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# ON INVOLUTIVITY OF $p$-SUPPORT

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Abstract. The $p$-support of a holonomic $\mathcal{D}$-module was introduced by Kontsevich. Thomas Bitoun in his PhD thesis proved several properties of $p$-support conjectured by Kontsevich. In this note we give an alternative proof for involutivity by reducing it to a slight extension of Gabber's theorem on the integrability of the characteristic variety. For the benefit of the reader we review how this extension follows from Kaledin's proof of Gabber's theorem.

## 1. Introduction

Let $X$ be a smooth affine variety over a field $K$ and let $N$ be a holonomic $\mathcal{D}$-module. The *singular support* of $N$ is a Langrangian conical subvariety of $T^*(X)$ whose construction is standard.

For a specialization $(X_k, N_k)$ of $(X, N)$ to a field of finite characteristic $k$ Kontsevich defines the *p-support* of $N$ as the support of $N_k$ considered as coherent sheaf on $T^*(X_k)^{(1)}$ (using the theory of crystalline differential operators in characteristic $p$, see [2]). He conjectured that if the specialization $K \to k$ is sufficiently generic then the $p$-support is Lagrangian. Since $p$-support is usually not conical it is potentially a finer invariant than singular support.

Kontsevich's conjecture was proven by Thomas Bitoun in his PhD thesis [3]. In particular involutivity required subtle geometric arguments based on Hodge theory. In this note we give an algebraic proof of involutivity by reducing it to a slight extension of Gabber's celebrated theorem on the integrability of the characteristic variety [5, 7]. To this end we use the observation by Belov-Kanel and Kontsevich that the Poisson bracket on $T^*(\mathbb{A}^n_{\mathbb{Z}/p\mathbb{Z}})^{(1)}$ is encoded in the lifting of differential operators on $\mathbb{A}^n_{\mathbb{Z}/p\mathbb{Z}}$ to differential operators on $\mathbb{A}^n_{\mathbb{Z}/p^2\mathbb{Z}}$.

Gabber's proof is elementary and very general but also quite intricate. On the other hand a conceptual proof of Gabber's theorem has been given by Kaledin in [7]. After some adaptations it applies to our setting as well. For the benefit of the reader this is explained in Appendix B.

## 2. Acknowledgement

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## 3. The $p$-support of a $\mathcal{D}$-module

We recall some notions introduced by Kontsevich in [9]. Let $X$ be a smooth affine variety over a field $K$ and let $\mathcal{D}(X)$ be the ring of differential operators on $X$.

Let $N$ be a finitely generated $\mathcal{D}(X)$-module. Then we may find a subring $R \subset K$ of finite type over $\mathbb{Z}$ such that $X$ has a smooth model $X_R/\operatorname{Spec} R$ and $N$ is obtained by base extension from a finitely generated $\mathcal{D}(X_R)$-module $N_R$, where $\mathcal{D}(X_R)$ is the ring of so-called $R$-linear "crystalline" differential operators [2]. I.e. the differential operators that are linear combinations of compositions of $R$-linear derivations.

For any prime $p$ and any maximal ideal $v$ in $R/pR$ with residue field $k_{p,v}$ one obtains a reduced scheme

$$\operatorname{Supp}_{p,v}(N_R) \stackrel{\text{def}}{=} \operatorname{Supp}(N_R \otimes_R k_{p,v}) \subset T^*(X_{k_{p,v}})^{(1)}$$

where we have used:

**Theorem 3.1.** [2] *$\mathcal{D}(X_{k_{p,v}})$ is an Azumaya algebra of rank $p^{2\dim X}$ over its center which is given by the global sections of $T^*(X_{k_{p,v}})^{(1)}$.*

Note that $T^*(X_{k_{p,v}})^{(1)}$ carries a canonical symplectic form inherited from $T^*(X_{k_{p,v}})$ by Frobenius pullback. In [9] Kontsevich made the following conjecture

**Conjecture 3.2.** If $N$ is holonomic then there exists a dense open subset $U$ in $\operatorname{Spec} R$ such that for all $(p, v) \in U$, $\operatorname{Supp}_{p,v}(N_R)$ is Lagrangian.

This conjecture was proved in [3] using subtle geometric arguments based on Hodge theory.

In the current note we give an algebraic proof of the involutivity part of Conjecture 3.2 by reducing it to a slight extension of Gabber's theorem on the integrability of the characteristic variety [5, 7].

We first use the observation made in [3, §4] that it is sufficient to understand the case $X_R = \mathbb{A}^n_R$.[1] Subsequently we use the following result by Belov-Kanel and Kontsevich:

**Lemma 3.3.** [1, Lemma 2] *Assume $X_R = \mathbb{A}^n_R$. Let $k'_{p,v}$ be a $\mathbb{Z}/p^2\mathbb{Z}$-flat quotient of $R/p^2R$ lifting of $k_{p,v}$. Then $\mathcal{D}(X_{k'_{p,v}})$ is a first order deformation of $\mathcal{D}(X_{k_{p,v}})$ (see §4 below) and the corresponding bracket (see below) on $T^*(X_{k_{p,v}})^{(1)}$ coincides with the natural Poisson bracket on $T^*(X_{k_{p,v}})^{(1)}$ used above, up to sign.*

*Proof.* $\mathcal{D}(X_{k'_{p,v}})$ and $\mathcal{D}(X_{k_{p,v}})$ are Weyl algebas over $k'_{p,v}$ and $k_{p,v}$ respectively. One may then invoke [1, Lemma 2]. □

This lemma will be combined with the following result by Thomas Bitoun

**Theorem 3.4.** [3, Theorem 5.3.2] *Assume that $X_R = \mathbb{A}^n_R$ and that $N$ is holonomic. There exists a dense open subset $U \subset \operatorname{Spec} R$ such that for all $(p, v) \in U$ the length of $N$ at the generic points of $\operatorname{Supp}_{p,v}(N_R)$ is bounded by $e(N)p^{\dim X}$ where $e(N)$ is a constant depending only on $N$.*

[1] To carry out the reduction to $\mathbb{A}^n$, Bitoun uses a closed embedding $X \subset \mathbb{A}^n$ combined with the formalism of $\mathcal{D}$-modules in finite characterictic. An alternative is to use etale local coordinates on $X$. This amounts to constructing an open affine covering $X = \bigcup_i U_i$ together with etale maps $U_i \to \mathbb{A}^{\dim X}$.

*Proof of Conjecture 3.2.* As before we assume $X_R = \mathbb{A}^n_R$ and $N$ holonomic. Bitoun has proved [3, Theorem 3.1.1] that $\operatorname{Supp}_{p,v}(N_R)$ is equidimensional of dimension $\dim X$ for $(p,v)$ in a dense open subset of $\operatorname{Spec} R$. Hence it remains to show that $\operatorname{Supp}_{p,v}(N_R)$ is coisotropic.

This follows from Lemma 3.3 and Proposition 4.2 below with $A = \mathcal{D}(X_{k_{p,v}})$, $A' = \mathcal{D}(X_{k'_{p,v}})$ (see Lemma 3.3), $M = N_{k_{p,v}}$, $M' = N_{k'_{p,v}}$, $k = k_{p,v}$, $n = \dim X$ and $p > e(M)$. □

## 4. Involutivity of $p$-support

Below $k$ is a field of characteristic $p$. Assume that $A$ is a $k$-algebra and $A'$ is a first order deformation of $A$ in the sense that $A'$ contains a central element $h$ such that $A'/hA' = A$ and $\operatorname{Ann}_{A'}(h) = hA'$. Then the center $Z = Z(A)$ of $A$ carries an anti-symmetric biderivation (a "bracket") given by

$$\tilde{a}\tilde{b} - \tilde{b}\tilde{a} = h\{a, b\}$$

where $a, b \in Z$ and $\tilde{a}$, $\tilde{b} \in A'$ are arbitrary liftings of $a, b$.

**Lemma 4.1.** *Let $n > 0$ and put $B' = M_n(A')$, $B = M_n(A)$. Then $B'$ is a first order deformation of $B$ and $Z(B) \cong Z$. The bracket induced on $Z(B)$ is the same as the one on $Z$.*

*Proof.* The center of $B$ consists of diagonal matrices with central, identical entries. These may be lifted to diagonal matrices with identical entries. From this the lemma easily follows. □

**Proposition 4.2.** *Assume in addition that $A$ is an Azumaya algebra over its center $Z$ and $Z/k$ is finitely generated and regular. Assume that $A$ has constant rank $r^2$ over its center with $r = p^n$. Assume that $M$ is a finitely generated left $A$-module and $M'$ is a first order $A'$-deformation of $M$ in the sense that $M'/hM' = M$ and $\operatorname{Ann}_{M'}(h) = hM'$. Let $I = \operatorname{rad} \operatorname{Ann}_Z M$.*

*Assume that the multiplicities (= length) of $M$ over the different primary components of $I$ are of the form $bp^n$ with $(b,p) = 1$. Then $I$ is coisotropic. That is*

$$\{I, I\} \subset I$$

*Proof.* The set of all lifts of the regular elements in $Z$ forms an Ore set in $A'$. Hence we may regard $A'$ as a sheaf on $\operatorname{Spec} Z$. It suffices to prove the statement in the generic points of $V(I)$. Hence we may assume that $Z$ is local with maximal ideal $m$ and $M$ has finite length equal to $bp^n$ as in the statement of the proposition.

From the conditions on the rank of $M$ we find that $A/mA$ is split. Without loss of generality we may replace $Z$ by its completion and then $A$ itself is split. Thus $A = M_r(Z)$. By lifting idempotents we find $A' = M_r(Z')$ where $Z'$ is a (normally non-commutative) first order deformation of $Z$ (note that we may assume $h \in Z'$) and by the above lemma it induces the same biderivation on $Z$ as $A'$.

We also have $M' = M_{r\times 1}(M'_0)$, $M = M_{r\times 1}(M_0)$ where $M'_0$ is a first order deformation of $M_0$ and the latter is a $Z$-module of finite length, equal to $b$.

Thus the length of $M_0$ is not divisible by $p$. To conclude we note that Gabber's proof in [5] yields that $m$ is involutive under these conditions. See [5, p468, first display].

After some adaptations Kaledin's proof of Gabber's theorem applies to our setting as well. This is explained in Appendix B. □

## Appendix A. Hochschild homology and the trace map

This is a preparatory section for the next one in which we discuss Kaledin's proof of Gabber's theorem. In this section we discuss some properties of the trace map. These properties are well-known to experts but for lack of a succinct reference we give proofs for some its properties here using results in [8].

Let $K$ be a field. If $\mathfrak{a}$ is a $K$-linear DG-category then we may define its Hochschild homology $\mathrm{HH}_*(\mathfrak{a})$ by the usual standard complex which we denote by $\mathrm{C}_*(\mathfrak{a})$. It will be convenient to think of $\mathrm{C}_*(\mathfrak{a})$ as the $\oplus$-total complex of a bicomplex $\mathrm{C}_*^*(\mathfrak{a})$ such that

$$\mathrm{C}_n^m(\mathfrak{a}) = \bigoplus_{A_0,\ldots,A_n\in\mathrm{Ob}(\mathfrak{a}),\sum_i m_i=m} \mathfrak{a}(A_{n-1},A_n)_{m_{n-1}} \otimes_K \cdots \otimes_K \mathfrak{a}(A_0,A_1)_{m_0}$$

with the standard differentials $d_{\mathrm{Hoch}} : \mathrm{C}_n^m(\mathfrak{a}) \to \mathrm{C}_{n-1}^m(\mathfrak{a})$, $d_{\mathfrak{a}} : \mathrm{C}_n^m(\mathfrak{a}) \to \mathrm{C}_n^{m+1}(\mathfrak{a})$.

Let $A \in \mathrm{Ob}(\mathfrak{a})$. The inclusion $\mathfrak{a}(A,A) \to \mathrm{C}_0^*(\mathfrak{a})$ defines a map of complexes

$$\mathfrak{a}(A,A) \to \mathrm{C}_*(\mathfrak{a})$$

which on the level of cohomology defines the so-called *trace* map

$$\mathrm{Tr}_A : H^*(\mathfrak{a}(A,A)) \to \mathrm{HH}_{-*}(\mathfrak{a})$$

Let $\mathrm{Mod}(\mathfrak{a})$ be the category of right $\mathfrak{a}$-modules[2]. We will identify $\mathfrak{a}$ with a full subcategory of $\mathrm{Mod}(\mathfrak{a})$ through the Yoneda embedding. Let $\tilde{\mathfrak{a}}$ be the category of perfect DG-modules in $\mathrm{Mod}(\mathfrak{a})$. By [8, Thm 2.4b)] the functor $\mathfrak{a} \to \tilde{\mathfrak{a}}$ induces an isomorphism $\mathrm{HH}_*(\mathfrak{a}) \cong \mathrm{HH}_*(\tilde{\mathfrak{a}})$. For $A \in \mathrm{Ob}(\tilde{\mathfrak{a}})$ we define the corresponding trace map as the composition

$$\mathrm{Tr}_A : \mathrm{Ext}^*_{\mathfrak{a}}(A,A) = H^*(\tilde{\mathfrak{a}}(A,A)) \to \mathrm{HH}_{-*}(\tilde{\mathfrak{a}}) \cong \mathrm{HH}_{-*}(\mathfrak{a})$$

The following properties follow directly from the definition

**Lemma A.1.** *(Functoriality) If $F : \mathfrak{a} \to \mathfrak{b}$ is a DG-functor then the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{Ext}^*_{\mathfrak{a}}(A,A) \arrow[r, "\mathrm{Tr}_A"] \arrow[d] & \mathrm{HH}_{-*}(\mathfrak{a}) \arrow[d] \\
\mathrm{Ext}^*_{\mathfrak{b}}(FA,FB) \arrow[r, "\mathrm{Tr}_{FA}"'] & \mathrm{HH}_{-*}(\mathfrak{b})
\end{tikzcd}$$

**Lemma A.2.** *(Symmetry) If $f \in \mathrm{Ext}^i_{\mathfrak{a}}(A,B)$, $g \in \mathrm{Ext}^j_{\mathfrak{a}}(B,A)$ then*

$$\mathrm{Tr}_B(fg) = (-1)^{ij}\,\mathrm{Tr}_B(gf)$$

*in* $\mathrm{HH}_*(\mathfrak{a})$.

From the previous lemma one obtains

**Lemma A.3.** *(Naturality) If $f \in \mathrm{Hom}_{\mathfrak{a}}(A,B)$ is invertible and $g \in \mathrm{Ext}^*_{\mathfrak{a}}(A,A)$ then*

$$\mathrm{Tr}_B(fgf^{-1}) = \mathrm{Tr}_A(g)$$

*In other words: up to the natural identification of* $\mathrm{Ext}^*_{\mathfrak{a}}(A,A)$ *and* $\mathrm{Ext}^*_{\mathfrak{a}}(B,B)$ *the traces* $\mathrm{Tr}_A$, $\mathrm{Tr}_B$ *coincide.*

[2]Right $\mathfrak{a}$-modules are contravariant functors $M : \mathfrak{a} \to \mathrm{Ab}$. In other words a collection of abelian groups $M(A)$, $A \in \mathfrak{a}$ which depends contravariantly on $A$.

Naturality implies that we may define the trace map on compact objects $C$ in $D(\mathfrak{a})$. We take an object $\tilde{C}$ in $\tilde{\mathfrak{a}}$ representing $C$ and put $\mathrm{Tr}_C = \mathrm{Tr}_{\tilde{C}}$. By naturality this yields a well defined map

$$\mathrm{Ext}^*_{\mathfrak{a}}(C, C) \to \mathrm{HH}_{-*}(\mathfrak{a})$$

Now we come to the additivity of traces. This is a slightly subtle problem. See [4].

**Lemma A.4.** *(Additivity) Assume that we have a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \tag{A.1}$$

*of closed maps in $\tilde{\mathfrak{a}}$ with $|u| = 0$. Let $C = \operatorname{cone} u$ and let $h : C \to C$ be the morphism obtained from $f, g$ by functoriality of cones in DG-categories. Then*

$$\mathrm{Tr}_B(g) = \mathrm{Tr}_A(f) + \mathrm{Tr}_C(h)$$

*Proof.* Let $\mathfrak{b}$ be the category consisting of triples $(A, B, u)$ with $A, B \in \tilde{\mathfrak{a}}$ and $u : A \to B$ a closed map of degree zero. It is easy to see that $\mathfrak{b}$ has a semi-orthogonal decomposition $(\tilde{\mathfrak{a}}, \tilde{\mathfrak{a}})$ and hence by [8, Thm 2.4b,c] we have

$$\mathrm{HH}_*(\mathfrak{b}) = \mathrm{HH}_*(\tilde{\mathfrak{a}}) \oplus \mathrm{HH}_*(\tilde{\mathfrak{a}}) = \mathrm{HH}_*(\mathfrak{a}) \oplus \mathrm{HH}_*(\mathfrak{a}) \tag{A.2}$$

There are three canonical functors $\pi_{1,2,3} : \mathfrak{b} \to \tilde{\mathfrak{a}}$ which send $(A, B, u)$ respectively to $A, B$ and $\operatorname{cone} u$. Using (A.2) one easily checks that

$$\mathrm{HH}_*(\pi_2) = \mathrm{HH}_*(\pi_1) + \mathrm{HH}_*(\pi_3) \tag{A.3}$$

Now (A.1) may be viewed as a morphism in $\mathfrak{b}$ which we denote by $(f, g)$. By functoriality we have

$$\begin{aligned} \mathrm{HH}_*(\pi_1)(\mathrm{Tr}_u(f,g)) &= \mathrm{Tr}_A(f) \\ \mathrm{HH}_*(\pi_2)(\mathrm{Tr}_u(f,g)) &= \mathrm{Tr}_B(g) \\ \mathrm{HH}_*(\pi_3)(\mathrm{Tr}_u(f,g)) &= \mathrm{Tr}_C(h) \end{aligned}$$

From (A.3) we then obtain what we want. □

We actually use the following variant of additivity.

**Lemma A.5.** *(Additivity) Assume that we have a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \tag{A.4}$$

*of closed maps in $\tilde{\mathfrak{a}}$ with $|u| = 0$. Assume that $u$ is injective and $C = \operatorname{coker} u$ is perfect. Let $h : C \to C$ be obtained by functoriality of cokernels. Then*

$$\mathrm{Tr}_B(g) = \mathrm{Tr}_A(f) + \mathrm{Tr}_C(h)$$

*Proof.* This follows easily from the fact that we have a natural commutative diagram

$$\begin{tikzcd} & & \operatorname{cone} u \arrow[dd] \\ A \arrow[r, "u"'] & B \arrow[ur] \arrow[dr] & \\ & & C \end{tikzcd}$$

where the vertical map is a homotopy equivalence, together with naturality. □

*Remark* A.6. Additivity holds in fact for morphisms of distinguished triangles in Neeman's alternative definition of the derived category. See [11].

*Remark* A.7. A natural way to formulate additivity is via the filtered derived category. See [6, V3.7.7, p.310].

## Appendix B. Gabber's theorem

The following result is a slight generalization of the main result of [7].

**Proposition B.1.** *Consider the following data*

- *$A$ is an equicharacteristic regular local ring with maximal ideal $m$ and residue field $K$ of characteristic $p$.*
- *$A'$ is a first order deformation of $A$.*
- *$M$ is a finite length $A$-module such that there exists a first order $A'$-deformation $M'$ of $M$.*
- *The characteristic of $K$ is either zero or prime to the length of $M$.*
- *$\{-,-\}$ is the induced bracket on $A$.*

*With these notations we have that $m$ is involutive. I.e. $\{m, m\} \subset m$.*

*Remark* B.2. This is more or less the setup [7, §3]. However there are some differences.

- We do not assume that $A'$ is defined over the same ground field as $A$.
- We do not assume $p = 0$.
- Kaledin does not assume that $(A, m)$ is local. However during the proof $A$ is localized at $m$. So it is sufficient to consider the local case.

We will now prove Proposition B.1 following the ideas in [7, §3]. However we will not use Hochschild cohomology of abelian categories. That part of Kaledin's argument does not generalize to our more general setting but luckily it is not needed.

First of all we may replace $A$ by its completion. Then $A$ contains a copy of its residue field which we will denote by $K$ as well. Let $\mathcal{C}$ be the category of finite length $A$-modules. We may view $\mathcal{C}$ as a $K$-linear category.

If $S \in \mathcal{C}$ then there is an obstruction [10] $o_S \in \operatorname{Ext}^2_{\mathcal{C}}(S, S)$ against the existence of a first order deformation of $S$ over $A'$. To construct it we choose projective resolutions $(P_S, d)$ of $S$ over $A$. Let $P'_S$ be a lifting of $P_S$ to $A'$ (as graded projective $A$-module) and let $d' : P'_S \to P'_S$ be a lifting of $d$ (as an endomorphism of a graded projective $A$-module). Then $o_S : P_S \to P_S[2]$ is defined by $h o_S = (d')^2$ (with obvious notation).

A priori we have $o_S \in \mathrm{Hom}_{D_{\mathcal{C}}(A)}(S, S[2])$. However it is well-known that in this case the canonical map $D(\mathcal{C}) \to D_{\mathcal{C}}(A)$ is an isomorphism so that we may interprete $o_S$ as an element of $\mathrm{Ext}^2_{\mathcal{C}}(S, S)$, if we so prefer.

If we have an exact sequence in $\mathcal{C}$

$$0 \to N \to S \to Q \to 0$$

then the above construction yields a commutative diagram of complexes

$$\begin{tikzcd} 0 \arrow[r] & P_N \arrow[r] \arrow[d, "o_N"'] & P_S \arrow[r] \arrow[d, "o_S"'] & P_Q \arrow[r] \arrow[d, "o_Q"] & 0 \\ 0 \arrow[r] & P_N[2] \arrow[r] & P_S[2] \arrow[r] & P_Q[2] \arrow[r] & 0 \end{tikzcd}$$

with the horizontal rows being exact. Thus by Lemma A.5

$$\mathrm{Tr}_S(o_S) = \mathrm{Tr}_N(o_N) + \mathrm{Tr}_Q(o_Q)$$

where $\mathrm{Tr}(-)$ is the trace map $\mathrm{Ext}^*_{\mathcal{C}}(-,-) \to \mathrm{HH}_{-*}(\mathcal{C})$ (see §A). Here in the expression $\mathrm{HH}_{-*}(\mathcal{C})$ we regard $\mathcal{C}$ as a $K$-linear DG-category via projective resolutions over $A$.

Applying this to $M$ as in the statement of Proposition B.1 we find

$$0 = \mathrm{Tr}_M(o_M) = (\mathrm{length}\, M)\, \mathrm{Tr}_K(o_K)$$

($\mathrm{Tr}_M(o_M) = 0$ since $M$ has a deformation $M'$). We conclude $\mathrm{Tr}_K(o_K) = 0$. Now $D_{\mathcal{C}}(A)$ has a compact generator $K$ and hence $D_{\mathcal{C}}(A) \cong D(\mathrm{RHom}_A(K, K))$ and by Koszul duality, $\mathrm{RHom}_A(K, K)$ is formal and quasi-isomorphic to $A^! = \mathrm{Ext}^*_A(K, K)$ which is commutative. From this we immediately deduce that

$$\mathrm{Tr}_K : \mathrm{Ext}^*_{\mathcal{C}}(K, K) \cong A^! \to \mathrm{HH}_{-*}(A^!) \cong \mathrm{HH}_{-*}(\mathcal{C})$$

is injective. Thus $o_K = 0$ and hence $K$ has a first order deformation. It is easily seen that this is equivalent to $\{m, m\} \subset m$.

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