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CYCLICITY OF THE ORIGIN IN SLOW-FAST CODIMENSION 3 SADDLE AND ELLIPTIC BIFURCATIONS

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Abstract. This paper is the continuation of our previous papers [16] and [17] where we studied small-amplitude limit cycles in slow-fast codimension 3 saddle and elliptic bifurcations. We find optimal upper bounds for the number of small-amplitude limit cycles in these slow-fast codimension 3 bifurcations. We use techniques from geometric singular perturbation theory.

1. **Introduction.** In [16] and [17] we started to study small-amplitude limit cycles (limit cycles near the origin $(x, y) = (0, 0)$) in singular perturbation problems occurring in planar slow-fast systems

$$X^{\pm}_{\bar\epsilon,b,\lambda} : \begin{cases} \dot x = y \\ \dot y = -xy + \bar\epsilon\Big(b_0 + b_1 x + b_2 x^2 \pm x^3 + x^4 \bar H(x, \lambda) + y^2 G(x, y, \lambda)\Big), \end{cases} \tag{1}$$

where G and $\bar H$ are smooth, $\bar\epsilon > 0$ is the singular parameter that is kept small, $b = (b_0, b_1, b_2)$ are regular perturbation parameters close to 0 and $\lambda \in \Lambda$, with Λ a compact subset of some euclidean space. The family $X^{+}_{\bar\epsilon,b,\lambda}$ represents slow-fast codimension 3 saddle bifurcations (in short, the saddle case) and $X^{-}_{\bar\epsilon,b,\lambda}$ represents slow-fast codimension 3 elliptic bifurcations (in short, the elliptic case). The goal of this paper is to finish the study of small-amplitude limit cycles of (1) which is initiated in [16] and [17]. More precisely, we show in the present paper that the cyclicity of the origin $(x, y) = (0, 0)$ is equal to 2 in both the saddle and elliptic cases, provided $\bar H(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$.

The small-amplitude limit cycle phenomenon in a planar slow-fast setting is typically observed when we deal with slow-fast limit cycles of canard type in planar slow-fast families of vector fields. Let us explain a well known scenario. Working with slow-fast systems in the (x, y)-plane with $\bar\epsilon$ as the singular perturbation parameter, one makes a rescaling in the $(x, y, \bar\epsilon)$-space near the origin. A new rescaled system, in rescaled variables (x_R, y_R), is less degenerate than the original one because the slow-fast structure is eliminated. A limit cycle in the (x_R, y_R)-space is called a small-amplitude limit cycle of the original system, since its size tends to 0 in the (x, y)-plane as $\bar\epsilon$ goes to 0. Besides these limit cycles, the (x, y)-plane may contain so-called detectable canard limit cycles, characterized by a slow movement along attractive and repelling parts of critical curve and a fast movement. Interested

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in studying limit cycles in a neighborhood of the origin in the (x, y)-plane that does not shrink to the origin when $\bar{\epsilon} \to 0$, one considers not only $O(\bar{\epsilon})$ small-amplitude limit cycles introduced above but also so-called "large" small-amplitude limit cycles in the (x, y)-plane that are unbounded in the (x_R, y_R)-plane and close to the origin in the (x, y)-plane. These large small-amplitude limit cycles are positioned between $O(\bar{\epsilon})$ small-amplitude limit cycles and detectable limit cycles of canard type, and can be examined by blowing up the origin $(x, y, \bar{\epsilon}) = (0, 0, 0)$ and studying dynamics in so-called family directional and phase-directional charts of the blow-up. The $O(\bar{\epsilon})$-limit cycles are seen in the family directional chart. The difficulty lies in the fact that these two types of small-amplitude limit cycles generally can not be studied in a uniform way, and one has to use different methods. It must be stressed that it remains a difficult problem even when one is interested in cyclicity results for each of the two types separately. To end this, one glues together the two different local results to obtain the cyclicity of the origin in the (x, y)-plane. We refer to [18] and [8]. In the first paper a codimension 1 Hopf case has been studied generalizing the Van der Pol system; in the second one, a slow-fast Hopf point of higher codimension in Liénard equations has been dealt with.

In the saddle and elliptic cases (1), an extra difficulty appears. Besides the singular parameter $\bar{\epsilon}$, there are 3 parameters $b = (b_0, b_1, b_2)$ that unfold the codimension 3 nilpotent singularity at the origin $(x, y) = (0, 0)$. Therefore, we have to combine two blow-up constructions (see [16] and [17]): a *primary blowing-up* where we blow up the phase coordinates (x, y) and the parameter b and a *secondary blowing-up* where we blow up the new phase coordinates $(\bar{x}, \bar{y})$ and the singular parameter $\bar{\epsilon}$. Here, the study of small-amplitude limit cycles near $(x, y) = (0, 0)$ is far more complex than the study presented in [18] and [8]. After the primary blow-up, we have $O(b)$ small-amplitude limit cycles in the (x, y)-plane and "large" small-amplitude limit cycles in the (x, y)-plane. The $O(b)$-limit cycles are seen in the $(\bar{x}, \bar{y})$-plane and originate from a system with a slow-fast structure, since the primary blow-up has nothing to do with $\bar{\epsilon}$. Hence, there are 3 different types of $O(b)$ limit cycles: detectable canard limit cycles in the $(\bar{x}, \bar{y})$-space, and after the secondary blow-up, large small-amplitude limit cycles in the $(\bar{x}, \bar{y})$-space and $O(\bar{\epsilon})$ small-amplitude limit cycles in the $(\bar{x}, \bar{y})$-plane. Putting this together, we obtain 4 different types of small-amplitude limit cycles in the (x, y)-space. Each of these 4 types has to be studied separately and then all the results have to be glued together to obtain the cyclicity of $(x, y) = (0, 0)$ in the saddle and elliptic bifurcations.

In Section 2, we describe the primary blow-up and the secondary blow-up in detail, and we give a survey of results obtained in [16] and [17]. In the present paper we focus first on those types of small-amplitude limit cycles in the saddle and elliptic cases that have not been studied in [16] and [17]. Then the global cyclicity result near $(x, y) = (0, 0)$ in (1) will be obtained by putting all local cyclicity results together. Structure of the paper and the main result are given in Section 2.

The motivation to study the very degenerate slow-fast codimension 3 saddle and elliptic bifurcations is twofold. We explain it in the remainder of this section.

First, let us consider the following open problem, closely related to Hilbert's 16th problem: Given any polynomial generalized Liénard equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x), \end{cases} \tag{2}$$

with $\deg f = n$ and $\deg g = m$, determine the uniform bound $\mathcal{Q}(m, n)$ on the number of limit cycles in terms of the two degrees. System (2) is a representation in the phase plane of the second-order scalar differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

At present, the bound $\mathcal{Q}(m,n)$ is only known in some very low-degree cases. It has been shown that $\mathcal{Q}(1,2)=1$ ([20]), $\mathcal{Q}(1,3)=1$ ([19]), $\mathcal{Q}(2,1)=1$ ([2]), $\mathcal{Q}(3,1)=1$ ([11] and [13]) and $\mathcal{Q}(2,2)=1$ ([12]). When $G=0$ and $\bar{H}$ is polynomial, the given family (1) of vector fields is of type (2) of degree $(m,1)$. For $n=1$, we point out that there is a relation between (2) and slow-fast type Liénard equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = -xy + \bar{\epsilon}(b_0 + b_1 x + ... + b_{l-1}x^{l-1} \pm x^l + x^{l+1}\widetilde{H}(x)), \end{cases} \tag{3}$$

where $\bar{\epsilon} > 0$ is the singular parameter kept small, $(b_0, b_1, ..., b_{l-1})$ are regular perturbation parameters close to 0 and where $\widetilde{H}$ is a smooth function. More precisely, if we want to contribute to finding $\mathcal{Q}(m,1)$ for $m \geq 4$, then we need to study (3) for $l = 0, 1, ..., m-1$, taking into account higher order terms in $\widetilde{H}$. For more details we refer to [10] and [21].

For $l = 0, 1, 2$, systems of type (3) have been studied in [8] and [5]. In [4] and [6], an arbitrary codimension $l \geq 3$ has been treated putting focus on detectable canard cycles. The study of small amplitude limit cycles of (3) for $l \geq 3$ is far more complex. The present paper, [16] and [17] give a complete study of the case $l = 3$. We point out that the methods introduced in these three papers and a recursive approach can surely be used to tackle a similar problem of an arbitrary codimension l.

When $\bar{\epsilon} = 0$, then (1) has a line of singular points given by $\{y=0\}$. All points of the line are normally hyperbolic, except for the origin where we deal with a nilpotent singularity. We call the origin a generic turning (or contact) point because $X^{\pm}_{0,b,\lambda}$ has a quadratic contact between the curve $\{y=0\}$ and the fast orbits. Though we are inspired by the generalized Liénard equations, we study more general systems (1) having an extra quadratic term $\bar{\epsilon}y^2 G(x,y,\lambda)$. In fact, the system (1) is a smooth local normal form for equivalence for slow-fast systems having a curve of singularities with a generic nilpotent contact point and having a singularity of order 2 in the slow dynamics located at the contact point (see [16]).

The second motivation can be found in [9] which deals with very delicate regular generic saddle, focus and elliptic bifurcations of three parameter families of planar vector fields around nilpotent singular points. Since $\bar{\epsilon} \sim 0$, the focus case can not occur in (1) (see [16]). We believe that the cyclicity result for the slow-fast codimension 3 saddle and elliptic bifurcations will help to finish the study of small limit cycles in the regular saddle and elliptic cases treated in [9]. This is a topic of further study.

2. **Statement of results.** If we introduce a new variable $Y = y + \frac{1}{2}x^2$, then (1) changes into

$$\begin{cases} \dot{x} = Y - \frac{1}{2}x^2 \\ \dot{Y} = \bar{\epsilon}\Big(b_0 + b_1 x + b_2 x^2 \pm x^3 + x^4 \bar{H}(x,\lambda) \\ \qquad +(Y - \frac{1}{2}x^2)^2 G(x, Y - \frac{1}{2}x^2, \lambda)\Big). \end{cases} \tag{4}$$

This is a representation of (1) in the so-called Liénard plane. In [16] we first have reparametrized the b-parameters, by introducing weighted spherical coordinates:

$$(b_0, b_1, b_2) = (r^3 \bar{B}_0, r^2 B_1, r B_2), \ r \geq 0, \ B = (\bar{B}_0, B_1, B_2) \in \mathbb{S}^2.$$

If we introduce this change in the parameter space in the family of vector fields (4), we obtain an $(\bar{\epsilon}, B, r, \lambda)$-family of vector fields in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = y - \frac{1}{2}x^2 \\ \dot{y} = \bar{\epsilon}\Big(r^3\bar{B}_0 + r^2B_1x + rB_2x^2 \pm x^3 + x^4\bar{H}(x,\lambda) \\ \qquad +(y-\frac{1}{2}x^2)^2G(x, y-\frac{1}{2}x^2,\lambda)\Big), \end{cases} \tag{5}$$

where we denote Y by y.

In [16] and [17], the calculations have been performed, as usual, in different charts of the sphere and, depending on the chart which one uses, one finds different configurations of limit cycles:

a) jump region: $\bar{B}_0 = \pm 1$ and (B_1, B_2) in an arbitrary compact subset of the plane; system (5) has no limit cycles in an arbitrary compact set in the (x, y)-plane for $\bar{\epsilon} \sim 0$, $r \sim 0$ and $\lambda \in \Lambda$,

b) slow-fast Hopf region: $\bar{B}_0$ close to 0, $B_1 = -1$ and B_2 in an arbitrary compact interval; if $B_2 \neq 0$ (uniformly), then system (5) has at most one hyperbolic limit cycle in an $(\bar{\epsilon}, \bar{B}_0, B_2, r, \lambda)$-uniform neighborhood of the origin $(x, y) = (0, 0)$, for $\bar{\epsilon} \sim 0$ and $r \sim 0$,

c) "$\mathbf{B_1 = 1}$"-region: $\bar{B}_0$ close to 0, $B_1 = 1$ and B_2 in an arbitrary compact interval; system (5) has no limit cycles in an arbitrary compact set in the (x, y)-plane for $\bar{\epsilon} \sim 0$, $r \sim 0$ and $\lambda \in \Lambda$,

d) slow-fast Bogdanov-Takens region: $\bar{B}_0$ and B_1 close to 0 and $B_2 = \pm 1$; system (5) has at most one hyperbolic limit cycle in an arbitrary compact set in the (x, y)-plane for $\bar{\epsilon} \sim 0$, $r \sim 0$ and $\lambda \in \Lambda$.

More precisely, let us write $P = (0, -1, 0) \in \mathbb{S}^2$. Define now

$$Q_{\delta'} = \{q \in \mathbb{S}^2 | d(q, P) < \delta'\} \subseteq \mathbb{S}^2,$$

where $\delta' > 0$ and $d(q, P) = \sqrt{q_1^2 + (q_2+1)^2 + q_3^2}$ with $q = (q_1, q_2, q_3)$. Putting these four cases together we obtain the following result (see [16]):

Theorem 2.1. (away from the point P) *Given $\delta' > 0$ arbitrary, there exist $\bar{\epsilon}_0 > 0$, $r_0 > 0$ and a neighborhood V of $(x, y) = (0, 0)$ such that for each $(\bar{\epsilon}, r, B, \lambda) \in [0, \bar{\epsilon}_0] \times [0, r_0] \times (\mathbb{S}^2 \backslash Q_{\delta'}) \times \Lambda$ the system (5) restricted to V has at most one hyperbolic limit cycle.*

Observe that Theorem 2.1 does not cover the slow-fast Hopf region for $B_2 \sim 0$ which corresponds to $(\bar{B}_0, B_1, B_2) \sim P$. In [16] and [17], we supposed that $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$ and we gave only partial results near P; e.g. we proved that for $\bar{\epsilon} > 0$ and $r > 0$ sufficiently small, system (5) contains a saddle-node bifurcation of limit cycles near P.

In the remainder of this section we suppose that $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. To better understand the results near P obtained in [16] and [17], we need to combine two blow-up constructions: one to unfold the codimension 3 singularity (a "primary blow-up"), and one to dissolve the slow-fast structure (a "secondary blow-up"). We focus on (5) with $\bar{B}_0 \sim 0$, $B_1 = -1$, $B_2 \sim 0$, and we introduce the following rescaling:

$$(\bar{\epsilon}, \bar{B}_0) = (\epsilon^2 E, \epsilon B_0),\ \epsilon \geq 0,\ \epsilon \sim 0,\ (E, B_0) \in \mathbb{S}^1,\ E \geq 0.$$

The calculations will be performed, as usual, in charts. When E is in an arbitrary compact interval in $[0, +\infty[$ and $B_0 = \pm 1$, then the system (5) has no small-amplitude limit cycles; for details see [16]. When $E = 1$ and $B_0 \sim 0$, then (5) changes into

$$X^{\pm}_{\epsilon,r,B_0,B_2,\lambda}:\begin{cases}\dot{x}=y-\frac{1}{2}x^2\\ \dot{y}=\epsilon^2\Big(r^3\epsilon B_0-r^2x+rB_2x^2\pm x^3+x^4\bar{H}(x,\lambda)\\ \qquad +(y-\frac{1}{2}x^2)^2G(x,y-\frac{1}{2}x^2,\lambda)\Big).\end{cases}\tag{6}$$

The idea in [16] was to introduce the following primary blow-up:

$$(x,y,r)=(u\bar{x},u^2\bar{y},u\bar{r}),\ u\geq 0,\ \bar{r}\geq 0,\ (\bar{x},\bar{y},\bar{r})\in\mathbb{S}^2,\tag{7}$$

and to study the dynamics of (6) in the blown-up coordinates in different charts. The family chart is obtained by taking $\bar{r}=1$ in (7) and keeping $(\bar{x},\bar{y})$ in some arbitrarily large disk in $\mathbb{R}^2$. In this family chart, the vector field (6) yields, after division by the positive factor u,

$$\begin{cases}\dot{\bar{x}}=\bar{y}-\frac{1}{2}\bar{x}^2\\ \dot{\bar{y}}=\epsilon^2\Big(\epsilon B_0-\bar{x}+B_2\bar{x}^2\pm\bar{x}^3+u\bar{x}^4\bar{H}(u\bar{x},\lambda)\\ \qquad +u(\bar{y}-\frac{1}{2}\bar{x}^2)^2G(u\bar{x},u^2(\bar{y}-\frac{1}{2}\bar{x}^2),\lambda)\Big).\end{cases}\tag{8}$$

For the study of (6) in the phase-directional charts "$\bar{x}=+1$, $\bar{x}=-1$, $\bar{y}=+1$, $\bar{y}=-1$" we refer to [16] and [17]. If we want to study limit cycles of (6) in a fixed neighborhood of the origin $(x,y)=(0,0)$, independent of r, then we need to study (6) in the family chart $\bar{r}=1$, and in the phase-directional chart $\bar{y}=+1$ where the blow-up map is

$$(x,y,r)=(U\bar{X},U^2,UR),$$

bearing in mind that $U\sim 0$, $U\geq 0$ and $(\bar{X},R)$ is in an arbitrarily large compact set in $\mathbb{R}\times[0,+\infty[$.

In order to desingularize (8), in [16] we have blown-up the origin $(\bar{x},\bar{y},\epsilon)=(0,0,0)$ using the blow-up transformation (secondary blow-up):

$$(\bar{x},\bar{y},\epsilon)=(\delta\widetilde{x},\delta^2\widetilde{y},\delta w),\tag{9}$$

where $\delta\geq 0$, $(\widetilde{x},\widetilde{y},w)\in\mathbb{S}^2_+$ and $\mathbb{S}^2_+$ is the half-sphere with $w\geq 0$. We have again the family chart "$w=+1$" and the phase-directional charts "$\widetilde{x}=+1$, $\widetilde{x}=-1$, $\widetilde{y}=+1$, $\widetilde{y}=-1$". For a detailed study of (8) in these charts we refer to [16] or Section 3 in this paper.

The goal of using the primary blow-up and the secondary blow-up was to detect all closed curves (so-called limit periodic sets) on the primary blow-up locus and the secondary blow-up locus that can generate small-amplitude limit cycles of (6) for $\epsilon>0$, $\epsilon\sim 0$, $r>0$, $r\sim 0$, $B_0\sim 0$ and $B_2\sim 0$. In [16] we have obtained Figure 1 in the saddle case and Figure 2 in the elliptic case. There are five different kinds of these limit periodic sets in the saddle case:

(i) and (ii) The singular point in the middle and the closed orbits L_h on the secondary blow-up locus;

(iii) The singular cycle L_0 consisting of singularities S_1,S_2 and the regular orbits that are heteroclinic to them;

(iv) The canard limit periodic sets $L_{\bar{y}}$, $\bar{y}\in]0,\frac{1}{2}[$, in the $(\bar{x},\bar{y})$-space;

(v) The slow-fast two-saddle-limit periodic set $L_{\frac{1}{2}}$.

There are five different kinds of such curves in the elliptic case:

(i), (ii) and (iii) The singular point in the middle, the closed orbits L_h and the singular cycle L_0 (see the saddle case);

(iv) The canard limit periodic sets $L_{\bar{y}}$, $\bar{y}\in]0,+\infty[$, in the $(\bar{x},\bar{y})$-plane;

(v) The singular cycle L_{00} consisting of singularities S_1, S_2, R_1, R_2 and the regular and singular sections that are connected (heteroclinic) to them.

For precise definitions of all limit periodic sets in the saddle and elliptic cases we refer to [16] and [17].

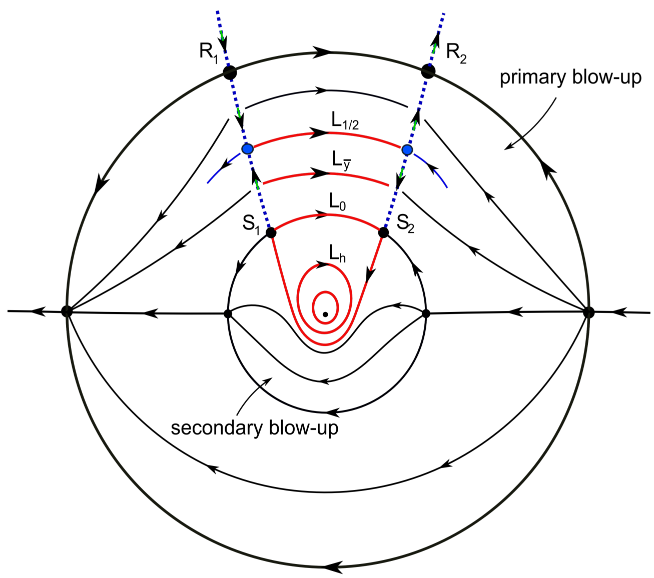

FIGURE 1. A bird's eye view of the primary and secondary blow-up with limit periodic sets in the saddle case for $(\epsilon, r, B_0, B_2) = (0, 0, 0, 0)$, with indication of (primary) slow dynamics.

In [16], we gave a complete study of the limit periodic sets $L_{\bar{y}}$ in both the saddle and elliptic cases, and the limit periodic set $L_{\frac{1}{2}}$ in the saddle case. We proved that the set $\cup_{\bar{y}\in[\bar{y}_1,\bar{y}_2]} L_{\bar{y}}$ can produce at most 2 limit cycles for any fixed $0 < \bar{y}_1 < \bar{y}_2 < \frac{1}{2}$ (resp. $0 < \bar{y}_1 < \bar{y}_2 < +\infty$) in the saddle case (resp. the elliptic case). The limit periodic set $L_{\frac{1}{2}}$ is shown to produce at most 2 limit cycles. The limit periodic set L_{00} in the elliptic case is shown to produce at most 2 limit cycles, based on the work in [17]. Let us recall that the paper [17] is devoted to the study of the transition from the limit cycles of $X^-_{\epsilon,r,B_0,B_2,\lambda}$ near (large) limit periodic sets $L_{\bar{y}}$ to the limit cycles of $X^-_{\epsilon,r,B_0,B_2,\lambda}$ near small (but detectable) canard limit periodic sets L_y in the (x, y)-space. This transition is referred to as primary birth of canards. The limit periodic sets L_y have been already studied in [4] and the limit cycles produced by L_y have nothing to do with the small-amplitude limit cycles of $X^-_{\epsilon,r,B_0,B_2,\lambda}$.

In Section 3 we prove that at most 3 limit cycles may occur near the polycycle L_0 in the saddle and elliptic cases. For a definition of L_0 we refer to Section 3. Near L_0 we have the transition from the limit cycles near the ovals L_h to the limit cycles near the small (but detectable) canard limit periodic sets $L_{\bar{y}}$ in the $(\bar{x}, \bar{y})$-space. We call this type of transition secondary birth of canards and it has been observed in [8]. In [8], they investigate the number of limit cycles that can appear near a slow-fast Hopf point in Liénard equations, and this under very general conditions. Such a slow-fast Hopf point occurs in system (8), at the origin. For a general smooth function G, system (8) is not necessary Liénard, and the results proven in [8] can not be used,

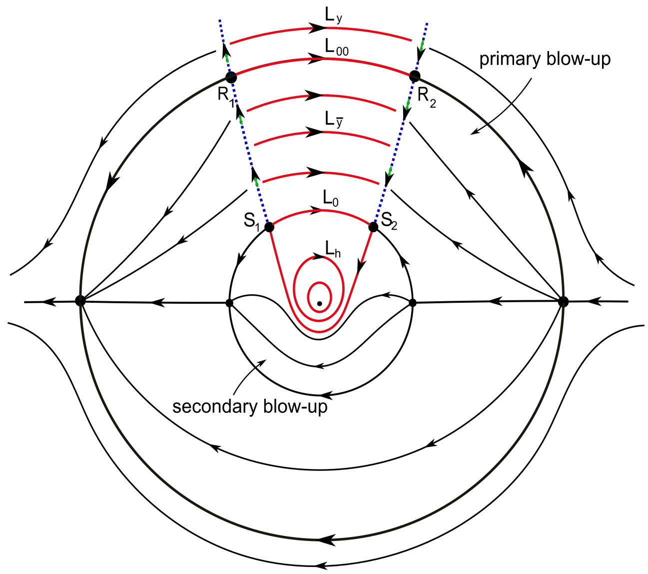

FIGURE 2. A bird's eye view of the primary and secondary blow-up with limit periodic sets in the elliptic case, for $(\epsilon, r, B_0, B_2) = (0, 0, 0, 0)$.

but the same type of methods as in [8] can be used to tackle this "non-Liénard" problem; one exploits symmetries that are present both in the system (8) and in the blow-up construction (9), one uses smooth normal forms near semi-hyperbolic singularities S_1 and S_2 to get the structure of the transition maps near S_1 and S_2, etc. Seen the length of this paper, in Section 3 we give only a sketch of the proof of the cyclicity result near L_0; we refer the interested reader to [15] for a detailed study of the non-Liénard equation (8) near L_0.

In Section 4 we investigate the limit cycles near the ovals L_h, for $h \in]0, 1]$, in the saddle and elliptic cases. We have $L_h = \{(\widetilde{x}, \widetilde{y});\ e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}\widetilde{x}^2 + 1) = h\}$ (see Section 3). Here L_1 represents the singular point in the middle in Figure 1 and Figure 2, and L_h tends to the polycycle L_0 as $h \to 0$. We suppose that L_h is oriented counter-clockwise. For any small but fixed $h_0 > 0$ we prove that the set $\cup_{h \in [h_0, 1]} L_h$ produces at most 2 limit cycles. We use the fact that the following conjecture up to codimension 2, formulated by Dumortier and Roussarie in [8], has been solved (see [14] and Section 4):

Conjecture. The system $\{\int_{L_h} \widetilde{x}^{-1} d\widetilde{y}, \int_{L_h} \widetilde{x} d\widetilde{y}, \int_{L_h} \widetilde{x}^3 d\widetilde{y}\}$ of analytic functions is a strict Chebyshev system on $[h_0, 1]$.

For a definition of a strict Chebyshev system we refer to [8] or Section 4. We point out that the work in [14] was motivated by [8] and the slow-fast codimension 3 saddle and elliptic bifurcations.

We say that the cyclicity of the origin $(x, y) = (0, 0)$ for $X^{\pm}_{\epsilon, b, \lambda}$, defined in (1), is bounded by N if there exist a neighborhood V of $(x, y) = (0, 0)$, a neighborhood W of $(0, 0, 0)$ in b-space and an $\bar{\epsilon}_0 > 0$ such that for each $(\bar{\epsilon}, b, \lambda) \in [0, \bar{\epsilon}_0] \times W \times \Lambda$ the systems $X^{\pm}_{\bar{\epsilon}, b, \lambda}$ have at most N limit cycles inside V. (The minimum of such N is the cyclicity of the origin). Now, based on the above discussion and Theorem 2.1, we get a finite cyclicity of $(x, y) = (0, 0)$ bounded by 9, in both the saddle and elliptic cases. Of course we do not expect that 9 is an optimal bound for the cyclicity of the origin. In Section 5 we discuss the relation of the formulas established in Section 3 with the abelian integrals in Section 4, and we find that the system (8) has at most 2 limit cycles in a fixed neighborhood of the origin $(\bar{x}, \bar{y}) = (0, 0)$, i.e. the set $\cup_{h\in[0,1]} L_h$ produces at most two limit cycles in the saddle and elliptic cases.

In Section 6 we study evolution of zeros in a divergence integral introduced in [16] and [17], and we find at most 2 limit cycles near $\left(\cup_{\bar{y}\in[\bar{y}_1,+\infty[} L_{\bar{y}}\right) \cup L_{00}$ in the elliptic case and at most 2 limit cycles near $\cup_{\bar{y}\in[\bar{y}_1,\frac{1}{2}]} L_{\bar{y}}$ in the saddle case where $\bar{y}_1 > 0$ is arbitrarily small.

In Section 7, based on Section 5 and Section 6, we finally prove our main result:

Theorem 2.2. *Consider (1) where $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. Then the cyclicity of the origin $(x, y) = (0, 0)$ is equal to 2, in both the saddle and elliptic cases.*

Remark 1. 1. Throughout the paper "smooth" stands for "C^∞ smoothness". We make no distinction between C^p-functions and C^q-functions where p and q are large natural numbers. In this paper we call them C^k-functions.
2. The notation $O(r_1, ..., r_j)$ where $r = (r_1, ..., r_j)$ is a function of $x = (x_1, ..., x_n)$ with values in $\mathbb{R}^j$ stands for a linear combination $\sum_{i=1}^{j} R_i r_i$, where R_i are also functions of x. When all these functions are smooth (respectively C^k) in x, then we say that the function $O(r_1, ..., r_j)$ is smooth (respectively C^k) in x. In the paper we allow x to be a function of $y = (y_1, ..., y_m)$.

3. Cyclicity of L_0.

Let $B_3^0 > 1$ be arbitrary, but fixed, real number. We write $\mathcal{B} = [-B_3^0, B_3^0]$ and consider smooth systems (8):

$$Z_{\epsilon,B_0,B_2,B_3,u,\lambda} : \begin{cases} \dot{\bar{x}} = \bar{y} - \frac{1}{2}\bar{x}^2 \\ \dot{\bar{y}} = \epsilon^2\Big(\epsilon B_0 - \bar{x} + B_2\bar{x}^2 + B_3\bar{x}^3 + u\bar{x}^4\bar{H}(u\bar{x}, \lambda) \\ \qquad +u(\bar{y} - \frac{1}{2}\bar{x}^2)^2 G(u\bar{x}, u^2(\bar{y} - \frac{1}{2}\bar{x}^2), \lambda)\Big), \end{cases} \tag{10}$$

where $\epsilon \geq 0$, $\epsilon \sim 0$, $B_0 \sim 0$, $B_2 \sim 0$, $B_3 \in \mathcal{B}$, $u \sim 0$ and $\lambda \in \Lambda$. Let us recall that $B_3 = +1$ represents the saddle case and $B_3 = -1$ represents the elliptic case. Our goal is to study limit cycles of (10) near L_0.

Remark 2. The upper bound for the cyclicity of L_0 that we obtain in this section will not depend on the coefficient B_3 kept in the compact set $\mathcal{B}$.

This section is structured as follows. In Section 3.1 we study system (10) in the family chart and in the phase directional charts of the secondary blow-up (9), and we detect L_0 on the secondary blow-up locus, for $B_0 = 0$ (Figure 3).

In Section 3.2 we define a difference map near L_0 which enables us to introduce the notion of cyclicity of L_0 as used in [17] and [8]. Then we state a result about the cyclicity of L_0 which provides us with an upper bound for the number of limit cycles near L_0 that is only one unit higher than a sharp upper bound that we expect to hold (Theorem 3.2).

In Section 3.3, we give a sketch of the proof of Theorem 3.2; the detailed proof of Theorem 3.2 can be found in [15].

3.1. Secondary blow-up.

If we add the equation $\dot{\epsilon} = 0$ to (10), we obtain a $\tau := (B_0, B_2, B_3, u, \lambda)$-family of vector fields on $\mathbb{R}^3$:

$$Z_\tau := Z_{\epsilon,B_0,B_2,B_3,u,\lambda} + 0\frac{\partial}{\partial\epsilon}.$$

We consider the blow-up map (9) (defining a singular change of coordinates):

$$\Theta_1 : \mathbb{R}^+ \times \mathbb{S}^2 \to \mathbb{R}^3 : (\delta, (\widetilde{x}, \widetilde{y}, w)) \mapsto (\bar{x}, \bar{y}, \epsilon) = (\delta\widetilde{x}, \delta^2\widetilde{y}, \delta w),\ w \geq 0.$$

The blown-up vector field is defined as the pullback of the original vector field Z_τ divided by δ:

$$\bar{Z}_\tau := \frac{1}{\delta}\Theta_1^* Z_\tau. \tag{11}$$

The calculations near the blow-up locus $\{0\} \times \mathbb{S}^2_+$ will be performed, as usual, in charts.

3.1.1. *Family directional chart.* We use the following family rescaling of (10):

$$(\bar{x}, \bar{y}) = (\epsilon\widetilde{x}, \epsilon^2\widetilde{y}), \tag{12}$$

with $(\widetilde{x}, \widetilde{y})$ in an arbitrarily large disk in $\mathbb{R}^2$ and $\epsilon \geq 0$. The blown-up field is a (ϵ, τ)-family of 2-dimensional vector fields

$$Z^{(1)}_{\epsilon,\tau} : \begin{cases} \dot{\widetilde{x}} = \widetilde{y} - \frac{1}{2}\widetilde{x}^2 \\ \dot{\widetilde{y}} = B_0 - \widetilde{x} + B_2\epsilon\widetilde{x}^2 + B_3\epsilon^2\widetilde{x}^3 + u\epsilon^3\widetilde{x}^4\bar{H}(u\epsilon\widetilde{x}, \lambda) \\ \qquad +u\epsilon^3(\widetilde{y} - \frac{1}{2}\widetilde{x}^2)^2 G(u\epsilon\widetilde{x}, u^2\epsilon^2(\widetilde{y} - \frac{1}{2}\widetilde{x}^2), \lambda), \end{cases} \tag{13}$$

where we treat ϵ as a parameter.

We give now some basic properties of the vector field (13). For $B_0 = \epsilon = 0$, $Z^{(1)}_{\epsilon,\tau}$ becomes independent of any parameter:

$$\begin{cases} \dot{\widetilde{x}} = \widetilde{y} - \frac{1}{2}\widetilde{x}^2 \\ \dot{\widetilde{y}} = -\widetilde{x}. \end{cases} \tag{14}$$

The vector field (14) is invariant under the symmetry $(\widetilde{x}, \widetilde{y}, t) \to (-\widetilde{x}, \widetilde{y}, -t)$. As a consequence, the vector field is of center type with the center at the origin $(\widetilde{x}, \widetilde{y}) = (0, 0)$. We also see that the vector field (14) is the dual of the differential 1-form:

$$\omega_0 = \widetilde{x}d\widetilde{x} + (\widetilde{y} - \tfrac{1}{2}\widetilde{x}^2)d\widetilde{y}$$

which admits the function $-e^{-\widetilde{y}}$ as integrating factor and the function $H(\widetilde{x}, \widetilde{y}) = e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}\widetilde{x}^2 + 1)$ as first integral. This means that

$$-e^{-\widetilde{y}}w_0 = dH.$$

Of course the integrating factor is not unique, but $-e^{-\widetilde{y}}$ has the advantage that the related Hamiltonian H is zero on $\gamma = \{\widetilde{y} = \frac{1}{2}\widetilde{x}^2 - 1\}$ and also at infinity ($e^{-\widetilde{y}}$ is flat for $\widetilde{y} = +\infty$). Notice that $\{H(\widetilde{x}, \widetilde{y}) = 1\}$ represents the center $(\widetilde{x}, \widetilde{y}) = (0, 0)$ (the singular point in the middle in Figure 1, Figure 2 and Figure 3 is denoted by L_1) and that $\{H(\widetilde{x}, \widetilde{y}) = h\} = L_h$ where $h \in]0, 1[$. The orbit γ separates the closed level curves of H, which are obtained for $h \in]0, 1[$, from the open ones, with $h < 0$; see Figure 3 for an illustration.

3.1.2. *Phase-directional charts.* To see what happens in the end points of the critical curve in the blown up space, we consider the phase-directional chart in the $\bar{y}$-direction:

$$(\bar{x}, \bar{y}, \epsilon) = (\delta v, \delta^2, \delta w). \tag{15}$$

We obtain a blown-up field which, after dividing by δ, can be written as

$$Z^{(2)}_\tau : \begin{cases} \dot{v} = 1 - \frac{1}{2}v^2 + \frac{1}{2}w^2 vD(\delta, w, v, \tau) \\ \dot{\delta} = -\frac{1}{2}\delta w^2 D(\delta, w, v, \tau) \\ \dot{w} = \frac{1}{2}w^3 D(\delta, w, v, \tau), \end{cases} \tag{16}$$

where $D(\delta,w,v,\tau) = -\Big(wB_0 - v + B_2\delta v^2 + B_3\delta^2v^3 + u\delta^3v^4\bar{H}(u\delta v,\lambda) + u\delta^3(1-\frac{1}{2}v^2)^2G(u\delta v, u^2\delta^2(1-\frac{1}{2}v^2),\lambda)\Big)$.

On $\{\delta=0, w=0\}$ (16) has singularities at $v=\pm\sqrt{2}$. The eigenvalues of the linear part at $v=\sqrt{2}$ are given by $(-\sqrt{2},0,0)$ and at $v=-\sqrt{2}$ by $(\sqrt{2},0,0)$. Hence (16) has at $S_2=(\sqrt{2},0,0)$ a semi-hyperbolic singularity with the v-axis as stable manifold and a two-dimensional center manifold, and at $S_1=(-\sqrt{2},0,0)$ a semi-hyperbolic singularity with the v-axis as unstable manifold and a two-dimensional center manifold. This is true for any τ. The points $S_{1,2}$ represent the end points of the critical curve $\{v=\pm\sqrt{2},\delta\geq 0, w=0\}$ in the blown-up space.

Other phase-directional charts are not relevant when studying L_0.

3.1.3. *Combining the family chart and the phase-directional chart in the* $\bar{\mathbf{y}}$*-direction for the secondary blow-up.* The secondary blow-up locus can be considered as a 2-dimensional closed disc which we denote by $\bar{S}$. We treat the blown-up vector field $\bar{Z}_\tau$ to be a τ-family of 3-dimensional vector fields defined in a neighborhood of $\bar{S}$. For $\epsilon=0$, this family only depends on the parameter $B_0\sim 0$.

For $B_0=0$, the vector field $\bar{Z}_\tau$ on $\bar{S}$ is represented by the vector field (14). It can be easily seen that the orbit γ of (14) connects S_2 and S_1 in such a way that the α-limit set of γ is S_2 and the ω-limit set is S_1. Let L_0 denote the singular cycle defined as the union of γ with the arc C of $\partial\bar{S}$ between S_1 and S_2. L_0 is a polycycle of the blown-up vector field $\bar{Z}_\tau$. We refer to Figure 3.

3.2. **Difference map and statement of results.** Inspired by [15], we introduce here a difference map near L_0. Take a small $w_0>0$ and a $v_0\in]0,\sqrt{2}[$ such that v_0 is close to $\sqrt{2}$. Choose (2-dimensional) sections $\Sigma_\pm=\{v=\pm v_0\}$ transverse to C and sections $T_\pm=\{w=w_0\}$ transverse to the orbit γ (Figure 4). Σ_-, T_- are chosen near S_1, while Σ_+, T_+ are chosen near S_2. The chosen sections $T_\pm$ are contained in the family directional chart of the secondary blow up (Section 3.1), and we parametrize $T_\pm$ by (h,ϵ) where h is the value of the corresponding Hamiltonian H introduced in Section 3.1.1. The sections $\Sigma_\pm$ are contained in the phase-directional chart $\{\widetilde{y}=1\}$ and can be parametrized by (δ,w), where (δ,w) are coordinates of (16) such that $(\delta,w)\sim(0,0)$ and $(\delta,w)\geq(0,0)$.

We define now regular transition maps and Dulac maps near L_0:
a) the regular transition map $\mathcal{F}_\tau$ near C from Σ_- to Σ_+, defined by the flow of $\bar{Z}_\tau$ (the differential equation for $\bar{Z}_\tau$ can be found by using (16)),
b) the Dulac maps $\mathcal{D}_\tau^\pm$ describing the passage from $\Sigma_\pm$ to $T_\pm$, defined by the flow of $\pm\bar{Z}_\tau$ (the differential equation for $\bar{Z}_\tau$ can be found by using (16)),
c) the regular transition map $\mathcal{G}_\tau$ near γ from T_- to T_+, defined by the flow of $-\bar{Z}_\tau$ (the differential equation for $\bar{Z}_\tau$ can be found by using (13)).

In order to study bifurcating canard limit cycles near L_0, we will study zeros of the difference map

$$\Omega_\tau(\delta,w) = \mathcal{D}_\tau^+\circ\mathcal{F}_\tau(\delta,w) - \mathcal{G}_\tau\circ\mathcal{D}_\tau^-(\delta,w), \tag{17}$$

where (δ,w) are the chosen regular parameters on Σ_-. This map $\Omega_\tau(\delta,w)$ takes its values in the (h,ϵ)-space. Since $\dot{\epsilon}=0$ in the vector field Z_τ, the ϵ-component of $\Omega_\tau(\delta,w)$ is identical to 0. We denote the h-component of $\Omega_\tau(\delta,w)$ by $\omega_\tau(\delta,w)$.

Let $B(d)$ be a ball of fixed radius $d>0$ at the origin $(\delta,w)=(0,0)$.

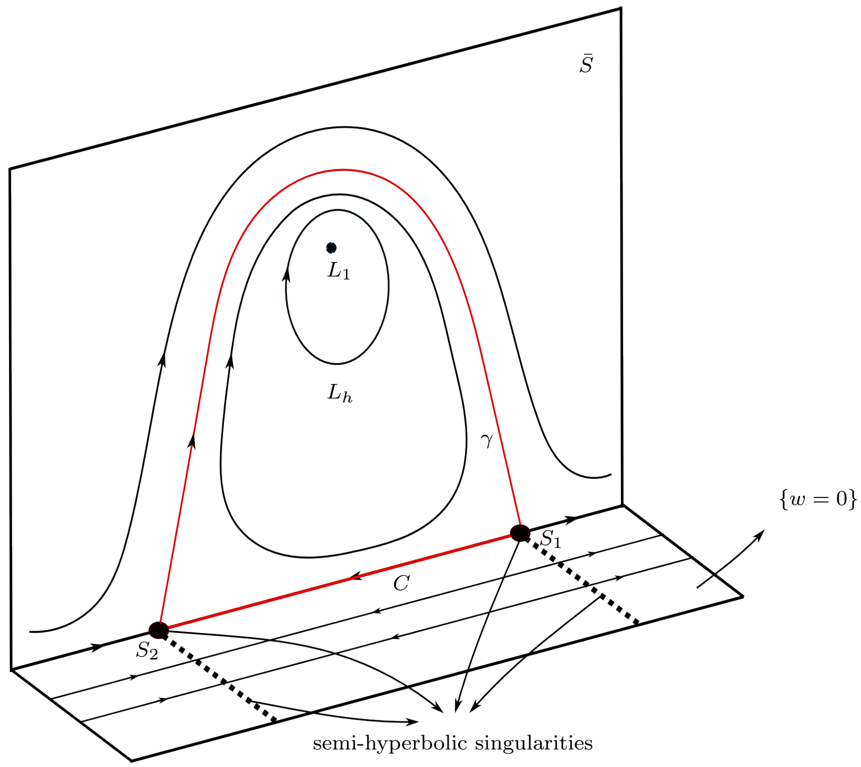

FIGURE 3. The dynamics of the vector field $\bar{Z}_\tau$ for $\epsilon = B_0 = 0$ and the singular cycle $L_0 = \gamma \cup C$.

Definition 3.1. a) $w_\tau(\delta, w)$ is said to have finite cyclicity if there exists $N \in \mathbb{N}_0$, $\epsilon_0 > 0$, $d > 0$, $\widetilde{B}_0 > 0$, $\widetilde{B}_2 > 0$ and $u_0 > 0$ such that for each fixed value of $(\epsilon, \tau) \in]0, \epsilon_0] \times [-\widetilde{B}_0, \widetilde{B}_0] \times [-\widetilde{B}_2, \widetilde{B}_2] \times \mathcal{B} \times [-u_0, u_0] \times \Lambda$, $(B_0, B_2, u) \neq (0,0,0)$, the number of zeros (counting multiplicity) of $w_\tau(\delta, w)$ on $s_\epsilon = \{(\delta, w); \delta w = \epsilon, (\delta, w) \in B(d), \delta \geq 0, w \geq 0\}$ is bounded by N (the number of zeros of a function with an empty domain is zero).
b) The minimum of such N is called cyclicity of w_τ. Cyclicity of L_0 is the cyclicity of w_τ.

The following theorem states that at most 3 limit cycles may occur in an (ϵ, τ)-uniform neighborhood of L_0 where $\epsilon \sim 0$, $B_0 \sim 0$, $B_2 \sim 0$, $B_3 \in \mathcal{B}$, $u \sim 0$, $\lambda \in \Lambda$ and $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$.

Theorem 3.2. *Suppose that $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. Then the cyclicity of L_0 is bounded by* 3.

3.3. **Sketch of the proof of Theorem 3.2.** In [15], a detailed study of the transition maps $\mathcal{D}_\tau^\pm$, $\mathcal{G}_\tau$ and $\mathcal{F}_\tau$ can be found, giving the following structure of the h-component $\omega_\tau(\delta, w)$:

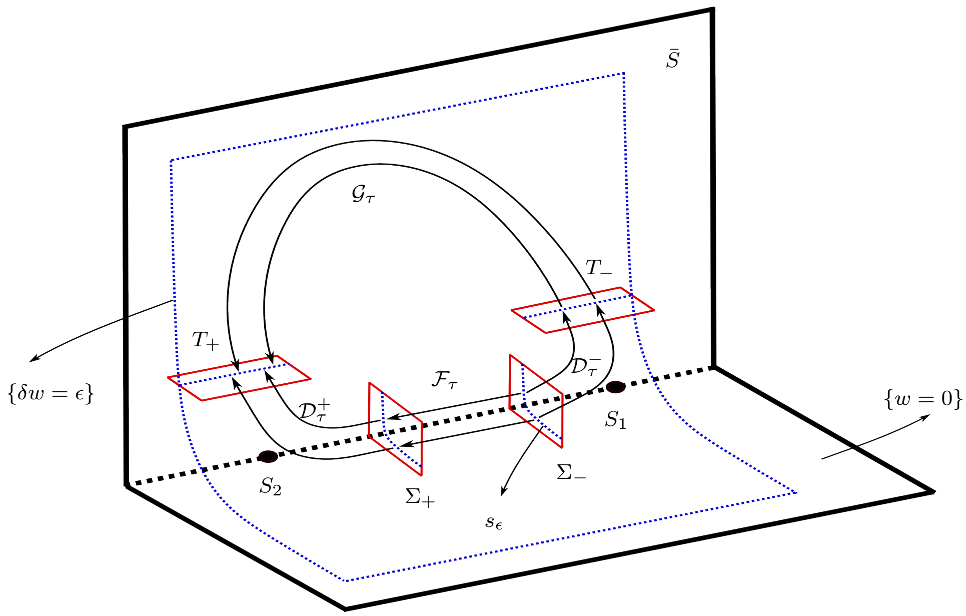

FIGURE 4. The maps $\mathcal{D}_\tau^\pm, \mathcal{F}_\tau, \mathcal{G}_\tau$.

Theorem 3.3.

$$\omega_\tau(\delta, w) = B_0\kappa_0(\tau,\epsilon) + \epsilon B_2\kappa_1(\tau,\epsilon) + \epsilon^3 u\kappa_3(\tau,\epsilon) \\ + \exp -\frac{1}{w^2}\Big(\widetilde{A}^+ + B_2\delta\widetilde{\Phi}_1^+ + u\delta^3\widetilde{\Phi}_3^+\Big) - \exp -\frac{1}{w^2}\Big(\widetilde{A}^- + B_2\delta\widetilde{\Phi}_1^- + u\delta^3\widetilde{\Phi}_3^-\Big)$$

where $\widetilde{A}^\pm(\delta, w, w^2\ln w, \tau)$ *and* $\widetilde{\Phi}_k^\pm(\delta, w, w^2\ln w, \tau)$ *are smooth functions in variable* $(\delta, w, w^2\ln w, \tau)$, κ_j *is a smooth function in variable* (τ, ϵ) *and* κ_0 *is strictly positive for* $(\epsilon, B_0) \sim (0,0)$. *Moreover, we have*

$$\widetilde{A}^\pm(0,0,0,\tau) = 1,\ \widetilde{\Phi}_1^\pm(0,0,0,\tau) = \pm 2\sqrt{2}/3,\ \widetilde{\Phi}_3^\pm(0,0,0,\tau) = (\pm 4\sqrt{2}/5)\bar{H}(0,\lambda)$$

and

$$\widetilde{A}^-(\delta, w, w^2\ln w, \tau) = \widetilde{A}^+(\delta, w, w^2\ln w, \tau) - 2\sqrt{2}B_0 w(1 + O(\delta^2, w)).$$

Proof. See [15], Theorem 4.20. □

3.3.1. *Lie-derivative.* For each fixed value of (ϵ, τ), with ϵ small and positive, the function ω_τ can be considered as 1-variable function defined on a segment $s_\epsilon = \{(\delta, w); \delta w = \epsilon, (\delta, w) \in B(d), \delta \geq 0, w \geq 0\}$. In order to be able to study the zeros of ω_τ on s_ϵ, we will consider its Lie-derivative $\mathcal{L}_\mathcal{Y}\omega_\tau = \delta\frac{\partial\omega_\tau}{\partial\delta} - w\frac{\partial\omega_\tau}{\partial w}$ along the vector field $\mathcal{Y} = \delta\frac{\partial}{\partial\delta} - w\frac{\partial}{\partial w}$. As the vector field $\mathcal{Y}$ is tangent to each segment s_ϵ and has no zero on it, Rolle's theorem will permit to find the cyclicity of ω_τ from the cyclicity of $\mathcal{L}_\mathcal{Y}\omega_\tau$. The reason to introduce this Lie-derivative is that the equation $\{\mathcal{L}_\mathcal{Y}\omega_\tau(\delta, w) = 0\}$ can be reduced to a simpler form than the equation $\{\omega_\tau(\delta, w) = 0\}$ which contains exponential terms.

In the remainder of this section we investigate $\mathcal{L}_\mathcal{Y}\omega_\tau$. As the functions $\widetilde{A}^\pm$ and $\widetilde{\Phi}_k^\pm$ given in Theorem 3.3 are smooth in $(\delta, w, w^2\ln w, \tau, \epsilon)$, we first give some properties of the Lie-derivation for functions which are smooth in $(\delta, w, w^2\ln w, \tau, \epsilon)$.

Lemma 3.4. *If $n, m \in \mathbb{Z}$, then $\mathcal{L}_{\mathcal{Y}}(\delta^n w^m) = (n-m)\delta^n w^m$. If $F(\delta, w, \tau, \epsilon) = f(\delta, w, w^2 \ln w, \tau, \epsilon)$ with f smooth, then $G(\delta, w, \tau, \epsilon) = \mathcal{L}_{\mathcal{Y}} F(\delta, w, \tau, \epsilon)$ is also smooth in $(\delta, w, w^2 \ln w, \tau, \epsilon)$ and $G(0, 0, \tau, \epsilon) \equiv 0$ (one can also write $G = o(1)$).*

Proof. See [8], Lemma 5.9. □

We have an easy consequence of Lemma 3.4 that will be useful later:

Lemma 3.5. *Suppose that $n, m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$.*
1. If $n \neq m$ and $F(\delta, w, \tau, \epsilon) = \delta^n w^m(\alpha + o(1))$ where $o(1)$ is smooth in variable $(\delta, w, w^2 \ln w, \tau, \epsilon)$ and $o(1) \equiv 0$ for $(\delta, w) = (0, 0)$, then

$$\mathcal{L}_{\mathcal{Y}} F(\delta, w, \tau, \epsilon) = (n-m)\delta^n w^m(\alpha + o(1)),$$

where the symbol $o(1)$ is also for a function smooth in $(\delta, w, w^2 \ln w, \tau, \epsilon)$ and $o(1) \equiv 0$ for $(\delta, w) = (0, 0)$.
2. If $n = m$, we have that

$$\mathcal{L}_{\mathcal{Y}}(\delta^n w^m(\alpha + o(1))) = o(\delta^n w^m).$$

Proof. As a consequence of Lemma 3.4, we have that

$$\begin{aligned}\mathcal{L}_{\mathcal{Y}}(\delta^n w^m(\alpha + o(1))) &= (\alpha + o(1))\mathcal{L}_{\mathcal{Y}}(\delta^n w^m) + \delta^n w^m \mathcal{L}_{\mathcal{Y}}(\alpha + o(1)) \\ &= (n-m)\delta^n w^m(\alpha + o(1)) + \delta^n w^m o(1).\end{aligned}$$

□

As a consequence of Lemma 3.4 ($\epsilon = \delta w$), we have that

$$\mathcal{L}_{\mathcal{Y}}(B_0\kappa_0(\tau, \epsilon) + \epsilon B_2 \kappa_1(\tau, \epsilon) + \epsilon^3 u \kappa_3(\tau, \epsilon)) = 0.$$

This means that it suffices to study the Lie-derivation of the exponential terms in the expression for $\omega_\tau(\delta, w)$. If we write $T^\pm(\delta, w, \tau, \epsilon) = \widetilde{A}^\pm + B_2\delta\widetilde{\Phi}_1^\pm + u\delta^3\widetilde{\Phi}_3^\pm$, then we obtain

$$\mathcal{L}_{\mathcal{Y}}\omega_\tau = -\mathcal{L}_{\mathcal{Y}}(\frac{1}{w^2}T^+) \exp -\frac{1}{w^2}T^+ + \mathcal{L}_{\mathcal{Y}}(\frac{1}{w^2}T^-) \exp -\frac{1}{w^2}T^-.$$

In order to push the terms $\mathcal{L}_{\mathcal{Y}}(\frac{1}{w^2}T^\pm)$ in the corresponding exponential terms, we have to study $\mathcal{L}_{\mathcal{Y}}(\frac{1}{w^2}T^\pm)$ carefully. Using Lemma 3.4 we obtain that

$$\begin{aligned}\mathcal{L}_{\mathcal{Y}}(\frac{1}{w^2}T^\pm) &= T^\pm \mathcal{L}_{\mathcal{Y}}(\frac{1}{w^2}) + \frac{1}{w^2}\mathcal{L}_{\mathcal{Y}}(T^\pm) \\ &= 2\frac{1}{w^2}T^\pm + \frac{1}{w^2}\mathcal{L}_{\mathcal{Y}}(T^\pm) = \frac{1}{w^2}\Big(2T^\pm + \mathcal{L}_{\mathcal{Y}}(T^\pm)\Big).\end{aligned}$$

Let us remind that $T^\pm$ is smooth in $(\delta, w, w^2 \ln w, \tau, \epsilon)$. Then, by Lemma 3.4, we get

$$2T^\pm(0, 0, \tau, \epsilon) + \mathcal{L}_{\mathcal{Y}}(T^\pm)(0, 0, \tau, \epsilon) = 2\widetilde{A}^\pm(0, 0, 0, \tau) + 0 = 2 > 0.$$

It follows that the function $P^\pm(\delta, w, \tau, \epsilon) := 2T^\pm(\delta, w, \tau, \epsilon) + \mathcal{L}_Y(T^\pm)(\delta, w, \tau, \epsilon)$ is strictly positive by taking a sufficiently small ball $B(d)$ at the origin $(\delta, w) = (0, 0)$. Since $P^\pm$ is smooth in $(\delta, w, w^2 \ln w, \tau, \epsilon)$, its logarithm is also a smooth function in $(\delta, w, w^2 \ln w, \tau, \epsilon)$. So, we can write

$$\begin{aligned}\mathcal{L}_{\mathcal{Y}}\omega_\tau &= -\frac{1}{w^2}P^+ \exp -\frac{1}{w^2}T^+ + \frac{1}{w^2}P^- \exp -\frac{1}{w^2}T^- \\ &= -\frac{1}{w^2}\Big(\exp -\frac{1}{w^2}(T^+ - w^2 \ln P^+) - \exp -\frac{1}{w^2}(T^- - w^2 \ln P^-)\Big). \qquad (18)\end{aligned}$$

The expression (18) implies that the equation $\{\mathcal{L}_{\mathcal{Y}}\omega_\tau = 0\}$ is equivalent to

$$T^+ - T^- + w^2 \ln \frac{P^-}{P^+} = 0, \tag{19}$$

for $w > 0$. Using the properties of $\widetilde{A}^\pm, \widetilde{\Phi}_1^\pm$ and $\widetilde{\Phi}_3^\pm$ given in Theorem 3.3 we have that

$$\begin{aligned} T^+ - T^- &= \widetilde{A}^+ - \widetilde{A}^- + B_2\delta(\widetilde{\Phi}_1^+ - \widetilde{\Phi}_1^-) + u\delta^3(\widetilde{\Phi}_3^+ - \widetilde{\Phi}_3^-) \\ &= 2\sqrt{2}B_0 w(1 + O(\delta^2, w)) + B_2\delta\frac{4\sqrt{2}}{3}(1 + o_1(1)) + u\delta^3\frac{8\sqrt{2}}{5}(\bar{H}(0,\lambda) + o_3(1)), \end{aligned} \tag{20}$$

where $O(\delta^2, w), o_1(1)$ and $o_3(1)$ are smooth in $(\delta, w, w^2 \ln w, \tau)$.

We look now at the remainder $w^2 \ln \frac{P^-}{P^+}$. We notice that

$$P^- - P^+ = 2(T^- - T^+) + \mathcal{L}_{\mathcal{Y}}(T^- - T^+).$$

Keeping in mind Lemma 3.5 and (20) we obtain that

$$\begin{aligned} \mathcal{L}_{\mathcal{Y}}(T^- - T^+) &= -2\sqrt{2}B_0\mathcal{L}_{\mathcal{Y}}(w(1 + O(\delta^2, w))) \\ &\quad - B_2\frac{4\sqrt{2}}{3}\mathcal{L}_{\mathcal{Y}}(\delta(1 + o_1(1))) - u\frac{8\sqrt{2}}{5}\mathcal{L}_{\mathcal{Y}}(\delta^3(\bar{H}(0,\lambda) + o_3(1))) \\ &= 2\sqrt{2}B_0 w(1 + O(\delta^2, w)) - B_2\frac{4\sqrt{2}}{3}\delta(1 + o_1(1)) - u\frac{24\sqrt{2}}{5}\delta^3(\bar{H}(0,\lambda) + o_3(1)), \end{aligned}$$

for some new functions $O(\delta^2, w), o_1(1), o_3(1)$ smooth in $(\delta, w, w^2 \ln w, \tau)$. We hence get

$$\begin{aligned} P^- &= P^+ + 2(T^- - T^+) + \mathcal{L}_Y(T^- - T^+) \\ &= P^+ - 2\sqrt{2}B_0 w(1 + O(\delta^2, w)) - B_2 4\sqrt{2}\delta(1 + o_1(1)) - u8\sqrt{2}\delta^3(\bar{H}(0,\lambda) + o_3(1)), \end{aligned}$$

for some new functions $O(\delta^2, w), o_1(1), o_3(1)$.

From the above expression and the fact that $P^-(0,0,\tau,\epsilon) = 2 > 0$ it follows that

$$\frac{P^-}{P^+} = 1 - \sqrt{2}B_0 w(1 + o(1)) - B_2 2\sqrt{2}\delta(1 + o_1(1)) - u4\sqrt{2}\delta^3(\bar{H}(0,\lambda) + o_3(1)),$$

for some functions $o(1), o_1(1), o_3(1)$ smooth in $(\delta, w, w^2 \ln w, \tau)$. Taking the logarithm of this expression we finally get

$$w^2 \ln \frac{P^-}{P^+} = B_0 w O(w^2) + B_2\delta O_1(w^2) + u\delta^3 O_3(w^2), \tag{21}$$

where $O(w^2), O_1(w^2)$ and $O_3(w^2)$ are smooth in $(\delta, w, w^2 \ln w, \tau)$.

If we use now the expressions (20) and (21) we obtain that

$$T^+ - T^- + w^2 \ln \tfrac{P^-}{P^+} =$$
$$2\sqrt{2}B_0 w(1 + O(\delta^2, w)) + B_2\delta\tfrac{4\sqrt{2}}{3}(1 + o_1(1)) + u\delta^3\tfrac{8\sqrt{2}}{5}(\bar{H}(0,\lambda) + o_3(1)),$$

where $O(\delta, w^2), o_1(1)$ and $o_3(1)$ are smooth in $(\delta, w, w^2 \ln w, \tau)$. Then, finally, the equation $\{\mathcal{L}_{\mathcal{Y}} w_\tau = 0\}$ is equivalent, for $w > 0$, to:

$$\begin{aligned} 2\sqrt{2}B_0 w(1 + O(\delta^2, w)) &+ B_2\delta\frac{4\sqrt{2}}{3}(1 + o_1(1)) \\ &+ u\delta^3\frac{8\sqrt{2}}{5}(\bar{H}(0,\lambda) + o_3(1)) = 0. \end{aligned} \tag{22}$$

3.3.2. *Upper bound for cyclicity of* $\mathbf{L_0}$. We use an algorithm of derivation-division and in (22) we introduce $(\bar{B}_0', \bar{B}_2, \bar{u})$, with $(\bar{B}_0', \bar{B}_2, \bar{u}) \in \mathbb{S}^2$ and

$$(B_0, B_2, u) = \rho(\bar{B}_0', \bar{B}_2, \bar{u})$$

where $\rho \geq 0$, $\rho \sim 0$. For $\rho > 0$, using this rescaling we can change (22) into

$$\bar{Q}(\delta, w, \rho, \bar{B}_0', \bar{B}_2, \bar{u}, B_3, \lambda) := 2\sqrt{2}\bar{B}_0' w(1 + O(\delta^2, w)) + \bar{B}_2\delta\frac{4\sqrt{2}}{3}(1 + o_1(1)) + \bar{u}\delta^3\frac{8\sqrt{2}}{5}(\bar{H}(0, \lambda) + o_3(1)) = 0. \tag{23}$$

If $\rho = 0$, then the left-hand side of (22) is identical to zero, corresponding to a system representing a center.

Instead of using coordinates on the sphere, it is customary to use different charts of the sphere.

(1) Case $\bar{u} = \pm 1, (\bar{B}_0', \bar{B}_2) \in K$, K is any compact subset of $\mathbb{R}^2$. We begin the derivation-division algorithm by dividing (23) by

$$T_1 = 2\sqrt{2}w(1 + O(\delta^2, w)),$$

the factor of the parameter $\bar{B}_0'$ in (23), which is strictly positive on segments s_ϵ, for each $\epsilon \sim 0$ and $\epsilon > 0$. Using Lemma 3.5 we obtain that

$$\mathcal{L}_\mathcal{Y}(\frac{\bar{Q}}{T_1}) = \bar{B}_2\delta w^{-1}k_1(1 + o_1(1)) + \bar{u}\delta^3 w^{-1}k_3(\bar{H}(0, \lambda) + o_3(1)), \tag{24}$$

for some positive constant k_1 and k_3 and some new $o_1(1)$ and $o_3(1)$. Hence, we have eliminated the parameter $\bar{B}_0'$. Let us divide now (24) by the positive factor $T_2 = \delta w^{-1}k_1(1 + o_1(1))$ of $\bar{B}_2$ in (24). Then, by Lemma 3.5,

$$\mathcal{L}_\mathcal{Y}(\frac{\mathcal{L}_\mathcal{Y}(\frac{\bar{Q}}{T_1})}{T_2}) = \bar{u}\delta^2\bar{k}_3(\bar{H}(0, \lambda) + o_3(1)), \tag{25}$$

where $\bar{k}_3$ is a positive constant and $o_3(1)$ is smooth in $(\delta, w, w^2 \ln w, \tau)$.

As $\bar{u} = \pm 1$ and $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$, the expression (25) is nonzero, for $(\delta, w) \sim (0, 0)$, $(\delta, w) > (0, 0)$, $\rho \sim 0$, $(\bar{B}_0', \bar{B}_2) \in K$, $B_3 \in \mathcal{B}$ and $\lambda \in \Lambda$. Now Rolle's Theorem implies that $\bar{Q}$ has at most 2 roots (counting multiplicity) on the segments s_ϵ under the given conditions on the parameters.

Remark 3. Since $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$, in each step of the above algorithm of derivation-division we deal with factors $\delta^n w^m(\alpha + o(1))$, where $\alpha \neq 0$ and $n, m \in \mathbb{Z}$, $n \neq m$. This enables us to have a well-defined algorithm in each step.

(2) Case $\bar{B}_0' = \pm 1$, $(\bar{B}_2, \bar{u}) \in K$, K is any compact set, and (3) Case $\bar{B}_2 = \pm 1$, $(\bar{B}_0', \bar{u}) \in K$, K is any compact set. The study of the case (2) and the case (3) is analogous to the study of the case (1); we can prove that $\bar{Q}$ has at most 2 roots (counting multiplicity) on the segments s_ϵ.

As the equation (22) is equivalent to $\{\mathcal{L}_\mathcal{Y}\omega_\tau = 0\}$ for $w > 0$, the Rolle's theorem implies that the cyclicity of ω_τ is bounded by 3 under the given condition on the function $\bar{H}$. Hence we arrive at the statement of Theorem 3.2.

4. **Limit cycles near** $\cup_{\mathbf{h} \in [\mathbf{h_0}, \mathbf{1}]}\mathbf{L_h}$**.** In this section we consider the blown-up vector field $\bar{Z}_\tau$, in the family directional chart, given by the expression (13):

$$Z_{\epsilon,\tau}^{(1)} : \begin{cases} \dot{\widetilde{x}} = \widetilde{y} - \frac{1}{2}\widetilde{x}^2 \\ \dot{\widetilde{y}} = B_0 - \widetilde{x} + B_2\epsilon\widetilde{x}^2 + B_3\epsilon^2\widetilde{x}^3 + u\epsilon^3\widetilde{x}^4\bar{H}(u\epsilon\widetilde{x}, \lambda) \\ \qquad +u\epsilon^3(\widetilde{y} - \frac{1}{2}\widetilde{x}^2)^2 G(u\epsilon\widetilde{x}, u^2\epsilon^2(\widetilde{y} - \frac{1}{2}\widetilde{x}^2), \lambda). \end{cases}$$

In Section 3.1.1, we defined a regular limit periodic set L_h, where $h \in]0,1]$. We take any value $h_0 \in]0,1[$ and consider the set $\cup_{h\in[h_0,1]}L_h$. In the remainder of this section we prove:

(1) If $\bar{H}(0,\lambda) \neq 0$ for all $\lambda \in \Lambda$, then $Z^{(1)}_{\epsilon,\tau}$ has at most two limit cycles near $\cup_{h\in[h_0,1]}L_h$ for each $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $B_3 \in \mathcal{B}$, $u \sim 0$ and $\lambda \in \Lambda$.

Our proof of (1) will be based on [8] and [14]. First we define a return map near $\cup_{h\in[h_0,1]}L_h$ as follows. We denote by $p_{\epsilon,\tau} = (\widetilde{x}_{\epsilon,\tau}, \widetilde{y}_{\epsilon,\tau}) \sim (0,0)$ the elliptic singular point of $Z^{(1)}_{\epsilon,\tau}$, for $\epsilon \sim 0$ and $B_0 \sim 0$. Let us choose $\widetilde{y}_1$ and $\widetilde{y}_2$ such that $0 < \widetilde{y}_1 < \widetilde{y}_2$ and a smooth function $i(\widetilde{y})$ on $\mathbb{R}$ such that $i(\widetilde{y}) \equiv 1$ for $\widetilde{y} \leq \widetilde{y}_1$ and $i(\widetilde{y}) \equiv 0$ for $\widetilde{y} \geq \widetilde{y}_2$.

We consider a smooth τ-family of two-dimensional sections $\mathcal{S}_\tau$, going from the segment $\{\epsilon \to p_{\epsilon,\tau}\}$ (in the family chart of the secondary blow-up) to the segment $\{v = 0, w = 0, \delta \geq 0\}$(in the phase-directional chart of the secondary blow-up). We refer to Figure 5. In the family chart, we suppose that $\mathcal{S}_\tau$ is equal to the section $\{\widetilde{x} = \widetilde{x}_{\epsilon,\tau}.i(\widetilde{y})\}$, parameterized by $\epsilon \sim 0$ and $\widetilde{y} \geq \widetilde{y}_{\epsilon,\tau}$. In the phase-directional chart, we suppose that $\mathcal{S}_\tau$ is equal to $\Sigma_B \subset \{v = 0\}$ where Σ_B is parametrized by $(\delta, w) \in [0,\delta_0] \times [0, w_0]$, for sufficiently small and strictly positive numbers δ_0 and w_0. For ϵ and B_0 small enough, this section $\mathcal{S}_\tau$ is transverse to $\bar{Z}_\tau$ and the return map of $-\bar{Z}_\tau$ is defined on it.

Now we can define an arbitrarily large subsection $\mathcal{S}_{\tau,h_0} \subset \mathcal{S}_\tau$ in the following way. We choose any value $h_0 \in]0,1[$, small enough, and define $\mathcal{S}_{\tau,h_0} = \{(\widetilde{x},\widetilde{y},\epsilon) \mid \epsilon \in [0,\epsilon_0],\ \widetilde{x} = \widetilde{x}_{\epsilon,\tau}.i(\widetilde{y}),\ \widetilde{y} \in [\widetilde{y}_{\epsilon,\tau}, \widetilde{y}_{h_0}]\}$ where $\widetilde{y}_{h_0}$ is the positive solution of the equation $e^{-\widetilde{y}}(\widetilde{y}+1) = h_0$ ($i(\widetilde{y}_{h_0}) = 0$, for h_0 small enough).

We can parametrize section $\mathcal{S}_{\tau,h_0}$ by $(h = e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}(\widetilde{x}_{\epsilon,\tau} i(\widetilde{y}))^2 + 1), \epsilon)$. Let us write $h^0_{\epsilon,\tau} = H(p_{\epsilon,\tau})$. Clearly we have $h^0_{\epsilon,\tau}|_{B_0=0} = 1$. Then $\mathcal{S}_{\tau,h_0} = \{(h,\epsilon) \mid h \in [h_0, h^0_{\epsilon,\tau}], \epsilon \in [0,\epsilon_0]\}$. If ϵ_0 is small enough, a return map $(P(h,\epsilon,\tau),\epsilon)$ of $-Z^{(1)}_{\epsilon,\tau}$ is defined from $\mathcal{S}_{\tau,h_0}$ to $\mathcal{S}_\tau$ where $\mathcal{S}_\tau$ is parametrized by (h,ϵ), in the family chart.

Let L_h be oriented counter-clockwise.

Proposition 4.1. *There exist smooth functions $\widetilde{L}_k(h,\epsilon,\tau)$, for $k = 0,1,3$, such that*

$$P(h,\epsilon,\tau) = h + B_0\widetilde{L}_0(h,\epsilon,\tau) + B_2\epsilon\widetilde{L}_1(h,\epsilon,\tau) + u\epsilon^3\widetilde{L}_3(h,\epsilon,\tau),$$

where $\widetilde{L}_0(h,0,0,B_2,B_3,u,\lambda) = -\int_{L_h} e^{-\widetilde{y}} d\widetilde{x} =: -J_0(h)$, $\widetilde{L}_1(h,0,0,B_2,B_3,u,\lambda) = -\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^2 d\widetilde{x} =: -J_1(h)$ *and* $\widetilde{L}_3(h,0,0,B_2,B_3,u,\lambda) = -\bar{H}(0,\lambda)\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^4 d\widetilde{x} - G(0,0,\lambda)\int_{L_h} e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}\widetilde{x}^2)^2 d\widetilde{x} =: -\mathcal{J}_2(h)$.

Proof. See [8], Proposition 6.1. □

Let $D_{\epsilon,\tau}(h) = P(h,\epsilon,\tau) - h$ be the displacement function. Fixed points of $h \to P(h,\epsilon,\tau)$, which are the roots of $\{D_{\epsilon,\tau} = 0\}$, correspond to the intersections of periodic orbits of $Z^{(1)}_{\epsilon,\tau}$ with the ϵ-level of S_{τ,h_0}.

We want to study the derivative $\frac{\partial}{\partial h}D_{\epsilon,\tau}$. We consider the derivative of $D_{\epsilon,\tau}$ and not directly the function itself, because this will permit to use results in [8] and [14]. We obtain

$$\frac{\partial}{\partial h}D_{\epsilon,\tau}(h) = B_0\bar{L}_0(h,\epsilon,\tau) + B_2\epsilon\bar{L}_1(h,\epsilon,\tau) + u\epsilon^3\bar{L}_3(h,\epsilon,\tau)$$

where $\bar{L}_k(h,\epsilon,\tau) = \frac{\partial}{\partial h}\widetilde{L}_k(h,\epsilon,\tau)$.

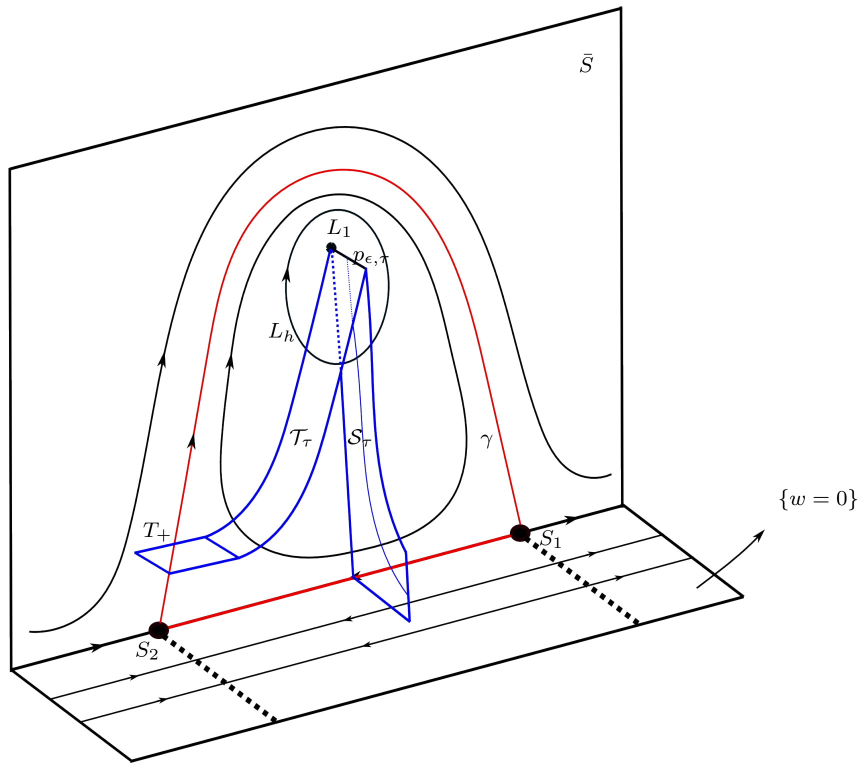

FIGURE 5. Sections $\mathcal{S}_\tau$ and $\mathcal{T}_\tau$ for $B_0 = 0$.

Let us now recall the notion of a strict Chebyshev system of C^∞-functions as it was introduced in [8]. We restrict to degree 1 and 2 pointing out that strict Chebyshev systems of degree 1 will be used in Section 6.

Definition 4.2. Let $\mathcal{F} = \{f_0, f_1\}$ (resp. $\mathcal{F} = \{f_0, f_1, f_2\}$) be a sequence of smooth functions defined on an interval $[a, b] \subset \mathbb{R}$. One says that $\mathcal{F}$ is a strict Chebyshev system on $[a, b]$ (in short ST-system) of degree 1 (resp. degree 2) if one has $f_0(z) \neq 0$ and $(\frac{f_1}{f_0})'(z) \neq 0$ (resp. $f_0(z) \neq 0$, $(\frac{f_1}{f_0})'(z) \neq 0$ and $\Big(\frac{(\frac{f_2}{f_0})'}{(\frac{f_1}{f_0})'}\Big)'(z) \neq 0$) for all $z \in [a, b]$.

Remark 4. We can see by applying the algorithm of derivation-division that, if $\mathcal{F} = \{f_0, f_1\}$ (resp. $\mathcal{F} = \{f_0, f_1, f_2\}$) is a ST-system on $[a, b]$, then any nontrivial linear combination $\alpha_0 f_0 + \alpha_1 f_1$ (resp. $\alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2$) has at most one zero (resp. two zeros) on $[a, b]$, counted with their multiplicity.

When $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$, then we will show that on the interval $[h_0, h^0_{\epsilon,\tau}]$ the system $\{\bar{L}_0, \bar{L}_1, \bar{L}_3\}$ is a ST-system of degree two, for each $\epsilon \sim 0$, $B_0 \sim 0$, $B_2 \sim 0$, $B_3 \in \mathcal{B}$, $u \sim 0$ and $\lambda \in \Lambda$. Remark 4 and Rolle's Theorem will then imply that the function $D_{\epsilon,\tau}$ has at most three zeros (counting multiplicity) on $[h_0, h^0_{\epsilon,\tau}]$, for $\epsilon > 0$ and $(B_0, B_2, u) \neq (0, 0, 0)$. But one zero corresponds to the focus/center point of our system ($D_{\epsilon,\tau}(h^0_{\epsilon,\tau}) = 0$). As a consequence, the function $D_{\epsilon,\tau}$ has at most two zeros (counting multiplicity) on $[h_0, h^0_{\epsilon,\tau}[$, for $\epsilon > 0$ and $(B_0, B_2, u) \neq (0, 0, 0)$. Hence, (1) is proven.

To prove that $\{\bar{L}_0, \bar{L}_1, \bar{L}_3\}$ is a ST-system, we have to consider

$$\{\bar{J}_0, \bar{J}_1, \bar{\mathcal{J}}_2\} := \{\tfrac{\partial}{\partial h} J_0, \tfrac{\partial}{\partial h} J_1, \tfrac{\partial}{\partial h} \mathcal{J}_2\}$$

on $[h_0, 1]$. First, we show that $\{\bar{J}_0, \bar{J}_1, \bar{\mathcal{J}}_2\}$ is a ST-system on $[h_0, 1]$ for each $\lambda \in \Lambda$, provided $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. We rely on [14]. In [14] the following conjecture, formulated by Dumortier and Roussarie in [8], has been solved for cases up to codimension 2 (for a definition of a ST-system of degree q we refer to [8]):

Conjecture. For each $j \geq 0$, we define an analytic function $I_j(h) = \int_{\bar{\gamma}_h} \widetilde{x}^{2j-1} d\widetilde{y}$, where $\bar{\gamma}_h \subset \{\mathcal{H}(\widetilde{x}, \widetilde{y}) := e^{-2\widetilde{y}}(\widetilde{y} - \widetilde{x}^2 + \frac{1}{2}) = h\}$ and $h \in]0, \frac{1}{2}]$. For each $q \geq 0$, the system $\{I_0, I_1, ..., I_q\}$ is a ST-system on $[h_1, \frac{1}{2}]$, for each $h_1 \in]0, \frac{1}{2}[$.

$\bar{\gamma}_h$ is a closed curve oriented counter-clockwise. In [14], it has been shown that the system $\{I_0, I_1, I_2\}$ is a ST-system on $[h_1, \frac{1}{2}]$, for each $h_1 \in]0, \frac{1}{2}[$. The following lemma gives the relation between the systems $\{\bar{J}_0, \bar{J}_1, \bar{\mathcal{J}}_2\}$ and $\{I_0, I_1, I_2\}$:

Lemma 4.3. *Given any $h_0 \in]0, 1[$. Then for all $h \in [h_0, 1]$ we have:*
i) $\bar{J}_0(h) = -I_0(\frac{h}{2})$;
ii) $\bar{J}_1(h) = -4I_1(\frac{h}{2})$;
iii) $\bar{\mathcal{J}}_2(h) = -16\bar{H}(0, \lambda)I_2(\frac{h}{2}) - 4G(0, 0, \lambda)I_1(\frac{h}{2})$.

Proof. First, we prove

$$\bar{J}_j(h) := \frac{d}{dh}\left(\int_{L_h} e^{-\widetilde{y}} \widetilde{x}^{2j} d\widetilde{x}\right) = -\int_{L_h} \widetilde{x}^{2j-1} d\widetilde{y}, \tag{26}$$

where $h \in]0, 1]$ and $j \geq 0$. Let us recall that $H(\widetilde{x}, \widetilde{y}) = e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}\widetilde{x}^2 + 1)$. We write $\omega_j = e^{-\widetilde{y}} \widetilde{x}^{2j} d\widetilde{x}$ and, by differentiation, we obtain that:

$$d\omega_j = e^{-\widetilde{y}} \widetilde{x}^{2j} d\widetilde{x} \wedge d\widetilde{y}.$$

If we define $\frac{d\omega_j}{dH} := -\widetilde{x}^{2j-1} d\widetilde{y}$, then it is easy to see that:

$$dH \wedge \tfrac{d\omega_j}{dH} = d\omega_j.$$

Then the formula in (26) follows from the formula of Picard-Lefchetz (see for example [1]): $\frac{d}{dh}\left(\int_{L_h} \omega_j\right) = \int_{L_h} \frac{d\omega_j}{dH}$. Taking into account (26) and using the change of variables $\{\widetilde{x} = 2\widetilde{X}, \widetilde{y} = 2\widetilde{Y}\}$, we obtain that:

$$\bar{J}_j(h) = -\int_{L_h} \widetilde{x}^{2j-1} d\widetilde{y} = -2^{2j} \int_{\bar{\gamma}_{\frac{h}{2}}} \widetilde{X}^{2j-1} d\widetilde{Y} = -2^{2j} I_j(\frac{h}{2}). \tag{27}$$

The expression (27) implies i) and ii).

It remains to prove iii). First, we show that

$$\int_{L_h} e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}\widetilde{x}^2)^2 d\widetilde{x} = \int_{L_h} e^{-\widetilde{y}} \widetilde{x}^2 d\widetilde{x} \tag{28}$$

for $h \in]0, 1]$. If $(\widetilde{x}, \widetilde{y}) \in L_h$, then $d\widetilde{x} = \frac{\frac{1}{2}\widetilde{x}^2 - \widetilde{y}}{\widetilde{x}} d\widetilde{y}$, and as a consequence, we obtain

$$\begin{aligned} &\int_{L_h} e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}\widetilde{x}^2)^2 d\widetilde{x} = \int_{L_h} e^{-\widetilde{y}}(\frac{1}{2}\widetilde{x}^2 - \widetilde{y})^3 \widetilde{x}^{-1} d\widetilde{y} \\ &= \frac{1}{8}\int_{L_h} e^{-\widetilde{y}} \widetilde{x}^5 d\widetilde{y} - \frac{3}{4}\int_{L_h} e^{-\widetilde{y}} \widetilde{y}\widetilde{x}^3 d\widetilde{y} + \frac{3}{2}\int_{L_h} e^{-\widetilde{y}} \widetilde{y}^2 \widetilde{x} d\widetilde{y} - \int_{L_h} e^{-\widetilde{y}} \widetilde{y}^3 \widetilde{x}^{-1} d\widetilde{y} \end{aligned} \tag{29}$$

and

$$\begin{aligned} d(f(\widetilde{y})\widetilde{x}^k) &= kf(\widetilde{y})\widetilde{x}^{k-1}d\widetilde{x} + f'(\widetilde{y})\widetilde{x}^k d\widetilde{y} \\ &= (\frac{k}{2}f(\widetilde{y}) + f'(\widetilde{y}))\widetilde{x}^k d\widetilde{y} - kf(\widetilde{y})\widetilde{y}\widetilde{x}^{k-2}d\widetilde{y}, \end{aligned} \tag{30}$$

where $k \geq 1$ and f is an analytic function. If we take $f(\widetilde{y}) = e^{-\widetilde{y}}\widetilde{y}^{m-1}$, $m \geq 1$, in (30) and if we use the fact that $\int_{L_h} d(f(\widetilde{y})\widetilde{x}^k) = 0$, then we find that:

$$\int_{L_h} e^{-\widetilde{y}}\widetilde{y}^m\widetilde{x}^{k-2}d\widetilde{y} = \frac{k-2}{2k}\int_{L_h} e^{-\widetilde{y}}\widetilde{y}^{m-1}\widetilde{x}^k d\widetilde{y} + \frac{m-1}{k}\int_{L_h} e^{-\widetilde{y}}\widetilde{y}^{m-2}\widetilde{x}^k d\widetilde{y}. \tag{31}$$

On account of (31) the integrals in (29) can be written as:
1) $\int_{L_h} e^{-\widetilde{y}}\widetilde{y}\widetilde{x}^3 d\widetilde{y} = \frac{3}{10}\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^5 d\widetilde{y}$,
2) $\int_{L_h} e^{-\widetilde{y}}\widetilde{y}^2\widetilde{x} d\widetilde{y} = \frac{1}{20}\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^5 d\widetilde{y} + \frac{1}{3}\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^3 d\widetilde{y}$,
3) $\int_{L_h} e^{-\widetilde{y}}\widetilde{y}^3\widetilde{x}^{-1} d\widetilde{y} = -\frac{1}{40}\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^5 d\widetilde{y} + \frac{1}{6}\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^3 d\widetilde{y}$.
Hence, we have that

$$\int_{L_h} e^{-\widetilde{y}}(\widetilde{y} - \frac{1}{2}\widetilde{x}^2)^2 d\widetilde{x} = \frac{1}{3}\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^3 d\widetilde{y}.$$

From the first equality in (30) for $k = 3$ and $f(\widetilde{y}) = e^{-\widetilde{y}}$ we obtain

$$\tfrac{1}{3}\int_{L_h} e^{-\widetilde{y}}\widetilde{x}^3 d\widetilde{y} = \int_{L_h} e^{-\widetilde{y}}\widetilde{x}^2 d\widetilde{x}.$$

We have shown (28). The formulas (27) and (28) imply iii). □

Proposition 4.4. *Take any $h_0 \in]0,1[$. If $\bar{H}(0,\lambda) \neq 0$ for all $\lambda \in \Lambda$, then the system $\{\bar{J}_0, \bar{J}_1, \bar{\mathcal{J}}_2\}$ is a ST-system on $[h_0, 1]$, for each $\lambda \in \Lambda$.*

Proof. Take any $h_0 \in]0,1[$. We are going to use Lemma 4.3 and the fact that the system $\{I_0, I_1, I_2\}$ is a ST-system on $[\frac{h_0}{2}, \frac{1}{2}]$. It is clear that $\bar{J}_0(h) = -I_0(\frac{h}{2}) \neq 0$ for all $h \in [h_0, 1]$. We have that

$$(\tfrac{\bar{J}_1}{\bar{J}_0})'(h) = 2(\tfrac{I_1}{I_0})'(\tfrac{h}{2}) \neq 0$$

for all $h \in [h_0, 1]$. Similarly, we obtain

$$\frac{(\frac{\bar{\mathcal{J}}_2}{\bar{J}_0})'(h)}{(\frac{\bar{J}_1}{\bar{J}_0})'(h)} = 4\bar{H}(0,\lambda)\frac{(\frac{I_2}{I_0})'(\frac{h}{2})}{(\frac{I_1}{I_0})'(\frac{h}{2})} + G(0,0,\lambda).$$

This implies

$$\Big(\frac{(\frac{\bar{\mathcal{J}}_2}{\bar{J}_0})'}{(\frac{\bar{J}_1}{\bar{J}_0})'}\Big)'(h) = 2\bar{H}(0,\lambda)\Big(\frac{(\frac{I_2}{I_0})'}{(\frac{I_1}{I_0})'}\Big)'(\tfrac{h}{2}).$$

If $\bar{H}(0,\lambda) \neq 0$ for all $\lambda \in \Lambda$, then this expression is nonzero for all $h \in [h_0, 1]$ and all $\lambda \in \Lambda$. □

To end this, we need the following stability property of *ST*-systems (see [8], Proposition 7.6.):

Proposition 4.5. *Let $\mathcal{F} = \{f_0, f_1\}$ (resp. $\mathcal{F} = \{f_0, f_1, f_2\}$) be a ST-system on $[a,b]$ of order 1 (resp. 2). Let $\mathcal{G} = \{g_0, g_1\}$ (resp. $\mathcal{G} = \{g_0, g_1, g_2\}$) be a second system of smooth functions. Then, if each g_i is sufficiently near f_i, for $i = 0, 1$ in the C^1-norm (resp. for $i = 0, 1, 2$ in the C^2-norm), the system $\mathcal{G}$ is also a ST-system on any interval $[a', b']$ for $a' \sim a$ and $b' \sim b$.*

If we suppose that $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\Lambda$, then Propositions 4.4 and 4.5 imply that on the interval $[h_0,h^0_{\epsilon,\tau}]$ the system $\{\bar{L}_0,\bar{L}_1,\bar{L}_3\}$ is a ST-system, for each $\epsilon\sim 0$, $B_0\sim 0$, $B_2\sim 0$, $B_3\in\mathcal{B}$, $u\sim 0$ and $\lambda\in\Lambda$.

5. **Limit cycles near $\cup_{\mathbf{h}\in[\mathbf{0},\mathbf{1}]}\mathbf{L_h}$.** In this section we consider system $Z_{\epsilon,\tau}$ near $(\bar{x},\bar{y})=(0,0)$, defined in (10), and show:
(1) If $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\Lambda$, then there exists a neighborhood V of $(\bar{x},\bar{y})=(0,0)$, independent of (ϵ,τ), such that $Z_{\epsilon,\tau}$ has at most two limit cycles in V for each $\epsilon\sim 0$, $\epsilon>0$, $B_0\sim 0$, $B_2\sim 0$, $B_3\in\mathcal{B}$, $u\sim 0$ and $\lambda\in\Lambda$.

We prove the proposition (1) by gluing results obtained in Sections 3 and 4 together. We merely consider "derivative of a global difference map", defined near the set $\cup_{h\in[0,1]}L_h$ on the secondary blow-up locus, that permits to use good Chebyshev properties of the derivative $\mathcal{L}_{\mathcal{Y}}\omega_\tau$ and the derivative $\frac{\partial}{\partial h}D_{\epsilon,\tau}(h)$ obtained respectively in Section 3 and Section 4. For some reasons that become clear in later sections we want the above mentioned global difference map to be a function of $(\bar{y},\epsilon,\tau)$ where $\bar{y}=\epsilon^2\widetilde{y}$.

We consider a smooth τ-family of two-dimensional sections $\mathcal{T}_\tau$ containing the segment $\{\epsilon\to p_{\epsilon,\tau}\}$ defined in Section 4 in its boundary and transversally cutting γ (see Figure 5). We suppose of course that for each τ we have $\mathcal{S}_\tau\cap\mathcal{T}_\tau=\{\epsilon\to p_{\epsilon,\tau}\}$ and also that $\mathcal{T}_\tau$ coincides with T_+ near γ where T_+ is defined in Section 3. We parameterize $\mathcal{T}_\tau$ by (h,ϵ).

We choose any value $h_0\in]0,1[$ small enough. Let $P^-_{\epsilon,\tau}(h)$ be the h-component of the transition map from $\mathcal{S}_{\tau,h_0}$ to $\mathcal{T}_\tau$ for $-Z^{(1)}_{\epsilon,\tau}$ and $P^+_{\epsilon,\tau}(h)$ be the h-component of the transition map from $\mathcal{S}_{\tau,h_0}$ to $\mathcal{T}_\tau$ for $Z^{(1)}_{\epsilon,\tau}$. We consider the difference map

$$\hat{D}_{\epsilon,\tau}(h)=P^+_{\epsilon,\tau}(h)-P^-_{\epsilon,\tau}(h). \tag{32}$$

Proposition 5.1. *There exist smooth functions $\hat{d}_k(h,\epsilon,\tau)$, for $k=0,1,3$, such that*

$$\hat{D}_{\epsilon,\tau}(h)=B_0\hat{d}_0(h,\epsilon,\tau)+B_2\epsilon\hat{d}_1(h,\epsilon,\tau)+u\epsilon^3\hat{d}_3(h,\epsilon,\tau),$$

where

$$\hat{d}_0(h,0,0,B_2,B_3,u,\lambda)=J_0(h),\ \hat{d}_1(h,0,0,B_2,B_3,u,\lambda)=J_1(h)$$

and

$$\hat{d}_3(h,0,0,B_2,B_3,u,\lambda)=\mathcal{J}_2(h).$$

Proof. We have $\hat{D}_{\epsilon,\tau}(h)=P^+_{\epsilon,\tau}(h)-P^+_{\epsilon,\tau}(P(h,\epsilon,\tau))$ where $P(h,\epsilon,\tau)$ is the h-component of the return map defined in Section 4. As $P^+_{\epsilon,\tau}(h)=h+O(\epsilon,B_0)$ the result follows from Proposition 4.1. □

Using the elimination $h=h_{\epsilon,\tau}(\bar{y}):=e^{-\frac{\bar{y}}{\epsilon^2}}(\frac{\bar{y}}{\epsilon^2}-\frac{1}{2}(\widetilde{x}_{\epsilon,\tau}.i(\frac{\bar{y}}{\epsilon^2}))^2+1)$, for $\epsilon>0$ and $\bar{y}\in[\epsilon^2\widetilde{y}_{\epsilon,\tau},\epsilon^2\widetilde{y}_{h_0}]$, and writing $\hat{D}^*_\tau(\bar{y},\epsilon)=\hat{D}_{\epsilon,\tau}(h_{\epsilon,\tau}(\bar{y}))$, we obtain that

$$\frac{\partial}{\partial\bar{y}}\hat{D}^*_\tau(\bar{y},\epsilon)=B_0\hat{d}^*_0(\bar{y},\epsilon,\tau)+B_2\epsilon\hat{d}^*_1(\bar{y},\epsilon,\tau)+u\epsilon^3\hat{d}^*_3(\bar{y},\epsilon,\tau),$$

with

$$\hat{d}^*_j(\bar{y},\epsilon,\tau)=(\frac{\partial\hat{d}_j}{\partial h})(h_{\epsilon,\tau}(\bar{y}),\epsilon,\tau).\frac{\partial}{\partial\bar{y}}h_{\epsilon,\tau}(\bar{y}), \tag{33}$$

for $j=0,1,3$. Using Section 4 and Proposition 5.1 we have that, if $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\lambda$, then the system $\{\frac{\partial\hat{d}_0}{\partial h},\frac{\partial\hat{d}_1}{\partial h},\frac{\partial\hat{d}_3}{\partial h}\}$ is a ST-system on the interval $[h_0,h^0_{\epsilon,\tau}]$, for each fixed $\epsilon \sim 0$, $B_0 \sim 0$, $B_2 \sim 0$, $B_3 \in \mathcal{B}$, $u \sim 0$ and $\lambda \in \Lambda$. Clearly, we have that

$$\frac{\partial}{\partial \bar{y}} h_{\epsilon,\tau}(\bar{y}) = -\frac{1}{\epsilon^2} e^{-\frac{\bar{y}}{\epsilon^2}} (\frac{\bar{y}}{\epsilon^2} - \frac{1}{2}(\widetilde{x}_{\epsilon,\tau}.i(\frac{\bar{y}}{\epsilon^2}))^2 + \widetilde{x}_{\epsilon,\tau}^2 i(\frac{\bar{y}}{\epsilon^2}) i'(\frac{\bar{y}}{\epsilon^2})) < 0$$

for $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$ and $\bar{y} \in]\epsilon^2 \widetilde{y}_{\epsilon,\tau}, \epsilon^2 \widetilde{y}_{h_0}]$. Putting all informations together, we obtain

Lemma 5.2. *If $\bar{H}(0,\lambda) \neq 0$ for all $\lambda \in \lambda$, then the system $\{\hat{d}_0^*, \hat{d}_1^*, \hat{d}_3^*\}$ is a ST-system in $\bar{y} \in]\epsilon^2 \widetilde{y}_{\epsilon,\tau}, \epsilon^2 \widetilde{y}_{h_0}]$, for each fixed $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $B_3 \in \mathcal{B}$, $u \sim 0$ and $\lambda \in \Lambda$.*

Remark 5. Suppose that $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a < b$. If one replaces the interval $[a, b]$ with $]a, b]$ in Definition 4.2, then one defines a ST-system on]a,b]. It is clear that the result for ST-systems on $[a, b]$ mentioned in Remark 4 remains true if one replaces $[a, b]$ with $]a, b]$.

In Section 3, by following the orbits of the blown-up vector field $\bar{Z}_\tau$ in forward and backward time, we defined transition maps $(\delta, w) \to \mathcal{D}_\tau^+ \circ \mathcal{F}_\tau(\delta, w)$ and $(\delta, w) \to \mathcal{G}_\tau \circ \mathcal{D}_\tau^-(\delta, w)$ from Σ_- to T_+, near L_0 on the secondary blow-up locus (Figure 4). If we replace the section Σ_- with the section Σ_B (Section 4) in the definition of $\mathcal{D}_\tau^+ \circ \mathcal{F}_\tau(\delta, w)$ and $\mathcal{G}_\tau \circ \mathcal{D}_\tau^-(\delta, w)$, then the form of the h-component of the new difference map $\mathcal{D}_\tau^+ \circ \mathcal{F}_\tau(\delta, w) - \mathcal{G}_\tau \circ \mathcal{D}_\tau^-(\delta, w)$ defined on Σ_B is given again in Theorem 3.3 with some new functions $\widetilde{A}^\pm$ and $\widetilde{\Phi}_k^\pm$ having the same properties as the old ones (see [15], Proposition 4.32). We denote the h-component of the difference map defined on Σ_B again by ω_τ.

We introduce the analytic function $K(\alpha, \beta) = \frac{\exp \alpha - \exp \beta}{\alpha - \beta}$ if $\alpha \neq \beta$ and $K(\alpha, \alpha) = \exp \alpha$. Using the notation introduced in Section 3.3.1, the expression (18) may be written as

$$\mathcal{L}_\mathcal{Y} \omega_\tau = \frac{1}{w^4} K(\alpha, \beta)(T^+ - T^- + w^2 \ln \frac{P^-}{P^+}) \tag{34}$$

with

$$\alpha = -\frac{1}{w^2}(T^+ - w^2 \ln P^+)$$

and

$$\beta = -\frac{1}{w^2}(T^- - w^2 \ln P^-).$$

In Section 3.3.1, using the elimination $(\delta, w) = (\sqrt{\bar{y}}, \frac{\epsilon}{\sqrt{\bar{y}}})$, we have obtained that

$$\begin{aligned} T^+ - T^- + (\frac{\epsilon}{\sqrt{\bar{y}}})^2 \ln \frac{P^-}{P^+} &= 2\sqrt{2} B_0 \frac{\epsilon}{\sqrt{\bar{y}}}(1 + o_0(1)) \\ &+ B_2 \sqrt{\bar{y}} \frac{4\sqrt{2}}{3}(1 + o_1(1)) + u(\sqrt{\bar{y}})^3 \frac{8\sqrt{2}}{5}(\bar{H}(0,\lambda) + o_3(1)) \end{aligned} \tag{35}$$

where $o_0(1), o_1(1)$ and $o_3(1)$ are smooth functions in $(\sqrt{\bar{y}}, \frac{\epsilon}{\sqrt{\bar{y}}}, (\frac{\epsilon}{\sqrt{\bar{y}}})^2 \ln \frac{\epsilon}{\sqrt{\bar{y}}}, \tau)$ which, uniformly in τ, tend to zero as $(\sqrt{\bar{y}}, \frac{\epsilon}{\sqrt{\bar{y}}}) \to (0, 0)$. If we define $\bar{\omega}_\tau(\bar{y}, \epsilon) := \omega_\tau(\sqrt{\bar{y}}, \frac{\epsilon}{\sqrt{\bar{y}}})$, then

$$\frac{\partial}{\partial \bar{y}} \bar{\omega}_\tau(\bar{y}, \epsilon) = \frac{1}{2\bar{y}}(\mathcal{L}_\mathcal{Y} \omega_\tau)(\sqrt{\bar{y}}, \frac{\epsilon}{\sqrt{\bar{y}}}).$$

Using (34) and (35), we obtain now an expression of $\frac{\partial}{\partial \bar{y}} \bar{\omega}_\tau$, for $(\sqrt{\bar{y}}, \frac{\epsilon}{\sqrt{\bar{y}}}) \sim (0, 0)$:

$$\frac{\partial}{\partial \bar{y}} \bar{\omega}_\tau(\bar{y}, \epsilon) = B_0 \hat{g}_0 + B_2 \epsilon \hat{g}_1 + u \epsilon^3 \hat{g}_3 \tag{36}$$

with

$\hat{g}_0(\bar{y},\epsilon,\tau)=\frac{1}{\epsilon^3}\sqrt{\bar{y}}K(\alpha,\beta)\sqrt{2}(1+o_0(1))$, $\hat{g}_1=\frac{1}{\epsilon^5}(\sqrt{\bar{y}})^3K(\alpha,\beta)\frac{2\sqrt{2}}{3}(1+o_1(1))$ and

$$\hat{g}_3=\frac{1}{\epsilon^7}(\sqrt{\bar{y}})^5K(\alpha,\beta)\frac{4\sqrt{2}}{5}(\bar{H}(0,\lambda)+o_3(1)), \tag{37}$$

recalling that $o_0(1),o_1(1)$ and $o_3(1)$ are introduced in (35). The function $K(\alpha,\beta)$ is a smooth in the variable $(\sqrt{\bar{y}},\frac{\epsilon}{\sqrt{\bar{y}}},\tau)$, flat at $w=\frac{\epsilon}{\sqrt{\bar{y}}}=0$ and strictly positive for $\frac{\epsilon}{\sqrt{\bar{y}}}>0$. (We say that a smooth function $f(x)$ is flat at $x=0$ when $j^\infty f(0)=0$.) Now it can be easily seen that

Lemma 5.3. *If $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\lambda$, then there exist sufficiently small $\delta_0>0$, $w_0>0$, $\epsilon_0>0$, $u_0>0$, $B_0^0>0$ and $B_2^0>0$ such that for each fixed $\epsilon\in]0,\epsilon_0]$, $B_0\in[-B_0^0,B_0^0]$, $B_2\in[-B_2^0,B_2^0]$, $B_3\in\mathcal{B}$, $u\in[0,u_0]$ and $\lambda\in\Lambda$ the system $\{\hat{g}_0,\hat{g}_1,\hat{g}_3\}$ is a ST-system in $\bar{y}\in[\frac{\epsilon^2}{w_0^2},\delta_0^2]$.*

We may suppose that $\widetilde{y}_{h_0}>\frac{1}{w_0^2}$. For $\epsilon\sim 0$ and $\epsilon>0$, on the segment $[\epsilon^2\widetilde{y}_{\epsilon,\tau},\epsilon^2\widetilde{y}_{h_0}]\cap[\frac{\epsilon^2}{w_0^2},\delta_0^2]=[\frac{\epsilon^2}{w_0^2},\epsilon^2\widetilde{y}_{h_0}]$ we have $\hat{D}^*_\tau(\bar{y},\epsilon)\equiv\bar{\omega}_\tau(\bar{y},\epsilon)$. Then we get a "global" difference map that we will write $\hat{D}^G_\tau(\bar{y},\epsilon)$ for $\bar{y}\in[\epsilon^2\widetilde{y}_{\epsilon,\tau},\delta_0^2]$. $\hat{D}^G_\tau(\bar{y},\epsilon)$ is equal to $\hat{D}^*_\tau(\bar{y},\epsilon)$ on $[\epsilon^2\widetilde{y}_{\epsilon,\tau},\epsilon^2\widetilde{y}_{h_0}]$ and equal to $\bar{\omega}_\tau(\bar{y},\epsilon)$ on $[\frac{\epsilon^2}{w_0^2},\delta_0^2]$. *We want to prove that the function $\frac{\partial}{\partial\bar{y}}\hat{D}^G_\tau$ can be written as:*

$$\frac{\partial}{\partial\bar{y}}\hat{D}^G_\tau(\bar{y},\epsilon)=B_0d_0^G(\bar{y},\epsilon,\tau)+B_2\epsilon d_1^G(\bar{y},\epsilon,\tau)+u\epsilon^3d_3^G(\bar{y},\epsilon,\tau), \tag{38}$$

for some functions d_j^G with the property that for each fixed $\epsilon\sim 0$, $\epsilon>0$, $B_0\sim 0$, $B_2\sim 0$, $B_3\in\mathcal{B}$, $u\sim 0$ and $\lambda\in\Lambda$ the functions d_j^G are smooth in the variable $\bar{y}\in[\epsilon^2\widetilde{y}_{\epsilon,\tau},\delta_0^2]$ and the system $\{d_0^G,d_1^G,d_3^G\}$ is a ST-system in the variable $\bar{y}\in]\epsilon^2\widetilde{y}_{\epsilon,\tau},\delta_0^2]$, provided $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\Lambda$.

To prove this, we choose $\widetilde{y}_3$ and $\widetilde{y}_4$ such that $\frac{1}{w_0^2}<\widetilde{y}_4<\widetilde{y}_3<\widetilde{y}_{h_0}$ and a smooth function $\widetilde{i}(\widetilde{y})$ on $\mathbb{R}$ such that $\widetilde{i}(\widetilde{y})\equiv 1$ for $\widetilde{y}\geq\widetilde{y}_3$, $\widetilde{i}(\widetilde{y})\equiv 0$ for $\widetilde{y}\leq\widetilde{y}_4$ and $0\leq\widetilde{i}(\widetilde{y})\leq 1$ for $\widetilde{y}\in[\widetilde{y}_4,\widetilde{y}_3]$. We now define for $j=0,1,3$, $\epsilon>0$ and $\bar{y}\in[\epsilon^2\widetilde{y}_{\epsilon,\tau},\delta_0^2]$:

$$d_j^G(\bar{y},\epsilon,\tau)=\widetilde{i}(\tfrac{\bar{y}}{\epsilon^2})\hat{g}_j(\bar{y},\epsilon,\tau)+(1-\widetilde{i}(\tfrac{\bar{y}}{\epsilon^2}))\hat{d}_j^*(\bar{y},\epsilon,\tau).$$

The system $\{d_0^G,d_1^G,d_3^G\}$ coincides with the system $\{\hat{g}_0,\hat{g}_1,\hat{g}_3\}$ (resp. $\{\hat{d}_0^*,\hat{d}_1^*,\hat{d}_3^*\}$) on the interval $[\epsilon^2\widetilde{y}_3,\delta_0^2]$ (resp. $[\epsilon^2\widetilde{y}_{\epsilon,\tau},\epsilon^2\widetilde{y}_4]$) and, by Lemma 5.3 (resp. Lemma 5.2), is a ST-system on $[\epsilon^2\widetilde{y}_3,\delta_0^2]$ (resp. $]\epsilon^2\widetilde{y}_{\epsilon,\tau},\epsilon^2\widetilde{y}_4]$), for $\epsilon\sim 0$, $\epsilon>0$ and $B_2\sim 0$.

Lemma 5.4. *If $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\Lambda$, then the system $\{d_0^G,d_1^G,d_3^G\}$ is a ST-system in the variable $\bar{y}\in[\epsilon^2\frac{1}{w_0^2},\epsilon^2\widetilde{y}_{h_0}]$, for each $\epsilon\sim 0$, $\epsilon>0$, $B_0\sim 0$, $B_2\sim 0$, $B_3\in\mathcal{B}$, $u\sim 0$ and $\lambda\in\Lambda$.*

Proof. We will use the change of coordinates $\bar{y}=\epsilon^2\widetilde{y}$, for $\epsilon>0$, and we will prove that the system $\{d_0^G,d_1^G,d_3^G\}$ is a ST-system in the variable $\widetilde{y}\in[\frac{1}{w_0^2},\widetilde{y}_{h_0}]$, provided $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\Lambda$. Then we will have that the system $\{d_0^G,d_1^G,d_3^G\}$ is a ST-system in the variable $\bar{y}\in[\epsilon^2\frac{1}{w_0^2},\epsilon^2\widetilde{y}_{h_0}]$, provided $\epsilon>0$ and $\bar{H}(0,\lambda)\neq 0$ for all $\lambda\in\Lambda$. Using (33) and (37), in the new variable $(\widetilde{y},\epsilon,\tau)$ we obtain that

$$\hat{d}_j^*=\tfrac{1}{\epsilon^2}\hat{d}_{jj}^*,\ \hat{g}_j=\tfrac{1}{\epsilon^2}\hat{g}_{jj},\ j=0,1,3,$$

where $\hat{d}_{jj}^*$ and $\hat{g}_{jj}$ are smooth functions in $(\widetilde{y},\epsilon,\tau)$, including $\epsilon=0$. As the functions $\hat{D}^*_\tau(\bar{y},\epsilon)$ and $\bar{\omega}_\tau(\bar{y},\epsilon)$ coincide on the intersection $[\epsilon^2\frac{1}{w_0^2},\epsilon^2\widetilde{y}_{h_0}]$, we have that

$$B_0\hat{d}^*_{00} + B_2\epsilon\hat{d}^*_{11} + u\epsilon^3\hat{d}^*_{33} \equiv B_0\hat{g}_{00} + B_2\epsilon\hat{g}_{11} + u\epsilon^3\hat{g}_{33}$$

for $\widetilde{y} \in [\frac{1}{w_0^2}, \widetilde{y}_{h_0}]$. Hence, we have $\hat{d}^*_{jj} \equiv \hat{g}_{jj}$ for $j = 0, 1, 3$ and $B_0 = B_2 = u = 0$. From this, it follows that d^G_j, in the variable $(\widetilde{y}, \epsilon, \tau)$, on the interval $[\frac{1}{w_0^2}, \widetilde{y}_{h_0}]$ can be written as

$$d^G_j = \tfrac{1}{\epsilon^2}(\hat{d}^*_{jj}|_{B_0=B_2=u=0} + O_j(B_0, B_2, u))$$

where $O_j(B_0, B_2, u)$ is a smooth function in $(\widetilde{y}, \epsilon, \tau)$, including $\epsilon = 0$. By Proposition 4.5, it suffices now to prove that the system $\{\hat{d}^*_{jj}|_{B_0=B_2=u=\epsilon=0}; j = 0, 1, 3\}$ is a ST-system in the variable $\widetilde{y} \in [\frac{1}{w_0^2}, \widetilde{y}_{h_0}]$. This follows directly from (33). □

To end this, we use the following lemma (see [8]):

Lemma 5.5. *Let $\mathcal{F} = \{f_0, f_1, f_2\}$ be a system of C^∞-functions on $[a, b]$ (or $]a, b]$). Let $\{I_k\}_{k=0,1,...,m}$, $m \geq 1$, be a family of intervals of $[a, b]$ (or $]a, b]$):*

$I_0 = [a, d_1]$ (or $]a, d_1]$), $I_k = [d_{2(k-1)}, d_{2k+1}]$, $k = 1, ..., m-1$ and $I_m = [d_{2m-2}, b]$ where $a < d_{2i} < d_{2i+1} < b$, $i = 0, ..., m-1$. Let us suppose that for each $k \in \{0, 1, ..., m\}$ the restriction of $\mathcal{F}$ to I_k is a ST-system of degree two on I_k. Then, $\mathcal{F}$ is a ST-system of degree two on $[a, b]$ (or $]a, b]$).

We can apply Lemma 5.5 to the partition

$$]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \delta_0^2] =]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \epsilon^2\widetilde{y}_4] \cup [\epsilon^2\tfrac{1}{w_0^2}, \epsilon^2\widetilde{y}_{h_0}] \cup [\epsilon^2\widetilde{y}_3, \delta_0^2]$$

as $\epsilon > 0$ and $\frac{1}{w_0^2} < \widetilde{y}_4 < \widetilde{y}_3 < \widetilde{y}_{h_0}$. Then we get (38). Using Rolle's theorem, we have that the difference map $\hat{D}^G_\tau(\bar{y}, \epsilon)$ has at most 3 zeros, counted with multiplicity, on the interval $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \delta_0^2]$ for $\epsilon > 0$, $(B_0, B_2, u) \sim (0, 0, 0)$ and $(B_0, B_2, u) \neq (0, 0, 0)$. As $\hat{D}^G_\tau(\epsilon^2\widetilde{y}_{\epsilon,\tau}, \epsilon) = 0$, Rolle's theorem implies that $\hat{D}^G_\tau(\bar{y}, \epsilon)$ can have at most 2 zeros, counted with multiplicity, on the interval $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \delta_0^2]$, for $\epsilon > 0$, $(B_0, B_2, u) \sim (0, 0, 0)$, $(B_0, B_2, u) \neq (0, 0, 0)$ and $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. Hence, we have proved (1).

6. **Further study of limit cycles near the primary blow-up locus in the elliptic and saddle cases.** In this section we consider $X^\pm_{\epsilon,r,B_0,B_2,\lambda}$ and assume that $\epsilon \sim 0$, $\epsilon > 0$, $r \sim 0$, $r > 0$, $B_0 \sim 0$, $B_2 \sim 0$ and $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$. (When $r = 0$, then systems $X^\pm_{\epsilon,r,B_0,B_2,\lambda}$ defined in (6) have no limit cycles near $(x, y) = (0, 0)$ (see [16]).) We give a key step toward finding the optimal upper bound on the number of limit cycles near $(x, y) = (0, 0)$, in both the saddle and elliptic cases. We prove that the set $(\cup_{\bar{y}\in[\bar{y}_1,+\infty[}L_{\bar{y}}) \cup L_{00}$, in the elliptic case, and the set $\cup_{\bar{y}\in[\bar{y}_1,\frac{1}{2}]}L_{\bar{y}}$, in the saddle case, each produce at most two limit cycles for any small $\bar{y}_1 > 0$. We merely glue together local cyclicity results for $L_{\bar{y}}$, $L_{\frac{1}{2}}$ and L_{00} proved in [16] and [17] by studying evolution of a simple zero in so-called full divergence integral as parameters of our system vary.

In Section 6.1 we will study in detail the elliptic case. The study of the saddle case will be similar to the study of the elliptic case and it will be given in Section 6.2. We point out that Theorem 6.5, proven in Section 6.1, and Theorem 6.7, proven in Section 6.2, will play an important role in Section 7 in which we will finally show that the cyclicity of $X^\pm_{\epsilon,r,B_0,B_2,\lambda}$ at the origin $(x, y) = (0, 0)$ is equal to 2.

First, let us recall the well known relation between the coordinates introduced in Sections 2 and 3:

$$r\bar{x} = U\bar{X},\ r^2\bar{y} = U^2,\ u = r = UR;$$
$$\bar{x} = \delta v,\ \bar{y} = \delta^2,\ \epsilon = \delta w.$$

From this, it follows that $\delta = \sqrt{\bar{y}}$, $w = \frac{\epsilon}{\sqrt{\bar{y}}}$, $R = \frac{1}{\sqrt{\bar{y}}}$ and $U = r\sqrt{\bar{y}}$.

6.1. **Limit cycles near $(\cup_{\bar{\mathbf{y}}\in[\bar{\mathbf{y}}_1,+\infty[}\mathbf{L}_{\bar{\mathbf{y}}}) \cup \mathbf{L}_{00}$ in the elliptic case.** In this section we suppose that $\bar{y}_1 > 0$ is arbitrarily small.

In [17], by following the orbits of $X^-_{\epsilon,r,B_0,B_2,\lambda}$ in forward and backward time, we defined C^k-transition maps from $\Sigma_P \subset \{\bar{X} = 0\}$ to T_+, near L_{00}, that we denote here by respectively (F_P, ϵ) and (C_P, ϵ). Section Σ_P is parametrized by $(U, R) \in [0, U_P] \times [0, R_P]$ where $U_P > 0$ and $R_P > 0$ are sufficiently small. Section T_+ is defined in Section 3 (see Figure 4). Recall that T_+ is parameterized by (h, ϵ). We define

$$h = \Delta_P(U, R, \epsilon, B_0, B_2, \lambda) := F_P(U, R, \epsilon, B_0, B_2, \lambda) - C_P(U, R, \epsilon, B_0, B_2, \lambda).$$

The consideration of Δ_P was sufficient to study the limit cycles bifurcating from L_{00} (see [17]).

For some reasons that will become clear later in this section it is better to parametrize $\Sigma_P \setminus \{UR = 0\}$ by the coordinates $(\bar{y}, r)$. Hence we can define

$$\widetilde{\Delta}_P(\bar{y}, r, \epsilon, B_0, B_2, \lambda) := \Delta_P(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \epsilon, B_0, B_2, \lambda) \tag{39}$$

where $r \sim 0$, $r > 0$, $\bar{y} \in [\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$, $(\epsilon, B_0, B_2) \sim (0, 0, 0)$ and $\lambda \in \Lambda$.

By following the orbits of (8) (with the $-$-sign in front of $\bar{x}^3$) in forward and backward time, we can define C^∞-transition maps from $\Sigma_D \subset \{\bar{x} = 0\}$ to T_+, near the set $\cup_{\bar{y}\in[\bar{y}_1,\bar{y}_2]} L_{\bar{y}}$ for any $\bar{y}_2$ such that $0 < \bar{y}_1 < \bar{y}_2 < +\infty$. We denote the transition maps by respectively (F_D, ϵ) and (C_D, ϵ). The section Σ_D is parametrized by $\epsilon \geq 0$ and $\bar{y} \in [\bar{y}_1, \bar{y}_2]$. For more details we refer to [16] and [3]. We define

$$h = \widetilde{\Delta}_D(\bar{y}, r, \epsilon, B_0, B_2, \lambda) := F_D(\bar{y}, r, \epsilon, B_0, B_2, \lambda) - C_D(\bar{y}, r, \epsilon, B_0, B_2, \lambda).$$

If we are interested in limit cycles bifurcating from $\cup_{\bar{y}\in[\bar{y}_1,\bar{y}_2]} L_{\bar{y}}$, it suffices to study $\widetilde{\Delta}_D$ (see [16]). We may suppose that $\frac{1}{R_p^2} < \bar{y}_2$.

For $\epsilon \sim 0$, $\epsilon > 0$, $r \sim 0$ and $r > 0$, on the intersection $[\frac{1}{R_P^2}, \frac{U_P^2}{r^2}] \cap [\bar{y}_1, \bar{y}_2] = [\frac{1}{R_P^2}, \bar{y}_2]$ $\widetilde{\Delta}_P$ and $\widetilde{\Delta}_D$ coincide. We have a "global" difference map $\widetilde{\Delta}_G(\bar{y}, r, \epsilon, B_0, B_2, \lambda)$, for $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $\bar{y} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$, such that $\widetilde{\Delta}_G$ is equal to $\widetilde{\Delta}_P$ on $[\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$ and equal to $\widetilde{\Delta}_D$ on $[\bar{y}_1, \bar{y}_2]$. *For each fixed $(r, \epsilon, B_0, B_2, \lambda)$, we are interested in the number of isolated zeros of $\widetilde{\Delta}_G$ on the segment $[\bar{y}_1, \frac{U_P^2}{r^2}]$.* We will study, as usual, the derivative of $\widetilde{\Delta}_G$ w.r.t. $\bar{y}$.

Being inspired by [16] and [17], first we want to get rid of the parameter B_0 in $\widetilde{\Delta}_G$. Based on [3], in [17] we proved that there exists a C^k-function $B_P(U, R, \epsilon, B_2, \lambda)$ such that solutions of $\Delta_P(U, R, \epsilon, B_0, B_2, \lambda) = 0$, for $\epsilon \sim 0$ and $B_0 \sim 0$, can only occur for $B_0 = B_P(U, R, \epsilon, B_2, \lambda)$. Moreover, B_P is identically zero when $B_2 = UR = 0$. We now define

$$\widetilde{B}_P(\bar{y}, r, \epsilon, B_2, \lambda) := B_P(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \epsilon, B_2, \lambda) \tag{40}$$

for $r > 0$, $r \sim 0$, $\epsilon \geq 0$, $\epsilon \sim 0$ and $\bar{y} \in [\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$. Now is clear that solutions of $\widetilde{\Delta}_P(\bar{y}, r, \epsilon, B_0, B_2, \lambda) = 0$, for $\epsilon \sim 0$ and $B_0 \sim 0$, can only occur for $B_0 = \widetilde{B}_P(\bar{y}, r, \epsilon, B_2, \lambda)$, and that

$$\widetilde{B}_P(\bar{y}, r, \epsilon, B_2, \lambda) = B_2 \widetilde{b}_P^1(\bar{y}, r, \epsilon, B_2, \lambda) + r\widetilde{b}_P^2(\bar{y}, r, \epsilon, B_2, \lambda)$$

where $\widetilde{b}_P^1$ and $\widetilde{b}_P^2$ are C^k-functions in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, r, \epsilon, B_2, \lambda)$.

Again based on [3], in [16] we proved existence of a C^∞-function $\widetilde{B}_D(\bar{y}, r, \epsilon, B_2, \lambda)$, for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$, such that solutions of $\widetilde{\Delta}_D(\bar{y}, r, \epsilon, B_0, B_2, \lambda) = 0$, for $\epsilon \sim 0$ and $B_0 \sim 0$, occur only for $B_0 = \widetilde{B}_D(\bar{y}, r, \epsilon, B_2, \lambda)$. We have $\widetilde{B}_D(\bar{y}, r, \epsilon, B_2, \lambda) = B_2 \widetilde{b}_D^1(\bar{y}, r, \epsilon, B_2, \lambda) + r\widetilde{b}_D^2(\bar{y}, r, \epsilon, B_2, \lambda)$ where $\widetilde{b}_D^1$ and $\widetilde{b}_D^2$ are C^∞-functions in the variable $(\bar{y}, r, \epsilon, B_2, \lambda)$.

Again we can consider a "global" function that we will write $\widetilde{B}_G(\bar{y}, r, \epsilon, B_2, \lambda)$ for $\bar{y} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$. $\widetilde{B}_G$ is equal to $\widetilde{B}_P$ on $[\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$ and equal to $\widetilde{B}_D$ on $[\bar{y}_1, \bar{y}_2]$. Solutions of $\widetilde{\Delta}_G(\bar{y}, r, \epsilon, B_0, B_2, \lambda) = 0$, for $\epsilon \sim 0$ and $B_0 \sim 0$, are only possible for $B_0 = \widetilde{B}_G(\bar{y}, r, \epsilon, B_2, \lambda)$. *Hence it suffices to study isolated zeros (w.r.t. $\bar{y} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$) of the following family:*

$$\bar{\Delta}_G(\bar{y}, r, \epsilon, B_2, \lambda, \hat{\bar{y}}) := \widetilde{\Delta}_G(\bar{y}, r, \epsilon, \widetilde{B}_G(\hat{\bar{y}}, r, \epsilon, B_2, \lambda), B_2, \lambda), \tag{41}$$

for each fixed $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$.

Proposition 6.1. *The function $\widetilde{B}_G$ can be written as*

$$\widetilde{B}_G(\bar{y}, r, \epsilon, B_2, \lambda) = B_2 \widetilde{b}_G^1(\bar{y}, r, \epsilon, B_2, \lambda) + r\widetilde{b}_G^2(\bar{y}, r, \epsilon, B_2, \lambda), \tag{42}$$

where $\widetilde{b}_G^1$ and $\widetilde{b}_G^2$ are C^k-functions in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \bar{y}, r, \epsilon, B_2, \lambda)$ and where $\bar{y} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$.

Proof. We glue $\widetilde{B}_P$ and $\widetilde{B}_D$ by a partition of unity. Let us choose $\bar{y}_P^1$ such that $\frac{1}{R_P^2} < \bar{y}_P^1 < \bar{y}_2$ and a smooth function $\xi(\bar{y})$ on the real line such that $\xi(\bar{y}) \equiv 1$ for $\bar{y} \geq \bar{y}_P^1$, $\xi(\bar{y}) \equiv 0$ for $\bar{y} \leq \frac{1}{R_P^2}$ and $0 \leq \xi(\bar{y}) \leq 1$ for $\frac{1}{R_P^2} \leq \bar{y} \leq \bar{y}_P^1$. It can be now easily seen that

$$\begin{aligned}\widetilde{B}_G &= \xi(\bar{y})\widetilde{B}_P + (1 - \xi(\bar{y}))\widetilde{B}_D \\ &= B_2\big(\xi(\bar{y})\widetilde{b}_P^1 + (1 - \xi(\bar{y}))\widetilde{b}_D^1\big) + r\big(\xi(\bar{y})\widetilde{b}_P^2 + (1 - \xi(\bar{y}))\widetilde{b}_D^2\big).\end{aligned}$$

□

Remark 6. The functions $\widetilde{b}_G^1$ and $\widetilde{b}_G^2$ in (42) are bounded by construction. More precisely, there exist $M > 0$, $\epsilon_0 > 0$, $r_0 > 0$, $B_2^0 > 0$ and $U_P > 0$ such that for each $\epsilon \in]0, \epsilon_0]$, $r \in]0, r_0]$, $B_2 \in [-B_2^0, B_2^0]$, $\lambda \in \Lambda$ and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$ we have $\left|\widetilde{b}_G^1(\hat{\bar{y}}, r, \epsilon, B_2, \lambda)\right| < M$ and $\left|\widetilde{b}_G^2(\hat{\bar{y}}, r, \epsilon, B_2, \lambda)\right| < M$. We will treat $\widetilde{b}_G^1$ and $\widetilde{b}_G^2$ as new bounded parameters bearing in mind that they depend on $(\hat{\bar{y}}, r, \epsilon, B_2, \lambda)$.

Let us finally study $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$. First, we consider the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ on each interval $[\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$ and $[\bar{y}_1, \bar{y}_2]$.

1. *On the interval* $[\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$. Let us define

$$\begin{aligned}\bar{\Delta}_P(\bar{y}, r, \epsilon, B_2, \lambda, \hat{\bar{y}}) &:= \widetilde{\Delta}_P(\bar{y}, r, \epsilon, \widetilde{B}_G(\hat{\bar{y}}, r, \epsilon, B_2, \lambda), B_2, \lambda), \\ \widetilde{F}_P(\bar{y}, r, \epsilon, B_0, B_2, \lambda) &:= F_P(r\sqrt{\bar{y}}, \tfrac{1}{\sqrt{\bar{y}}}, \epsilon, B_0, B_2, \lambda), \\ \widetilde{C}_P(\bar{y}, r, \epsilon, B_0, B_2, \lambda) &:= C_P(r\sqrt{\bar{y}}, \tfrac{1}{\sqrt{\bar{y}}}, \epsilon, B_0, B_2, \lambda),\end{aligned}$$

where $\bar{y} \in [\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$ and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$.

We write $\mathcal{L}_Y f = U\frac{\partial f}{\partial U} - R\frac{\partial f}{\partial R}$, where f is a C^k-function in $(U, R, \epsilon, B_0, B_2, \lambda)$. Again we invoke [17] and obtain that

$$\begin{aligned} \frac{\partial \widetilde{F}_P}{\partial \bar{y}} &= \frac{1}{2\bar{y}}(\mathcal{L}_Y F_P)(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, ...) = -\frac{1}{2\bar{y}\epsilon^4} \exp \frac{1}{\epsilon^2}\widetilde{F}_P^1, \\ \frac{\partial \widetilde{C}_P}{\partial \bar{y}} &= \frac{1}{2\bar{y}}(\mathcal{L}_Y C_P)(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, ...) = -\frac{1}{2\bar{y}\epsilon^4} \exp \frac{1}{\epsilon^2}\widetilde{C}_P^1, \end{aligned}$$

with

$$\widetilde{F}_P^1 = \widetilde{F}_P^2 + \widetilde{F}_P^3 \ln \frac{1}{\sqrt{\bar{y}}},\ \widetilde{C}_P^1 = \widetilde{C}_P^2 + \widetilde{C}_P^3 \ln \frac{1}{\sqrt{\bar{y}}},$$

where $\widetilde{F}_P^2, \widetilde{F}_P^3, \widetilde{C}_P^2$ and $\widetilde{C}_P^3$ are C^k in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \frac{1}{\sqrt{\bar{y}}} \ln \frac{1}{\sqrt{\bar{y}}}, \epsilon, \epsilon^2 \ln \epsilon, B_0, B_2, \lambda)$ and $\widetilde{F}_P^3$ and $\widetilde{C}_P^3$ are strictly positive. Furthermore, [17] implies that

$$\widetilde{F}_P^1 - \widetilde{C}_P^1 = B_0.\rho_0 + B_2(-\frac{\pi}{2} + \rho_1) + r\sqrt{\bar{y}}(-2\sqrt{2}\bar{H}(0, \lambda) + \rho_2), \tag{43}$$

where ρ_0, ρ_1 and ρ_2 are C^k-functions in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \frac{1}{\sqrt{\bar{y}}} \ln \frac{1}{\sqrt{\bar{y}}}, \epsilon, \epsilon^2 \ln \epsilon, B_0, B_2, \lambda)$ which, uniformly in (B_0, λ), tend to zero when $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, B_2, \epsilon) \to (0, 0, 0, 0)$.

We finally get

$$\frac{\partial \widetilde{\Delta}_P}{\partial \bar{y}} = \frac{\partial \widetilde{F}_P}{\partial \bar{y}} - \frac{\partial \widetilde{C}_P}{\partial \bar{y}} = -\frac{1}{2\bar{y}\epsilon^6} K(\frac{1}{\epsilon^2}\widetilde{F}_P^1, \frac{1}{\epsilon^2}\widetilde{C}_P^1)(\widetilde{F}_P^1 - \widetilde{C}_P^1) \tag{44}$$

where $K(\alpha, \beta) = \frac{\exp \alpha - \exp \beta}{\alpha - \beta}$ if $\alpha \neq \beta$ and $K(\alpha, \alpha) = \exp \alpha$.

Lemma 6.2. *$K(\frac{1}{\epsilon^2}\widetilde{F}_P^1, \frac{1}{\epsilon^2}\widetilde{C}_P^1)$ is a C^k-function in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \epsilon, B_0, B_2, \lambda)$ and strictly positive for $\epsilon > 0$ and $\frac{1}{\sqrt{\bar{y}}} > 0$. Moreover, for any $m \in \mathbb{N}$ we can take ϵ sufficiently small such that $K(\frac{1}{\epsilon^2}\widetilde{F}_P^1, \frac{1}{\epsilon^2}\widetilde{C}_P^1) = O(\epsilon^m(\frac{1}{\sqrt{\bar{y}}})^m)$.*

Proof. We have

$$K(\frac{1}{\epsilon^2}\widetilde{F}_P^1, \frac{1}{\epsilon^2}\widetilde{C}_P^1) = \int_0^1 \exp \frac{1}{\epsilon^2}\big(t\widetilde{F}_P^2 + (1-t)\widetilde{C}_P^2 + (t\widetilde{F}_P^3 + (1-t)\widetilde{C}_P^3) \ln \frac{1}{\sqrt{\bar{y}}}\big)dt.$$

Let us recall that $\widetilde{F}_P^3$ and $\widetilde{C}_P^3$ are strictly positive. By choosing ϵ small enough, the expression $\frac{t\widetilde{F}_P^3 + (1-t)\widetilde{C}_P^3}{\epsilon^2}$ can be chosen arbitrarily high, so that the integrand in the above integral can be written as $\epsilon^m.(\frac{1}{\sqrt{\bar{y}}})^m.f(t, \bar{y}, r, \epsilon, B_0, B_2, \lambda)$ where f is a C^k-function in $(t, r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \epsilon, B_0, B_2, \lambda)$. □

Taking into account (42), (43), (44), Lemma 6.2 and Remark 6 we obtain the following expression for the derivative $\frac{\partial \bar{\Delta}_P}{\partial \bar{y}}$:

$$\begin{aligned} \frac{\partial \bar{\Delta}_P}{\partial \bar{y}} &= K_P.\Big(B_2(-\frac{\pi}{2} + \bar{\rho}_1) + r\sqrt{\bar{y}}(-2\sqrt{2}\bar{H}(0, \lambda) + \bar{\rho}_2)\Big) \\ &=: K_P(B_2 f_P^1 + r f_P^2) =: K_P.I_P \end{aligned} \tag{45}$$

where $\bar{\rho}_1$ and $\bar{\rho}_2$ are C^k-functions in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \frac{1}{\sqrt{\bar{y}}} \ln \frac{1}{\sqrt{\bar{y}}}, \epsilon, \epsilon^2 \ln \epsilon, r, B_2, \lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$ which, uniformly in $(\lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$, tend to zero when $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, B_2, \epsilon) \to (0, 0, 0, 0)$, and K_P is a C^k-function in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \epsilon, r, B_2, \lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$ and strictly negative for $\epsilon > 0$ and $\frac{1}{\sqrt{\bar{y}}} > 0$.

In Definition 4.2, we can replace C^∞ with C^k. Then we deal with ST-systems of C^k-functions. Properties of ST-systems of C^∞-functions mentioned in Sections 4 and 5 remain true if we replace C^∞ with C^k.

Proposition 6.3. *There exist $U_P>0$, $R_P>0$, $r_0>0$, $\epsilon_0>0$ and $B_2^0>0$ sufficiently small such that, for each $r\in]0,r_0]$, $\epsilon\in]0,\epsilon_0]$, $B_2\in[-B_2^0,B_2^0]$, $\lambda\in\Lambda$ and $\hat{\bar{y}}\in[\bar{y}_1,\frac{U_P^2}{r^2}]$, the system $\{f_P^1,f_P^2\}=\{-\frac{\pi}{2}+\bar\rho_1,$ $\sqrt{\bar y}(-2\sqrt2\bar H(0,\lambda)+\bar\rho_2)\}$ is a (C^k) ST-system in the variable $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{r^2}]$.*

Proof. Let us recall that $(r\sqrt{\bar y},\frac{1}{\sqrt{\bar y}})\sim(0,0)$ if $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{r^2}]$ where $R_P>0$ and $U_P>0$ are sufficiently small. Remark 6 and the above mentioned property of $\bar\rho_1$ imply that $f_P^1\neq0$ for each $r\sim0$, $r>0$, $\epsilon\sim0$, $\epsilon>0$, $B_2\sim0$, $\lambda\in\Lambda$, $\hat{\bar y}\in[\bar y_1,\frac{U_P^2}{r^2}]$ and $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{r^2}]$ where w_0, U_P and R_P are sufficiently small. It can be easily seen that

$$\frac{\partial}{\partial\bar y}(\frac{f_P^2}{f_P^1})=\frac{1}{\sqrt{\bar y}}(\frac{2\sqrt2\bar H(0,\lambda)}{\pi}+\bar\rho_3)$$

where $\bar\rho_3$ is a C^k-function in $(r\sqrt{\bar y},\frac{1}{\sqrt{\bar y}},\frac{1}{\sqrt{\bar y}}\ln\frac{1}{\sqrt{\bar y}},\epsilon,\epsilon^2\ln\epsilon,r,B_2,\lambda,\widetilde b_G^1,\widetilde b_G^2)$ which, uniformly in $(r,\lambda,\widetilde b_G^1,\widetilde b_G^2)$, tends to zero when $(r\sqrt{\bar y},\frac{1}{\sqrt{\bar y}},B_2,\epsilon)\to(0,0,0,0)$. Since $\bar H(0,\lambda)\neq0$ for all $\lambda\in\Lambda$, we have that $\frac{\partial}{\partial\bar y}(\frac{f_P^2}{f_P^1})\neq0$ for each $r\sim0$, $r>0$, $\epsilon\sim0$, $\epsilon>0$, $B_2\sim0$, $\lambda\in\Lambda$, $\hat{\bar y}\in[\bar y_1,\frac{U_P^2}{r^2}]$ and $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{r^2}]$ where w_0, U_P and R_P are sufficiently small. □

Since the function K_P is nonzero for $\epsilon>0$ and $\frac{1}{\sqrt{\bar y}}>0$, Proposition 6.3 implies that for each $r\sim0$, $r>0$, $\epsilon\sim0$, $\epsilon>0$, $B_2\sim0$, $\lambda\in\Lambda$ and $\hat{\bar y}\in[\bar y_1,\frac{U_P^2}{r^2}]$ the derivative $\frac{\partial\bar\Delta_P}{\partial\bar y}$ has at most one zero (counting multiplicity) on the segment $[\frac{1}{R_P^2},\frac{U_P^2}{r^2}]$.

2. *On the interval* $[\bar y_1,\bar y_2]$. Let us define

$$\bar\Delta_D(\bar y,r,\epsilon,B_2,\lambda,\hat{\bar y}):=\widetilde\Delta_D(\bar y,r,\epsilon,\widetilde B_G(\hat{\bar y},r,\epsilon,B_2,\lambda),B_2,\lambda),$$

where $\bar y\in[\bar y_1,\bar y_2]$ and $\hat{\bar y}\in[\bar y_1,\frac{U_P^2}{r^2}]$.

Using [16] and [3] we obtain

$$\tfrac{\partial F_D}{\partial\bar y}=-\tfrac{1}{\epsilon^4}\exp\tfrac{1}{\epsilon^2}\widetilde F_D^1,\ \tfrac{\partial C_D}{\partial\bar y}=-\tfrac{1}{\epsilon^4}\exp\tfrac{1}{\epsilon^2}\widetilde C_D^1,$$

with

$$\begin{aligned}\widetilde F_D^1&=\textstyle\int_0^{\sqrt{2\bar y}}\frac{\varrho d\varrho}{-1+B_2\varrho-\varrho^2+r\varrho^3\bar H(r\varrho,\lambda)}+\widetilde F_D^2<0,\\ \widetilde C_D^1&=\textstyle-\int_{-\sqrt{2\bar y}}^0\frac{\varrho d\varrho}{-1+B_2\varrho-\varrho^2+r\varrho^3\bar H(r\varrho,\lambda)}+\widetilde C_D^2<0,\end{aligned}$$

where $\widetilde F_D^2$ and $\widetilde C_D^2$ are C^∞-functions in $(\bar y,r,\epsilon,\epsilon^2\ln\epsilon,B_0,B_2,\lambda)$ and identically equal to zero when $\epsilon=0$. Furthermore, [16] implies that

$$\begin{aligned}\widetilde F_D^1-\widetilde C_D^1=B_0.\rho_D^0+B_2(&\int_{-\sqrt{2\bar y}}^{\sqrt{2\bar y}}\frac{-\varrho^2d\varrho}{(1+\varrho^2)^2}+\rho_D^1)\\ &+r(\bar H(0,\lambda)\int_{-\sqrt{2\bar y}}^{\sqrt{2\bar y}}\frac{-\varrho^4d\varrho}{(1+\varrho^2)^2}+\rho_D^2),\end{aligned}\tag{46}$$

where ρ_D^0, ρ_D^1 and ρ_D^2 are C^∞-functions in $(\bar{y}, r, \epsilon, \epsilon^2 \ln \epsilon, B_0, B_2, \lambda)$ and identically equal to zero when $B_2 = r = \epsilon = 0$.

We finally get

$$\frac{\partial \widetilde{\Delta}_D}{\partial \bar{y}} = \frac{\partial F_D}{\partial \bar{y}} - \frac{\partial C_D}{\partial \bar{y}} = -\frac{1}{\epsilon^6} K(\frac{1}{\epsilon^2}\widetilde{F}_D^1, \frac{1}{\epsilon^2}\widetilde{C}_D^1)(\widetilde{F}_D^1 - \widetilde{C}_D^1) \tag{47}$$

where K is introduced in (44). The function $K(\frac{1}{\epsilon^2}\widetilde{F}_D^1, \frac{1}{\epsilon^2}\widetilde{C}_D^1)$ is, by [3], a C^∞-function, flat at $\epsilon = 0$ and strictly positive for $\epsilon > 0$. Taking into account (42), (46), (47) and Remark 6 we obtain the following expression for the derivative $\frac{\partial \bar{\Delta}_D}{\partial \bar{y}}$:

$$\begin{aligned}\frac{\partial \bar{\Delta}_D}{\partial \bar{y}} &= K_D.\Big(B_2(\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{-\varrho^2 d\varrho}{(1+\varrho^2)^2} + \bar{\rho}_D^1) + r(\bar{H}(0,\lambda)\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{-\varrho^4 d\varrho}{(1+\varrho^2)^2} + \bar{\rho}_D^2)\Big)\\ &=: K_D.(B_2 f_D^1 + r f_D^2) =: K_D.I_D \end{aligned}\tag{48}$$

where $\bar{\rho}_D^1$ and $\bar{\rho}_D^2$ are C^∞-functions in $(\bar{y}, r, \epsilon, \epsilon^2 \ln \epsilon, B_2, \lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$, identically equal to zero when $B_2 = r = \epsilon = 0$, and K_D is a C^∞-function in variable $(\bar{y}, r, \epsilon, B_2, \lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$ and strictly negative for $\epsilon > 0$.

Lemma 6.4. *The system* $\{\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^2 d\varrho}{(1+\varrho^2)^2}, \bar{H}(0,\lambda)\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^4 d\varrho}{(1+\varrho^2)^2}\}$ *is a ST-system on* $[\bar{y}_1, \bar{y}_2]$*, for all* $\lambda \in \Lambda$.

Proof. Let us write $f_0 = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^2 d\varrho}{(1+\varrho^2)^2}$ and $f_1 = \bar{H}(0,\lambda)\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^4 d\varrho}{(1+\varrho^2)^2}$. It is clear that $f_0(\bar{y}) > 0$ for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$. We have

$$f_0(\bar{y}) = -\frac{\sqrt{2\bar{y}}}{1+2\bar{y}} + \arctan(\sqrt{2\bar{y}}) \tag{49}$$

and

$$f_1(\bar{y}) = \bar{H}(0,\lambda)\Big(\frac{\sqrt{2\bar{y}}(3+4\bar{y})}{1+2\bar{y}} - 3\arctan(\sqrt{2\bar{y}})\Big). \tag{50}$$

Combining (49) and (50) we get

$$(\frac{f_1}{f_0})'(\bar{y}) = \bar{H}(0,\lambda)\frac{-12\bar{y} + 2\sqrt{2\bar{y}}(3+2\bar{y})\arctan(\sqrt{2\bar{y}})}{(\sqrt{2\bar{y}} - (1+2\bar{y})\arctan(\sqrt{2\bar{y}}))^2}.$$

Since $\bar{H}(0,\lambda) \neq 0$ for all $\lambda \in \Lambda$, it suffices now to prove that $-12\bar{y} + 2\sqrt{2\bar{y}}(3 + 2\bar{y})\arctan(\sqrt{2\bar{y}}) > 0$ for all $\bar{y} > 0$. Using $x = \sqrt{2\bar{y}}$, it suffices to prove that $f_2(x) := -6x^2 + 2x(3+x^2)\arctan(x) > 0$ for all $x > 0$.

Since $f_2(0) = 0$, it suffices to prove that $f_2'(x) > 0$ for all $x > 0$. This will be true if and only if $f_3(x) := -x(3+5x^2) + 3(1+x^2)^2\arctan(x) > 0$ for all $x > 0$.

Since $f_3(0) = 0$, it suffices to prove that $f_3'(x) > 0$ for all $x > 0$. $f_3'(x) > 0$ for all $x > 0$ if and only if $f_4(x) := -x + (1+x^2)\arctan(x) > 0$ for all $x > 0$. The function f_4 is strictly positive for $x > 0$ because $f_4(0) = 0$ and $f_4'(x) = 2x\arctan(x) > 0$ for all $x > 0$. □

Since the functions $\bar{\rho}_D^1$ and $\bar{\rho}_D^2$ in (48) are $O(B_2, r, \epsilon, \epsilon^2 \ln \epsilon)$ and smooth in $(\bar{y}, r, \epsilon, \epsilon^2 \ln \epsilon, B_2, \lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$, all their derivatives w.r.t. $\bar{y}$ are $O(B_2, r, \epsilon, \epsilon^2 \ln \epsilon)$ functions which are smooth in the variable $(\bar{y}, r, \epsilon, \epsilon^2 \ln \epsilon, B_2, \lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$. Lemma 6.4, Proposition 4.5 and Remark 6 now imply that for each $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$ the system $\{f_D^1, f_D^2\}$ is a smooth ST-system in variable $\bar{y} \in [\bar{y}_1, \bar{y}_2]$. As K_D is smooth in $(\bar{y}, r, \epsilon, B_2, \lambda, \widetilde{b}_G^1, \widetilde{b}_G^2)$ and strictly negative for $\epsilon > 0$, we have that for each $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $\hat{y} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$ the derivative $\frac{\partial \bar{\Delta}_D}{\partial \bar{y}}$ has at most one zero (counting multiplicity) on the segment $[\bar{y}_1, \bar{y}_2]$.

We know that $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ is equal to $\frac{\partial \bar{\Delta}_D}{\partial \bar{y}}$ on $[\bar{y}_1, \bar{y}_2]$ and equal to $\frac{\partial \bar{\Delta}_P}{\partial \bar{y}}$ on $[\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$. Putting all informations together, we have that for each $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $\hat{y} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$ the derivative $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has at most 2 simple zeros on the segment $[\bar{y}_1, \frac{U_P^2}{r^2}]$. Simple zeros in $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ correspond to saddle-node bifurcations of limit cycles (see [7]).

We will prove that $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has at most one simple zero on the segment $[\bar{y}_1, \frac{U_P^2}{r^2}]$ (see Theorem 6.5). To see the intuition behind this result, we first make rescaling in the parameter space $(B_2, r) = \varpi(\bar{B}_2, \bar{r})$, where $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{r} > 0$, $\varpi \sim 0$ and $\varpi > 0$. Next, we find sets Λ_1 and Λ_2 such that $\Lambda = \Lambda_1 \cup \Lambda_2$, $\bar{H}(0, \lambda) > 0$ for all $\lambda \in \Lambda_1$ and $\bar{H}(0, \lambda) < 0$ for all $\lambda \in \Lambda_2$. Suppose that $\lambda \in \Lambda_1$. When $(\bar{B}_2, \bar{r})$ is between $(1, 0) \in \mathbb{S}^1$ and $(0, 1) \in \mathbb{S}^1$, then $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has no zeros on the segment $[\bar{y}_1, \frac{U_P^2}{r^2}]$. Hence, no saddle-node bifurcation of limit cycles can occur near the set $(\cup_{\bar{y} \in [\bar{y}_1, +\infty[} L_{\bar{y}}) \cup L_{00}$. When $(\bar{B}_2, \bar{r}) \sim (0, 1)$, then the $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has no zeros on $[\bar{y}_1, \frac{U_P^2}{r^2}]$ and may have one simple zero near $\bar{y} = 0$. Hence, a saddle-node bifurcation of limit cycles may occur near the limit periodic set $\cup_{h \in [0,1]} L_h$, when B_0 varies. When $(\bar{B}_2, \bar{r})$ goes from $(0, 1)$ to $(-1, 0)$, then a simple zero of $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ travels from $\bar{y} = 0$ to $\bar{y} = +\infty$. Hence, the $\bar{y}$-value, near which a saddle-node bifurcation of limit cycles occurs, travels from $\bar{y} = 0$ to $\bar{y} = +\infty$. A similar elaboration is possible when $\lambda \in \Lambda_2$. We make this more precise in Theorem 6.5.

Let us write $g_0(\bar{y}) = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{-\varrho^2 d\varrho}{(1+\varrho^2)^2}$ and $g_1(\bar{y}) = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{-\varrho^4 d\varrho}{(1+\varrho^2)^2}$. Using (49) and (50) we have that

$$\lim_{\bar{y} \to +\infty} g_0(\bar{y}) = -\frac{\pi}{2}, \lim_{\bar{y} \to 0^+} \frac{1}{(\sqrt{\bar{y}})^3} g_0(\bar{y}) = -\frac{4\sqrt{2}}{3} \tag{51}$$

and

$$\lim_{\bar{y} \to +\infty} \frac{1}{\sqrt{\bar{y}}} g_1(\bar{y}) = -2\sqrt{2}, \lim_{\bar{y} \to 0^+} \frac{1}{(\sqrt{\bar{y}})^5} g_1(\bar{y}) = -\frac{8\sqrt{2}}{5}. \tag{52}$$

Using (51), (52) and the proof of Lemma 6.4 once more we obtain that the quotient $\frac{g_1}{g_0}$ is a strictly increasing analytic diffeomorphism on $[0, +\infty[$, $\lim_{\bar{y} \to 0^+} \frac{g_1(\bar{y})}{g_0(\bar{y})} = 0$ and $\lim_{\bar{y} \to +\infty} \frac{g_1(\bar{y})}{g_0(\bar{y})} = +\infty$. Note that the quotient $\frac{g_1}{g_0}$ has an analytic extension to $\bar{y} = 0$ because g_0 and g_1 are odd functions in variable $\sqrt{\bar{y}}$. Moreover $(\frac{g_1}{g_0})'(0) = \frac{6}{5}$.

Functions I_D and I_P defined respectively in (48) and (45) can be written as

$$I_D = \varpi.\bar{I}_D, I_P = \varpi.\bar{I}_P,$$

where $\bar{I}_D = \bar{B}_2 f_D^1 + \bar{r} f_D^2$ and $\bar{I}_P = \bar{B}_2 f_P^1 + \bar{r} f_P^2$. As $\bar{r} > 0$ and $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, we have $\bar{r} = \sqrt{1 - \bar{B}_2^2}$.

Theorem 6.5. *The following propositions are true:*
(a) Take any small $\bar{y}_1 > 0$. There exist sufficiently small $\mu_0 > 0$, $R_P > 0$, $U_P > 0$, $\epsilon_0 > 0$ and $\varpi_0 > 0$ and sufficiently large $\bar{y}_2 > 0$ such that $\frac{1}{R_P^2} < \bar{y}_2$ and for each fixed $\varpi \in]0, \varpi_0]$, $\bar{B}_2 \in [-\mu_0, 1[$ (resp. $\bar{B}_2 \in]-1, \mu_0]$), $\epsilon \in]0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $\hat{y} \in [\bar{y}_1, \frac{U_P^2}{(\varpi \bar{r})^2}]$ we have $I_D < 0$ (resp. $I_D > 0$) for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and

$I_P<0$ (resp. $I_P>0$) for $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{(\varpi\bar r)^2}]$.
(b) For each small $\mu_0>0$ and $\bar y_1^0>0$ there exist sufficiently small $\bar y_1>0$, $R_P>0$, $U_P>0$, $\epsilon_0>0$ and $\varpi_0>0$ and sufficiently large $\bar y_2>0$ such that $\bar y_1<\bar y_1^0$, $\frac{1}{R_P^2}<\bar y_2$ and for each fixed $\varpi\in]0,\varpi_0]$, $\bar B_2\in[-1+\mu_0,-\mu_0]$ (resp. $\bar B_2\in[\mu_0,1-\mu_0]$), $\epsilon\in]0,\epsilon_0]$, $\lambda\in\Lambda_1$ (resp. $\lambda\in\Lambda_2$) and $\hat{\hat y}\in[\bar y_1,\frac{U_P^2}{(\varpi\bar r)^2}]$ I_D has precisely one simple zero on the interval $[\bar y_1,\bar y_2]$, I_D is strictly positive (resp. strictly negative) at $\bar y=\bar y_1$ and $I_P<0$ (resp. $I_P>0$) for $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{(\varpi\bar r)^2}]$.
(c) Take any small $\bar y_1>0$. There exist sufficiently small $\mu_0>0$, $R_P>0$, $U_P>0$, $\epsilon_0>0$ and $\varpi_0>0$ and sufficiently large $\bar y_2>0$ such that $\frac{1}{R_P^2}<\bar y_2$ and for each fixed $\varpi\in]0,\varpi_0]$, $\bar B_2\in]-1,-1+\mu_0]$ (resp. $\bar B_2\in[1-\mu_0,1[$), $\epsilon\in]0,\epsilon_0]$, $\lambda\in\Lambda_1$ (resp. $\lambda\in\Lambda_2$) and $\hat{\hat y}\in[\bar y_1,\frac{U_P^2}{(\varpi\bar r)^2}]$ we have that $I_D>0$ (resp. $I_D<0$) for $\bar y\in[\bar y_1,\bar y_2]$ and I_P has at most one simple zero on the interval $[\frac{1}{R_P^2},\frac{U_P^2}{(\varpi\bar r)^2}]$.

Proof. Using (48) (resp. (45)) we see that $\bar I_D=0$ (resp. $\bar I_P=0$) if and only if

$$-\frac{\bar B_2}{\bar r}=\frac{f_D^2}{f_D^1}=\frac{\bar H(0,\lambda)g_1+\bar\rho_D^2}{g_0+\bar\rho_D^1}$$
$$(resp.\ -\frac{\bar B_2}{\bar r}=\frac{f_P^2}{f_P^1}=\frac{\sqrt{\bar y}(-2\sqrt{2}\bar H(0,\lambda)+\bar\rho_2)}{-\frac{\pi}{2}+\bar\rho_1}\ ).$$

The proposition (a) for $\bar B_2\in[0,1[$ (resp. $\bar B_2\in]-1,0]$) and $\lambda\in\Lambda_1$ (resp. $\lambda\in\Lambda_2$) follows directly from (45) and (48). It remains to prove the proposition (a) for $\bar B_2\sim 0$, $\bar B_2<0$ and $\lambda\in\Lambda_1$ (resp. $\bar B_2\sim 0$, $\bar B_2>0$ and $\lambda\in\Lambda_2$). Take any small $\bar y_1>0$. We observe that $\frac{\bar B_2}{\bar r}\to 0$ as $(\bar B_2,\bar r)\to(0,1)$. As the quotient $\frac{g_1}{g_0}$ is a strictly increasing function and $\frac{g_1(0)}{g_0(0)}=0$, there exist small $\mu_0>0$, $R_P>0$, $U_P>0$, $\epsilon_0>0$ and $\varpi_0>0$, and $\bar y_2>0$ large enough such that $\frac{1}{R_P^2}<\bar y_2$ and for each $\varpi\in]0,\varpi_0]$, $\bar B_2\in[-\mu_0,0]$ (resp. $\bar B_2\in[0,\mu_0]$), $\epsilon\in]0,\epsilon_0]$, $\lambda\in\Lambda_1$ (resp. $\lambda\in\Lambda_2$) and $\hat{\hat y}\in[\bar y_1,\frac{U_P^2}{(\varpi\bar r)^2}]$ we have that $-\frac{\bar B_2}{\bar r}<\frac{f_D^2}{f_D^1}$ (resp. $-\frac{\bar B_2}{\bar r}>\frac{f_D^2}{f_D^1}$) for $\bar y\in[\bar y_1,\bar y_2]$ and $-\frac{\bar B_2}{\bar r}<\frac{f_P^2}{f_P^1}$ (resp. $-\frac{\bar B_2}{\bar r}>\frac{f_P^2}{f_P^1}$) for $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{(\varpi\bar r)^2}]$.

Let us prove the proposition (b). Take any small $\mu_0>0$ and $\bar y_1^0>0$. For $\bar B_2\in[-1+\mu_0,-\mu_0]$ (resp. $\bar B_2\in[\mu_0,1-\mu_0]$, the quotient $\frac{\bar B_2}{\bar r}$ takes values in a compact set $C\subset]-\infty,0[$ (resp. $C\subset]0,+\infty[$). Since $\lim_{\bar y\to 0^+}\frac{g_1(\bar y)}{g_0(\bar y)}=0$, there exist now sufficiently small $\bar y_1>0$, $U_P>0$, $\epsilon_0>0$ and $\varpi_0>0$ such that $\bar y_1<\bar y_1^0$ and, for each fixed $\varpi\in]0,\varpi_0]$, $\epsilon\in]0,\epsilon_0]$, $\bar B_2\in[-1+\mu_0,-\mu_0]$ (resp. $\bar B_2\in[\mu_0,1-\mu_0]$), $\lambda\in\Lambda_1$ (resp. $\lambda\in\Lambda_2$) and $\hat{\hat y}\in[\bar y_1,\frac{U_P^2}{(\varpi\bar r)^2}]$, $-\frac{\bar B_2}{\bar r}>\frac{f_D^2}{f_D^1}$ (resp. $-\frac{\bar B_2}{\bar r}<\frac{f_D^2}{f_D^1}$) for $\bar y=\bar y_1$. In other words, I_D is strictly positive (resp. strictly negative) for $\bar y=\bar y_1$. Moreover, there exists sufficiently small $R_P>0$ such that for each fixed $\varpi\in]0,\varpi_0]$, $\bar B_2\in[-1+\mu_0,-\mu_0]$ (resp. $\bar B_2\in[\mu_0,1-\mu_0]$), $\epsilon\in]0,\epsilon_0]$, $\lambda\in\Lambda_1$ (resp. $\lambda\in\Lambda_2$) and $\hat{\hat y}\in[\bar y_1,\frac{U_P^2}{(\varpi\bar r)^2}]$ we have that $-\frac{\bar B_2}{\bar r}<\frac{f_P^2}{f_P^1}$ (resp. $-\frac{\bar B_2}{\bar r}>\frac{f_P^2}{f_P^1}$) for $\bar y\in[\frac{1}{R_P^2},\frac{U_P^2}{(\varpi\bar r)^2}]$, up to shrinking ϵ_0, ϖ_0 and U_P if necessary. Now we choose $\bar y_2>0$ such that $\frac{1}{R_P^2}<\bar y_2$. Recall that $\frac{\partial\bar\Delta_P}{\partial\bar y}\equiv\frac{\partial\bar\Delta_D}{\partial\bar y}$ for $\bar y\in[\frac{1}{R_P^2},\bar y_2]$ (hence I_D is strictly negative (resp. strictly positive) for $\bar y=\bar y_2$). The result after Lemma 6.4 now implies that $\bar I_D$ has precisely one simple zero on the interval $[\bar y_1,\bar y_2]$, for each fixed $\varpi\in]0,\varpi_0]$,

$\bar{B}_2 \in [-1+\mu_0, -\mu_0]$ (resp. $\bar{B}_2 \in [\mu_0, 1-\mu_0]$), $\epsilon \in]0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{(\varpi\bar{r})^2}]$, up to shrinking ϵ_0 and ϖ_0 if necessary.

It remains to prove the proposition (c). Take any small $\bar{y}_1 > 0$. On account of Proposition 6.3 there exist $U_P > 0$, $R_P > 0$, $\varpi_0 > 0$ and $\epsilon_0 > 0$ sufficiently small such that, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in]0, \epsilon_0]$, $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{r} > 0$, $\lambda \in \Lambda$ and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{(\varpi\bar{r})^2}]$, $\bar{I}_P$ has at most one simple zero on the interval $[\frac{1}{R_P^2}, \frac{U_P^2}{(\varpi\bar{r})^2}]$.

Let us recall that $\bar{r} > 0$. We observe that the quotient $\frac{\bar{B}_2}{\bar{r}} \to -\infty$ (resp. $\frac{\bar{B}_2}{\bar{r}} \to +\infty$) as $(\bar{B}_2, \bar{r}) \to (-1, 0)$ (resp. $(\bar{B}_2, \bar{r}) \to (1, 0)$). Taking the well known properties of the quotient $\frac{g_1}{g_0}$ into account, we see that there exist $\mu_0 > 0$ small enough and $\bar{y}_2 > 0$ large enough such that $\frac{1}{R_P^2} < \bar{y}_2$ and for each fixed $\varpi \in]0, \varpi_0]$, $\bar{B}_2 \in]-1, -1+\mu_0]$ (resp. $\bar{B}_2 \in [1-\mu_0, 1[$), $\epsilon \in]0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{(\varpi\bar{r})^2}]$ we have $-\frac{\bar{B}_2}{\bar{r}} > \frac{f_D^2}{f_D^1}$ (resp. $-\frac{\bar{B}_2}{\bar{r}} < \frac{f_D^2}{f_D^1}$) for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$, up to shrinking ϵ_0 and ϖ_0 if necessary. □

In each proposition in Theorem 6.5 at least one of the two functions I_D and I_P is nonzero on its domain. Hence, Theorem 6.5 implies that there exist $r_0 > 0$, $\epsilon_0 > 0$, $B_2^0 > 0$ and $U_P > 0$ small enough such that for each $r \in]0, r_0]$, $\epsilon \in]0, \epsilon_0]$, $B_2 \in [-B_2^0, B_2^0]$, $\lambda \in \Lambda$ and $\hat{\bar{y}} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$ the derivative $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has at most one zero (counting multiplicity) in the variable $\bar{y} \in [\bar{y}_1, \frac{U_P^2}{r^2}]$. Using Rolle's Theorem we obtain that the set $L_{00} \cup (\cup_{\bar{y} \in [\bar{y}_1, +\infty[} L_{\bar{y}})$ produces at most two limit cycles.

6.2. **Limit cycles near $\cup_{\bar{\mathbf{y}} \in [\bar{\mathbf{y}}_1, \frac{1}{2}]} \mathbf{L}_{\bar{\mathbf{y}}}$ in the saddle case.** We choose a small real number $\bar{y}_1$ such that $0 < \bar{y}_1 < \frac{1}{2}$. In this section we prove that the set $\cup_{\bar{y} \in [\bar{y}_1, \frac{1}{2}]} L_{\bar{y}}$ can produce at most two limit cycles (see Theorem 6.7). Our attention goes to a study of evolution of limit cycles in the $(\bar{x}, \bar{y})$-space born near the very delicate slow-fast two-saddle-limit periodic set $L_{\frac{1}{2}}$ as parameters of our system vary. Notations that we use in this section has nothing to do with the notations used in Section 6.1.

In [16], it has been shown that near $(\bar{x}, \bar{y}) = (\pm 1, \frac{1}{2})$ the family (8) (with the $+$-sign in front of $\bar{x}^3$ and $u = r$) has a persistent hyperbolic saddle which we denote here by $s_\pm$. Of course, $s_\pm$ depends on $\epsilon > 0$, $\epsilon \sim 0$, $r \sim 0$, $B_0 \sim 0$, $B_2 \sim 0$ and $\lambda \in \Lambda$. We define

$$\bar{y}_{\max}(r, \epsilon, B_0, B_2, \lambda) := \min\{\bar{y}_{unst}(r, \epsilon, B_0, B_2, \lambda), \bar{y}_{st}(r, \epsilon, B_0, B_2, \lambda)\}$$

where $\bar{y}_{unst}(r, \epsilon, B_0, B_2, \lambda)$ (resp. $\bar{y}_{st}(r, \epsilon, B_0, B_2, \lambda)$) represents the smooth (including $\epsilon = 0$) intersection of the unstable manifold at the point s_- (resp. the stable manifold at the point s_+), at the $(r, \epsilon, B_0, B_2, \lambda)$-level, and section $\{\bar{x} = 0\}$ parameterized by $\bar{y}$. It can be easily seen that $\bar{y}_{\max}(0, 0, B_0, 0, \lambda) = \frac{1}{2}$ ([16]).

By following the orbits of the system (8) in forward and backward time we define transition maps from $\Sigma_S \subset \{\bar{x} = 0\}$ to T_+, near the slow-fast two-saddle-limit periodic set $L_{\frac{1}{2}}$, which we denote here by respectively (F_S, ϵ) and (C_S, ϵ). Section T_+ is defined in Section 3 (see Figure 4) and parameterized by (h, ϵ). Section Σ_S is parametrized by $\bar{y} \in [\bar{y}_S, \bar{y}_{\max}(r, \epsilon, B_0, B_2, \lambda)]$ where we choose a sufficiently large $\bar{y}_S$ such that $\bar{y}_S < \frac{1}{2}$. The two maps F_S and C_S are smooth with C^k-extensions to $\epsilon = 0$ and $\bar{y} = \bar{y}_{\max}$ (see [16]). We define

$$h = \widetilde{\Delta}_S(\bar{y}, r, \epsilon, B_0, B_2, \lambda) := F_S(\bar{y}, r, \epsilon, B_0, B_2, \lambda) - C_S(\bar{y}, r, \epsilon, B_0, B_2, \lambda).$$

Following the orbits of (8) in forward and backward time we can define transition maps from $\Sigma_C \subset \{\bar{x}=0\}$ to T_+ near the set $\cup_{\bar{y}\in[\bar{y}_1,\bar{y}_2]} L_{\bar{y}}$, for any $\bar{y}_2$ such that $0<\bar{y}_1<\bar{y}_2<\frac{1}{2}$. We denote transition maps by respectively (F_C,ϵ) and (C_C,ϵ). Section Σ_C is parametrized by $\bar{y}\in[\bar{y}_1,\bar{y}_2]$. The two transition maps F_C and C_C are smooth, including $\epsilon=0$ (see [3]). We define

$$h=\widetilde{\Delta}_C(\bar{y},r,\epsilon,B_0,B_2,\lambda):=F_C(\bar{y},r,\epsilon,B_0,B_2,\lambda)-C_C(\bar{y},r,\epsilon,B_0,B_2,\lambda).$$

We may suppose that $\bar{y}_S<\bar{y}_2$.

As in Section 6.1 we can construct a single difference map using $\widetilde{\Delta}_S$ and $\widetilde{\Delta}_C$. We will write it $\widetilde{\Delta}_G(\bar{y},r,\epsilon,B_0,B_2,\lambda)$ where $r\sim 0$, $r>0$, $\epsilon\sim 0$, $\epsilon>0$, $B_0\sim 0$, $B_2\sim 0$, $\lambda\in\Lambda$ and $\bar{y}\in[\bar{y}_1,\bar{y}_{\max}(r,\epsilon,B_0,B_2,\lambda)]$. $\widetilde{\Delta}_G$ is equal to $\widetilde{\Delta}_C$ on $[\bar{y}_1,\bar{y}_2]$ and equal to $\widetilde{\Delta}_S$ on Σ_S. *Clearly, the number of isolated zeros of* $\widetilde{\Delta}_G$ *on* $[\bar{y}_1,\bar{y}_{\max}(r,\epsilon,B_0,B_2,\lambda)[$ *has to be studied.*

First, we want to eliminate the breaking parameter B_0. [16] implies that there is a C^k-function $\widetilde{B}_S(\bar{y},r,\epsilon,B_2,\lambda)$ such that solutions of $\widetilde{\Delta}_S(\bar{y},r,\epsilon,B_0,B_2,\lambda)=0$, for $\bar{y}\sim\frac{1}{2}$, $r\sim 0$, $\epsilon\sim 0$, $B_0\sim 0$, $B_2\sim 0$ and $\lambda\in\Lambda$, can only be possible for $B_0=\widetilde{B}_S(\bar{y},r,\epsilon,B_2,\lambda)$. $\widetilde{B}_S$ is identically zero when $B_2=r=0$. In [16], it has been proved that solutions of $\widetilde{\Delta}_C(\bar{y},r,\epsilon,B_0,B_2,\lambda)=0$, for $\epsilon\sim 0$, $B_0\sim 0$ and $\bar{y}\in[\bar{y}_1,\bar{y}_2]$, can only occur for $B_0=\widetilde{B}_C(\bar{y},r,\epsilon,B_2,\lambda)$ where $\widetilde{B}_C(\bar{y},r,\epsilon,B_2,\lambda)$ is a C^∞-function. $\widetilde{B}_C$ is identically zero when $B_2=r=0$. We consider a global function $\widetilde{B}_G$ that is equal to $\widetilde{B}_C$ on $[\bar{y}_1,\bar{y}_2]$ and equal to $\widetilde{B}_S$ for $\bar{y}\geq\bar{y}_S$ and $\bar{y}\sim\frac{1}{2}$. Clearly, roots of $\widetilde{\Delta}_G(\bar{y},r,\epsilon,B_0,B_2,\lambda)=0$ can only be possible for $B_0=\widetilde{B}_G(\bar{y},r,\epsilon,B_2,\lambda)$. *Thus, we study zeros on* $[\bar{y}_1,\bar{y}_{\max}(r,\epsilon,\widetilde{B}_G(c,r,\epsilon,B_2,\lambda),B_2,\lambda)[=:\Sigma_G$ *of:*

$$\bar{\Delta}_G(\bar{y},r,\epsilon,B_2,\lambda,c):=\widetilde{\Delta}_G(\bar{y},r,\epsilon,\widetilde{B}_G(c,r,\epsilon,B_2,\lambda),B_2,\lambda), \tag{53}$$

for each $r\sim 0$, $r>0$, $\epsilon\sim 0$, $\epsilon>0$, $B_2\sim 0$, $\lambda\in\Lambda$ *and* $c\in[\bar{y}_1,c_0]$ *where* c_0 *is a fixed real number sufficiently close to* $\frac{1}{2}$ *and* $c_0>\frac{1}{2}$.

We can write $\widetilde{B}_G(c,r,\epsilon,B_2,\lambda)$, for $c\in[\bar{y}_1,c_0]$, as

$$\widetilde{B}_G(c,r,\epsilon,B_2,\lambda)=B_2\widetilde{b}_G^1(c,r,\epsilon,B_2,\lambda)+r\widetilde{b}_G^2(c,r,\epsilon,B_2,\lambda) \tag{54}$$

where $\widetilde{b}_G^1$ and $\widetilde{b}_G^2$ are C^k-functions in $(c,r,\epsilon,B_2,\lambda)$.

We study $\frac{\partial\bar{\Delta}_G}{\partial\bar{y}}$ on $[\bar{y}_1,\bar{y}_2]$ and

$$[\bar{y}_S,\bar{y}_{\max}(r,\epsilon,\widetilde{B}_G(c,r,\epsilon,B_2,\lambda),B_2,\lambda)[=:\Sigma_S^c.$$

1. *On the segment* $[\bar{y}_1,\bar{y}_2]$. Let us define

$$\bar{\Delta}_C(\bar{y},r,\epsilon,B_2,\lambda,c):=\widetilde{\Delta}_C(\bar{y},r,\epsilon,\widetilde{B}_G(c,r,\epsilon,B_2,\lambda),B_2,\lambda),$$

where $\bar{y}\in[\bar{y}_1,\bar{y}_2]$ and $c\in[\bar{y}_1,c_0]$. It is clear that $\bar{\Delta}_G\equiv\bar{\Delta}_C$ for $\bar{y}\in[\bar{y}_1,\bar{y}_2]$. Using (54) and [16], we have the following expression for the derivative $\frac{\partial\bar{\Delta}_C}{\partial\bar{y}}$:

$$\begin{aligned}\frac{\partial\bar{\Delta}_C}{\partial\bar{y}}&=K_C.\Big(B_2\big(\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}}\frac{-\varrho^2d\varrho}{(1-\varrho^2)^2}+\bar{\rho}_C^1\big)+r\big(\bar{H}(0,\lambda)\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}}\frac{-\varrho^4d\varrho}{(1-\varrho^2)^2}+\bar{\rho}_C^2\big)\Big)\\&=:K_C.(B_2f_C^1+rf_C^2)=:K_C.I_C\end{aligned} \tag{55}$$

where $\bar{\rho}_C^1$ and $\bar{\rho}_C^2$ are C^k-functions in $(\bar{y},r,\epsilon,\epsilon^2\ln\epsilon,B_2,\lambda,c)$, identically equal to zero when $B_2=r=\epsilon=0$, and where K_C is a C^k-function in variable $(\bar{y},r,\epsilon,B_2,\lambda,c)$ and strictly negative for $\epsilon>0$.

Lemma 6.6. *For any* $0 < \bar{y}_1 < \bar{y}_2 < \frac{1}{2}$, $\{\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^2 d\varrho}{(1-\varrho^2)^2}, \bar{H}(0,\lambda)\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^4 d\varrho}{(1-\varrho^2)^2}\}$ *is a ST-system on* $[\bar{y}_1, \bar{y}_2]$*, for all* $\lambda \in \Lambda$.

Proof. Let us write $f_0 = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^2 d\varrho}{(1-\varrho^2)^2}$ and $f_1 = \bar{H}(0,\lambda)\int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{\varrho^4 d\varrho}{(1-\varrho^2)^2}$. It is clear that $f_0(\bar{y}) > 0$ for all $y \in [\bar{y}_1, \bar{y}_2]$. We have

$$f_0(\bar{y}) = \frac{\sqrt{2\bar{y}}}{1-2\bar{y}} + \frac{1}{2}\ln\frac{1-\sqrt{2\bar{y}}}{1+\sqrt{2\bar{y}}} \tag{56}$$

and

$$f_1(\bar{y}) = \bar{H}(0,\lambda)\left(\frac{\sqrt{2\bar{y}}(-3+4\bar{y})}{2\bar{y}-1} + \frac{3}{2}\ln\frac{1-\sqrt{2\bar{y}}}{1+\sqrt{2\bar{y}}}\right). \tag{57}$$

Combining (56) and (57) we get

$$(\frac{f_1}{f_0})'(\bar{y}) = \bar{H}(0,\lambda)\frac{4\sqrt{\bar{y}}\big(-12\sqrt{\bar{y}} + \sqrt{2}(-3+2\bar{y})\ln\frac{1-\sqrt{2\bar{y}}}{1+\sqrt{2\bar{y}}}\big)}{\big(2\sqrt{2\bar{y}} + (1-2\bar{y})\ln\frac{1-\sqrt{2\bar{y}}}{1+\sqrt{2\bar{y}}})\big)^2}.$$

Since $\bar{H}(0,\lambda) \neq 0$ for all $\lambda \in \Lambda$, it suffices to prove that $-12\sqrt{\bar{y}} + \sqrt{2}(-3+2\bar{y})\ln\frac{1-\sqrt{2\bar{y}}}{1+\sqrt{2\bar{y}}} > 0$ for all $\bar{y} \in]0, \frac{1}{2}[$. Using the change of variable $z = \sqrt{2\bar{y}}$, it suffices to prove that $f_2(z) := -6z + (-3+z^2)\ln\frac{1-z}{1+z} > 0$ for all $z \in]0,1[$. Since $f_2(0) = 0$, it suffices to prove that $f_2'(z) > 0$ for all $z \in]0,1[$. $f_2'(z) > 0$ for all $z \in]0,1[$ if and only if $f_3(z) := 2z + (1-z^2)\ln\frac{1-z}{1+z} > 0$ for all $z \in]0,1[$. The function f_3 is strictly positive on $]0,1[$ because $f_3(0) = 0$ and $f_3'(z) = 2z\ln\frac{1+z}{1-z} > 0$ for all $z \in]0,1[$. □

Using Lemma 6.6, we obtain that $\frac{\partial\bar{\Delta}_C}{\partial\bar{y}}$ has at most one zero (counting multiplicity) on $[\bar{y}_1, \bar{y}_2]$, for each $\epsilon \sim 0$, $\epsilon > 0$, $r \sim 0$, $r > 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $c \in [\bar{y}_1, c_0]$.

Let us write $g_0(\bar{y}) = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{-\varrho^2 d\varrho}{(1-\varrho^2)^2}$ and $g_1(\bar{y}) = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{-\varrho^4 d\varrho}{(1-\varrho^2)^2}$. Using (56) and (57) we get

$$g_0(\bar{y}) = (\sqrt{\bar{y}})^3(-\tfrac{4\sqrt{2}}{3} + O((\sqrt{\bar{y}})^2)),\ g_1(\bar{y}) = (\sqrt{\bar{y}})^5(-\tfrac{8\sqrt{2}}{5} + O((\sqrt{\bar{y}})^2))$$

and $\lim_{\bar{y}\to\frac{1}{2}^-} \frac{g_1(\bar{y})}{g_0(\bar{y})} = 1$. Combining this with the proof of Lemma 6.6 we obtain that the quotient $\frac{g_1}{g_0}$ is a (strictly) increasing analytic diffeomorphism on $[0, \frac{1}{2}[$, $\lim_{\bar{y}\to 0^+} \frac{g_1(\bar{y})}{g_0(\bar{y})} = 0$ and $\lim_{\bar{y}\to\frac{1}{2}^-} \frac{g_1(\bar{y})}{g_0(\bar{y})} = 1$.

2. *On* Σ_S^c. Let us write

$$\bar{\Delta}_S(\bar{y}, r, \epsilon, B_2, \lambda, c) = \widetilde{\Delta}_S(\bar{y}, r, \epsilon, \widetilde{B}_G(c, r, \epsilon, B_2, \lambda), B_2, \lambda),$$

where $\bar{y} \in \Sigma_S^c$ and $c \in [\bar{y}_1, c_0]$. It is clear that $\bar{\Delta}_G \equiv \bar{\Delta}_S$ for $\bar{y} \in \Sigma_S^c$. In [16], it has been proven that the derivative $\frac{\partial\bar{\Delta}_S}{\partial\bar{y}}$ can be written as:

$$\frac{\partial\bar{\Delta}_S}{\partial\bar{y}}(\bar{y}, r, \epsilon, B_2, \lambda, c) = K_S(\bar{y}, r, \epsilon, B_2, \lambda, c).I_S(\bar{y}, r, \epsilon, B_2, \lambda, c)$$

where K_S is a C^k-function in $(\bar{y}, r, \epsilon, B_2, \lambda, c)$, strictly negative for $\epsilon > 0$ and $O(\epsilon^m)$ for m arbitrarily large, and where

$$I_S(\bar{y}, r, \epsilon, B_2, \lambda, c) = m(r, \epsilon, B_2, \lambda, c).\Big(\ln\varsigma_+ - \big(1 + B_2(1 + O_1(B_2, r, \epsilon)) + r(3\bar{H}(0,\lambda) + O_2(B_2, r, \epsilon))\big)\ln\varsigma_- + O_3(B_2, r)\Big). \tag{58}$$

In the expression (58), we suppose that m is a strictly positive C^k- function, O_1 and O_2 are C^k- functions not depending on $\bar{y}$, O_3 is an ϵ-regularly C^k-function, i.e. all its derivatives w.r.t. $(\bar{y}, r, B_2, \lambda, c)$ are continuous including $\epsilon = 0$, ς_+ and ς_- are C^k-functions, $\varsigma_\pm = O(\bar{y} - \frac{1}{2}, B_2, r, \epsilon)$, $\frac{\partial \varsigma_\pm}{\partial \bar{y}} = -1 + O(\bar{y} - \frac{1}{2}, B_2, r, \epsilon)$ and $\varsigma_+ - \varsigma_- = B_2(-1 + O(\bar{y} - \frac{1}{2}, B_2, r, \epsilon)) + r(-\bar{H}(0, \lambda) + O(\bar{y} - \frac{1}{2}, B_2, r, \epsilon))$. It has been shown in [16] that for each fixed $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon \geq 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $c \in [\bar{y}_1, c_0]$ the function I_S has at most one zero (counting multiplicity) on Σ_S^c. Hence, the derivative $\frac{\partial \bar{\Delta}_S}{\partial \bar{y}}$ has at most one zero (counting multiplicity) on Σ_S^c, for each fixed $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $c \in [\bar{y}_1, c_0]$.

Putting all informations together, we obtain that, for each $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $c \in [\bar{y}_1, c_0]$, the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}(\bar{y}, r, \epsilon, B_2, \lambda, c)$ has at most two (simple) zeros on Σ_G. This result can be improved. As a simple consequence of Theorem 6.7, we have that the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has at most one zero (counting multiplicity) on Σ_G, under a similar condition on the parameters. Rolle's Theorem will then imply that the set $\cup_{\bar{y} \in [\bar{y}_1, \frac{1}{2}]} L_{\bar{y}}$ produces at most two limit cycles.

Like in the elliptic case, we give the intuition behind the result stated below as Theorem 6.7. We use the rescaling in the (B_2, r)-parameter space, $r > 0$, and the decomposition of Λ into the disjoint union $\Lambda = \Lambda_1 \cup \Lambda_2$ defined in Section 6.1. Suppose that $\lambda \in \Lambda_1$, i.e. $\bar{H}(0, \lambda) > 0$. When $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$ is between $(1, 0)$ and $(0, 1)$, then the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has no zeros on Σ_G. Hence, we have at most one limit cycle near $\cup_{\bar{y} \in [\bar{y}_1, \frac{1}{2}]} L_{\bar{y}}$. When $(\bar{B}_2, \bar{r}) \sim (0, 1)$, then again the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has no zeros on Σ_G, and two limit cycles may occur near $(\bar{x}, \bar{y}) = (0, 0)$, created in a Hopf bifurcation of codimension 2 at $B_0 = 0$. When $(\bar{B}_2, \bar{r})$ is between $(0, 1)$ and $(-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}}, \frac{1}{\sqrt{1+\bar{H}(0,\lambda)^2}})$, then the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has no zeros on Σ_S^c and has precisely one (simple) zero on $[\bar{y}_1, \bar{y}_2]$. Hence, a saddle-node bifurcation of limit cycles occurs near $\cup_{\bar{y} \in [\bar{y}_1, \bar{y}_2]} L_{\bar{y}}$ when B_0 varies (see [7]). When $(\bar{B}_2, \bar{r}) \sim (-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}}, \frac{1}{\sqrt{1+\bar{H}(0,\lambda)^2}})$, then the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has no zeros on $[\bar{y}_1, \bar{y}_2]$ and has one (simple) zero on Σ_S^c. This simple zero corresponds to two limit cycles that appear near the slow-fast two-saddle-limit periodic set $L_{\frac{1}{2}}$, for an appropriate value of B_0. When $(\bar{B}_2, \bar{r})$ is between $(-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}}, \frac{1}{\sqrt{1+\bar{H}(0,\lambda)^2}})$ and $(-1, 0)$, then the function $\frac{\partial \bar{\Delta}_G}{\partial \bar{y}}$ has no zeros on Σ_G.

While the proof of Theorem 6.5 is based on the study of divergence integrals I_D and I_P (hence, we do not need to consider derivatives of I_D and I_P w.r.t. $\bar{y}$), in the proof of Theorem 6.7 we use the divergence integral I_C and the derivative $\frac{\partial I_S}{\partial \bar{y}}$ due to the fact that the expression (58) for I_S is not suitable for the study of evolution of a simple zero in I_S as the parameter $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$ varies.

Theorem 6.7. *The following propositions are true:*
(a) Take any small $\bar{y}_1 > 0$. There exist sufficiently small $\mu_0 > 0$, $\varpi_0 > 0$ and $\epsilon_0 > 0$ and sufficiently large $\bar{y}_S < \frac{1}{2}$ and $\bar{y}_2 < \frac{1}{2}$ such that $\bar{y}_S < \bar{y}_2$ and for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in]0, \epsilon_0]$, $\bar{B}_2 \in [-\mu_0, 1]$ (resp. $\bar{B}_2 \in [-1, \mu_0]$), $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $c \in [\bar{y}_1, c_0]$ we have that $I_C < 0$ (resp. $I_C > 0$) for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$, and $I_S < 0$ (resp. $I_S > 0$) for $\bar{y} \in \Sigma_S^c$.
(b) Take any small $\mu_0 > 0$ and $\bar{y}_1^0 > 0$. Then there exist sufficiently small $\bar{y}_1 > 0$, $\varpi_0 > 0$ and $\epsilon_0 > 0$ and sufficiently large $\bar{y}_S < \frac{1}{2}$ and $\bar{y}_2 < \frac{1}{2}$ such that $\bar{y}_S < \bar{y}_2$,

$\bar{y}_1 < \bar{y}_1^0$ *and for each fixed* $\varpi \in]0, \varpi_0]$, $\epsilon \in]0, \epsilon_0]$, $\lambda \in \Lambda_1$ *(resp.* $\lambda \in \Lambda_2$*)*, $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ *(resp.* $\bar{B}_2 \in [\mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$*) and* $c \in [\bar{y}_1, c_0]$ I_C *has precisely one zero (counting multiplicity) on the interval* $[\bar{y}_1, \bar{y}_2]$, I_C *is strictly positive (resp. strictly negative) at* $\bar{y} = \bar{y}_1$ *and* $I_S < 0$ *(resp.* $I_S > 0$*) for* $\bar{y} \in \Sigma_S^c$.
(c) Take any small $\bar{y}_1 > 0$. *There exist sufficiently small* $\mu_0 > 0$, $\varpi_0 > 0$ *and* $\epsilon_0 > 0$ *and sufficiently large* $\bar{y}_S < \frac{1}{2}$ *and* $\bar{y}_2 < \frac{1}{2}$ *such that* $\bar{y}_S < \bar{y}_2$ *and for each fixed* $\varpi \in]0, \varpi_0]$, $\epsilon \in]0, \epsilon_0]$, $\lambda \in \Lambda_1$ *(resp.* $\lambda \in \Lambda_2$*)*, $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0]$ *and* $c \in [\bar{y}_1, c_0]$ $I_C > 0$ *(resp.* $I_C < 0$*) for* $\bar{y} \in [\bar{y}_1, \bar{y}_2]$, *and* I_S *has at most one zero (counting multiplicity) on the interval* Σ_S^c.
(d) Take any small $\mu_0 > 0$ *and* $\bar{y}_1 > 0$. *There exist sufficiently small* $\varpi_0 > 0$ *and* $\epsilon_0 > 0$ *and sufficiently large* $\bar{y}_S < \frac{1}{2}$ *and* $\bar{y}_2 < \frac{1}{2}$ *such that* $\bar{y}_S < \bar{y}_2$ *and for each fixed* $\varpi \in]0, \varpi_0]$, $\epsilon \in]0, \epsilon_0]$, $\lambda \in \Lambda_1$ *(resp.* $\lambda \in \Lambda_2$*)*, $\bar{B}_2 \in [-1, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$ *(resp.* $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, 1]$*) and* $c \in [\bar{y}_1, c_0]$ *we have that* $I_C > 0$ *(resp.* $I_C < 0$*) for* $\bar{y} \in [\bar{y}_1, \bar{y}_2]$, *and* $I_S > 0$ *(resp.* $I_S < 0$*) for* $\bar{y} \in \Sigma_S^c$.

Proof. Using (58) it can be easily seen (see [16]) that

$$\frac{\partial I_S}{\partial \bar{y}} = m.\varpi.\Big(\frac{-\bar{B}_2 - \bar{H}(0,\lambda)\bar{r} + O(\bar{y} - \frac{1}{2}, \varpi, \epsilon)}{\varsigma_+\varsigma_-} + O(1)\Big). \tag{59}$$

Let us prove the proposition (a). Take any small $\bar{y}_1 > 0$. Since $\bar{H}(0,\lambda) \neq 0$ for all $\lambda \in \Lambda$, we can find a sufficiently small $\mu_0 > 0$ such that $\bar{B}_2 + \bar{H}(0,\lambda)\bar{r} > 0$ for all $\bar{B}_2 \in [-\mu_0, 1]$, $\bar{r} = \sqrt{1 - \bar{B}_2^2}$ and $\lambda \in \Lambda_1$, and $\bar{B}_2 + \bar{H}(0,\lambda)\bar{r} < 0$ for all $\bar{B}_2 \in [-1, \mu_0]$, $\bar{r} = \sqrt{1 - \bar{B}_2^2}$ and $\lambda \in \Lambda_2$. As a simple consequence of (59), we can find a sufficiently large $\bar{y}_S < \frac{1}{2}$ and sufficiently small $\varpi_0 > 0$ and $\epsilon_0 > 0$ such that, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $\bar{B}_2 \in [-\mu_0, 1]$ (resp. $B_2 \in [-1, \mu_0]$), $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $c \in [\bar{y}_1, c_0]$, $\frac{\partial I_S}{\partial \bar{y}} < 0$ (resp. $\frac{\partial I_S}{\partial \bar{y}} > 0$) for all $\bar{y} \in \Sigma_S^c$.

Let us fix $\bar{y}_2 \in]\bar{y}_S, \frac{1}{2}[$. Using (55) we see that $I_C < 0$ (resp. $I_C > 0$) on the interval $[\bar{y}_1, \bar{y}_2]$, for each $\bar{B}_2 \in [0, 1]$ (resp. $\bar{B}_2 \in [-1, 0]$), $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $c \in [\bar{y}_1, c_0]$. Let us recall that the quotient $\frac{\bar{B}_2}{\bar{r}} \to 0$ as $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$ goes to $(0, 1)$, the function $\frac{g_1}{g_0}$ is strictly increasing on $]0, \frac{1}{2}[$ and $\frac{g_1(\bar{y}_1)}{g_0(\bar{y}_1)} > 0$. From this, it follows that $-\frac{\bar{B}_2}{\bar{r}} < \frac{f_C^2}{f_C^1}$, i.e. $I_C < 0$ (resp. $-\frac{\bar{B}_2}{\bar{r}} > \frac{f_C^2}{f_C^1}$, i.e. $I_C > 0$), on the interval $[\bar{y}_1, \bar{y}_2]$, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $\bar{B}_2 \in [-\mu_0, 0]$ (resp. $\bar{B}_2 \in [0, \mu_0]$), $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $c \in [\bar{y}_1, c_0]$, up to shrinking μ_0, ϖ_0 and ϵ_0 if necessary. Now, as a consequence of the fact that $\frac{\partial \bar{\Delta}_C}{\partial \bar{y}} \equiv \frac{\partial \bar{\Delta}_S}{\partial \bar{y}}$ for $\bar{y} \in [\bar{y}_S, \bar{y}_2]$, we have that $I_S(\bar{y}_S) < 0$ and $\frac{\partial I_S}{\partial \bar{y}} < 0$ (resp. $I_S(\bar{y}_S) > 0$ and $\frac{\partial I_S}{\partial \bar{y}} > 0$) for all $\bar{y} \in \Sigma_S^c$, for each fixed $\epsilon \sim 0$, $\epsilon > 0$, $\bar{B}_2 \in [-\mu_0, 1]$ and $\lambda \in \Lambda_1$ (resp. $\bar{B}_2 \in [-1, \mu_0]$ and $\lambda \in \Lambda_2$). From this, it follows that $I_S < 0$ (resp. $I_S > 0$) on Σ_S^c, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in]0, \epsilon_0]$, $\bar{B}_2 \in [-\mu_0, 1]$ (resp. $\bar{B}_2 \in [-1, \mu_0]$), $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $c \in [\bar{y}_1, c_0]$.

Let us prove the proposition (b). Take any small $\mu_0 > 0$ and $\bar{y}_1^0 > 0$. It follows that $-\frac{\bar{B}_2}{\bar{r}} \geq \frac{\mu_0}{\sqrt{1-\mu_0^2}}$ (resp. $-\frac{\bar{B}_2}{\bar{r}} \leq -\frac{\mu_0}{\sqrt{1-\mu_0^2}}$) for all $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ (resp. $\bar{B}_2 \in [\mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$), $\bar{r} = \sqrt{1 - \bar{B}_2^2}$ and $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$). Since $\lim_{\bar{y} \to 0^+} \frac{g_1(\bar{y})}{g_0(\bar{y})} = 0$, there exist sufficiently small $\bar{y}_1 > 0$, $\varpi_0 > 0$ and $\epsilon_0 > 0$

such that $\bar{y}_1 < \bar{y}_1^0$ and, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$), $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ (resp. $\bar{B}_2 \in [\mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$) and $c \in [\bar{y}_1, c_0]$, I_C is strictly positive (resp. strictly negative) at $\bar{y} = \bar{y}_1$.

We have that $\bar{B}_2 + \bar{H}(0,\lambda)\bar{r} \geq \mu_0$ (resp. $\bar{B}_2 + \bar{H}(0,\lambda)\bar{r} \leq -\mu_0$) for all $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ (resp. $\bar{B}_2 \in [\mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$) and $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$). Using (59) we find a sufficiently large $\bar{y}_S < \frac{1}{2}$ such that, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$), $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ (resp. $\bar{B}_2 \in [\mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$) and $c \in [\bar{y}_1, c_0]$, $\frac{\partial I_S}{\partial \bar{y}} < 0$ (resp. $\frac{\partial I_S}{\partial \bar{y}} > 0$) for all $\bar{y} \in \Sigma_S^c$, up to shrinking $\varpi_0 > 0$ and $\epsilon_0 > 0$ if necessary.

Since $\lim_{\bar{y} \to \frac{1}{2}^-} \bar{H}(0,\lambda) \frac{g_1(\bar{y})}{g_0(\bar{y})} = \bar{H}(0,\lambda)$ for all $\lambda \in \Lambda$ and $-\frac{\bar{B}_2}{\bar{r}} \leq \bar{H}(0,\lambda) - \frac{\mu_0}{\bar{r}}$ (resp. $-\frac{\bar{B}_2}{\bar{r}} \geq \bar{H}(0,\lambda) + \frac{\mu_0}{\bar{r}}$) for all $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ (resp. $\bar{B}_2 \in [\mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$), $\bar{r} = \sqrt{1 - \bar{B}_2^2}$ and $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$), we can choose $\bar{y}_2$ such that $\bar{y}_S < \bar{y}_2 < \frac{1}{2}$ and $-\frac{\bar{B}_2}{\bar{r}} < \frac{f_C^2}{f_C^1}$ (resp. $-\frac{\bar{B}_2}{\bar{r}} > \frac{f_C^2}{f_C^1}$) for $\bar{y} = \bar{y}_2$, $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$), $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ (resp. $\bar{B}_2 \in [\mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$) and $c \in [\bar{y}_1, c_0]$, up to shrinking ϖ_0 and ϵ_0 if necessary. Hence, by taking $\bar{y}_S$ sufficiently close to $\bar{y}_2$, we have $I_S < 0$ (resp. $I_S > 0$) on Σ_S^c, under the same conditions on the parameters and $\epsilon > 0$, due to the above result on $\frac{\partial I_S}{\partial \bar{y}}$ (see the proof of the prooposition (a)).

Using Lemma 6.6 and putting all informations together, we have that I_C has precisely one simple zero on the interval $[\bar{y}_1, \bar{y}_2]$, for each $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{r} \geq 0$, $\lambda \in \Lambda$ and $c \in [\bar{y}_1, c_0]$, up to shrinking ϖ_0 and ϵ_0 if necessary.

Let us prove the proposition (c). Take a small $\bar{y}_1 > 0$. As mentioned above, by [16], we can find sufficiently small $\varpi_0 > 0$ and $\epsilon_0 > 0$ and a sufficiently large $\bar{y}_S < \frac{1}{2}$ such that, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{r} \geq 0$, $\lambda \in \Lambda$ and $c \in [\bar{y}_1, c_0]$, the function I_S has at most one zero (counting multiplicity) on Σ_S^c. Fix $\bar{y}_2 \in]\bar{y}_S, \frac{1}{2}[$. When $\bar{B}_2 \sim -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}}$, then $-\frac{\bar{B}_2}{\bar{r}} \sim \bar{H}(0,\lambda)$. Combining this with the fact that the function $\frac{g_1}{g_0}$ is strictly positive and increasing on $]0, \frac{1}{2}[$ with the property that $\lim_{\bar{y} \to \frac{1}{2}^-} \bar{H}(0,\lambda) \frac{g_1(\bar{y})}{g_0(\bar{y})} = \bar{H}(0,\lambda)$, we observe that there exists sufficiently small $\mu_0 > 0$ such that, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$), $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0]$ and $c \in [\bar{y}_1, c_0]$, $-\frac{\bar{B}_2}{\bar{r}} > \frac{f_C^2}{f_C^1}$ for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ (resp. $-\frac{\bar{B}_2}{\bar{r}} < \frac{f_C^2}{f_C^1}$ for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$), up to shrinking ϖ_0 and ϵ_0 if necessary.

Let us prove the proposition (d). Take any small $\mu_0 > 0$ and $\bar{y}_1 > 0$. It follows that $\bar{B}_2 + \bar{H}(0,\lambda)\bar{r} < 0$ (resp. $\bar{B}_2 + \bar{H}(0,\lambda)\bar{r} > 0$) for all $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$) and $\bar{B}_2 \in [-1, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$ (resp. $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, 1]$). Using (59) we find a sufficiently large $\bar{y}_S < \frac{1}{2}$ and sufficiently small $\varpi_0 > 0$ and $\epsilon_0 > 0$ such that, for each fixed $\varpi \in]0, \varpi_0]$, $\epsilon \in [0, \epsilon_0]$, $\lambda \in \Lambda_1$ (resp. $\lambda \in \Lambda_2$), $\bar{B}_2 \in [-1, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$ (resp. $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, 1]$) and $c \in [\bar{y}_1, c_0]$, $\frac{\partial I_S}{\partial \bar{y}} > 0$ (resp. $\frac{\partial I_S}{\partial \bar{y}} < 0$) for all $\bar{y} \in \Sigma_S^c$.

Fix any $\bar{y}_2 \in]\bar{y}_S, \frac{1}{2}[$. Again we use the above mentioned property of the quotient $\frac{g_1}{g_0}$ and see that $I_C > 0$ (resp. $I_C < 0$) on the interval $[\bar{y}_1, \bar{y}_2]$, under the same conditions on the parameters, up shrinking ϖ_0 and ϵ_0 if necessary. For the rest of the proof we refer to the proof of the proposition (a). □

7. **Cyclicity of $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$ in the elliptic and saddle cases.** We consider smooth families $X^{\pm}_{\epsilon,r,B_0,B_2,\lambda}$ defined in (6) near the origin $(x, y) = (0, 0)$, for $\epsilon \sim 0$, $\epsilon > 0$, $r \sim 0$, $r > 0$, $B_0 \sim 0$, $B_2 \sim 0$ and $\lambda \in \Lambda$. *In this section we suppose that $\bar{H}(0, \lambda) \neq 0$ for all $\lambda \in \Lambda$, and we will prove that the systems $X^{\pm}_{\epsilon,r,B_0,B_2,\lambda}$ have a cyclicity of $(x, y) = (0, 0)$ equal to* 2. This result and Theorem 2.1 will then imply Theorem 2.2.

In Section 7.1, a complete study of the elliptic case will be given. We will use results obtained in Section 5 and the properties of $\frac{\partial}{\partial \bar{y}} \bar{\Delta}_G$ on segment $[\bar{y}_1, \frac{U_P^2}{r^2}]$ obtained in Theorem 6.5.

In Section 7.2, we will study the saddle case. As the study of the saddle case is similar to the study of the elliptic case, in Section 7.2 we will not enter into all details, but we will rather show how to adapt the arguments used in the elliptic case.

In [16] we proved that the cyclicity of $(x, y) = (0, 0)$ is at least 2 in both the elliptic and saddle cases.

Let us write $J_i(h) = \int_{L_h} e^{-\widetilde{y}} \widetilde{x}^{2i} d\widetilde{x}$, for each $i \geq 0$ and $h \in]0, 1[$. The following lemma plays an important role in both the elliptic and saddle cases.

Lemma 7.1. *We have* $\lim_{h \to 0} \frac{J_1(h)}{J_0(h)} = \frac{J_1(0)}{J_0(0)} = 1$ *and* $\lim_{h \to 0} \frac{J_2(h)}{J_0(h)} = \frac{J_2(0)}{J_0(0)} = 3$.

Proof. Knowing that

$$J_i(0) = e \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\widetilde{x}^2} \widetilde{x}^{2i} d\widetilde{x}, \text{ for } i \geq 0,$$

we get $J_1(0) = J_0(0)$ and $J_2(0) = 3J_0(0)$, by integrating by parts. □

System (10) has a singularity $(\bar{x}, \bar{y}) = (\epsilon \widetilde{x}_{\epsilon,\tau}, \epsilon^2 \widetilde{y}_{\epsilon,\tau})$ with eigenvalues

$$\epsilon \frac{-\widetilde{x}_{\epsilon,\tau} \pm \sqrt{\widetilde{x}^2_{\epsilon,\tau} - 4(1 + O(\epsilon))}}{2},$$

where $p_{\epsilon,\tau} = (\widetilde{x}_{\epsilon,\tau}, \widetilde{y}_{\epsilon,\tau})$ is introduced in Section 4. As $\widetilde{x}_{\epsilon,\tau} = B_0(1 + O(\epsilon))$, $(\epsilon \widetilde{x}_{\epsilon,\tau}, \epsilon^2 \widetilde{y}_{\epsilon,\tau})$ is a hyperbolically stable focus (respectively a hyperbolically unstable focus) for $\epsilon > 0$ and $B_0 > 0$ (respectively $B_0 < 0$). We refer to Figure 6.

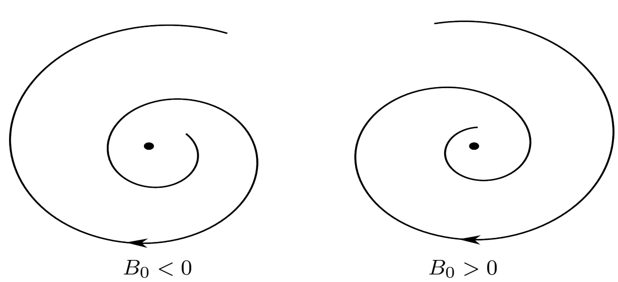

FIGURE 6. The dynamics of (10) near $(\bar{x}, \bar{y}) = (\epsilon \widetilde{x}_{\epsilon,\tau}, \epsilon^2 \widetilde{y}_{\epsilon,\tau})$, for $\epsilon > 0$.

In Section 5, we have proved that there exists $\bar{y}_{00} > 0$ small enough such that the function $\hat{D}^G_\tau$ and its derivative $\frac{\partial}{\partial \bar{y}} \hat{D}^G_\tau$ have at most two zeros (counting multiplicity) on the interval $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{00}]$, for each $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $r = u \sim 0$, $(B_0, B_2, r) \neq (0,0,0)$, $\lambda \in \Lambda$ and $B_3 \in \mathcal{B}$. Hence, $\bar{y}_{00}$ is independent of (ϵ, τ). In this section we suppose that $B_3 \in \{-1, 1\}$.

We know that $\hat{D}^G_\tau(\epsilon^2\widetilde{y}_{\epsilon,\tau}, \epsilon) = 0$, for $\epsilon > 0$ and $\epsilon \sim 0$. Suppose that $\bar{y}_{SM} \in]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{00}]$ is the smallest value such that $\hat{D}^G_\tau(\bar{y}_{SM}, \epsilon) = 0$. Since the section $\mathcal{T}_\tau$ defined in Section 5 is parameterized by h and h increases as we approach the singularity $(\widetilde{x}, \widetilde{y}) = p_{\epsilon,\tau}$, Figure 5 and Figure 6 imply that $\hat{D}^G_\tau(\bar{y}, \epsilon) > 0$ (respectively $\hat{D}^G_\tau(\bar{y}, \epsilon) < 0$) for all $\bar{y} \in]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{SM}[$ when $B_0 > 0$ (respectively $B_0 < 0$).

7.1. **Study of the elliptic case.** We keep in mind the notation used in Section 5 and Section 6.1. Suppose that $\tau = (B_0, B_2, -1, r, \lambda)$.

For $\epsilon > 0$ and $r > 0$, on $[\epsilon^2\widetilde{y}_{\epsilon,\tau}, \delta_0^2] \cap [\bar{y}_1, \frac{U_P^2}{r^2}] = [\bar{y}_1, \delta_0^2]$ $\hat{D}^G_\tau$ and $\widetilde{\Delta}_G$, defined respectively in Section 5 and Section 6.1, coincide ($\bar{y}_1 > 0$ can be arbitrarily small but fixed). Now we have a global difference map near $(x, y) = (0,0)$ that we write $\Delta_{El}(\bar{y}, r, \epsilon, B_0, B_2, \lambda)$, for $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $\bar{y} \in [\epsilon^2\widetilde{y}_{\epsilon,\tau}, \frac{U_P^2}{r^2}]$. *In the remainder of this section we prove that the difference map Δ_{El} has at most two zeros (counting multiplicity) on the interval $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \frac{U_P^2}{r^2}]$, under the above-mentioned conditions on the parameters.* This implies that the cyclicity of the origin $(x, y) = (0, 0)$ in the elliptic case is bounded by two.

In Section 6.1, the two control curves $B_0 = \widetilde{B}_P$ and $B_0 = \widetilde{B}_D$ are introduced. Based on the following lemma, we know when these control curves are positive or negative, depending on the parameter $\bar{B}_2 \neq 0$ defined in Section 6.1.

Lemma 7.2. *We have that*
(a) $\widetilde{B}_P = B_2\epsilon(-1 + O_1(\epsilon, B_2, r)) + r\epsilon^3(-3\bar{H}(0,\lambda) - G(0,0,\lambda) + O_2(\epsilon, B_2, r))$ *where O_1 and O_2 are C^k-functions in* $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, r, \epsilon, B_2, \lambda)$,
(b) $\widetilde{B}_D = B_2\epsilon(-1 + O_1(\epsilon, B_2, r)) + r\epsilon^3(-3\bar{H}(0,\lambda) - G(0,0,\lambda) + O_2(\epsilon, B_2, r))$ *where O_1 and O_2 are C^∞-functions in* $(\bar{y}, r, \epsilon, B_2, \lambda)$.

Proof. Taking into account Theorem 3.3 and Section 5, the h-component of the difference map defined on Σ_B, near L_0, is given by:

$$\begin{aligned}\omega_\tau(\delta, w) = {} & B_0\kappa_0(\tau, \epsilon) + B_2\epsilon\kappa_1(\tau, \epsilon) + r\epsilon^3\kappa_3(\tau, \epsilon) \\ & + \exp -\frac{1}{w^2}\left(\widetilde{A}^+ + B_2\delta\widetilde{\Phi}_1^+ + r\delta^3\widetilde{\Phi}_3^+\right) \\ & - \exp -\frac{1}{w^2}\left(\widetilde{A}^- + B_2\delta\widetilde{\Phi}_1^- + r\delta^3\widetilde{\Phi}_3^-\right). \end{aligned} \tag{60}$$

We know that $\epsilon = \delta w$, $\widetilde{A}^- = \widetilde{A}^+ + O(B_0 w)$ where $O(B_0 w)$ is a C^∞-function in variable $(\delta, w, w^2 \ln w, \tau)$. It can be easily seen that $\exp -\frac{1}{w^2}\left(\widetilde{A}^+ + B_2\delta\widetilde{\Phi}_1^+ + r\delta^3\widetilde{\Phi}_3^+\right) = \exp -\frac{1}{w^2}\widetilde{A}^+ + O(B_2\delta, r\delta^3)$ and $\exp -\frac{1}{w^2}\left(\widetilde{A}^- + B_2\delta\widetilde{\Phi}_1^- + r\delta^3\widetilde{\Phi}_3^-\right) = \exp -\frac{1}{w^2}\widetilde{A}^+ + O(B_0 w, B_2\delta, r\delta^3)$ where $O(B_2\delta, r\delta^3)$ and $O(B_0 w, B_2\delta, r\delta^3)$ are C^∞-functions in variable (δ, w, τ) and are flat at $w = 0$ (see [15], Theorem 4.15). Hence (60) can be written as

$$\begin{aligned}\omega_\tau(\delta, w) = {} & B_0\kappa_0(\tau, \epsilon) + B_2\epsilon\kappa_1(\tau, \epsilon) + r\epsilon^3\kappa_3(\tau, \epsilon) \\ & + O(B_0 w, B_2\delta, r\delta^3), \end{aligned} \tag{61}$$

for a new C^∞-function $O(B_0 w, B_2\delta, r\delta^3)$, flat at $w = 0$. Clearly, $\omega_\tau(0, w)|_{B_0=0} = 0$. Since $\kappa_0(\tau, \epsilon) \neq 0$, the implicit function theorem and (61) imply existence of unique smooth function $B_B(\delta, w, r, B_2, \lambda)$, given by

$$B_B(\delta, w, r, B_2, \lambda) = B_2\epsilon(\kappa_1^0(\lambda) + O_1(w, B_2, r)) + r\epsilon^3(\kappa_3^0(\lambda) + O_2(w, B_2, r)) \quad (62)$$

where O_1 and O_2 are smooth functions and $\kappa_i^0(\lambda) = -\frac{\kappa_i(\tau,\epsilon)}{\kappa_0(\tau,\epsilon)}|_{\epsilon=B_0=B_2=r=0}$, such that solutions of $\{\omega_\tau(\delta, w) = 0\}$, for $\delta \sim 0$, $w \sim 0$ and $B_0 \sim 0$, can only occur for $B_0 = B_B(\delta, w, r, B_2, \lambda)$.

Since $\bar{y}_1 < \delta_0^2$, $\omega_\tau(\sqrt{\bar{y}}, \frac{\epsilon}{\sqrt{\bar{y}}})$ is well-defined for $\bar{y} = \bar{y}_1$ and $\omega_\tau(\sqrt{\bar{y}_1}, \frac{\epsilon}{\sqrt{\bar{y}_1}}) = \widetilde{\Delta}_D(\bar{y}_1, ...)$ where $\widetilde{\Delta}_D$ is defined in Section 6.1. For $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ we obtain

$$\begin{aligned}\widetilde{\Delta}_D(\bar{y}, ...) &= \widetilde{\Delta}_D(\bar{y}_1, ...) + \int_{\bar{y}_1}^{\bar{y}} \frac{\partial \widetilde{\Delta}_D}{\partial \bar{y}'}(\bar{y}', ...)d\bar{y}' \\ &= \omega_\tau(\sqrt{\bar{y}_1}, \frac{\epsilon}{\sqrt{\bar{y}_1}}) + \int_{\bar{y}_1}^{\bar{y}} \frac{\partial \widetilde{\Delta}_D}{\partial \bar{y}'}(\bar{y}', ...)d\bar{y}' \\ &= B_0\kappa_0(\tau, \epsilon) + B_2\epsilon\kappa_1(\tau, \epsilon) + r\epsilon^3\kappa_3(\tau, \epsilon) + O(B_0\epsilon, B_2, r) \\ &\quad + \int_{\bar{y}_1}^{\bar{y}} \frac{\partial \widetilde{\Delta}_D}{\partial \bar{y}'}(\bar{y}', ...)d\bar{y}'. \end{aligned} \quad (63)$$

where $O(B_0\epsilon, B_2, r)$ is a smooth function in (ϵ, τ), flat at $\epsilon = 0$. In the last step in (63) we used (61). We also have that $\frac{\partial \widetilde{\Delta}_D}{\partial \bar{y}'}(\bar{y}', ...)$ is a smooth function in $(\bar{y}', \epsilon, \tau)$ that is flat at $\epsilon = 0$ and identically equal to 0 when $B_0 = B_2 = r = 0$ (see (46) and (47)). Hence (63) can be written as

$$\begin{aligned}\widetilde{\Delta}_D(\bar{y}, r, \epsilon, B_0, B_2, \lambda) = B_0\kappa_0(\tau, \epsilon) + B_2\epsilon\kappa_1(\tau, \epsilon) + r\epsilon^3\kappa_3(\tau, \epsilon) \\ + O(B_0\epsilon, B_2\epsilon, r\epsilon^3)\end{aligned} \quad (64)$$

where $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and $O(B_0\epsilon, B_2\epsilon, r\epsilon^3)$ is a smooth function, flat at $\epsilon = 0$. As a simple consequence of (64), we have that

$$\widetilde{B}_D = B_2\epsilon(\kappa_1^0(\lambda) + O_1(\epsilon, B_2, r)) + r\epsilon^3(\kappa_3^0(\lambda) + O_2(\epsilon, B_2, r)), \quad (65)$$

where O_1 and O_2 are smooth functions in $(\bar{y}, r, \epsilon, B_2, \lambda)$ and where $\kappa_1^0(\lambda)$ and $\kappa_3^0(\lambda)$ are introduced above.

Let us find a similar expression for $\widetilde{B}_P$ where $\widetilde{B}_P$ is defined for $\bar{y} \in [\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$ (see Section 6.1). We can take $\frac{1}{R_P^2} < \bar{y}_2$. For $r \sim 0$, $r > 0$ and $\bar{y} \in [\frac{1}{R_P^2}, \frac{U_P^2}{r^2}]$, the difference map $\widetilde{\Delta}_P$, defined in Section 6.1, can be written

$$\begin{aligned}\widetilde{\Delta}_P(\bar{y}, ...) &= \widetilde{\Delta}_P(\frac{1}{R_P^2}, ...) + \int_{\frac{1}{R_P^2}}^{\bar{y}} \frac{\partial \widetilde{\Delta}_P}{\partial \bar{y}'}(\bar{y}', ...)d\bar{y}' \\ &= \widetilde{\Delta}_D(\frac{1}{R_P^2}, ...) + \int_{\frac{1}{R_P^2}}^{\bar{y}} \frac{\partial \widetilde{\Delta}_P}{\partial \bar{y}'}(\bar{y}', ...)d\bar{y}' \\ &= B_0\kappa_0(\tau, \epsilon) + B_2\epsilon\kappa_1(\tau, \epsilon) + r\epsilon^3\kappa_3(\tau, \epsilon) \\ &\quad + O(B_0\epsilon, B_2\epsilon, r\epsilon^3) + \int_{\frac{1}{R_P^2}}^{\bar{y}} \frac{\partial \widetilde{\Delta}_P}{\partial \bar{y}'}(\bar{y}', ...)d\bar{y}' \end{aligned} \quad (66)$$

where $O(B_0\epsilon, B_2\epsilon, r\epsilon^3)$ is a smooth function in (ϵ, τ), flat at $\epsilon = 0$. In the last equality in (66), we used (64).

From (43), (44) and Lemma 6.2, it follows that the derivative $\frac{\partial \widetilde{\Delta}_P}{\partial \bar{y}'}$ is equal to 0 when $B_0 = B_2 = r = 0$ and

$$\frac{\partial \widetilde{\Delta}_P}{\partial \bar{y}'}(\bar{y}', r, \epsilon, B_0, B_2, \lambda) = O(\epsilon^m \frac{1}{(\sqrt{\bar{y}'})^m}),$$

where $m \in \mathbb{N}$ is large enough and where $O(\epsilon^m \frac{1}{(\sqrt{\bar{y}'})^m})$ is a C^k-function in variable $(r\sqrt{\bar{y}'}, \frac{1}{\sqrt{\bar{y}'}}, \epsilon, B_0, B_2, \lambda)$. Since $\widetilde{\Delta}_P(\bar{y}, ...)$ is a C^k-function in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, \epsilon, B_0, B_2, \lambda)$ (Section 6.1), (66) can be written now as

$$\widetilde{\Delta}_P(\bar{y}, r, \epsilon, B_0, B_2, \lambda) = B_0 \kappa_0(\tau, \epsilon) + B_2 \epsilon \kappa_1(\tau, \epsilon) + r\epsilon^3 \kappa_3(\tau, \epsilon) + \epsilon^m . O(B_0, B_2, r) \tag{67}$$

where $O(B_0, B_2, r)$ is a C^k-function in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, r, \epsilon, B_0, B_2, \lambda)$. If we replace B_0 with $\widetilde{B}_P$ in (67), then (67) is equal to zero. Hence we get

$$\widetilde{B}_P = B_2 \epsilon (\kappa_1^0(\lambda) + O_1(\epsilon, B_2, r)) + r\epsilon^3 (\kappa_3^0(\lambda) + O_2(\epsilon, B_2, r)) \tag{68}$$

where O_1 and O_2 are C^k-functions in $(r\sqrt{\bar{y}}, \frac{1}{\sqrt{\bar{y}}}, r, \epsilon, B_2, \lambda)$.

To end the proof of (a) and (b), we show that $\kappa_1^0 \equiv -1$ and $\kappa_3^0 \equiv -3\bar{H}(0, \lambda) - G(0, 0, \lambda)$ in the expressions (65) and (68).

We study the coefficient κ_1^0; the study of the coefficient κ_3^0 is completely analogous. Suppose the contrary. Then there exists $\lambda_0 \in \Lambda$, $m_0 > 0$ and $w_1 > 0$ such that $|\kappa_1^0(\lambda_0) + O_1(w, B_2, r) + 1| \geq m_0$, for $(\delta, B_2, r) \sim 0$ and $w \in [0, w_1]$, where $O_1(w, B_2, r)$ is introduced in (62). By Lemma 7.1, there exists $w_2 > 0$ such that $w_2 < w_1$ and $| - \frac{J_1(h_0)}{J_0(h_0)} + 1| \leq \frac{m_0}{2}$, for $0 < h_0 = e^{-\frac{1}{w_2^2}}(\frac{1}{w_2^2} + 1) < 1$. As $J_0(h) > 0$ for $h \sim h_0$, Proposition 5.1 and the implicit function theorem imply existence of unique smooth function $B_{00}(h, \epsilon, r, B_2, \lambda)$ such that solutions of $\hat{D}_{\epsilon,\tau}(h) = 0$, for $\epsilon \sim 0$, $B_0 \sim 0$ and $h \sim h_0$, can only occur for $B_0 = B_{00}(h, \epsilon, r, B_2, \lambda)$. Using the elimination $h = h(w) := e^{-\frac{1}{w^2}}(\frac{1}{w^2} + 1)$ and Proposition 5.1 once more, we obtain that

$$B_{00} = B_2 \epsilon \Big(- \frac{J_1(h(w))}{J_0(h(w))} + O_1(\delta w) \Big) + r\epsilon^3 \Big(- \frac{\mathcal{J}_2(h(w))}{J_0(h(w))} + O_2(\delta w) \Big). \tag{69}$$

For $w \sim w_2$ the functions B_B and B_{00} coincide and then, using (62) and (69), we have that

$$m_0 \leq |\kappa_1^0(\lambda_0) + O_1(w, B_2, r) + 1| = | - \tfrac{J_1(h(w))}{J_0(h(w))} + O_1(\delta w) + 1| = | - \tfrac{J_1(h_0)}{J_0(h_0)} + 1| \leq \tfrac{m_0}{2},$$

for $w = w_2$, $\delta = B_2 = r = 0$ and $\lambda = \lambda_0$. This gives a contradiction. □

Lemma 7.2 implies

$$\begin{cases} \widetilde{B}_P = \varpi \epsilon (-\bar{B}_2 + O_1(\epsilon, \varpi)) \\ \widetilde{B}_D = \varpi \epsilon (-\bar{B}_2 + O_2(\epsilon, \varpi)) \end{cases} \tag{70}$$

where $(B_2, r) = \varpi(\bar{B}_2, \bar{r})$, $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{r} > 0$, $\varpi \sim 0$ and $\varpi > 0$.

The difference map Δ_{El} and its derivative $\frac{\partial}{\partial \bar{y}} \Delta_{El}$ have at most two zeros (counting multiplicity) on the interval $]\epsilon^2 \widetilde{y}_{\epsilon,\tau}, \bar{y}_{00}]$, for each $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$ and $\lambda \in \Lambda$. The constant $\bar{y}_{00}$ is introduced after Lemma 7.1. We choose $\bar{y}_{11}$ such that $0 < \bar{y}_{11} < \bar{y}_{00}$ and we restrict ourselves to the case where $\lambda \in \Lambda_1$; the study of the case where $\lambda \in \Lambda_2$ is analogous. Let us recall that $K_P < 0$

and $K_D < 0$ for $\epsilon > 0$, where K_P and K_D are defined respectively in (45) and (48). Based on Theorem 6.5, we have 3 subcases.

(1) $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{B}_2 \in [-\mu_0, 1[$, $\mu_0 > 0$ *and* $\mu_0 \sim 0$. By Theorem 6.5 (a), there exist sufficiently small $\mu_0 > 0$ and $U_P > 0$ such that, for each $\varpi \sim 0$, $\varpi > 0$, $\bar{B}_2 \in [-\mu_0, 1[$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$ and $\hat{\bar{y}} \in [\bar{y}_{11}, \frac{U_P^2}{(\varpi\bar{r})^2}]$, we have

$$\frac{\partial}{\partial \bar{y}}\Delta_{El}(\bar{y}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(\hat{\bar{y}}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda) = \frac{\partial}{\partial \bar{y}}\bar{\Delta}_G(\bar{y}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda, \hat{\bar{y}}) > 0$$

for all $\bar{y} \in [\bar{y}_{11}, \frac{U_P^2}{(\varpi\bar{r})^2}]$. If we suppose that $\Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0)$ has at least three zeros (counting multiplicity) on the interval $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$ for some $\varpi^0 \sim 0$, $\varpi^0 > 0$, $\bar{B}_2^0 \in [-\mu_0, 1[$, $\epsilon^0 \sim 0$, $\epsilon^0 > 0$, $B_0^0 \sim 0$, $\lambda^0 \in \Lambda_1$ and $\tau^0 = (B_0^0, \varpi^0\bar{B}_2^0, -1, \varpi^0\bar{r}^0, \lambda^0)$, then at least one zero has to be in $[\bar{y}_{00}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$, i.e. $B_0^0 = \widetilde{B}_G(\hat{\bar{y}}^0, \varpi^0\bar{r}^0, \epsilon^0, \varpi^0\bar{B}_2^0, \lambda^0)$ for some $\hat{\bar{y}}^0 \in [\bar{y}_{00}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$ (Section 6.1), and then, as a consequence of the last inequality, we have

$$\frac{\partial}{\partial \bar{y}}\Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0) > 0 \text{ for all } \bar{y} \in [\bar{y}_{00}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}].$$

Using now Rolle's theorem we have that $\Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0)$ has precisely one zero $\bar{y} = \hat{\bar{y}}^0$ (counting multiplicity) on the interval $[\bar{y}_{00}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$. Hence, the function $\Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0)$ has precisely two zeros (counting multiplicity) on the interval $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \bar{y}_{00}[$. Using the fact that $\hat{D}_\tau^G(\epsilon^2\widetilde{y}_{\epsilon,\tau}, \epsilon) = 0$ and Rolle's theorem once more we obtain that $\frac{\partial}{\partial \bar{y}}\Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0)$ has at least three zeros (counting multiplicity) in $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \hat{\bar{y}}^0[$. Since we have $\frac{\partial}{\partial \bar{y}}\Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0) > 0$ for all $\bar{y} \in [\bar{y}_{00}, \hat{\bar{y}}^0[$, we see that the derivative $\frac{\partial}{\partial \bar{y}}\Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0)$ has at least three zeros (counting multiplicity) in $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \bar{y}_{00}[$. This is in clear contradiction to the fact that $\frac{\partial}{\partial \bar{y}}\Delta_{El}$ has at most two zeros (counting multiplicity) on the interval $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{00}]$ for $\epsilon > 0$ and $r > 0$. Hence the difference map $\Delta_{El}(\bar{y}, \varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)$ has at most two zeros (counting multiplicity) on $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \frac{U_P^2}{(\varpi\bar{r})^2}]$ for each $\varpi \sim 0$, $\varpi > 0$, $\bar{B}_2 \in [-\mu_0, 1[$, $B_0 \sim 0$, $\epsilon \sim 0$, $\epsilon > 0$ and $\lambda \in \Lambda_1$.

(2) $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $(\bar{B}_2, \bar{r}) \sim (-1, 0)$, $\bar{B}_2 > -1$. By Theorem 6.5 (c), there exist small $U_P > 0$ and $\mu_0 > 0$ such that, for each $\varpi \sim 0$, $\varpi > 0$, $\bar{B}_2 \in]-1, -1+\mu_0]$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$ and $\hat{\bar{y}} \in [\bar{y}_{11}, \frac{U_P^2}{(\varpi\bar{r})^2}]$, $\frac{\partial}{\partial \bar{y}}\Delta_{El}(\bar{y}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(\hat{\bar{y}}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda)$ has at most one zero (counting multiplicity) on the interval $[\bar{y}_{11}, \frac{U_P^2}{(\varpi\bar{r})^2}]$ and

$$\frac{\partial}{\partial \bar{y}}\Delta_{El}(\bar{y}_{11}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(\hat{\bar{y}}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda) < 0.$$

Suppose that $\Delta_{El}^0(\bar{y}) := \Delta_{El}(\bar{y}, \varpi^0\bar{r}^0, \epsilon^0, B_0^0, \varpi^0\bar{B}_2^0, \lambda^0)$ has at least three zeros (counting multiplicity) on the interval $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$ for some $\varpi^0 \sim 0$, $\varpi^0 > 0$, $\bar{B}_2^0 \in]-1, -1+\mu_0]$, $\epsilon^0 \sim 0$, $\epsilon^0 > 0$, $B_0^0 \sim 0$, $\lambda^0 \in \Lambda_1$ and $\tau^0 = (B_0^0, \varpi^0\bar{B}_2^0, -1, \varpi^0\bar{r}^0, \lambda^0)$. Then $B_0^0 = \widetilde{B}_G(\hat{\bar{y}}^0, \varpi^0\bar{r}^0, \epsilon^0, \varpi^0\bar{B}_2^0, \lambda^0)$ for some $\hat{\bar{y}}^0 \in [\bar{y}_{11}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$, and

$$\frac{\partial}{\partial \bar{y}}\Delta_{El}^0(\bar{y}_{11}) < 0. \tag{71}$$

Taking into account (70) we have that $B_0^0 > 0$. There are two possibilities, either $\Delta_{El}^0(\bar{y}_{11}) < 0$ or $\Delta_{El}^0(\bar{y}_{11}) \geq 0$. First, we suppose that $\Delta_{El}^0(\bar{y}_{11}) < 0$. Then, as a simple consequence of (71), Rolle's theorem and of the fact that $\frac{\partial}{\partial \bar{y}}\Delta_{El}^0(\bar{y})$ has at most one zero counting multiplicity on the interval $[\bar{y}_{11}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$, $\Delta^0_{El}(\bar{y})$ has at most one zero counting multiplicity on the interval $[\bar{y}_{11}, \frac{U_P^2}{(\varpi^0\bar{r}^0)^2}]$. Hence, the function $\Delta^0_{El}(\bar{y})$ has precisely two zeros (counting multiplicity) on the interval $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \bar{y}_{11}[$. Since $\Delta^0_{El}((\epsilon^0)^2\widetilde{y}_{\epsilon_0,\tau^0}) = 0$, $\Delta^0_{El}(\bar{y}_{11}) < 0$ and $\Delta^0_{El}(\bar{y}) > 0$ for $\bar{y}$ strictly between $(\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}$ and the smallest zero of $\Delta^0_{El}(\bar{y})$ $(B_0^0 > 0)$, Rolle's theorem implies that the derivative $\frac{\partial}{\partial\bar{y}}\Delta^0_{El}(\bar{y})$ has at least three zeros (counting multiplicity) in $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \bar{y}_{11}[$. Again this gives a contradiction. Suppose now that $\Delta^0_{El}(\bar{y}_{11}) \geq 0$. It suffices to prove that $\Delta^0_{El}(\bar{y}) > 0$ for all $\bar{y} \in](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \bar{y}_{11}[$. To prove this, we use (71), $\Delta^0_{El}((\epsilon^0)^2\widetilde{y}_{\epsilon_0,\tau^0}) = 0$, Rolle's theorem, the fact that $\Delta^0_{El}(\bar{y}) > 0$ for $\bar{y}$ strictly between $(\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}$ and the smallest zero of $\Delta^0_{El}(\bar{y})$ $(B_0^0 > 0)$ and the fact that $\Delta^0_{El}(\bar{y})$ and $\frac{\partial}{\partial\bar{y}}\Delta^0_{El}(\bar{y})$ have at most two zeros counting multiplicity on $](\epsilon^0)^2\widetilde{y}_{\epsilon^0,\tau^0}, \bar{y}_{11}[$. Hence, the difference map $\Delta_{El}(\bar{y}, \varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)$ has at most two zeros (counting multiplicity) on $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \frac{U_P^2}{(\varpi\bar{r})^2}]$ for each $\varpi \sim 0$, $\varpi > 0$, $\bar{B}_2 \in]-1, -1+\mu_0]$, $B_0 \sim 0$, $\epsilon \sim 0$, $\epsilon > 0$ and $\lambda \in \Lambda_1$.

(3) $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{B}_2 \in C$, $C \subset]-1, 0[$ *is an arbitrarily large compact subset of* $\mathbb{R}$. We suppose that $C = [-1+\mu_0, -\mu_0]$ where $\mu_0 > 0$ is arbitrarily small and fixed. By Theorem 6.5 (b), there exist $U_P > 0$ and $\bar{y}_{22} > 0$ such that $\bar{y}_{22} < \bar{y}_{00}$ and, for each $\varpi \sim 0$, $\varpi > 0$, $\bar{B}_2 \in C$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$ and $\hat{\bar{y}} \in [\bar{y}_{22}, \frac{U_P^2}{(\varpi\bar{r})^2}]$, $\frac{\partial}{\partial\bar{y}}\Delta_{El}(\bar{y}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(\hat{\bar{y}}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda)$ has at most one zero (counting multiplicity) on the interval $[\bar{y}_{22}, \frac{U_P^2}{(\varpi\bar{r})^2}]$ and

$$\frac{\partial}{\partial\bar{y}}\Delta_{El}(\bar{y}_{22}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(\hat{\bar{y}}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda) < 0.$$

As in the subcase (2) we can see that the difference map $\Delta_{El}(\bar{y}, \varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)$ has at most two zeros (counting multiplicity) on $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \frac{U_P^2}{(\varpi\bar{r})^2}]$ for each $\varpi \sim 0$, $\varpi > 0$, $\bar{B}_2 \in C$, $B_0 \sim 0$, $\epsilon \sim 0$, $\epsilon > 0$ and $\lambda \in \Lambda_1$.

Combining (1), (2) and (3), we obtain that there exists $U_P > 0$ such that the difference map $\Delta_{El}(\bar{y}, r, \epsilon, B_0, B_2, \lambda)$ has at most 2 zeros (counting multiplicity) on $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \frac{U_P^2}{r^2}]$ for each $r \sim 0$, $r > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $\epsilon \sim 0$, $\epsilon > 0$ and $\lambda \in \Lambda$. Hence the cyclicity of the origin $(x, y) = (0, 0)$ in the elliptic case is bounded by 2.

7.2. **Study of the saddle case.** We keep in mind the notation used in Section 5 and Section 6.2, and we suppose that $\tau = (B_0, B_2, 1, r, \lambda)$.

As in Section 7.1 we can consider a difference map near the origin $(x, y) = (0, 0)$ that we will write $\Delta_{Sa}(\bar{y}, r, \epsilon, B_0, B_2, \lambda)$, for $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $\lambda \in \Lambda$ and $\bar{y} \in [\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{\max}(r, \epsilon, B_0, B_2, \lambda)]$. $\Delta_{Sa} \equiv \widetilde{\Delta}_G$ on $[\bar{y}_1, \bar{y}_{\max}(r, \epsilon, B_0, B_2, \lambda)]$ and equal to $\Delta_{Sa} \equiv \hat{D}^G_\tau$ on $[\epsilon^2\widetilde{y}_{\epsilon,\tau}, \delta_0^2]$ where the two maps $\hat{D}^G_\tau$ and $\widetilde{\Delta}_G$ are defined respectively in Section 5 and Section 6.2. We may suppose that $\bar{y}_1 < \delta_0^2$. *In this section we will prove that the difference map* Δ_{Sa} *has at most two zeros (counting multiplicity) on* $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{\max}(r, \epsilon, B_0, B_2, \lambda)[$, *for each* $r \sim 0$, $r > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$ *and* $\lambda \in \Lambda$. This will imply that the cyclicity of the origin $(x, y) = (0, 0)$ in the saddle case is bounded by two.

Using the same trick as used in the elliptic case, in the saddle case is also possible to show that

$$\begin{cases} \widetilde{B}_S = \varpi\epsilon(-\bar{B}_2 + O_1(\epsilon, \varpi)) \\ \widetilde{B}_C = \varpi\epsilon(-\bar{B}_2 + O_2(\epsilon, \varpi)) \end{cases} \tag{72}$$

where $(B_2, r) = \varpi(\bar{B}_2, \bar{r})$, $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{r} > 0$, $\varpi \sim 0$ and $\varpi > 0$.

Similarly, the functions Δ_{Sa} and $\frac{\partial}{\partial \bar{y}}\Delta_{Sa}$ each have at most 2 zeros (counting multiplicity) on the interval $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{00}]$, for each $\epsilon \sim 0$, $\epsilon > 0$, $B_0 \sim 0$, $B_2 \sim 0$, $r \sim 0$, $(B_0, B_2, r) \neq (0, 0, 0)$ and $\lambda \in \Lambda$. Again we choose $\bar{y}_{11}$ such that $0 < \bar{y}_{11} < \bar{y}_{00}$, and we consider $\lambda \in \Lambda_1$. The functions K_S and K_C, introduced in Section 6.2, are strictly negative for $\epsilon > 0$. We have 4 subcases.

(1) $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$*,* $\bar{B}_2 \in [-\mu_0, 1[$. By Theorem 6.7 (a), there exists a sufficiently small $\mu_0 > 0$ such that for each fixed $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\bar{B}_2 \in [-\mu_0, 1[$, $\lambda \in \Lambda_1$ and $c \in [\bar{y}_{11}, c_0]$ we have that

$$\frac{\partial}{\partial \bar{y}}\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda) = \frac{\partial}{\partial \bar{y}}\bar{\Delta}_G(\bar{y}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda, c) > 0$$

for all $\bar{y} \in [\bar{y}_{11}, \bar{y}_{\max}(\varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda)[=: \Sigma_{GG}$. Recall that the constant c_0 is defined in Section 6.2. Like in the subcase (1) in the elliptic case, we obtain that the difference map $\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)$ has at most two zeros (counting multiplicity) on $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{\max}(\varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)[$, for each $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\bar{B}_2 \in [-\mu_0, 1[$, $\lambda \in \Lambda_1$ and $B_0 \sim 0$.

(2) $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$*,* $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0]$. By Theorem 6.7 (c), there exists a sufficiently small $\mu_0 > 0$ such that, for each fixed $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$, $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0]$ and $c \in [\bar{y}_{11}, c_0]$, the derivative $\frac{\partial}{\partial \bar{y}}\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda)$ has at most one zero (counting multiplicity) on the interval Σ_{GG} and

$$\frac{\partial}{\partial \bar{y}}\Delta_{Sa}(\bar{y}_{11}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda) < 0.$$

Clearly, the study of this case is analogous to the study of the subcase (2) in the elliptic case. We obtain that the difference map $\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)$ has at most two zeros (counting multiplicity) on $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{\max}(\varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)[$, for each $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$, $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0]$ and $B_0 \sim 0$.

(3) $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$*,* $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$*,* $\mu_0 > 0$ *is arbitrarily small and fixed.* By Theorem 6.7 (b), there exists a sufficiently small $\bar{y}_{22} > 0$ such that $\bar{y}_{22} < \bar{y}_{00}$ and, for each fixed $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$, $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ and $c \in [\bar{y}_{22}, c_0]$, the derivative

$$\frac{\partial}{\partial \bar{y}}\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda)$$

has at most one zero (counting multiplicity) on

$$[\bar{y}_{22}, \bar{y}_{\max}(\varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda)[$$

and

$$\frac{\partial}{\partial \bar{y}}\Delta_{Sa}(\bar{y}_{22}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda) < 0.$$

The study of this subcase is analogous to the study of the subcase (2) in the saddle case (or, equivalently, of the subcase (2) in the elliptic case). The difference map $\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)$ has at most two zeros (counting multiplicity) on

$]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{\max}(\varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)[$, for each $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$, $\bar{B}_2 \in [-\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} + \mu_0, -\mu_0]$ and $B_0 \sim 0$.

(4) $(\bar{B}_2, \bar{r}) \in \mathbb{S}^1$, $\bar{B}_2 \in]-1, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$, *$\mu_0 > 0$ is arbitrarily small and fixed.* By Theorem 6.7 (d), for each $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$, $\bar{B}_2 \in]-1, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$ and $c \in [\bar{y}_{11}, c_0]$,

$$\frac{\partial}{\partial \bar{y}}\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, \widetilde{B}_G(c, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda), \varpi\bar{B}_2, \lambda) = \frac{\partial}{\partial \bar{y}}\bar{\Delta}_G(\bar{y}, \varpi\bar{r}, \epsilon, \varpi\bar{B}_2, \lambda, c) < 0.$$

for all $\bar{y} \in \Sigma_{GG}$. The study of this case is analogous to the study of the subcase (1) in both the elliptic and saddle cases. The difference map $\Delta_{Sa}(\bar{y}, \varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)$ has at most two zeros (counting multiplicity) on $]\epsilon^2\widetilde{y}_{\epsilon,\tau}, \bar{y}_{\max}(\varpi\bar{r}, \epsilon, B_0, \varpi\bar{B}_2, \lambda)[$, for each $\varpi \sim 0$, $\varpi > 0$, $\epsilon \sim 0$, $\epsilon > 0$, $\lambda \in \Lambda_1$, $\bar{B}_2 \in]-1, -\frac{\bar{H}(0,\lambda)}{\sqrt{1+\bar{H}(0,\lambda)^2}} - \mu_0]$ and $B_0 \sim 0$.

REFERENCES

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps. Volume 2*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2012, Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous and revised by the authors and James Montaldi, Reprint of the 1988 translation.

[2] W. A. Coppel, Some quadratic systems with at most one limit cycle, in *Dynamics reported, Vol. 2*, vol. 2 of Dynam. Report. Ser. Dynam. Systems Appl., Wiley, Chichester, (1989), 61–88.

[3] P. De Maesschalck and F. Dumortier, Time analysis and entry-exit relation near planar turning points, *J. Differential Equations*, **215** (2005), 225–267.

[4] P. De Maesschalck and F. Dumortier, Singular perturbations and vanishing passage through a turning point, *J. Differential Equations*, **248** (2010), 2294–2328.

[5] P. De Maesschalck and F. Dumortier, Slow-fast Bogdanov-Takens bifurcations, *J. Differential Equations*, **250** (2011), 1000–1025.

[6] P. De Maesschalck and F. Dumortier, Detectable canard cycles with singular slow dynamics of any order at the turning point, *Discrete Contin. Dyn. Syst.*, **29** (2011), 109–140.

[7] F. Dumortier, Slow divergence integral and balanced canard solutions, *Qual. Theory Dyn. Syst.*, **10** (2011), 65–85.

[8] F. Dumortier and R. Roussarie, Birth of canard cycles, *Discrete Contin. Dyn. Syst. Ser. S*, **2** (2009), 723–781.

[9] F. Dumortier, R. Roussarie, J. Sotomayor and H. Zoladek, *Bifurcations of Planar Vector Fields*, vol. 1480 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1991, Nilpotent singularities and Abelian integrals.

[10] F. Dumortier, Compactification and desingularization of spaces of polynomial Liénard equations, *J. Differential Equations*, **224** (2006), 296–313.

[11] F. Dumortier and C. Li, On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations, *Nonlinearity*, **9** (1996), 1489–1500.

[12] F. Dumortier and C. Li, Quadratic Liénard equations with quadratic damping, *J. Differential Equations*, **139** (1997), 41–59.

[13] F. Dumortier and C. Rousseau, Cubic Liénard equations with linear damping, *Nonlinearity*, **3** (1990), 1015–1039.

[14] J.-L. Figueras, W. Tucker and J. Villadelprat, Computer-assisted techniques for the verification of the Chebyshev property of Abelian integrals, *J. Differential Equations*, **254** (2013), 3647–3663.

[15] R. Huzak, *Limit Cycles in Slow-Fast Codimension 3 Bifurcations. Dissertation*, Hasselt University, Belgium, 2013.

[16] R. Huzak, P. De Maesschalck and F. Dumortier, Limit cycles in slow-fast codimension 3 saddle and elliptic bifurcations, *J. Differential Equations*, **255** (2013), 4012–4051.

[17] R. Huzak, P. De Maesschalck and F. Dumortier, Primary birth of canard cycles in slow-fast codimension 3 elliptic bifurcations, *Communications on Pure and Applied Analysis*, **13** (2014), 2641–2673.

[18] M. Krupa and P. Szmolyan, Relaxation oscillation and canard explosion, *J. Differential Equations*, **174** (2001), 312–368.

[19] C. Li and J. Llibre, Uniqueness of limit cycles for Liénard differential equations of degree four, *J. Differential Equations*, **252** (2012), 3142–3162.

[20] A. Lins, W. de Melo and C. C. Pugh, On Liénard's equation, in *Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976)*, Lecture Notes in Math., Springer, Berlin, **597** (1977), 335–357.

[21] R. Roussarie, Putting a boundary to the space of Liénard equations, *Discrete Contin. Dyn. Syst.*, **17** (2007), 441–448.

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