus n}$. Then $\Pi_R(S)$ is isomorphic to the preprojective algebra over R associated to the quiver with one central vertex and n outgoing arrows. (See [3, Lemma 1.5])

We shall use the following result from [3]:

Theorem 3.11. *Let S/R be relative Frobenius of rank 4 and assume R is noetherian, then $\Pi_R(S)$ is noetherian as well.*

Throughout, we shall make use of the following lemma, well-known to experts:

Lemma 3.12. *Let $\mathcal{L}$ be a line bundle on Y and $p \in X$. Then there is an open subset $U \subset X$ containing p, such that $\mathcal{L}|_{f^{-1}(U)} \cong \mathcal{O}_{f^{-1}(U)}$.*

Proof. We can reduce to the case where $X = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(S)$ are affine schemes where S is finitely generated over R and $\mathcal{L} = \tilde{L}$ for some invertible S-module L. Let $\mathfrak{p}$ be the prime ideal in $\operatorname{Spec}(R)$ corresponding to $f(p) \in X$, then $S_\mathfrak{p} := S \otimes_R R_\mathfrak{p}$ is a semilocal ring, hence every finitely generated projective of constant rank is free and in particular the Picard group is trivial. Consequently, there exists an $l \in L$ such that

$$S_\mathfrak{p} \xrightarrow{\cdot l} L_\mathfrak{p}$$

is an isomorphism.

Now consider the morphism $S \xrightarrow{\cdot l} L$ with kernel K and cokernel C. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow S \xrightarrow{\cdot l} L \longrightarrow C \longrightarrow 0 \tag{12}$$

K is a finitely generated R-submodule of S by the noetherianity of R. L is finitely generated over R, being an invertible S-module. It follows that C is finitely generated over R as a quotient of L.

Now let $\alpha_1, \ldots, \alpha_n$ be a set of generators for K, then as $K \otimes R_\mathfrak{p} = 0$ there exist elements $x_1, \ldots, x_n \in R \backslash \mathfrak{p}$ such that $\alpha_1 x_1 = \ldots = \alpha_n x_n = 0$. Set $x := x_1 \cdot \ldots \cdot x_n \in R \backslash \mathfrak{p}$, then $\alpha \cdot x = 0$ for all $\alpha \in K$. Similarly there is a $x' \in R \backslash \mathfrak{p}$ such that $\beta \cdot x' = 0$ for all $\beta \in C$. Now define $z = x \cdot x'$, then $K \otimes R_z = C \otimes R_z = 0$ implying that $\cdot l$ defines an isomorphism

$$S \otimes R_z \xrightarrow{\cong} L \otimes R_z$$

$U = \operatorname{Spec}(R_z)$ then is the desired open subset. □

We can now prove the main result of this subsection

Theorem 3.13. *Write $\mathcal{E} = {}_f(\mathcal{L})_{Id}$ as in lemma 3.2 . There is a finite cover $X = \bigcup_l U_l$ of affine open subsets $U_l = \operatorname{Spec}(R_l)$ such that:*

i) $\mathcal{L}|_{f^{-1}(U_l)}$ *is a trivial* $\mathcal{O}_{f^{-1}(U_l)}$*-module*

ii) $\omega_Y|_{f^{-1}(U_l)}$ *is a trivial* $\mathcal{O}_{f^{-1}(U_l)}$*-module*

iii) $\omega_X|_{U_l}$ *is a trivial* $\mathcal{O}_{U_l}$*-module*
iv) $f^{-1}(U_l) = \mathrm{Spec}(S_l)$ *where* S_l/R_l *is relative Frobenius of rank 4.*

Proof. We first note the following two facts:

- Let $\mathrm{Spec}(R)$ be an affine open subset on which i), ii), iii) or iv) holds. Then the same statement holds for any standard open $\mathrm{Spec}(R_f) \subset \mathrm{Spec}(R)$. This is obvious for i), ii) and iii). For iv) it follows from [3, Lemma 3.1].
- Let $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R')$ be affine open subsets of X, then their intersection is covered by affine open subsets of the form $\mathrm{Spec}(R_f) = \mathrm{Spec}(\mathrm{R}'_g)$

By these two facts it suffices to find affine open covers for i), ii), iii) and iv) separately. For i) and ii) such a cover exists by Lemma 3.12 and the fact that ω_Y is a line bundle on the smooth variety Y. The existence for a cover satisfying iii) is immediate from the fact that ω_X is a line bundle. Hence the proof reduces to finding a cover satisfying iv).
As $f : Y \longrightarrow X$ is a finite, by Lemma 3.12: $f^!\omega_X$ is completely determined by $f_*\left(f^!\omega_X\right)$ and we have an isomorphism of $f_*\mathcal{O}_Y$-modules

$$f_*\left(f^!\omega_X\right) := \mathcal{H}om_X(f_*\mathcal{O}_Y, \omega_X) \cong f_*\omega_Y \tag{13}$$

As moreover f is also surjective and flat, there is a cover $X = \bigcup_l U_l$ with $U_l = \mathrm{Spec}(R_l)$ and $f^{-1}(U_l) = \mathrm{Spec}(S_l)$ where S_l is a free R_l-module of rank 4 for each l. By the previous arguments we can assume that ii) and iii) are also satisfied on this cover. In this case, replacing f by its restriction $f^{-1}(U_l) \longrightarrow U_l$, (13) reads

$$f_*\left(f^!\mathcal{O}_{U_l}\right) := \mathcal{H}om_{U_l}(f\mathcal{O}_{f^{-1}(U_l)}, \mathcal{O}_{U_l}) \cong f_*\mathcal{O}_{f^{-1}(U_l)}$$

and taking sections yields the required isomorphism of S_l-modules:

$$\mathrm{Hom}_{R_l}(S_l, R_l) \cong S_l$$

□

3.3. **A local description of** $\mathbb{S}(\mathcal{E})$**.** In this sectin, we shall show that for affine schemes satsifying the conditions of 3.13, it is possible to describe $\mathbb{S}(\mathcal{E})$ using generalized preprojective algebras from 3.9. We shall assume that X and Y are smooth affine varietes over some base field k, say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ such that S/R is relative Frobenius of rank 4 and $\omega_X \cong \mathcal{O}_X$, $\omega_Y \cong \mathcal{L} \cong \mathcal{O}_Y$. We introduce some auxiliary notations: for the be the symmetric sheaf-$\mathbb{Z}$-algebra over X and Y in standard form $\mathbb{S}(\mathcal{E})$, there is a $\mathbb{Z}$-algebra over k, $\Gamma(\mathcal{A})$ defined by

$$\Gamma(\mathcal{A})_{m,n} := \Gamma(X_m \times X_n, \mathcal{A}_{m,n})$$

since each component $\Gamma(\mathcal{A})_{m,n}$ is an $R - S$ or $S - R$ bimodule depending on the indices, $\Gamma(\mathcal{A})$ is in fact a $\mathbb{Z}$-algebra over the ring $R \oplus S$ as in the discussion following remark 3.8. The classical equivalence between quasi-coherent modules and global sections can easily be adapted to our setting to obtain an equivalence:

$$\Gamma : \mathrm{Gr}(\mathcal{A}) \xrightarrow{\cong} \mathrm{Gr}(\Gamma(\mathcal{A})) : \{\mathcal{M}_n\}_{n\in\mathbb{Z}} \mapsto \{\Gamma(X_n, \mathcal{M}_n)\}_{n\in\mathbb{Z}}$$

The following is an immediate consequence of the assumptions of this section:

Lemma 3.14. *The* $\mathbb{Z}$*-algebra* $\Gamma(\mathbb{S}(\mathcal{E}))$ *is 2-periodic in the sense that*

$$\Gamma(\mathbb{S}(\mathcal{E}))_{m,n} = \Gamma(\mathbb{S}(\mathcal{E}))_{m+2,n+2}$$

Proof. By 2.13, there are isomorphisms $\mathbb{S}(\mathcal{E}))_{i+2,j+2} \cong \omega_i^{-1} \otimes \mathbb{S}(\mathcal{E})) \otimes \omega_j$. By the assumptions in the beginning of this sections, both canonical bundles are trivial, implying that $\mathbb{S}(\mathcal{E})_{m,n} = \mathbb{S}(\mathcal{E})_{m+2,n+2}$. The result follows after applying $\Gamma(-)$. □

Using the methods in appendix A (Lemma A.3), the 2-periodic $\mathbb{Z}$-algebra $\Gamma(\mathbb{S}(\mathcal{E}))$ gives rise to a graded algebra $\overline{\Gamma(\mathbb{S}(\mathcal{E}))}$.

Lemma 3.15. *Let $X = \operatorname{Spec}(R)$ and $Y = \operatorname{Spec}(S)$ be affine schemes such that S/R is relative Frobenius of rank 4. Let $f : Y \to X$ be the induced morphism and $\mathcal{E} = {}_f(\mathcal{O}_Y)_{Id}$. Then $\overline{\Gamma(\mathbb{S}(\mathcal{E}))} \cong \Pi_R(S)$.*

Proof. Consider the quotient map

$$\mathbb{T}(\mathcal{E}) \twoheadrightarrow \mathbb{S}(\mathcal{E})$$

Taking global sections in each component $\Gamma(X_m \times X_n, (-)_{m,n})$ yields a surjection

$$\Gamma(\mathbb{T}(\mathcal{E})) \twoheadrightarrow \Gamma(\mathbb{S}(\mathcal{E})).$$

because $X_m \times X_n$ is affine.
Since the functor $\overline{(-)}$ preserves surjectivity (see Proposition A.6), we obtain a map

$$\pi : \overline{\Gamma(\mathbb{T}(\mathcal{E}))} \twoheadrightarrow \overline{\Gamma(\mathbb{S}(\mathcal{E}))}.$$

We first show that there is a canonical isomorphism of $R \oplus S$-modules

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))} \cong T(R, S) \tag{14}$$

For this (as $\Gamma(\mathbb{S}(\mathcal{E})$ is clearly generated in degrees 0 and 1) it suffices to show the following three facts

- $\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_0 \cong T(R, S)_0 = R \oplus S$ as rings
- $\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_1 \cong T(R, S)_1 \cong {}_R S_S \oplus {}_S S_R$ as $R \oplus S$ modules
- the multiplication map yields isomorphisms

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_1 \otimes \overline{\Gamma(\mathbb{T}(\mathcal{E}))}_n \xrightarrow{\cong} \overline{\Gamma(\mathbb{T}(\mathcal{E}))}_{n+1}$$

For the first item, we compute:

$$\begin{aligned} \overline{\Gamma(\mathbb{T}(\mathcal{E}))}_0 &= \begin{pmatrix} \Gamma(\mathbb{T}(\mathcal{E}))_{0,0} & 0 \\ 0 & \Gamma(\mathbb{T}(\mathcal{E}))_{1,1} \end{pmatrix} \\ &= \begin{pmatrix} \Gamma\left(X \times X, {}_{Id}\left(\mathcal{O}_X\right)_{Id}\right) & 0 \\ 0 & \Gamma\left(Y \times Y, {}_{Id}\left(\mathcal{O}_Y\right)_{Id}\right) \end{pmatrix} \end{aligned}$$

next, we have

$$\begin{aligned} \Gamma\left(X \times X, {}_{Id}\left(\mathcal{O}_X\right)_{Id}\right) &= \operatorname{Hom}\left(\mathcal{O}_{X\times X}, \Delta_*\left(\mathcal{O}_X\right)\right) \\ &= \operatorname{Hom}\left(\Delta^*\left(\mathcal{O}_{X\times X}\right), \mathcal{O}_X\right) \\ &= \operatorname{Hom}\left(\mathcal{O}_X, \mathcal{O}_X\right) \\ &\cong R \end{aligned}$$

And similarly $\Gamma\left(Y \times Y, {}_{Id}\left(\mathcal{O}_Y\right)_{Id}\right) \cong S$ such that

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_0 \cong \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \cong R \oplus S$$

In a completely similar fashion, we check the second condition:

$$\begin{aligned}
\overline{\Gamma(\mathbb{T}(\mathcal{E}))_1} &= \begin{pmatrix} 0 & \Gamma(\mathbb{T}(\mathcal{E}))_{0,1} \\ \Gamma(\mathbb{T}(\mathcal{E}))_{1,2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Gamma\left(X\times Y,\mathcal{E}\right) \\ \Gamma\left(Y\times X,\mathcal{E}^*\right) & \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Gamma\left(X\times Y,{}_f(\mathcal{O}_Y)_{Id}\right) \\ \Gamma\left(Y\times X,{}_{Id}(\mathcal{O}_Y)_f\right) & 0 \end{pmatrix} \\
&\cong \begin{pmatrix} 0 & {}_RS_S \\ {}_SS_R & 0 \end{pmatrix} \cong {}_RS_S \oplus {}_SS_R
\end{aligned}$$

To check the final condition, we have the isomorphisms

$$\mathbb{T}(\mathcal{E})_{i,i+1} \otimes \mathbb{T}(\mathcal{E})_{i+1,i+n+1} \longrightarrow \mathbb{T}(\mathcal{E})_{i,i+n+1}$$

We now apply the $\Gamma(X_i \times X_{i+n+1}, -)$. Note that as all schemes are affine, the tensor product and $\Gamma(-)$ commute, resulting in an isomorphism

$$\Gamma(\mathbb{T}(\mathcal{E}))_{i,i+1} \otimes \Gamma(\mathbb{T}(\mathcal{E}))_{i+1,i+n+1} \longrightarrow \Gamma(\mathbb{T}(\mathcal{E}))_{i,i+n+1}$$

and finally, application of the functor $\overline{(-)}$ yields the required isomorphism

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))_1} \otimes \overline{\Gamma(\mathbb{T}(\mathcal{E}))_n} \xrightarrow{\cong} \overline{\Gamma(\mathbb{T}(\mathcal{E}))_{n+1}}$$

proving the isomorphism (14). Finally, we prove that the relations defining $\Pi_R S$ coincide with the kernel of π, i.e. there is a commutative diagram:

$$\begin{array}{ccc}
\overline{\Gamma(\mathbb{T}(\mathcal{E}))} & \xrightarrow{\pi} & \overline{\Gamma(\mathbb{S}(\mathcal{E})} \\
\cong\downarrow & & \downarrow\cong \\
T(R,S) & \xrightarrow[\overline{\pi}]{} & \Pi_R(S)
\end{array}$$

By the isomorphisms in the previous step and the construction of Γ, there are isomorphisms:

$$\zeta_0 : \mathrm{Hom}_{X\times X}({}_{Id}\left(\mathcal{O}_X\right)_{Id}, \mathcal{E}\otimes\mathcal{E}^*) \xrightarrow{\cong} \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)$$

$$\zeta_1 : \mathrm{Hom}_{Y\times Y}({}_{Id}\left(\mathcal{O}_Y\right)_{Id}, \mathcal{E}^*\otimes\mathcal{E}) \xrightarrow{\cong} \mathrm{Hom}_S(S, {}_SS_R \otimes_R S_S)$$

$\mathbb{S}(\mathcal{E})$ is defined as a quotient of $\mathbb{T}(\mathcal{E})$ by the relations given by the unit morphisms $i_0 \in \mathrm{Hom}_{X\times X}({}_{Id}\left(\mathcal{O}_X\right)_{Id}, \mathcal{E}\otimes\mathcal{E}^*), i_1 \in \mathrm{Hom}_{Y\times Y}({}_{Id}\left(\mathcal{O}_Y\right)_{Id}, \mathcal{E}^*\otimes\mathcal{E})$ as in (2). Similarly $\Pi_R(S)$ is defined as a quotient of $T_R(S)$ by elements $\eta_0 \in \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)$, $\eta_1 \in \mathrm{Hom}_S(S, {}_SS_R \otimes_R S_S)$. Hence we must prove $\zeta_0(i_0) = \eta_0$ and $\zeta_1(i_1) = \eta_1$. First note that there is a commutative diagram of isomorphisms

$$\begin{array}{ccc}
\mathrm{Hom}_{X\times Y}(\mathcal{E},\mathcal{E}) & \longrightarrow & \mathrm{Hom}_{X\times X}({}_{Id}\left(\mathcal{O}_X\right)_{Id}, \mathcal{E}\otimes\mathcal{E}^*) \\
\downarrow & & \downarrow\zeta_0 \\
\mathrm{Hom}_{R\otimes S}({}_RS_S, {}_RS_S) & \xrightarrow{\varphi_0} & \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)
\end{array}$$

where φ_0 is given by the adjunction $- \otimes_R S_S \dashv - \otimes_S S_R = (-)_R$. Hence $\zeta_0(i_0) = \varphi_0(Id_{_R S_S}) : 1_R \mapsto 1_S \otimes 1_S$, which coincides with η_0. Similarly the existence of the dual bases $(e_i)_i$, $(f_j)_j$ implies there is an adjunction
$- \otimes_S S_R = (-)_R \dashv - \otimes_R S_S$ given by

$$\varphi_1 : \mathrm{Hom}_R(M \otimes_S S_R, N) \longrightarrow \mathrm{Hom}_S(M, N \otimes_R S_S) : \psi \mapsto \left(\psi' : m \mapsto \sum_i \psi(me_i) \otimes f_i\right)$$

Where we used Lemma 3.16 to see that the morphisms in the image of φ_1 indeed have an S-module structure. A commutative diagram as above shows that $\zeta_1(i_1) = \varphi_1(Id_{_S S_R}) : 1_S \mapsto \sum_i e_i \otimes f_i$ which coincides with η_1.

□

Lemma 3.16. *$\sum_i e_i \otimes f_i$ is central in the S-bimodule $S \otimes_R S$. I.e. for all $a \in S$ we have*

$$\sum_i ae_i \otimes f_i = \sum_i e_i \otimes f_i a$$

Proof. It is sufficient to prove that for all j, k we have

$$\sum_i \lambda(ae_i f_j)\lambda(f_i e_k) = \sum_i \lambda(e_i f_j)\lambda(f_i a e_k)$$

which is clear since both sides are equal to $\lambda(ae_k f_j)$. □

3.4. **Proof of Theorem 3.1.** As X and Y are noetherian we know that $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are locally noetherian categories and hence there exist collections of noetherian generating objects for these categories, say $\mathcal{N}^X := \{\mathcal{N}_i^X\}_{i \in I}$ and $\mathcal{N}^Y := \{\mathcal{N}_j^Y\}_{j \in J}$. For each $n \in \mathbb{Z}$ we define $\mathcal{N}^n$ in $\mathrm{QCoh}(X_n)$ as:

$$\mathcal{N}^n = \begin{cases} \mathcal{N}^X & \text{if } n \text{ is even} \\ \mathcal{N}^Y & \text{if } n \text{ is odd} \end{cases}$$

We shall prove that the collection

$$\{\mathcal{N} \otimes e_n \mathbb{S}(\mathcal{E}) \mid n \in \mathbb{Z}, \mathcal{N} \in \mathcal{N}^n\} \tag{15}$$

forms a set of noetherian generators for $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. Not that the collection is easily seen to generate as for each $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ there is a surjective morphism

$$\oplus_{n \in \mathbb{Z}} \mathcal{M}_n \otimes e_n \mathcal{A} \twoheadrightarrow \mathcal{M}$$

and for each $n \in \mathbb{Z}$ there is a surjective morphism

$$\oplus_\alpha (\mathcal{N}_\alpha^n)^{m_\alpha} \twoheadrightarrow \mathcal{M}_n$$

where $\mathcal{N}_\alpha^n \in \mathcal{N}^n$. Hence we only need to show that the elements of (15) are noetherian objects in $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. By Lemma 3.6 and Corollary 3.5 this can be checked locally for any open cover $X = \bigcup_l U_l$. By Theorem 3.13 we may hence assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are affine schemes such that

i) $\mathcal{L} \cong \mathcal{O}_Y \cong \omega_Y$
ii) $\omega_X \cong \mathcal{O}_X$
iii) S/R is relative Frobenius of rank 4.

With these assumptions there are functors

$$\begin{array}{c} \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \\ \cong \big\downarrow \\ \mathrm{Gr}(\Gamma\left(\mathbb{S}(\mathcal{E})\right)) \\ \big\downarrow \text{Proposition A.6} \\ \mathrm{Gr}\left(\overline{\Gamma\left(\mathbb{S}(\mathcal{E})\right)}\right) \\ \cong \big\downarrow \text{Lemma 3.15} \\ \mathrm{Gr}(\Pi_R(S)) \end{array} \tag{16}$$

Let $F : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{Gr}(\Pi_R(S))$ be the composition. Then the above diagram shows that F is an exact embedding of categories. Hence $\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E})$ is a noetherian object in $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ if $F(\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E}))$ is a noetherian object in $\mathrm{Gr}(\Pi_R(S))$. On the other hand, as $\mathcal{N}$ is noetherian in $\mathrm{QCoh}(X_n)$ there is an $m \in \mathbb{N}$ and an surjection $\mathcal{O}_{X_n}^{\oplus m} \twoheadrightarrow \mathcal{N}$ giving rise to an surjection

$$F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E}))^{\oplus m} \twoheadrightarrow F(\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E}))$$

Hence it suffices to show that $F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E}))$ is a Noetherian object in $\Pi_R(S)$. This is however obvious as

$$F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E})) = \begin{cases} R \cdot \Pi_R(S)(-n) & \text{if } n \text{ is even} \\ S \cdot \Pi_R(S)(-n) & \text{if } n \text{ is odd} \end{cases}$$

As both $R \cdot \Pi_R(S)$ and $S \cdot \Pi_R(S)$ are direct summands of $\Pi_R(S)$, which is a noetherian ring by Theorem 3.11, we have proven the theorem. □

Since the proof of Theorem 3.1 exhibits an explicit set of generators, we can also prove the following:

Theorem 3.17. *The category* $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ *is Grothendieck.*

Proof. Let $(\mathcal{M}_i, f_{ij})$ be a direct system of graded $\mathbb{S}(\mathcal{E})$-modules. In each degree d, we obtain a direct system of quasicoherent X_d-modules $(\mathcal{M}^d, f_{ij}^d)$. Since $\mathrm{QCoh}(X_n)$ is Grothendieck, we can form the direct limit in each degree to obtain a sequence of X_n-modules $\mathcal{L}_n := \varinjlim(\mathcal{M}_i^n, f_{ij}^n)$. If we fix a couple (n, m), the universality of the direct limit naturally defines a map

$$\mathbb{S}(\mathcal{E})_{n,m} \otimes \mathcal{L}_n = \mathbb{S}(\mathcal{E})_{n,m} \otimes \varinjlim(X_i^n, f_{ij}^n) \longrightarrow \varinjlim(X_i^m, f_{ij}^m) = \mathcal{L}_m$$

showing that $\mathcal{L}$ is in fact a graded $\mathbb{S}(\mathcal{E})$-module. The fact that $\mathcal{L}$ is a direct limit and that the formation of $\mathcal{L}$ is exact is an easy consequence of the construction. Finally a collection of generators for $\mathrm{Gr}(\mathbb{S}(\mathcal{E})$ is given by (15). □

4. Homological Properties of Sheaf $\mathbb{Z}$-Algebras

This section is dedicated to adapting the results in [9] and [5] to obtain a formula to compute certain Ext-groups. Throughout X and Y will denote smooth curves over an algebraically closed field k of characteristic zero. Let $\mathcal{A}$ be a sheaf $\mathbb{Z}$ algebra over $(X_i \longrightarrow \mathrm{Spec}(k))_i$. To keep the geometric intuition (as in 2.14) we denote the truncation functors $(\omega(-))_m : \mathrm{Proj}(\mathcal{A}) \longrightarrow \mathrm{QCoh}(X_m)$ by Π_{m*}. The left adjoints, which are given explicitly by $p((-)\otimes e_m\mathcal{A})$, are in turn denoted by Π_m^*. We shall use the notations X_n and Q_n as in (3).

If $\mathcal{E}\in \mathrm{bimod}(X-X)$ is locally free of rank (2,2) and $\mathcal{A}=\mathbb{S}(\mathcal{E})$, [5] computes the Euler characteristics $\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\rangle$ for two locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. In this section, we develop the machinery to perform an analogous calculation in the case of a bimodule $\mathcal{E}\in \mathrm{bimod}_S(X-Y)$ of rank (4,1). As mentioned in the introduction and motivated by Proposition 2.13 our main goal is to understand Euler characteristics of the form $\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\rangle$ with $n-m=1,0$ or -1. The goal of this section will be to prove the following, slightly stronger theorem:

Theorem 4.1. *Let $\mathcal{E}\in \mathrm{bimod}(X,Y)$ be locally free of rank (4,1). Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves on X_m respectively X_n for $m,n\in\mathbb{Z}$ such that $m\geq n-1$. Then*

$$\mathrm{Ext}^i_{\mathrm{Proj}(\mathcal{A})}\left(\Pi_m^*\mathcal{F},\Pi_n^*\mathcal{G}\right)\cong \mathrm{Ext}^i_{X_m}\left(\mathcal{F},\mathcal{G}\otimes\mathbb{S}(\mathcal{E})_{n,m}\right)$$

for all $i\geq 0$.

From this we immediately have the following corollaries:

Corollary 4.2. *With the above assumptions, the Euler characteristics of the pulled back sheaves on* $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ *are given by*

$$\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\rangle = \langle \mathcal{F}, \mathcal{G}\otimes\mathbb{S}(\mathcal{E})_{n,m}\rangle$$

Corollary 4.3. *Let $\{\mathcal{F}_1,\ldots,\mathcal{F}_a\}$ and $\{\mathcal{G}_1,\ldots,\mathcal{G}_b\}$ be exceptional sequences of locally free sheaves on X_n and X_{n+1} respectively.*
Then $\Pi_{n+1}^*\mathcal{G}_1,\ldots\Pi_{n+1}^*\mathcal{G}_b,\Pi_n^*\mathcal{F}_1,\ldots,\Pi_n^*\mathcal{F}_b$ is an exceptional sequence on $\mathrm{Proj}(\mathcal{A})$.

As an immediate application of Corollary 4.2 we can construct a noncommutative Del-Pezzo surface with the desired Gram matrix (see introduction):

Corollary 4.4. *Let $\mathcal{E}$ be the following $\mathbb{P}^1$-bimodule ${}_f(\mathcal{O}_{\mathbb{P}^1})_{Id}$ and $\mathbb{S}(\mathcal{E})$ be the associated symmetric sheaf $\mathbb{Z}$-algebra. Then*

$$\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}),\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)),\Pi_0^*(\mathcal{O}_{\mathbb{P}^1}),\Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))$$

is an exceptional sequence of graded $\mathbb{S}(\mathcal{E})$-modules for which the Gram matrix of the Euler form is given by

$$\begin{bmatrix}1&2&1&5\\0&1&0&4\\0&0&1&2\\0&0&0&1\end{bmatrix}$$

Throughout this section $\mathcal{E}$ will be a locally free $X-Y$-bimodule of rank $(4,1)$ and we let $\mathcal{A}:=\mathbb{S}(\mathcal{E})$ denote the associated symmetric sheaf $\mathbb{Z}$-algebra in standard form. The proof of Theorem 4.1 is based on a chain of lemmas and the following technical result of which the proof will be the subject of appendix B:

Theorem 4.5. *There is an exact sequence (in $\mathrm{bimod}(\mathcal{O}_{X_m}-\mathcal{A})$, see* [9, Section 3.2.] *for the definition of this category)*

$$0\longrightarrow \mathcal{Q}_m\otimes e_{m+2}\mathcal{A}\longrightarrow \mathcal{E}^{*m}\otimes e_{m+1}\mathcal{A}\longrightarrow e_m\mathcal{A}\longrightarrow {}_{Id}(\mathcal{O}_{X_m})_{Id}\longrightarrow 0 \quad (17)$$

Proof. By the nature of the relations this sequence is known to be right exact. The proof of the left exactness is based on point modules and is given in Appendix B. □

As an immediate corollary of this theorem and its proof we find:

Corollary 4.6. *for each $i, j \in \mathbb{Z}$, the bimodule $\mathcal{A}_{ij}$ is locally free on the left and on the right. Its rank is equal to*

$$\operatorname{rk}(\mathcal{A})_{m,n} := \begin{cases} (n-m+1, n-m+1) & m \equiv n \, mod \, 2 \\ \left(\dfrac{n-m+1}{2}, 2(n-m+1)\right) & m \, odd, \, n \, even \\ \left(2(n-m+1), \dfrac{n-m+1}{2}\right) & m \, even, \, n \, odd \end{cases}$$

An immediate application of this result is the following lemma that will be very conventient in the rest of our discussion.

Lemma 4.7. *For each $m \in \mathbb{Z}$, the functor $\Pi_m^* : \operatorname{QCoh}(X_m) \longrightarrow \operatorname{Gr}(\mathcal{A})$ is an exact functor*

Proof. For each $n \geq m$, $\mathcal{A}_{m,n}$ is locally free by Corollary 4.6, hence the functor $- \otimes \mathcal{A}_{m,n} : \operatorname{QCoh}(X_m) \longrightarrow \operatorname{QCoh}(X_n)$ is exact. As taking direct limits in $\operatorname{QCoh}(X)$ and $\operatorname{QCoh}(Y)$ is exact, the result follows. □

Lemma 4.8. *There is a natural isomorphism for all $\mathcal{F} \in \operatorname{QCoh}(X_m)$ and $\mathcal{C} \in \mathcal{D}^+(\operatorname{Gr}(\mathcal{A}))$:*

$$\operatorname{RHom}_{\operatorname{Proj}(\mathcal{A})}(\Pi_m^* \mathcal{F}, \mathcal{C}) \cong \operatorname{RHom}_{X_m}(\mathcal{F}, \operatorname{R}\Pi_{m*}\mathcal{C})$$

Proof. This is follows from lemma 4.7 using a spectral sequence argument, see for example [5, Lemma 4.2]. □

We are especially interested in the case where $\mathcal{C} = \Pi_n^* \mathcal{G}$ for a locally free sheaf $\mathcal{G}$ on X_n. Hence we need to understand complexes of the form $R\Pi_{m*}(\Pi_n^* \mathcal{G})$. The strategy for computing its homology is as follows: by Lemma 4.10 it suffices to understand the derived functors of τ. These in terms follow from the derived functors of an internal Hom-functor $\underline{\mathcal{H}om}$ (Lemma 4.12).

Lemma 4.9. *We have the following facts for the derived functors of the torsion functor $\tau : \operatorname{Gr}(\mathcal{A}) \longrightarrow \operatorname{Tors}(\mathcal{A})$:*

i) for $i \geq 1$, there is an isomorphism of functors

$$\operatorname{R}^{i+1} \tau \cong (\operatorname{R}^i \omega) \circ p$$

ii) For each $\mathcal{M} \in \operatorname{Gr}(\mathcal{A})$ there is an exact sequence:

$$0 \longrightarrow \tau(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \omega(p(\mathcal{M})) \longrightarrow \operatorname{R}^1 \tau(\mathcal{M}) \longrightarrow 0$$

Proof. Since $\operatorname{Gr}(\mathcal{A})$ is locally noetherian, by [7, Lemma 2.12], any essential extension of a torsion module remains a torsion module. In particular, the category $\operatorname{Tors}(\mathcal{A})$ is closed under injective enveloppes, the result now follows from [8, Theorem 2.14.15]. □

Lemma 4.10. *For $i \geq 1$, there is an isomorphism*

$$\operatorname{R}^i \Pi_{m*}(\Pi_n^* \mathcal{V}) \cong \operatorname{R}^{i+1} \tau(\mathcal{V} \otimes e_n \mathcal{A})_m$$

Proof. As the functors p and $(-)_m$ are exact there is a functorial isomorphism

$$\operatorname{R}^i (\Pi_{m*}(p(-)) \cong \operatorname{R}^i \omega(p(-))_m$$

Combining this isomorphism with the one in Lemma 4.9 we obtain for each $i \geq 1$:

$$\operatorname{R}^i \Pi_{m*}(\Pi_n^* \mathcal{V}) := \operatorname{R}^i \Pi_{m*}(p(\mathcal{V} \otimes e_n \mathcal{A})) \cong \operatorname{R}^i \omega(p(\mathcal{V} \otimes e_n \mathcal{A}))_m \cong \operatorname{R}^{i+1} \tau(\mathcal{V} \otimes e_n \mathcal{A})_m$$

□

The following is based on [6, Section 3.2]:
Let $\mathrm{BiMod}(\mathcal{A}-\mathcal{A})$ denote the category whose objects are of the form

$$\{\mathcal{B}_{m,n} \in \mathrm{BiMod}(X_m - X_n)\}_{m,n}$$

such that the left and right multiplications $\mathcal{A}_{l,m} \otimes \mathcal{B}_{m,n} \longrightarrow \mathcal{B}_{l,n}$ resp $\mathcal{B}_{m,n} \otimes \mathcal{A}_{n,l} \longrightarrow \mathcal{B}_{m,l}$ are compatible in the obvious sense. We denote by $\mathbb{B}$ for the subcategory for which all $\mathcal{B}_{m,n}$ are coherent and locally free. There are an Hom-functors

$$\underline{\mathcal{H}om} : \mathbb{B}^{op} \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A})$$

$$\mathcal{H}om : \mathrm{BiMod}(\mathcal{O}_{X_n} - \mathcal{A}) \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{QCoh}(X_n)$$

satisfying the following properties:

Proposition 4.11. *i)* $\underline{\mathcal{H}\mathrm{om}}(\mathcal{B}, \mathcal{M})_m = \mathcal{H}\mathrm{om}(e_m \otimes \mathcal{B}, \mathcal{M})$ *for all* $\mathcal{B} \in \mathbb{B}$ *and* $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$
ii) $\underline{\mathcal{H}\mathrm{om}} : \mathbb{B}^{op} \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A})$ *is a bifunctor, left exact in both its arguments*
iii) $\mathcal{H}\mathrm{om} : \mathrm{BiMod}(\mathcal{O}_{X_n} - \mathcal{A}) \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{QCoh}(X_n)$ *is a bifunctor, left exact in both its arguments*
iv) $\mathcal{H}\mathrm{om}(\mathcal{Q} \otimes e_m\mathcal{A}, \mathcal{M}) \cong \mathcal{M}_m \otimes \mathcal{Q}^*$ *for all* $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ *and* $\mathcal{Q} \in \mathrm{coh}(X_m)$ *locally free*

Proof. i) This follows immediately by checking the exact definitions in [6, Section 3.2]
ii) [6, Proposition 3.11, Theorem 3.16(1)]
iii) [6, Theorem 3.16(3)]
iv) [6, Theorem 3.16(4)] □

By ii. and iii. in the above proposition it makes sense to define the right derived functors $\underline{\mathcal{E}xt}^i$ and $\mathcal{E}xt^i$ for all $i \geq 0$. Moreover we use the notation $\mathcal{A}_{\geq l}$ to denote the object in $\mathbb{B}$ given by

$$(\mathcal{A}_{\geq l})_{m,n} = \begin{cases} \mathcal{A}_{m,n} & \text{if } n-m \geq l \\ 0 & \text{else} \end{cases}$$

and $\mathcal{A}_0 := \mathcal{A}/\mathcal{A}_{\geq 1}$. Then we have the following relation between the derived functors of τ and the $\underline{\mathcal{E}xt}^i$:

Lemma 4.12. $R^i\tau(-) \cong \lim\limits_{l\to\infty} \underline{\mathcal{E}\mathrm{xt}}^i_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$

Proof. By [7, Proposition 3.19], we have an isomorphism of functors

$$\tau \cong \lim_{l\to\infty} \underline{\mathcal{H}om}_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$$

Since $\mathrm{Gr}(\mathcal{A})$ is a Grothendieck category, the direct limit of an exact sequence remains exact and the isomorphism descends to an isomorphism

$$\mathrm{R}\,\tau \cong \lim_{l\to\infty} \mathrm{R}\,\underline{\mathcal{H}om}_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$$

and taking homology yields

$$R^i\tau(-) \cong \lim_{l\to\infty} \underline{\mathcal{E}xt}^i_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$$ □

Lemma 4.13. *Let $\mathcal{B} \in \mathbb{B}$ be concentrated in degree $l \geq 0$ (i.e. $\mathcal{B}_{m,n} = 0$ whenever $m + l \neq n$) and $\mathcal{V}$ a locally free sheaf. Then for $n - l - 1 \leq m$ and for all $i \geq 0$:*

$$\underline{\mathcal{E}\mathrm{xt}}^i(\mathcal{B}, \mathcal{V} \otimes e_n\mathcal{A})_m = 0$$

Proof. Using a classical δ-functor argument, one sees that Proposition 4.11(iv) gives rise to an isomorphism

$$\underline{\mathcal{E}xt}^i(\mathcal{B}, \mathcal{V} \otimes e_n\mathcal{A})_m \cong \underline{\mathcal{E}xt}^i(\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A})_{m+l} \otimes \mathcal{B}^*_{m,m+l}$$

which easily reduces the proof to the case $\mathcal{B} = \mathcal{A}_0$ for which $l = 0$.
By Proposition 4.11(4) we see that the exact sequence from Theorem 4.5 is a resolution of $e_m\mathcal{A}_0 = {}_{Id}(\mathcal{O}_{X_m})_{Id}$ by $\mathcal{H}om(-, \mathcal{V} \otimes e_n\mathcal{A})$-acyclic sheaves. In particular we can calculate $\underline{\mathcal{E}xt}^i(A_0, \mathcal{V} \otimes e_n\mathcal{A})_m = \mathcal{E}xt^i(e_mA_0, \mathcal{V} \otimes e_n\mathcal{A})$ by taking homology of the complex

$$\begin{aligned}
&0 \longrightarrow \mathcal{H}om(e_m\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \xrightarrow{d_0} \mathcal{H}om(\mathcal{E}^{*m} \otimes e_{m+1}\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \\
&\xrightarrow{d_1} \mathcal{H}om(\mathcal{Q}_m \otimes e_{m+2}\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \longrightarrow 0
\end{aligned}$$

again using Proposition 4.11(iv), this complex becomes

$$0 \longrightarrow \mathcal{V} \otimes \mathcal{A}_{n,m} \xrightarrow{d_0} \mathcal{V} \otimes \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{m*} \xrightarrow{d_1} \mathcal{V} \otimes \mathcal{A}_{n,m+2} \otimes \mathcal{Q}^*_m \longrightarrow 0 \tag{18}$$

Hence we have

- $\mathcal{E}xt^{\,0}(e_mA_0, \mathcal{V} \otimes e_n\mathcal{A}) = \ker(d_0)$
- $\mathcal{E}xt^{\,1}(e_mA_0, \mathcal{V} \otimes e_n\mathcal{A}) = \ker(d_1)/\operatorname{im}(d_0)$
- $\mathcal{E}xt^{\,2}(e_mA_0, \mathcal{V} \otimes e_n\mathcal{A}) = \operatorname{coker}(d_1)$
- $\mathcal{E}xt^i(e_mA_0, \mathcal{V} \otimes e_n\mathcal{A}) = 0$ for all $i \geq 3$

We defer to the proof of [6, Theorem 4.4] for showing that $\ker(d_0) = 0$ and $ker(d_1) = \operatorname{im}(d_0)$. To show that d_1 is surjective, recall that each stalk of the structure sheaf of a smooth curve is a PID. In particular the stalk of $\operatorname{im}(d_1)$ at any point p is a free $\mathcal{O}_p$-module as a submodule of $(\mathcal{V} \otimes \mathcal{A}_{n,m+2} \otimes \mathcal{Q}^*)_p$. Hence $\operatorname{im}(d_1)$ is a locally free subsheaf of $\mathcal{V} \otimes \mathcal{A}_{n,m+2} \otimes \mathcal{Q}^*$ and to show that d_1 is surjective, it suffices to show that the rank of $\operatorname{im}(d_1)$ equals the rank of $\mathcal{V} \otimes \mathcal{A}_{n,m+2} \otimes \mathcal{Q}^*$. By exactness of (18) at the first and middle term we know the rank of $\operatorname{im}(d_1)$ is given by:

$$\begin{aligned}
\operatorname{rk}(\operatorname{im}(d_1)) &= \operatorname{rk}\left(\mathcal{V} \otimes \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1}\right) - \operatorname{rk}(\ker(d_1)) \\
&= \operatorname{rk}\left(\mathcal{V} \otimes \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1}\right) - \operatorname{rk}(\operatorname{im}(d_0)) \\
&= \operatorname{rk}\left(\mathcal{V} \otimes \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1}\right) - \operatorname{rk}\left(\mathcal{V} \otimes \mathcal{A}_{n,m}\right)
\end{aligned}$$

Hence we define

$$d_{n,m} := \operatorname{rk}\left(\mathcal{V} \otimes \mathcal{A}_{n,m+2} \otimes \mathcal{Q}^*\right) - \operatorname{rk}\left(\mathcal{V} \otimes \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1}\right) + \operatorname{rk}\left(\mathcal{V} \otimes \mathcal{A}_{n,m}\right)$$

and show $d_{n,m} = 0$ for all n, m. As $d_{n,m}$ is obviously is linear in $\operatorname{rk}(\mathcal{V})$, so we can reduce to the case that $\mathcal{V} = \mathcal{L}$ is a line bundle. We can then use Corollary 4.6 to compute the ranks whenever $n \leq m$:

parity	$\operatorname{rk}(\mathcal{L}\otimes\mathcal{A}_{n,m})$	$\operatorname{rk}\left(\mathcal{L}\otimes\mathcal{A}_{n,m+1}\otimes\mathcal{E}^{*m+1}\right)$	$\operatorname{rk}(\mathcal{L}\otimes\mathcal{A}_{n,m+2}\otimes\mathcal{Q}_m^*)$	$d_{n,m}$
n even, m even	$m-n+1$	$\frac{(m+1)-n+1}{2}\cdot 4$	$m+2-n+1$	0
n odd, m even	$2(m-n+1)$	$(m+1-n+1)\cdot 4$	$2((m+2)-n+1)$	0
n even, m odd	$\frac{m-n+1}{2}$	$(m+1)-n+1$	$\frac{m+2-n+1}{2}$	0
n odd, m odd	$m-n+1$	$2(m+1-n+1)$	$m+2-n+1$	0

In the case where $n=m+1$ (which by the assumption of the theorem is the only case with $n\geq m$) we have

$$\begin{aligned}\operatorname{rk}(\operatorname{im}(d_1)) &= \operatorname{rk}\left(\mathcal{L}\otimes\mathcal{A}_{m+1,m+1}\otimes\mathcal{E}^{*m+1}\right)-\operatorname{rk}(\mathcal{L}\otimes\mathcal{A}_{m+1,m})\\ &= \operatorname{rk}\left(\mathcal{L}\otimes\mathcal{E}^{*m+1}\right)\\ &= \operatorname{rk}(\mathcal{L}\otimes\mathcal{A}_{m+1,m+2}\otimes\mathcal{Q}_m^*)\end{aligned}$$

again showing surjectivity of d_1. □

Lemma 4.14. *$\underline{\mathcal{E}\mathrm{xt}}^i(\mathcal{A}/\mathcal{A}_{\geq l},\mathcal{V}\otimes e_n\mathcal{A})_m=0$ for $m\geq n-1$ and $i\geq 0$*

Proof. Consider the short exact sequence

$$0\longrightarrow\mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1}\longrightarrow\mathcal{A}/\mathcal{A}_{\geq l+1}\longrightarrow\mathcal{A}/\mathcal{A}_{\geq l}\longrightarrow 0$$

Applying $\underline{\mathcal{H}om}(-,\mathcal{V}\otimes e_n\mathcal{A})$ gives rise to a long exact sequence for each $m\geq n-1$

$$\begin{aligned}\ldots\longrightarrow\underline{\mathcal{E}xt}^i(\mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1},\mathcal{V}\otimes e_n\mathcal{A})_m &\longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l+1},\mathcal{V}\otimes e_n\mathcal{A})_m\\ \longrightarrow\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l},\mathcal{V}\otimes e_n\mathcal{A})_m &\longrightarrow \underline{\mathcal{E}xt}^{i+1}(\mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1},\mathcal{V}\otimes e_n\mathcal{A})_m\longrightarrow\ldots\end{aligned}$$

As $m\geq n-1$ it follows from Lemma 4.13 that for each $i\geq 0$ we have an exact sequence

$$0\longrightarrow\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l+1},\mathcal{V}\otimes e_n\mathcal{A})_m\longrightarrow\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l},\mathcal{V}\otimes e_n\mathcal{A})_m\longrightarrow 0$$

Hence

$$\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l},\mathcal{V}\otimes e_n\mathcal{A})_m\cong\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq 0},\mathcal{V}\otimes e_n\mathcal{A})_m=\underline{\mathcal{E}xt}^i(0,\mathcal{V}\otimes e_n\mathcal{A})_m=0$$

□

We can now finish the proof of Theorem 4.1

Proof. of Theorem 4.1
Take $m,n\in\mathbb{Z}$ with $m\geq n-1$. Let $\mathcal{F}$ be locally free on X_m and $\mathcal{G}$ locally free on X_n, then by Corollary 4.8:

$$\begin{aligned}\operatorname{Ext}^i_{\operatorname{Proj}(\mathcal{A})}(\Pi_m^*\mathcal{F},\Pi_n^*\mathcal{G}) &= h^i\left(\operatorname{R}\operatorname{Hom}_{\operatorname{Proj}(\mathcal{A})}(\Pi_m^*\mathcal{F},\Pi_n^*\mathcal{G})\right)\\ &\cong h^i\left(\operatorname{R}\operatorname{Hom}_{X_m}(\mathcal{F},\operatorname{R}\Pi_{m*}\Pi_n^*\mathcal{G})\right)\end{aligned}$$

Now for $i\geq 1$ we have

$$\begin{aligned}\operatorname{R}^i\Pi_{m*}\Pi_n^*\mathcal{G} &\cong \operatorname{R}^{i+1}\tau(\mathcal{G}\otimes e_n\mathcal{A})_m\\ &\cong \lim_{l\to\infty}\underline{\mathcal{E}xt}^{i+1}(\mathcal{A}/\mathcal{A}_{\geq l},\mathcal{G}\otimes e_n\mathcal{A})_m\\ &= 0\end{aligned}$$

by Lemmas 4.10, 4.12 and 4.14 respectively.
In particular the complex $\operatorname{R}\Pi_{m*}\Pi_n^*\mathcal{G}$ is quasi-isomorphic to the complex that is

equal to $\Pi_{m*}\Pi_n^*\mathcal{G}$ concentrated in position zero. Finally we can conclude by noticing that $\Pi_{m*}\Pi_n^*\mathcal{G} = (\omega p(\mathcal{G} \otimes e_n\mathcal{A}))_m$ and by Lemma 4.9 there is an exact sequence

$$0 = \tau(\mathcal{G} \otimes e_n\mathcal{A})_m \longrightarrow \mathcal{G} \otimes \mathcal{A}_{n,m} \xrightarrow{\cong} \omega(p(\mathcal{G} \otimes e_n\mathcal{A}))_m \longrightarrow \mathrm{R}^1\,\tau(\mathcal{G} \otimes e_n\mathcal{A})_m = 0$$

where the first term equals zero because $\mathcal{G} \otimes e_n\mathcal{A}$ is torsion free and the last term is zero because $\mathrm{R}^1\,\tau(\mathcal{G} \otimes e_n\mathcal{A})_m \cong \lim_{l\to\infty} \underline{\mathcal{E}xt}^{\,1}(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{G} \otimes e_n\mathcal{A})_m = 0$.

Hence we can conclude that for $m \geq n-1$ we have

$$\begin{aligned} \mathrm{Ext}^i_{\mathrm{Proj}(\mathcal{A})}(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}) &\cong h^i(\mathrm{R}\,\mathrm{Hom}_{X_m}(\mathcal{F}, \mathrm{R}\,\Pi_{m*}\Pi_n^*\mathcal{G})) \\ &\cong h^i(\mathrm{R}\,\mathrm{Hom}_{X_m}(\mathcal{F}, \mathcal{G} \otimes \mathcal{A}_{n,m})) \\ &= \mathrm{Ext}^i_{X_m}(\mathcal{F}, \mathcal{G} \otimes \mathcal{A}_{n,m}) \quad \square \end{aligned}$$

Appendix A. From periodic $\mathbb{Z}$-algebras to graded algebras

In this section, we show how a periodic $\mathbb{Z}$-algebras A gives rise to a graded algebra $\overline{A}$ such that $\mathrm{Gr}(A)$ is a direct summand of the category $\mathrm{Gr}(\overline{A})$. We shall consider a slightly more general version of $\mathbb{Z}$-algebra to fit our needs

Definition A.1. Let $(R_i)_{i\in\mathbb{Z}}$ be a collection of commutative k-algebras. A *bimodule $\mathbb{Z}$-algebra* is a collection of $R_i - R_j$-bimodules A_{ij} together with multiplication maps

$$A_{ij} \otimes_{R_j} A_{jl} \longrightarrow A_{il}$$

and R_i-linear unit maps $R_i \longrightarrow A_{ii}$ satisfying the usual $\mathbb{Z}$-algebra axioms.

To ease notation we shall omit the word *bimodule* whenever there is no confusion. We briefly recall what we mean by periodicity of a $\mathbb{Z}$-algebra.

Definition A.2. Let A be a $\mathbb{Z}$-algebra over $(R_i)_{i\in\mathbb{Z}}$ and $d > 0$ an integer.
Assume that for each i ,we have $R_{i+d} = R_i$. We say A is d-periodic if there is an isomorphism of $\mathbb{Z}$-algebras $\varphi : A \xrightarrow{\sim} A(d)$. I.e. there is a collection of $R_i - R_j$-bimodule isomorphisms $\{\varphi_{ij} : A_{i,j} \xrightarrow{\sim} A_{i+d,j+d}\}_{i,j}$ compatible with the multiplication and unit maps.

Let A be d-periodic and let $R := \bigoplus_{i=0}^{d-1} R_i$. We shall construct a graded R-algebra $\overline{A}$ as follows: let $\overline{A}_n$ be a $d \times d$ matrix with entries

$$\left(\overline{A}_n\right)_{i,j} = \begin{cases} A_{i,i+n} & \text{if } j - i \equiv n \ (mod\ d) \\ 0 & \text{else} \end{cases}$$

(Where we use the convention that the numbering of rows and columns of the matrix starts at 0 instead of 1.)
By way of example, we have

$$\overline{A}_1 = \begin{pmatrix} 0 & A_{0,1} & 0 & \dots & 0 \\ 0 & 0 & A_{1,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{d-2,d-1} \\ A_{d-1,d} & 0 & 0 & \dots & 0 \end{pmatrix}$$

Each $\overline{A}_n$ is naturally a left (resp. right) R-module by letting a d-tuple $(r_0, \dots r_{d-1})$ act as a diagonal matrix D with entries $D_{ii} := r_i$ on the left (resp. right).
Moreover, there is a canonical multiplication map

$$\overline{A}_n \otimes_R \overline{A}_m \longrightarrow \overline{A}_{n+m}$$

given by the ordinary matrix multiplication and applying the periodicity isomorphisms ϕ_{ij} whenever necessary. The $(R_i)_{i\in\mathbb{Z}}$-linearity of the $\mathbb{Z}$-algebra multiplication implies that the above maps are indeed R-bilinear.

Lemma A.3. *Suppose A is d-periodic, then the above maps define a graded R-algebra structure on the R-module $\overline{A} := \oplus_{i\in\mathbb{Z}}\overline{A}_i$*

Proof. The reader checks that the compatibilty of the periodicity isomorphisms with the $\mathbb{Z}$-algebra multiplication maps implies that the multiplication is associative. The multiplication is distributive by construction and the algebra has a unit given by

$$1 = \begin{pmatrix} e_0 & 0 & \dots & 0 \\ 0 & e_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{d-1} \end{pmatrix} \in \overline{A}_0$$

where e_i is the unit in A_{ii}. □

There is a convenient description of graded right $\overline{A}$-modules as follows: let $M \in \mathrm{Gr}(\overline{A})$. Then $M = \oplus_{i\in\mathbb{Z}} M_i$. Moreover, each R-module M_i in turn has a direct sum decomposition given by $M_i = \oplus_{j=0}^{d-1} M_i e_j$. We define $M_i^j := M_i e_j$.
The $\overline{A}$-module structure has a nice description in these terms. For a matrix $\overline{a} \in \overline{A}_m$, $e_j.\overline{a}$ only has one nonzero entry at position $(j, j+m)$. It follows from the right R-structure on A_m that $e_j \overline{a} = \overline{a}.e_{j+m}$ (where we consider $j+m \bmod d$). Thus the right action of $\overline{A}_m$ on M_i^j becomes a map of the form $M_i^j \otimes A_{j,j+m} \longrightarrow M_{i+m}^{j+m}$ or equivalently for $l = j+m$,

$$M_i^j \otimes A_{j,l} \longrightarrow M_{i+l-j}^l$$

we now have:

Lemma A.4. *Suppose A is d-periodic and let $\mathcal{C}$ be the category defined as follows:*

1. *an object is a collection of R-modules $(M_i^j)_{i\in\mathbb{Z}, 0\le j\le d-1}$, such that M_i^j is an R_j-module together with multiplication maps*
$$\mu_{i,j,l}^M : M_i^j \otimes A_{j,l} \longrightarrow M_{i+l-j}^l$$
for each i, j, l (where l and $i+l-j$ should be interpreted modulo d) satisfying the obvious compatibility condition for multiplication and unit.
2. *a morphism is a collection $f_{i,j}$ of R_j- linear maps $M_i^j \longrightarrow N_i^j$ such that*
$$f_{i+l-j,l} \circ \mu_{i,j,l}^M = \mu_{i,j,l}^N \circ (f_{i,j} \otimes A_{j,l})$$

Then there is a canonical isomorphism of categories $\mathcal{C} \cong \mathrm{Gr}(\overline{A})$

Proof. The above discussion shows that the assignment $M \longrightarrow (M_l e_i)_{l\in\mathbb{Z}, 0\le i\le n-1}$ is well defined and essentially surjective. A morphism of graded modules $f : M \longrightarrow N$ will satisfy $f(M_i e_j) \subset N_i e_j$ and we can define $f_{i,j}$ as the restriction to these submodules. The A-linearity guarantees that $(f_{i,j})_{i,j}$ indeed defines a morphism in $\mathcal{C}$ and since $\oplus M_i e_j = M$ it is clear that this assignment is faithful. Since any collection of maps $f_{i,j}$ satisfying the above compatibility with the multiplication will sum up to an $\overline{A}$-linear map, the assignment is also full. □

Lemma A.5. *There exists a decomposition*

$$\mathcal{C} = \mathcal{C}_0 \oplus \ldots \oplus \mathcal{C}_{d-1}$$

where $\mathcal{C}_n$ is the full subcategory of $\mathcal{C}$ whose objects are collections of R-modules $(M_i^j)_{i\in\mathbb{Z}, 0\le j\le d-1}$ where $M_i^j = 0$ unless $j - i \equiv n$ (mod d).

Proof. This follows immediately from the construction of $\mathcal{C}$ and the fact that $j-i = l-(l+i-j)$. Hence, if $(\mathcal{M}_i^j)_{ij}$ is a non-zero object in $\mathcal{C}_n$, then so is $(\mathcal{M}_{l+i-j}^l)_{ij}$ for all l. □

Proposition A.6. *There is an exact embedding of categories*

$$\overline{(-)} : \mathrm{Gr}(A) \hookrightarrow \mathrm{Gr}(\overline{A})$$

moreover the essential image is a direct summand of $\mathrm{Gr}(\overline{A})$*.*

Proof. Let M be an A-module with multiplication maps $\mu_{i,m} : M_i \otimes_R A_m \longrightarrow M_{i+m}$ and let $\mathcal{C}$ be as above. We define an object $\overline{M}$ in $\mathcal{C}$ by

$$\overline{M}_i^j = \begin{cases} M_i & \text{if } j \equiv i \bmod d \\ 0 & \text{else} \end{cases}$$

where the multiplication is given by

$$\overline{\mu}_{i,j,l} = \begin{cases} \mu_{i,l-j} & \text{if } j \equiv i \bmod d \\ 0 & \text{else} \end{cases}$$

This assignment clearly defines an exact embedding $\mathrm{Gr}(A) \xrightarrow{\cong} \mathcal{C}_0 \hookrightarrow \mathcal{C}$, finishing the proof by Lemmas A.4 and A.5. □

Appendix B. Point Modules and proof of Theorem 4.5

In this section we will assume $\mathcal{A} = \mathbb{S}(\mathcal{E})$ is a symmetric sheaf $\mathbb{Z}$-algebra in standard form with $\mathcal{E} \in \mathrm{bimod}(X - Y)$ locally free of rank (4,1), in particular $\mathcal{E} = {}_f(\mathcal{L})_{Id}$ as in Lemma 3.2. As before we assume X, Y are smooth curves over an algebraically closed field $k$and let $\alpha : X \longrightarrow \mathrm{Spec}(k)$, $\beta : Y \longrightarrow \mathrm{Spec}(k)$ be the structure morphisms. As always we will write

$$(X_n, \alpha_n) = \begin{cases} (X, \alpha) & \text{if } n \text{ is even} \\ (Y, \beta) & \text{if } n \text{ is odd} \end{cases}$$

We say $P_n \in \mathrm{coh}(X_n)$ is locally free over k of rank l if $\alpha_{n,*} P_n$ is free of rank l.

A module $P \in \mathrm{Gr}(\mathcal{A})$ is said to be generated in degree m if $P_n = 0$ for all $n < m$ and $P_m \otimes \mathcal{A}_{m,n} \longrightarrow P_n$ is surjective for all $n \geq m$. As $\mathcal{A}$ is generated in degree one as an algebra, we have surjectivity of $P_{n_1} \otimes \mathcal{A}_{n_1,n_2} \longrightarrow P_{n_2}$ for all $n_2 \geq n_1 \geq m$ by the following commuting diagram

$$\begin{array}{ccc} P_m \otimes \mathcal{A}_{m,n_1} \otimes \mathcal{A}_{n_1,n_2} & \twoheadrightarrow & P_{n_1} \otimes \mathcal{A}_{n_1,n_2} \\ \downarrow\!\!\!\!\downarrow & & \downarrow \\ P_m \otimes \mathcal{A}_{m,n_2} & \twoheadrightarrow & P_{n_2} \end{array}$$

Remark B.1. An obvious example of a module generated in degree m is $e_m\mathcal{A}$. The above diagram implies that the maps $\mathcal{A}_{m,n} \otimes e_n\mathcal{A} \longrightarrow e_m\mathcal{A}$ are surjective for all $m \geq n$.

An m-shifted point-module over $\mathcal{A}$ is defined in [9] as an object $P \in \mathrm{Gr}(\mathcal{A})$ such that P is generated in degree m and for which P_n is locally free of rank one over k for all $n \geq m$. As the next Lemma shows, this definition is not desirable in the current situation.

Lemma B.2. *Let* $i \in \mathbb{Z}$ *and* $P \in \mathrm{Gr}(\mathcal{A})$ *generated in degree* $2i$ *such that* P_{2i} *and* P_{2i+1} *are locally free of rank one over* k*. Then* $P_n = 0$ *for all* $n \geq 2i + 2$*.*

Proof. Recall that the following composition

$$P_{2i} \longrightarrow P_{2i} \otimes \mathcal{E}^{*2i} \otimes \mathcal{E}^{*2i+1} \longrightarrow P_{2i+1} \otimes \mathcal{E}^{*2i+1} \longrightarrow P_{2i+2}$$

must be zero as it represents the action of $\mathcal{Q}_{2i}$. By [9, Lemma 4.3.2.] this composition equals

$$P_{2i} \xrightarrow{\varphi_{2i}^*} P_{2i+1} \otimes \mathcal{E}^{*2i+1} \xrightarrow{\varphi_{2i+1}} P_{2i+2}$$

where φ_{2i}^* is obtained by adjointness from $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \longrightarrow P_{2i+1}$. Remark that as P_{2i} and $P_{2i+1} \otimes \mathcal{E}^{*2i+1}$ are locally free of rank one over k we have that either φ_{2i}^* is an isomorphism or it is zero. Similarly φ_{2i+1} is monic or zero. Hence the only way the composition can be zero is when $\varphi_{2i}^* = 0$ or $\varphi_{2i+1} = 0$. The first option cannot happen as $\varphi_{2i} \neq 0$ (because P was generated in degree $2i$ and $P_{2i+1} \neq 0$). Hence we must have $\varphi_{2i+1} = 0$. However φ_{2i+1} must be surjective (again because P was generated in degree $2i$), hence $P_{2i+2} = 0$. Using surjectivity of $P_{2i+2} \otimes \mathcal{A}_{2i+2,n} \longrightarrow P_n$ for all $n \geq 2i+2$ the result follows. □

The following definition will make more sense:

Definition B.3. A shifted point module is an object $P \in \mathrm{Gr}(\mathcal{A})$ which is generated in degree $2i$ for some integer i and such that for all $n \geq 2i$, P_n is locally free over k of rank one if n is even and rank two if n is odd. We will often use the short hand notation $\dim_k(P_n) = length_{\mathrm{Spec}(k)}(\alpha_{n,*}(P_n))$ whenever the latter is finite. So we could say P is a shifted point module if is generated in degree $2i$ and:

$$\dim_k(P_jn) = \begin{cases} 0 & \text{if } n < 2i \\ 1 & \text{if } n \geq 2i \text{ is even} \\ 2 & \text{if } n > 2i \text{ is odd} \end{cases}$$

The following Lemma shows that this new definition of point modules behaves way better than the naive one:

Lemma B.4. *Let $P \in \mathrm{Gr}(\mathcal{A})$ be a graded module and $i \in \mathbb{Z}$ such that:*

- *P is generated in degree $2i$*
- $\dim_k(P_{2i}) = 1$
- $\dim_k(P_{2i+1}) = 2$

Then for all $n \geq 2i+2$ fixed, we have

$$\dim_k(P_n) \leq \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 2 & \textit{if } n \textit{ is odd} \end{cases} \tag{19}$$

Moreover if equality holds in (19)*, then P_n is defined up to unique isomorphism by the data $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \longrightarrow P_{2i+1}$.*
If on the other hand (19) *is a strict inequality, then $P_l = 0$ for all $l > n$.*

Proof. We prove all facts by induction on n. So suppose (19) and the subsequent claims hold for $n = 2i, \ldots, m$. We distinguish several cases depending on whether the inequalities are in fact equalities or not.

Case 1: Equality holds in (19) for $n = 2i, \ldots, m$.
The following composition is zero:

$$P_{m-1} \xrightarrow{\varphi_{m-1}^*} P_m \otimes \mathcal{E}^{*m} \xrightarrow{\varphi_m} P_{m+1}$$

φ_m is surjective, hence one can easily check that (19) holds if we can prove φ_{m-1}^* is injective. Moreover if the equality holds for $\dim_k(P_{m+1})$, then $P_{m+1} \cong \mathrm{coker}(\varphi_{m-1}^*)$

and is hence defined up to unique isomorphism.

Case 1a: m is odd
$\dim_k(P_{m-1}) = 1$ hence it suffices to prove $\varphi^*_{m-1} \neq 0$ and this holds because $\varphi_{m-1} \neq 0$
Case 1b: m is even
If φ^*_{m-1} were not injective, then there is a $W \subset P_{m-1}$, $\dim_k(W) = 1$ such that the composition

$$W \hookrightarrow P_{m-1} \xrightarrow{\varphi^*_{m-1}} P_m \otimes \mathcal{E}^{*m}$$

is zero. This implies that there is an $\overline{W} \in \mathrm{Gr}(\mathcal{A})$ given by $\overline{W}_{m-1} = W$ and $\overline{W}_l = 0$ for $l \neq m-1$. By construction there is an embedding $\chi : \overline{W} \hookrightarrow P_{\geq m-2}$ and let $C = \mathrm{coker}(\chi)$. Then C is generated in degree $m-2$ (which is even!) and $\deg_k(C_{m-2}) = \deg_k(C_{m-1}) = \deg_k(C_m) = 1$ contradicting Lemma B.2.

Case 2: There is an integer $n \in \{2i+2, \ldots, m\}$ such that there is a strict inequality for $\dim_k(P_n)$ in (19)
Let n_0 be the smallest such n. We have to show $P_l = 0$ for all $l > n_0$.
Assume that $P_{n_0} = 0$, then $P_l = 0$ by surjectivity of $P_{n_0} \otimes \mathcal{A}_{n_0,l} \longrightarrow P_l$.
The only nontrivial case is when n_0 is odd and $\dim_k(P_{n_0}) = 1$. In this case $\dim_k(P_{n_0-1}) = 1$ as well and the result follows from Lemma B.2. □

Remark B.5. The above Lemma also shows that any data $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \twoheadrightarrow P_{2i+1}$ with $\dim_k(P_{2i}) = 1$ and $\dim_k(P_{2i+1}) = 2$ can be extended to a shifted point module which is unique up to unique isomorphism.

From now on we use the following short hand notation:

$$L_{n,p} := \mathcal{O}_p \otimes e_n \mathcal{A}$$

where p is any point on X_n.

Proof. of Theorem 4.5
Exactness of the sequence (17) can be checked for each degree n separately:

$$0 \longrightarrow \mathcal{Q}_m \otimes \mathcal{A}_{m+2,n} \longrightarrow \mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n} \longrightarrow \mathcal{A}_{m,n} \longrightarrow 0 \tag{20}$$

As all terms in this sequence are elements of $\mathrm{bimod}(X_m - X_n)$, applying $\pi_{m,*}$ gives a sequence of coherent sheaves on X_m:

$$0 \longrightarrow \pi_{m,*}(\mathcal{Q}_m \otimes \mathcal{A}_{m+2,n}) \longrightarrow \pi_{m,*}(\mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n}) \longrightarrow \pi_{m,*}(\mathcal{A}_{m,n}) \longrightarrow 0 \tag{21}$$

and (21) is exact if and only if (20) is. The structure of the relations on $\mathcal{A}$ implies that (17) and hence also (20) and (21) are right exact. Now for any point $p \in X_m$ the following will be right exact as well:

$$\begin{aligned} 0 \to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{Q}_m \otimes \mathcal{A}_{m+2,n}) &\to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n}) \to \ldots \\ \ldots &\to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{A}_{m,n}) \to 0 \end{aligned} \tag{22}$$

and as all terms (22) are locally free over k, its left exactness can be checked numerically. If for all terms in (22) $\dim_k$ does not depend on the point p_m, then the terms in (21) are locally free and exactness of (21) follows from exactness of (22). Hence in order to prove the Lemma we show that the terms in (22) have the "correct" constant length (see (27)). From this it follows that (20) is exact and its terms are locally free on the left. The locally freeness on the right then follows from [9, Proposition 3.1.6.].

So we are left with finding the length of the objects in (22). Any object in $\mathrm{bimod}(X_m - X_n)$ is of the form ${}_u\mathcal{U}_v$ for finite maps u and v. As taking the direct image through a finite morphism does not change the length of sheaves, we have for such a bimodule:

$$\begin{aligned}
\dim_k(\mathcal{O}_p \otimes \pi_{m,*}({}_u\mathcal{U}_v)) &= \dim_k(\mathcal{O}_p \otimes u_*\mathcal{U}) \\
&= \dim_k(u_*(u^*(\mathcal{O}_p) \otimes \mathcal{U})) \\
&= \dim_k(u^*(\mathcal{O}_p) \otimes \mathcal{U}) \\
&= \dim_k(v_*(u^*(\mathcal{O}_p) \otimes \mathcal{U})) \\
&= \dim_k(\mathcal{O}_p \otimes {}_u\mathcal{U}_v))
\end{aligned}$$

Hence the length of the terms in (22) can be calculated from

$$0 \to \mathcal{O}_p \otimes \mathcal{Q}_m \otimes \mathcal{A}_{m+2,n} \to \mathcal{O}_p \otimes \mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n} \to \mathcal{O}_p \otimes \mathcal{A}_{m,n} \to 0 \tag{23}$$

Now $\dim_k(\mathcal{O}_p \otimes \mathcal{Q}_m) = 1$, hence there is a point $\widetilde{p} \in X_{m+2}$ such that $\mathcal{O}_p \otimes \mathcal{Q}_m = \mathcal{O}_{\widetilde{p}}$. Similarly, in the case where $m = 2i-1$ there must be a $q \in X_{2i}$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*2i-1} = \mathcal{O}_q$.In the case where $m = 2i$, we have $\dim_k(\mathcal{O}_p \otimes \mathcal{E}^{*2i}) = 4$, hence there must be points $\widetilde{q^a} \in X_{2i+1}$, $a = 1, \dots, 4$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*2i}$ is an extension of the $\mathcal{O}_{\widetilde{q^a}}$. We denote the corresponding extension of the $L_{2i+1,\widetilde{q^a}}$ by $M_{2i+1,p}$. The sequence (23) now gives rise to the following right exact sequences

$$L_{2i+1,\widetilde{p}} \longrightarrow L_{2i,q} \longrightarrow L_{2i-1,p} \longrightarrow 0 \tag{24}$$

$$L_{2i+2,\widetilde{p}} \longrightarrow M_{2i+1,p} \longrightarrow L_{2i,p} \longrightarrow 0 \tag{25}$$

Finally there also is a right exact sequence:

$$L_{2i+1,p'} \longrightarrow L_{2i-1,p} \longrightarrow P_p \longrightarrow 0 \tag{26}$$

where the morphism $L_{2i+1,p'} \longrightarrow L_{2i-1,p}$ comes from the fact that $\dim_k(\mathcal{O}_p \otimes \mathcal{A}_{2i-1,2i+1}) = 3 > 0$ such that there is a $p' \in X_{2i+1}$ with a nonzero morphism $\mathcal{O}_{p'} \longrightarrow \mathcal{O}_p \otimes \mathcal{A}_{2i-1,2i+1}$. P_p is defined as the cokernel of this morphism.

We now prove the following by induction on j (similtaneously for all p and all i):

$$\begin{aligned}
\dim_k((P_p)_{2i+2j}) &= 1 \\
\dim_k((P_p)_{2i+2j+1}) &= 2 \\
\dim_k((L_{2i,p})_{2i+2j}) &= 2j+1 \\
\dim_k((L_{2i,p})_{2i+2j+1}) &= 4j+4 \\
\dim_k((L_{2i-1,p})_{2i+2j}) &= j+1 \\
\dim_k((L_{2i-1,p})_{2i+2j+1}) &= 2j+3
\end{aligned} \tag{27}$$

By construction these facts are known to hold for $j = 0$. So by induction we now suppose they hold for $j = 0, \dots, l$, for all points and for all $i \in \mathbb{Z}$. We prove that these facts then also hold for $j = l + 1$.

By (25) we can see:

$$\begin{aligned}\dim_k((L_{2i,p})_{2i+2l+2}) &\geq \dim_k((M_{2i+1,p})_{2i+2l+2}) - \dim_k((L_{2i+2,\widetilde{p}})_{2i+2l+2})\\ &= \sum_{a=1}^{4}\dim_k((L_{2i+2,\widetilde{q^a_{2i+1}}})_{2i+2l+2}) - \dim_k((L_{(2i+2,\widetilde{p}})_{2i+2l+2})\\ &= 4\cdot(l+1)-(2l+1)\\ &= 2l+3\end{aligned}$$

Where the last equality follows from the induction hypothesis. Similarly we find a lower bound $\dim_k((L_{2i+1,p})_{2i+2l+2}) \geq 4l+4$. This can be written schematically as:

$$\begin{array}{cccccccccc} 0 \\ \uparrow \\ L_{2i,p} & 0 & 1 & 4 & 3 & \dots & 2l+1 & 4l+4 & \underline{2l+3} & \underline{4l+8} \\ \uparrow \\ M_{2i+1,p} & 0 & 0 & 4 & 4 & \dots & 4l & 8l+4 & 4l+4 & 8l+12 \\ \uparrow \\ L_{2i+2,\widetilde{p}} & 0 & 0 & 0 & 1 & \dots & 2l-1 & 4l & 2l+1 & 4l+4 \end{array} \tag{28}$$

Where the numbers on the right of a module denote $\dim_k((-)_x)$ for $x = 2i-1,\dots,2i+2l+3$. And an underlined number implies a lower bound for $\dim_k$. Similarly we will write $\overline{N}$ to denote that N is an upperbound for a certain $\dim_k$.

Now consider the module $P_{p,\geq 2i+2l}$. This is generated in degree $2i+2l$ because P_p is a quotient of $L_{2i-1,p}$. Moreover $\dim_k((P_p)_{2i+2l}) = 1$ and $\dim_k((P_p)_{2i+2l+1}) = 2$, so Lemma B.4 implies $\dim_k((P_p)_{2i+2l+2}) \leq 1$ and $\dim_k((P_p)_{2i+2l+3}) \leq 2$. Together with the right exact sequence (26) this gives us the following upper bounds:

$$\begin{array}{ccccccccc} 0 \\ \uparrow \\ P & 1 & 1 & 2 & \dots & 1 & 2 & \overline{1} & \overline{2} \\ \uparrow \\ L_{2i-1,p} & 1 & 1 & 3 & \dots & l+1 & 2l+3 & \overline{l+2} & \overline{2l+5} \\ \uparrow \\ L_{2i+1,p'} & 0 & 0 & 1 & \dots & l & 2l+1 & l & +12l+3 \end{array} \tag{29}$$

Combining the bounds found in (28) and (29) and using (24) we have found

$$\begin{array}{ccccccccc} 0 \\ \uparrow \\ L_{2i-1,p} & 1 & 1 & 4 & \dots & l+1 & 2l+3 & \overline{l+2} & \overline{2l+5} \\ \uparrow \\ L_{2i,q} & 0 & 1 & 4 & \dots & 2l+1 & 4l+4 & \underline{2l+3} & \underline{4l+8} \\ \uparrow \\ L_{2i+1,\widetilde{p}} & 0 & 0 & 1 & \dots & l-1 & 2l+1 & l+1 & 2l+3 \end{array} \tag{30}$$

Right exactness of (24) implies that the bounds in (30) are in fact equalities, because for example we find the upper bound

$$\begin{aligned}\dim_k((L_{2i,q})_{2i+2l+2}) &\leq \dim_k((L_{2i-1,p})_{2i+2l+2}) + \dim_k((L_{2i+1,\widetilde{p}})_{2i+2l+2})\\ &\leq l+2+l+1\\ &= 2l+3\end{aligned}$$

which equals the already known lower bound for $\dim_k((L_{2i,q})_{2i+2l+2}$. Hence we have found the exact value for $\dim_k((L_{2i+1,q})$. A priori the above right exact sequence only gives this exact value for the points $q \in X_{2i}$ for which there is a $p \in X_{2i-1}$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*2i-1} = \mathcal{O}_q$. But as $\mathcal{E}^{*2i-1}$ is of the form ${}_{Id}(\mathcal{L}_{i-1})_f$ as in (12) we have $q = f(p)$ and surjectivity of f implies that q runs through all points of X_{2i} as p runs through all points of X_{2i-1}. For $L_{2i,p}$ no such problems arrise.

Hence we have proven (27) for all $i, j \in \mathbb{Z}$ and for all points p. As these values do not depend on p we have that the terms in (21) are locally free on the left (and hence also on the right). Filling in these values for (22) we find that the sequences must be exact. □

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