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DOCTORAATSPROEFSCHRIFT

Brauer groups of braided fusion categories

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Samenvatting

De hoofdzaak van deze thesis is het karakterizeren van Brauer groepen van gevlochten fusie categorieën, door gebruik te maken van technieken van zwakke Hopf algebras.

Zij (H, R) een quasi-triangulaire zwakke Hopf algebra. Ten eerste tonen we aan dat het vol centrum van het eenheidsobject (zoals in [26]) isomorf is met een centralizer deelalgebra van H. We bewijzen dat dit vol centrum is uitgerust met de structuur van een gevlochten Hopf algebra. Ten tweede gaan we deze gevlochten Hopf algebra ${}_RH$ gebruiken om aan te tonen dat de categorie van Yetter-Drinfeld modulen equivalent is met de categorie van linkse ${}_RH$-comodulen in de categorie (als gevlochten monoïdale categorieën). Ten derde, zij A een gevlochten bi-Galois objecten over ${}_RH$, de functor $A\Box-$ is een gevlochten auto-equivalentie van de categorie van Yetter-Drinfeld modulen als en slechts als A quantum commutatief is. Ten slotte definiëren we de groep bestaande uit quantum commutatieve Galois objecten.

Zij H een eindig dimensionale coquasi-triangulaire zwakke Hopf algebra over een veld $\Bbbk$, we beschrijven een verband tussen de Brauer groep en de groep van quantum commutatieve Galois objecten aan de hand van een rij van groepen. Dit veralgemeent de exact rij in [88] voor zwakke Hopf algebras. Indien $\Bbbk$ een algebraïsch gesloten veld is van karakteristiek nul en indien H cosemi-eenvoudige en co-aangesloten is, bekomen we een isomorfisme tussen de Brauer groep $Br(\mathscr{M}^H)$ en de groep van quantum commutatieve Galois objecten. Als toepassing verkrijgen we dat, voor een gevlochten fusie categorie $\mathscr{C}$, de Brauer groep $Br(\mathscr{C})$ isomorf is met de groep van quantum commutatieve Galois objecten over een zekere gevlochten Hopf algebra. Als voorbeeld bestuderen we de Brauer groep van alle modulaire categorieën verkregen uit $SU(N)_L$-SOS modellen.

Daarenboven gaan we na dat voor een zwakke Hopf algebra H, de verzameling van isomorfisme klassen van H-bi-Galois objecten een groep vormt, met als bewerking het cotensor product. Dit veralgemeent het werk van Schauenburg in [66].

Introduction

In [83] F. Van Oystaeyen and Y. H. Zhang defined and studied the Brauer group $Br(\mathscr{C})$ of a braided monoidal category $\mathscr{C}$, which unifies the Brauer group of a Hopf algebra in [18, 19], the Brauer-Long group in [45, 46] and other known Brauer groups of structured algebras.

When $\mathscr{C}$ is a braided fusion category, the Brauer group $Br(\mathscr{C})$ plays a vital role in the core (an important invariant of $\mathscr{C}$) studied in [30], and in the conjecture of V.G Drinfeld, which states that the pair $(Br(\mathscr{C}), Aut^{br}(\mathscr{C}))$ forms a crossed module. In [28] A. Davydov and D. Nikshych proved this conjecture by applying the Picard crossed module of a braided fusion category $\mathscr{C}$ defined in [31]. They interpret the group in terms of braided autoequivalences of the Drinfeld center $\mathscr{Z}(\mathscr{C})$ of $\mathscr{C}$. What they used as a key point in the proof is an isomorphism between the Brauer group of $\mathscr{C}$ defined in [83] and the subgroup $Aut^{br}(\mathscr{Z}(\mathscr{C}), \mathscr{C})$ consisting of isomorphisms classes of braided autoequivalences of $\mathscr{Z}(\mathscr{C})$ trivializable on $\mathscr{C}$, see Theorem 4.2 in [28]. Thus determining the Picard crossed module and the Brauer group of a braided fusion category $\mathscr{C}$ can be converted to computing or characterizing the group $Aut^{br}(\mathscr{Z}(\mathscr{C}), \mathscr{C})$. This thesis is to investigate this group via braided bi-Galois objects.

The group of bi-Galois objects plays an important role in the study of Brauer groups, see [3, 21, 22, 24, 79, 84]. For example, if H is a finite dimensional Hopf algebra which is commutative and cocommutative, K. H. Ulbrich gave an exact sequence :

$$1 \longrightarrow Br(\Bbbk) \longrightarrow BD(\theta, H^*) \longrightarrow D(\theta, H^*),$$

where θ is a Hopf map from H to H^* and the group $D(\theta, H^*)$ consists of isomorphism classes of certain H^*-Galois objects, see [79]. When $\theta = \varepsilon$, this is just the well-known Beattie's exact sequence in [3]. However, when H is noncommutative and noncocommutative, the group of bi-Galois objects (see [66, 81]) does not fit into the exact sequence as above. In order to deal with the case of a finite dimensional (co)quasi-triangular Hopf algebra H, Y.H. Zhang constructed in [88] a special group $Gal^{qc}({}_RH^*)$ and also proved that there exists an exact sequence of groups

$$1 \longrightarrow Br(\Bbbk) \longrightarrow Br(\mathscr{M}^H) \longrightarrow Gal^{qc}({}_RH^*),$$

where the group $Gal^{qc}({}_RH^*)$ is actually the group of braided Galois objects over some braided Hopf algebra. It was seen in [88] that the group $Gal^{qc}({}_RH^*)$ is relatively easier to be computed. So the Brauer group $Br(\mathscr{M}^H)$ can be studied by investigating the group $Gal^{qc}({}_RH^*)$. Our project is to introduce the group of quantum commutative Galois objects to characterize the Brauer group of a braided fusion category.

However, not every braided fusion category is the category of modules over some finite dimensional Hopf algebra. By [31, Thm 8.33], a modular category $\mathscr{C}$ (a braided fusion category with some additional structures) with $End(1) = \mathbb{C}$ is the category of modules over some finite dimensional quasi-Hopf algebra with a quasi-triangular structure if and only if each simple object of $\mathscr{C}$ has an integer Frobenius-Perron dimension. But there exist many interesting examples of modular categories that contain simple objects of non-integer Frobenius-Perron dimensions. For example, H. H. Andersen provided a construction of a modular category as a quotient of the category of tilting modules over a quantum group $U_q(\mathfrak{g})$ at a root of unity, see [1]. The resulting modular category is in general no longer the category of modules over a Hopf algebra. Therefore, the exact sequence in [88] can not apply to the Brauer groups of these well-known modular categories in [1, 43, 64, 65, 74, 75, 76, 77].

In order to deal with the Brauer groups of these important categories, we need to generalize the exact sequence in [88] to the case of a braided fusion category. Here, the first difficulty that we meet is how to construct the group of so-called braided Galois objects such that it is closely related to the Brauer group $Br(\mathscr{C})$ of a braided fusion category $\mathscr{C}$.

It was proved that there exist special relations between (multi-) fusion categories and representation categories of weak Hopf algebras defined in [8, 9], see [31, 39, 60]. For any fusion category $\mathscr{C}$, V. Ostrik showed in [60] that there exists a weak Hopf algebra $H_{\mathscr{C}}$ such that $\mathscr{C}$ is equivalent to the category of finite dimensional left $H_{\mathscr{C}}$-modules, also see [39]. By [57], if $\mathscr{C}$ is additionally braided, then $H_{\mathscr{C}}$ can be equipped with a quasi-triangular structure such that $\mathscr{C}$ is equivalent to the category of finite dimensional left $H_{\mathscr{C}}$-modules as a braided fusion category. These facts inspire us to construct the desired group by using the techniques from weak Hopf algebras.

Our project is divided into the following four steps.

The first step is to construct a braided Hopf algebra from a quasi-triangular weak Hopf algebra. Let (H, R) be such a weak Hopf algebra. Unlike the Hopf algebra case, the original algebra H can not be deformed into a Hopf algebra in the category of modules over H by Majid's transmutation theory. Here, our method is to consider the full center of the unit object in the sense of [26]. This full center indicates that our braided Hopf algebra ${}_RH$ should be based on some centralizer subalgebra of H, instead of the original algebra H. This is explained in Chapter 2, where the main result reads as follow:

Theorem 1. [Theorem 2.2.7] *Let (H, R) be a quasi-triangular weak Hopf algebra. Then the centralizer subalgebra $C_H(H_s)$ is a braided Hopf algebra (or Hopf algebra in the category $({}_H\mathcal{M}, \otimes_t, H_t, l, r)$) with the following structures:*

The multiplication $\overline{\mu}$ and unit $\overline{\eta}$ are defined by:

$$\overline{\mu} : C_H(H_s) \otimes_t C_H(H_s) \longrightarrow C_H(H_s), \quad a \otimes_t b \longmapsto (1_1 \cdot a)(1_2 \cdot b),$$
$$\overline{\eta} = Id_{H_t} : H_t \longrightarrow C_H(H_s), \quad x \longmapsto x.$$

The comultiplication $\overline{\Delta}$ and counit $\overline{\varepsilon}$ are given by:

$$\overline{\Delta} : C_H(H_s) \longrightarrow C_H(H_s) \otimes_t C_H(H_s), \quad x \longmapsto x_1 S(R^2) \otimes R^1 \cdot x_2,$$
$$\overline{\varepsilon} = \varepsilon_t : C_H(H_s) \longrightarrow H_t, \quad x \longmapsto \varepsilon_t(x).$$

The antipode $\overline{S}$ is given by

$$\overline{S} : C_H(H_s) \longrightarrow C_H(H_s), \quad x \longmapsto R^2 R'^2 S(R^1 x S(R'^1)).$$

The second step is to use the braided Hopf algebra ${}_RH$ to re-describe the Drinfeld center of the category of left H-modules by the category of left ${}_RH$-comodules. Then we discuss the relation between braided autoequivalences of the Drinfeld center and braided bi-Galois objects. Finally, we construct the group of quantum commutative Galois objects. These will appear in Chapter 3. The main result in Chapter 3 is written in the language of weak Hopf algebras as follows:

Theorem 2. [Corollary 3.1.7 and Theorem 3.2.8] *Let (H, R) be a finite dimensional quasi-triangular weak Hopf algebra over a field $\Bbbk$. Then there is a braided monoidal equivalence between the category of Yetter-Drinfeld modules over H and the category of comodules over ${}_RH$. Moreover, if A is a braided bi-Galois object, then the functor $A\Box-$ is a braided autoequivalence of the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules if and only if A is quantum commutative.*

Applying Theorem 2 to a braided fusion category, we obtain the following statement:

Theorem 3. [Corollary 3.2.9] *Let $\mathscr{C}$ be a braided fusion category. Then the Drinfeld center of $\mathscr{C}$ is equivalent to the category of finite dimensional left comoduels over some braided Hopf algebra $H_{\mathscr{C}}$. Moreover, if A is a braided bi-Galois object over $H_{\mathscr{C}}$, then the functor $A\Box-$ is a braided autoequivalence of the Drinfeld center of $\mathscr{C}$ if and only if A is quantum commutative.*

The third step is to relate the Brauer group to the group of quantum commutative Galois objects that we constructed in Chapter 3. This is to generalize the exact sequence in [88] to the case of a weak Hopf algebra. The main result in Chapter 4 reads as follows:

Theorem 4. [Theorem 4.3.9 and Corollary 4.4.6] *Let H be a finite dimensional coquasi-triangular weak Hopf algebra over a field $\Bbbk$. Let ${}_RH^*$ be the associated braided Hopf algebra (apply Theorem 1 to the dual H^*). Then there exists a sequence of groups*

$$Br(\mathscr{M}^{H_m}) \hookrightarrow Br(\mathscr{M}^H) \longrightarrow Gal^{qc}({}_RH^*),$$

where H_m is the minimal weak Hopf algebra of H and $Gal^{qc}({}_RH^*)$ is the group of quantum commutative Galois objects over ${}_RH^*$.

The last part is to show that the Brauer group $Br(\mathscr{C})$ of a braided fusion category $\mathscr{C}$ is isomorphic to the group of quantum commutative Galois objects when the base field is algebraically closed:

Theorem 5. [Theorem 5.2.6] *Let H be a finite dimensional coquasi-triangular weak Hopf algebra over an algebraically closed field k of characteristic zero such that it is cosemisimple and co-connected. Then there is an isomorphism between the Brauer group $Br(\mathscr{M}^H)$ and the group $Gal^{qc}({}_RH^*)$ of quantum commutative Galois objects.*

In the language of a braided fusion category, we have the following:

Theorem 6. [Corollary 5.2.7] *Let $\mathscr{C}$ be a braided fusion category. Then the Brauer group $Br(\mathscr{C})$ of $\mathscr{C}$ is isomorphic to the group of quantum commutative Galois objects over some braided Hopf algebra.*

Since a ribbon (modular) category is a braided fusion category with some additional structures (see Section 1.2), Theorem 6 also holds in the ribbon (modular) case, see Corollary 5.2.8 (5.2.9). As an application, we compute the Brauer groups of all modular categories obtained from $SU(N)_L$-SOS models by showing that they are isomorphic to the groups of Galois objects over the corresponding Hopf algebras, see Theorem 5.3.24 and 5.3.27.

As the direct sum of Hopf algebras is a weak Hopf algebra, the Brauer group of a modular categories obtained from $SU(N)_L$-SOS models can be characterized by the group of Galois objects over a weak Hopf algebra (the direct sum of Hopf algebras). This motivates us to consider how to form the group of bi-Galois objects over a weak Hopf algebra. This generalizes Schauenburg's work in [66] as follows:

Theorem 7. [Theorem 6.2.14] *Let H be a faithfully flat weak Hopf algebra. Let $Gal(H, H_t)$ be the set of isomorphism classes of H- bi-Galois objects. Then $Gal(H, H_t)$ forms a group under the cotensor product $\square_H$.*

Notations and conventions

In this thesis, $\Bbbk$ is a fixed field. All vector spaces, algebras and coalgebras are assumed to be over $\Bbbk$. If not stated otherwise, all algebras are associative $\Bbbk$- algebras with unities. The unabored tensor $\otimes$ means $\otimes_{\Bbbk}$. A module over an algebra always means a unitary module. $(-)^*$ denotes the functor $\mathrm{Hom}_{\Bbbk}(-, \Bbbk)$.

A category in this thesis means an abelian category. All functors between such categories are additive. We will use the symbol $\cong$ for an equivalence between categories and the symbol $\simeq$ for an isomorphism between two objects (algebras, modules, sets and vector spaces). The symbols id and Id will mean the identity map or functor.

A braided fusion category means over an algebraically closed field $\Bbbk$ of characteristic 0. In some proofs, we often use morphisms to replace diagrammatic methods for the sake of simplification.

Without otherwise stated, a weak Hopf algebra always means a weak Hopf algebra over a filed $\Bbbk$ with a bijective antipode.

We use the (sumless) Sweedler's notation for the comultiplication and coaction (see [71]):

(1) For a weak Hopf algebra H, for $h \in H$, $\Delta(h) = h_1 \otimes h_2$. For a left H-comodule M, ρ^L stands for the left H-coaction, $\rho^L(m) = m_{[-1]} \otimes m_{[0]}$ for all $m \in M$. For a right H-comodule N, ρ^Rstands for the right H-coaction, $\rho^R(n) = n_{[0]} \otimes n_{[1]}$ for all $n \in N$.

(2) For a braided Hopf algebra H, we let $\Delta'(a) = a_{(1)} \otimes a_{(2)}$. For a left H-comodule M', we use ρ^l for the left H-coaction, $\rho^l(m') = m'_{(-1)} \otimes m'_{(0)}$ for all $m' \in M'$. For a right H-comodule N', ρ^r means the right H-coaction, $\rho^r(n') = n'_{(0)} \otimes n'_{(1)}$ for all $n' \in N'$.

Chapter 1

Preliminaries

In this chapter, we will present some basic definitions and properties that are needed in this thesis. The notions of a braided monoidal (fusion) category, a braided Hopf algebra and the full center of an algebra will be first recalled. Second, we will follow the process in [83] to give the definition of the Brauer group of a braided monoidal category. Finally, the theory of weak Hopf algebras, our main tool in this thesis, will be briefly recalled.

1.1 Braided monoidal categories

We will briefly recall the notion of a braided (rigid) monoidal category. For more detailed discussion about braided monoidal categories, we refer to [42].

1.1.1 Monoidal categories

Definition 1.1.1. A *monoidal category* $(\mathscr{C}, \otimes, I, a, l, r)$ is a category $\mathscr{C}$ equipped with

- a tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$;
- an object I, called the *unit* of the monoidal category;
- a natural isomorphism $a : \otimes(\otimes \,\mathrm{id}) \to \otimes(\mathrm{id} \otimes)$, called the *associativity constraint*

- a natural isomorphism $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$, called the *left unit constraint* with respect to I;
- a natural isomorphism $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$, called the *right unit constraint* with respect to I;

such that the *Pentagon Axiom* and the *Triangle Axiom* are satisfied. That is, the following two diagrams are commutative for all objects U, V, W and X in $\mathscr{C}$.

$$
\begin{array}{ccc}
(U \otimes (V \otimes W)) \otimes X & \xleftarrow{a_{U,V,W} \otimes \mathrm{id}_X} & ((U \otimes V) \otimes W) \otimes X \\
 & & \downarrow{\scriptstyle a_{U \otimes V, W, X}} \\
\downarrow{\scriptstyle a_{U, V \otimes W, X}} & & (U \otimes V) \otimes (W \otimes X) \\
 & & \downarrow{\scriptstyle a_{U,V,W \otimes X}} \\
U \otimes ((V \otimes W) \otimes X) & \xrightarrow{\mathrm{id} \otimes a_{V,W,X}} & U \otimes (V \otimes (W \otimes X))
\end{array}
$$

$$
\begin{array}{ccc}
(V \otimes I) \otimes W & \xrightarrow{a_{V,I,W}} & V \otimes (I \otimes W) \\
 & {\scriptstyle r_V \otimes \mathrm{id}_W} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_V \otimes l_W} & \\
 & V \otimes W &
\end{array}
$$

Definition 1.1.2. Let $(\mathscr{C}, \otimes, I, a, l, r)$ and $(\mathscr{D}, \otimes, I, a, l, r)$ be monoidal categories. A *lax monoidal functor* from $\mathscr{C}$ to $\mathscr{D}$ is a triple $(F, \varphi_0, \varphi_2)$, where $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor, and φ_0 is a morphism from $I_{\mathscr{D}}$ to $F(I_{\mathscr{D}})$, and

$$\varphi_2 : F(U) \otimes F(V) \longrightarrow F(U \otimes V)$$

is a family of natural transformations indexed by all couples (U, V) of objects of $\mathscr{C}$ such that the following three diagrams are commutative for all objects (U, V, W) in $\mathscr{C}$.

$$
\begin{array}{ccc}
(F(U) \otimes F(V)) \otimes F(W) & \xrightarrow[a_{F(U),F(V),F(W)}]{} & F(U) \otimes ((F(V) \otimes F(W)) \\
\downarrow{\scriptstyle \varphi_2(U,V) \otimes id_{F(W)}} & & \downarrow{\scriptstyle id_{F(U)} \otimes \varphi_2(V,W)} \\
F(U \otimes V) \otimes F(W) & & F(U) \otimes F(V \otimes W) \\
\downarrow{\scriptstyle \varphi_2(U \otimes V, W)} & & \downarrow{\scriptstyle \varphi_2(U, V \otimes W)} \\
F((U \otimes V) \otimes W) & \xrightarrow{F(a_{U,V,W})} & F(U \otimes (V \otimes W))
\end{array}
$$

$$\begin{array}{ccc} I\otimes F(U) & \xrightarrow{\;\; l_{F(U)} \;\;} & F(U) \\ {\scriptstyle \varphi_0\otimes id_{F(U)}}\downarrow & & \downarrow{\scriptstyle F(l_U^{-1})} \\ F(I)\otimes F(U) & \xrightarrow{\;\; \varphi_2(I,U) \;\;} & F(I\otimes U) \end{array}$$

$$\begin{array}{ccc} F(U)\otimes I & \xrightarrow{\;\; r_{F(U)} \;\;} & F(U) \\ {\scriptstyle id_{F(U)}\otimes\varphi_0}\downarrow & & \downarrow{\scriptstyle F(r_U^{-1})} \\ F(U)\otimes F(I) & \xrightarrow{\;\; \varphi_2(U,I) \;\;} & F(U\otimes I) \end{array}$$

In particular, if φ_0 and φ_2 are additionally natural isomorphisms, then we call the lax monoidal functor $(F, \varphi_0, \varphi_2)$ a *monoidal functor*.

A *natural monoidal transformation* $\eta : (F, \varphi_0, \varphi_2) \longrightarrow (F', \varphi_0', \varphi_2')$ between monoidal functors from $\mathscr{C}$ to $\mathscr{D}$ is a natural transformation $\eta : F \longrightarrow F'$ such that the following diagrams commute for each couple (U, V) of objects in $\mathscr{C}$.

$$\begin{array}{ccc} F(U)\otimes F(V) & \xrightarrow[\;\; \varphi_2(U,V) \;\;]{} & F(U\otimes V) \\ {\scriptstyle \eta_{F(U)}\otimes\eta_{F(V)}}\downarrow & & {\scriptstyle \eta_{F(U\otimes V)}}\downarrow \\ F'(U)\otimes F'(V) & \xrightarrow{\;\; \varphi_2(U,V) \;\;} & F'(U\otimes V) \end{array}$$

$$\begin{array}{ccc} I_{\mathscr{D}} & \xrightarrow{\;\; \varphi_0 \;\;} & F(I_{\mathscr{C}}) \\ & {\scriptstyle \varphi_0'}\searrow \quad \swarrow{\scriptstyle \eta(I)} & \\ & F'(I_{\mathscr{C}}) & \end{array}$$

A *monoidal equivalence* between monoidal categories is a monoidal functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ such that there exist a monoidal functor $F' : \mathscr{D} \longrightarrow \mathscr{C}$ and natural monoidal isomorphisms $\eta : Id_{\mathscr{D}} \cong FF'$ and $\eta' : Id_{\mathscr{C}} \cong F'F$. In case there exists a monoidal equivalence between $\mathscr{C}$ and $\mathscr{D}$, we say that $\mathscr{C}$ and $\mathscr{D}$ are *monoidal equivalent*.

The monoidal functor $(F, \varphi_0, \varphi_2)$ is said to be *strict* if the isomorphisms φ_0 and φ_2 are identities in $\mathscr{D}$.

A monoidal category is said to be *strict* if the associativity and the unit constraints $a, 1, r$ are all identities of the category. In particular, a monoidal category is always equivalent to a strict one, see Proposition XI.5.1 in [42].

Definition 1.1.3. (1) A *left dual* of an object V in a monoidal category is a triple

(V^*, b_V, d_V), where V^* is another object and $b_V : I \to V \otimes V^*$, $d_V : V^* \otimes V \to I$, are morphisms such that the compositions

$$V \xrightarrow{l_V^{-1}} I \otimes V \xrightarrow{b_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes d_V} V \otimes I \xrightarrow{r_V} V$$

and

$$V^* \xrightarrow{r_{V^*}^{-1}} V^* \otimes I \xrightarrow{\mathrm{id}_{V^*} \otimes b_V} V^* \otimes V \otimes V^* \xrightarrow{d_V \otimes \mathrm{id}_{V^*}} I \otimes V^* \xrightarrow{l_{V^*}} V^*$$

are identities of V and V^* respectively.

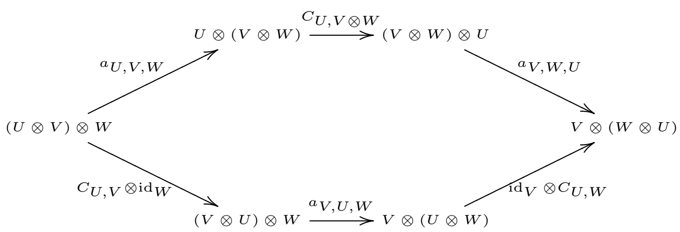
(2) A monoidal category $\mathscr{C}$ is called *rigid* if any object V in $\mathscr{C}$ admits a left dual.

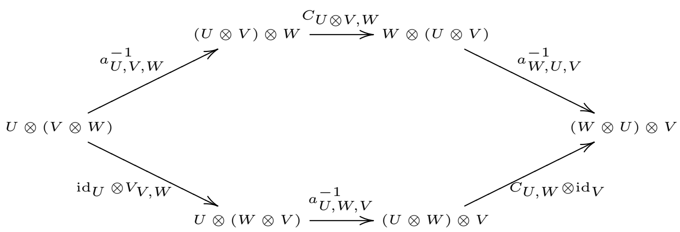
1.1.2 Braided monoidal categories

Let $\mathscr{C}$ be a monoidal category with a tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. Denote by $\tau : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ the *flip functor*. That is, $\tau(V, W) = (W, V)$ for any V, W in $\mathscr{C}$. A *commutativity constraint* C is a natural isomorphism $C : \otimes \to \otimes\tau$.

Definition 1.1.4. Let $(\mathscr{C}, \otimes, I, a, l, r)$ be a monoidal category.

(1) A *braiding* is a commutativity constraint C satisfying the *Hexagon Axiom*. That is, the following two diagrams commute for any objects U, V and W in $\mathscr{C}$.





(2) A *braided monoidal category* $(\mathscr{C}, \otimes, I, a, l, r, C)$ is a monoidal category with a braiding.

Definition 1.1.5. A monoidal functor $(F, \varphi_0, \varphi_2)$ from a braided monoidal category $(\mathscr{C}, \otimes, C)$ to a braided monoidal category $(\mathscr{D}, \otimes, D)$ is called *braided* if for any pair (U, V) of objects in $\mathscr{C}$, the square

$$\begin{array}{ccc} F(U) \otimes F(V) & \xrightarrow{\varphi_2(U,V)} & F(U \otimes V) \\ {\scriptstyle D_{F(U),F(V)}} \downarrow & & \downarrow {\scriptstyle F(C_{U,V})} \\ F(V) \otimes F(U) & \xrightarrow{\varphi_2(V,U)} & F(V \otimes U) \end{array}$$

commutes.

A braided monoidal functor $(F, \varphi_0, \varphi_2)$ is called a *braided monoidal equivalence* if the functor F is a monoidal equivalence.

Note that if C is a braiding, so is the inverse C^{-1}. We denote by $(\mathscr{C}^{rev}, \otimes, I, C^{-1})$ the braided monoidal category $(\mathscr{C}, \otimes, I)$ with the braiding C^{-1}.

The most fundamental example of a braided monoidal category is the following Drinfeld center of a monoidal category.

Example 1.1.6. The *right Drinfeld center* $\mathscr{Z}_r(\mathscr{C})$ of a monoidal category $\mathscr{C}$ is defined to be the category whose objects are pairs $(M, \nu_{-,M})$, where M is an object of $\mathscr{C}$ and $\nu_{-,M}$ is a family of natural isomorphisms, called half braidings:

$$\nu_{M,N} : M \otimes N \longrightarrow N \otimes M, \quad \forall\ N \in \mathcal{C}$$

satisfying Hexagon Axiom, see [42]. Similarly, the *left Drinfeld center* $\mathscr{Z}_l(\mathscr{C})$ is the category, whose objects are pairs $(U, \nu_{U,-})$, where U is an object of $\mathscr{C}$ and $\nu_{U,-}$ is a family of natural isomorphisms satisfying Hexagon Axiom. Then the left center $\mathscr{Z}_l(\mathscr{C})$ and the right center $\mathscr{Z}_r(\mathscr{C})$ are two braided monoidal categories. By Proposition 1.1 in [13], there exist braided equivalences of braided monoidal categories

$$\mathscr{Z}_l(\mathscr{C}) \cong \mathscr{Z}_r(\mathscr{C})^{rev}, \quad \mathscr{Z}_r(\mathscr{C}) \cong \mathscr{Z}_l(\mathscr{C})^{rev}.$$

Definition 1.1.7. A braided monoidal category $\mathscr{C}$ is called *closed* if the functor $- \otimes X : \mathscr{C} \longrightarrow \mathscr{C}$ has a right adjoint for all objects X in $\mathscr{C}$. The right adjoint, called the *inner hom functor*, will be denoted by $[X, -] : \mathscr{C} \longrightarrow \mathscr{C}$.

Definition 1.1.8. An object P in a closed braided monoidal category is a finite object of $\mathscr{C}$ if the canonical morphism

$$[P, P] \otimes [P, I] \longrightarrow [P, P]$$

is an isomorphism. This is equivalent to the existence of a 'dual basis'.

Definition 1.1.9. A braided monoidal category $\mathscr{C}$ is called *rigid* if $\mathscr{C}$ is rigid as a monoidal category.

Remark 1.1.10. (1) A rigid braided monoidal category $\mathscr{C}$ is closed because the functor $- \otimes X^*$ is a right adjoint of $- \otimes X$ for any object in $\mathscr{C}$.

(2) The category of finite-dimensional vector spaces over a field is rigid.

(3) Any object in a rigid braided monoidal category is finite.

1.2 Braided fusion categories

In this thesis, a (braided) fusion category will always mean over an algebraically closed field $\Bbbk$ of characteristic 0. We refer the reader to [2, 30, 31] for a general theory of braided fusion categories. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Here we give the notion of a (braided, ribbon, modular) fusion category.

Definition 1.2.1. (1) An object U in an abelian category $\mathscr{C}$ is called *simple* if any injection $V \hookrightarrow U$ is either 0 or an isomorphism.

(2) An abelian category $\mathscr{C}$ is called *semisimple* if any object V is isomorphic to a direct sum of simple objects:

$$V \simeq \bigoplus_{V_i \in J} N_i V_i,$$

where V_i are simple objects, J is the set of isomorphism classes of non-zero simple objects in $\mathscr{C}$, $N_i \in \mathbb{Z}_+$ and only a finite number of N_i is non-zero.

Definition 1.2.2. An abelian category $\mathscr{C}$ is $\Bbbk$-*linear* if each hom set is a vector space and the composition of morphisms is bilinear.

Definition 1.2.3. A $\Bbbk$-linear abelian category $\mathscr{C}$ is called *finite* if

(1) the morphism spaces are finite dimensional;

(2) every object is of finite length;

(3) $\mathscr{C}$ has enough projectives, i.e., every simple object has a projective cover;

(4) there are only finitely many simple objects (up to isomorphism) in $\mathscr{C}$.

Remark 1.2.4. A finite $\Bbbk$-linear abelian category is equivalent to the representation category of some finite dimensional algebra.

Definition 1.2.5. [34] Let $\mathscr{C}$ be a rigid monoidal category.

(1) A finite *multi-tensor category* $\mathscr{C}$ is a finite abelian $\Bbbk$-linear rigid monoidal category, where the tensor product is $\Bbbk$-bilinear.

(2) A finite multi-tensor category $\mathscr{C}$ is called a finite *tensor category* if the unit object I is simple.

Definition 1.2.6. [30, 31] Let $\mathscr{C}$ be a finite tensor category.

(1) The category $\mathscr{C}$ is called a *fusion category* if $\mathscr{C}$ is semisimple.

(2) A fusion category $\mathscr{C}$ is a *braided fusion* category if $\mathscr{C}$ is additionally braided.

For any two simple object V_i and V_j in a fusion category, we have

$$Hom(V_i, V_j) = \begin{cases} \Bbbk & if \quad i = j, \\ 0 & if \quad i \neq j. \end{cases}$$

The theory of Frobenius-Perron dimensions or modules over fusion categories can be found in [31] and [60]. Now we follow [2] to define a ribbon category and a modular category.

Definition 1.2.7. A braided fusion category $(\mathscr{C}, \otimes, I, a, l, r, C, (-)^*, b, d)$ is called a *ribbon category* if there exist natural isomorphisms (*ribbon twist*) $\theta_X : X \longrightarrow X$ such that

$$\theta_{X\otimes Y} = C_{Y,X} \circ C_{X,Y} \circ (\theta_X \otimes \theta_Y)$$

and

$$(\theta_X \otimes id_{X^*}) \circ b_X = (id_X \otimes \theta_X) \circ b_X.$$

Following [75], the quantum trace of an endomorphism $f \in End(V)$ in a ribbon category is defined to be

$$tr_q(f) = d_V C_{V,V^*}(\theta f \otimes id_{V^*}) b_v$$

with values in $End(I)$. In particular, the quantum trace $tr_q(id_V)$ is called the quantum dimension of V.

Definition 1.2.8. A ribbon category $(\mathscr{C}, \otimes, I, a, l, r, C, (-)^*, b, d, \theta)$ is called a *modular category* if the following conditions are satisfied:

(1) The number of isomorphism classes of simple objects is finite;

(2) The square matrix $S = \{S_{i,j}\}_{i,j\in J} = \{tr_q(C_{V_i,V_j}C_{V_j,V_i})\}$ is invertible over $\Bbbk$.

Remark 1.2.9. Many authors consider a ribbon category by not necessarily requiring semisimplicity in our definition, see [75]. We refer the reader to [44, 49] for the non-semisimple case.

1.3 Braided Hopf algebras

A Hopf algebra in a braided monoidal category is also called a braided Hopf algebra or a braided group, see [50, 51, 52]. The general construction of Hopf algebras in a braided monoidal category can be found in [51, 52]. Here, we briefly recall from [52] a braided Hopf algebra.

1.3.1 Braided Hopf algebras

Definition 1.3.1. Let $(\mathscr{C}, \otimes, I, a, l, r)$ be a monoidal category.

(1) An *algebra* in $\mathscr{C}$ is a triple $A = (A, \eta_A, \mu_A)$, where A is an object in $\mathscr{C}$ and $\eta_A : I \longrightarrow A$ (unit) and $\mu_A : A \otimes A \longrightarrow A$ are morphisms in $\mathscr{C}$ such that $\mu_A \circ (id_A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes id_A)$, $\mu_A \circ (id_A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes id_A)$.

(2) A *coalgebra* in $\mathscr{C}$ is a triple $D = (D, \varepsilon_D, \Delta_D)$, where D is an object in $\mathscr{C}$ and $\varepsilon_D : D \longrightarrow I$ (counit) and $\Delta_D : D \longrightarrow D \otimes D$ are morphisms in $\mathscr{C}$ such that $(id_D \otimes \varepsilon_D) \circ \Delta_D = id_D = (\varepsilon_D \otimes id_D) \circ \Delta_D$, $(id_D \otimes \Delta_D) \circ \Delta_D = (\Delta_D \otimes id_D) \circ \Delta_D$.

Let (A, μ_A) and (B, μ_B) be two algebras in $\mathscr{C}$. An object M in $\mathscr{C}$ is called an *A-B-bimodule* if there are morphisms in $\mathscr{C}$

$$m_A : A \otimes M \longrightarrow M, \quad m_B : M \otimes B \longrightarrow M$$

satisfying the coherence conditions:

$$m_A(\mu_A \otimes id_M) = m_A(id_A \otimes m_A) \text{ on } A \otimes A \otimes M,$$
$$m_B(id_M \otimes \mu_B) = m_B(m_B \otimes id_B) \text{ on } M \otimes B \otimes B,$$
$$m_A(id_A \otimes m_B) = m_B(m_A \otimes id_B) \text{ on } A \otimes M \otimes B.$$

The bimodule category ${}_A\mathscr{C}_B$ consists of objects in $\mathscr{C}$ which have A-B-bimodule structures, and morphisms in $\mathscr{C}$ which are A-B-bilinear. It is easy to see that I is an algebra. Write $\mathscr{C}_B$ and ${}_A\mathscr{C}$ for the categories ${}_I\mathscr{C}_B$ and ${}_A\mathscr{C}_I$ respectively.

Dually, for a coalgebra D, we define a left D-comodule, a right D-comodule and a D-bicomodule. We denote by ${}^D\mathscr{C}$ ($\mathscr{C}^D$, ${}^D\mathscr{C}^D$) the category of left D-comodules (right D-comodules, D-bicomodules).

Now assume that a monoidal category $(\mathscr{C}, \otimes, I, a, l, r, C)$ has a braiding C.

An algebra (A, η_A, μ_A) in $\mathscr{C}$ is called *braided-commutative* if

$$\mu_A = \mu_A \circ C_{A,A}.$$

Definition 1.3.2. We call $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H, S_H)$ *a bialgebra in a braided monoidal category* $\mathscr{C}$ if (H, μ_H, η_H) is an algebra in $\mathscr{C}$ and $(H, \Delta_H, \varepsilon_H)$ is a coalgebra in $\mathscr{C}$ such that the map ε_H is an algebra map and

$$\Delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (id_H \otimes C_{H,H} \otimes id_H) \circ (\Delta_H \otimes \Delta_H).$$

A bialgebra $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ in $\mathscr{C}$ is called a *braided Hopf algebra* or a Hopf algebra in $\mathscr{C}$ if there exists additionally a morphism $S_H : H \longrightarrow H$ in $\mathscr{C}$, called the *antipode*, satisfying

$$\mu_H \circ (S_H \otimes id_H) \circ \Delta_H = \varepsilon_H \circ \eta_H = \mu_H \circ (id_H \otimes S_H) \circ \Delta_H.$$

Definition 1.3.3. Let $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H, S_H)$ be a braided Hopf algebra. An algebra (A, μ_A, η_A) is called a *right H-comodule algebra* in $\mathscr{C}$ if there exists a right coaction $\rho^r : A \longrightarrow A \otimes H$ such that (A, ρ^r) is a right H-comodule satifying

$$\rho^r \circ \mu_A = (\mu_A \otimes \mu_H) \circ (id_A \otimes C_{H,A} \otimes id_H) \circ (\rho^r \otimes \rho^r).$$

Similarly, one can define a *left H-comodule algebra* in $\mathscr{C}$.

Remark 1.3.4. Let H be a braided Hopf algebra. By [69] the right H-comodule category is a monoidal category, where for any two right H-comodules (M, ρ^r_M) and (N, ρ^r_M), $M \otimes N$ is a right H-comodule with the following structure:

$$\rho^r_{M\otimes N} = (id_M \otimes id_N \otimes \mu_H) \circ (id_M \otimes C_{H,N} \otimes id_N) \circ (\rho^r_M \otimes \rho^r_N).$$

Definition 1.3.5. Let H be a baided Hopf algebra and A a right H-comodule algebra $(A, \mu_A, \eta_A, \rho^r_A)$. A right A-module is called a Doi-Hopf module if there exists a right H-coaction ρ^r_M such that (A, ρ^r) is a right H-comodule and

$$\rho \circ m_A = (m_A \otimes \mu_H) \circ (id_M \otimes C_{H,A} \otimes id_H) \circ (\rho^r_M \otimes \rho^r_A).$$

Denote by $\mathscr{C}^H_A$ the category of Doi-Hopf modules in $\mathscr{C}$. A morphism in $\mathscr{C}^H_A$ is a morphism in $\mathscr{C}$ such that it is right A-linear and H-colinear.

1.3.2 Braided bi-Galois objects

Let $(\mathscr{C}, \otimes, I, a, l, r, C)$ be a braided monoidal category. An object P in $\mathscr{C}$ is called *flat* if the functor $P \otimes - : \mathscr{C} \longrightarrow \mathscr{C}$ preserves equalizers. If it, in addition, reflects isomorphisms, then P is called *faithfully flat*.

Definition 1.3.6. [25, 70] Let H be a Hopf algebra in $\mathscr{C}$. A right H-comodule algebra $(A, \mu_A, \eta_A, \rho^r)$ in $\mathscr{C}$ is a right H-Galois object if A is faithfully flat and the morphism

$$\beta = (\mu_A \otimes id_H) \circ (id_A \otimes \rho^r) : A \otimes A \longrightarrow A \otimes H$$

is an isomorphism in $\mathscr{C}$.

Similarly, one can define a left H-Galois object and an H-bi-Galois object, see [25, 70]. The relation between a right Galois object and the category of Doi-Hopf modules is as follows:

Theorem 1.3.7. *[25, 70] Let H be a flat braided Hopf algebra in $\mathscr{C}$ and A a right H-comodule algebra in $\mathscr{C}$. Then A is a right H-Galois object if and only if the functor $- \otimes A : \mathscr{C} \longrightarrow \mathscr{C}^H_A$ defines an equivalence.*

1.4 The Brauer group of a braided monoidal category

For the general theory of the Brauer groups of braided monoidal categories, the reader is referred to [83]. In this thesis, we only consider the Brauer group of a rigid braided monoidal category. Let $(\mathscr{C}, \otimes, I, a, l, r, C)$ be a rigid braided monoidal category.

Definition 1.4.1. [61] An object P in $\mathscr{C}$ is called *faithfully projective* if the morphism

$$[P, I] \otimes_{[P,P]} P \longrightarrow I$$

induced by the evaluation, is an isomorphism.

Let (A, μ_A) be an algebra in $\mathscr{C}$. There is another algebra $(\overline{A}, \mu_{\overline{A}})$, where $\overline{A} = A$ as an object in $\mathscr{C}$ and the multiplication is defined by $\mu_{\overline{A}} = \mu_A C_{A,A}$. The algebra $(\overline{A}, \mu_{\overline{A}})$ is called the *opposite algebra* of A.

Let (A, μ_A) and (B, μ_B) be two algebras in $\mathscr{C}$. We have an algebra $(A\sharp B, \mu_{A\sharp B})$, where $A\sharp B = A \otimes B$ as an object in $\mathscr{C}$ and the multiplication is defined by $\mu_{A\sharp B} = (\mu_A \otimes \mu_B)(id_A \otimes C_{B,A} \otimes id_B)$. In particular, the algebras $A\sharp\overline{A}$ and $\overline{A}\sharp A$ are called $\mathscr{C}$*-enveloping algebras* of A and denoted by A^e and ${}^eA$ respectively.

An object M in ${}_A\mathscr{C}_B$ is called B*-coflat* if for all algebras $C, D \in \mathscr{C}$ and objects $X \in {}_B\mathscr{C}_C$, $M \otimes_B X$ exists and the natural morphism $M \otimes_B (X \otimes Y) \longrightarrow (M \otimes_B X) \otimes Y$ in ${}_A\mathscr{C}_D$ is an isomorphism for any $Y \in \mathscr{C}_D$. In particular, M is called *bicoflat* if it is both A-coflat and B-coflat.

Definition 1.4.2. A *Morita context* in $\mathscr{C}$ is a sextuple $(A, B, {}_AP_B, {}_BQ_A, f, g)$ consisting of algebras $A, B \in \mathscr{C}$, an A-B-bimodule $P \in {}_A\mathscr{C}_B$, a B-A-bimodule $Q \in {}_B\mathscr{C}_A$, and bilinear morphisms

$$f : P \otimes_B Q \longrightarrow A, \quad g : Q \otimes_A P \longrightarrow B$$

such that the following diagrams commute:

$$\begin{array}{ccc} P \otimes_B Q \otimes_A P & \xrightarrow{f \otimes id} & A \otimes_A P \\ {\scriptstyle id \otimes g}\downarrow & & \downarrow \\ P \otimes_B B & \longrightarrow & P \end{array}$$

$$\begin{array}{ccc} Q \otimes_A P \otimes_B Q & \xrightarrow{id \otimes f} & Q \otimes_A A \\ {\scriptstyle g \otimes id} \downarrow & & \downarrow \\ B \otimes_B Q & \longrightarrow & Q. \end{array}$$

Theorem 1.4.3. *[83, Thm. 2.1] Let* $(A, B, {}_AP_B, {}_BQ_A, f, g)$ *be a Morita context in* $\mathscr{C}$*. If* P *and* Q *are bi-coflat,* f *and* g *are bijective, then the following hold:*

1. $A \simeq [P, P]_B \simeq {}_B\{Q, Q\}$ *and* $B \simeq [Q, Q]_A \simeq_A \{P, P\}$ *as algebras in* $\mathscr{C}$*;*
2. $P \simeq [Q, A]_A \simeq_B \{Q, B\}$ *and* $Q \simeq [P, B]_B \simeq_A \{P, A\}$ *as bi-modules in in* $\mathscr{C}$*;*
3. $A_P, P_B, {}_BQ$ *and* Q_A *are faithfully projective.*

In particular, if P is a faithfully projective object in $\mathscr{C}$, then as algebras in $\mathscr{C}$,

$$[P, P] \simeq \{P^*, P^*\},\ \{P, P\} \simeq [P^*, P^*],\ \overline{[P, P]} \simeq \{P, P\}.$$

More detail about Morita contexts can be found in the subsection 2.4 of [25].

Definition 1.4.4. An algebra (A, μ_A) in $\mathscr{C}$ is called an *Azumaya algebra* in $\mathscr{C}$ if A is faithfully projective in $\mathscr{C}$ and the following canonical morphisms are isomorphisms:

$$F : A\sharp\overline{A} \longrightarrow [A, A],\ F(a\sharp\overline{b})(c) = a\mu_{\overline{A}}(b \otimes c),$$
$$G : \overline{A}\sharp A \longrightarrow \overline{[A, A]},\ G(\overline{a}\sharp b)(c) = \mu_{\overline{A}}(c \otimes a)b.$$

Following Corollary 2.2 in [83], the following statements hold:

1. If P is faithfully projective in $\mathscr{C}$, then $[P, P]$ is an Azumaya algebra in $\mathscr{C}$.
2. If A is an Azumaya algebra in $\mathscr{C}$, so is the opposite algebra $\overline{A}$.
3. If A and B are Azumaya algebras in $\mathscr{C}$, so is $A\sharp B$.

Two Azumaya algebras A and B in $\mathscr{C}$ are called *Brauer equivalent*, denoted by $A \sim B$, if there exist faithfully projective objects M and N in $\mathscr{C}$ such that

$$A\sharp[M, M] \simeq B\sharp[N, N]$$

as algebras. This defines an equivalence relation in the set $B(\mathscr{C})$ of isomorphism classes of Azumaya algebras. The quotient set $B(\mathscr{C})/\sim$ is denoted by $Br(\mathscr{C})$. For an Azumaya algebra in $\mathscr{C}$, we will always use $[A]$ to represent the equivalent class of A.

The product $\sharp$ induces an operation on the quotient set $Br(\mathscr{C})$, that is, $[A][B] = [A\sharp B]$ for two Azumaya algebras A and B. Then by [83] the set $Br(\mathscr{C})$ forms a group with the product induced by $\sharp$, where the identity element is $[I]$ or the class of $[P, P]$ for a faithfully projective object $[P]$, and the inverse of $[A]$ is given by $[\overline{A}]$. This group is called the *Brauer group* of $\mathscr{C}$.

Remark 1.4.5. (1) By Theorem 3.1 in [83], the algebra A is an Azumaya algebra in $\mathscr{C}$ if and only if the following two functors are equivalent:

$$A \otimes - : \mathscr{C} \longrightarrow {}_{A\sharp\overline{A}}\mathscr{C}, \quad - \otimes A : \mathscr{C} \longrightarrow \mathscr{C}_{\overline{A}\sharp A}.$$

(2) For an Azumaya algebra A, the inverse functor of $A \otimes -$ is given by

$${}_{A\sharp\overline{A}}[A, -] : {}_{A\sharp\overline{A}}\mathscr{C} \longrightarrow \mathscr{C},$$

which is isomorphic to the functor $(-)^A$, see Proposition 3.4 in [25].

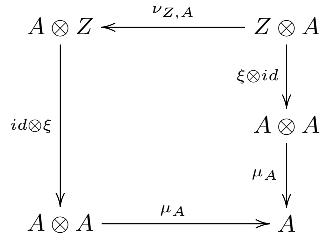
(3) This categorical construction of Brauer groups is quite general. All known Brauer groups can be derived from this unifying definition, see Example 3.6-3.12 in [83].

1.5 The full center of an algebra

Let $(\mathscr{C}, \otimes, I, a, l, r)$ be a monoidal category. We will recall the *full center* of an algebra from Section 4 of [26], which plays an important role in this thesis.

Definition 1.5.1. Let (A, μ_A) be an algebra in a monoidal category $\mathscr{C}$. The *full center* $\mathcal{Z}(A)$ of A is an object in the left Drinfeld center $\mathscr{Z}_l(\mathscr{C})$ together with a morphism $\mathcal{Z}(A) \longrightarrow A$ in $\mathscr{C}$ such that the following universal property holds:

For any pair (Z, ξ) with $Z \in \mathscr{Z}_l(\mathscr{C})$ and $\xi : Z \longrightarrow A$ is a morphism in $\mathscr{C}$ such that the diagram



commutes in $\mathscr{C}$, where $\nu_{Z,-}$ is the half-braiding of Z as an object of $\mathscr{Z}_l(\mathscr{C})$, there exists a unique morphism $Z \longrightarrow \mathcal{Z}(A)$ in $\mathscr{Z}_l(\mathscr{C})$, such that the following diagram is commutative:

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Z}(A) \\ & {\scriptstyle\xi}\searrow \quad \swarrow & \\ & A. & \end{array}$$

Proposition 1.5.2. *[26, Prop. 4.1] Let A be an algebra in a monoidal category $\mathscr{C}$. The full centre $\mathcal{Z}(A)$ has a unique structure of an algebra in $\mathscr{Z}_l(\mathscr{C})$ such that the morphism $\mathcal{Z}(A) \longrightarrow A$ is a homomorphism of algebras in $\mathscr{C}$. Moreover $\mathcal{Z}(A)$ is a braided-commutative algebra in $\mathscr{Z}_l(\mathscr{C})$.*

In addition, the full centre is an invariant under Morita equivalence, see Corollary 6.3 in [26]. Now assume that $(\mathscr{C}, \otimes, I, a, l, r)$ has a braiding C. We follows [60, 83] to define the left center $C_l(B)$ of an algebra B in $\mathscr{C}$.

Definition 1.5.3. The *left center* $C_l(B)$ of an algebra B in $\mathscr{C}$ is the terminal object in the category of morphisms $y : Y \longrightarrow B$ such that the following diagram commutes

$$\begin{array}{ccc} B \otimes Y & \xleftarrow{C_{Y,B}} & Y \otimes B \\ & & \downarrow {\scriptstyle y \otimes id} \\ \downarrow {\scriptstyle id \otimes y} & & B \otimes B \\ & & \downarrow {\scriptstyle \mu_B} \\ B \otimes B & \xrightarrow{\mu_B} & B. \end{array}$$

Similarly, we can define a *right center* $C_r(B)$ of B.

Proposition 1.5.4. *[26, Prop.5.1] Let $\mathscr{C}$ be a braided monoidal category and B an algebra in $\mathscr{C}$. Then the left center $C_l(B)$ has a unique structure of algebra in $\mathscr{C}$ such that the morphism $C_l(B) \longrightarrow B$ is a homomorphism of algebras in $\mathscr{C}$. Moreover, $C_l(B)$ is a braided-commutative algebra in $\mathscr{C}$.*

Assume that the forgetful functor $F : \mathscr{Z}_l(\mathscr{C}) \longrightarrow \mathscr{C}$ has a right adjoint $R : \mathscr{C} \longrightarrow \mathscr{Z}_l(\mathscr{C})$ with the natural transformations of the adjunction:

$$\alpha_U : U \longrightarrow R \circ F(U), \ \beta_X : F \circ R(X) \longrightarrow X$$

for all $U \in \mathscr{Z}_l(\mathscr{C})$ and $X \in \mathscr{C}$. Note that the functor R is automatically lax monoidal, where the morphism φ_0 is given by the composite

$$I \longrightarrow R \circ F(I) \longrightarrow R(I);$$

and the natural transformation φ_2 is the following composition:

$$\begin{array}{ccc} R \circ F(R(X) \otimes R(Y)) & \xleftarrow{\alpha_{R(X)\otimes R(Y)}} & R(X) \otimes R(Y) \\ {\scriptstyle R(F_{R(X),R(Y)})}\big\downarrow & & \\ R(F \circ R(X) \otimes F \circ R(Y)) & \xrightarrow{R(\beta_X \otimes \beta_y)} & R(X \otimes Y). \end{array}$$

The lax monoidal structure of R can transport algebras from $\mathscr{C}$ to $\mathscr{Z}_l(\mathscr{C})$. If (A, μ_A, η_A) is an algebra in $\mathscr{C}$, $R(A)$ is an algebra in $\mathscr{Z}_l(\mathscr{C})$ with the unit map

$$R(\eta_A) \circ \varphi_0 : I \longrightarrow R(I) \longrightarrow R(A)$$

and the multiplication

$$R(\mu_A) \circ \varphi_2 : R(A) \otimes R(A) \longrightarrow R(A \otimes A) \longrightarrow R(A).$$

The following theorem states the relation between a full center and a left center.

Theorem 1.5.5. *[26, Thm. 5.4] Assume that the forgetful functor $F : \mathscr{Z}_l(\mathscr{C}) \longrightarrow \mathscr{C}$ has a right adjoint $R : \mathscr{C} \longrightarrow \mathscr{Z}_l(\mathscr{C})$ with the natural transformations α and β as above. If the natural transformation β of the adjunction is epic, then*

$$\mathcal{Z}(A) \simeq C_l(R(A)),$$

for any algebra A in a monoidal category $\mathscr{C}$.

For a general theory of full centers, the reader is referred to [26].

1.6 Weak Hopf algebras

In this thesis, we will mainly use the techniques from weak Hopf algebras to deal with the problems with braided fusion categories. In this section, we will recall some basic definitions and properties of weak Hopf algebras. For more detail about weak Hopf algebras, we refer to [8, 55, 56].

1.6.1 Weak Hopf algebras

Definition 1.6.1. [8, Defn. 2.1] A weak Hopf algebra H is a $\Bbbk$-algebra (H, m, μ) and $\Bbbk$-coalgebra (H, Δ, ε) such that the following axioms hold:

- $\Delta(hk) = \Delta(h)\Delta(k)$,
- $\Delta^2(1) = 1_1 \otimes 1_2 1_{(1')} \otimes 1_{2'} = 1_1 \otimes 1_{1'} 1_2 \otimes 1_{2'}$,
- $\varepsilon(hkl) = \varepsilon(hk_1)\varepsilon(k_2 l) = \varepsilon(hk_2)\varepsilon(k_1 l)$,
- There exists a $\Bbbk$-linear map $S : H \longrightarrow H$, called the *antipode*, satisfying

  $h_1 S(h_2) = \varepsilon(1_1 h)1_2, \qquad S(h_1)h_2 = 1_1 \varepsilon(h 1_2),$

  $S(h) = S(h_1)h_2 S(h_3),$

for all $h, k, l \in H$.

For a weak Hopf algebra H, we have idempotent maps $\varepsilon_t, \varepsilon_s$: $H \longrightarrow H$ defined by

$$\varepsilon_t(h) = \varepsilon(1_1 h)1_2, \qquad \varepsilon_s(h) = 1_1 \varepsilon(h 1_2).$$

The maps ε_t and ε_s are called the *target map* and the *source map* respectively, and their images H_t and H_s are called the *target space* and *source space* respectively. In particular, H_t and H_s are Frobenius-separable subalgebras of H. Moreover, the following equations hold:

$$h_1 \otimes h_2 S(h_3) = 1_1 h \otimes 1_2, \tag{1.1}$$
$$S(h_1)h_2 \otimes h_3 = 1_1 \otimes h 1_2, \tag{1.2}$$
$$h_1 \otimes S(h_2)h_3 = h 1_1 \otimes S(1_2), \tag{1.3}$$
$$h_1 S(h_2) \otimes h_3 = S(1_1) \otimes 1_2 h, \tag{1.4}$$
$$\varepsilon(g \varepsilon_t(h)) = \varepsilon(gh) = \varepsilon(\varepsilon_s(g)h), \tag{1.5}$$
$$y 1_1 \otimes S(1_2) = 1_1 \otimes S(1_2) y, \tag{1.6}$$
$$z S(1_1) \otimes 1_2 = S(1_1) \otimes 1_2 z, \tag{1.7}$$

for $g, h \in H, y \in H_s$ and $z \in H_t$.

If H is a finite dimensional weak Hopf algebra with an antipode S, then S is automatically bijective. Moreover, the dual $H^* = Hom_{\Bbbk}(H, \Bbbk)$ of H is also a weak Hopf algebra with the unit ε and the counit $\widehat{\varepsilon}\colon\ \phi \mapsto \langle 1, \phi\rangle$ respectively. Other structures on H^* are given as follows:

$$\langle h, \phi\psi\rangle = \langle \Delta(h), \phi \otimes \psi\rangle, \quad \langle g \otimes h, \widehat{\Delta}(\phi)\rangle = \langle gh, \phi\rangle, \quad \langle h, \widehat{S}(\phi)\rangle = \langle S(h), \phi\rangle,$$

for all $\phi, \psi \in H^*$ and $g, h \in H$. The target subalgebra of H^* is denoted by H_t^*.

Any weak Hopf algebra H contains a canonical *minimal weak Hopf algebra* H_{min}, which is generated, as an algebra, by H_t and H_s. All minimal weak Hopf algebras were classified in [55]. The following definitions can be found in [38, 56, 57, 62].

Definition 1.6.2. Let H be a weak Hopf algebra and $Z(H)$ the center of H.

(1) H is called *connected* if $Z(H) \cap H_t = \Bbbk 1_H$.

(2) H is called *regular* if $S^2(x) = x$ for all $x \in H_{min}$.

(3) If H is finite dimensional, H is called *co-connected* if H^* is connected.

(4) H is called a face algebra if H_s is a commutative algebra.

(5) H is called *semisimple* if H is semisimple as an algebra.

(6) H is called *cosemisimple* if H is cosemisimple as a coalgebra.

Remark 1.6.3. (1) A weak Hopf algebra H is an ordinary Hopf algebra if and only if $\Delta(1) = 1 \otimes 1$, if and only if ε is an algebra homomorphism if and only if $H_t = H_s = \Bbbk 1_H$.

(2) Let H be a weak Hopf algebra with an antipode S. Then S is an anti-algebra isomorphism between H_t and H_s.

(3) Every weak Hopf algebra can be obtained by twisting a regular weak Hopf algebra with the same algebra structure, see [55].

In what follows, a weak Hopf algebra will always mean a regular weak Hopf algebra.

Definition 1.6.4. [57, Defn. 6.1] Let H be a weak Hopf algebra with a bijective antipode S. A *quasi-triangular weak Hopf algebra* is a pair (H, R), where

$$R = R^1 \otimes R^2 \in \Delta^{cop}(1)(H \otimes H)\Delta(1),$$

satisfying the following conditions:

$$(id \otimes \Delta)R = R_{13}R_{12}, \tag{1.8}$$
$$(\Delta \otimes id)R = R_{13}R_{23}, \tag{1.9}$$
$$\Delta^{cop}(h)R = R\Delta(h), \tag{1.10}$$

where $h \in H$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, etc., and there exists $\overline{R} \in \Delta(1)(H \otimes H)\Delta^{cop}(1)$ with $R\overline{R} = \Delta^{op}(1)$ and $\overline{R}R = \Delta(1)$. R is often called an *R-matrix.* In particular, (H, R) is called a *triangular weak Hopf algebra* if $\overline{R} = R^2 \otimes R^1$.

For $y \in H_s, z \in H_t$, the following equations hold (see [57]):

$$(1 \otimes z)R = R(z \otimes 1), \qquad (y \otimes 1)R = R(1 \otimes y), \tag{1.11}$$
$$(z \otimes 1)R = (1 \otimes S(z))R, \qquad (1 \otimes y)R = (S(y) \otimes 1)R, \tag{1.12}$$
$$R(y \otimes 1) = R(1 \otimes S(y)), \qquad R(1 \otimes z) = R(S(z) \otimes 1), \tag{1.13}$$
$$(\varepsilon_s \otimes id)(R) = \Delta(1), \qquad (id \otimes \varepsilon_s)(R) = (S \otimes id)\Delta^{cop}(1), \tag{1.14}$$
$$(\varepsilon_t \otimes id)(R) = \Delta^{cop}(1), \qquad (id \otimes \varepsilon_t)(R) = (S \otimes id)\Delta(1). \tag{1.15}$$

Definition 1.6.5. A *coquasi-triangular weak Hopf algebra* is a pair (H, σ), where H is a weak Hopf algebra with a bijective antipode, and a $\Bbbk$-linear map $\sigma : H \otimes H \to \Bbbk$ satisfies:

$$\sigma(ab, c) = \sigma(a, c_1)\sigma(b, c_2), \quad \sigma(a, bc) = \sigma(a_1, c)\sigma(a_2, b),$$
$$\sigma(a_1, b_1)a_2b_2 = b_1a_1\sigma(a_2, b_2), \ \sigma(b, a) = \varepsilon(a_1b_1)\sigma(b_2, a_2)\varepsilon(a_3b_3),$$

for any $a, b, c \in H$, and there exists $\sigma^{-1} : H \otimes H \to \Bbbk$ such that

$$\sigma(a_1, b_1)\sigma^{-1}(a_2, b_2) = \epsilon(ba), \ \sigma^{-1}(a_1, b_1)\sigma(a_2, b_2) = \epsilon(ba)$$
$$\epsilon(a_1b_1)\sigma^{-1}(a_2, b_2)\epsilon(b_3a_3) = \sigma^{-1}(a, b)$$

for all $a, b, c \in H$, where σ^{-1} is called a *weak inverse* of σ, see Section 2 in [38].

Similar to a Hopf algebra, if H is a finite dimensional coquasi-triangular (quasi-triangular) weak Hopf algebra, then the dual H^* is quasi-triangular (coquasi-triangular).

1.6.2 Categories of modules and braided fusion categories

Let H be a weak Hopf algebra and ${}_H\mathcal{M}$ denote the category of left H-modules. By [59, 7], we have a monoidal category $({}_H\mathcal{M}, \otimes_t, H_t, a, l, r)$ as follows:

- For any two objects M and N in ${}_H\mathcal{M}$,

  $$M \otimes_t N = \{m \otimes n \in M \otimes N | \Delta(1)(m \otimes n) = m \otimes n\}.$$

  Clearly, $M \otimes_t N = \Delta(1)(M \otimes N) \subseteq M \otimes N$;

- For any two objects M and N in ${}_H\mathcal{M}$, the H-module structure on $M \otimes_t N$ is as follows: $h \cdot (m \otimes_t n) = h_1 m \otimes_t h_2 n$ for all $h \in H$ and $m \in M$ and $n \in N$;

- H_t is the unit object with H-action $h \cdot z = \varepsilon_t(hz)$, where $h \in H, z \in H_t$, and the $\Bbbk$-linear maps l_M, r_M and their inverses are given by

  $$\begin{aligned} l_M(1_1 \cdot z \otimes 1_2 \cdot m) &= z \cdot m, \quad l_M^{-1}(m) = 1_1 \cdot 1_H \otimes 1_2 \cdot m \\ r_M(1_1 \cdot m \otimes 1_2 \cdot z) &= S(z) \cdot m, \quad r_M^{-1}(m) = 1_1 \cdot m \otimes 1_2, \end{aligned}$$

  for any $z \in H_t$ and $m \in M$, where M is an object in ${}_H\mathcal{M}$.

If (H, R) is a quasi-triangular weak Hopf algebra, then the category ${}_H\mathcal{M}$ can be equipped with a braiding C as follows:

$$C_{M,N}(m \otimes_t n) = R^2 \cdot n \otimes_t R^1 \cdot m, \text{ for all } m \in M \text{ and } n \in N,$$

where M and N are any objects in ${}_H\mathcal{M}$.

Lemma 1.6.6. *[57, Prop.5.2] Let H be a weak Hopf algebra. If H has a quasi-triangular structure R, then there exists a braiding C on the category ${}_H\mathcal{M}$:*

$$C_{M,N}(m \otimes_t n) = R^2 \cdot n \otimes_t R^1 \cdot m, \tag{1.16}$$

where $R = R^1 \otimes R^2$, $m \in M$ and $n \in N$. Conversely, if the category ${}_H\mathcal{M}$ is braided, then H can be equipped with a quasi-triangular structure R such that the category ${}_H\mathcal{M}$ has a braiding as (1.16).

Let H be a finite dimensional weak Hopf algebra. Let ${}_H\mathscr{M}$ denote the category of finite dimensional left H-modules. Then $({}_H\mathscr{M}, \otimes_t, H_t, a, l, r)$ forms a rigid monoidal category with the following rigid structure:

- For any object M in ${}_H\mathscr{M}$, define an action of H on M^* by $(h\cdot\phi)(m) = \phi(S(h)\cdot m)$, for any $h \in H$, $\phi \in M^*$ and $m \in M$. The duality morphisms are defined as follow:

$$d_M : M^* \otimes_t M \longrightarrow H_t, \quad \phi \otimes m \longmapsto \phi(1_1 \cdot m)1_2$$
$$b_M : H_t \longrightarrow M \otimes_t M^*, \quad z \longmapsto z \cdot (\sum_i e_i \otimes f^i),$$

  where e_i and f^i are dual bases of M and M^* respectively.

In particular, the category $({}_H\mathscr{M}, \otimes_t, H_t, a, l, r)$ is a braided monoidal category if H is finite dimensional and quasi-triangular. In this case, we have the Brauer group $Br({}_H\mathscr{M})$.

Remark 1.6.7. (1) Let H be a finite dimensional weak Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. By Proposition 3.1.5 in [56], if H is a biconnected weak Hopf algebra, then H is semisimple if and only if H^* is semisimple.

(2) The category ${}_H\mathscr{M}$ is a fusion category if and only if H is semisimple and connected, see Corollary 2.22 in [31]. By Lemma 1.6.6, a braided fusion category is equivalent to the category of finite dimensional modules over some quasi-triangular weak Hopf algebra, which is semisimple and connected.

Dually, let $\mathscr{M}^H$ denote the category of finite dimensional right H-comodules. Then $(\mathscr{M}^H, \otimes_s, H_s)$ is also a rigid monoidal category, see [38, 59, 62]. Here, a right H-comodule M has an induced H_s-bimodule structure as follows,

$$y \cdot m = m_{[0]}\varepsilon(ym_{[1]}), \ m \cdot y = m_{[0]}\varepsilon(m_{[1]}y),$$

where $\rho^R(m) = m_{[0]} \otimes m_{[1]}$, for all $m \in M$ and $y \in H_s$.

For finite dimensional right H-comodules M and N, denote by $Hom_{-H_s}(M, N)$ the set of right H_s-linear maps from M to N. Then $Hom_{-H_s}(M, N)$ is a right H-comodule with ρ^R defined by

$$\rho^R(f)(m) = f(m_{[0]})_{[0]} \otimes_s f(m_{[0]})_{[1]}S(m_{[1]}),$$

for any $f \in Hom_{-H_s}(M,N)$ and $m \in M$. Moreover, for any object M in $\mathscr{M}^H$, the functor $-\otimes_s M$ has a right adjoint functor $Hom_{-H_s}(M,-)$, which is isomorphic to the functor $(-)^*$. Let $End_{-H_s}(M)$ be the set of right H_s-linear maps from M to itself.

When H is coquasi-triangular, the category $\mathscr{M}^H$ has a braiding C:

$$C_{U,V} : U \otimes_s V \to V \otimes_s U,\ u \otimes_s v \mapsto v_{[0]} \otimes_s u_{[0]}\sigma(u_{[1]}, v_{[1]}),$$

where U and V are any objects in $\mathscr{M}^H$. Following Section 1.4, we can form the Brauer group $Br(\mathscr{M}^H)$, which will be mainly investigated in this thesis.

1.6.3 Yetter-Drinfeld modules

Definition 1.6.8. Let H be a weak Hopf algebra. A left H-module M is called a *left-left Yetter-Drinfeld module* if (M, ρ^L) is a left H-comodule such that the conditions

- $\rho^L(m) = m_{[-1]} \otimes m_{[0]} \in H \otimes_t V,$
- $(h \cdot m)_{[-1]} \otimes (h \cdot m)_{[0]} = h_1 m_{[-1]} S(h_3) \otimes h_2 \cdot m_{[0]},$

are satisfied for all $h \in H$ and $m \in M$.

Let M be a left-left Yetter-Drinfeld module. For all $m \in M$,

$$m_{[-1]} \otimes m_{[0]} = m_{[-1]} S(1_2) \otimes 1_1 \cdot m_{[0]}. \tag{1.17}$$

We denote by ${}^H_H\mathscr{YD}$ the category of finite dimensional left-left Yetter-Drinfeld modules and the morphisms that are both left H-linear and left H-colinear . Furthermore, if S is bijective, then ${}^H_H\mathscr{YD}$ is a braided monoidal category with a braiding given by

$$C_{V,W}(v \otimes w) = v_{[-1]} \cdot w \otimes v_{[0]},$$

where $v \in V \in {}^H_H\mathscr{YD}$ and $w \in W \in {}^H_H\mathscr{YD}$. In particular, if (H, R) is a quasi-triangular weak Hopf algebra, then every left H-mudule M becomes a left-left Yetter-Drinfeld module in a natural way:

$$\rho^L(m) = R^2 \otimes R^1 \cdot m,\ \forall\, m \in M.$$

Clearly, the category ${}_H\mathscr{M}$ is a braided monoidal subcategory of the category ${}_H^H\mathscr{Y}\mathscr{D}$.

A left H-module M is called a *left-right Yetter-Drinfeld module* if (M, ρ^R) is a right H-comodule such that the following conditions

- $\rho^L(m) = m_{[0]} \otimes m_{[1]} \in H \otimes_t V$,
- $(h \cdot m)_{[0]} \otimes (h \cdot m)_{[1]} = h_2 \cdot m_{[0]} \otimes h_3 m_{[1]} S^{-1}(h_1)$,

are satisfied for all $h \in H$ and $m \in M$.

Similarly, one can define a *right-right Yetter-Drinfeld module* and a *right-left Yetter-Drinfeld module*. The category of finite dimensional left-right(right-right, right-left) Yetter-Drinfeld modules is denoted by ${}_H\mathscr{Y}\mathscr{D}^H$ $(\mathscr{Y}\mathscr{D}_H^H, {}^H\mathscr{Y}\mathscr{D}_H)$. The relations between the Drinfeld centers and the categories of Yetter-Drindeld modules are given in the following theorem:

Theorem 1.6.9. *[20] Let H be a weak Hopf algebra with a bijective antipode. Then the following two statements hold:*

1. *The category $\mathscr{Z}_l({}_H\mathscr{M})$ is equivalent to ${}_H^H\mathscr{Y}\mathscr{D}$ as a braided monoidal category;*
2. *The category $\mathscr{Z}_r({}_H\mathscr{M})$ is equivalent to ${}_H\mathscr{Y}\mathscr{D}^H$ as a braided monoidal category.*

Theorem 1.6.10. *[20] Let H be a weak Hopf algebra with a bijective antipode. Then the category ${}_H^H\mathscr{Y}\mathscr{D}^{rev}$ is equivalent to ${}_H\mathscr{Y}\mathscr{D}^H$ as a braided monoidal category.*

Proof. Note that $\mathscr{Z}_r({}_H\mathscr{M}) \cong \mathscr{Z}_l({}_H\mathscr{M})^{rev}$. Then we get a braided monoial functor

$$E : {}_H\mathscr{Y}\mathscr{D}^H \longrightarrow {}_H^H\mathscr{Y}\mathscr{D}^{rev},$$

where for any left-right Yetter-Drinfeld module (M, ρ^R), $E(M) = M$ is a left-left Yetter-Drinfeld module with the original action and the following coaction:

$$\rho^L(m) = S(m_{[1]}) \otimes m_{[0]},$$

where $\rho^R(m) = m_{[0]} \otimes m_{[1]}$ for all $m \in M$. □

Definition 1.6.11. An algebra A in ${}_H^H\mathscr{Y}\mathscr{D}$ is called *quantum commutative* if A is braided-commutative in ${}_H^H\mathscr{Y}\mathscr{D}$. Similarly, one can define a *quantum commutative* algebra in $\mathscr{Y}\mathscr{D}_H^H$.

1.6.4 Smash product algebras

Definition 1.6.12. Let H be a weak Hopf algebra. An algebra A with unity is called a *left H-module algebra* if A is a left H-module such that

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_A = \varepsilon_t(h) \cdot 1_A.$$

for all $a, b \in A$ and $h \in H$.

Similarly, one can define a right H-module algebra. Now take a subspace $A \boxtimes H$ of $A \otimes H$ as

$$\{1_1 \cdot a \otimes 1_2 h | \ \forall\ a \in A, \forall\ h \in H\}.$$

Then $A \boxtimes H$ is an associative algebra with the unity $1_1 \cdot 1_A \otimes 1_2$, and the multiplication given by

$$(a \boxtimes h)(a' \boxtimes h') = a(h_1 \cdot a') \boxtimes h_2 h', \ \ \forall\ a, a' \in A,\ \forall\ h, h' \in H.$$

We call $A \boxtimes H$ the *smash product algebra* of A with H, see [54].

Definition 1.6.13. Let H be a weak Hopf algebra. An algebra with unity is called a *right H-comodule algebra* if A is a right H-comodule such that the coaction ρ^R satisfies

$$\rho(1_A)(a \otimes 1) = (id_A \otimes \varepsilon_t) \circ \rho(a), \quad \rho(ab) = \rho(a)\rho(b),$$

for all $a, b \in A$, see [15].

Define the right coinvariants of a right H-comodule algebra A as

$$A^{coH} = \{a \in A \mid \rho(a) \in A \otimes H_t\}.$$

It follows from [16] that $A_0 = A^{coH} = \{a \in A \mid \rho(a) = a1_{[0]} \otimes 1_{[1]}\}$, where $\rho^R(1) = 1_{[0]} \otimes 1_{[1]}$. Moreover, we know form [87] that

$$\{a \in A | a_{[0]} \otimes a_{[1]} = a1_{[0]} \otimes 1_{[1]}\} = \{a \in A | a_{[0]} \otimes a_{[1]} = 1_{[0]} a \otimes 1_{[1]}\}.$$

The *centralizer* of A_0 in A is denoted by $C_A(A_0)$, i.e.,

$$C_A(A_0) = \{a \in A | ab = ba, \ \forall b \in A_0\}.$$

Lemma 1.6.14. *[87] Let H be a weak Hopf algebra, and A a right H-comodule algebra. If there is a right H-comodule map ϕ from H to A such that ϕ is an algebra map, then the right H-comodule algebra A is isomorphic to a smash product algebra $A_0 \boxtimes H$, where the coinvariant subalgebra A_0 is a left H-module algebra with the action defined by $h \rightharpoonup a = \phi(h_1)a\phi[S(h_2)]$, for any $h \in H, a \in A_0$.*

Let A be a right H-comodule algebra. Consider a projection

$$p : A \otimes H \to A \otimes H, p(a \otimes h) = a1_{[0]} \otimes h1_{[1]}.$$

The canonical map $\gamma : A \otimes_{A_0} A \to Im(p)$ is defined by $\gamma(a \otimes_{A_0} b) = ab_{[0]} \otimes b_{[1]}$.

Definition 1.6.15. [16] A right H-comodule algebra A is a right weak H-Galois extension of A_0 if $\gamma : A \otimes_{A_0} A \to Im(p)$ is bijective.

Let A be a right weak H-Galois extension. It is easy to see that the centralizer $C_A(A_0)$ with the *MUV action* (Miyashita-Ulbrich-Van Oystaeyen ([53, 78, 80]))

$$a \leftharpoondown h = h^{[1]} a h^{[2]},$$

is a right H-module algebra, where $\gamma^{-1}(1_{[0]} \otimes h1_{[1]}) := h^{[1]} \otimes h^{[2]}$. In particular, the centralizer $C_A(A_0)$ with the MUV action $\leftharpoondown$ is a quantum commutative algebra in the category of right-right Yetter-Drinfeld modules. So we have a map π from the set of weak H-Galois extensions to quantum commutative algebras in the right Yetter-Drinfeld modules

$$\pi(A) = C_A(A_0).$$

This map will play a key role in this thesis.

Example 1.6.16. Let A be a left H-module algebra. Then the smash product algebra $A \boxtimes H$ is a right weak H-Galois extension with the coinvariant subalgebra $A \boxtimes 1$ ($\simeq A$). The canonical map is given by

$$(A \boxtimes H) \otimes_A (A \boxtimes H) \longrightarrow (A \boxtimes H) \otimes H, \ (a \boxtimes h) \otimes (a' \boxtimes h') \longmapsto a(h_1 \cdot a') \boxtimes h_2 h'_1 \otimes h'_2.$$

Its inverse is given by

$$(A \boxtimes H) \otimes H \longrightarrow (A \boxtimes H) \otimes_A (A \boxtimes H),\ (a \boxtimes h) \otimes h' \longmapsto (a \boxtimes hS(h'_1)) \otimes (1 \boxtimes h'_2).$$

So $h^{[1]} \otimes h^{[2]} = (1 \boxtimes S(h_1)) \otimes (1 \boxtimes h_2)$. The MUV action is given by

$$(a \boxtimes g) \leftharpoondown h = h^{[1]}(a \boxtimes g)h^{[2]} = (1 \boxtimes S(h_1))(a \boxtimes g)(1 \boxtimes h_2) = (S(h_2) \cdot a) \boxtimes S(h_1)gh_3,$$

for all $a \boxtimes g \in A \boxtimes H$ and $h \in H$.

Let H be a weak Hopf algebra and A a right weak H-Galois extension. Then the following identities hold:

$$h^{[1]}h^{[2]}_{[0]} \otimes h^{[2]}_{[1]} = 1_{[0]} \otimes h1_{[1]}, \tag{1.18}$$
$$h^{[1]}h^{[2]} = 1_{[0]}\varepsilon(h1_{[1]}), \tag{1.19}$$
$$a_{[0]}a^{[1]}_{[1]} \otimes_B a^{[2]}_{[1]} = 1 \otimes_B a, \tag{1.20}$$
$$bh^{[1]} \otimes_B h^{[2]} = h^{[1]} \otimes_B h^{[2]}b, \tag{1.21}$$
$$h^{[1]} \otimes_B h^{[2]}_{[0]} \otimes h^{[2]}_{[1]} = h^{[1]}_1 \otimes_B h^{[1]}_1 \otimes h_2, \tag{1.22}$$
$$h^{[1]}_{[0]} \otimes_B h^{[2]} \otimes h^{[1]}_{[1]} = h^{[1]}_2 \otimes_B h^{[2]}_2 \otimes S(h_1), \tag{1.23}$$
$$(hg)^{[1]} \otimes_B (hg)^{[2]} = g^{[1]}h^{[1]} \otimes_B h^{[1]}g^{[1]}, \tag{1.24}$$
$$h^{[1]}_1 \otimes_B h^{[2]}_1 h^{[1]}_2 \otimes_B h^{[1]}_2 = h^{[1]} \otimes_B 1 \otimes_B h^{[2]}, \tag{1.25}$$
$$S^{-1}(a_{[1]})^{[1]} \otimes S^{-1}(a_{[1]})^{[2]}a_{[0]} = a \otimes 1, \tag{1.26}$$

for $g, h \in H$, $a \in A$ and $b \in B$, see [41].

Remark 1.6.17. A left H-module (right H-comodule) algebra is the same as an algebra in the category of left H-modules (right H-comodules).

1.6.5 Weak Hopf algebras and Hopf algebroids

Let A be an associative $\Bbbk$-algebra with unity. An algebra U is called an A-*algebra* if there is an algebra map $i : A \longrightarrow U$, see [72, 73]. In particular, the algebra $End(A)$ is an A^e-algebra with structure $a \otimes \overline{b} \longmapsto (c \longmapsto acb)$. Denote by A^{op} the opposite algebra of A, the map $op : A \longrightarrow A^{op}$ ($a \longmapsto \overline{a}$) is anti-isomorphic. Write A^e for the enveloping algebra $A \otimes A^{op}$ and $a\overline{b}$ for $a \otimes \overline{b}$ for all $a, b \in A$.

We use the following Sweedler-Takeuchi's notations (see [72, 73]):

$$\int_a {}_{\overline{a}}M \otimes_a N := M \otimes N / \langle \overline{a}m \otimes n - m \otimes an | a \in A, m \in M, n \in N \rangle,$$
$$\int^a M_{\overline{a}} \otimes N_a := \{ m \otimes n \in M \otimes N | m\overline{a} \otimes n = m \otimes na, \forall a \in A \},$$

for any two A^e-bimodules M and N. The $\times_A$-product in [73] is defined as

$$M \times_A N := \int^b \int_a {}_{\overline{a}}M_{\overline{b}} \otimes_a N_b.$$

Let (U, i) and (V, j) be two A^e-algebras. An A^e-algebra map $f : U \longrightarrow V$ is an *algebra map*from U to V such that $f \circ i = j$.

If (U, i) and (V, j) are A^e-algebras, so is $U \times_A V$, with the A^e-algebra structure given by

$$(\sum u \otimes v)(\sum u' \otimes v') = \sum uu' \otimes vv',$$
$$h : A^e \longrightarrow U \times_A V, a \otimes \overline{b} \longmapsto i(a) \otimes j(\overline{b}).$$

For an A^e-algebra (U, i), we can replace the map i by its restrictions

$$s := i(- \otimes 1) : A \longrightarrow U, \quad t := i(1 \otimes -) : A^{op} \longrightarrow U,$$

which are $\Bbbk$-algebra maps with commuting ranges in U, see [11]. The maps s and t are called the *source* map and the *target* map respectively. In what follows, an A^e-algebra (U, i) will be given sometimes by a triple (U, s, t), where s and t are as above.

Let A be a $\Bbbk$-algebra. An *A-coring* is a triple (C, Δ, ε), where C is an A-bimodule, $\Delta : C \longrightarrow C \otimes_A C$ and $\varepsilon : C \longrightarrow A$ are A-bimodule maps, satisfying the coassociativity and the counit conditions. A detailed study about the theory of corings can be found in [12].

Definition 1.6.18. Let A be an associative $\Bbbk$-algebra. A left *bialgebroid* $\mathcal{B} = (B, A, s, t, \Delta, \varepsilon)$ consists of an A^e-algebra (B, s, t) and an A-coring (B, Δ, ε) on the same $\Bbbk$-linear space B subject to the following compatibility axioms:

(i) The bimodule structure in the A-coring (B, Δ, ε) is related to the A^e-algebra

(B, s, t) via
$$r \cdot b \cdot r' := s(r)t(r')b, \ \forall b \in B, r, r' \in A.$$

(ii) Consider B as an A-bimodule as in (i). The coproduct Δ corestricts to a $\Bbbk$-algebra map from B to $B \times_A B$.

(iii) The counit ε satisfies $\varepsilon(1_B) = 1_A$ and
$$\varepsilon(bs(\varepsilon(b'))) = \varepsilon(bb') = \varepsilon(bt(\varepsilon(b')))$$
for all $b, b' \in B$.

Remark 1.6.19. The notions of Takeuchi's $\times_A$-bialgebra in [73], Lu's bialgebroid in [48], Xu's bialgebroid with an anchor in [85] have been shown to be equivalent in [11]. Definition 1.6.18 is just the one in [48]. Similarly, we can define a right bialgebroid, see [4, 40]. In particular, when A is Frobenius-Separable, a left (right) bialgebroid is equivalent to a weak bialgebra, see [67].

Definition 1.6.20. [10, Defn.4.1] A *Hopf algebroid* $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ consists of a left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \Delta_L, \varepsilon_L)$, a right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \Delta_R, \varepsilon_R)$ and a $\Bbbk$-module map $S : H \longrightarrow H$, called the *antipode*, such that the following conditions hold:

(1) $s_L \circ \varepsilon_L \circ t_R = t_R, \ t_L \circ \varepsilon_L \circ s_R = s_R,$
$s_R \circ \varepsilon_R \circ t_L = t_L, \ t_R \circ \varepsilon_R \circ s_L = s_L,$

(2) $(\Delta_L \otimes id) \circ \Delta_R, \quad (id \otimes \Delta_R) \circ \Delta_L,$

(3) S is both L-linear and R-linear,

(4) $m_H \circ (id \otimes S) \circ \Delta_L = s_R \circ \varepsilon_R, \ m_H \circ (S \otimes id) \circ \Delta_R = s_L \circ \varepsilon_L.$

Example 1.6.21. Let H be a weak Hopf algebra with a bijective antipode. By [10] we get a Hopf algebroid H consisting of a left bialgebroid H_l and a right bialgebroid H_r:
$$H_l = (H, H_t, id_{H_t}, S^{-1}|_{H_t}, \Delta, \varepsilon_t), \quad H_r = (H, H_s, id_{H_s}, S^{-1}|_{H_s}, \Delta, \varepsilon_s).$$

Remark 1.6.22. The other equivalent notions of a Hopf algebroid with a bijective antipode have been given by Böhm and Szlachányi, see [10, Prop.4.2]. Moreover, Galois theory of Hopf algebroids can be found in [4, 5].

Chapter 2

Constructions of braided Hopf algebras

Let (H, R) be a quasi-triangular weak Hopf algebra over a field $\Bbbk$. In this chapter, we will discuss how to construct a braided Hopf algebra from (H, R). In Section 2.1, we will mainly work out the full center of the unit object. This full center will be equipped with the structure of a braided Hopf algebra in Section 2.2.

2.1 The full centre of the unit object

Let H be a Hopf algebra. There exists an adjoint pair of functors between the category of H-modules and the category of Yetter-Drinfeld modules, see [17]. Now we generalize this to the case of a weak Hopf algebra.

Lemma 2.1.1. *Let H be a weak Hopf algebra with a bijective antipode S. Then the forgetful functor $F' : {}_H^H\mathscr{Y}\mathscr{D} \longrightarrow {}_H\mathscr{M}$ has a right adjoint functor*

$$I'(-) : {}_H\mathscr{M} \longrightarrow {}_H^H\mathscr{Y}\mathscr{D},$$

where $I'(N) = 1_1 H S(1'_2) \otimes 1_2 1'_1 \cdot N$ with the following H-action and H-coaction:

$$h \cdot (g \otimes n) = h_1 g S(h_3) \otimes h_2 \cdot n, \quad \rho^L(g \otimes n) = g_1 \otimes g_2 \otimes n.$$

The adjunction morphisms

$$\alpha_M : M \longrightarrow I'F'(M), \quad \beta_L : F'I'(L) \longrightarrow L$$

are given by the comodule structure on M and by the counit respectively:

$$\alpha_M(m) = m_{[-1]} \otimes m_{[0]}, \quad \beta_L(h \otimes n) = \varepsilon_t(h) \cdot n.$$

In particular, the adjunction morphism β is epic.

Proof. Although it is similar to the proof of Corollary 2.8 in [17] or Proposition. 5.1 in [27], we write down the detail for the sake of completeness. Denote by $H \odot N$ the object $I'(N)$ for $N \in {}_H\mathscr{M}$.

(1) We first show that for a left H-module N, $I'(N)$ is a left-left Yetter-Drinfeld module. The right H-action as above is well-defined since

$$h_1gS(h_3) \otimes h_2 \cdot n \quad = \quad 1_1h_1gS(h_3)S(1_3) \otimes 1_2h_2 \cdot n \in H \odot N.$$

For all $g \otimes n \in H \odot N$, we have $g \otimes n = 1_1gS(1_3) \otimes 1_2 \cdot n = 1_1gS(1'_2) \otimes 1_21'_1 \cdot n$.

$$\rho^L(g \otimes n) = \rho(1_1gS(1'_2) \otimes 1_21'_1 \cdot n) = g_1 \otimes 1_1g_2S(1'_2) \otimes 1_21'_1 \cdot n \in H \otimes (H \odot N).$$

Note that $\Delta(x) = 1_1x \otimes 1_2$ for any $x \in H_t$. So

$$1''_1g_1 \otimes 1''_2 \cdot (g_2 \otimes n) \quad = \quad 1''_1g_1 \otimes 1''_21_1g_2S(1'_2) \otimes 1_21'_1 \cdot n = g_1 \otimes (g_2 \otimes n).$$

Thus $\rho^L(g \otimes n) \in 1_1H \otimes 1_2 \cdot (H \odot N)$. The compatible condition holds since

$$\begin{aligned} \rho[h \cdot (g \otimes n)] &= \rho(h_1gS(h_3) \otimes h_2 \cdot n) \\ &= h_1g_2S(h_5) \otimes h_2g_2S(h_4) \otimes h_3 \cdot n \\ &= h_1g_2S(h_3) \otimes h_2 \cdot (g_2 \otimes n), \end{aligned}$$

for any $h \in H$. Hence, the functor $I'(-) : {}_H\mathscr{M} \longrightarrow {}^H_H\mathscr{YD}$ is well-defined.

(2) Next we verify that α is natural. We have that for all $m \in M \in {}^H_H\mathscr{YD}$,

$$m_{[-1]} \otimes m_{[0]} = 1_1m_{[-1]}S(1_3) \otimes 1_2 \cdot m_{[0]}.$$

So α is well-defined. For any morphism f from M to N in the category of Yetter-Drinfeld modules, the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\quad Id(f)\quad} & N \\ \big\downarrow{\scriptstyle \alpha_M} & & {\scriptstyle \alpha_N}\big\downarrow \\ H \odot F(M) & \xrightarrow{\quad I'F'(f)\quad} & H \odot F(N). \end{array}$$

The coassociativity implies the H-colinearity of α. So $\alpha : Id \longrightarrow I'F'$ is a natural transformation.

(3) β is also a natural transformation. Indeed, for any left H-module L,

$$\begin{aligned} \beta_L[h \cdot (g \otimes l)] &= \beta_L(h_1 g S(h_3)) \otimes h_2 \cdot l) \\ &= \varepsilon_t(h_1 g S(h_3)) \cdot (h_2 \cdot l) \\ &= [\varepsilon_t(h_1 g S(h_3)) h_2] \cdot l \\ &= \varepsilon(h_1 g S(h_3)) h_2 \cdot l \\ &= \varepsilon(\varepsilon_s(h_1) g S(h_3)) h_2 \cdot l \\ &= \varepsilon(1_1 g S(h_2)) h_1 1_2 \cdot l \\ &= \varepsilon(1_1 g S(\varepsilon_s(h_2))) h_1 1_2 \cdot l \\ &= \varepsilon(1_1 g S(S(1'_2))) h 1'_1 1_2 \cdot l \\ &= \varepsilon(1_1 g S(1'_2)) h \cdot (1'_1 1_2 \cdot l) \\ &= h \cdot (\varepsilon(1_1 g S(1'_2)) 1'_1 1_2 \cdot l) \\ &= h \cdot (\varepsilon_t(1_1 g S(1'_2)) 1'_1 1_2 \cdot l) \\ &= h \cdot [\beta_L(1_1 g S(1'_2) \otimes 1'_1 1_2 \cdot l)] \\ &= h \cdot [\beta_L(g \otimes l)], \end{aligned}$$

for any $g \otimes l \in H \odot L$ and $h \in H$. So β_L is left H-linear. Given a morphism g from L to U in the category of left H-modules. Then g is left H_t- and H_s-linear. Thus,

$$g(\varepsilon_t(a) \cdot l) = \varepsilon_t(a) \cdot g(l).$$

for any $a \otimes l \in H \odot L$. Hence, the diagram defining a natural transformation holds.

$$\begin{array}{ccc} F(H \odot L) & \xrightarrow{F'I'(g)} & F'(H \odot U) \\ \downarrow \beta_L & & \beta_U \downarrow \\ L & \xrightarrow{Id(g)} & U \end{array}$$

(4) We check the adjunction axioms for α and β. The composition

$$\beta F' \circ F'\alpha : F' \longrightarrow F'I'F' \longrightarrow F'$$

is the identity. Let M be a left-left Yetter-Drinfeld module, and $m \in M$. We have

$$\beta F' \circ F'\alpha(m) = \beta F'(m_{[-1]} \otimes m_{[0]}) = \varepsilon_t(m_{[-1]}) \cdot m_{[0]} = m.$$

On the other hand, for any left H-module L and all $l \in L$, we have

$$I'\beta \circ \alpha I'(l) = I'\beta \circ \alpha(l^1 \otimes l^0) = I'\beta(l^1_1 \otimes l^1_2 \otimes l^0) = (l^1 \otimes l^0) = I'(l),$$

here $I'(l) = l^1 \otimes l^0 \in H \odot L$. So the composition $I'\beta \circ \alpha I$ is also the identity.

(5) Finally, we verify that the map β is epic. For all $n \in N$,

$$\beta_L(n_{[-1]} \otimes n_{[0]}) = \varepsilon_t(n_{[-1]}) \cdot n_{[0]} = n. \qquad \square$$

Remark 2.1.2. (1) Since the braiding of the category ${}^H_H\mathscr{YD}$ is not involved in the proof of Lemma 2.1.1, this adjoint relation holds also for the category ${}^H_H\mathscr{YD}^{rev}$. Namely, the forgetful functor $F' : {}^H_H\mathscr{YD}^{rev} \longrightarrow {}_H\mathscr{M}$ has a right adjoint functor $I'(-) : {}_H\mathscr{M} \longrightarrow {}^H_H\mathscr{YD}^{rev}$, where F' and I' are the same as in Lemma 2.1.1.

(2) The functor $I'(-) := H \odot -$ is just the $\times_{H_t}$-product in the sense of [73]. In particular, the functor $I'(-)$ is lax monoidal, see Section 1.5.

In what follows, we will mainly consider $H \odot H_t$.

Lemma 2.1.3. *Let H be a weak Hopf algebra. Then $H \odot H_t = 1_1 H 1'_1 \otimes 1_2 1'_2$.*

Proof. It is clear that $H \odot H_t \supseteq 1_1 H 1'_1 \otimes 1_2 1'_2$. For all $x \in H_t$ and $h \in H$, we have

$$1_1 hS(1'_2) \otimes 1_2 1'_1 \cdot x = 1_1 hS(1'_2) \otimes 1_2 x S(1'_1) = 1_1 S^{-1}(x) h 1'_1 \otimes 1_2 1'_2 \in 1_1 H 1'_1 \otimes 1_2 1'_2.$$

So $H \odot H_t \subseteq 1_1 H 1'_1 \otimes 1_2 1'_2$. □

Now we work out the module structure on $H \odot H_t$. For all $g, h \in H$,

$$\begin{aligned} h \cdot (1_1 g 1'_1 \otimes 1_2 1'_2) &= h_1 1_1 g 1'_1 S(h_3) \otimes h_2 \cdot (1_2 1'_2) \\ &= h_1 1_1 g 1'_1 S(h_3) \otimes \varepsilon_t(h_2 1_2 1'_2) \\ &= h_1 g 1'_1 S((h_2 1'_2)_2) \otimes \varepsilon_t[(h_2 1'_2)_1] \\ &= h_1 g 1'_1 S(1''_2 h_2 1'_2) \otimes S(1''_1) \\ &= h_1 g S(h_2) 1_1 \otimes 1_2. \end{aligned}$$

Since H_t is an algebra in the category of left H-modules, Theorem 5.4 in [26] implies that $I'(H_t)$ is an algebra in the category of left-left Yetter-Drinfeld modules with the following structure

$$(1_1 g 1'_1 \otimes 1_2 1'_2)(1''_1 h 1'''_1 \otimes 1''_2 1'''_2) = (1_1 g 1'_1 h 1''_1 \otimes 1_2 1'_2 1''_2),$$

for all $g, h \in H$.

Let H be a weak Hopf algebra. Consider the adjoint action on H, for all $g, h \in H$,

$$H \otimes H \longrightarrow H, \quad g \cdot h = g_1 h S(g_2).$$

In general, the equation $h = 1_1 h S(1_2)$ is not necessarily true, see Example 2.1.4 below. So H with the adjoint action is not necessarily a left H-module.

Now define a $\Bbbk$-linear map $p : H \longrightarrow H, \quad h \longmapsto 1_1 h S(1_2)$ and so we have a decomposition $H = H_0 \oplus H_1$, where $p(H_0) = 0$ and $p(h) = h$, for all $h \in H_1$. It is easy to prove $H_1 = C_H(H_s)$, where $C_H(H_s)$ is the centralizer of H_s in H, see [57].

Example 2.1.4. Böhm and Szlach*á*nyi described first the following weak Hopf algebra H of dimension 13, whose structures were given by the matrix unit basis, see Section 5 in [9]. Now we present H in the form of generators as in [58]: Let $\Bbbk = \mathbb{C}$ and $\lambda^{-1} = 4cos^2\frac{\pi}{5}$. As an algebra, H is generated by $\{1, e_1, e_2, e_3\}$ with the following relations:

$$\begin{aligned} &e_i^2 = e_i, \; i = 1, 2, 3, \; e_1 e_3 = e_3 e_1, \\ &e_1 e_2 e_1 = \lambda e_1, \; e_2 e_3 e_2 = \lambda e_2, \\ &e_3 e_2 e_3 = \lambda e_3, \; e_2 e_1 e_2 = \lambda e_2. \end{aligned}$$

The formulas for the comultiplication are as follows:

$$\begin{aligned}
\Delta(1) = &\quad e_3 \otimes e_1 + (1-e_3)\otimes(1-e_1),\\
\Delta(e_1) = &\quad e_1e_3 \otimes e_1 + e_1(1-e_3)\otimes(1-e_1),\\
\Delta(e_3) = &\quad e_3 \otimes e_1e_3 + (1-e_3)\otimes(1-e_1)e_3,\\
\Delta(e_2) = &\quad (1-\tfrac{(e_3-e_2)^2}{1-\lambda})\otimes(1-\tfrac{(e_1-e_2)^2}{1-\lambda}) + \lambda e_3\otimes e_1\\
&+(1-\lambda)\tfrac{\lambda e_3 - e_2e_3}{\sqrt{\lambda-\lambda^2}}\otimes\tfrac{\lambda e_1-e_2e_1}{\sqrt{\lambda-\lambda^2}} + \lambda\tfrac{\lambda e_3-e_3e_2}{\sqrt{\lambda-\lambda^2}}\otimes\tfrac{\lambda e_1-e_1e_2}{\sqrt{\lambda-\lambda^2}}\\
&+(1-\lambda)(\tfrac{(e_3-e_2)^2}{1-\lambda}-e_3)\otimes(\tfrac{(e_1-e_2)^2}{1-\lambda}-e_1)
\end{aligned}$$

The counit is given by :

$$\begin{aligned}
\epsilon(1) = \epsilon(e_2) = 2,\\
\epsilon(e_1) = \epsilon(e_3) = \epsilon(e_1e_3) = 1,\\
\epsilon(e_1e_2) = \epsilon(e_3e_2) = \epsilon(e_1e_3e_2) = 1,\\
\epsilon(e_1e_2e_3) = \epsilon(e_3e_2e_1) = \lambda,\\
\epsilon(e_2e_1) = \epsilon(e_2e_3) = \epsilon(e_2e_1e_3) = \epsilon(e_2e_1e_3e_2) = 2\lambda.
\end{aligned}$$

The antipode is determined by

$$S(1) = 1,\ \ S(e_1) = e_3,\ \ S(e_3) = e_1$$
$$S(e_2) = (\lambda e_3 + (1-\lambda)(1-e_3))e_2(\lambda e_3 + (1-\lambda)(1-e_3))^{-1}.$$

It is easy to see that H_t and H_s are generated by $\{1, e_1\}$ and $\{1, e_3\}$ respectively. Now we work out the centralizer subalgebra $C_H(H_s)$. Clearly,

$$1_1 1_H S(1_2) = 1_H,\ \ 1_1 e_1 S(1_2) = e_1,\ \ 1_1 e_3 S(1_2) = e_1.$$

However, $1_1e_2S(1_2) \neq e_2$. In fact, $1_1e_2S(1_2)$ is of the form:

$$\begin{aligned}
1_1e_2S(1_2) &= e_3e_2S(e_1) + (1-e_3)e_2S(1-e_1)\\
&= e_3e_2e_3 + (1-e_3)e_2(1-e_3)\\
&= e_3e_2e_3 + (e_2 - e_3e_2)(1-e_3)\\
&= e_3e_2e_3 + e_2 - e_3e_2 - e_2e_3 + e_3e_2e_3
\end{aligned}$$

$$\begin{aligned} &= 2\lambda e_3 + e_2 - e_3e_2 - e_2e_3 \\ &= (2\lambda - 1)e_3 + (e_2 - e_3)^2. \end{aligned}$$

Suppose that $1_1 e_2 S(1_2) = e_2$. Then $2\lambda e_3 + e_2 - e_3e_2 - e_2e_3 = e_2$, which means $2\lambda e_3 - e_3e_2 = e_2e_3$. So we have $2\lambda e_3e_3 - e_3e_2e_3 = e_2e_3e_3$. Since ${e_3}^2 = e_3$ and $e_3e_2e_3 = \lambda e_3$, then $\lambda e_3 = e_2e_3$ and so $\lambda e_3e_2 = e_2e_3e_2$. Using ${e_2}^2 = e_2$ and $e_2e_3e_2 = \lambda e_2$, we can get $\lambda e_3e_2 = \lambda e_2$ and so $\lambda e_2e_3e_2 = \lambda {e_2}^2$, which implies that $\lambda^2 e_2 = \lambda e_2$ and $\lambda^2 = \lambda$. Thus $\lambda = 1$, which is a contradiction since $\lambda^{-1} = 4cos^2 \frac{\pi}{5}$.

Therefore, $C_H(H_s) = 1_1 H S(1_2)$ is generated by $\{1, e_1, e_3, (2\lambda - 1)e_3 + (e_2 - e_3)^2\}$.

Lemma 2.1.5. *Let H be a weak Hopf algebra. Then $C_H(H_s)$ is a left H-module with the left adjoint action $h \cdot x = h_1 x S(h_2)$ for all $x \in C_H(H_s)$ and $h \in H$.*

Proof. Straightforward. □

For any $x, y \in C_H(H_s)$, by $\Delta(1) \in H_s \otimes H_t$, $1_1 xy S(1_2) = xy$. Now we have the following lemma:

Lemma 2.1.6. *Let H be a weak Hopf algebra. Then the multiplication of H induces the following morphism $\overline{\mu}$ on $C_H(H_s)$ in the category of left H-modules:*

$$\begin{aligned} \overline{\mu} : C_H(H_s) \otimes_t C_H(H_s) &\longrightarrow C_H(H_s), \\ a \otimes_t b &\longmapsto ab. \end{aligned}$$

Proof. Note that $(1_1 \cdot x)(1_2 \cdot y) = xS(1_1)1_2 y = xy$ for all $x, y \in C_H(H_s)$. So the map $\overline{\mu}$ is well-defined. For all $h \in H$, we have

$$\begin{aligned} h \cdot (xy) &= h_1(xy)S(h_2) \overset{(1.3)}{=} h_1 \varepsilon_s(h_2) xy S(h_3) \\ &\overset{(1.6)}{=} h_1 x \varepsilon_s(h_2) y S(h_3) = h_1 x S(h_2) h_3 y S(h_4) \\ &= (h_1 \cdot x)(h_2 \cdot y). \end{aligned}$$

Thus the map $\overline{\mu}$ is a morphism in the category of left H-modules. □

Lemma 2.1.7. *Let H be a weak Hopf algebra. Then $(C_H(H_s), \overline{\mu})$ is an algebra in the category of left H-modules with the following unit:*

$$\overline{\eta} = Id_{H_t} : H_t \longrightarrow C_H(H_s).$$

Proof. It is clear that $\overline{\mu}$ is associative. We only need to check the axioms for the unit. It follows from the definition of l^{-1} that

$$\begin{aligned}\overline{\mu}(\overline{\eta}\otimes 1)l^{-1}(x) &= \overline{\mu}(\overline{\eta}\otimes 1)(S(1_1)\otimes 1_2\cdot x)\\ &= \overline{\mu}(S(1_1)\otimes 1_2\cdot x)\\ &= S(1_1)(1_2\cdot x) = x.\end{aligned}$$

for any $x\in C_H(H_s)$. So $\overline{\mu}(\overline{\eta}\otimes 1) = id$. Similarly, we also have

$$\overline{\mu}(1\otimes\overline{\eta})r^{-1}(x) = \overline{\mu}(1\otimes\overline{\eta})(1_1\cdot x\otimes 1_2) = \overline{\mu}(1_1\cdot x\otimes 1_2) = x.$$

□

Similarly, $C_H(H_s)$ with the comultiplication Δ is an algebra in the category of left-left Yetter-Drinfeld modules.

Lemma 2.1.8. *Let H be a weak Hopf algebra. Then $I'(H_t)$ is isomorphic to $C_H(H_s)$ as an algebra in the category ${}^H_H\mathscr{YD}$ of left-left Yetter-Drinfeld modules, where $C_H(H_s)$ has the following action and coaction:*

$$h\cdot g = h_1 g S(h_2),\quad \rho^L(g) = g_1\otimes g_2,$$

for all $g\in C_H(H_s)$ and $h\in H$. In particular, $I'(H_t)$ is quantum commutative.

Proof. First consider a k-linear map $r: H\otimes H\longrightarrow H,\ g\otimes h\longmapsto S^{-1}(h)g$. We get

$$\begin{aligned}r(1_1hS(1'_2)\otimes 1_21'_1\cdot x) &= S^{-1}((1_21'_1\cdot x))[1_1hS(1'_2)]\\ &= S^{-1}(1_2xS(1'_1))[1_1hS(1'_2)]\\ &= 1'_1S^{-1}(x)hS(1'_2)\in C_H(H_s).\end{aligned}$$

So we have a well-defined map:

$$r: H\odot H_t\longrightarrow C_H(H_s),\quad 1_1hS(1'_2)\otimes 1_21'_1\cdot x\longmapsto 1'_1S^{-1}(x)hS(1'_2).$$

Define another map:

$$r': C_H(H_s)\longrightarrow H\odot H_t,\ h\longmapsto 1_1h1'_1\otimes 1_21'_2.$$

We then show that the map r' is the inverse of the map r. In fact, for $h \in H$, we have

$$\begin{aligned} r'r(1_1h1'_1 \otimes 1_21'_2) &= r'(1_1hS(1_2)) \\ &= 1_11''_1hS(1''_2)1'_1 \otimes 1_21'_2 \\ &= 1_11''_1h1'_1 \otimes 1_21''_21'_2 \\ &= 1_1h1'_1 \otimes 1_21'_2. \end{aligned}$$

So $r'r = id_{H \odot H_t}$. On the other hand, we have

$$rr'(h) = r(1_1h1'_1 \otimes 1_21'_2) = 1_1hS(1_2) = h.$$

Thus $rr' = id_{C_H(H_s)}$.

Now we claim that r is an algebra isomorphism between $H \odot H_t$ and $C_H(H_s)$ as algebras in the category of Yetter-Drinfeld modules. It is easy to see that r is H-colinear. For all $g, h \in H$, we have

$$\begin{aligned} & r[h \cdot (1_1g1'_1 \otimes 1_21'_2)] \\ = \ & r(h_1gS(h_2)1_1 \otimes 1_2) \\ = \ & h_1gS(h_2) = h \cdot (1_1gS(1_2)) \\ = \ & h \cdot [r(1_1g1'_1 \otimes 1_21'_2)]. \end{aligned}$$

So r is H-linear. For all $g, g' \in H$,

$$\begin{aligned} & r[(1_1g1'_1 \otimes 1_21'_2)(1_1g'1'_1 \otimes 1_21'_2)] \\ = \ & r(1_1g1'_1g'1''_1 \otimes 1_21'_21''_2) \\ = \ & r(1_1g1'_1g'1''_1 \otimes 1_21'_21''_2) \\ = \ & 1_11'_1gS(1'_2)g'S(1_2) \\ = \ & 1'_1gS(1'_2)1_1g'S(1_2) \\ = \ & r(1_1g1'_1 \otimes 1_21'_2)r(1_1g'1'_1 \otimes 1_21'_2). \end{aligned}$$

Finally, the quantum commutativity of $C_H(H_s)$ follows from

$$(a_1 \cdot b)a_2 = a_1bS(a_2)a_3 = a_1S(a_2)a_3b = ab.$$

for all $a, b \in C_H(H_s)$. □

Now we give the relation between the full center of the algebra H_t and $C_H(H_s)$.

Lemma 2.1.9. *Let H be a weak Hopf algebra. Then the full center $\mathcal{Z}(H_t)$ of the algebra H_t in the category of left H-modules is isomorphic to the algebra $C_H(H_s)$ in the category of left-left Yetter-Drinfeld modules.*

Proof. By Theorem 5.4 in [26], the full center $\mathcal{Z}(H_t)$ of H_t in the category of left H-modules is isomorphic to the left center $C_l(I'(H_t))$ of $I'(H_t)$ in the category of left-left Yetter-Drinfeld modules. By Lemma 2.1.8, $I'(H_t)$ is quantum commutative. So $C_l(I'(H_t)) = I'(H_t)$. Thus $\mathcal{Z}(H_t) \simeq C_H(H_s)$ as algebras. □

2.2 Constructing a braided Hopf algebra

It is well-known that if H is a quasi-triangular Hopf algebra, then H can be deformed into a Hopf algebra in the category of left modules, see [50, 52]. In general, a weak Hopf algebra H with a usual adjoint action is not necessarily a left (unitary) H-module, see Example 2.1.4. So one should not expect that a quasi-triangular weak Hopf algebra with the usual adjoint action can be deformed. Based on the full center $C_H(H_s)$ of the unit object H_t, however, we can still construct a Hopf algebra in the category of left H-modules in a similar way to Majid's method in [50].

Let (H, R) be a quasi-triangular weak Hopf algebra. Consider a map $\overline{\Delta}$:

$$\overline{\Delta} : C_H(H_s) \longrightarrow H \otimes H, \quad x \longmapsto x_1 S(R^2) \otimes R^1 \cdot x_2,$$

where the R-matrix $R = R^1 \otimes R^2$.

Lemma 2.2.1. *Let (H, R) be a quasi-triangular weak Hopf algebra. Then the map $\overline{\Delta}$ is a $\Bbbk$-linear map from H to $C_H(H_s) \otimes C_H(H_s)$.*

Proof. It is easy to see that $\overline{\Delta}(H) \subset H \otimes C_H(H_s)$. Moreover,

$$\begin{aligned} \overline{\Delta}(x) &= x_1 S(R^2) \otimes R^1 \cdot x_2 = x_1 S(R^2) \otimes R^1{}_1 x_2 S(R^1{}_2) \\ &= x_1 S(R^2 r^2) \otimes R^1 x_2 S(r^1) \overset{(1.16)}{=} x_1 r^2 S(R^2) \otimes R^1 x_2 r^1 \\ &= r^2 x_2 S(R^2) \otimes R^1 r^1 x_1 = R^2 \cdot x_2 \otimes R^1 x_1 \in C_H(H_s) \otimes H. \end{aligned}$$

Thus $\overline{\Delta}$ is well-defined. □

Lemma 2.2.2. *Let (H, R) be a quasi-triangular weak Hopf algebra. Then the following hold:*

1. *The map $\overline{\Delta}$ is a $\Bbbk$-linear map from H to $C_H(H_s) \otimes_t C_H(H_s)$;*
2. *In particular, $\overline{\Delta}|_{C_H(H_s)}$ is a morphism in the category of left H-modules;*
3. *The morphism $\overline{\Delta}$ is coassociative.*

Proof. By Lemma 2.2.1, the first statement is true if

$$1_1 \cdot (x_1 S(R^2)) \otimes 1_2 \cdot (R^1 \cdot x_2) = \overline{\Delta}(x),$$

for all $x \in H$. Indeed,

$$\begin{aligned}
& 1_1 \cdot (x_1 S(R^2)) \otimes 1_2 \cdot (R^1 \cdot x_2) \\
= \; & 1_1 x_1 S(R^2) S(1_2) \otimes 1_3 R^1 \cdot x_2 \\
= \; & 1_1 x_1 S(R^2) S(1'_1) S(1_2) \otimes 1'_2 R^1 \cdot x_2 \\
= \; & 1_1 x_1 S(1'_1 R^2) S(1_2) \otimes 1'_2 R^1 \cdot x_2 \\
\overset{(1.12)}{=} \; & 1_1 x_1 S(1'_1 S(1'_2) R^2) S(1_2) \otimes R^1 \cdot x_2 \\
= \; & 1_1 x_1 S(R^2) S(1_2) \otimes R^1 \cdot x_2 = x_1 S(R^2) \otimes R^1 \cdot x_2,
\end{aligned}$$

where the last equality was given by $\overline{\Delta}(x) \in C_H(H_s) \otimes H$.

Now we show that $\overline{\Delta}$ restricts to a morphism in the category of left modules. For any $h \in H$, we have

$$\begin{aligned}
h_1 \cdot (x_1 S(R^2)) \otimes h_2 \cdot (R^1 \cdot x_2) &= h_1 x_1 S(R^2) S(h_2) \otimes h_3 R^1 \cdot x_2 \\
&= h_1 x_1 S(h_2 R^2) \otimes h_3 R^1 \cdot x_2 \\
&= h_1 x_1 S(R^2 h_3) \otimes R^1 h_2 \cdot x_2 \\
&= h_1 x_1 S(h_4) S(R^2) \otimes R^1 \cdot (h_2 x_2 S(h_3)) \\
&= (h \cdot x)_1 S(R^2) \otimes R^1 \cdot (h \cdot x)_2 = \overline{\Delta}(h \cdot x).
\end{aligned}$$

Finally, we verify that the map $\overline{\Delta}$ is coassociative. Indeed, for all $x \in {}_R H$,

$$\begin{aligned}
(\overline{\Delta} \otimes 1) \circ \overline{\Delta}(x) &= \overline{\Delta}(x_1 S(R^2)) \otimes R^1 \cdot x_2 \\
&= (x_1 S(R^2))_1 S(r^2) \otimes r^1 \cdot (x_1 S(R^2))_2 \otimes R^1 \cdot x_2
\end{aligned}$$

$$\begin{aligned}
&= x_1S(R^2)S(r^2)\otimes r^1\cdot(x_2S(R'^2))\otimes R^1R'^1\cdot x_3\\
&= x_1S(r^2R^2)\otimes r^1\cdot(x_2S(R'^2))\otimes R^1R'^1\cdot x_3\\
&= x_1S(R^2)\otimes R^1{}_1\cdot(x_2S(R'^2))\otimes R^1{}_2R'^1\cdot x_3\\
&= x_1S(R^2)\otimes R^1{}_1\cdot(x_2S(R'^2))\otimes R^1{}_2\cdot(R'^1\cdot x_3)\\
&= x_1S(R^2)\otimes\overline{\Delta}(R^1\cdot x_2).
\end{aligned}$$

Thus the proof is completed. □

Lemma 2.2.3. *Let (H,R) be a quasi-triangular weak Hopf algebra. Then the map ε_t is a left H-linear map from $C_H(H_s)$ to H_t.*

Proof. For all $h\in H, x\in C_H(H_s)$, we have

$$\begin{aligned}
\varepsilon_t(h\cdot x) &= \varepsilon_t(h_1xS(h_2)) \overset{(1.5)}{=} \varepsilon_t(h_1x\varepsilon_tS(h_2))\\
&= \varepsilon_t(h_1xS\varepsilon_s(h_2)) \overset{(1.3)}{=} \varepsilon_t(h1_1xS^2(1_2))\\
&= \varepsilon_t(hx1_1S^2(1_2)) = \varepsilon_t(hx1_1S(1_2))\\
&= \varepsilon_t(hx) = \varepsilon_t(h\varepsilon_t(x)).
\end{aligned}$$

Thus ε_t is left H-linear. □

Now the coalgebraic structure on $C_H(H_s)$ can be given as follows:

Lemma 2.2.4. *Let (H,R) be a quasi-triangular weak Hopf algebra. Then $C_H(H_s)$ is a coalgebra in the category of left H-modules with the following structures:*

$$\begin{aligned}
&\overline{\Delta}: C_H(H_s)\longrightarrow C_H(H_s)\otimes_t C_H(H_s),\\
&\overline{\varepsilon}=\varepsilon_t: C_H(H_s)\longrightarrow H_t.
\end{aligned}$$

Proof. Using Lemma 2.2.2 and 2.2.3, we only need to check the axioms of the counit, $(\varepsilon_t\otimes 1)\circ\overline{\Delta}=Id=(1\otimes\varepsilon_t)\circ\overline{\Delta}$. For all $x\in C_H(H_s)$, we compute the following:

$$\begin{aligned}
r(1\otimes\varepsilon_t)\circ\overline{\Delta}(x) &= S(\varepsilon_t(R^1\cdot x_2))\cdot(x_1S(R^2))\\
&= 1_1(x_1S(R^2))S(1_2)S^2(\varepsilon_t(R^1\cdot x_2))\\
&= [1\cdot(x_1S(R^2))]S^2(\varepsilon_t(R^1\cdot x_2))\\
&= (x_1S(R^2))\varepsilon_t(R^1\cdot x_2)
\end{aligned}$$

$$
\begin{aligned}
&= (x_1 S(R^2))\varepsilon(1_1 R^1{}_1 x_2 S(R^1{}_2))1_2 \\
&= (x_1 S(R^2 R'^2))\varepsilon(S^{-1}(1_1) R^1 x_2 S(R'^1))1_2 \\
&\overset{(1.12)}{=} (x_1 S(1_1 R^2 R'^2))\varepsilon(R^1 x_2 S(R'^1))1_2 \\
&= x_1 S(R^2)\varepsilon(R^1{}_1 x_2 S(R^1{}_2))S(1_1)1_2 \\
&= x_1 S(R^2)\varepsilon(R^1{}_1 x_2 S(R^1{}_2)) \\
&= x_1 S(R^2)\varepsilon(\varepsilon_s(R^1{}_1) x_2 S(R^1{}_2)) \overset{(1.2)}{=} x_1 S(R^2)\varepsilon(1_1 x_2 S(R^1 1_2)) \\
&= x_1 S(R^2)\varepsilon(1_1 x_2 S(1_2) S(R^1)) \overset{(1.16)}{=} x_1 R^2 \varepsilon(1_1 x_2 S(1_2) R^1) \\
&\overset{(1.5)}{=} x_1 R^2 \varepsilon(1_1 x_2 S(1_2)\varepsilon_t(R^1)) \overset{(1.15)}{=} x_1 1'_1 \varepsilon(1_1 x_2 S(1_2) 1'_2) \\
&= x_1 1'_1 \varepsilon(1_1 x_2 1'_2 S(1_2)) = (1 \cdot x)_1 \varepsilon((1 \cdot x)_2) = x.
\end{aligned}
$$

Similarly, by the equations (2.1)-(2.7), we have

$$
\begin{aligned}
l(\overline{\Delta} \otimes 1) \circ \overline{\Delta}(x) &= [\varepsilon_t(x_1 S(R^2))R^1] \cdot x_2 \overset{(1.5)}{=} [\varepsilon_t(x_1 \varepsilon_t S(R^2))R^1] \cdot x_2 \\
&= [\varepsilon_t(x_1 S \varepsilon_s(R^2))R^1] \cdot x_2 \overset{(1.14)}{=} [\varepsilon_t(x_1 S(1_1)) S(1_2)] \cdot x_2 \\
&= [\varepsilon_t(x_1 1_2) 1_1] \cdot x_2 = 1'_1[\varepsilon_t(x_1 1_2) x_2 S(1_1)] S(1'_2) \\
&= 1'_1[\varepsilon_t(x_1 S^{-1}(1_1)) x_2 1_2] S(1'_2) = 1'_1[\varepsilon_t(x_1 S(1_1)) x_2 1_2] S(1'_2) \\
&\overset{(1.4)}{=} 1'_1[\varepsilon_t(x_1 \varepsilon_t(1_1)) x_2 1_2] S(1'_2) = 1'_1[\varepsilon_t(x_1 1_1) x_2 1_2] S(1'_2) \\
&= 1'_1[\varepsilon_t(x_1) x_2] S(1'_2) \overset{(1.4)}{=} 1 \cdot x = x.
\end{aligned}
$$

Thus $(C_H(H_s), \overline{\Delta}, \overline{\varepsilon})$ is a coalgebra in the category of left H-modules. □

Now we show that $C_H(H_s)$ can be equipped with a bialgebraic structure.

Lemma 2.2.5. *Let (H, R) be a quasi-triangular weak Hopf algebra. Then $C_H(H_s)$ is a bialgebra in the category of left H-modules.*

Proof. For all $x, y \in C_H(H_s)$, we have

$$
\begin{aligned}
\varepsilon_t(xy) &= \varepsilon(1_1 xy)1_2 \overset{(1.5)}{=} \varepsilon(1_1 x \varepsilon_t(y))1_2 \\
&= \varepsilon(x 1_1 \varepsilon_t(y))1_2 = \varepsilon(x \varepsilon_t(y) 1_1)1_2 \\
&\overset{(1.5)}{=} \varepsilon(x \varepsilon_t(y) S(1_1))1_2 \overset{(1.7)}{=} \varepsilon(x S(1_1))1_2 \varepsilon_t(y) \\
&\overset{(1.5)}{=} \varepsilon(x 1_1)1_2 \varepsilon_t(y) = \varepsilon(1_1 x)1_2 \varepsilon_t(y) = \varepsilon_t(x)\varepsilon_t(y),
\end{aligned}
$$

which shows that $\overline{\varepsilon}$ is an algebra map. By Lemma 2.2.1 and 2.2.4, it remains only to verify the equation (3.1). Note that $R\overline{R} = 1_2 \otimes 1_1$. The compatible condition on $\overline{\Delta}$ is shown by the following computation:

$$
\begin{aligned}
& (\overline{\mu}\otimes\overline{\mu})(1\otimes C\otimes 1)(\overline{\Delta}\otimes\overline{\Delta})(1_1\cdot x\otimes 1_2\cdot y)\\
= \;& (\overline{\mu}\otimes\overline{\mu})(1\otimes C\otimes 1)(\overline{\Delta}\otimes\overline{\Delta})(xS(1_1)\otimes 1_2 y)\\
= \;& (\overline{\mu}\otimes\overline{\mu})(1\otimes C\otimes 1)(x_1S(1_1)S(R^2)\otimes R^1\cdot x_2\otimes 1_2y_1S(R'^2)\otimes R'^1\cdot y_2)\\
= \;& (\overline{\mu}\otimes\overline{\mu})(x_1S(R^21_1)\otimes r^2\cdot(1_2y_1S(R'^2))\otimes r^1R^1\cdot x_2\otimes R'^1\cdot y_2)\\
= \;& (\overline{\mu}\otimes\overline{\mu})(x_1S(R^21_1)\otimes r^21_2\cdot(y_1S(R'^2))\otimes r^1R^1\cdot x_2\otimes R'^1\cdot y_2)\\
\overset{(1.11)}{=} \;& (\overline{\mu}\otimes\overline{\mu})(x_1S(R^2)\otimes r^21_2\cdot(y_1S(R'^2))\otimes r^11_1R^1\cdot x_2\otimes R'^1\cdot y_2)\\
= \;& (\overline{\mu}\otimes\overline{\mu})(x_1S(R^2)\otimes r^2\cdot(y_1S(R'^2))\otimes r^1R^1\cdot x_2\otimes R'^1\cdot y_2)\\
= \;& (x_1S(R^2))(r^2\cdot(y_1S(R'^2)))\otimes(r^1R^1\cdot x_2)(R'^1\cdot y_2)\\
= \;& (x_1S({R^2}_1))({R^2}_2\cdot(y_1S(R'^2)))\otimes(R^1\cdot x_2)(R'^1\cdot y_2)\\
= \;& (x_1S({R^2}_1))({R^2}_2(y_1S(R'^2))S({R^2}_3))\otimes(R^1\cdot x_2)(R'^1\cdot y_2)\\
= \;& (x_1\varepsilon_s({R^2}_1))(y_1S(R'^2))S({R^2}_2)\otimes(R^1\cdot x_2)(R'^1\cdot y_2)\\
\overset{(1.2)}{=} \;& (x_11_1)(y_1S(R'^2))S(R^21_2)\otimes(R^1\cdot x_2)(R'^1\cdot y_2)\\
\overset{(1.13)}{=} \;& (x_11_1)(y_1S(R'^2))S(R^2)\otimes(R^1S(1_2)\cdot x_2)(R'^1\cdot y_2)\\
= \;& (x_11_1)(y_1S(R'^2))S(R^2)\otimes(R^1\cdot(x_2S^2(1_2))(R'^1\cdot y_2)\\
= \;& x_11_1y_1S(R^2)\otimes R^1\cdot(x_2S^2(1_2)y_2)\\
= \;& x_11_11'_1y_1S(R^2)\otimes R^1\cdot(x_2S^2(1_2)1'_2y_2)\\
= \;& x_11'_1y_1S(R^2)\otimes R^1\cdot(x_2S^2(1_2)S^{-1}(S(1'_2)1_1)y_2)\\
= \;& x_1y_1S(R^2)\otimes R^1\cdot(x_2S^2(1_2)S^{-1}(1_1)y_2)\\
= \;& x_1y_1S(R^2)\otimes R^1\cdot(x_2S(1_1)1_2y_2)\\
= \;& x_1y_1S(R^2)\otimes R^1\cdot(x_2y_2),
\end{aligned}
$$

for all $x, y \in C_H(H_s)$. Thus the proof is completed. □

Let (H, R) be a quasi-triangular weak Hopf algebra. Let $u = S(R^2)R^1$ and $v = R^1S(R^2)$, where $R = R^1 \otimes R^2$. Then by [57, Prop.5.7] u and v are two invertible elements in H, and for all $h \in H$, we have

$$u^{-1} = R^2S^2(R^1), \quad v^{-1} = S^2(R^1)R^2, \quad S^2(h) = uhu^{-1}, \quad S^{-2}(h) = vhv^{-1}.$$

Similar to [51, Thm. 4.1], define a $\Bbbk$-linear map:

$$\overline{S} : C_H(H_s) \longrightarrow H, \quad x \longmapsto R^2 R'^2 S(R^1 x S(R'^1)).$$

Since $R \in \Delta^{op}(1)(H \otimes H)\Delta(1)$, we have for all $x \in {}_R H$,

$$\begin{aligned} 1_1 R^2 R'^2 S(R^1 x S(R'^1)) S(1_2) &= 1_1 R^2 R'^2 S(1_2 R^1 x S(R'^1)) \\ &= R^2 R'^2 S(R^1 x S(R'^1)), \end{aligned}$$

which implies that $\overline{S}$ is also a well-defined map from $C_H(H_s)$ to itself.

Lemma 2.2.6. *Let (H, R) be a quasi-triangular weak Hopf algebra. Then the map $\overline{S} : C_H(H_s) \longrightarrow C_H(H_s)$ is left H-linear.*

Proof. Take $x \in C_H(H_s)$. We have

$$\begin{aligned} \overline{S}(x) &= R^2 R'^2 S(R^1 x S(R'^1)) \\ &= R^2 R'^2 S^2(R'^1) S(R^1 x) \\ &= R^2 u^{-1} S(R^1 x). \end{aligned}$$

Using the properties of R, we get

$$\begin{aligned} \overline{S}(h \cdot x) &= R^2 u^{-1} S(R^1 h_1 x S(h_2)) \\ &= R^2 u^{-1} S^2(h_2) S(R^1 h_1 x) \\ &= R^2 h_2 u^{-1} S(R^1 h_1 x) \\ &= h_1 R^2 u^{-1} S(h_2 R^1 x) = h \cdot \overline{S}(x), \end{aligned}$$

for all $h \in H$. Hence, $\overline{S}$ is left H-linear. $\square$

Now we state the main result in this chapter.

Theorem 2.2.7. *Let (H, R) be a quasi-triangular weak Hopf algebra. Then $C_H(H_s)$ is a Hopf algebra in the category $({}_H\mathcal{M}, \otimes_t, H_t, l, r)$ with the following structures:*

1. *the multiplication $\overline{\mu}$ and the unit $\overline{\eta}$ are defined by:*

$$\overline{\mu} : C_H(H_s) \otimes_t C_H(H_s) \longrightarrow C_H(H_s), \quad a \otimes_t b \longmapsto (1_1 \cdot a)(1_2 \cdot b),$$
$$\overline{\eta} = Id_{H_t} : H_t \longrightarrow C_H(H_s), \quad x \longmapsto x.$$

2. the comultiplication $\overline{\Delta}$ and the counit $\overline{\varepsilon}$ are given by:

$$\overline{\Delta}: C_H(H_s) \longrightarrow C_H(H_s) \otimes_t C_H(H_s), \quad x \longmapsto x_1 S(R^2) \otimes R^1 \cdot x_2,$$
$$\overline{\varepsilon} = \varepsilon_t : C_H(H_s) \longrightarrow H_t, \qquad x \longmapsto \varepsilon_t(x).$$

3. the antipode $\overline{S}$ is given by

$$\overline{S}: C_H(H_s) \longrightarrow C_H(H_s), \quad x \longmapsto R^2 R'^2 S(R^1 x S(R'^1)).$$

Proof. Using Lemma 2.2.5 and 2.2.6, it is sufficient to verify the axiom of the antipode $\overline{S}$. Note that $\overline{\Delta}(x) \in C_H(H_s) \otimes_t C_H(H_s)$ and $\overline{R}R = \Delta(1)$. For all $x \in C_H(H_s)$, we directly compute the following:

$$\begin{aligned}
\overline{\mu} \circ (1 \otimes \overline{S}) \circ \overline{\Delta}(x) &= x_1 S(r^2) R^2 R'^2 S(R^1(r^1 \cdot x_2) S(R'^1)) \\
&= x_1 S(r^2) R^2 S(R^1{}_1(r^1 \cdot x_2) S(R^1{}_2)) \\
&= x_1 S(r^2) R^2 S(R^1 r^1 \cdot x_2)) \\
&= x_1 S(R^2{}_1) R^2{}_2 S(R^1 \cdot x_2)) \\
&= x_1 \varepsilon_s(R^2) S(R^1 \cdot x_2)) \\
&\overset{(1.14)}{=} x_1 1_1 S(S(1_2) \cdot x_2) \\
&= x_1 1_1 S(x_2 S^2(1_2)) \\
&= x_1 1_1 S(x_2 1_2) = \varepsilon_t(x).
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\overline{\mu} \circ (\overline{S} \otimes 1) \circ \overline{\Delta}(x) &= R^2 u^{-1} S(R^1 x_1 S(r^2))(r^1 \cdot x_2) \\
&= R^2 u^{-1} S^2(r^2) S(R^1 x_1)(r^1 \cdot x_2) \\
&= R^2 r^2 u^{-1} S(R^1 x_1)(r^1 \cdot x_2) \\
&= R^2 u^{-1} S(R^1{}_1 x_1)(R^1{}_2 \cdot x_2) \\
&= R^2 u^{-1} S(x_1) S(R^1{}_1) R^1{}_2 x_2 S(R^1{}_3) \\
&\overset{(1.2)}{=} R^2 u^{-1} S(x_1) 1_1 x_2 S(R^1 1_2) \\
&= R^2 u^{-1} S(x_1) 1_1 x_2 S(1_2) S(R^1) \\
&= R^2 u^{-1} \varepsilon_s(1 \cdot x) S(R^1) \\
&= R^2 u^{-1} S(R^1 S^{-1} \varepsilon_s(x))
\end{aligned}$$

$$\begin{aligned}
&\overset{(1.11)}{=} S^{-1}\varepsilon_s(x)R^2u^{-1}S(R^1)\\
&= S^{-1}(1_1)\varepsilon(x1_2)R^2R'^2S^2(R'^1)S(R^1)\\
&= 1_2\varepsilon(xS(1_1))R^2R'^2S(R^1S(R'^1))\\
&\overset{(1.4)}{=} 1_2\varepsilon(x\varepsilon_t(1_1))R^2S(R^1{}_1S(R^1{}_2))\\
&\overset{(1.5)}{=} 1_2\varepsilon(x1_1)R^2S\varepsilon_t(R^1)\\
&\overset{(1.15)}{=} 1_2\varepsilon(x1_1)1'_1S(1'_2) = \varepsilon_t(x).
\end{aligned}$$

Therefore, $\overline{S}$ is the antipode of $C_H(H_s)$. □

Now we apply Theorem 2.2.7 to the quantum double of a weak Hopf algebra.

Example 2.2.8. Let H be a finite dimensional regular weak Hopf algebra. We use the following Sweedler hit actions:

$$h \rightharpoonup \phi = \phi_1\langle h, \phi_2\rangle, \quad \phi \leftharpoonup h = \langle h, \phi_1\rangle\phi_2,$$

for all $h \in H, \phi \in H^*$. Nikshych et. al showed in [57] that the linear span J of the following elements

$$\begin{aligned}
&\phi \otimes yh - (y \rightharpoonup \varepsilon)\phi \otimes h,\ y \in H_s,\\
&\phi \otimes zh - (\varepsilon \leftharpoonup z)\phi \otimes h,\ \ z \in H_t,
\end{aligned}$$

is a two-sided ideal in $(H^{op})^* \otimes H$. Denote by $D(H)$ the factor-algebra $(H^{op})^* \otimes H/J$. We write $[\phi \otimes h]$ for the class of $\phi \otimes h$ in $D(H)$. Then $D(H)$ is a weak Hopf algebra with the following structures:

$$\begin{aligned}
&[\phi \otimes h][\psi \otimes g] = [\psi_2\phi \otimes h_2g]\langle S(h_1), \psi_1\rangle\langle h_3, \psi_3\rangle, \quad 1_{D(H)} = [\varepsilon \otimes 1]\\
&\Delta([\phi \otimes h]) = [\phi_1 \otimes h_1] \otimes [\phi_2 \otimes h_2], \quad \varepsilon([\phi \otimes h]) = \langle \varepsilon_t(h), \phi\rangle,\\
&S([\phi \otimes h]) = [S^{-1}(\phi_2) \otimes S(h_2)]\langle h_1, \phi_1\rangle\langle S(h_3), \psi_3\rangle,\\
&D(H)_s = [H^*{}_s \otimes 1], \quad D(H)_t = [\varepsilon \otimes H_t].
\end{aligned}$$

for all $\phi, \psi \in (H^{op})^*$ and $h, g \in H$. In addition, $D(H)$ has the following quasi-triangular structure:

$$R = \sum_i [e^i \otimes 1] \otimes [\varepsilon \otimes e_i], \quad \overline{R} = \sum_j [S^{-1}(e^j) \otimes 1] \otimes [\varepsilon \otimes e_j],$$

where $\{e_i\}$ and $\{e^i\}$ are dual basis of H and H^*.

By Theorem 2.2.7, the centralizer subalgebra $C_{D(H)}(D(H)_s)$ is a Hopf algebra in the category of left $D(H)$-modules.

Remark 2.2.9. (1) When $H_t = H_s = k$, it is clear that $C_H(H_s) = H$ is a Hopf algebra in the category of left H-modules, see [50].

(2) Similarly, we can also construct another braided Hopf algebra based on $C_H(H_t)$.

(3) It is easy to check that the Hopf algebra $C_H(H_s)$ in the category $({}_H\mathcal{M}, \otimes_t, H_t, l, r)$ is cocommutative in the sense of [50], see Definition 2.3 in [50].

(4) In Example 2.2.8, by taking H as a special weak Hopf algebra like a groupoid algebra, a face algebra in [38] or a generalized Kac algebra in [86], we can easily get many detailed Hopf algebras in corresponding braided monoidal categories.

Let (H, R) be a quasi-triangular weak Hopf algebra. In what follows, a braided Hopf algebra will always mean the Hopf algebra $C_H(H_s)$ in the category of left H-modules in Theorem 2.2.7 and be denoted by ${}_RH$.

Now we look at the dual case, the construction of a braided Hopf algebra from a coquasi-triangular weak Hopf algebra. Let (H, σ) be a coquasi-triangular weak Hopf algebra with a bijective antipode S. H contains an idempotent element $e = 1_1 1_2$. So there exists a decomposition $H = eH \oplus (1 - e)H$. We can prove that eH is a Hopf algebra in the category of H-comodules, see [23].

Theorem 2.2.10. *Let (H, σ) be a coquasi-triangular weak Hopf algebra. Then eH is a Hopf algebra in the category of right H-comodules with the following structures:*

1. *the multiplication $\overline{m}$ and unit $\overline{\eta}$ are defined by,*

$$\overline{m} : eH \otimes eH \to eH,\ \overline{m}(eh \otimes eg) = eh_2 g_2 \sigma(S(h_1)h_3, S(g_1))$$

$$\overline{\eta} : H_s \to eH, \ \ \overline{\eta}(x) = ex$$

2. *the comultiplication $\overline{\Delta}$ and counit ε_s are given by,*

$$\overline{\Delta} : eH \to eH \otimes eH, \ \ \overline{\Delta}(eh) = eh_1 \otimes eh_2,$$

$$\overline{\varepsilon} = \varepsilon_s : eH \to H_s.$$

3. the antipode $\overline{S}$ is given by,

$$\overline{S} : eH \to eH,\ \overline{S}(eh) = eS(h_2)\sigma(S(h_1), h_5)\sigma(S^2(h_3), (h_4).$$

2.3 The case of the right Drinfeld center

Now we consider the case of the right Drinfeld center that will be used in Chapter 5.

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. Assume that the functor $F : \mathscr{D} \longrightarrow \mathscr{C}$ has a right adjoint $I : \mathscr{C} \longrightarrow \mathscr{D}$ with a counit-unit adjunction $(\beta, \alpha) : F \longrightarrow G$. Then we can construct a hom-set adjunction

$$\Phi : hom_{\mathscr{C}}(F-, -) \longrightarrow hom_{\mathscr{D}}(-, I-)$$

in the following steps: For each $f : FY \longrightarrow X$ and each $g : Y \longrightarrow IX$, define

$$\Phi_{Y,X}(f) = I(f) \circ \alpha_Y$$

and its inverse $\Psi_{Y,X}(g) = \beta_X \circ F(g)$. Conversely, if the functor $F : \mathscr{D} \longrightarrow \mathscr{C}$ has a right adjoint $I : \mathscr{C} \longrightarrow \mathscr{D}$ with a hom-set adjunction $\Phi : hom_{\mathscr{C}}(F-, -) \longrightarrow hom_{\mathscr{D}}(-, I-)$, then we can construct its counit-unit adjunction $(\beta, \alpha) : F \longrightarrow G$ with

$$\beta_X = \Phi^{-1}_{GX,X}(1_{GX}) \in hom_{\mathscr{C}}(FGX, X), \quad \alpha_Y = \Phi_{Y,FY}(1_{FY}) \in hom_{\mathscr{D}}(Y, GFY).$$

Let $\mathscr{C}$ be a monoidal category. By Example 1.1.6, there exists a braided monoidal equivalence

$$E : \mathscr{Z}_r(\mathscr{C}) \longrightarrow \mathscr{Z}_l(\mathscr{C})^{rev}$$

with the inverse functor E^{-1}. Note that $\mathscr{Z}_l(\mathscr{C}) = \mathscr{Z}_l(\mathscr{C})^{rev}$ as monoidal categories. We have the following lemma:

Lemma 2.3.1. *Let $\mathscr{C}$ be a monoidal category. Assume that there is a functor $R : \mathscr{C} \longrightarrow \mathscr{Z}_r(\mathscr{C})$. Then R is a right adjoint functor of the forgetful functor $F : \mathscr{Z}_r(\mathscr{C}) \longrightarrow \mathscr{C}$ if and only if the composition functor $E \circ R : \mathscr{C} \longrightarrow \mathscr{Z}_l(\mathscr{C})$ is a right adjoint functor of the forgetful functor $F' : \mathscr{Z}_l(\mathscr{C}) \longrightarrow \mathscr{C}$.*

Let $\mathscr{C}$ be a monoidal category. Assume that the forgetful functor $F' : \mathscr{Z}_l(\mathscr{C}) \longrightarrow \mathscr{C}$ has a right adjoint functor $R' : \mathscr{C} \longrightarrow \mathscr{Z}_l(\mathscr{C})$ with the natural transformations of the adjunction:

$$\alpha_U : U \longrightarrow R' \circ F(U), \ \beta_X : F \circ R'(X) \longrightarrow X$$

for all $U \in \mathscr{Z}_l(\mathscr{C})$ and $X \in \mathscr{C}$. By Section 1.5 the functor R' is automatically lax monoidal, where the natural transformation φ_2 is given by the composite

$$R'(\beta_X \otimes \beta_Y) \circ R'(\varphi_2^{-1}(R'(X), R'(Y))) \circ \alpha_{R'(X)\otimes R'(Y)},$$

for all X and Y in $\mathscr{C}$. Note that E gives a monoidal equivalence. By Lemma 2.3.1, we have the following lemma:

Lemma 2.3.2. *Let $\mathscr{C}$ be a monoidal category. Assume that the forgetful functor $F' : \mathscr{Z}_l(\mathscr{C}) \longrightarrow \mathscr{C}$ has a right adjoint functor $R' : \mathscr{C} \longrightarrow \mathscr{Z}_l(\mathscr{C})$. Then the composition functor $R := E^{-1} \circ R' : \mathscr{C} \longrightarrow \mathscr{Z}_r(\mathscr{C})$ has a lax monoidal structure induced by E^{-1} and R', where E is the same as above.*

Remark 2.3.3. By [32, 33, 60], the functor R in Lemma 2.3.2 has another lax monoidal structure defined as follows:
Let $\mu_X : F(R_{\mathscr{C}}(X)) \longrightarrow X$ be the image of $id_{R(X)}$ under the canonical isomorphism

$$hom(R(X), R(X)) \simeq hom(F(R_{\mathscr{C}}(X)), X),$$

for all X in $\mathscr{C}$. Let $\mu_{X,Y}$ be the image of $\mu_X \otimes \mu_Y : R_{\mathscr{C}}(X) \otimes R_{\mathscr{C}}(Y) \longrightarrow R_{\mathscr{C}}(X \otimes Y)$ under the canonical isomorphism

$$\begin{aligned} hom(F(R_{\mathscr{C}}(X)) \otimes F(R_{\mathscr{C}}(Y)), X \otimes Y) &\simeq hom(F(R_{\mathscr{C}}(X) \otimes R_{\mathscr{C}}(Y)), X \otimes Y) \\ &\simeq hom(R_{\mathscr{C}}(X) \otimes R_{\mathscr{C}}(Y), R_{\mathscr{C}}(X \otimes Y)), \end{aligned}$$

for all X and Y in $\mathscr{C}$. If α' and β' are the counit and the unit of (F, R) respectively, by the relation between its counit-unit adjunction and its hom-set adjunction, the map $\mu_{X,Y}$ is given by the composition $R(\beta_X \otimes \beta_Y) \circ R(\varphi^{-1}_{R(X),R(Y)}) \circ \alpha'_{R(X)\otimes R(Y)}$:

$$R(X) \otimes R(Y) \longrightarrow R \circ F(R(X) \otimes R(Y)) \longrightarrow R[F(R(X)) \otimes F(R(Y))] \longrightarrow R(X \otimes Y).$$

By the counit $\alpha' = E^{-1} \circ \alpha \circ E$, this lax monoidal structure on R coincides with the lax monoidal structure induced by E^{-1} and R'.

By Theorem 1.6.10, there exists a braided monoidal equivalence functor from ${}_H\mathscr{Y}\mathscr{D}^H$ to ${}^H_H\mathscr{Y}\mathscr{D}^{rev}$. Here we still denote this braided monoidal functor and its inverse by E and E^{-1} respectively.

Corollary 2.3.4. *Assume that there is a functor $R : {}_H\mathscr{M} \longrightarrow {}_H\mathscr{Y}\mathscr{D}^H$. Then the functor R is a right adjoint functor of the forgetful functor $F : {}_H\mathscr{Y}\mathscr{D}^H \longrightarrow {}_H\mathscr{M}$ if and only if the functor $E \circ R$ is isomorphic to the functor I'.*

Proof. Follows from Theorem 1.6.9, Lemma 2.1.1 and 2.3.1. □

Corollary 2.3.5. *Assume that the forgetful functor $F : {}_H\mathscr{Y}\mathscr{D}^H \longrightarrow {}_H\mathscr{M}$ has a right adjoint functor $R : {}_H\mathscr{M} \longrightarrow {}_H\mathscr{Y}\mathscr{D}^H$. Then*

$$E[R(H_t)] \simeq I'(H_t)$$

as algebras in the category ${}^H_H\mathscr{Y}\mathscr{D}$, where E and R are as above.

Proof. By Corollary 2.3.4, the functor $E \circ R$ is also a right adjoint functor of the forgetful functor F'. By the uniqueness of a full center, $E[R(H_t)] \simeq I'(H_t)$ as algebras in the category ${}^H_H\mathscr{Y}\mathscr{D}$. □

Remark 2.3.6. (1) Since E is an equivalent functor, we have

$$R(H_t) \simeq E^{-1}(I'(H_t))$$

as algebras in the category ${}_H\mathscr{Y}\mathscr{D}^H$. By Remark 2.3.2, the algebra $R(H_t)$ was the one constructed in [28, 32], which is of importance in the study of braided fusion categories, see [28, 32]. Thus, this motives us to relate the theory in [28, 32] to the full center of the unit object, see Chapter 5.

(2) Similar to [27], we can describe the full center of an algebra in the Drinfeld center of H-modules by using the centralizer subalgebras of the smash product algebra.

Chapter 3

Braided autoequivalences and bi-Galois objects

Let H be a quasi-triangular weak Hopf algebra over a field $\Bbbk$. In the previous chapter, we equipped the full center of the unit object with the structure of a braided Hopf algebra. In this chapter, this braided Hopf algebra is used to re-describe the Drinfeld center by showing that there exists a braided monoidal equivalence between the category of Yetter-Drinfeld modules over H and the category of comodules over this braided Hopf algebra. This equivalence motivates us to discuss the relation between braided autoequivalences and bi-Galois objects. Finally, we form the group of quantum commutative Galois objects.

3.1 The Drinfeld center

Let (H, R) be a quasi-triangular weak Hopf algebra. By Theorem 2.2.7, we have the braided Hopf algebra ${}_RH$. In this section, we will investigate the relation between the category of left comodules over ${}_RH$ and the category of left-left Yetter-Drinfeld modules over H. Now we need a special case of Definition 1.3.5.

Definition 3.1.1. Let H be a quasi-triangular weak Hopf algebra with a bijective antipode S. Let M be a left H-module. We call (M, ρ^l) a left ${}_RH$-comodule in the category of left H-modules if (M, ρ^l) is a left ${}_RH$-comodule such that ρ^l is left

H-linear, i.e.,

$$\rho^l(h\cdot m) = h_1\cdot m_{(-1)}\otimes h_2\cdot m_{(0)},\ \forall h\in H,\ m\in M.$$

In the sequel, a left ${}_RH$-comodule will always mean a left ${}_RH$-comodule in the category of left H-modules. Similarly, we have a right ${}_RH$-comodule and an ${}_RH$-bicomodule.

For any two left ${}_RH$-comdules (M,ρ^l) and (N,ρ^l), $M\otimes_t N$ is a left ${}_RH$-comdule with the following structure:

$$h\cdot(m\otimes n) = h_1\cdot m\otimes h_2\cdot n,$$
$$\rho^l(m\otimes n) = (\overline{\mu}\otimes 1\otimes 1)(1\otimes C\otimes 1)(\rho^l\otimes\rho^l)(m\otimes n),$$

where $m\in M$, $n\in N$, $h\in H$ and C is the braiding of the category ${}_H\mathscr{M}$.

Remark 3.1.2. We denote by ${}^{{}_RH}({}_H\mathscr{M})$ the category of left ${}_RH$-comodules and the morphisms that are both left H-linear and left ${}_RH$-colinear. By [70], the category ${}^{{}_RH}({}_H\mathscr{M})$ is a monoidal category with the unit object H_t.

Now we discuss the relation between the category ${}^{{}_RH}({}_H\mathscr{M})$ and the category of left-left Yetter-Drinfeld modules.

Lemma 3.1.3. *Let H be a quasi-triangular weak Hopf algebra with a bijective antipode S. Let (M,ρ^l) be a left ${}_RH$-comodule. Then M is a left-left Yetter-Drinfeld module with the following H-comodule structure:*

$$\rho^L(m) = m_{(-1)}R^2\otimes R^1\cdot m_{(0)},$$

where $\rho^l(m) = m_{(-1)}\otimes m_{(0)}$ for all $m\in M$.

Proof. For any $m\in M$, we have

$$\begin{aligned} 1_1 m_{(-1)}R^2\otimes 1_2R^1\cdot m_{(0)} &= m_{(-1)}1_1R^2\otimes 1_2R^1\cdot m_{(0)} \\ &= m_{(-1)}R^2\otimes R^1\cdot m_{(0)}. \end{aligned}$$

So $\rho^L(M)\in H\otimes_t M$.

Next let us check that (M, ρ^L) is a left H-comodule. For the coassociativity,

$$\begin{aligned}
(1\otimes\rho^L)\rho^L &= (1\otimes\rho^L)(m_{(-1)}R^2\otimes R^1\cdot m_{(0)})\\
&= m_{(-1)}R^2\otimes(R^1\cdot m_{(0)})_{(-1)}q^2\otimes q^1\cdot(R^1\cdot m_{(0)})_{(0)}\\
&= m_{(-1)}R^2\otimes(R^1_1\cdot m_{(0)_{(-1)}})q^2\otimes q^1\cdot(R^1_2\cdot m_{(0)})\\
&= m_{(-1)_1}S(r^2)R^2\otimes(R^1_1r^1\cdot m_{(-1)_2})q^2\otimes q^1R^1_2\cdot m_{(0)}\\
&\overset{(1.9)}{=} m_{(-1)_1}S(r^2)p^2R^2\otimes(p^1r^1\cdot m_{(-1)_2})q^2\otimes q^1R^1\cdot m_{(0)}\\
&\overset{(1.8)}{=} m_{(-1)_1}\varepsilon_s(r^2)R^2\otimes(r^1\cdot m_{(-1)_2})q^2\otimes q^1R^1\cdot m_{(0)}\\
&\overset{(1.14)}{=} m_{(-1)_1}1_1R^2\otimes(S(1_2)\cdot m_{(-1)_2})q^2\otimes q^1R^1\cdot m_{(0)}\\
&= m_{(-1)_1}1_1R^2\otimes(m_{(-1)_2}S^2(1_2))q^2\otimes q^1R^1\cdot m_{(0)}\\
&= m_{(-1)_1}1_1R^2\otimes(m_{(-1)_2}1_2)q^2\otimes q^1R^1\cdot m_{(0)}\\
&= m_{(-1)_1}R^2\otimes m_{(-1)_2}q^2\otimes q^1R^1\cdot m_{(0)}\\
&= (m_{(-1)_1}R^2_1\otimes(m_{(-1)_2}R^2_2\otimes R^1\cdot m_{(0)})\\
&= (\Delta\otimes 1)(m_{(-1)}R^2\otimes R^1\cdot m_{(0)})\\
&= (\Delta\otimes 1)\rho^L(m).
\end{aligned}$$

The counit axiom holds as we have

$$\begin{aligned}
(\varepsilon\otimes 1)\rho^L(m) &= \varepsilon(m_{(-1)}R^2)(R^1\cdot m_{(0)})\\
&\overset{(1.5)}{=} \varepsilon(m_{(-1)}\varepsilon_t(R^2))(R^1\cdot m_{(0)})\\
&\overset{(1.15)}{=} \varepsilon(m_{(-1)}1_2)(S(1_1)\cdot m_{(0)})\\
&= \varepsilon(m_{(-1)}S(1_1))(1_2\cdot m_{(0)})\\
&= \varepsilon(m_{(-1)}1_1)(1_2\cdot m_{(0)})\\
&= \varepsilon(1_1m_{(-1)})(1_2\cdot m_{(0)})\\
&= \varepsilon_t(m_{(-1)})\cdot m_{(0)}) = m,
\end{aligned}$$

where the last equality followed from the counit of a left ${}_RH$-comodule.

Finally, the compatible condition holds since

$$\begin{aligned}
&h_1(m_{(-1)}R^2)\otimes h_2\cdot[R^1\cdot m_{(0)}]\\
=\ &h_11_1m_{(-1)}S(1_2)R^2\otimes h_2R^1\cdot m_{(0)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(1.3)}{=} && h_1 m_{(-1)} S(h_2) h_3 R^2 \otimes h_4 R^1 \cdot m_{(0)} \\
&\overset{(1.10)}{=} && h_1 m_{(-1)} S(h_2) R^2 h_4 \otimes R^1 h_3 \cdot m_{(0)} \\
&= && (h_1 \cdot m)_{(-1)} R^2 h_2 \otimes R^1 \cdot (h_1 \cdot m)_{(0)}.
\end{aligned}$$

for all $m \in M$ and $h \in H$. □

The following lemma tells us that the converse of Lemma 3.1.3 also holds.

Lemma 3.1.4. *Let H be a quasi-triangular weak Hopf algebra with a bijective antipode S. Let (N, ρ^L) be a left-left Yetter-Drinfeld module. Then N is a left ${}_RH$-comodule with the following structure:*

$$\rho^l(n) = n_{[-1]} S(R^2) \otimes R^1 \cdot n_{[0]},$$

where $\rho^L(n) = n_{[-1]} \otimes n_{[0]}$ for all $n \in N$.

Proof. First of all, we need to check that ρ^l is well-defined. For all $n \in N$,

$$\begin{aligned}
1_1 \cdot [n_{[-1]} S(R^2)] \otimes 1_2 R^1 \cdot n_{[0]} &= [n_{[-1]} S(R^2)] S(1_1) \otimes 1_2 R^1 \cdot n_{[0]} \\
&= [n_{[-1]} S(1_1 R^2)] \otimes 1_2 R^1 \cdot n_{[0]} \\
&= [n_{[-1]} S(R^2)] \otimes R^1 \cdot n_{[0]} \\
1_1 [n_{[-1]} S(R^2) S(1_2) \otimes R^1 \cdot n_{[0]} &= 1_1 n_{[-1]} S(1_2 R^2) \otimes R^1 \cdot n_{[0]} \\
&\overset{(1.11)}{=} 1_1 n_{[-1]} S(R^2) \otimes R^1 1_2 \cdot n_{[0]} \\
&= 1_1 n_{[-1]} S(R^2) \otimes R^1 \cdot (1_2 \cdot n_{[0]}) \\
&= n_{[-1]} S(R^2) \otimes R^1 \cdot n_{[0]}.
\end{aligned}$$

So $\rho^l(N) \subset {}_RH \otimes_t N$. The H-linearity of the map ρ^l follows from

$$\begin{aligned}
& && h_1 \cdot [n_{[-1]} S(R^2)] \otimes h_2 R^1 \cdot n_{[0]} \\
&= && h_1 n_{[-1]} S(R^2) S(h_2) \otimes h_3 R^1 \cdot n_{[0]} \\
&= && h_1 n_{[-1]} S(h_2 R^2) \otimes h_3 R^1 \cdot n_{[0]} \\
&\overset{(1.10)}{=} && h_1 n_{[-1]} S(R^2 h_3) \otimes R^1 h_2 \cdot n_{[0]}
\end{aligned}$$

$$\begin{aligned}
&= (h_1 n_{[-1]} S(h_3))S(R^2) \otimes R^1 \cdot (h_2 \cdot n_{[0]}) \\
&= (h \cdot n)_{[-1]} S(R^2) \otimes R^1 \cdot (h \cdot n)_{[0]} \\
&= \rho^l(h \cdot n),
\end{aligned}$$

for all $h \in H$.

Now let us show that (N, ρ^l) is a left ${}_RH$-comodule. For any $n \in N$,

$$\begin{aligned}
(1 \otimes \rho^l)\rho^l(n) &= n_{[-1]} S(R^2) \otimes (R^1 \cdot n_{[0]})_{[-1]} S(r^2) \otimes r^1 \cdot (R^1 \cdot n_{[0]})_{[0]} \\
&= n_{[-1]} S(R^2) \otimes R^1_1 n_{[0]_{[-1]}} S(R^1_3) S(r^2) \otimes r^1 \cdot (R^1_2 \cdot n_{[0]_{[0]}}) \\
&= n_{[-1]_1} S(R^2) \otimes R^1_1 n_{[-1]_2} S(r^2 R^1_3) \otimes r^1 R^1_2 \cdot n_{[0]} \\
&\overset{(1.10)}{=} n_{[-1]_1} S(R^2) \otimes R^1_1 n_{[-1]_2} S(R^1_2 r^2) \otimes R^1_3 r^1 \cdot n_{[0]} \\
&\overset{(1.9)}{=} n_{[-1]_1} S(R^2 q^2) \otimes R^1 \cdot [n_{[-1]_2} S(r^2)] \otimes q^1 r^1 \cdot n_{[0]} \\
&\overset{(1.8)}{=} [n_{[-1]_1} S(r^2_2)] S(R^2) \otimes R^1 \cdot [n_{[-1]_2} S(r^2_1)] \otimes r^1 \cdot n_{[0]} \\
&= [n_{[-1]} S(r^2)]_1 S(R^2) \otimes R^1 \cdot [n_{[-1]} S(r^2)]_2 \otimes r^1 \cdot n_{[0]} \\
&= \overline{\Delta}[n_{[-1]} S(r^2)] \otimes r^1 \cdot n_{[0]} = (\overline{\Delta} \otimes 1)\rho^l(n).
\end{aligned}$$

So the coassociativity holds. The counit axiom is given by the following

$$\begin{aligned}
& \varepsilon_t(n_{[-1]} S(R^2)) \cdot (R^1 \cdot n_{[0]}) \\
&= (\varepsilon_t(n_{[-1]} S(R^2)) R^1) \cdot n_{[0]} \\
&\overset{(1.9)}{=} (1_2 R^1) \cdot n_{[0]} \varepsilon(1_1 n_{[-1]} \varepsilon_t[S(R^2)]) \\
&= (1_2 R^1) \cdot n_{[0]} \varepsilon(1_1 n_{[-1]} S[\varepsilon_s(R^2)]) \\
&\overset{(1.14)}{=} (1_2 S(1'_2)) \cdot n_{[0]} \varepsilon(1_1 n_{[-1]} S(1'_1)) \\
&= (1_2 1'_1) \cdot n_{[0]} \varepsilon(1_1 n_{[-1]} 1'_2) \\
&= (1_2 1'_1) \cdot n_{[0]} \varepsilon(1_1 n_{[-1]} S(1'_2)) \\
&= 1_2 \cdot n_{[0]} \varepsilon(1_1 n_{[-1]} S(1_3)) = n.
\end{aligned}$$

Therefore, (N, ρ^l) is a left ${}_RH$-comodule. □

By Lemma 3.1.3 and 3.1.4, we obtain the following:

Theorem 3.1.5. *Let (H, R) be a quasi-triangular weak Hopf algebra with a bijective antipode S. Then there exists a monoidal equivalence $\mathcal{F}$ from the category ${}^{_RH}({}_H\mathscr{M})$*

of left $_RH$-comodules to the category ${}^H_H\mathscr{YD}$ of left-left Yetter-Drinfeld modules:

$$\mathcal{F}:\ {}^{_RH}({}_H\mathscr{M}) \longrightarrow {}^H_H\mathscr{YD},\ (M,\rho^l)\longmapsto (M,\rho^L),$$

where ρ^L is defined in Lemma 3.1.3 and the inverse of $\mathcal{F}$ is given by

$$\mathcal{G}:\ {}^H_H\mathscr{YD} \longrightarrow {}^{_RH}({}_H\mathscr{M}),\ (N,\rho^L)\longmapsto (N,\rho^l),$$

where ρ^l is also defined in Lemma 3.1.4.

Proof. We first claim that $\mathcal{GF}(M) = M$ for any object M of ${}^{_RH}({}_H\mathscr{M})$. It is enough to check that $\rho^l(m) \equiv m_{(-1)}\otimes m_{(0)}$ for all $m\in M$. In fact,

$$\begin{aligned}
\rho^l(m) &= m_{[-1]}S(R^2)\otimes R^1\cdot m_{[0]}\\
&= m_{(-1)}r^2S(R^2)\otimes R^1\cdot[r^1\cdot m_{(0)}]\\
&= m_{(-1)}r^2S(R^2)\otimes (R^1r^1)\cdot m_{(0)}\\
&\overset{(1.8)}{=} m_{(-1)}\varepsilon_t(R^2)\otimes R^1\cdot m_{(0)}\\
&\overset{(1.15)}{=} m_{(-1)}1_2\otimes S(1_1)\cdot m_{(0)}\\
&= S^{-1}(1_2)\cdot m_{(-1)}\otimes S(1_1)\cdot m_{(0)}\\
&= S(1_2)\cdot m_{(-1)}\otimes S(1_1)\cdot m_{(0)}\\
&= 1_1\cdot m_{(-1)}\otimes 1_2\cdot m_{(0)}\\
&= m_{(-1)}\otimes m_{(0)}.
\end{aligned}$$

Next we show that $\mathcal{FG}(N) = N$ for any object N of ${}^H_H\mathscr{YD}$. For all $n\in N$,

$$\begin{aligned}
\rho^L(n) &= n_{(-1)}R^2\otimes R^1\cdot n_{(0)}\\
&= n_{[-1]}S(r^2)R^2\otimes R^1\cdot(r^1\cdot n_{[0]})\\
&= n_{[-1]}S(r^2)R^2\otimes (R^1r^1)\cdot n_{[0]}\\
&\overset{(1.8)}{=} n_{[-1]}\varepsilon_s(R^2)\otimes R^1\cdot n_{[0]}\\
&\overset{(1.14)}{=} n_{[-1]}1_1\otimes S(1_2)\cdot n_{[0]}\\
&= n_{[-1]}S(1_2)\otimes 1_1\cdot n_{[0]}\\
&= 1'_1n_{[-1]}S(1_2)\otimes 1_1\cdot(1'_2\cdot n_{[0]})\\
&= 1_1n_{[-1]}S(1_3)\otimes 1_2\cdot n_{[0]}
\end{aligned}$$

$$= \quad n_{[-1]} \otimes n_{[0]}.$$

Finally, we verify that the triple $(\mathcal{G}, id, Id)$ is monoidal. It is clear that $\mathcal{G}(H_t) = H_t$. For any two left-left Yetter-Drinfeld modules U and V, the left ${}_RH$-comodule structure on $\mathcal{G}(U) \otimes \mathcal{G}(V)$ is as follows:

$$\begin{aligned}
& (\mu \otimes 1 \otimes 1)(1 \otimes C \otimes 1)(\rho^L \otimes \rho^L)(u \otimes v) \\
= \; & (\mu \otimes 1 \otimes 1)(1 \otimes C \otimes 1)(u_{(-1)} \otimes u_{(0)} \otimes n_{(-1)} \otimes v_{(0)}) \\
= \; & (\mu \otimes 1 \otimes 1)(u_{(-1)} \otimes R^2 \cdot v_{(-1)} \otimes R^1 \cdot u_{(0)} \otimes v_{(0)}) \\
= \; & u_{(-1)}(R^2 \cdot v_{(-1)}) \otimes R^1 \cdot u_{(0)} \otimes v_{(0)},
\end{aligned}$$

where $u \in U$ and $v \in V$. Now we have

$$\begin{aligned}
& u_{(-1)}(R^2 \cdot v_{(-1)}) \otimes R^1 \cdot u_{(0)} \otimes v_{(0)} \\
= \; & (u_{[-1]}S(p^2))R^2{}_1(v_{[-1]}S(q^2))S(R^2{}_2) \otimes R^1 \cdot (p^1 \cdot u_{[0]}) \otimes q^1 \cdot v_{[0]} \\
\overset{(1.8)}{=} \; & (u_{[-1]}S(p^2))r^2(v_{[-1]}S(q^2))S(R^2) \otimes (R^1 r^1 p^1) \cdot u_{[0]}) \otimes q^1 \cdot v_{[0]} \\
\overset{(1.8)}{=} \; & u_{[-1]}\varepsilon_s(r^2)(v_{[-1]}S(q^2))S(R^2) \otimes (R^1 r^1) \cdot u_{[0]}) \otimes q^1 \cdot v_{[0]} \\
\overset{(1.14)}{=} \; & u_{[-1]}S(1_2)(v_{[-1]}S(q^2))S(R^2) \otimes (R^1 1_1) \cdot u_{[0]}) \otimes q^1 \cdot v_{[0]} \\
= \; & u_{[-1]}S(1_2)(v_{[-1]}S(q^2))S(R^2) \otimes R^1 \cdot (1_1 \cdot u_{[0]}) \otimes q^1 \cdot v_{[0]} \\
= \; & u_{[-1]}(v_{[-1]}S(q^2))S(R^2) \otimes R^1 \cdot u_{[0]} \otimes q^1 \cdot v_{[0]} \\
= \; & (u_{[-1]}v_{[-1]})S(R^2 q^2) \otimes R^1 \cdot u_{[0]} \otimes q^1 \cdot v_{[0]} \\
\overset{(1.9)}{=} \; & (u_{[-1]}v_{[-1]})S(R^2) \otimes R^1 \cdot (u_{[0]} \otimes v_{[0]}) \\
= \; & (u \otimes_t v)_{[-1]}S(R^2) \otimes R^1 \cdot (u \otimes_t v)_{[0]} \\
= \; & \rho^l(u \otimes_t v).
\end{aligned}$$

Hence, $\mathcal{G}(U \otimes V) = \mathcal{G}(U) \otimes \mathcal{G}(V)$. The other axioms of a monoidal functor are obviously true. □

Since the category of left-left Yetter-Dinfeld modules is braided, the equivalence in Theorem 3.1.5 can induce a braiding in the category of left ${}_RH$-comodules such that the equivalence becomes braided.

Corollary 3.1.6. *Let (H, R) be a quasi-triangular weak Hopf algebra a bijective antipode S. Then the category of left ${}_RH$-comodules is a braided monoidal category with* *the following braiding, for all $u \in U$ and $v \in V$,*

$$\widetilde{C}(u \otimes v) = u_{(-1)}R^2 \cdot v \otimes R^1 \cdot u_{(0)},$$

$\forall u \in U$ and $v \in V$, where U and V are any two left ${}_RH$-comodules. The inverse of $\widetilde{C}$ is given by

$$\widetilde{C}^{-1}(v \otimes u) = R^1 \cdot u_{(0)} \otimes S^{-1}(u_{(-1)}R^2) \cdot v,$$

Moreover, the functor $\mathcal{G}$ in Theorem 3.1.5 is a braided monoidal equivalence.

Proof. Using Theorem 3.1.5 and [42] we work out the stated braiding using the following diagram

$$\begin{array}{ccc} \mathcal{G}(U) \otimes_t \mathcal{G}(V) & \longrightarrow & \mathcal{G}(U \otimes_t V)) \\ \downarrow C_{\mathcal{G}(U),\mathcal{G}(V)} & & \mathcal{G}(C_{U,V}) \downarrow \\ \mathcal{G}(V) \otimes_t \mathcal{G}(U) & \longleftarrow & \mathcal{G}(V \otimes_t U). \end{array}$$

By $Id : \mathcal{G}(U) \otimes \mathcal{G}(V) = \mathcal{G}(U \otimes V)$, the braiding $\widetilde{C}$ is the composition $Id \circ C_{U,V} \circ Id$. For any $u \in U$ and $v \in V$, we have

$$\begin{aligned} \widetilde{C}_{U,V}(u \otimes v) &= Id \circ C_{U,V} \circ Id(u \otimes v) \\ &= Id \circ C_{U,V}(u \otimes v) \\ &= Id(u_{[-1]} \cdot v \otimes u_{[0]}) \\ &= u_{(-1)}R^2 \cdot v \otimes R^1 \cdot u_{(0)}. \end{aligned}$$

Similarly, one can obtain the inverse of $\widetilde{C}$. □

Remark 3.1.7. (1) When H is a finite dimensional quasi-triangular Hopf algebra, the functor $\mathcal{G}$ was first proved in [88] to have a right adjoint functor.

(2) In Theorem 3.1.5 and Corollary 3.1.6, we do not need H to be finite dimensional. In fact, our result holds for any infinite dimensional quasi-triangular weak Hopf algebra over any field (or over a commutative ring).

Remark 3.1.8. Let (H, R) be a finite dimensional quasi-triangular Hopf algebra. By Radford's biproduct theorem in [63], there exists naturally a Hopf algebra ${}_RH\sharp H$. In particular, the category ${}_{RH}({}_H\mathscr{M})$ of left ${}_RH$-modules (in the category ${}_H\mathscr{M}$ of left H-modules) is equivalent to the category ${}_{RH\sharp H}\mathscr{M}$ of left ${}_RH\sharp H$-modules as a monoidal category.

By [25, 52], an induced structure $\overline{R}$ $(:= (\iota\otimes\iota)(R)$ $)$ on the Hopf algebra ${}_RH\sharp H$ is quasi-triangular if and only if the braiding of ${}_H\mathscr{M}$ is ${}_RH$-linear in ${}_H\mathscr{M}$. In fact, as we know, the Hopf algebra ${}_RH\sharp H$ is in general not quasi-triangular. So it does not seem for the above equivalence naturally to become a braided equivalence between the left Drinfeld center of ${}_H\mathscr{M}$ and the category ${}_{RH}({}_H\mathscr{M})$.

Following Corollary 3.1.6, however, we can naturally view the left Drinfeld center of the category ${}_H\mathscr{M}$ as the category of left ${}_RH$-comodules. This motivates us to consider the relation between braided bi-Galois objects and braided autoequivalences of the Drinfeld center of the category ${}_H\mathscr{M}$ in the next section.

3.2 Braided autoequivalences and braided bi-Galois objects

In this section, let H be a finite dimensional quasi-triangular weak Hopf algebra. We will construct braided autoequivalences from braided bi-Galois objects.

Remark 3.2.1. Note that H is finite dimensional. So is ${}_RH$. By [57], ${}_RH$ has a dual object. Thus ${}_RH$ is a finite object in the the category ${}_H\mathscr{M}$. It follows from [25] that ${}_RH$ is flat in the category ${}_H\mathscr{M}$.

Now we need some detailed definitions from Section 1.3.

Definition 3.2.2. An algebra A in the braided monoidal category ${}_H\mathscr{M}$ is called a left ${}_RH$-comodule algebra if A with a left ${}_RH$-coaction ρ^l is a left ${}_RH$-comodule such that

$$\rho^l(ab) = a_{(-1)}(R^2\cdot b_{(-1)})\otimes(R^1\cdot a_{(0)})b_{(0)},$$

for all $a, b\in A$, where $\rho^r(a) = a_{(0)}\otimes a_{(1)}$. Namely, ρ^l is an algebra map in ${}_H\mathscr{M}$.

Similarly, an algebra A in the category ${}_H\mathscr{M}$ is called a right ${}_RH$-comodule algebra if A with a right ${}_RH$-coaction ρ^r is a right ${}_RH$-comodule such that

$$\rho^r(ab) = a_{(0)}(R^2\cdot b_{(0)})\otimes(R^1\cdot a_{(1)})b_{(1)},$$

where $a, b \in A$ and $\rho^r(a) = a_{(0)} \otimes a_{(1)}$. An ${}_RH$-bicomodule algebra can be defined similarly, see [69].

Let A be a right ${}_RH$-comodule algebra. Define the coinvariant subalgebra of A as

$$A_\circ = \{a \in A | \rho^r(a) = a \otimes_t 1\}.$$

Similarly, the coinvariant subalgebra of a left ${}_RH$-comodule algebra can be defined.

Definition 3.2.3. [70, Defn 2.1] Let A be a right ${}_RH$-comodule algebra. We call A a right braided ${}_RH$-Galois object if A is faithfully flat and the morphism

$$\beta : A \otimes_t A \longrightarrow A \otimes_t {}_RH,\ a \otimes b \longmapsto ab_{(0)} \otimes b_{(1)}$$

is an isomorphism in ${}_H\mathscr{M}$.

Similarly, one can define a left braided ${}_RH$-Galois object and a braided bi-Galois object, see [70].

Remark 3.2.4. For a right ${}_RH$-Galois object A, the coinvariant subalgebra $A_\circ$ is trivial. Similarly, the coinvariant subalgebra of a left ${}_RH$-Galois object A is trivial.

It is easy to see that $({}_RH, \tau_{{}_RH,-})$ is an object in the left Drinfeld center of the category of left H-modules, where $\tau_{{}_RH,-}$ is a half-braiding defined by

$$\tau_{{}_RH,M} : {}_RH \otimes M \longrightarrow M \otimes {}_RH,\ h \otimes m \longmapsto r^2R^1 \cdot m \otimes r^1hR^2.$$

Since ${}_RH$ is cocommutative cocentral, for any left ${}_RH$-comodule (M, ρ^l), by [70], there is a natural right comodule structure induced by the half-braiding $\tau_{{}_RH,M}$:

$$\rho^r = \tau_{{}_RH,M} \circ \rho^l : M \longrightarrow {}_RH \otimes M \longrightarrow M \otimes {}_RH.$$

Let (M, ρ^l, ρ^r) be an ${}_RH$-bicomodule. By [70], we call M *cocommutative* if the right coaction ρ^r is induced by the left coaction ρ^l.

Definition 3.2.5. A cocommutative braided bi-Galois object A is called a *quantum commutative Gaols object* if A is quantum commutative as an algebra in the category of left-left Yetter-Drinfeld modules.

For any two left-left Yetter-Drinfeld modules M and N, by Theorem 3.1.5 and Corollary 3.1.6, the cotensor product $M\Box N$ over ${}_RH$ is defined as

$$\{m\otimes n\in M\otimes_t N|\rho^r(m)\otimes n=m\otimes\rho^l(n)\},$$

which is equal to

$$\{m\otimes n\in M\otimes_t N|r^2\cdot m_{[0]}\otimes r^1m_{[-1]}\otimes n=m\otimes n_{[-1]}S(R^2)\otimes R^1\cdot n_{[0]}\}. \qquad (3.1)$$

If A is a braided ${}_RH$-bi-Galois object, by [69] we have an isomorphism

$$\xi:(A\Box M)\otimes_t(A\Box N)\simeq A\Box(M\otimes_t N)$$

given by

$$\xi((a\otimes m)\otimes(b\otimes n))=a(R^2\cdot b)\otimes R^1\cdot m\otimes n,$$

for all $a,b\in A$, $m\in M$ and $b\in N$. Following [70], the cotensor functor $A\Box-$ is a monoidal autoequivalence of the category ${}^{_RH}({}_H\mathscr{M})$.

Lemma 3.2.6. *Let (H,R) be a finite dimensional quasi-triangular weak Hopf algebra. If A is a quantum commutative Galois object, then the functor $A\Box-$ is a braided autoequivalence of the category ${}^{_RH}({}_H\mathscr{M})$.*

Proof. Let A be a quantum commutative Galois object. By Theorem 3.1.5 and [42], we only need to check that the diagram:

$$\begin{array}{ccc} (A\Box M)\otimes_t(A\Box N) & \longrightarrow & A\Box(M\otimes_t N)) \\ \Big\downarrow \widetilde{C}_{A\Box M,A\Box N} & & A\Box\widetilde{C}_{M,N}\Big\downarrow \\ (A\Box N)\otimes_t(A\Box M) & \longrightarrow & A\Box(N\otimes_t M) \end{array} \qquad (*)$$

is commutative. On one hand, for any $a\otimes m\in A\Box M$ and $b\otimes n\in A\Box N$, we have

$$\begin{aligned} & [\xi\circ(\widetilde{C}_{A\Box M,A\Box N})][(a\otimes m)\otimes(b\otimes n)] \\ = \;& \xi[(a\otimes m)_{(-1)}r^2\cdot(b\otimes n)\otimes r^1\cdot(a\otimes m)_{(0)}] \\ = \;& \xi[a_{(-1)}r^2\cdot(b\otimes n)\otimes r^1\cdot(a_{(0)}\otimes m)] \end{aligned}$$

$$
\begin{aligned}
&= \xi[a_{(-1)_1}r_1^2\cdot b\otimes a_{(-1)_2}r_2^2\cdot n\otimes r_1^1\cdot a_{(0)}\otimes r_2^1\cdot m)]\\
&= [a_{(-1)_1}r_1^2\cdot b][R^2r_1^1\cdot a_{(0)}]\otimes R^1a_{(-1)_2}r_2^2\cdot n\otimes r_2^1\cdot m\\
&= [a_{[-1]_1}S(q_2^2)]r_1^2\cdot b][R^2r_1^1\cdot[q^1\cdot a_{[0]}]]\otimes R^1[a_{[-1]_2}S(q_1^2)]r_2^2\cdot n\otimes r_2^1\cdot m\\
&= [a_{[-1]_1}[S(q^2)r^2]_1\cdot b][R^2r_1^1q^1\cdot a_{[0]}]]\otimes R^1[a_{[-1]_2}[S(q^2)r^2]_2]\cdot n\otimes r_2^1\cdot m\\
&\overset{(1.9)}{=} [a_{[-1]_1}[S(q^2)r^2p^2]_1\cdot b][R^2r^1q^1\cdot a_{[0]}]]\otimes R^1[a_{[-1]_2}[S(q^2)r^2p^2]_2]\cdot n\otimes p^1\cdot m\\
&\overset{(1.8)}{=} [a_{[-1]_1}[\varepsilon_s(r^2)p^2]_1\cdot b][R^2r^1\cdot a_{[0]}]]\otimes R^1[a_{[-1]_2}[\varepsilon_s(r^2)p^2]_2]\cdot n\otimes p^1\cdot m\\
&\overset{(1.14)}{=} [a_{[-1]_1}[1_1p^2]_1\cdot b][R^2S(1_2)\cdot a_{[0]}]]\otimes R^1[a_{[-1]_2}[1_1p^2]_2]\cdot n\otimes p^1\cdot m\\
&= [a_{[-1]_1}p_1^2\cdot b][R^2S(1_2)\cdot a_{[0]}]]\otimes R^1[a_{[-1]_2}1_1p_2^2]\cdot n\otimes p^1\cdot m\\
&\overset{(1.16)}{=} [a_{[-1]_1}p_1^2\cdot b][R^2\cdot a_{[0]}]\otimes R^1[a_{[-1]_2}p_2^2]\cdot n\otimes p^1\cdot m,
\end{aligned}
$$

where Corollary 3.1.6 and Lemma 3.1.4 were used in the first and the fifth equality respectively. On the other hand, we have

$$
\begin{aligned}
& (1\otimes\widetilde{C})\circ\xi[(a\otimes m)\otimes(b\otimes n)]\\
&= a(r^2\cdot b)\otimes\widetilde{C}(r^1\cdot m\otimes n)\\
&= a(r^2\cdot b)\otimes(r^1\cdot m)_{(-1)}W^2\cdot n\otimes W^1\cdot(r^1\cdot m)_{(0)}\\
&= a(r^2\cdot b)\otimes(r_1^1\cdot m_{(-1)})W^2\cdot n\otimes W^1r_2^1\cdot m_{(0)}\\
&= a(r^2\cdot b)\otimes r_1^1m_{(-1)}S(r_2^1)W^2\cdot n\otimes W^1r_3^1\cdot m_{(0)}\\
&= a(r^2\cdot b)\otimes r_1^1m_{[-1]}S(R^2)S(r_2^1)W^2\cdot n\otimes W^1r_3^1R^1\cdot m_{[0]}\\
&= a(r^2\cdot b)\otimes r_1^1m_{[-1]}S(r_2^1R^2)W^2\cdot n\otimes W^1r_3^1R^1\cdot m_{[0]}\\
&= a(r^2\cdot b)\otimes r_1^1m_{[-1]}S(r_3^1)S(R^2)W^2\cdot n\otimes W^1R^1r_2^1\cdot m_{[0]}\\
&= a(r^2\cdot b)\otimes r_1^1m_{[-1]}S(r_3^1)\cdot n\otimes r_2^1\cdot m_{[0]}\\
&= a(r^2\cdot b)\otimes r_1^1m_{[-1]}S(1_2)S(r_3^1)\cdot n\otimes r_2^11_1\cdot m_{[0]}\\
&\overset{(1.14)}{=} a(r^2\cdot b)\otimes r_1^1m_{[-1]}S(R^2)p^2S(r_3^1)\cdot n\otimes r_2^1p^1R^1\cdot m_{[0]}\\
&\overset{(3.1)}{=} (R^2\cdot a_{[0]})(r^2\cdot b)\otimes r_1^1R^1a_{[-1]}p^2S(r_3^1)\cdot n\otimes r_2^1p^1\cdot m\\
&= [(R^2\cdot a_{[0]})_{[-1]}\cdot(r^2\cdot b)](R^2\cdot a_{[0]})_{[0]}\otimes r_1^1R^1a_{[-1]}p^2S(r_3^1)\cdot n\otimes r_2^1p^1\cdot m\\
&= [(R_1^2a_{[-1]_2}S(R_3^2)r^2\cdot b)](R_2^2\cdot a_{[0]})\otimes r_1^1R^1a_{[-1]_1}p^2S(r_3^1)\cdot n\otimes r_2^1p^1\cdot m\\
&\overset{(1.8)}{=} [(R^2a_{[-1]_2}S(Q_2^2)r^2\cdot b)](Q_1^2\cdot a_{[0]})\otimes r_1^1Q^1R^1a_{[-1]_1}p^2S(r_3^1)\cdot n\otimes r_2^1p^1\cdot m\\
&\overset{(1.10)}{=} [(a_{[-1]_1}R^2S(Q_2^2)r^2\cdot b)](Q_1^2\cdot a_{[0]})\otimes r_1^1Q^1a_{[-1]_2}R^1p^2S(r_3^1)\cdot n\otimes r_2^1p^1\cdot m
\end{aligned}
$$

$$\begin{aligned}
&\overset{(1.9)}{=} [(a_{[-1]_1}R^2S(U^2)V^2r^2\cdot b)](Q^2\cdot a_{[0]})\otimes V^1U^1Q^1a_{[-1]_2}R^1p^2S(r_2^1)\cdot n\otimes r_1^1p^1\cdot m\\
&\overset{(1.14)}{=} [(a_{[-1]_1}R^21_1r^2\cdot b)](Q^2\cdot a_{[0]})\otimes S(1_2)Q^1a_{[-1]_2}R^1p^2S(r_2^1)\cdot n\otimes r_1^1p^1\cdot m\\
&\overset{(1.11)}{=} [(a_{[-1]_1}R^2r^2\cdot b)](Q^2S(1_2)\cdot a_{[0]})\otimes Q^1a_{[-1]_2}1_1R^1p^2S(r_2^1)\cdot n\otimes r_1^1p^1\cdot m\\
&\overset{(1.16)}{=} [(a_{[-1]_1}R^2r^2\cdot b)](Q^2\cdot a_{[0]})\otimes Q^1a_{[-1]_1}R^1p^2S(r_2^1)\cdot n\otimes r_1^1p^1\cdot m\\
&\overset{(1.9)}{=} [(a_{[-1]_1}R^2\cdot b)](Q^2\cdot a_{[0]})\otimes Q^1a_{[-1]_1}R_1^1p^2S(R_3^1)\cdot n\otimes R_2^1p^1\cdot m\\
&\overset{(1.10)}{=} [(a_{[-1]_1}R^2\cdot b)](Q^2\cdot a_{[0]})\otimes Q^1a_{[-1]_1}p^2R_2^1S(R_3^1)\cdot n\otimes p^1R_1^1\cdot m\\
&= [(a_{[-1]_1}R^2\cdot b)](Q^2\cdot a_{[0]})\otimes Q^1a_{[-1]_1}p^21_2\cdot n\otimes p^11_1R^1\cdot m\\
&= [(a_{[-1]_1}R^2\cdot b)](Q^2\cdot a_{[0]})\otimes Q^1a_{[-1]_1}p^2\cdot n\otimes p^1R^1\cdot m,
\end{aligned}$$

where Corollary 3.1.6 and Lemma 3.1.4 were used in the second and the fifth equality respectively; the twelfth and the thirteenth equations stemmed from the compatible condition and the quantum-commutativity respectively. Thus

$$\xi\circ(\widetilde{C}_{A\Box M,A\Box N})=(1\otimes\widetilde{C})\circ\xi.$$

Therefore, the proof is completed. □

Lemma 3.2.7. *Let (H,R) be a finite dimensional quasi-triangular weak Hopf algebra. Assume that A is a braided bi-Galois object. If the functor $A\Box-$ is a braided autoequivalence of the category ${}^{_RH}({}_H\mathscr{M})$, then A is quantum commutative.*

Proof. Assume that the functor $A\Box-$ is a braided autoequivalence. We first have the commutative diagram $(*)$. Let M and N be any two left ${}_RH$-comodules. By the proof of Lemma 3.2.6, we get the following equation:

$$\begin{aligned}
& a_{(0)}(r^2\cdot b)\otimes r_1^1a_{(1)}p^2S(r_3^1)\cdot n\otimes r_2^1p^1\cdot m\\
=\;& [a_{(-1)_1}r_1^2\cdot b][R^2r_1^1\cdot a_{(0)}]\otimes R^1a_{(-1)_2}r_2^2\cdot n\otimes r_2^1\cdot m, \qquad (3.2)
\end{aligned}$$

for all $a\otimes m\in A\Box M$ and $b\otimes n\in A\Box N$. In particular, for any $a,b\in A$, we have $a_{(0)}\otimes a_{(1)},b_{(0)}\otimes b_{(1)}\in A\Box_RH$ and so

$$\begin{aligned}
& a_{(0)}(r^2\cdot b_{(0)})\otimes(r^1\cdot a_{(1)})_{[-1]}\cdot b_{(1)}\otimes(r^1\cdot a_{(1)})_{[0]}\\
=\;& [a_{(-1)_1}r_1^2\cdot b_{(0)}][R^2r_1^1\cdot a_{(0)}]\otimes R^1a_{(-1)_2}r_2^2\cdot b_{(1)}\otimes r_2^1\cdot a_{(1)}.
\end{aligned}$$

Applying the map $1 \otimes \varepsilon_t \otimes \varepsilon_t$ to the above equality, we get

$$\begin{aligned} & [a_{(-1)_1} r_1^2 \cdot b_{(0)}][R^2 r_1^1 \cdot a_{(0)}] \otimes \varepsilon_t(R^1 a_{(-1)_2} r_2^2 \cdot b_{(1)}) \varepsilon_t(r_2^1 \cdot a_{(1)}) \\ = \; & a_{(0)}(r^2 \cdot b_{(0)}) \otimes \varepsilon_t[(r^1 \cdot a_{(1)})_{[-1]} \cdot b_{(1)}] \varepsilon_t[(r^1 \cdot a_{(1)})_{[0]}]. \end{aligned}$$

Since ε_t is an algebra map in the category ${}_H\mathscr{M}$ and A is a right ${}_RH$-comodule algebra, we obtain

$$[a_{(-1)} r^2 \cdot b][r^1 \cdot a_{(0)}] = ab,$$

which is equivalent to $ab = (a_{[-1]} \cdot b) a_{[0]}$. Thus, A is a quantum commutative algebra.

Now we show that A is cocommutative. Namely, we need to verify that the right coaction ρ^r on A is induced by its left coaction ρ^r and the half-braiding. Note that the regular left H-module H has an induced Yetter-Drinfeld module structure, where the comodule structure is given by

$$\rho^L(h) = R^2 \otimes R^1 h := h_{[-1]} \otimes h_{[0]}, \quad \forall h \in H,$$

Then by Lemma 3.1.4 we have a left ${}_RH$-comodule on H, where $\rho^l(h) = 1 \otimes_t h$ for any $h \in H$. Namely, (H, ρ^l) is a trivial left ${}_RH$-comodule. Now we consider $A \square_R H$ and $A \square H$. Note that $1_A \otimes_t 1_H \in A \square H$ and $a_{(0)} \otimes a_{(1)} \in A \square_R H$. By the equality (3.2), we easily get

$$\begin{aligned} & a_{(0)}(r^2 \cdot 1) \otimes r_1^1 a_{(1)} p^2 S(r_3^1) \otimes r_2^1 p^1 \cdot a_{(2)} \\ = \; & [a_{(-1)_1} r_1^2 \cdot 1][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)_2} r_2^2 \otimes r_2^1 \cdot a_{(1)}. \end{aligned}$$

One one hand, we have

$$\begin{aligned} & a_{(0)}(r^2 \cdot 1_A) \otimes r_1^1 a_{(1)} p^2 S(r_3^1) \otimes r_2^1 p^1 \cdot a_{(2)} \\ = \; & a_{(0)}(\varepsilon_t(r^2) \cdot 1_A) \otimes r_1^1 a_{(1)} p^2 S(r_3^1) \otimes r_2^1 p^1 \cdot a_{(2)} \\ \overset{(1.15)}{=} \; & a_{(0)}(1'_2 \cdot 1_A) \otimes S(1'_1) 1_1 a_{(1)} p^2 S(1_3) \otimes 1_2 p^1 \cdot a_{(2)} \\ = \; & 1'_1 \cdot a_{(0)} \otimes 1'_2 1_1 a_{(1)} p^2 S(1_3) \otimes 1_2 p^1 \cdot a_{(2)} \\ = \; & a_{(0)} \otimes 1_1 a_{(1)} p^2 S(1_3) \otimes 1_2 p^1 \cdot a_{(2)} \\ = \; & a_{(0)} \otimes a_{(1)} p^2 S(1_2) \otimes 1_1 p^1 \cdot a_{(2)} \\ \overset{(1.11)}{=} \; & a_{(0)} \otimes a_{(1)} p^2 \otimes p^1 \cdot a_{(2)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& [a_{(-1)_1} r_1^2 \cdot 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)_2} r_2^2 \otimes r_2^1 \cdot a_{(1)} \\
= \; & [\varepsilon_t(a_{(-1)_1} r_1^2) \cdot 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)_2} r_2^2 \otimes r_2^1 \cdot a_{(1)} \\
= \; & [\varepsilon_t(a_{(-1)_1} \varepsilon_t(r_1^2)) \cdot 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)_2} r_2^2 \otimes r_2^1 \cdot a_{(1)} \\
\overset{(1.4)}{=} \; & [\varepsilon_t(a_{(-1)_1} S(1_1)) \cdot 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)_2} 1_2 r^2 \otimes r_2^1 \cdot a_{(1)} \\
= \; & [\varepsilon_t(a_{(-1)_1}) \cdot 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 a_{(-1)_2} r^2 \otimes r_2^1 \cdot a_{(1)} \\
= \; & [1_1 \cdot 1_A][R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 1_2 a_{(-1)} r^2 \otimes r_2^1 \cdot a_{(1)} \\
= \; & S(1_1) R^2 r_1^1 \cdot a_{(0)}] \otimes R^1 1_2 a_{(-1)} r^2 \otimes r_2^1 \cdot a_{(1)} \\
\overset{(1.11)}{=} \; & R^2 r_1^1 \cdot a_{(0)} \otimes R^1 a_{(-1)} r^2 \otimes r_2^1 \cdot a_{(1)}.
\end{aligned}$$

Then the following equation holds:

$$a_{(0)} \otimes a_{(1)} p^2 \otimes p^1 \cdot a_{(2)} = R^2 r_1^1 \cdot a_{(0)} \otimes R^1 a_{(-1)} r^2 \otimes r_2^1 \cdot a_{(1)} \in A \otimes H \otimes H.$$

Applying $(1 \otimes 1 \otimes \varepsilon)$ to right side of the above equation, we get

$$\begin{aligned}
& (1 \otimes 1 \otimes \varepsilon)(R^2 r_1^1 \cdot a_{(0)} \otimes R^1 a_{(-1)} r^2 \otimes r_2^1 \cdot a_{(1)}) \\
\overset{(1.5)}{=} \; & R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(\varepsilon_s(r_2^1) a_{(1)} S(r_3^1)) \\
\overset{(1.2)}{=} \; & R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(a_{(1)} S(r_2^1)) \\
= \; & R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(a_{(1)} \varepsilon_t(S(r_2^1))) \\
= \; & R^2 r_1^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(a_{(1)} S(\varepsilon_s(r_2^1))) \\
\overset{(1.3)}{=} \; & R^2 r^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(a_{(1)} S^2(1_2))) \\
= \; & R^2 r^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(a_{(1)} S(1_2))) \\
= \; & R^2 r^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(S(1_2) a_{(1)})) \\
= \; & R^2 r^1 1_1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \varepsilon(1_2 a_{(1)}) \\
= \; & R^2 r^1 S(\varepsilon_t(a_{(1)})) \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \\
= \; & R^2 r^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2],
\end{aligned}$$

where the counit of a right ${}_R H$-comodule A was used in the last equality. Now we obtain

$$R^2 r^1 \cdot a_{(0)} \otimes [R^1 a_{(-1)} r^2] \quad = \quad (1 \otimes 1 \otimes \varepsilon)(a_{(0)} \otimes a_{(1)} p^2 \otimes p^1 \cdot a_{(2)})$$

$$
\begin{aligned}
&= a_{(0)} \otimes a_{(1)} p^2 \varepsilon(p^1 \cdot a_{(2)}) \\
&= a_{(0)} \otimes a_{(1)} p^2 \varepsilon(\varepsilon_s(p^1_1) a_{(2)} S(p^1_2))] \\
&= a_{(0)} \otimes a_{(1)} p^2 \varepsilon(a_{(2)} S(p^1)] \\
&= a_{(0)} \otimes a_{(1)} p^2 \varepsilon(a_{(2)} S(\varepsilon_s(p^1))] \\
&\overset{(1.14)}{=} a_{(0)} \otimes a_{(1)} 1_2 \varepsilon(a_{(2)} S(1_1)] \\
&= a_{(0)} \otimes a_{(1)} 1_2 \varepsilon(1_1 a_{(2)}) \\
&= a_{(0)} \otimes a_{(1)} \varepsilon_t(a_{(2)}) \\
&= a_{(0)} \otimes a_{(1)},
\end{aligned}
$$

where the counit of ${}_R H$ was applied to the last equality. This means that a right ${}_R H$-comodule structure on A is induced by its left ${}_R H$-coaction. Therefore, A is a quantum commutative Galois object. □

Theorem 3.2.8. *Let (H, R) be a finite dimensional quasi-triangular weak Hopf algebra. Assume that A is a braided bi-Galois object. Then the functor $A\square-$ is a braided autoequivalence of the category ${}^H_H\mathscr{YD}$ of left-left Yetter-Drinfeld modules if and only if A is quantum commutative.*

Proof. Assume that A is a braided bi-Galois object. By Lemma 3.2.6 and 3.2.7, the functor $A\square-$ is a braided autoequivalence of the category ${}^{{}_R H}({}_H\mathscr{M})$ if and only if A is quantum commutative. Following Corollary 3.1.6, the category ${}^{{}_R H}({}_H\mathscr{M})$ is equivalent to the category ${}^H_H\mathscr{YD}$ of Yetter-Drinfeld modules as a braided monoidal category. Thus the functor $A\square-$ is a braided autoequivalence of the category ${}^H_H\mathscr{YD}$ of Yetter-Drinfeld modules if and only if A is quantum commutative. □

When H is a finite dimensional quasi-triangular weak Hopf algebra over an algebraically closed field of characteristic 0 such that it is semisimple and connected, we have the following statement:

Corollary 3.2.9. *Let $\mathscr{C}$ be a braided fusion category. Then the Drinfeld center of $\mathscr{C}$ is equivalent to the category of finite dimensional left $H_{\mathscr{C}}$-comoduels over some braided Hopf algebra $H_{\mathscr{C}}$. Moreover, if A is a braided bi-Galois object over $H_{\mathscr{C}}$, then the cotensor functor $A\square-$ is a braided autoequivalence of the Drinfeld center of $\mathscr{C}$ if and only if A is quantum commutative.*

Proof. Let $\mathscr{C}$ be a braided fusion category. By [60], there is a semisimple connected weak Hopf algebra such that $\mathscr{C}$ is equivalent to the category of finite dimensional left $H_{\mathscr{C}}$-modules. By [57], $H_{\mathscr{C}}$ can be equipped with a quasi-triangular structure such that $\mathscr{C}$ is equivalent to the category of finite dimensional left $H_{\mathscr{C}}$-modules as a braided fusion category. Thus the proof is completed by applying Corollary 3.1.6,Theorem 3.2.8 and [33, Thm. 3.1]. □

Similarly, we have the following corollary:

Corollary 3.2.10. *Let $\mathscr{C}$ be a braided fusion category. Then the Drinfeld center of $\mathscr{C}$ is equivalent to the category of finite dimensional left $H_{\mathscr{C}}$-comoduels over some braided Hopf algebra $H_{\mathscr{C}}$. Moreover, if A is a braided bi-Galois object over $H_{\mathscr{C}}$, then the cotensor functor $-\Box A$ is a braided autoequivalence of the Drinfeld center of $\mathscr{C}$ if and only if A is quantum commutative.*

3.3 The group of quantum commutative Galois objects

Corollary 3.3.1. *Let (H, R) be a finite dimensional quasi-triangular weak Hopf algebra. If A and B are quantum commutative Galois objects, so is $A\Box B$.*

Proof. Let A and B be quantum commutative Galois objects. Then we have that $A\Box-$ and $B\Box-$ are braided autoequivalences on the category ${}_H^H\mathscr{Y}\mathscr{D}$. So is the composition $(A\Box B)\Box-$. By Proposition 3.2.3, $A\Box B$ is quantum commutative. □

For a bi-Galois object A, by [69] there exists a unique braided bi-Galois object A^{-1} (up to isomorphism) such that $A\Box A^{-1} \simeq {}_RH$ and $A^{-1}\Box A \simeq {}_RH$. By Theorem 6.6 in [69], A^{-1} is isomorphic to $\overline{A}$ as ${}_RH$-bicomodule algebras.

Corollary 3.3.2. *Let (H, R) be a finite dimensional quasi-triangular weak Hopf algebra. If A is a quantum commutative Galois object, so is A^{-1}.*

Proof. Suppose that A is a quantum commutative Galois object. We know that the functor $A\Box-$ is a braided autoequivalence. So is the inverse functor $A^{-1}\Box-$. Thus the Galois object A^{-1} is quantum commutative. □

Theorem 3.3.3. *Let (H, R) be a finite dimensional quasi-triangular weak Hopf algebra. Denote by $Gal^{qc}({}_RH)$ the set of isomorphism classes of quantum commutative*

Galois objects over ${}_RH$. *Then* $Gal^{qc}({}_RH)$ *forms a group under the cotensor product* $\square$. *This group is called the group of quantum commutative Galois objects over* ${}_RH$.

Proof. Follows from Theorem 3.2.8, Corollary 3.3.1 and 3.3.2. $\square$

Remark 3.3.4. For any left H-module M, define a left H-comodule as follows:

$$\rho^L(m) = R^2 \otimes R^1 \cdot m := m_{[-1]} \otimes m_{[0]},\ \forall m \in M.$$

It is easy to see that (M, ρ^L) is a left Yetter-Drinfeld module. So we have an embedding ${}_H\mathscr{M} \hookrightarrow {}^H_H\mathscr{YD}$. By Lemma 3.1.4, there is a left ${}_RH$-comodule structure on M $\rho^l(m) = 1 \otimes_t m$, for any $m \in M$. Namely, (M, ρ^l) is a trivial left ${}_RH$-comodule. If A is a braided bi-Galois object, then $(A \square M) \simeq M$, which means that the functor $A\square-$ is isomorphic to the identity functor on the category of left H-modules.

Definition 3.3.5. [28, Defn 2.1] A braided autoequivalence $\mathcal{F}$ of ${}^H_H\mathscr{YD}$ is called *trivializable* on ${}_H\mathscr{M}$ if the restriction $\mathcal{F}|_{{}_H\mathscr{M}}$ is isomorphic to the identity functor as a braided monoidal functor. We denote by $Aut^{br}({}^H_H\mathscr{YD}, {}_H\mathscr{M})$ the group of isomorphism classes of braided autoequivalences of ${}^H_H\mathscr{YD}$ trivializable on ${}_H\mathscr{M}$.

Corollary 3.3.6. *Let* (H, R) *be a finite dimensional quasi-triangular weak Hopf algebra. Then the group* $Gal^{qc}({}_RH)$ *is a subgroup of the group* $Aut^{br}({}^H_H\mathscr{YD}, {}_H\mathscr{M})$.

Proof. Follows from Theorem 3.2.8 and 3.3.3 and Remark 3.3.4. $\square$

Remark 3.3.7. In Chapter 5, we will see that the group $Gal^{qc}({}_RH)$ will play a fundamental role in the characterization of the Brauer group of a braided fusion category.

Now we write down the left center and the right center for an algebra in the category of left H-modules, see Section 1.5 for the definiton.

Let A be an algebra in the category of left H-modules. The *left center* of A is defined as

$$Z_l(A) = \{a \in A | ab = (R^2 \cdot b)(R^1 \cdot a),\ \forall b \in A\}.$$

Similarly, the *right center* of A is

$$Z_r(A) = \{c \in A | bc = (R^2 \cdot c)(R^1 \cdot b),\ \forall c \in A\}.$$

By Section 1.5, we have the following lemma:

Lemma 3.3.8. *Let (H, R) be a quasi-triangular weak Hopf algebra. Then $Z_l(A)$ and $Z_r(A)$ are objects in the category of right H-comodules.*

Proof. For the sake of completeness, we also give here a detailed proof. For all $a \in Z_l(A)$, $b \in A$ and $h \in H$, we have

$$\begin{aligned}
(R^2 \cdot b)(R^1 h \cdot a) &= (R^2 1_2 \cdot b)(R^1 1_1 h \cdot a) \\
&= (R^2 \varepsilon_t(h_2) \cdot b)(R^1 h_1 \cdot a) \\
&= (R^2 h_2 S(h_3) \cdot b)(R^1 h_1 \cdot a) \\
&= (h_1 R^2 S(h_3) \cdot b)(h_2 R^1 \cdot a) \\
&= h_1 \cdot [(R^2 S(h_3) \cdot b)(R^1 \cdot a)] \\
&= h_1 \cdot [(R^2 \cdot (S(h_3) \cdot b))(R^1 \cdot a)] \\
&= h_1 \cdot [a(S(h_3) \cdot b)] \\
&= (h_1 \cdot a)(h_2 S(h_3) \cdot b) \\
&= (1_1 h \cdot a)(1_2 \cdot b) = (h \cdot a)b.
\end{aligned}$$

So $(h \cdot a) \in Z_l(A)$. Similarly, for all $h \in H$ and $b \in Z_l(A)$,

$$\begin{aligned}
(R^2 h \cdot b)(R^1 \cdot a) &= (R^2 h_3 \cdot b)(R^1 h_2 S^{-1}(h_1) \cdot a) \\
&= (h_2 R^2 \cdot b)(h_3 R^1 S^{-1}(h_1) \cdot a) \\
&= h_2 \cdot [(R^2 \cdot b)(R^1 S^{-1}(h_1) \cdot a)] \\
&= h_2 \cdot [(S^{-1}(h_1) \cdot a)b] \\
&= (h_2 S^{-1}(h_1) \cdot a)(h_3 \cdot b) \\
&= (1_1 a)(1_2 h \cdot b) = a(h \cdot b).
\end{aligned}$$

Thus $Z_l(A)$ and $Z_r(A)$ are objects in the category of left H-modules. □

Now let A be a left ${}_R H$-comodule algebra in the category of left H-modules. Then the coinvariant subalgebra is:

$$ {}_{\circ}A = \{a \in A | a_{(-1)} \otimes a_{(0)} = S(1_1) \otimes 1_2 \cdot a\}.$$

Similarly, if A is a right ${}_R H$-comodule algebra in the category of left H-modules, its coinvariant subalgebra is:

$$A_\circ = \{a \in A | a_{(0)} \otimes a_{(1)} = 1_1 \cdot a \otimes 1_2\}.$$

3.4 The coquasi-triangular case

Let (H, σ) be a finite dimensional coquasi-triangular weak Hopf algebra. Naturally, the dual H^* is quasi-triangular. Since the category $\mathscr{YD}_H^H$ of right-right Yetter-Drinfeld modules is equivalent to the category ${}_{H^*}^{H^*}\mathscr{YD}$ of left-left Yetter-Drinfeld modules, by Corollary 3.1.6 there exists an equivalence between the category $\mathscr{YD}_H^H$ of right-right Yetter-Drinfeld modules and the category ${}^{_RH^*}({}_{H^*}\mathscr{M})$ of left ${}_RH^*$-comodules. For two right-right Yetter-Drinfeld modules X and Y, we can define a cotensor product $X \square Y$ over ${}_RH^*$. Now we collect some facts needed in the next chapter.

Let M be a right H-comodule. There are two right H-module structures on M:

$$m \triangleleft h = m_{[0]}\sigma(m_{[1]}, h), \ \forall\ h \in H, m \in M.$$
$$m \blacktriangleleft h = m_{[0]}\sigma(h, S^{-1}(m_{[1]})), \ \forall h \in H, m \in M.$$

We have the following characterization of the cotensor product $\square_{{}_RH^*}$:

Lemma 3.4.1. *Let (H, σ) be a finite dimensional coquasi-triangular weak Hopf algebra. Let X and Y be right-right Yetter-Drinfeld modules. Then*

$$X \square Y = \{x \otimes y \in X \otimes Y | (x \blacktriangleleft h_1) \otimes y \cdot h_2 = x \cdot h_1 \otimes y \triangleleft h_2), \ \forall\ h \in H\}.$$

Proof. Similar to the proof of [88, Lemma 2.9].

Lemma 3.4.2. *Let (H, σ) be a finite dimensional coquasi-triangular weak Hopf algeba and A an algebra in the category of right-right Yetter-Drinfeld modules. Then*

$${}_\circ A = \{a \in A | a \cdot h = a \triangleleft h, \quad \forall\ h \in H\},$$
$$A_\circ = \{a \in A | a \cdot h = a \blacktriangleleft h, \quad \forall\ h \in H\}.$$

Proof. Similar to the proof of [88, Lemma 2.5]. □

Chapter 4

The Brauer group $Br(\mathscr{M}^H)$ and the group $Gal^{qc}({}_RH^*)$

For a finite dimensional quasi-triangular weak Hopf algebra (H, R), the category ${}_H\mathscr{M}$ of finite dimensional left H-modules is equivalent to the category $\mathscr{M}^{H^*}$ of finite dimensional right H^*-comodules. So there exists an isomorphism between the Brauer groups $Br({}_H\mathscr{M})$ and $Br(\mathscr{M}^{H^*})$. For the sake of convenience, we investigate directly the case of a finite dimensional coquasi-triangular weak Hopf algebra (H, σ).

In this chapter, H and H^* will always mean a finite dimensional coquasi-triangular weak Hopf algebra (H, σ) with an antipode S and the dual (H^*, R) respectively. Moreover, the antipode of the dual (H^*, R) will be still denoted by S. Since the dual (H^*, R) is quasi-triangular, by Theorem 2.2.7, we have the braided Hopf algebra ${}_RH^*$. By Theorem 3.3.3, we get the group $Gal^{qc}({}_RH^*)$ of quantum commutative Galois objects over ${}_RH^*$. In this chapter, we will study the relation between the Brauer group $Br(\mathscr{M}^H)$ and the group $Gal^{qc}({}_RH^*)$ by generalizing the exact sequence in [88] to the case of a weak Hopf algebra.

4.1 Galois-Azumaya algebras

An Azumaya algebra in the category $\mathscr{M}^H$ is called *H-Azumaya*.

Definition 4.1.1. [88] An H-Azumaya algebra A is called *Galois-Azumaya* if A is a right weak H-Galois extension of its coinvariant subalgebra.

Let M be a finite dimensional right H-comodule. Then the dual space M^* of M is a right H-comodule with the coaction ρ^R given by

$$\rho^R(m^*) = m^*_{[0]} \otimes m^*_{[1]}, \quad m^*_{[0]}(m)m^*_{[1]} = m^*(m_{[0]})S(m_{[1]}), \forall\ m \in M.$$

Now we work out the induced right H-action $\triangleleft$ on M^*. In fact,

$$\begin{aligned}(\alpha \triangleleft h)(m) &= \alpha_{[0]}(m)\sigma(\alpha_{[1]}, h) = \alpha(m_{[0]})\sigma(S(m_{[1]}), h)\\ &= \alpha(m_{[0]})\sigma(m_{[1]}, S^{-1}(h)) = \alpha(m \triangleleft S^{-1}(h)),\end{aligned}$$

for all $\alpha \in M^*$, $h \in H$ and $m \in M$.

Note that H is a regular right H-comodule with the comultiplication Δ. So $(H, \triangleleft)$ becomes a right H-module. For all $h \in H$, $x \in H_t$ and $y \in H_s$, we have

$$\begin{aligned}h \triangleleft x &= h_1\sigma(h_2, x) = h_1\varepsilon(S(x)h_2) = S(x)h\\ h \triangleleft y &= h_1\sigma(h_2, y) = h_1\varepsilon(h_2S(y)) = hy.\end{aligned}$$

Moreover, we have two right H-comodules H^* and $End_{-H_s}(H^*)$, see Remark 1.6.7.

Lemma 4.1.2. *We have a right H-comodule algebra map θ from H to $End_{-H_s}(H^*)$ defined by*

$$\theta : H \longrightarrow End_{-H_s}(H^*), \quad \langle\theta(h)(\alpha), a\rangle = \langle\alpha, aS^{-2}(h)\rangle.$$

Proof. We first verify that the map θ is right H_s-linear. Indeed,

$$\begin{aligned}\langle\theta(h)(\alpha \cdot y), a\rangle &= \langle\alpha \cdot y, aS^{-2}(h)\rangle = \langle\alpha, (aS^{-2}(h)) \cdot S^{-1}(y)\rangle\\ &= \langle\alpha, S(S^{-1}(y))(aS^{-2}(h))\rangle = \langle\alpha, (ya)S^{-2}(h))\rangle\\ &= \langle\theta(h)(\alpha), ya\rangle = \langle\theta(\alpha), a \cdot S^{-1}(y))\rangle = \langle\theta(h)(\alpha) \cdot y, a\rangle.\end{aligned}$$

Next we show that θ is an algebra map. For any $a, g, h \in H$, $\alpha \in H^*$,

$$\langle\theta(gh)(\alpha), a\rangle = \langle\alpha, aS^{-2}(gh)\rangle = \langle\alpha, aS^{-2}(g)S^{-2}(h)\rangle$$

$$= \langle \theta(g)[\theta(h)(\alpha)], a\rangle = \langle [\theta(g)\theta(h)](\alpha), a\rangle.$$

Finally, we claim that the map θ is right H-colinear. Since the right H-comodule structures on H^* and $End_{-,H_s}(H^*)$ are induced by a regular right H-comodule (H,Δ), we have $\langle \alpha_{[0]}, a\rangle\alpha_{[1]} = \langle \alpha, a_1\rangle S(a_2)$ and $\theta(h)_{[0]}(\alpha)\otimes\theta(h)_{[1]} = \theta(h)(\alpha_{[0]})_{[0]}\otimes \theta(h)(\alpha_{[0]})_{[1]}S(\alpha_{[1]})$ for all $\alpha\in H^*$ and $a,h\in H$. So

$$\begin{aligned}
\langle \theta(h)_{[0]}(\alpha), a\rangle\theta(h)_{[1]} &= \langle \theta(h)(\alpha_{[0]})_{[0]}, a\rangle\theta(h)(\alpha_{[0]})_{[1]}S(\alpha_{[1]})\\
&= \langle \theta(h)(\alpha_{[0]}), a_1\rangle S(a_2)S(\alpha_{[1]})\\
&= \langle \alpha_{[0]}, a_1S^{-2}(h)\rangle S(a_2)S(\alpha_{[1]})\\
&= \langle \alpha, a_1S^{-2}(h_1)\rangle S(a_3)S(S(a_2S^{-2}(h_2)))\\
&= \langle \alpha, a_1S^{-2}(h_1)\rangle S(a_3)S^2(a_2)h_2\\
&= \langle \alpha, a_1S^{-2}(h_1)\rangle S(\varepsilon_s(a_2))h_2\\
&\overset{(1.3)}{=} \langle \alpha, a1_1S^{-2}(h_1)\rangle S(S(1_2))h_2\\
&= \langle \alpha, aS^{-2}(h_1)\rangle h_2 = \langle \theta(h_1)(\alpha), a\rangle h_2.
\end{aligned}$$

Therefore, θ is right H-colinear. □

Corollary 4.1.3. *$End_{-H_s}(H^*)$ is a smash product algebra.*

Proof. Follows from Lemma 1.6.14 and 4.1.2. □

Note that $1_{[0]}f\otimes 1_{[1]} = f_{[0]}\otimes f_{[1]} = f1_{[0]}\otimes 1_{[1]}$ for any $f\in End_{-H_s}(M)^{\ coH}$. We have

$$(1_{[0]}f)(m)\otimes 1_{[1]} = f_{[0]}(m)\otimes f_{[1]} = [f1_{[0]}](m)\otimes 1_{[1]}, \forall m\in M,$$

which is equivalent to

$$f(m)_{[0]}\otimes\varepsilon_t(f(m)_{[1]}) = f(m_{[0]})_{[0]}\otimes f(m_{[0]})_{[1]}S(m_{[1]}) = f(m_{[0]})\otimes\varepsilon_t(m_{[1]})$$

since $1_{[0]}(m)\otimes 1_{[1]} = m_{[0]}\otimes m_{[1]}S(m_{[2]}) = m_{[0]}\otimes\varepsilon_t(m_{[1]})$. Let $End^H_{-H_s}(M)$ denote the set of all elements in $End^H_{-H_s}(M)$ which are right H-colinear.

Lemma 4.1.4. *Let M be a finite dimensional right H-comodule. Then*

$$End_{-H_s}(M)^{\ coH} = End^H_{-H_s}(M).$$

Proof. Note that the following equation holds:

$$f(m)_{[0]} \otimes \varepsilon_t(f(m)_{[1]}) = f(m_{[0]})_{[0]} \otimes f(m_{[0]})_{[1]} S(m_{[1]}),$$

for any $f \in End_{-H_s}(M)^{\ coH}$ and $m \in M$. On one hand, we have

$$\begin{aligned}
&[(1 \otimes \mu\tau(1 \otimes S^{-1}))(\rho^R \otimes 1)][f(m_{[0]})_{[0]} \otimes f(m_{[0]})_{[1]} S(m_{[1]})].\\
&= (1 \otimes \mu\tau(1 \otimes S^{-1}))[f(m_{[0]})_{[0]} \otimes f(m_{[0]})_{[1]} \otimes f(m_{[0]})_{[2]} S(m_{[1]})]]\\
&= f(m_{[0]})_{[0]} \otimes m_{[1]} S^{-1}[f(m_{[0]})_{[2]}] f(m_{[0]})_{[1]}\\
&= f(m_{[0]})_{[0]} \otimes m_{[1]} S^{-1}[\varepsilon_s(f(m_{[0]})_{[1]})]\\
&= f(m_{[0]})_{[0]} \otimes m_{[1]} S^{-1}[1_1]\varepsilon(f(m_{[0]})_{[1]} 1_2)\\
&= f(m_{[0]})_{[0]} 1'_1 \otimes m_{[1]} S^{-1}[1_1]\varepsilon(f(m_{[0]})_{[1]} 1'_2 1_2)\\
&= f(m_{[0]})_{[0]} 1'_1 S^{-1}[1_2] \otimes m_{[1]} S^{-1}[1_1]\varepsilon(f(m_{[0]})_{[1]} 1'_2 1_2)\\
&= f(m_{[0]})_{[0]} 1_1 \otimes m_{[1]} 1_2 \varepsilon(f(m_{[0]})_{[1]})\\
&= f(m_{[0]}) 1_1 \otimes m_{[1]} 1_2\\
&= f(m_{[0]} 1_1) \otimes m_{[1]} 1_2 = f(m_{[0]}) \otimes m_{[1]},
\end{aligned}$$

where τ is a flip map. On the other hand, we have

$$\begin{aligned}
&[(1 \otimes \mu\tau(1 \otimes S^{-1}))(\rho^R \otimes 1)][f(m)_{[0]} \otimes \varepsilon_t(f(m)_{[1]})]\\
&= f(m)_{[0]} \otimes S^{-1}[\varepsilon_t(f(m)_{[2]})] f(m)_{[1]} = f(m)_{[0]} \otimes f(m)_{[1]}.
\end{aligned}$$

So $f \in End^H_{-,H_s}(M)$.

Conversely, we get

$$\begin{aligned}
g(m_{[0]})_{[0]} \otimes g(m_{[0]})_{[1]} S(m_{[1]}) &= g(m_{[0]_{[0]}}) \otimes m_{[0]_{[1]}} S(m_{[1]})\\
&= g(m_{[0]}) \otimes \varepsilon_t(m_{[1]}),
\end{aligned}$$

for all $g \in End^H_{-H_s}(M)$ and $m \in M$. Thus $g \in End_{-H_s}(M)^{\ coH}$. □

Lemma 4.1.5. *H^* is a left H-module algebra with the following H-action:*

$$h \cdot h^* = h^*{}_1 \langle h^*{}_2, S^{-2}(h) \rangle, \quad \forall\ h \in H, \quad h^* \in H^*.$$

Proof. It is easy to check that H^* with the given H-action is a left H-module. The equation $h \cdot 1^* = \varepsilon_t(h) \cdot 1^*$ holds since

$$\begin{aligned}\langle h \cdot 1^*, a\rangle &= \langle 1^*{}_1, a\rangle\langle 1^*{}_2, S^{-2}(h)\rangle = \langle 1^*{}_1, a\rangle\langle 1^*{}_2, S^{-2}(h)\rangle \\ &= \langle \varepsilon, aS^{-2}(h)\rangle = \langle \varepsilon, aS^{-2}(\varepsilon_t(h))\rangle \\ &= \langle \varepsilon_t(h) \cdot 1^*, a\rangle,\end{aligned}$$

for any $a, h \in H$. Moreover, we have

$$\begin{aligned}(h_1 \cdot g^*)(h_2 \cdot h^*) &= g^*{}_1\langle g^*{}_2, S^{-2}(h_1)\rangle h^*{}_1\langle h^*{}_2, S^{-2}(h_2)\rangle \\ &= g^*{}_1 h^*{}_1\langle g^*{}_2 h^*{}_2, S^{-2}(h)\rangle \\ &= h \cdot (g^* h^*),\end{aligned}$$

for all $g^*, h^* \in H^*$ and $h \in H$. Thus H^* is a left H-module algebra. □

Lemma 4.1.6. *$End_{-H_s}(H^*) \simeq H^* \boxtimes H$, where H^* is a left H-module algebra with the H-action given in Lemma 4.1.5.*

Proof. Consider an algebra map:

$$\varsigma : H^* \longrightarrow End(H^*), \quad h^* \longmapsto [\varsigma(h^*)](g^*) = h^* g^*.$$

It is easy to see that $\varsigma(h^*)$ is right H_s-linear for all $h^* \in H^*$. If the map $\varsigma(h^*)$ is also right H-colinear, then the map ς is well-defined from H^* to $End^H_{-H_s}(H^*)$. In fact, the colinearity of $\varsigma(h^*)$ follows from

$$\begin{aligned}(\varsigma(h^*) \otimes 1)\rho^R(g^*)(h) &= \varsigma(h^*)(g^*{}_{[0]})(h) \otimes g^*{}_{[1]} = h^* g^*{}_{[0]}(h) \otimes g^*{}_{[1]} \\ &= \langle h^* g^*{}_{[0]}, h\rangle g^*{}_{[1]} = \langle h^*, h_1\rangle\langle g^*{}_{[0]}, h_2\rangle g^*{}_{[1]} \\ &= \langle h^*, h_1\rangle\langle g^*, h_2\rangle S(h_3) = \langle h^* g^*, h_1\rangle S(h_2) \\ &= \rho^R(h^* g^*)(h) = \rho^R(\varsigma(h^*)(g^*))(h),\end{aligned}$$

for all $g^*, h^* \in H^*$ and $h \in H$.

Note that $End^H(H^*) = End_{H^*-}(H^*)$, where the left H^*-module structure on H^* is naturally induced by the right H-comodule structure. Namely,

$$h^* \cdot g^* = g^*{}_{[0]}\langle h^*, g^*{}_{[1]}\rangle \quad \forall g^*, h^* \in H^*.$$

Now for all $h \in H$, we have

$$\langle {g^*}_{[0]}, h\rangle\langle h^*, {g^*}_{[1]}\rangle = \langle g^*, h_1\rangle\langle h^*, S(h_2)\rangle = \langle g^*, h_1\rangle\langle S(h^*), h_2\rangle = \langle g^*S(h^*), h\rangle.$$

So $h^* \cdot g^* = g^*S(h^*)$. Moreover, any $\alpha \in End^H(H^*)$ is left H^*-linear, i.e.,

$$\alpha(g^*S(h^*)) = \alpha(g^*)S(h^*), \quad \forall \;\; g^*, h^* \in H^*.$$

Now we prove that ς has an inverse ς' defined by $\varsigma'(\alpha) = \alpha(1_{H^*})$. Indeed,

$$\begin{aligned}
\varsigma\varsigma'(\alpha)(g^*) &= \alpha(1_{H^*})g^* = \alpha(1_{H^*})S(S^{-1}(g^*)) \\
&= \alpha(1_{H^*}S(S^{-1}(g^*))) = \alpha(g^*) \\
\varsigma'\varsigma(h^*) &= \varsigma(h^*)(1_{H^*}) = h^*.
\end{aligned}$$

Finally, we show that ς is H-linear. By Lemma 4.1.4, $End_{-H_s}(H^*) \simeq End^H_{-H_s}(H^*) \boxtimes H$, where the left H-module structure on $End^H_{-H_s}(H^*)$ is given by $h\cdot\alpha = \theta(h_1)\alpha\theta(S(h_2))$ for all $\alpha \in End^H_{-H_s}(H^*)$ and $h \in H$. Now we have

$$\begin{aligned}
\langle[\theta(h_1)\varsigma(h^*)\theta(S(h_2))](g^*), a\rangle &= \langle[\varsigma(h^*)\theta(S(h_2))](g^*), aS^{-2}(h_1)\rangle \\
&= \langle[h^*\theta(S(h_2))(g^*), aS^{-2}(h_1)\rangle \\
&= \langle h^*, a_1S^{-2}(h_1)\rangle\langle\theta(S(h_3))(g^*), a_2S^{-2}(h_2)\rangle \\
&= \langle h^*, a_1S^{-2}(h_1)\rangle\langle g^*, a_2S^{-2}(h_2)S^{-2}(S(h_3))\rangle \\
&= \langle h^*, a_1S^{-2}(h_1)\rangle\langle g^*, a_2S^{-2}(\varepsilon_t(h_2))\rangle \\
&\overset{(1.1)}{=} \langle h^*, a_1S^{-2}(h)\rangle\langle g^*, a_2\rangle \\
&= \langle {h^*}_1, a_1)\rangle\langle {h^*}_2, S^{-2}(h)\rangle\langle g^*, a_2\rangle \\
&= \langle {h^*}_1g^*, a)\rangle\langle {h^*}_2, S^{-2}(h)\rangle \\
&= \langle\varsigma({h^*}_1)(g^*), a)\rangle\langle {h^*}_2, S^{-2}(h)\rangle \\
&= \langle\varsigma(h\cdot h^*)(g^*), a)\rangle,
\end{aligned}$$

for all $a, h \in H$ and $g^* \in H^*$. So $\varsigma(h\cdot h^*) = h\cdot\varsigma(h^*)$. Thus the proof is completed. □

For a finite dimensional coquasi-triangular Hopf algebra H, any H-Azumaya algebra is equivalent to a Galois-Azumaya algebra. In fact, any H-Azumaya algebra is equivalent to a smash product Azumaya algebra. Now we extend this result to the case of a weak Hopf algebra.

Proposition 4.1.7. *Let (H,σ) be a finite dimensional coquasi-triangular weak Hopf algebra. Then any element of the Brauer group $Br(\mathscr{M}^H)$ can be represented by a smash product Azumaya algebra.*

Proof. For any H-Azumaya algebra A, we have $[A] = [A\sharp End(H^*)]$ in the Brauer group $Br(\mathscr{M}^H)$. Note that the composition:

$$H \longrightarrow End_{-H_s}(H^*) \hookrightarrow A\sharp End(H^*)$$

is right H-colinear. By Lemma 4.1.6, the H-Azumaya algebra $A\sharp End_{-,H_s}(H^*)$ is isomorphic to a smash product algebra. $\square$

4.2 Centralizer subalgebras

In this section, we show that the algebra $C_H(H_s)$, the centralizer subalgebra discussed in Section 2.1 is isomorphic to some centralizer subalgebra of $End_{-H_s}(H^*)$.

By Lemma 4.1.6, $End^H_{-H_s}(H^*)$ is isomorphic to H^* as an algebra. We have that $\varsigma(h^*) \in End^H_{-H_s}(H^*)$ for any $h^* \in H^*$. Let $\alpha \in End_{-H_s}(H^*)$ and $l^* \in H^*$. Then

$$\begin{aligned} \varsigma(h^*)\alpha = \alpha\varsigma(h^*) &\iff [\varsigma(h^*)\alpha](l^*) = [\alpha\varsigma(h^*)](l^*) \\ &\iff h^*\alpha(l^*) = \alpha(h^*l^*). \end{aligned}$$

Hence, the centralizer subalgebra $\pi(End_{-H_s}(H^*))$ (see Section 1.6.4) can be characterized as

$$\{\alpha \in End_{-H_s}(H^*) | h^*\alpha(l^*) = \alpha(h^*l^*),\ \forall\ h^*, l^* \in H^*\}.$$

Clearly, we have an anti-algebra map

$$\omega : H^* \longrightarrow End_{H^*-}(H^*), \quad h^* \longmapsto [\omega(h^*)](g^*) = g^*h^*,$$

where $End_{H^*}(H^*, -)$ is

$$\{\alpha \in End(H^*) | h^*\alpha(l^*) = \alpha(h^*l^*),\ \forall\ h^*, l^* \in H^*\}.$$

Moreover, ω has the inverse ω' given by $\omega'(\alpha) = \alpha(1_{H^*})$.

Lemma 4.2.1. *The map $\omega(h^*)$ is right H_s-linear if and only if*

$$S(1^*_1)h^*1^*_2 = h^*$$

for all $h^* \in H^*$, where $\Delta(1^*) = 1^*_1 \otimes 1^*_2$.

Proof. We need to show that for any $g^*, h^* \in H^*$ and $y \in H_s$,

$$[\omega(h^*)](g^* \triangleleft y) = [\omega(h^*)](g^*) \triangleleft y \Longleftrightarrow S(1^*_1)h^*1^*_2 = h^*.$$

First of all, we have

$$\begin{aligned}
\langle [\omega(h^*)](g^*) \triangleleft y, h\rangle &= \langle g^*h^*, yh\rangle = \langle g^*, h_1\rangle\langle h^*, yh_2\rangle;\\
\langle [\omega(h^*)](g^* \triangleleft y), h\rangle &= \langle (g^* \triangleleft y)h^*, h\rangle\\
&= \langle (g^* \triangleleft y), h_1\rangle\langle h^*, h_2\rangle\\
&= \langle g^*, h_1 \triangleleft S^{-1}(y)\rangle\langle h^*, h_2\rangle\\
&= \langle g^*, yh_1\rangle\langle h^*, h_2\rangle,
\end{aligned}$$

for all $h \in H$.

If $[\omega(h^*)](g^* \triangleleft y) = [\omega(h^*)](g^*) \triangleleft y$ for any $y \in H_s$, by taking $g^* = 1^*$, we obtain

$$\langle h^*, S(y)h\rangle = \langle h^*, yh\rangle.$$

Then $S(1^*_1)h^*1^*_2 = h^*$ follows from

$$\langle h^*, 1_1 1_2 h\rangle = \langle h^*, S(1_1)1_2 h\rangle = \langle h^*, h\rangle.$$

Conversely, if $S(1^*_1)h^*1^*_2 = h^*$, then for all $h \in H$, we have

$$\langle h^*, h\rangle = \langle S(1^*_1)h^*1^*_2, h\rangle = \langle h^*, 1_1 1_2 h\rangle.$$

The equation $[\omega(h^*)](g^*) \triangleleft y = [\omega(h^*)](g^* \triangleleft y)$ holds since

$$\begin{aligned}
&\langle [\omega(h^*)](g^*) \triangleleft y, h\rangle\\
= {}& \langle g^*, h_1\rangle\langle h^*, yh_2\rangle = \langle g^*, h_1\rangle\langle h^*, 1_1 1_2 y h_2\rangle\\
= {}& \langle g^*, h_1\rangle\langle h^*, 1_1 1_2 S^{-1}(y)h_2\rangle = \langle g^*, yh_1\rangle\langle h^*, 1_1 1_2 h_2\rangle
\end{aligned}$$

$$= \quad \langle g^*, yh_1\rangle\langle h^*, h_2\rangle = \langle[\omega(h^*)](g^* \lhd y), h\rangle,$$

for all $h \in H$. □

Corollary 4.2.2. *The centralizer subalgebra $\pi(End_{-H_s}(H^*))$ is isomorphic to ${}_RH^*$ as a right H-comodule algebra.*

Proof. By Lemma 4.2.1, we have an algebra anti-isomorphism:

$$\omega : S(1^*_1)H^*1^*_2 \longrightarrow \pi(End_{-H_s}(H^*)).$$

Since there is an anti-algebra isomorphism $S : {}_RH^* \longrightarrow S(1^*_1)H^*1^*_2$, we get an algebra isomorphism:

$${}_RH^* \longrightarrow \pi(End_{-H_s}(H^*)),\ h^* \longmapsto \omega[S(h^*)].$$

By the proof of Lemma 4.1.6, the left H^*-module structure on H^* is given by

$$h^* \cdot g^* = {g^*}_{[0]}\langle h^*, {g^*}_{[1]}\rangle \text{ or } \ h^* \cdot g^* = g^*S(h^*),\ \forall g^* \in H^*.$$

For any $\alpha \in End^H(H^*)$, α is left H^*-linear, i.e., $\alpha(g^*S(h^*)) = \alpha(g^*)S(h^*)$.

Now we prove that for any $h^* \in H^*$ and $g^* \in {}_RH^*$,

$$\omega[S(h^* \cdot g^*)] = h^* \cdot \omega[S(g^*)] = \omega[S(g^*)]_{[0]}\langle h^*, \omega[S(g^*)]_{[1]}\rangle,$$

where the left H^*-action on ${}_RH^*$ is the left adjoint action. Indeed,

$$\begin{aligned}
&\langle\omega[S(g^*)]_{[0]}(\beta), h\rangle\langle h^*, \omega[S(g^*)]_{[1]}\rangle\\
&= \langle[\omega[S(g^*)](\beta_{[0]})]_{[0]}, h\rangle\langle h^*, [\omega[S(g^*)](\beta_{[0]})]_{[1]}S(\beta_{[1]}\rangle\\
&= \langle[\beta_{[0]}S(g^*)]_{[0]}, h\rangle\langle h^*, [\beta_{[0]}S(g^*)]_{[1]}S(\beta_{[1]})\rangle\\
&= \langle\beta_{[0]}S(g^*), h_1\rangle\langle h^*, S(h_2)S(\beta_{[1]})\rangle\\
&= \langle\beta, h_1\rangle\langle S(g^*), h_3\rangle\langle h^*, S(h_4)S^2(h_2)\rangle\\
&= \langle\beta, h_1\rangle\langle h^*_1g^*S(h^*_2), S(h_2)\rangle\\
&= \langle\beta, h_1\rangle\langle S(h^*_1g^*S(h^*_2)), h_2\rangle\\
&= \langle\beta S(h^*_1g^*S(h^*_2), h\rangle
\end{aligned}$$

$$= \langle \omega[S(h_1^* g^* S(h_2^*))](\beta), h\rangle$$
$$= \langle \omega[S(h^* \cdot g^*)](\beta), h\rangle,$$

for all $\beta \in H^*$ and $h \in H$. Thus $\pi(End_{-H_s}(H^*)) \simeq {}_RH^*$ as right H-comodule algebras. □

If there is an H-comodule algebra morphism θ from H to A, then by Lemma 1.6.14, $A \simeq A_0 \boxtimes H$ as right H-comodule algebras. By Example 1.6.16, $C_A(A_0)$ is a right H-module algebra with the MUA action, where the MUA action is given by

$$a \leftharpoondown h = \theta(S(h_1))a\theta(h_2),\ \forall\ a \in A, h \in H.$$

Note that ${}_RH^*$ is a left H^*-comodule algebra with the comultiplication of H^*. Thus ${}_RH^*$ is a right H-module algebra with the following action:

$$h^* \cdot h = {h^*}_2 \langle {h^*}_1, h\rangle,\quad h^* \in {}_RH^*,\ h \in H.$$

Proposition 4.2.3. *The centralizer subalgebra $\pi(End_{-H_s}(H^*))$ is isomorphic to ${}_RH^*$ as an algebra in the category of right-right Yetter-Drinfeld modules.*

Proof. By Lemma 4.2.1 and 4.2.2, it is sufficient to show that the map ω is right H-linear. In fact, we have

$$\begin{aligned}
& \langle [\omega[S(h^*)] \leftharpoondown h](g^*), g\rangle \\
= & \langle [\theta(S(h_1))\omega[S(h^*)]\theta(h_2)](g^*), g\rangle \\
= & \langle [\omega[S(h^*)]\theta(h_2)](g^*), gS^{-1}(h_1)\rangle \\
= & \langle \theta(h_2)(g^*)S(h^*), gS^{-1}(h_1)\rangle \\
= & \langle \theta(h_3)(g^*), g_1S^{-1}(h_2)\rangle\langle S(h^*), g_2S^{-1}(h_1)\rangle \\
= & \langle g^*, g_1S^{-1}(h_2)S^{-2}(h_3)\rangle\langle S(h^*), g_2S^{-1}(h_1)\rangle \\
= & \langle g^*, g_1S^{-2}(S(h_2)h_3)\rangle\langle S(h^*), g_2S^{-1}(h_1)\rangle \\
\overset{(1.3)}{=} & \langle g^*, g_1\rangle\langle S(h^*), g_2S^{-1}(h)\rangle \\
= & \langle g^*, g_1\rangle\langle h^*, hS(g_2)\rangle \\
= & \langle g^*, g_1\rangle\langle h_1^*, h\rangle\langle S(h_2^*), g_2\rangle \\
= & \langle g^*S(h_2^*), g\rangle\langle h_1^*, h\rangle \\
= & \langle \omega[S(h_2^*)](g^*), g\rangle\langle h_1^*, h\rangle
\end{aligned}$$

$$= \quad \langle \omega[S(h^* \cdot h)](g^*), g \rangle,$$

for all $h^* \in {}_R H^*,\ h \in H$ and $g^* \in H^*$. Thus ω is right H-linear. $\square$

Lemma 4.2.4. *Let M be a finite dimensional right H-comodule. Then there exists an algebra map*

$$\lambda : {}_R H^* \longrightarrow C_{End_{-H_s}(M)}((End_{-H_s}(M))_0),$$

where λ is defined by

$$[\lambda(h^*)](m) = m_{[0]} \langle h^*, m_{[1]} \rangle,$$

for all $h^* \in {}_R H^*$ and $m \in M$.

Proof. Note that there is an algebra map

$$\lambda : {}_R H^* \hookrightarrow H^* \longrightarrow End_{-H_s}(M), \quad [\lambda(h^*)](m) = m_{[0]} \langle h^*, m_{[1]} \rangle.$$

Now we verify that the map $\lambda(h^*)$ is right H_s-linear for all $h^* \in {}_R H^*$. Note that

$$\langle h^* h \rangle = \langle 1^*{}_1 h^* S(1^*{}_2), h \rangle = \langle h^*, h 1_1 1_2 \rangle,$$

for all $h^* \in {}_R H^*$ and $h \in H$. So we have

$$\begin{aligned}
[\lambda(h^*)](m \lhd y) &= h^* m_{[0]} \sigma(m_{[1]}, y) = m_{[0]} \langle h^*, m_{[1]} \rangle \sigma(m_{[2]}, y) \\
&= m_{[0]} \langle h^*, m_{[1]} \rangle \sigma(m_{[2]}, y) = m_{[0]} \langle h^*, m_{[1]} y \rangle \\
&= m_{[0]} \langle h^*, m_{[1]} y 1_1 1_2 \rangle = m_{[0]} \langle h^*, m_{[1]} S^{-1}(y) 1_1 1_2 \rangle \\
&= m_{[0]} \langle h^*, m_{[1]} S^{-1}(y) \rangle = m_{[0]} \langle h^*, m_{[2]} \rangle \sigma(m_{[1]}, y) \\
&= [\lambda(h^*)(m)] \lhd y.
\end{aligned}$$

Hence, the map $\lambda(h^*)$ is right H_s-linear. Moreover, we have

$$\begin{aligned}
[\lambda(h^*) f](m) &= \lambda(h^*)[f(m)] = f(m)_{[0]} \langle h^*, f(m)_{[1]} \rangle \\
&= f(m_{[0]}) \langle h^*, m_{[1]} \rangle = f[\lambda(h^*)(m)],
\end{aligned}$$

for all $f \in (End_{-H_s}(M))_0$. So $\lambda(h^*) \in C_{End_{-H_s}(M)}((End_{-H_s}(M))_0)$. $\square$

Although the proof of the following lemma looks a bit similar to Lemma 4.8 in [88], we will write down its detail for the sake of completeness.

Lemma 4.2.5. *Let M be a finite dimensional right H-comodule such that $A = End_{-H_s}(M)$ is a smash product Azumaya algebra. Then λ in Lemma 4.2.4 is a morphism in the category of right-right Yetter-Drinfeld modules.*

Proof. By lemma 4.2.4, it is enough to prove that the map λ is a morphism in the category of Yetter-Drinfeld modules. Note that ${}_RH^*$ is a left-left Yetter-Drinfeld module over H^*. So ${}_RH^*$ is a right-right Yetter-Drinfeld module over H^* with the action and the coation given by:

$$h^* \cdot h = {h^*}_2\langle {h^*}_1, h\rangle, \quad \rho^R(m) := m_{[0]} \otimes m_{[1]} = \sum_i e^i \cdot m \otimes e_i,$$

where e_i and e^i are dual bases of H and H^* respectively.

We first show that the map λ is right H-colinear. Indeed,

$$\begin{aligned}
(\lambda \otimes 1)\rho(h^*)(m) &= \sum \lambda(e^i \cdot h^*)(m) \otimes e_i \\
&= \sum m_{[0]} \otimes e_i \langle e^i \cdot h^*, m_{[1]}\rangle \\
&= \sum m_{[0]} \otimes e_i \langle h^*, m_{[2]}\rangle \langle e^i, m_{[1]}S(m_{[3]})\rangle \\
&= m_{[0]} \otimes m_{[1]}S(m_{[3]})\langle h^*, m_{[2]}\rangle \\
&= \lambda(h^*)(m_{[0]})_{[0]} \otimes \lambda(h^*)(m_{[0]})_{[1]}S(m_{[1]}) = \rho\lambda(h^*)(m).
\end{aligned}$$

Next we verify that $f_{[0]}\lambda(h^* \cdot f_{[1]}) = \lambda(h^*)f$ for all $f \in End_{-H_s}(M)$. Note that

$$\lambda(h^* \cdot h)(m) = \lambda({h^*}_2)(m)\langle {h^*}_1, h\rangle = m_{[0]}\langle h^*, hm_{[1]}\rangle,$$

for any $h \in H$ and $m \in M$. We have

$$\begin{aligned}
f_{[0]}\lambda(h^* \cdot f_{[1]})(m) &= f_{[0]}(m_{[0]})\langle h^* \cdot f_{[1]}, m_{[1]}\rangle = f_{[0]}(m_{[0]})\langle h^*, f_{[1]}m_{[1]}\rangle \\
&= [f(m_{[0]})]_{[0]}\langle h^*, [f(m_{[0]})]_{[1]}S(m_{[1]})m_{[2]}\rangle \\
&= [f(m_{[0]})]_{[0]}\langle h^*, [f(m_{[0]})]_{[1]}\varepsilon_s(m_{[1]})\rangle \\
&= [f(m_{[0]}) \triangleleft \varepsilon_s(m_{[1]}]_{[0]}\langle h^*, [f(m_{[0]}) \triangleleft \varepsilon_s(m_{[1]}]_{[1]})\rangle \\
&= [f(m)]_{[0]}\langle h^*, [f(m)]_{[1]})\rangle = \lambda(h^*)f(m),
\end{aligned}$$

for all $f \in End_{-H_s}(M)$ and $m \in M$.

Finally, we verify that

$$\lambda(h^*) \leftharpoonup h = \lambda(h^* \cdot h) \Longleftrightarrow \lambda(h^*)f = f_{[0]}\lambda(h^* \cdot f_{[1]}),\ \forall f \in A.$$

If $\lambda(h^*) \leftharpoonup h = \lambda(h^* \cdot h)$, then $\lambda(h^*)f = f_{[0]}\lambda(h^* \cdot f_{[1]})$ follows from the equation (1.18). The converse also holds since

$$\lambda(h^*) \leftharpoonup h = h^{[1]}\lambda(h^*)h^{[2]} = h^{[1]}h^{[1]}{}_{[0]}\lambda(h^*\cdot h^{[2]}{}_{[1]}) = 1_{[0]}\lambda(h^*\cdot h_2)\varepsilon(h_1 1_{[1]})(m) = \lambda(h^*\cdot h)(m),$$

where the last equality is obtained by the following equations:

$$\begin{aligned} 1_{[0]}\lambda(h^* \cdot h_2)\varepsilon(h_1 1_{[1]})(m) &= 1_{[0]}\lambda(h^* \cdot h1_{[1]})(m) \\ &= 1_{[0]}(m_{[0]})\langle h^* \cdot h1_{[1]}, m_{[1]}\rangle \\ &= 1_{[0]}(m_{[0]})\langle h^*, h1_{[1]}m_{[1]}\rangle \\ &= m_{[0]}\langle h^*, hm_{[1]}\varepsilon_s(m_{[2]})\rangle \\ &= m_{[0]}\langle h^*, hm_{[1]}\rangle = \lambda(h^* \cdot h)(m). \end{aligned}$$

Thus λ is right H-linear. □

4.3 From $Br(\mathscr{M}^H)$ to $Gal^{qc}({}_RH^*)$

In this section, we will construct a group homomorphism from the Brauer group to the group of quantum commutative Galois objects.

Let (H, R) be a finite dimensional quasi-triangular weak Hopf algebra. We have the braided Hopf algebra ${}_RH$. If A is a right ${}_RH$-comodule algebra with $A_\circ \simeq H_t$, then $M^{co_RH} \otimes_t A$ is a Doi-Hopf module with the following structures:

$$\begin{aligned} &h \cdot (m \otimes_t a) = h_1 \cdot m \otimes_t h_2 \cdot a, \\ &(m \otimes_t a) \cdot b = m \otimes_t ab, \\ &\rho^r(m \otimes_t a) = m \otimes_t a_{(0)} \otimes a_{(1)}, \end{aligned}$$

where $a, b \in A$, $h \in H$, $m \in M$ and $M^{co_RH} = \{m \in M | \rho^r(m) = m \otimes_t 1\}$.

Lemma 4.3.1. *Let (H, R) be a quasi-triangular weak Hopf algebra and (A, μ) a right*

${}_RH$-comodule algebra with $A_\circ \simeq H_t$. If there exists a right ${}_RH$-comodule algebra morphism ψ from ${}_RH$ to A, then the map

$$\alpha' : M^{co_R H} \otimes_t A \longrightarrow M,\ m \otimes a \longmapsto m \cdot a$$

is a right Doi-Hopf module isomorphism with the inverse

$$\beta' : M \longrightarrow M^{co_R H} \otimes_t A,\ n \longmapsto n_{(0)} \cdot \psi\overline{S}(n_{(1)}) \otimes \psi(n_{(2)}).$$

for all $m \in M^{co_R H}$ and $n \in M$.

Proof. It is easy to see that the map α' is left H-linear and right A-linear. It is also right ${}_RH$-colinear since

$$\begin{aligned}
\rho^r[\alpha'(m \otimes a)] &= \rho(ma) = m_{(0)}(R^2 \cdot a_{(0)}) \otimes (R^1 \cdot m_{(1)})a_{(1)} \\
&= m(R^2 \cdot a_{(0)}) \otimes (R^1 \cdot 1)a_{(1)} \\
&= 1_1 \cdot m(R^2 \cdot a_{(0)}) \otimes \varepsilon_t(R^1 1_2)a_{(1)} \\
&= 1_1 \cdot m(1_2 R^2 \cdot a_{(0)}) \otimes \varepsilon_t(R^1)a_{(1)} \\
&= 1_1 \cdot m(1_2 1'_1 \cdot a_{(0)}) \otimes 1'_2 a_{(1)} \\
&= m \cdot a_{(0)} \otimes a_{(1)} = \alpha'(m \otimes a_{(0)}) \otimes a_{(1)},
\end{aligned}$$

for all $a \in A$ and $m \in M^{co_R H}$.

Now we show that β' is a well-defined morphism. Since $\overline{S}$ and ψ are morphisms, we have

$$\begin{aligned}
&\rho^r \circ \mu \circ (1 \otimes \psi\overline{S}) \circ \rho^r \\
&= (\mu \otimes \overline{\mu})(1 \otimes C \otimes 1)(\rho^r \otimes \rho^r)(1 \otimes \psi\overline{S}) \circ \rho^r \\
&= (\mu \otimes \overline{\mu})(1 \otimes C \otimes 1)(\rho^r \otimes 1 \otimes 1)(1 \otimes \rho^r \circ \psi)(1 \otimes \overline{S}) \circ \rho \\
&= (\mu \otimes \overline{\mu})(1 \otimes C \otimes 1)(1 \otimes 1 \otimes \rho \circ \psi)(\rho \otimes \overline{S}) \circ \rho \\
&= (\mu \otimes \overline{\mu})(1 \otimes C \otimes 1)(1 \otimes 1 \otimes \psi \otimes 1)(1 \otimes 1 \otimes \overline{\Delta})(\rho \otimes \overline{S}) \circ \rho \\
&= (\mu \otimes \overline{\mu})(1 \otimes \psi \otimes 1 \otimes 1)(1 \otimes C \otimes 1)(1 \otimes 1 \otimes \overline{\Delta})(\rho \otimes \overline{S}) \circ \rho \\
&= (\mu \otimes \overline{\mu})(1 \otimes \psi \otimes 1 \otimes 1)(1 \otimes C \otimes 1)(1 \otimes 1 \otimes \overline{\Delta S})(\rho \otimes 1) \circ \rho \\
&= (\mu \circ \psi \otimes \overline{\mu})(1 \otimes C \otimes 1)(1 \otimes 1 \otimes (\overline{S} \otimes \overline{S}) \circ C \circ \overline{\Delta})(\rho \otimes 1) \circ \rho \\
&= (\mu \circ \psi \otimes \overline{\mu})(1 \otimes C \otimes 1)(1 \otimes 1 \otimes \overline{S} \otimes \overline{S})(1 \otimes 1 \otimes C \circ \overline{\Delta})(\rho \otimes 1) \circ \rho
\end{aligned}$$

$$
\begin{aligned}
&= (\mu \circ \psi \otimes \overline{\mu})(1 \otimes \overline{S} \otimes 1 \otimes \overline{S})(1 \otimes C \otimes 1)(1 \otimes 1 \otimes C \circ \overline{\Delta})(\rho \otimes 1) \circ \rho \\
&= (\mu(1 \otimes \psi\overline{S}) \otimes \overline{\mu})(1 \otimes 1 \otimes 1 \otimes \overline{S})(1 \otimes C \otimes 1)(1 \otimes 1 \otimes C \circ \overline{\Delta})(\rho \otimes 1) \circ \rho \\
&= (\mu(1 \otimes \psi\overline{S}) \otimes 1)[1 \otimes (1 \otimes \overline{\mu})(1 \otimes 1 \otimes \overline{S})(C \otimes 1)(1 \otimes C \circ \overline{\Delta})](\rho \otimes 1) \circ \rho \\
&= (\mu(1 \otimes \psi\overline{S}) \otimes 1)[1 \otimes C(1 \otimes \overline{\mu})(1 \otimes \overline{S} \otimes 1)(1 \otimes \overline{\Delta})](1 \otimes \overline{\Delta}) \circ \rho \\
&= (\mu(1 \otimes \psi\overline{S}) \otimes 1)(1 \otimes C)[1 \otimes 1 \otimes \overline{\mu}(\overline{S} \otimes 1)\overline{\Delta}](1 \otimes \overline{\Delta}) \circ \rho \\
&= \mu(1 \otimes \psi\overline{S})\rho \otimes 1,
\end{aligned}
$$

where 1 denotes the identity morphism and the twelfth equality is given by

$$
\begin{aligned}
&(1 \otimes \mu)(1 \otimes 1 \otimes \overline{S})(C \otimes 1)(1 \otimes C \circ \overline{\Delta}) \\
&= (1 \otimes \mu)(C \otimes 1)(1 \otimes 1 \otimes \overline{S})(1 \otimes C)(1 \otimes \overline{\Delta}) \\
&= (1 \otimes \mu)(C \otimes 1)(1 \otimes C)(1 \otimes \overline{S} \otimes 1)(1 \otimes \overline{\Delta}) \\
&= C(1 \otimes \mu)(1 \otimes \overline{S} \otimes 1)(1 \otimes \overline{\Delta}).
\end{aligned}
$$

So β' is well-defined.

Finally, we verify that β is the inverse of α'. Indeed,

$$
\begin{aligned}
\beta'\alpha'(m \otimes a) &= (m \cdot a)_{(0)} \cdot \psi\overline{S}((m \cdot a)_{(1)}) \otimes \psi((m \cdot a)_{(2)}) \\
&= m \cdot a_{(0)}\psi\overline{S}(a_{(1)}) \otimes \psi(a_{(2)}) \\
&= m \otimes a_{(0)}\psi\overline{S}(a_{(1)})\psi(a_{(2)}) = m \otimes a,
\end{aligned}
$$

for any $m \otimes a \in M^{co_R H} \otimes_t A$. Similarly, $\alpha'\beta'(n) = n_{(0)} \cdot \psi\overline{S}(n_{(1)})\psi(n_{(2)}) = n$ for all $n \in M$. □

Now we come back to the coquasi-triangular case.

Lemma 4.3.2. *Let A be a smash product Azumaya algebra. Then ${}_\circ\pi(A)$ is the left center of $(A, \lhd)$. Similarly, $\pi(A)_\circ$ is the right center.*

Proof. (1) If a is an element in ${}_\circ\pi(A)$, we have

$$
ab \overset{(1.20)}{=} b_{[0]}{b_{[1]}}^{[1]}a{b_{[1]}}^{[2]} = b_{[0]}(a \leftharpoondown b_{[1]}) = b_{[0]}(a \lhd b_{[1]}), \forall b \in A.
$$

Conversely, if a' is an element in the left center, $a'b = b_{[0]}(a' \lhd b_{[1]})$ for any $b \in A$.

Since $1_{[0]}(a' \lhd 1_{[1]}) = 1_{[0]}a'_{[0]}\sigma(a'_{[1]}, 1_{[1]}) = 1_{[0]}a'_{[0]}\varepsilon(S(1_{[1]})a'_{[1]}) = a'$, we have

$$\begin{aligned} a' \leftharpoondown h &= h^{[1]}a'h^{[2]} = h^{[1]}h^{[2]}{}_{[0]}(a' \lhd h^{[2]}{}_{[1]}) \\ &\overset{(1.18)}{=} 1_{[0]}(a' \lhd (h1_{[1]})) = a' \lhd h. \end{aligned}$$

(2) If c is an element in $\pi(A)_\circ$, we have

$$\begin{aligned} c_{[0]}(b \lhd c_{[1]}) &= c_{[0]}b_{[0]}\sigma(b_{[1]}, c_{[1]}) = c_{[0]}b_{[0]}\sigma(S^{-1}(b_{[1]}), S^{-1}(c_{[1]})) \\ &= [c \blacktriangleleft S^{-1}(b_{[1]})]b_{[0]} = S^{-1}(b_{[1]})^{[1]}cS^{-1}(b_{[1]})^{[2]}b_{[0]} \overset{(1.26)}{=} bc. \end{aligned}$$

Conversely, if c' is an element in the right center, $bc' = c'_{[0]}(b \lhd c'_{[1]})$ for any $b \in A$.

$$\begin{aligned} c' \leftharpoondown h &= c'_{[0]}(h^{[1]} \lhd c'_{[1]})h^{[2]} \\ &= c'_{[0]}h^{[1]}{}_{[0]}h^{[2]}\sigma(h^{[1]}{}_{[1]}, c'_{[1]}) \\ &\overset{(1.23)}{=} c'_{[0]}h^{[1]}{}_2h_2^{[2]}\sigma(S(h_1), c'_{[1]}) \\ &\overset{(1.19)}{=} c'_{[0]}1_{[0]}b\varepsilon(h_2 1_{[1]})\sigma(S(h_1), c'_{[1]}) \\ &= c'_{[0]}1_{[0]}\sigma(S(h1_{[1]}), c'_{[1]}) \\ &= c'_{[0]}1_{[0]}\sigma(S(1_{[1]}), c'_{[1]})\sigma(S(h), c'_{[2]}) \\ &= c'_{[0]}1_{[0]}\varepsilon(S(1_{[1]})c'_{[1]})\sigma(S(h), c'_{[2]}) \\ &= c'_{[0]}\sigma(S(h), c'_{[1]}) = c' \blacktriangleleft h. \end{aligned}$$

Thus the proof is completed. □

Let A and B be two right H-comodule algebras. We have the algebra $A\sharp B$ with the braided product, see Section 1.4. For simplifying the notation, we write $x\sharp y$, instead of $\sum_i x_i\sharp y_i$, for an element in $A\sharp B$.

Lemma 4.3.3. *Let M be a faithfully projective object such that $A = End_{-H_s}(M)$ is a smash product Azumaya algebra. Then $\pi(A)_\circ$ is isomorphic to H_s.*

Proof. Let $End_{A^e-H_s}(A, A)$ be a subspace consisting of all elements in $End_{-H_s}(A, A)$ which are left A^e-linear, where A is a left A^e-module with the following structure:

$$(a\sharp\bar{b}) \cdot c = ac_{[0]}(b \cdot c_{[1]}), \quad \forall c \in A.$$

By Proposition 3.4 in [25], the right center of A is isomorphic to $End_{A^e-H_s}(A,A)$. Using Theorem 2.1 in [83] (or Subsection 2.4 in [25]), $End_{A^e-H_s}(A,A)$ is isomorphic to H_s. Thus Lemma 4.3.2 implies that $\pi(A)_\circ$ is isomorphic to H_s. □

Proposition 4.3.4. *Let M be a faithfully projective object such that $A = End_{-H_s}(M)$ is a smash product Azumaya algebra. Then ${}_RH^* \simeq \pi(A)$.*

Proof. Let $\mathscr{C} :=_{H^*} \mathscr{M}$. By [70], we only need to show that there is an equivalence

$$\mathscr{C} \longrightarrow \mathscr{C}_{\pi(A)}^{{}_RH^*}, \quad V \longmapsto V \otimes_s \pi(A)$$

with the inverse functor:

$$\mathscr{C}_{\pi(A)}^{{}_RH^*} \longrightarrow \mathscr{C}, \quad W \longmapsto W^{co_RH}.$$

It is easy to see that the functor $(-)^{co_RH}$ is a right adjoint functor of $-\otimes_s \pi(A)$. So it is sufficient o check that $(V \otimes_s \pi(A))^{co_RH} \simeq V$ and $W^{co_RH} \otimes_s \pi(A) \simeq W$, for any object V in $\mathscr{C}$ and any object W in $\mathscr{C}_{\pi(A)}^{{}_RH^*}$. By Lemma 4.3.3, $\pi(A)_\circ \simeq H_s$. So

$$(V \otimes_s \pi(A))^{co_RH} \simeq V \otimes_s H_s \simeq V.$$

Following lemma 4.2.5, the map λ is an algebra map from ${}_RH^*$ to $\pi(A)$ in the category of right-right Yetter-Drinfeld modules. By Theorem 3.3.7, the map λ is an ${}_RH^*$-comodule algebra morphism from ${}_RH^*$ to $\pi(A)$. It follows from Lemma 4.3.1 and 4.3.3 that $W^{co_RH} \otimes_s \pi(A) \simeq W$ for any object W in $\mathscr{C}_{\pi(A)}^{{}_RH^*}$.

From Proposition 3.2 in [70], we know that $\pi(A)$ is a faithfully flat ${}_RH^*$-Galois object. By Lemma 4.2.5, there is a morphism between braided bi-Galois objects ${}_RH^*$ and $\pi(A)$. Thus $\pi(A) \simeq {}_RH^*$ from [25, Prop. 4.6]. □

Lemma 4.3.5. *Let A and B be two smash product Azumaya algebras. Then*

$$\pi(A\sharp B) = \pi(A)\Box\pi(B).$$

Proof. Observe that $A_0\sharp B_0 \subseteq (A\sharp B)_0$ implies that $\pi(A\sharp B) \subseteq \pi(A)\sharp\pi(B)$. In fact, we have

$$(a\otimes b)(a'\otimes b') = (a\otimes 1)[(1\otimes b)(a'\otimes b')] = (a\otimes 1)[(a'\otimes b')(1\otimes b)] = aa'\otimes b'b,$$
$$(a'\otimes b')(a\otimes b) = a'a_{[0]}\otimes(b'\cdot a_{[1]})b = a'a1_{[0]}\otimes(b'\cdot 1_{[1]})b = a'a\otimes bb',$$

for any $a\otimes b\in A_0\sharp B_0$ and $a'\otimes b'\in\pi(A\sharp B)$. Note that $A\sharp B$ is a smash product algebra. Let

$$\beta^{-1}(1_{[0]}\otimes h1_{[1]})=h^{[1]_A}\otimes h^{[2]_A},\ \ \beta^{-1}(1_{[0]}\otimes h1_{[1]})=h^{[1]_B}\otimes h^{[2]_B}.$$

Then the canonical map

$$\beta:(a\sharp b)\otimes(c\sharp d)\longmapsto(a\sharp b)(c_{[0]}\sharp d_{[0]})\otimes c_{[1]}d_{[1]}$$

has an inverse given by

$$\beta':(a\sharp b)\otimes h\longmapsto(a\sharp b)(1\sharp h^{[1]_B})\otimes(1\sharp h^{[2]_B}).$$

Indeed, we have

$$\begin{aligned}\beta\beta'((a\sharp b)\otimes h) &= (a\sharp b)(1\sharp h^{[1]_B})(1\sharp h^{[2]_B}{}_{[0]})\otimes h^{[2]_B}{}_{[1]}\\ &= (a\sharp b)(1\sharp h^{[1]_B}h^{[2]_B}{}_{[0]})\otimes h^{[2]_B}{}_{[1]}\\ &= (a\sharp b1_{[0]})\otimes h1_{[1]}=(a\sharp b)\otimes h.\end{aligned}$$

Then the canonical map is surjective. It follows from the Galois theory of Hopf algebroids in [4, 41] that the canonical map is bijective.

Now we write: $\beta^{-1}(1\sharp 1)1_{[0]}\otimes h1_{[1]})=h^{[1]_{A\sharp B}}\otimes h^{[2]_{A\sharp B}}$. By the map β,

$$(h^{[1]_A}\sharp 1)\otimes(h^{[2]_A}\sharp 1)=h^{[1]_{A\sharp B}}\otimes h^{[2]_{A\sharp B}}=(1\sharp h^{[1]_B})\otimes(1\sharp h^{[2]_B}).$$

This implies that the MUV action on $\pi(A\sharp B)$ can be given by

$$(a\otimes b)\leftharpoondown h \;=\; (h^{[1]_A}\sharp 1)(a\otimes b)(h^{[2]_A}\sharp 1)=(1\sharp h^{[1]_B})(a\otimes b)(1\sharp h^{[2]_B}).$$

On one hand, we have

$$\begin{aligned}(a\otimes b)\leftharpoondown h &= (h^{[1]_A}\sharp 1)(a\otimes b)(h^{[2]_A}\sharp 1)\\ &= (h^{[1]_A}\sharp 1)(a(h^{[2]_A})_{[0]}\otimes(b\lhd(h^{[2]_A})_{[1]}))\\ &= h^{[1]_A}a(h^{[2]_A})_{[0]}\otimes(b\lhd(h^{[2]_A})_{[1]}))\\ &= h_1{}^{[1]_A}ah_1{}^{[2]_A}\otimes(b\lhd h_2)\\ &= a\leftharpoondown h_1\otimes(b\lhd h_2).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(a\otimes b)\leftharpoondown h &= (1\sharp h^{[1]_B})(a\otimes b)(1\sharp h^{[2]_B})\\
&= (a_{[0]}\sharp(h^{[1]_B}\triangleleft a_{[1]})b)(1\sharp h^{[2]_B})\\
&= a_{[0]}\sharp(h^{[1]_B}\triangleleft a_{[1]})b)h^{[2]_B}\\
&= a_{[0]}\sharp(h^{[1]_B})_{[0]}bh^{[2]_B}\sigma((h^{[1]_B})_{[1]},a_{[1]})\\
&= a_{[0]}\sharp h_2{}^{[1]_B}bh_2{}^{[2]_B}\sigma((S(h_1),a_{[1]})\\
&= a_{[0]}\sharp h_2{}^{[1]_B}bh_2{}^{[2]_B}\sigma((h_1,S^{-1}(a_{[1]}))\\
&= a\blacktriangleleft h_1\sharp b\leftharpoondown h_2.
\end{aligned}$$

So $a\sharp b\in\pi(A)\Box\pi(B)$. Namely, $\pi(A\sharp B)\subseteq\pi(A)\Box\pi(B)$.

Conversely, for all $x\sharp y\in(A\sharp B)_0$, we have

$$x_{[0]}\sharp y_{[0]}\otimes x_{[1]}y_{[1]}=(x\sharp y)1_{[0]}\otimes 1_{[1]}. \tag{4.1}$$

Applying $[(1\otimes 1\otimes m(1\otimes S))\circ(1\otimes\rho^R\otimes 1)]$ to the two sides of the equation (4.1), the left side is computed as follow:

$$\begin{aligned}
&[(1\otimes 1\otimes m(1\otimes S))\circ(1\otimes\rho^R\otimes 1)](x_{[0]}\sharp y_{[0]}\otimes x_{[1]}y_{[1]})\\
&=(1\otimes 1\otimes m(1\otimes S))(x_{[0]}\sharp y_{[0]}\otimes y_{[1]}\otimes x_{[1]}y_{[2]})\\
&=x_{[0]}\sharp y_{[0]}\otimes y_{[1]}S(y_{[2]})S(x_{[1]})=x_{[0]}\sharp 1_{[0]}y\otimes 1_{[1]}S(x_{[1]}).
\end{aligned}$$

Similarly, we have the right side:

$$\begin{aligned}
&[(1\otimes 1\otimes m(1\otimes S))\circ(1\otimes\rho^R\otimes 1)][(x\sharp y)1_{[0]}\otimes 1_{[1]}]\\
&=[(1\otimes 1\otimes m(1\otimes S))\circ(1\otimes\rho^R\otimes 1)][(x\sharp y)\cdot 1_1\otimes 1_2]\\
&=[(1\otimes 1\otimes m(1\otimes S))\circ(1\otimes\rho^R\otimes 1)][(x\sharp y\cdot 1_1)\otimes 1_2]\\
&=(1\otimes 1\otimes m(1\otimes S))[(x\sharp y_{[0]})\otimes y_{[1]}1_1\otimes 1_2]=(x\sharp y_{[0]})\otimes y_{[1]}.
\end{aligned}$$

Therefore, we obtain the following equation:

$$x_{[0]}\sharp 1_{[0]}y\otimes x_{[1]}S^{-1}(1_{[1]})=(x\sharp y_{[0]})\otimes S^{-1}(y_{[1]}). \tag{4.2}$$

Using the quantum commutativity of $\pi(A)$, we have

$$\begin{aligned}
(a\sharp b)(x\sharp y) &= ax_{[0]}\sharp(b\lhd x_{[1]})y\\
&= x_{[0]}(a\leftharpoondown x_{[1]}1_1)\sharp(b\lhd x_{[2]}1_2)y\\
&= x_{[0]}(a\blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})(b\leftharpoondown x_{[2]})y\\
&= x_{[0]}(a\blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})y_{[0]}(b\leftharpoondown x_{[2]}y_{[1]})\\
&= x_{[0]}(a\blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})y1_{[0]}(b\leftharpoondown 1_{[1]})\\
&= x_{[0]}(a\blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})yb\\
&= x_{[0]}(a\blacktriangleleft x_{[1]}S^{-1}(1_2))\sharp 1'_{[0]}\varepsilon(S^{-1}(1_1)1'_{[1]})yb\\
&= x_{[0]}(a\blacktriangleleft x_{[1]}S^{-1}(1'_{[1]}))\sharp 1'_{[0]}yb\\
&\overset{(4.2)}{=} x(a\blacktriangleleft S^{-1}(y_{[1]}))\sharp y_{[0]}b\\
&= xa_{[0]}\sigma(S^{-1}(y_{[1]}),S^{-1}(a_{[1]}))\sharp y_{[0]}b\\
&= xa_{[0]}\sharp y_{[0]}b\sigma(y_{[1]},a_{[1]})\\
&= xa_{[0]}\sharp(y\lhd a_{[1]})b,
\end{aligned}$$

for all $a\sharp b\in\pi(A)\Box\pi(B)$, where Lemma 3.4.1 was applied to the third equality. So $\pi(A)\Box\pi(B)\subseteq\pi(A\sharp B)$. Thus $\pi(A)\Box\pi(B)=\pi(A\sharp B)$. $\Box$

Lemma 4.3.6. *If A and B are two smash product Azumaya algebras such that A is Brauer equivalent to B, then $\pi(A)\simeq\pi(B)$.*

Proof. Suppose that $[A]=[B]$. Then there are two faithfully projective objects M and N such that

$$A\sharp End_{-H_s}(M)\simeq B\sharp End_{-H_s}(N).$$

So $(A\sharp End_{-H_s}(M))\sharp End_{-H_s}(H^*)\simeq(B\sharp End_{-H_s}(N))\sharp End_{-H_s}(H^*)$. By Proposition 3.2 in [83], we have

$$End_{-H_s}(M))\sharp End_{-H_s}(N)\simeq End_{-H_s}(M\otimes_s N).$$

Since $End_{-H_s}(N))\sharp End_{-H_s}(H^*)$ is a smash product algebra, so is $End_{-H_s}(M\otimes_s H^*)$. It follows from the associativity that

$$A\sharp End_{-H_s}(M\otimes_s H^*)\simeq B\sharp End_{-H_s}(N\otimes_s H^*).$$

Now applying Lemma 4.3.4, we obtain

$$\begin{aligned}\pi(A) &\simeq \pi(A)\Box_R H^* \simeq \pi(A)\Box\pi(End_{-H_s}(M\otimes_s H^*))\\ &\simeq \pi(A\sharp End_{-H_s}(M\otimes_s H^*)) \simeq \pi(B\sharp End_{-H_s}(N\otimes_s H^*))\\ &\simeq \pi(B)\Box\pi(End_{-H_s}(N\otimes_s H^*)) \simeq \pi(B)\Box_R H^* \simeq \pi(B).\end{aligned}$$

Thus $\pi(A)\simeq\pi(B)$. □

Lemma 4.3.7. *Let A and B be two H-Azumaya algebras. Then*

$$[(A\sharp End_{-H_s}(H^*))\sharp(B\sharp End_{-H_s}(H^*))] = [(A\sharp B)\sharp End_{-H_s}(H^*))].$$

Proof. Since A is an H-Azumaya algebra, $[A]=[A\sharp End_{-H_s}(H^*)]$. So we have

$$\begin{aligned}&[(A\sharp End_{-H_s}(H^*))\sharp(B\sharp End_{-H_s}(H^*))]\\ =\ &[A\sharp End_{-H_s}(H^*)][B\sharp End_{-H_s}(H^*)]\\ =\ &[A][B]=[A\sharp B]=[(A\sharp B)\sharp End_{-H_s}(H^*))].\end{aligned}$$

□

Lemma 4.3.8. *Let A be an H-Azumaya algebra. Then $\pi(A\sharp End_{-H_s}(H^*))$ is a quantum commutative Galois object.*

Proof. For any faithfully projective object M, we have $[\overline{End_{-H_s}(M)}]=[End_{-H_s}(M)]$. Following Proposition 4.3.3, Lemma 4.3.5 and 4.3.6, we obtain

$$\pi[\overline{End_{-H_s}(M)}\sharp End_{-H_s}(H^*)] \simeq \pi[End_{-H_s}(M)\sharp End_{-H_s}(H^*)] \simeq {}_RH^*.$$

Assume that A is an H-Azumaya algebra. Then $A\sharp End_{-H_s}(H^*)$ is equivalent to A as an H-Azumaya algebra. By Lemma 4.3.5 and 4.3.6,

$$\begin{aligned}&\pi[A\sharp End_{-H_s}(H^*)]\Box\pi[\overline{A}\sharp End_{-H_s}(H^*)]\\ =\ &\pi[(A\sharp End_{-H_s}(H^*))\sharp(\overline{A}\sharp End_{-H_s}(H^*))]\\ \simeq\ &\pi[(A\sharp\overline{A})\sharp End_{-H_s}(H^*))]\\ \simeq\ &\pi[End_{-H_s}(A)\sharp End_{-H_s}(H^*)] \simeq {}_RH^*.\end{aligned}$$

Similarly, we have

$$\begin{aligned} & \pi[\overline{A}\sharp End_{-H_s}(H^*)]\Box\pi[A\sharp End_{-H_s}(H^*)] \\ = & \ \pi[(\overline{A}\sharp End_{-H_s}(H^*))\sharp(A\sharp End_{-H_s}(H^*))] \\ \simeq & \ \pi[(\overline{A}\sharp A)\sharp End_{-H_s}(H^*))] \\ \simeq & \ \pi[\overline{End_{-H_s}(A)}\sharp End_{-H_s}(H^*)] \simeq {}_RH^*. \end{aligned}$$

It follows from Proposition 3.4 in [70] that $\pi(A\sharp End_{-H_s}(H^*))$ is a braided bi-Galois object. It is clear that $\pi(A\sharp End_{-H_s}(H^*))$ is quantum commutative. □

Now we state our main result in this section.

Theorem 4.3.9. *Let (H,σ) be a finite dimensional coquasi-triangular weak Hopf algebra. Then π induces a group homomorphism*

$$\begin{aligned} \Pi : Br(\mathscr{M}^H) &\longrightarrow Gal^{qc}({}_RH^*), \\ [A] &\longmapsto \pi(A\sharp End_{-H_s}(H^*)), \end{aligned}$$

where A is an H-Azumaya algebra.

Proof. Suppose that A and B are equivalent as H-Azumaya algebras. It is obvious that $A\sharp End_{-,H_s}(H^*)$ and $B\sharp End_{-H_s}(H^*)$ are also equivalent. By Lemma 4.3.5,

$$\pi(A\sharp End_{-H_s}(H^*)) \simeq \pi(B\sharp End_{-H_s}(H^*)).$$

So the map Π is well-defined. Now Proposition 4.3.3 implies that

$$\pi(End_{-H_s}(M)\sharp End_{-H_s}(H^*)) \simeq \pi(End_{-H_s}(M\otimes_s H^*)) \simeq {}_RH^*,$$

where M is any faithfully projective object. For any two H-Azumaya algebras C and D, by Lemma 4.3.4 and 4.3.6, we have

$$\begin{aligned} \pi((C\sharp D)\sharp End_{-H_s}(H^*))) &\simeq \pi((C\sharp End_{-H_s}(H^*))\sharp(D\sharp End_{-H_s}(H^*))) \\ &= \pi(C\sharp End_{-H_s}(H^*))\Box\pi(D\sharp End_{-H_s}(H^*)). \end{aligned}$$

Thus

$$\Pi[C\sharp D] = \pi(C\sharp End_{-H_s}(H^*))\Box\pi(D\sharp End_{-H_s}(H^*)) = \Pi[C]\Box\Pi[D].$$ □

4.4 A subgroup of the kernel of Π

In Theorem 4.3.9, if H is a Hopf algebra, then we know form [88] that the kernel $Ker\Pi$ of Π is equal to the Brauer group $Br(\mathbb{k})$. This is not the case when H is a (real) weak Hopf algebra. In fact, in the case of a weak Hopf algebra, an Azumaya algebra with a trivial coaction is not necessarily an H-Azumaya algebra. Thus, $Br(\mathbb{k})$ is not necessarily contained in the kernel of Π. In this section, we will work out a subgroup of $Ker\Pi$. Although we expect it to be the full kernel, we are not able to prove it at this moment.

Let (H, σ) be a finite dimensional coquasi-triangular weak Hopf algebra. It is clear that the minimal weak Hopf algebra H_m is also coquasi-triangular. There exists natural embedding:

$$\iota : \mathscr{M}^{H_m} \hookrightarrow \mathscr{M}^H, \quad M \longmapsto M,$$

which induces a group homomorphism $\iota : Br(\mathscr{M}^{H_m}) \longrightarrow Br(\mathscr{M}^H), \quad [A] \longmapsto [A]$. So we have the composition $\Pi \circ \iota$:

$$Br(\mathscr{M}^{H_m}) \longrightarrow Br(\mathscr{M}^H) \longrightarrow Gal^{qc}({}_R H^*).$$

In this section, we will show that $Im\iota \subseteq Ker\Pi$.

Lemma 4.4.1. *Let A be a left H_m-module algebra. If $A \boxtimes H_m$ is an Azumaya algebra in the category $\mathscr{M}^{H_m}$. Then $(A \boxtimes H_m)\sharp(H^* \boxtimes H)$ is a smash product Azumaya algebra.*

Proof. Assume that $A \boxtimes H_m$ is an Azumaya algebra in the category $\mathscr{M}^{H_m}$. Then $A \boxtimes H_m$ is an H-Azumaya algebra. Note that $H^* \boxtimes H$ is a smash product Azumaya algebra. Thus, $(A \boxtimes H_m)\sharp(H^* \boxtimes H)$ is a smash product Azumaya algebra □

Let A be a left H-module algebra. Clearly, A is also a left H_m-module algebra. Define the following sets:

$$A \boxtimes H_s := \{1_1 \cdot a \otimes 1_2 y | \forall\, a \in A,\ y \in H_s\},$$
$$(A \boxtimes 1)\sharp(H^* \boxtimes 1) := \{(1_1 \cdot a \otimes 1_2) \triangleleft 1''_1 \otimes (1'_1 \cdot \alpha \otimes 1'_2) \triangleleft 1''_2 | \ \forall a \in A,\ \alpha \in H^*\},$$

$$(A \boxtimes 1)\sharp(H^* \boxtimes H) := \{(1_1 \cdot a \otimes 1_2) \triangleleft 1''_1 \otimes (1'_1 \cdot \alpha \otimes 1'_2 h) \triangleleft 1''_2 | \ \forall a \in A,\ \alpha \in H^*,\ h \in H\},$$

where the left action · and the right action ◁ on $A \boxtimes H$ are induced by the left H-module structure on A and the right H-comodule structure on $A \boxtimes H$ respectively.

Namely, we have

$$(a \boxtimes h) \triangleleft y = (a \boxtimes h_1)\sigma(h_2, y) = (a \boxtimes hy),$$
$$(a \boxtimes h) \triangleleft x = (a \boxtimes h_1)\sigma(h_2, x) = a \boxtimes S(x)h,$$

for all $x \in H_t$ and $y \in H_s$.

In the sequel, we will write $a \boxtimes h$ for $1_1 \cdot a \otimes 1_2 h$, sometimes for the simplicity.

Lemma 4.4.2. *Let A be a left H_m-module algebra. Then*

$$A \boxtimes H_m = A \boxtimes H_s.$$

Proof. It is clear that $A \boxtimes H_m \supset A \boxtimes H_s$. For any $a \in A$ and $xy \in H_m$ such that $x \in H_t$ and $y \in H_s$, we have

$$1_1 \cdot a \otimes 1_2 xy = 1_1 S^{-1}(x) \cdot a \otimes 1_2 y = 1_1 \cdot (S^{-1}(x) \cdot a) \otimes 1_2 y \in A \boxtimes H_s.$$

So $A \boxtimes H_m \subset A \boxtimes H_s$. □

Lemma 4.4.3. *Let A be a left H_m-module algebra. Then*

$$[(A \boxtimes H_m)\sharp(H^* \boxtimes H)]^{coH} = (A \boxtimes 1)\sharp(H^* \boxtimes 1).$$

Proof. We first show that

$$(A \boxtimes H_m)\sharp(H^* \boxtimes H) = (A \boxtimes 1)\sharp(H^* \boxtimes H).$$

By Lemma 4.4.2, $A\sharp H_m = A \boxtimes H_s$. Clearly, $(A \boxtimes H_m)\sharp(H^* \boxtimes H) \supset (A \boxtimes 1)\sharp(H^* \boxtimes H)$. For any $a \in A$, $y \in H_s$ and $\alpha \boxtimes h \in H^* \boxtimes H$, we have

$$\begin{aligned}
&(1_1 \cdot a \otimes 1_2 y) \triangleleft 1'_1 \otimes (\alpha \boxtimes h) \triangleleft 1'_2 \\
&= (1_1 \cdot a \otimes 1_2 1''_1)\sigma(1''_2 y, 1'_1) \otimes (\alpha \boxtimes h_1)\sigma(h_2, 1'_2) \\
&= (1_1 \cdot a \otimes 1_2 y 1'_1) \otimes (\alpha \boxtimes h_1)\varepsilon(S(1'_2)h_2) \\
&= (1_1 \cdot a \otimes 1_2 1'_1) \otimes (\alpha \boxtimes h_1)\varepsilon(S(1'_2)yh_2) \\
&= (1_1 \cdot a \otimes 1_2) \triangleleft 1'_1 \otimes (\alpha \boxtimes yh) \triangleleft 1'_2.
\end{aligned}$$

So $(A \boxtimes H_m)\sharp(H^* \boxtimes H) \subset (A \boxtimes 1)\sharp(H^* \boxtimes H)$.

Now we claim that $[(A \boxtimes H_m)\sharp(H^* \boxtimes H)]^{coH} = (A \boxtimes 1)\sharp(H^* \boxtimes 1)$. Applying $(A \boxtimes H_m)\sharp(H^* \boxtimes H) = (A \boxtimes 1)\sharp(H^* \boxtimes H)$, we only need to show that

$$[(A \boxtimes 1)\sharp(H^* \boxtimes H)]^{coH} = (A \boxtimes 1)\sharp(H^* \boxtimes 1).$$

It is clear that $[(A \boxtimes H_m)\sharp(H^* \boxtimes H)]^{coH} \supset (A \boxtimes 1)\sharp(H^* \boxtimes 1)$. So it is sufficient to verify that

$$(1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (\alpha \boxtimes h) \lhd 1'_2 \in (A \boxtimes 1)\sharp(H^* \boxtimes H),$$

for any $(1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (\alpha \boxtimes h) \lhd 1'_2 \in [(A \boxtimes H_m)\sharp(H^* \boxtimes H)]^{coH}$. By the definition of $[(A \boxtimes H_m)\sharp(H^* \boxtimes H)]^{coH}$, we have

$$\begin{aligned} & (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (\alpha \boxtimes h_1) \lhd 1'_2 \otimes h_2 \\ = \;& [(1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (\alpha \boxtimes h) \lhd 1'_2] \lhd 1''_1 \otimes 1''_2 \\ = \;& [(1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (\alpha \boxtimes h) \lhd 1'_2 1''_1] \otimes 1''_2. \end{aligned}$$

Following the definition of $\sharp$, we get

$$\begin{aligned} & (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha \otimes 1'''_2 h_1) \lhd 1'_2 \otimes S(h_2) \\ = \;& (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha \otimes 1'''_2 S(1'_2) h_1) \otimes S(h_2) \\ = \;& (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes [1'''_2 S(1'_2) h_1) \otimes S(h_2)]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & [(1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha \otimes 1'''_2 h) \lhd 1'_2 1''_1] \otimes S(1''_2) \\ = \;& (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes [1'''_2 S(1'_2) h 1''_1) \otimes S(1''_2)]. \end{aligned}$$

So we obtain that

$$\begin{aligned} & (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes [1'''_2 S(1'_2) h_1 \otimes S(h_2)] \\ = \;& (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes [1'''_2 S(1'_2) h 1''_1 \otimes S(1''_2)]. \end{aligned}$$

Applying $id \otimes id \otimes \mu$ to both sides of the above equation, where μ is the multiplication of H, we get the following equation:

$$\begin{aligned} & (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes [1'''_2 S(1'_2) h_1 S(h_2)] \\ = \;& (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes [1'''_2 S(1'_2) h 1''_1 S(1''_2)]. \end{aligned}$$

We compute the left side of the equation:

$$\begin{aligned}
&(1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes [1'''_2 S(1'_2)h_1 S(h_2)] \\
&= 1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha) \otimes 1'''_2 S(1'_2)\varepsilon_t(h) \\
&= 1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 S^{-1}(\varepsilon_t(h)) \cdot \alpha) \otimes 1'''_2 S(1'_2) \\
&= 1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes [(1'''_1 S^{-1}(\varepsilon_t(h)) \cdot \alpha) \otimes 1'''_2] \lhd 1'_2.
\end{aligned}$$

That $(1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (\alpha \sharp h) \lhd 1'_2 \in (A \boxtimes 1)\sharp_R(H^* \boxtimes 1)$ follows from

$$\begin{aligned}
& (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (\alpha \sharp h) \lhd 1'_2 \\
= \;& (1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes (1'''_1 \cdot \alpha \otimes 1'''_2 S(1'_2)h) \\
= \;& 1_1 \cdot a \otimes 1_2) \lhd 1'_1 \otimes [(1'''_1 S^{-1}(\varepsilon_t(h)) \cdot \alpha) \otimes 1'''_2] \lhd 1'_2.
\end{aligned}$$

Thus $[(A \boxtimes H_m)\sharp(H^* \boxtimes H)]^{coH} = (A \boxtimes 1)\sharp(H^* \boxtimes 1)$. □

Lemma 4.4.4. *Let A be a left H_m-module algebra such that $A \boxtimes H_m$ is an Azumaya algebra in the category $\mathscr{M}^{H_m}$. Then there exists an H-comodule algebra homomorphism from $\pi(H^* \boxtimes H)$ to $\pi[(A \boxtimes H_m)\sharp(H^* \boxtimes H)]$.*

Proof. We first prove that $(1 \boxtimes 1)\sharp(\beta \boxtimes h) \in \pi[(A \boxtimes H_m)\sharp_R(H^* \boxtimes H)]$ for any $\beta \otimes h \in \pi(H^* \boxtimes H)$. We have the following computation:

$$\begin{aligned}
&[(1 \boxtimes 1)\sharp(\beta \otimes h)][(a \boxtimes 1)\sharp(\alpha \otimes 1)] \\
&= (a \boxtimes 1)_{[0]}\sharp[(\beta \otimes h) \lhd [(a \boxtimes 1)_{[1]}](\alpha \otimes 1)] \\
&= (a \boxtimes 1_1)\sharp[((\beta \otimes h) \lhd 1_2)(\alpha \otimes 1)] \\
&= (a \boxtimes 1_1)\sharp[(\beta \otimes h)_{[0]}(\alpha \otimes 1)]\sigma((\beta \otimes h)_{[1]}, 1_2) \\
&= (a \boxtimes 1_1)\sharp[(\beta \otimes h_1)(\alpha \otimes 1)]\varepsilon(1_2 h_2) \\
&= (a \boxtimes 1_1)\sharp[(\beta \otimes S(1_2)h)(\alpha \otimes 1)] \\
&= (a \boxtimes 1_1)\sharp[(1 \otimes S(1_2))(\beta \otimes h)(\alpha \otimes 1)] \\
&= (a \boxtimes 1_1)\sharp[(1 \otimes S(1_2))(\alpha \otimes 1)(\beta \otimes h)] \\
&= (a \boxtimes 1_1)\sharp[(1 \otimes S(1_2))(\alpha\beta \otimes h)] \\
&= (a \boxtimes 1_1)\sharp[\alpha\beta \otimes S(1_2)h] \\
&= (a \boxtimes 1) \lhd 1_1 \sharp[\alpha\beta \otimes h] \lhd 1_2
\end{aligned}$$

$$= (a \boxtimes 1)\sharp[\alpha\beta \otimes h]$$
$$= [(a \boxtimes 1)\sharp(\alpha \otimes 1)][(1 \boxtimes 1)\sharp(\beta \otimes h)],$$

where the seventh equality follows from the definition of $\pi(H^* \boxtimes H)$ and the sixth equality stems from

$$(1 \otimes y)(\beta \otimes h) = (1_1 \cdot \beta) \otimes 1_2 yh = b \otimes yh, \ , \forall y \in H_s.$$

Now we have a well-defined map:

$$\iota' : \pi(H^*\sharp H) \longrightarrow \pi[(A\sharp H_m)\sharp_R(H^*\sharp H)], \ \beta \boxtimes h \longmapsto (1 \boxtimes 1)\sharp(\beta \boxtimes h).$$

It is easy to see that ι' is a homomorphism between right H-comodule algebras. □

Theorem 4.4.5. *Let A be a left H_m-module algebra such that $A \boxtimes H_m$ is an Azumaya algebra in the category $\mathscr{M}^{H_m}$. Then*

$$\pi[(A \boxtimes H_m)\sharp(H^* \boxtimes H)] \simeq {}_R H^*.$$

Proof. By [25], we only need to show that ι' is a morphism from Galois object $\pi(H^* \boxtimes H)$ to Galois object $\pi[(A \boxtimes H_m)\sharp(H^* \boxtimes H)]$, or equivalent to say, ι' is a morphism between the two Yetter- Drinfeld modules under the MUV action. Using Lemma 4.4.4, it is sufficient to prove that the map ι' in Lemma 4.4.4 is right H-linear under the MUV action $\leftharpoondown$.

We first work out the canonical map γ and its inverse. In fact,

$$\begin{aligned} &\gamma[(a \boxtimes 1)\sharp(\alpha \boxtimes h)] \otimes [(b \boxtimes 1)\sharp(\beta \boxtimes g)] \\ &= [(a \boxtimes 1)\sharp(\alpha \boxtimes h)][(b \boxtimes 1)\sharp(\beta \boxtimes g)]_{[0]} \otimes [(b \boxtimes 1)\sharp(\beta \boxtimes g)]_{[1]} \\ &= [(a \boxtimes 1)\sharp(\alpha \boxtimes h)][(b \boxtimes 1)\sharp(\beta \boxtimes g_1)] \otimes g_2. \end{aligned}$$

Since $(A \boxtimes H_m)\sharp(H^* \boxtimes H)$ is a right weak H-Galois extension, the canonical map is bijective. Moreover, the canonical map has the inverse:

$$\begin{aligned} &\gamma^{-1}[((a \boxtimes 1)\sharp(\alpha \boxtimes l))1_{[0]} \otimes h1_{[1]}] \\ &= [(a \boxtimes 1)\sharp(\alpha \boxtimes l)][(1 \boxtimes 1)\sharp(1 \boxtimes S(h_1))] \otimes [(1 \boxtimes 1)\sharp(1 \boxtimes h_2)] \\ &= [(a \boxtimes 1)\sharp(\alpha \boxtimes l)(1 \boxtimes S(h_1))] \otimes [(1 \boxtimes 1)\sharp(1 \boxtimes h_2)]. \end{aligned}$$

We verify it as follows. On one hand,

$$\begin{aligned}
&\gamma^{-1}\gamma[[(a\boxtimes 1)\sharp(\alpha\boxtimes h)][(b\boxtimes 1)\sharp(\beta\boxtimes g)]]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes h)][(b\boxtimes 1)\sharp(\beta\boxtimes g_1)][(1\boxtimes 1)\sharp(1\boxtimes S(g_2))]\otimes[(1\boxtimes 1)\sharp(1\boxtimes g_3)]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes h)][(b\boxtimes 1)\sharp(\beta\boxtimes g_1)(1\boxtimes S(g_2))]\otimes[(1\boxtimes 1)\sharp(1\boxtimes g_3)]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes h)][(b\boxtimes 1)\sharp(\beta\boxtimes\varepsilon_t(g_1))]\otimes[(1\boxtimes 1)\sharp(1\boxtimes g_2)]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes h)][(b\boxtimes 1)\sharp(S^{-1}(\varepsilon_t(g_1)))\cdot\beta\boxtimes 1]\otimes[(1\boxtimes 1)\sharp(1\boxtimes g_2)]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes h)]\otimes[(b\boxtimes 1)\sharp(S^{-1}(\varepsilon_t(g_1)))\cdot\beta\boxtimes 1][(1\boxtimes 1)\sharp(1\boxtimes g_2)]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes h)]\otimes[(b\boxtimes 1)\sharp(S^{-1}(\varepsilon_t(g_1))\cdot\beta\boxtimes 1)(1\boxtimes g_2)]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes h)]\otimes[(b\boxtimes 1)\sharp(\beta\boxtimes g)].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\gamma\gamma^{-1}[((a\boxtimes 1)\sharp(\alpha\boxtimes l))1_{[0]}\otimes h1_{[1]}]\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes l)(1\boxtimes S(h_1))][(1\boxtimes 1)\sharp(1\boxtimes h_2)]\otimes h_3\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes l)(1\boxtimes S(h_1))(1\boxtimes h_2)]\otimes h_3\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes l)(1\boxtimes\varepsilon_s(h_1))]\otimes h_2\\
&=[(a\boxtimes 1)\sharp(\alpha\boxtimes l1_1)]\otimes h1_2=((a\boxtimes 1)\sharp(\alpha\boxtimes l))1_{[0]}\otimes h1_{[1]}.
\end{aligned}$$

So we have obtained

$$\gamma^{-1}(1_{[0]}\otimes h1_{[1]})=[(1\boxtimes 1)\sharp(1\boxtimes S(h_1))]\otimes[(1\boxtimes 1)\sharp(1\boxtimes h_2)],\ \forall h\in H.$$

Next we show that the map ι' in Lemma 4.4.4 is right H-linear under the MUV action $\leftharpoondown$. For any $h\in H$ and $\beta\boxtimes g\in H^*\boxtimes H$, we have

$$\begin{aligned}
&[(1\boxtimes 1)\sharp(\beta\boxtimes g)]\leftharpoondown h\\
&=[(1\boxtimes 1)\sharp(1\boxtimes S(h_1))][(1\boxtimes 1)\sharp(\beta\boxtimes g)][(1\boxtimes 1)\sharp(1\boxtimes h_2)]\\
&=[(1\boxtimes 1)\sharp(1\boxtimes S(h_1))(\beta\boxtimes g)(1\boxtimes h_2)]\\
&=\iota[(1\boxtimes S(h_1))(\beta\boxtimes g)(1\boxtimes h_2)]=\iota[(\beta\boxtimes g)\leftharpoondown h].
\end{aligned}$$

Thus ι' is a Yetter-Drinfeld module morphism.

Finally, note that Lemma 4.3.3 implies that $\pi(End_{-,H_s}(H^*))\simeq\pi(H^*\boxtimes H)\simeq {}_RH^*$. Therefore, $\pi[(A\boxtimes H_m)\sharp(H^*\boxtimes H)]\simeq {}_RH^*$. □

Corollary 4.4.6. *Let (H, σ) be a finite dimensional coquasi-triangular weak Hopf algebra. Then the Brauer group $Br(\mathscr{M}^{H_m})$ is a subgroup of the kernel $Ker\Pi$ of the map Π.*

Proof. By Proposition 4.1.7, any element in the Brauer group $Br(\mathscr{M}^{H_m})$ can be represented by an Azumaya algebra A in the category $\mathscr{M}^{H_m}$ such that A is isomorphic to a smash product algebra $A' \boxtimes H_m$. It follows from Theorem 4.4.5 that $Im\iota \subseteq Ker\Pi$. Therefore, the group $Im\iota$ is a subgroup of $Ker\Pi$. □

Combining Theorem 4.3.9 and Corollary 4.4.6, we obtain the main result in this chapter:

Theorem 4.4.7. *Let H be a finite dimensional coquasi-triangular weak Hopf algebra over a field $\Bbbk$. Let ${}_RH^*$ be the associated braided Hopf algebra. Then there exists a sequence of group homomorphisms:*

$$Br(\mathscr{M}^{H_m}) \hookrightarrow Br(\mathscr{M}^H) \longrightarrow Gal^{qc}({}_RH^*),$$

where H_m is the minimal weak Hopf algebra of H and $Gal^{qc}({}_RH^*)$ is the group of quantum commutative Galois objects over ${}_RH^*$.

Remark 4.4.8. Note that a weak Hopf algebra is a Hopf algebra if and only if its minimal weak Hopf algebra H_m is trivial, i.e., $H_m = \Bbbk 1_H$. When $H_m = \Bbbk 1_H$, the category $\mathscr{M}^{H_m}$ is just the category of finite dimensional vector spaces. Therefore, the group $Br(\mathscr{M}^{H_m})$ also becomes the group $Br(\Bbbk)$ if H is a finite dimensional coquasi-triangular Hopf algebra. In this case, the sequence is the exact sequence in [88].

In general, the minimal weak Hopf algebra of a weak Hopf algebra H is not necessarily trivial, for example, see [57, Example. 8.3]. So an Azumaya algebra with the trivial coaction is not necessarily an Azumaya algebra in the category $\mathscr{M}^{H_m}$.

Chapter 5

Brauer groups of braided fusion categories

In the previous chapter, we constructed a group homomorphism Π from the Brauer group of a finite dimensional coquasi-triangular weak Hopf algebra (H, σ) to the group of quantum commutative Galois objects over ${}_RH^*$. The map Π will be studied further in this chapter. To be precise, we will show that the map Π is an isomorphism in case H is a finite dimensional, cosemisimple, co-connected and coquasi-triangular weak Hopf algebra over an algebraically closed field k of characteristic 0. That is, $\mathscr{M}^H$ is a braided fusion category. So the computation of the Brauer group of a braided fusion category will be transferred to the computation of the group of quantum commutative Galois objects, which are easier to deal with. This method will help us to characterize effectively the Brauer groups of some modular categories.

In this chapter, if not stated otherwise, (H, σ) will always mean a finite dimensional coquasi-triangular weak Hopf algebra over an algebraically closed field k of characteristic 0 such that it is cosemisimple and co-connected.

5.1 The surjectivity of Π

In this section, we will use the result of [28] to show that the map Π is surjective. Since the dual H^* is quasi-triangular, by Theorem 2.2.7, the full center of H_t^* or ${}_RH^*$

is a braided Hopf algebra.

Lemma 5.1.1. *Let A be a quantum commutative algebra in the category of right-right Yetter-Drinfeld modules over H. Then $C_A(A_0) = A$.*

Proof. For any $a \in A_0$ and $b \in A$, we have $ba = a_{[0]}(b \cdot a_{[1]}) = ab$. □

Lemma 5.1.2. *Let A be a right H-comodule algebra and B a smash product algebra. Then $A\sharp B$ is isomorphic to a smash product algebra $(A\sharp B)_0 \boxtimes H$.*

Proof. Note that there exists a right H-comodule algebra morphism

$$H \hookrightarrow B \hookrightarrow A\sharp B.$$

The proof follows from Lemma 1.6.14. □

Lemma 5.1.3. *Let A be a quantum commutative algebra in the category of right-right Yetter-Drinfeld modules. If B is a smash product algebra, then $A\Box\pi(B)$ is a subalgebra of $\pi(A\sharp B)$ in the category of right-right Yetter-Drinfeld modules.*

Proof. By the definition of the cotensor product, we have that $A\Box\pi(B)$ is a comodule subalgebra of $A\sharp\pi(B)$. Since $A\sharp\pi(B)$ is a comodule subalgebra of $A\sharp B$, so is $A\Box\pi(B)$. Since Lemma 5.1.2 implies that $A\sharp B$ is a smash product algebra, $\pi(A\sharp B)$ makes sense.

Now we show that $A\Box\pi(B) \subseteq \pi(A\sharp B)$ as a right H-comodule algebra. Denote by $\bullet$ the right H-action on A. For any $x\sharp y \in (A\sharp B)_0$ and $a\sharp b \in A\Box\pi(B)$, we have

$$\begin{aligned}
(a\sharp b)(x\sharp y) &= ax_{[0]}\sharp(b \triangleleft x_{[1]})y \\
&= x_{[0]}(a \bullet x_{[1]})\sharp(b \triangleleft x_{[2]})y \\
&= x_{[0]}(a \bullet x_{[1]}1_1)\sharp(b \triangleleft x_{[2]}1_2)y \\
&= x_{[0]}(a \blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})(b \leftharpoondown x_{[2]})y \\
&= x_{[0]}(a \blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})y_{[0]}(b \leftharpoondown x_{[2]}y_{[1]}) \\
&\overset{(4.1)}{=} x_{[0]}(a \blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})y1_{[0]}(b \leftharpoondown 1_{[1]}) \\
&= x_{[0]}(a \blacktriangleleft x_{[1]}1_1)\sharp 1'_{[0]}\varepsilon(1_2 1'_{[1]})yb \\
&= x_{[0]}(a \blacktriangleleft x_{[1]}S^{-1}(1_2))\sharp 1'_{[0]}\varepsilon(S^{-1}(1_1)1'_{[1]})yb \\
&= x_{[0]}(a \blacktriangleleft x_{[1]}S^{-1}(1'_{[1]}))\sharp 1'_{[0]}yb \\
&\overset{(4.2)}{=} x(a \blacktriangleleft S^{-1}(y_{[1]}))\sharp y_{[0]}b
\end{aligned}$$

$$\begin{aligned} &= xa_{[0]}\sigma(S^{-1}(y_{[1]}), S^{-1}(a_{[1]}))\sharp y_{[0]}b \\ &= xa_{[0]}\sharp y_{[0]}b\sigma(y_{[1]}, a_{[1]}) \\ &= xa_{[0]}\sharp(y \lhd a_{[1]})b, \end{aligned}$$

where the quantum commutativity of A and the definition of the cotensor product were used in the second and the fourth equalities respectively.

Finally, we verify that the right H-module structure on $A\Box\pi(B)$ is just the MUV action from a smash product algebra. Let $B = B_0 \boxtimes H$ and $D = A\sharp(B_0 \boxtimes H)$. By Example 1.6.16, we get

$$\begin{aligned} (x \otimes y \otimes z) \leftharpoondown h &= (1 \otimes 1 \otimes S(h_1))[x \otimes (y \otimes z)](1 \otimes 1 \otimes h_2) \\ &= [1 \otimes (1 \otimes S(h_1)][x \otimes (y \otimes z)(1 \otimes h_2)] \\ &= x_{[0]} \otimes [(1 \otimes S(h_1)) \lhd x_{[1]}](y \otimes z)(1 \otimes h_2)], \end{aligned}$$

for all $h \in H$ and $x \otimes y \otimes z \in C_D(D_0)$. In particular, if $x \otimes y \otimes z \in A\Box\pi(B)$, then

$$\begin{aligned} (x \otimes y \otimes z) \leftharpoondown h &= x_{[0]} \otimes [(1 \otimes S(h_1)) \lhd x_{[1]}](y \otimes z)(1 \otimes h_2)] \\ &= x_{[0]} \otimes (1 \otimes S(h_2))(y \otimes z)(1 \otimes h_3)]\sigma(S(h_1), x_{[1]}) \\ &= x_{[0]} \otimes [(y \otimes z) \leftharpoondown h_2]\sigma(S(h_1), x_{[1]}) \\ &= (x \blacktriangleleft h_1) \otimes [(y \otimes z) \leftharpoondown h_2]. \end{aligned}$$

Therefore, $A\Box\pi(B)$ is a subalgebra of $\pi(A\sharp B)$ in the category of right-right Yetter-Drinfeld modules. □

Corollary 5.1.4. *Let A be a quantum commutative algebra in the category of right-right Yetter-Drinfeld modules. Then A is a subalgebra of $\pi(A\sharp End_{-,H_s}(H^*))$.*

Proof. The proof follows from Lemma 5.1.2 and 5.1.3, since $\pi(End_{-,H_s}(H^*)) \simeq {}_RH^*$ and $A \simeq A\Box_R H^*$ as algebras in the category of right-right Yetter-Drinfeld modules. □

Now let A^1 and A^2 be two right H-comodule algebras. It is clear that the direct sum $A^1 \bigoplus A^2$ is also a right H-comodule algebra.

Lemma 5.1.5. *Let $A = \bigoplus_{i\in J} A^i$ be a right H-comodule algebra such that every A^i*

is a right H-comodule subalgebra of A, where J is a finite index set. Then

$$A_0 = \bigoplus_{i\in J} A_0^i \quad and \quad C_A(A_0) = \bigoplus_{i\in J} C_{A^i}(A_0^i).$$

Proof. Note that $\rho^R(A) \subset (\bigoplus_{i\in J} A^i)\otimes H$. For $a \in A^i$ such that $\rho^R(a) \in A^j \otimes H$, where $i \neq j$, we have $a \in A^j$ and so $a = 0$. □

Lemma 5.1.6. *Let $A = \bigoplus_{i\in J} A_i$ be a right H-comodule algebra such that A_i is a right H-comodule subalgebra of A, where J is a finite index set. If B is a right H-comodule algebra, then*

$$A\sharp B = \bigoplus_{i\in J}(A_i\sharp B).$$

Proof. Since every A^i is a right H-comodule algebra, so is $A_i\sharp B$. It is easy to see that $A\sharp B = \sum_i A_i\sharp B$. For any $i, j \in J$ such that $i \neq j$, $A_iA_j = 0$. We can get

$$(a\otimes b)(a'\otimes b') = aa'_{[0]} \otimes (b \triangleleft a'_{[1]})b' = 0,$$

for all $a\otimes b \in A_i\sharp B$ and $a'\otimes b' \in A_j\sharp B$, where $a'_{[0]}\otimes a'_{[1]} \in A_j \otimes H$. Thus $A\sharp B = \bigoplus_{i\in J}(A_i\sharp B)$. □

Corollary 5.1.7. *Let $A = \bigoplus_i A_i$ be a right H-comodule algebra such that A_i is a right H-comodule subalgebra of A, where J is a finite index set. If B is a right H-comodule algebra, then*

$$C_{A\sharp B}((A\sharp B)_0) = \bigoplus_{i\in J} C_{A_i\sharp B}((A_i\sharp B)_0).$$

Proof. Follows from Lemma 5.1.5 and 5.1.6. □

Remark 5.1.8. It is easy to see that the above lemmas and corollaries hold for any coquasitriangular weak Hopf algebra.

Now let $(\mathscr{C}, \otimes, I, C)$ be a braided monoidal category. Denote by $Aut^{br}(\mathscr{C})$ the group of isomorphism classes of braided autoequivalences of $\mathscr{C}$.

Lemma 5.1.9. *Let $(\mathscr{C}, \otimes, I, C)$ be a braided monoidal category. Then*

$$Aut^{br}(\mathscr{C}) = Aut^{br}(\mathscr{C}^{rev}).$$

Proof. If α is a braided autoequivalence of $\mathscr{C}$, then we have the following commutative diagram:

$$\begin{array}{ccc} \alpha(M)\otimes\alpha(N) & \longrightarrow & \alpha(M\otimes N)) \\ \downarrow C_{\alpha(M),\alpha(N)} & & \alpha(C_{M,N}) \downarrow \\ \alpha(N)\otimes\alpha(M) & \longrightarrow & \alpha(N\otimes M). \end{array}$$

Since all the morphisms in the above diagram are isomorphisms, the following diagram commutes:

$$\begin{array}{ccc} \alpha(N)\otimes\alpha(M) & \longrightarrow & \alpha(N\otimes M)) \\ \downarrow C^{-1}_{\alpha(M),\alpha(N)} & & \alpha(C^{-1}_{M,N}) \downarrow \\ \alpha(M)\otimes\alpha(N) & \longrightarrow & \alpha(M\otimes N). \end{array}$$

So α is a braided autoequivalence of $\mathscr{C}^{rev}$. The proof of the converse is similar. □

Lemma 5.1.10. *Let $\mathcal{S}$ be a braided monoidal equivalence between $\mathscr{C}$ and $\mathscr{D}$. Then*

$$Aut^{br}(\mathscr{C}) \simeq Aut^{br}(\mathscr{D}).$$

Proof. For any braided autoequivalence $\alpha \in Aut^{br}(\mathscr{C})$, define a braided autoequivalence of $\mathscr{D}$ as follows:

$$S(\alpha) = \mathcal{S}\alpha\mathcal{S}^{-1} : \mathscr{D} \longrightarrow \mathscr{C} \longrightarrow \mathscr{C} \longrightarrow \mathscr{D}.$$

It is easy to see that the map S is a well-defined group homomorphism from $Aut^{br}(\mathscr{C})$ to $Aut^{br}(\mathscr{C})$ and has the inverse $S^{-1}(\beta) = \mathcal{S}^{-1}\beta\mathcal{S}$ for all $\beta \in Aut^{br}(\mathscr{C})$. □

We have seen that $Aut^{br}(\mathscr{C})$ is a group invariant of $\mathscr{C}$.

Lemma 5.1.11. *Let $\mathscr{C}$ be a braided monoidal category. Then the following hold:*

1. *An algebra in $\mathscr{C}$ is indecomposable if and only it is indecomposable in $\mathscr{C}^{rev}$;*

2. *An algebra A in $\mathscr{C}$ is a direct sum $\bigoplus_{i\in J} A^i$ in $\mathscr{C}$ if and only it is a direct sum $\bigoplus_{i\in J} A^i$ in $\mathscr{C}^{rev}$;*

3. *An algebra (A,μ) in $\mathscr{C}$ is braided-commutative in $\mathscr{C}$ if and only it is braided-commutative in $\mathscr{C}^{rev}$.*

Proof. The statement (1) and (2) are evident since they are not involved with the braiding and its inverse. The last one follows from the fact that $\mu = \mu C_{A,A}$ if and only if $\mu C_{A,A}^{-1} = \mu$, where C is the braiding of $\mathscr{C}$. □

By [56], the category $\mathscr{M}^H$ of finite dimensional right H-comodules is a braided fusion category. Note that ${}_{H^*}\mathscr{M} \cong \mathscr{M}^H$ and ${}_{H^*}\mathscr{YD}^{H^*} \cong \mathscr{Z}_r({}_{H^*}\mathscr{M})$. We have

$$Aut^{br}(\mathscr{Z}_r({}_{H^*}\mathscr{M}), {}_{H^*}\mathscr{M}) \simeq Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M}) = Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, \mathscr{M}^H).$$

By Lemma 2.1.1 and Corollary 2.3.4, the forgetful functor $F : {}_{H^*}\mathscr{YD}^{H^*} \longrightarrow {}_{H^*}\mathscr{M} \cong \mathscr{M}^H$ has a right adjoint functor $R : {}_{H^*}\mathscr{M} \cong \mathscr{M}^H \longrightarrow {}_{H^*}\mathscr{YD}^{H^*}$. Following the proof of theorem 4.1 of [28] or Section 5 in [32], we have the following lemma:

Lemma 5.1.12. *[32] Let α be a braided autoequivalence in $Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, \mathscr{M}^H)$. Then $F[\alpha^{-1}(R(H_t^*))] = \bigoplus_{i\in J} L_\alpha^i$ such that L_α^i is equivalent to L_α^j as H-Azumaya algebras for all $i, j \in J$, where J is a finite index set.*

Proof. For any braided autoequivalence $\alpha \in Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, \mathscr{M}^H)$, we know from [32] that $\alpha^{-1}(R(H_t^*))$ is an indecomposable algebra in ${}_{H^*}\mathscr{YD}^{H^*}$. However, the algebra $L_\alpha := F(\alpha^{-1}R(H_t^*))$ may be decomposable in $\mathscr{M}^H$.

Now assume that $L_\alpha = \bigoplus_{i\in J} L_\alpha^i$, where every L_α^i is an exact invertible and indecomposable algebra in the category of finite dimensional right comodules. By [28, Sec. 3.2], every L_α^i is an H-Azumaya algebra. It follows from [32, Sec.5] that L^i and L^j are equivalent as H-Azumaya algebras. □

For any braided autoequivalence $\alpha \in Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, \mathscr{M}^H)$, we will always use the same notation as in [32]:

$$L_\alpha := F[\alpha^{-1}(R(H_t^*))] = \bigoplus_{i\in J} L_\alpha^i.$$

Corollary 5.1.13. *Let α be an element in $Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, \mathscr{M}^H)$. Then*

$$\pi(L_\alpha \sharp End_{-,H_s}(H^*)) = \bigoplus_{i\in J} G^i$$

such that for all $i, j \in J$, $G^i \simeq G^j$ as braided Galois objects, where J is a finite index set.

Proof. By Lemma 5.1.12, any L^i_α in the above is an H-Azumaya algebra. We know from Lemma 4.3.8 that the centralizer subalgebra $\pi(L^i_\alpha \sharp End_{-,H_s}(H^*))$ is a braided Galois object over ${}_RH^*$, which we denote by G^i. Using Corollary 5.1.7 we obtain

$$\pi((L\sharp End_{-,H_s}(H^*)) = \bigoplus_{i\in J} \pi(L^i_\alpha \sharp End_{-,H_s}(H^*)) = \bigoplus_{i\in J} G^i.$$

By Lemma 5.1.12, L^i is equivalent to L^j as H-Azumaya algebras. It follows from Lemma 4.3.6 that $G^i \simeq G^j$ as braided Galois objects for any $i, j \in J$. □

Let A be a quantum commutative Galois object over ${}_RH^*$. By Corollary 3.2.10 and Lemma 5.1.9, the functor $-\Box A \in Aut^{br}({}^{H^*}_{H^*}\mathscr{Y}\mathscr{D}^{re}, \mathscr{M}^H)$. Following Lemma 5.1.10 , the composition functor $E \circ (-\Box A) \circ E^{-1} \in Aut^{br}({}_{H^*}\mathscr{Y}\mathscr{D}^{H^*}, \mathscr{M}^H)$. Let α_A denote the composition functor $E \circ (-\Box A) \circ E^{-1}$.

Lemma 5.1.14. *Let A be a quantum commutative Galois object. Then the following statements hold:*

1. $\alpha_A(R(H^*_t)) \simeq E(A)$;
2. *As left H^*-module algebras, $E(A) = A$;*
3. *The algebra $\alpha_A(R(H^*_t))$ is indecomposable in the category ${}_{H^*}\mathscr{Y}\mathscr{D}^{H^*}$ if and only if A is indecomposable in the category ${}^{H^*}_{H^*}\mathscr{Y}\mathscr{D}$.*

Proof. Note that $E^{-1}(R(H^*_t)) \simeq I'(H^*_t) \simeq {}_RH^*$. We have

$$\begin{aligned}
[E \circ (-\Box A) \circ E^{-1}](R(H^*_t)) &= [E \circ (-\Box A)[E^{-1}(R(H^*_t))] \\
&= E[[E^{-1}(R(H^*_t))]\Box A] \\
&\simeq E[{}_RH^*\Box A] \simeq E(A).
\end{aligned}$$

Since E is an equivalent functor, the algebra $\alpha_A(R(H^*_t))$ is indecomposable in the category ${}_{H^*}\mathscr{Y}\mathscr{D}^{H^*}$ if and only if A is indecomposable in the category ${}^{H^*}_{H^*}\mathscr{Y}\mathscr{D}^{rev}$. Thus, the statement (3) follows from Lemma 5.1.11. □

Lemma 5.1.15. *Let A be a quantum commutative Galois object. Then there exists an H-Azumaya algebra L' such that $\pi(L'\sharp End_{-,H_s}(H^*)) \simeq A$ as braided Galois objects.*

Proof. Assume that A is a quantum commutative Galois object, whose inverse we denote by A^{-1}. Clearly, the functor $-\Box A^{-1}$ is the inverse of $-\Box A$ in the group

$Aut^{br}({}^{H^*}_{H^*}\mathscr{Y}\mathscr{D}^{rev}, {}_{H^*}\mathscr{M})$. Lemma 5.1.14 implies that $\alpha_{A^{-1}}(R(H_t^*)) \simeq E(A^{-1})$. In particular, $E(A^{-1}) = A^{-1}$ as left H^*-module algebras.

We know from Lemma 5.1.12 that $A^{-1} = \bigoplus_{i\in J} L^i_{\alpha_{A^{-1}}}$ such that $L^i_{\alpha_{A^{-1}}}$ is equivalent to $L^j_{\alpha_{A^{-1}}}$ as H-Azumaya algebras for any $i, j \in J$. It follows from Lemma 5.1.3 and Corollary 5.1.13 that

$$A^{-1} \simeq A^{-1}\Box_R H^* \subset \pi(A^{-1}\sharp End_{-,H_s}(H^*)) \simeq \bigoplus_{i\in J} G^i.$$

Since $E(A^{-1})$ is indecomposable in the category of left-right Yetter-Drinfeld modules, A^{-1} is indecomposable in the category of left-left Yetter-Drinfeld modules by Lemma 5.1.14. So A^{-1} is a subalgebra of some Galois object G^i. By [25, Prop. 4.6], $A^{-1} \simeq G^i$. From Corollary 5.1.13, we derive that $A^{-1} \simeq G^i$ for all $i \in J$.

Now choose an $i \in J$, and let L' be the opposite algebra $\overline{L^i_{\alpha_{A^{-1}}}}$ of $L^i_{\alpha_{A^{-1}}}$. We get

$$\pi(\overline{L^i_{-\Box A^{-1}}}\sharp End_{-,H_s}(H^*)) \simeq (A^{-1})^{-1} \simeq A$$

as braided Galois objects over ${}_RH^*$. □

Corollary 5.1.16. *Let (H, σ) be a finite dimensional coquasi-triangular weak Hopf algebra over an algebraically closed field k of characteristic 0 such that it is cosemisimple and co-connected. Then the map Π in Theorem 4.3.9 is surjective.*

Proof. Follows from Lemma 5.1.15. □

5.2 The trivial kernel

In this section we will show that the kernel of the map Π is trivial.

Let $aut^{br}({}_{H^*}\mathscr{Y}\mathscr{D}^{H^*}, {}_{H^*}\mathscr{M})$ be the following set:

$$\{\alpha \in Aut^{br}({}_{H^*}\mathscr{Y}\mathscr{D}^{H^*}, {}_{H^*}\mathscr{M}) | (E\circ\alpha\circ E^{-1}({}_RH^*) \simeq {}_RH^* \}.$$

Proposition 5.2.1. *$aut^{br}({}_{H^*}\mathscr{Y}\mathscr{D}^{H^*}, {}_{H^*}\mathscr{M})$ is a subgroup of $Aut^{br}({}_{H^*}\mathscr{Y}\mathscr{D}^{H^*}, {}_{H^*}\mathscr{M})$.*

Proof. It is clear that $id \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$. For all $\alpha, \beta \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$,

$$\begin{aligned}(E \circ \alpha\beta \circ E^{-1}({}_RH^*) &\simeq [(E \circ \alpha \circ E^{-1}\circ)(E \circ \beta \circ E^{-1})]({}_RH^*)\\ &\simeq (E \circ \alpha \circ E^{-1}\circ)({}_RH^*) \simeq {}_RH^*.\end{aligned}$$

So $\alpha\beta \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$. Similarly, $\alpha^{-1} \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$. Thus $aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$ is a subgroup. □

Lemma 5.2.2. $\pi(L^i_{id}\sharp End_{-H_s}(H^*)) \simeq {}_RH^*$.

Proof. By Lemma 5.1.14, we have $\alpha_{{}_RH^*} = E \circ id \circ E^{-1} = id$. Similar to the proof of Lemma 5.1.15, we obtain $\pi(L^i_{id}\sharp End_{-H_s}(H^*)) \simeq {}_RH^*$. □

Lemma 5.2.3. *Let α be an element in $Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$ with inverse α^{-1}. Then*

$$\pi(L^i_\alpha\sharp End_{-H_s}(H^*)) \simeq {}_RH^* \Longleftrightarrow \pi(L^i_{\alpha^{-1}}\sharp End_{-H_s}(H^*)) \simeq {}_RH^*.$$

Proof. By Theorem 4.3.8, if A and B are two H-Azumaya algebras, we have

$$\pi[(A\sharp B)\sharp End_{-H_s}(H^*)] \simeq \pi(A\sharp End_{-H_s}(H^*)\Box\pi(B\sharp End_{-H_s}(H^*).$$

Moreover, for $\alpha, \beta \in Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$, by [28, 32] the algebra $L^i_\alpha\sharp L^i_\beta$ is equivalent to $L^i_{\alpha\beta}$ as an H-Azumaya algebra. So

$$\begin{aligned}\pi(L^i_{\alpha\beta}\sharp End_{-H_s}(H^*)) &\simeq \pi[(L^i_\alpha\sharp L^i_\beta)\sharp End_{-H_s}(H^*)]\\ &\simeq \pi(L^i_\alpha\sharp End_{-H_s}(H^*))\Box\pi(L^i_\beta\sharp End_{-H_s}(H^*)).\end{aligned}$$

Now assume that $\alpha \in Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$ such that there exists some H-Azumaya algebra L^i_α satisfying $\pi(L^i_\alpha\sharp End_{-H_s}(H^*)) \simeq {}_RH^*$. Then

$$\begin{aligned}\pi(L^i_{\alpha^{-1}}\sharp End_{-H_s}(H^*)) &\simeq \pi(L^i_{\alpha^{-1}}\sharp End_{-H_s}(H^*))\Box_RH^*\\ &\simeq \pi(L^i_{\alpha^{-1}}\sharp End_{-H_s}(H^*))\Box\pi(L^i_\alpha\sharp End_{-H_s}(H^*))\\ &\simeq \pi[(L^i_{\alpha^{-1}}\sharp L^i_\alpha)\sharp End_{-H_s}(H^*)]\\ &\simeq \pi(L^i_{\alpha\alpha^{-1}}\sharp End_{-H_s}(H^*)) \simeq {}_RH^*,\end{aligned}$$

where Lemma 5.2.2 was applied to the last equality. □

Lemma 5.2.4. *Let α be an element in $Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$. Then there exists some H-Azumaya algebra L^i_α satisfying $\pi(L^i_\alpha \sharp End_{-H_s}(H^*)) \simeq {}_R H^*$ if and only if*

$$\alpha \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M}).$$

Proof. If $\pi(L^i_\alpha \sharp End_{-H_s}(H^*)) \simeq {}_R H^*$, then we have

$$(E \circ \alpha^{-1} \circ E^{-1})({}_R H) \subset \pi(L_\alpha \sharp End_{-H_s}(H^*)) = \bigoplus_{i \in J} {}_R H^*.$$

The indecomposability of L_α in the category of Yetter-Dinfeld modules implies that

$$(E \circ \alpha^{-1} \circ E^{-1})({}_R H^*) \subset {}_R H^*.$$

It follows from Lemma 5.2.3 that $\pi(L^i_{\alpha^{-1}} \sharp End_{-H_s}(H^*)) \simeq {}_R H^*$. Similarly, we have

$$(E \circ \alpha \circ E^{-1})({}_R H^*) \subset {}_R H^*.$$

The foregoing inclusion means that $\alpha^{-1}({}_R H^*)$ is a subobject of ${}_R H^*$. So the embedding $i : (E \circ \alpha^{-1} \circ E^{-1})({}_R H^*) \hookrightarrow {}_R H^*$ is a monomorphism. Since $(E \circ \alpha \circ E^{-1})$ is a braided autoequivalence, $(E \circ \alpha \circ E^{-1})(i)$ is a monomorphism. Then we have

$${}_R H^* = (E \circ \alpha \circ E^{-1})[(E \circ \alpha^{-1} \circ E^{-1})({}_R H^*)] \subset (E \circ \alpha \circ E^{-1})({}_R H^*) \subset {}_R H^*.$$

Thus, $(E \circ \alpha \circ E^{-1})({}_R H^*) = {}_R H^*$. The proof of the converse is easy. □

Proposition 5.2.5. *The subgroup $aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$ is trivial.*

Proof. From $E(R(H^*_t)) \simeq I'(H^*_t) \simeq {}_R H^*$, we derive that

$$\begin{aligned} \alpha(R(H^*_t)) \simeq I(H^*_t) \quad &\Longleftrightarrow \quad (E \circ \alpha \circ E^{-1}(I'(H^*_t)) \simeq (E \circ E^{-1}(I'(H^*_t)) \\ &\Longleftrightarrow \quad (E \circ \alpha \circ E^{-1}({}_R H^*) \simeq {}_R H^*. \end{aligned}$$

If there is another $\beta \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$, then

$$\alpha^{-1}(R(H^*_t)) \simeq E^{-1}({}_R H^*) \simeq \beta^{-1}(R(H^*_t)).$$

It follows from [28, 32] that α is uniquely determined by $\alpha^{-1}(R(H^*_t))$ for any $\alpha \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$. So $\alpha \cong \beta$. Thus the subgroup $aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$ is trivial. □

Theorem 5.2.6. *The map* Π *is an isomorphism from the Brauer group* $Br(\mathscr{M}^H)$ *to the group* $Gal^{qc}({}_RH^*)$*.*

Proof. Using Theorem 4.3.9 we have a group homomorphism

$$\Pi : Br(\mathscr{M}^H) \longrightarrow Gal^{qc}({}_RH^*), \quad [A] \longmapsto \pi(A\sharp End_{-,H_s}(H^*)).$$

Moreover, the surjectivity of Π follows from Corollary 5.1.16. By Theorem 4.3 in [28], we get another group isomorphism:

$$\psi : Br(\mathscr{M}^H) \simeq Br({}_{H^*}\mathscr{M}) \simeq Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M}).$$

Let A' be any H-Azumaya algebra satisfying $\pi(A'\sharp End_{-,H_s}(H^*)) \simeq {}_RH^*$. So the braided autoequivalence $\psi(A')$ lies in $Aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$ and the H-Azumaya algebra $L^i_{\psi(A')}$ is equivalent to A'. Hence, we have

$$\pi(L^i_{\psi(A')}\sharp End_{-,H_s}(H^*)) \simeq \pi(A'\sharp End_{-,H_s}(H^*)) \simeq {}_RH^*,$$

which means that $\psi(A') \in aut^{br}({}_{H^*}\mathscr{YD}^{H^*}, {}_{H^*}\mathscr{M})$. But $\psi(ker\Pi) = 1$ by Proposition 5.2.5. Therefore, $ker\Pi$ is trivial. Thus Π is an isomorphism. □

Now we state our result in the language of a braided fusion category.

Corollary 5.2.7. *Let* $\mathscr{C}$ *be a braided fusion category. Then the Brauer group* $Br(\mathscr{C})$ *of* $\mathscr{C}$ *is isomorphic to the group of quantum commutative Galois objects over some braided Hopf algebra.*

In [62], Pfeiffer gave a detailed construction of a finite dimensional cosemisimple coquasi-triangular weak Hopf algebra $H_{\mathscr{C}}$ for any ribbon category $\mathscr{C}$ over a algebraically closed field k and showed that $\mathscr{C}$ is equivalent to the category of finite dimensional right $H_{\mathscr{C}}$-comodules as a ribbon category. As a consequence, we obtain:

Corollary 5.2.8. *Let* $\mathscr{C}$ *be a ribbon category* $\mathscr{C}$ *over an algebraically closed field* k *of characteristic zero. Assume that* $H_{\mathscr{C}}$ *is the reconstructed weak Hopf algebra in [62]. Then the Brauer group* $Br(\mathscr{C})$ *is isomorphic to the group* $Gal^{qc}({}_RH^*_{\mathscr{C}})$*.*

As a modular category is also equivalent to the category of finite dimensional comodules over some coquasi-triangular weak Hopf algebra, we have the following corollary:

Corollary 5.2.9. *Let $\mathscr{C}$ be a modular category $\mathscr{C}$ over an algebraically closed field k. Assume that $H_{\mathscr{C}}$ is the reconstructed weak Hopf algebra in [62]. Then the Brauer group $Br(\mathscr{C})$ is isomorphic to the group $Gal^{qc}({}_RH^*_{\mathscr{C}})$.*

5.3 The Brauer groups of a class of modular categories

By investigating the algebraic structure of the lattice models of face type, Hayashi found a class of quantum groups, called Hopf face algebras, see [36]. A Hopf face algebra is a finite dimensional cosemisimple coquasi-triangular weak Hopf algebra. In [38], Hayashi used face algebras to construct modular tensor categories with positive definite inner product. The fusion rules and S-matrices of those fusion categories are the same as (or slightly different from) those obtained from $U_q(\mathfrak{sl}_N)$ at roots of unity. In this section, we use Theorem 5.2.6 to characterize the Brauer groups of this class of modular categories.

5.3.1 Face algebras and their comodules

For the detailed study of face algebras, the reader is referred to [36, 37, 38, 39]. Here we avoid the complicated construction of a face algebra and reduce some parameters by taking the case of $\epsilon = -1$. Now we write down Hayashi's face algebra $A_{N,t}$.

Let $N \geq 2$ be an integer and $\mathbb{V}$ the cyclic group $\mathbb{Z}/N\mathbb{Z}$. Let $t \in \mathbb{C}$ be a primitive N^{th} root of unity. The $\mathbb{V}$-face algebra $A_{N,t}$ is defined to be the $\mathbb{C}$-linear span of symbols $e^i_j(m)$ $(i, j, m \in \mathbb{V})$ equipped with the structure given by

$$\begin{aligned}
e^i_j(p)e^k_l(q) &= \delta_{i+p,k}\delta_{j+p,l}\ e^i_j(p+q),\\
\Delta(e^i_j(m)) &= \sum_k e^i_k(m)\otimes e^k_j(m),\\
\varepsilon(e^i_j(m)) &= \delta_{i,j},\\
S(e^i_j(p)) &= e^{j+p}_{i+p}(-p).
\end{aligned}$$

Then the algebra $A_{N,t}$ is a coquasi-triangular weak algebra with the coquasi-triangular structure:

$$\sigma(e^i_j(p), e^k_l(q)) = \delta_{i,k+q}\delta_{j,k}\delta_{i+p,l+q}\delta_{j+p,l}t^{-pq},$$

where $i, j, k, l, p, q \in \mathbb{V}$. Denote by e_λ the sum $\sum_{i\in\mathbb{V}} e^i_\lambda(0)$, for any $\lambda \in \mathbb{V}$. The source subalgebra is the $\mathbb{C}$-linear span of $\{e_\lambda|\lambda \in \mathbb{V}\}$.

It is not hard to check that the dual $A^*_{N,t}$ of $A_{N,t}$ is spanned by $\{X^i_j(s)|i, j, s \in \mathbb{V}\}$ and a quasi-triangular weak Hopf algebra equipped with the following structures:

$$\begin{aligned}
&\Delta(X^i_j(s)) = \sum_{p+q=s} X^i_j(p) \otimes X^{i+p}_{j+p}(q),\ \ \varepsilon(X^i_j(s)) = \delta_{s,0},\\
&X^i_j(p)X^k_l(q) = \delta_{j,k}\delta_{p,q}X^i_l(p), \quad 1 = \sum_{i,p} X^i_i(p),\\
&S(X^i_j(p)) = X^{j+p}_{i+p}(-p),\\
&R_1 \otimes R_2 = \sum_{i,j,p} X^i_j(p) \otimes X^j_{j+p}(i-j)t^{-p(i-j)},\\
&R'_1 \otimes R'_2 = \sum_{i,j,p} X^{j+p}_{i+p}(-p) \otimes X^j_{j+p}(i-j)t^{-p(i-j)},
\end{aligned}$$

where $\{X^i_j(p)\}$ is the dual basis of $\{e^i_j(p)\}$. Moreover, the target subalgebra of $A^*_{N,t}$ is the $\mathbb{C}$-linear span of $\{\sum_p X^i_i(p)|i \in \mathbb{V}\}$. We denote by 1^i the sum $\sum_p X^i_i(p)$ for all $i \in \mathbb{V}$. Then the target subalgebra of $A^*_{N,t}$ is the direct sum $\bigoplus_{i\in\mathbb{V}} \mathbb{C}1^i$.

Now we recall the comodule theory of face algebras. Let H be a $\mathbb{V}$-face algebra and U a right H-comodule. By [38], there exists a face space decomposition

$$U = \bigoplus_{\lambda,\nu\in\mathbb{V}} U(\lambda, \nu),$$

where $U(\lambda, \nu) = \{u_{[0]}\varepsilon(e_\lambda u_{[1]} e_\nu)|u \in U\}$. If U and V are two right H-comodules, then

$$U \otimes_s V = \bigoplus_{\lambda,,k,l\in\mathbb{V}} U(\lambda, k) \otimes V(k, l).$$

Lemma 5.3.1. *If $f : U \longrightarrow V$ is a right H-comodule isomorphism, then*

$$f(U(\lambda, \nu)) = V(\lambda, \nu), \quad \forall\ \lambda, \nu \in \mathbb{V}.$$

Proof. Since f is right H-colinear, the map f is H_s-linear. We need to show that

$\rho(e_\lambda f(u)e_\nu) = \rho(f(u))$ for any $u \in U(\lambda,\nu)$ and $\lambda,\nu \in \mathbb{V}$. Indeed,

$$\begin{aligned}\rho(e_\lambda f(u)e_\nu) &= f(u)_{[0]} \otimes e_\lambda f(u)_{[1]}e_\nu \\ &= f(u_{[0]}) \otimes e_\lambda u_{[1]}e_\nu \\ &= f(u_{[0]}) \otimes u_{[1]} \\ &= f(u)_{[0]} \otimes f(u)_{[1]}.\end{aligned}$$

Thus, $f(U(\lambda,\nu)) = V(\lambda,\nu)$ for any $\lambda,\nu \in \mathbb{V}$. □

Lemma 5.3.2. *Let U, V and W be finite dimensional right H-comodules such that $U \otimes_s V \simeq U \otimes_s W$ as right H-comodules. Then $dim(V) = dim(W)$.*

Proof. By Lemma 5.3.1, we have $U(\lambda,k)\otimes V(k,l) \simeq U(\lambda,k)\otimes W(k,l)$ for any $\lambda,k,l \in \mathbb{V}$. Since their dimensions are finite, $dim(V(k,l)) = dim(W(k,l))$. Thus $dim(V) = dim(W)$. □

5.3.2 Hopf algebras and braided Hopf algebras

In the sequel, H will always denote the dual of $A_{N,t}$. Let ${}_RH$ be the braided Hopf algebra constructed in Theorem 2.2.7. It is easy to see that the target subalgebra $H_t = \bigoplus_{i\in\mathbb{V}} \mathbb{C}1^i$ is commutative. In this section, we will show that ${}_RH$ can be viewed as the direct sum of some ordinary Hopf algebras. First, we need to work out ${}_RH$.

Lemma 5.3.3. *The braided Hopf algebra ${}_RH$ is the $\mathbb{C}$-linear span of $\{X_i^i(p)|i,p \in \mathbb{V}\}$ equipped with the following structures:*

$$\begin{aligned}&\Delta'(X_k^k(s)) = \sum_{w+q=s} X_k^k(w) \otimes X_k^k(q)\\ &\varepsilon_t(X_i^i(s)) = \delta_{s,0}\sum_p X_i^i(p),\\ &X_i^i(p)X_k^k(q) = \delta_{i,k}\delta_{p,q}X_i^i(p),\ 1 = \sum_{i,p} X_i^i(p),\\ &S(X_k^k(s)) = X_k^k(-s).\end{aligned}$$

Proof. Note that $\Delta(1_H) = \Delta(\sum_{i,s} X_i^i(s)) = \sum_{i,s} \sum_{p+q=s} X_i^i(p) \otimes X_{i+p}^{i+p}(q)$. We have

$$\begin{aligned} 1_1 X_n^m(r) S(1_2) &= \sum_{i,s} \sum_{p+q=s} X_i^i(p) X_n^m(r) S(X_{i+p}^{i+p}(q)) \\ &= \sum_{i,s} \sum_{p+q=s} X_i^i(p) X_n^m(r) X_{i+p+q}^{i+p+q}(-q) \\ &= \sum_{i,s} \sum_{p+q=s} \delta_{i,m} \delta_{n,i+p+q} \delta_{p,r} \delta_{-q,r} X_{i+p+q}^i(p) \\ &= \sum_i \delta_{i,m} \delta_{n,i} X_i^i(r) = \delta_{m,n} X_n^m(r), \end{aligned}$$

for all $m, n, r \in \mathbb{V}$. So ${}_R H$ is the $\mathbb{C}$-linear span of $\{X_i^i(p) | i, p \in V\}$.

Using $\Delta(R^1) \otimes R^2 = \sum_{i,j,p} \sum_{u+v=p} X_j^i(u) \otimes X_{j+u}^{i+u}(v) \otimes X_{j+p}^j(i-j) t^{-p(i-j)}$, we compute the deformed comultiplication as follows:

$$\begin{aligned} & \Delta'(X_k^k(s)) \\ = & \sum_{w+q=s} X_k^k(w) S(R^2) \otimes R^1 \cdot X_{k+w}^{k+w}(q) \\ = & \sum_{w+q=s} X_k^k(w) S(R^2) \otimes R_1^1 X_{k+w}^{k+w}(q) S(R_2^1) \\ = & \sum_{w+q=s} \sum_{i,j,p} \sum_{u+v=p} X_k^k(w) S(X_{j+p}^j(i-j)) \otimes X_j^i(u) X_{k+w}^{k+w}(q) S(X_{j+u}^{i+u}(v)) t^{-p(i-j)} \\ = & \sum_{w+q=s} \sum_{i,j,p} \sum_{u+v=p} X_k^k(w) X_i^{i+p}(j-i)) \otimes X_j^i(u) X_{k+w}^{k+w}(q) X_{i+u+v}^{j+u+v}(-v)) t^{-p(i-j)} \\ = & \sum_{w+q=s} \sum_{i,j,p} \sum_{u+v=p} \delta_{w,j-i} \delta_{k,i+p} X_i^k(w) \otimes \delta_{u,q} \delta_{q,-v} \delta_{j,k+w} \delta_{k+w,j+u+v} X_{i+u+v}^i(u) t^{-p(i-j)} \\ = & \sum_{w+q=s} \sum_{i,j} \delta_{w,j-i} \delta_{k,i} \delta_{j,k+w} X_i^k(w) \otimes X_i^i(q) \\ = & \sum_{w+q=s} \sum_j \delta_{w,j-k} \delta_{j,k+w} X_k^k(w) \otimes X_k^k \\ = & \sum_{w+q=s} X_k^k(w) \otimes X_k^k(q). \end{aligned}$$

By Theorem 2.2.7, the antipode is given by $\overline{S}(x) = R^2 R'^2 S^2(R'^1) S(R^1 x)$. For convenience, we first compute $R'^2S^2(R'^1)$. Indeed,

$$\begin{aligned} R'^2S^2(R'^1) &= \sum_{i,j,p} X^j_{j+p}(i-j)S^2(X^i_j(p))t^{-p(i-j)} \\ &= \sum_{i,j,p} X^j_{j+p}(i-j)X^i_j(p)t^{-p(i-j)} \\ &= \sum_{i,j,p} \delta_{j+p,i}X^j_j(i-j)t^{-p(i-j)}. \end{aligned}$$

Now we have

$$\begin{aligned} \overline{S}(X^k_k(-s)) &= R^2R'^2S(R^1X^k_k(s)S(R'^1)) \\ &= \sum_{i,j,p}\sum_{i',j',p'} \delta_{j'+p',i'}X^j_{j+p}(i-j)X^{j'}_{j'}(i'-j')S(X^i_j(p)X^k_k(s))t^{-[p(i-j)+p'(i'-j')]} \\ &= \sum_{i,j,p}\sum_{i',j',p'} \delta_{j'+p',i'}\delta_{j,k}\delta_{p,s}X^j_{j+p}(i-j)X^{j'}_{j'}(i'-j')S(X^i_k(s))t^{-[p(i-j)+p'(i'-j')]} \\ &= \sum_{i}\sum_{i',j',p'} \delta_{j'+p',i'}X^k_{k+s}(i-k)X^{j'}_{j'}(i'-j')S(X^i_k(s))t^{-[s(i-k)+p'(i'-j')]} \\ &= \sum_{i}\sum_{i',j',p'} \delta_{j'+p',i'}X^k_{k+s}(i-k)X^{j'}_{j'}(i'-j')X^{k+s}_{i+s}(-s)t^{-[s(i-k)+p'(i'-j')]} \\ &= \sum_{i}\sum_{i',j',p'} \delta_{j'+p',i'}\delta_{i-k,i'-j'}\delta_{i'-j',-s}\delta_{k+s,j'}X^k_{i+s}(-s)t^{-[s(i-k)+p'(i'-j')]} \\ &= \sum_{i}\sum_{j',p'} \delta_{i-k,p'}\delta_{p',-s}\delta_{k+s,j'}X^k_{i+s}(-s)t^{-[s(i-k)+p'p']} \\ &= \sum_{i}\sum_{j'} \delta_{i-k,-s}\delta_{k+s,j'}X^k_{i+s}(-s)t^{-[s(i-k)+(-s)(-s)]} \\ &= \sum_{i} \delta_{i-k,-s}X^k_{i+s}(-s)t^{-[s(i-k)+(-s)(-s)]} \\ &= X^k_k(-s)t^{-[s(-s)+(-s)(-s)]} = X^k_k(-s). \end{aligned}$$

Thus the proof is completed. □

Take $i \in \mathbb{V}$. Define H^i to be the $\mathbb{C}$-linear span of $\{X^i_i(p) | p \in \mathbb{V}\}$. It is obvious that H^i is a subalgebra of ${}_RH$ with unity 1^i. Moreover, ${}_RH$ is the direct sum of all these H^i, i.e.,

$${}_RH = \bigoplus_{i\in\mathbb{V}} H^i.$$

We will show that every H^i is also an ordinary Hopf algebra and so ${}_RH$ is actually the direct sum of all these Hopf algebras. In order to verify that every H^i can be equipped with a coalgebraic structure, we need to give the vector space ${}_RH \otimes_t {}_RH$.

Lemma 5.3.4. ${}_RH \otimes_t {}_RH = \bigoplus_{i\in\mathbb{V}}(H^i \otimes H^i)$.

Proof. It is equivalent to show that

$$1_1 \cdot X_a^a(b) \otimes 1_2 \cdot X_u^u(w) = \delta_{u,a} X_a^a(b) \otimes X_u^u(w),$$

for all $a, b, u, w \in \mathbb{V}$. Indeed, we have

$$\begin{aligned}
&1_1 \cdot X_a^a(b) \otimes 1_2 \cdot X_u^u(w)\\
&= \sum_{i,s} \sum_{p+q=s} X_i^i(p) \cdot X_a^a(b) \otimes X_{i+p}^{i+p}(q) \cdot X_u^u(w)\\
&= \sum_{i,s} \sum_{p+q=s} \delta_{i,a}\delta_{p,0} X_i^i(b) \otimes \delta_{i+p,u}\delta_{q,0} X_{i+p}^{i+p}(w)\\
&= \sum_{i} \delta_{i,a} X_i^i(b) \otimes \delta_{i,u} X_i^i(w)\\
&= \delta_{u,a} X_a^a(b) \otimes X_u^u(w),
\end{aligned}$$

for all $a, b, u, w \in \mathbb{V}$. □

Lemma 5.3.5. *For all $i \in \mathbb{V}$, H^i is a coalgebra over $\mathbb{C}1^i$ with the following structures:*

$$\begin{aligned}
\Delta'(X_i^i(s)) &= \sum_{w+q=s} X_i^i(w) \otimes X_i^i(q),\\
\varepsilon_t(X_i^i(s)) &= \delta_{s,0} \sum_{p} X_i^i(p).
\end{aligned}$$

Proof. Follows from Lemma 5.3.3 and 5.3.4. □

Proposition 5.3.6. *For all $i \in \mathbb{V}$, H^i is a Hopf algebra over $\mathbb{C}1^i$ equipped with the following structures:*

$$\begin{aligned}
&X_i^i(p)X_i^i(q) = \delta_{p,q} X_i^i(p),\ 1_{H^i} = 1^i,\\
&\Delta'(X_i^i(s)) = \sum_{w+q=s} X_i^i(w) \otimes X_i^i(q),\\
&\varepsilon_t(X_i^i(s)) = \delta_{s,0} \sum_{p} X_i^i(p),\\
&S(X_i^i(s)) = X_i^i(-s).
\end{aligned}$$

Proof. We know already that H^i is both an algebra and a coalgebra, it remains to be proved that Δ' and ε_t are multiplicative, and that the axioms of the antipode S hold. We first check that Δ'' is multiplicative. Indeed,

$$\begin{aligned}
\Delta'(X_i^i(s))\Delta''(X_i^i(t)) &= [\sum_{p+q=s} X_i^i(p)\otimes X_i^i(q)][\sum_{p'+q'=t} X_i^i(p')\otimes X_i^i(q')] \\
&= \sum_{p+q=s}\sum_{p'+q'=t} [X_i^i(p)X_i^i(p')\otimes X_i^i(q)X_i^i(q')] \\
&= \sum_{p+q=s}\sum_{p'+q'=t} \delta_{p,p'}\delta_{q,q'}[X_i^i(p)\otimes X_i^i(q)] \\
&= \delta_{s,t}\sum_{p+q=s} X_i^i(p)\otimes X_i^i(q) \\
&= \Delta'(X_i^i(s)X_i^i(t)),
\end{aligned}$$

for all $i, s, u, t \in \mathbb{V}$.

Note that $\Delta'(1) = 1 \otimes_t 1$. It follows from Lemma 5.3.4 that $\Delta'(1^i) = 1^i \otimes 1^i$.

Next we verify that ε_t is an algebra map. For all $s, t \in \mathbb{V}$, we have

$$\begin{aligned}
\varepsilon_t(X_i^i(s))\varepsilon_t(X_i^i(t)) &= \delta_{s,0}\delta_{t,0}(\sum_p X_i^i(p))(\sum_q X_i^i(q)) \\
&= \delta_{s,0}\delta_{t,0}(\sum_p X_i^i(p)) = \delta_{s,t}\delta_{s,0}\varepsilon_t(X_i^i(s)) \\
&= \varepsilon_t(X_i^i(s)X_i^i(t)).
\end{aligned}$$

Finally, we prove that the antipode axioms hold. Indeed,

$$\begin{aligned}
m(1\otimes S)\Delta''(X_i^i(s)) &= \sum_{p+q=s} X_i^i(p)S(X_i^i(q)) = \sum_{p+q=s} X_i^i(p)X_i^i(-q) \\
&= \delta_{p,-q}\sum_{p+q=s} X_i^i(p) = \delta_{s,0}\sum_{p\in\mathbb{V}} X_i^i(p) = \varepsilon_t(X_i^i(s)).
\end{aligned}$$

for any $s \in \mathbb{V}$. Similarly, we also have

$$\begin{aligned}
\sum_{w+q=s} S(X_i^i(w)X_i^i(q) &= \sum_{w+q=s} X_i^i(-w)X_i^i(q)) = \sum_{w+q=s} \delta_{-w,q}X_i^i(q) \\
&= \sum_q \delta_{s,0}X_i^i(q) = \varepsilon_t(X_i^i(s)).
\end{aligned}$$

Hence, H^i is an ordinary Hopf algebra over $\mathbb{C}1^i$. □

Corollary 5.3.7. *The braided Hopf algebra $_RH$ has a decomposition:*

$$_RH = \bigoplus_{i\in\mathbb{V}} H^i,$$

where H^i is a Hopf algebra over $\mathbb{C}1^i$ with unity 1^i. Moreover, there exists a Hopf algebra isomorphism from H^i to H^j defined by

$$\iota_i^j : X_i^i(p) \longmapsto X_j^j(p),$$

for all $i, j, p \in \mathbb{V}$.

Proof. Follows from Proposition 5.3.6. □

5.3.3 Galois objects and Brauer groups

In this section, we compute the braided Galois objects over $_RH$ and the Brauer group of the weak Hopf algebra $A_{N,t}$. Before we study the braided Galois $_RH$-Galois objects, let us look at the H-module structure on $_RH$ (see Lemma 2.1.5), which we need later on.

Lemma 5.3.8. *The left adjoint action of H on $_RH$ is given by the following formula:*

$$X_j^i(p) \cdot X_k^k(s) = \delta_{j,k}\delta_{p,0}X_i^i(s),$$

for all $i, j, p, k, s \in \mathbb{V}$.

Proof. We compute the left adjoint action as follows:

$$\begin{aligned}
X_j^i(p) \cdot X_k^k(s) &= \sum_{u+v=p} X_j^i(u)X_k^k(s)S(X_{j+u}^{i+u}(v)) \\
&= \sum_{u+v=p} X_j^i(u)X_k^k(s)X_{i+u+v}^{j+u+v}(-v) \\
&= \delta_{j,k} \sum_{u+v=p} \delta_{u,s}X_k^i(s)X_{i+p}^{j+p}(-v)
\end{aligned}$$

$$
\begin{aligned}
&= \delta_{j,k} \sum_{u+v=p} \delta_{u,s}\delta_{s,-v}\delta_{j+p,k} X^i_k(s) X^{j+p}_{i+p}(-v) \\
&= \delta_{j,k}\delta_{p,0} X^i_i(s),
\end{aligned}
$$

for all $i, j, p, k, s \in \mathbb{V}$. □

Let A be a right braided ${}_RH$-Galois object. Then $A_\circ \simeq H_t$. By Corollary 5.3.7, ${}_RH = \bigoplus_{i\in\mathbb{V}} H^i$. Then we have a vector space decomposition: $A = \bigoplus_{i\in\mathbb{V}} A^i$, where $\rho^r(A^i) \in A^i \otimes H^i$. Since $H_t = \bigoplus_{i\in\mathbb{V}} \mathbb{C}1^i$, $A_\circ = \bigoplus_{i\in\mathbb{V}} A^i_\circ$. In what follows, we will prove that every A^i is a Galois object over H^i, where H^i is the same as in Corollary 5.3.7. Note that H is spanned by $\{X^i_j(p) | i, j, p \in \mathbb{V}\}$. We first find an idempotent element of A^i by the left H-module structure on A.

Lemma 5.3.9. *If A is a right braided ${}_RH$-Galois object, then for any $k \in \mathbb{V}$, there exists an idempotent element e^k in A^k such that the action of H on e^k is given by*

$$X^i_j(p) \cdot e^k = \delta_{p,0}\delta_{j,k} e^i,$$

for all $i, j, p \in \mathbb{V}$. In particular, $A_\circ$ is the $\mathbb{C}$-linear span of $\{e^k\}_{k\in\mathbb{V}}$.

Proof. Note that $A_\circ \simeq H_t$. Let f be an isomorphism from H_t to $A_\circ$ in the category of finite dimensional left H-modules. For all $u \in V$, we have $1^u \neq 0$ and so $f(1^u) \neq 0$. Moreover, the H_t-linearity implies that $f(1^u) = 1^u \cdot 1_A$. Since f is an algebra map, $f(1^u) = f(1^u)f(1^u)$. Given $a \in A^u$, we have

$$f(1^u)a = (1^u \cdot 1_A)a = 1 \cdot ((1^u \cdot 1_A)a) = (1_1 1^u \cdot 1_A)(1_2 \cdot a) = 1^u \cdot a.$$

Note that $h \cdot 1_A = \varepsilon_t(h) \cdot 1_A$ for any $h \in H$. So for all $p \neq 0$ and $i, j \in \mathbb{V}$,

$$X^i_j(p) \cdot 1_A = 0.$$

For any $i, j, k \in \mathbb{V}$, we have

$$
\begin{aligned}
X^i_j(0) \cdot (1^k \cdot 1_A) &= \sum_p (X^i_j(0) X^k_k(p)) \cdot 1_A \delta_{j,k} X^i_k(0) \cdot 1_A \\
&= \delta_{j,k} \varepsilon_t(X^i_k(0)) \cdot 1_A = \delta_{j,k} 1^i \cdot 1_A.
\end{aligned}
$$

Now let $e^k := 1^k \cdot 1_A$. Then

$$X^i_j(p) \cdot e^k = \delta_{p,0}\delta_{j,k}e^i.$$

Thus $A_\circ$ is also the $\mathbb{C}$-linear span of $\{e^i\}_{i\in\mathbb{V}}$. □

Now we study the algebraic structure of a braided ${}_RH$-Galois object.

Lemma 5.3.10. *If A is a right braided ${}_RH$-Galois object, then for each $l \in \mathbb{V}$, A^l is a left ideal of A.*

Proof. It is sufficient to show that $A^kA^l \subseteq A^l$, $\forall\ k \in \mathbb{V}$. For any $u^k \in A^k$, we have $\rho^r(u^k) = u^k_{(0)} \otimes u^k_{(1)} \in A^k \otimes H^k$. By Lemma 5.3.8,

$$\begin{aligned}\rho^r(u^ku^l) &= \sum_{i,j,p} u^k_{(0)}(X^j_{j+p}(i-j)\cdot u^l_{(0)}) \otimes (X^i_j(p)\cdot u^k_{(1)})u^l_{(1)}t^{-p(i-j)}\\ &= \sum_{i,j} u^k_{(0)}(X^j_j(i-j)\cdot u^l_{(0)}) \otimes (X^i_j(0)\cdot u^k_{(1)})u^l_{(1)}\\ &= \sum_{i} u^k_{(0)}(X^k_k(i-k)\cdot u^l_{(0)}) \otimes (X^i_k(0)\cdot u^k_{(1)})u^l_{(1)}\\ &= u^k_{(0)}(X^k_k(l-k)\cdot u^l_{(0)}) \otimes (X^l_k(0)\cdot u^k_{(1)})u^l_{(1)},\end{aligned}$$

for all $u^l \in A^l$. So $A^kA^l \subseteq A^l$. □

Lemma 5.3.11. *Let A be a right braided ${}_RH$-Galois object. Then the following two statements hold for all $i, j, p, k \in \mathbb{V}$ and $u^k \in A^k$:*

1. *$X^i_j(p) \cdot u^k \in A^{i+p}$. Moreover, if $j + p \neq k$, then $X^i_j(p) \cdot u^k = 0$.*
2. *$X^i_j(p) \cdot u^k = 0$ if and only if $X^l_j(p) \cdot u^k = 0$ for any $l \in \mathbb{V}$.*

Proof. Following the left H-linearity, we obtain

$$\begin{aligned}\rho(X^i_j(p)\cdot u^k) &= \sum_{w+v=p} X^i_j(w)\cdot u^k_{(0)} \otimes X^{i+u}_{j+u}(v)\cdot u^k_{(1)}\\ &= X^i_j(p)\cdot u^k_{(0)} \otimes X^{i+p}_{j+p}(0)\cdot u^k_{(1)},\end{aligned}$$

for all $i, j, p, k \in \mathbb{V}$ and $u^k \in A^k$. Moreover, Lemma 5.3.9 implies that

$$\rho(X^i_j(p)\cdot u^k) = X^i_j(p)\cdot u^k_{(0)} \otimes X^{i+p}_{j+p}(0)\cdot u^k_{(1)} \in A \otimes H^{i+p}.$$

Then $X^i_j(p) \cdot u^k \in A^{i+p}$. In particular, when $j+p \neq k$, we have $\rho(X^i_j(p) \cdot u^k) = 0$. Thus $X^i_j(p) \cdot u^k = 0$.

The second statement follows from

$$X^l_j(p) \cdot u^k = X^l_i(p) \cdot (X^i_j(p) \cdot u^k) = 0,$$

for any $i, l \in \mathbb{V}$. □

Lemma 5.3.12. *Let A be a quantum commutative Galois object. Then*

$$A^k A^l = 0,$$

for all $k, l \in \mathbb{V}$ such that $k \neq l$.

Proof. First of all, we recall from [70] how to construct a left ${}_RH$-comodule from a right ${}_RH$-comodule. Let M be a right ${}_RH$-comodule in ${}_H\mathscr{M}$. It is not hard to check that the map

$$\tau_{{}_RH,M} : {}_RH \otimes M \longrightarrow M \otimes {}_RH, \; h \otimes m \longmapsto r^2R^1 \cdot m \otimes r^1hR^2$$

is a haif-braiding, whose inverse is given by

$$\tau'_{M,{}_RH} : M \otimes {}_RH \longrightarrow {}_RH \otimes M, \; m \otimes h \longmapsto r^1hR^2 \otimes S^{-1}(r^2R^1) \cdot m.$$

Then the induced left ${}_RH$-comodule is given by

$$\begin{aligned} \rho^l(a) &= \tau'(a_{(0)} \otimes a_{(1)}) \\ &= r^1a_{(1)}R^2 \otimes S^{-1}(r^2R^1) \cdot a_{(0)}, \end{aligned}$$

where for all $a \in A$ and $\rho^r(a) = a_{(0)} \otimes a_{(1)}$.

Next note that Lemma 3.1.3 implies that a left ${}_RH$-comodule induces a left-left Yetter-Drinfeld module with the coaction ρ^L, where

$$\rho^L(m) = m_{(-1)}R^2 \otimes R^1 \cdot m_{(0)} \in H \otimes M.$$

Let us work out this left H-comodule. For all $a \in A$,

$$\begin{aligned}
\rho^L(a) &= r^1 a_{(1)} R^2 p^2 \otimes p^1 S^{-1}(r^2 R^1) \cdot a_{(0)} \\
&= r^1 a_{(1)} R^2 p^2 \otimes S^{-1}[R^1 S(p^1)] S^{-1}(r^2) \cdot a_{(0)} \\
&= r^1 a_{(1)} R^2 \otimes S^{-1}[\varepsilon_t(R^1)] S^{-1}(r^2) \cdot a_{(0)} \\
&= r^1 a_{(1)} 1_1 \otimes S^{-1}(1_2) S^{-1}(r^2) \cdot a_{(0)} \\
&= r^1 1_1 a_{(1)} \otimes S^{-1}(r^2 1_2) \cdot a_{(0)} \\
&= r^1 a_{(1)} \otimes S^{-1}(r^2) \cdot a_{(0)}.
\end{aligned}$$

Now we claim that for all $a^k \in A^k$ and $b^l \in A^l$,

$$X_v^u(q) \cdot (ab) = \delta_{v,l} [(X_k^u(l-k) a_{(1)}^k) \cdot b^l][X_l^{u+l-k}(q+k-l)) \cdot a_{(0)}^k], \tag{5.1}$$

for any $u, v, q \in \mathbb{V}$. Using the quantum commutativity of A, we first have

$$\begin{aligned}
a^k b^l &= [(r^1 a_{(1)}^k) \cdot b^l][S^{-1}(r^2) \cdot a_{(0)}^k] \\
&= \sum_{i,j,p} [(X_j^i(p) a_{(1)}^k) \cdot b^l][S^{-1}(X_{j+p}^j(i-j)) \cdot a_{(0)}^k] t^{-p(i-j)} \\
&= \sum_{i,j,p} [(X_j^i(p) a_{(1)}^k) \cdot b^l][X_i^{i+p}(j-i)) \cdot a_{(0)}^k] t^{-p(i-j)} \\
&= \sum_{i,p} [(X_k^i(p) a_{(1)}^k) \cdot b^l][X_i^{i+p}(k-i)) \cdot a_{(0)}^k] t^{-p(i-k)} \\
&= \sum_{i} [(X_k^i(l-k) a_{(1)}^k) \cdot b^l][X_i^{i+l-k}(k-i)) \cdot a_{(0)}^k] t^{-(l-k)(i-k)},
\end{aligned}$$

for all $a^k \in A^k$ and $b^l \in A^l$. Then

$$\begin{aligned}
X_v^u(q) \cdot (a^k b^l) &= X_v^u(q) \cdot [\sum_i (X_k^i(l-k) a_{(1)}^k) \cdot b^l][X_i^{i+l-k}(k-i)) \cdot a_{(0)}^k] \\
&= \sum_{c+d=q} \sum_i [(X_v^u(c) X_k^i(l-k) a_{(1)}^k) \cdot b^l][(X_{v+c}^{u+c}(d) X_i^{i+l-k}(k-i)) \cdot a_{(0)}^k] \\
&= \sum_i [(X_v^u(l-k) X_k^l(l-k) a_{(1)}^k) \cdot b^l][(X_{v+l-k}^{u+l-k}(q+k-l) X_l^{l+l-k}(k-i))) \cdot a_{(0)}^k] \\
&= \sum_i \delta_{q+k-l,k-i} \delta_{v,l} [(X_k^u(l-k) a_{(1)}^k) \cdot b^l][X_l^{u+l-k}(q+k-l)) \cdot a_{(0)}^k] \\
&= \delta_{v,l} [(X_k^u(l-k) a_{(1)}^k) \cdot b^l][X_l^{u+l-k}(q+k-l)) \cdot a_{(0)}^k],
\end{aligned}$$

for any $u, v, q \in \mathbb{V}$.

Finally, we verify that $A^k A^l = 0$ for all $k, l \in \mathbb{V}$ such that $k \neq l$. If $q \neq 0$, then $l + q + k - l \neq k$. It follows from (1) in Lemma 5.3.10 and the equation (5.1) that $X^u_v(q) \cdot (a^k b^l) = 0$. So

$$X^u_v(q) \cdot (a^k b^l) = \delta_{q,0}\delta_{v,l}[(X^u_k(l-k)a^k_{(1)}) \cdot b^l][X^{u+l-k}_l(k-l)) \cdot a^k_{(0)}].$$

Now we show that $X^u_v(q) \cdot b^l = 0$ when $q \neq 0$. Note that $1_A = \sum_k e^k$. We get

$$\begin{aligned} X^u_v(q) \cdot b^l &= X^u_v(q) \cdot (1b^l) = \sum_k X^u_v(q) \cdot (e^k b^l) \\ &= \delta_{q,0} \sum_k \delta_{v,l}[(X^u_k(l-k)e^k_{(1)}) \cdot b^l][X^{u+l-k}_l(k-l)) \cdot e^k_{(0)}] = 0. \end{aligned}$$

Similar to the proof in Lemma 5.3.9, we have

$$\rho(u^k u^l) = u^k_{(0)}(X^k_k(l-k) \cdot u^l_{(0)}) \otimes (X^l_k(0) \cdot u^k_{(1)})u^l_{(1)} = 0,$$

for all $a^k \in A^k$ and $u^l \in A^l$. So $u^k u^l = 0$ when $l \neq k$. Thus $A^k A^l = 0$. □

Corollary 5.3.13. *If A is a quantum commutative Galois object, then $1_A = \bigoplus_{k \in \mathbb{V}} e^k$ such that every e^k is an identity element of A^k, where $e^k = 1^k \cdot 1_A$.*

Proof. For all $u^k \in A^k$, we have $u^k = 1_A u^k = \sum_i e^i u^k$. By Lemma 5.3.12, $u^k = e^k u^k$. Similarly, $u^k = u^k 1_A = \sum_i u^k e^i = u^k e^k$. □

Lemma 5.3.14. *Let A be a quantum commutative Galois object. Then there exists a basis of A given by a union:*

$$\bigcup_{l \in \mathbb{V}} \{u^l_0, u^l_1, ..., u^l_{N-1}\},$$

where $\{u^l_0, u^l_1, ..., u^l_{N-1}\}$ is a basis of A^l such that $e^l = u^l_0$ and

$$X^i_j(p) \cdot u^l_m = \delta_{p,0}\delta_{j,l}u^i_m,$$

for all $i, j, p, l, m \in \mathbb{V}$.

Proof. First of all, we show that $X^i_k(0) \cdot u^k \neq 0$ for any $0 \neq u^k \in A^k$ and $i \in \mathbb{V}$. For all $u^k \in A^k$, we have

$$\begin{aligned} u^k &= e^k u^k = f(1^k)u^k = (1^k \cdot f(1))u^k \\ &= (1^k \cdot 1_A)u^k = (1_1 1^k \cdot 1_A)(1_2 \cdot u^k) \\ &= 1^k \cdot u^k = \sum_p X_k^k(p) \cdot u^k \\ &= X_k^k(0) \cdot u^k. \end{aligned}$$

Using the associativity of H-action we get

$$u^k = X_k^k(0) \cdot u^k = (X_i^k(0) X_k^i(0)) \cdot u^k = X_i^k(0) \cdot ((X_k^i(0) \cdot u^k),$$

for any $i \in \mathbb{V}$. If $u^k \neq 0$, then $X_k^i(0) \cdot u^k \neq 0$ for any $i \in \mathbb{V}$.

Next we claim that if any two elements u^k and v^k in A^k are linearly independent, then $X_k^i(0) \cdot u^k$ and $X_k^i(0) \cdot v^k$ in A^i are linearly independent. Indeed, if there exist l_u and l_v in $\mathbb{C}$ such that $l_u X_k^i(0) \cdot u^k + l_v X_k^i(0) \cdot v^k = 0$, we have

$$\begin{aligned} 0 &= X_i^k(0)(l_u X_k^i(0) \cdot u^k + l_v X_k^i(0) \cdot v^k) \\ &= l_u (X_i^k(0) X_k^i(0)) \cdot u^k + l_v (X_i^k(0) X_k^i(0)) \cdot v^k) \\ &= l_u X_k^k(0) \cdot u^k + l_v X_k^k(0) \cdot v^k \\ &= l_u u^k + l_v v^k. \end{aligned}$$

So $l_u = 0$ and $l_v = 0$. Thus $X_k^i(0) \cdot u^k$ and $X_k^i(0) \cdot v^k$ are linearly independent.

By the above, if $\{u_0^k, u_1^k, ...\}$ is a basis of A^k such that $u_0^k = e^k$, then $\{e^i, X_i^k(0) \cdot u_1^k, X_i^k(0) \cdot u_2^k, ...\} \subseteq A^i$ is linearly independent. Conversely, if $\{u_0^i, u_1^i, ...\}$ is a basis of A^i such that $u_0^i = e^i$, then $\{e^k, X_k^i(0) \cdot u_1^i, X_k^i(0) \cdot u_2^i, ...\} \subseteq A^k$ are also linearly independent. Since A^i and A^k are finite dimensional, $dim(A^i) = dim(A^k)$. So

$$dim(A) = \sum_i dim(A^i) = N \ dim(A^i).$$

Note that $dim(A) = dim(_R H) = N^2$. Then $dim(A^i) = N$ for any $i \in \mathbb{V}$.

Now fix a $k \in \mathbb{V}$. We construct a basis of A from some basis of A^k. First choose a basis of A^k: $\{u_0^k, u_1^k, ..., u_{N-1}^k\}$ such that $u_0^k = e^k$. Then $\{X_i^k(0) \cdot u_0^k, X_i^k(0) \cdot u_1^k, X_i^k(0) \cdot u_2^k, ..., X_i^k(0) \cdot u_{n-1}^k\}$ as above is a basis of A^i such that $X_i^k(0) \cdot u_0^k = e^i$, for any $i \in V$.

Denote $X_i^k(0) \cdot u_p^k$ by u_p^i. Now the induced basis of A^i can be expressed by

$$\{u_0^i, u_1^i, ..., u_{N-1}^i\}.$$

Thus, we obtain a basis of A by taking the union:

$$\bigcup_{l \in \mathbb{V}} \{u_0^l, u_1^l, ..., u_{N-1}^l\}.$$

Finally, we verify that $X_j^i(p) \cdot u_m^l = \delta_{p,0}\delta_{j,l} u_m^i$. Indeed,

$$\begin{aligned} X_j^i(p) \cdot u_m^l &= X_j^i(p) \cdot (X_k^l(0) \cdot u_m^k) = (X_j^i(p) X_k^l(0)) \cdot u_m^k) \\ &= \delta_{p,0}\delta_{j,l} X_k^i(0) \cdot u_m^k = \delta_{p,0}\delta_{j,l} u_m^i, \end{aligned}$$

for any $i, j, p, l, m \in V$. □

Corollary 5.3.15. *If A is a quantum commutative Galois object, then*

$$dim(A^i) = N, \forall\ i \in \mathbb{V}.$$

In particular, $dim(H^i) = N, \forall\ i \in \mathbb{V}$.

Proof. The proof follows from Lemma 5.3.4. □

We have seen that $dim(A^i)$ is equal to N for each $i \in \mathbb{V}$. In fact, $A^i \simeq A^k, \forall i, k \in \mathbb{V}$.

Corollary 5.3.16. *If A is a quantum commutative Galois object, then $A^i \simeq A^k$ as algebras, for all $i, k \in \mathbb{V}$.*

Proof. Take $i, k \in \mathbb{V}$. It is easy to see that the map

$$\mu_k^i : A^k \longrightarrow A^i,\ u^k \longmapsto X_k^i(0) \cdot u^k$$

is bijective. Since $X_k^i(0)$ is a group-like element of H, we have

$$X_k^i(0) \cdot (u^k v^k) = (X_k^i(0) \cdot u^k)(X_k^i(0) \cdot v^k) = u^i v^i.$$

for any $u^k, v^k \in A^k$. Hence, μ_k^i is an algebra isomorphism. □

Remark 5.3.17. (1) The isomorphism μ_k^i in Corollary 5.3.16 is neither left H-linear nor right ${}_RH$-colinear. Indeed, if $i \neq k$,

$$X_k^i(0) \cdot (X_k^k(0) \cdot u^k) = (X_k^i(0) X_k^k(0)) \cdot u^k = u^i.$$

But we have

$$X_k^k(0) \cdot (X_k^i(0) \cdot u^k) = (X_k^k(0) X_k^i(0)) \cdot u^k) = 0.$$

Moreover,

$$\rho(X_k^i(0)(u^k)) = u^i_{(0)} \otimes u^i_{(1)} \in A^i \otimes H^i.$$

However,

$$(f \otimes 1)\rho(u^k) = X_k^i(0) \cdot u^i_{(0)} \otimes u^i_{(1)} \in A^k \otimes H^i.$$

(2) Let $\rho(u^k) = \sum_{j,p} \alpha_{j,p} u_j^k \otimes X_k^k(p)$ for all $u^k \in A^k$. By the left H-linearity,

$$\begin{aligned} \rho(u_l^i) &= \rho(X_k^i(0) \cdot u_l^k) = X_k^i(0) \cdot u^k_{(0)} \otimes X_k^i(0) \cdot u^k_{(1)} \\ &= \sum_{j,p} \alpha_{j,p} X_k^i(0) \cdot u_j^k \otimes X_k^i(0) \cdot X_k^k(p) = \sum_{j,p} \alpha_{j,p} u_j^i \otimes X_i^i(p). \end{aligned}$$

That is to say, the ${}_RH$-comodule structure on A^i is uniquely determined by the comodule structure on A^k. This motivates us to introduce the following definition.

Definition 5.3.18. Let f be a Hopf algebra isomorphism from H to H'. Let A and A' be a right H-comodule algebra and a right H'-comodule algebra respectively. If there is an algebra map g from A to B such that

$$(g \otimes f)\rho_A = \rho_{A'} \circ g,$$

we say that (g, f) is an (H, H')-comodule algebra map from A to A'. In particular, g is called an (H, H')-comodule algebra isomorphism if g is additionally bijective.

Example 5.3.19. If A is a quantum commutative Galois object, then every μ_k^i in Corollary 5.3.16 is an (H^k, H^i)-comodule algebra isomorphism from A^k to A^i, where ι_k^i is a Hopf algebra isomorphism from H^k to H^i, see Corollary 5.3.7.

Lemma 5.3.20. *Let f be a Hopf algebra isomorphism from H to H'. Let A be an H-Gaolis object and A' an H'-Galois object. If there is an (H, H')-comodule algebra map g from A to A', then g is an (H, H')-comodule algebra isomorphism.*

Proof. Similar to the Hopf case, see [14]. □

Following Corollary 5.3.16, we know that a quantum commutative Galois object A is actually the direct sum of subalgebras. Next we will show that A is the direct sum of Galois objects over H^i.

Corollary 5.3.21. *If A is a quantum commutative Galois object, then for any $i \in \mathbb{V}$, the canonical map restricted on $A^i \otimes A^i$:*

$$\gamma : A^i \otimes A^i \longrightarrow A^i \otimes \ H^i,\ a \otimes b \longmapsto ab_{(0)} \otimes b_{(1)}$$

is bijective. Namely, A^i is a Galois object over H^i.

Proof. First let us show that $A \otimes_t A = \bigoplus_i (A^i \otimes A^i)$. Similar to Lemma 5.3.4, we only need to show that

$$1_1 \cdot u^k \otimes 1_2 \cdot u^l = \delta_{k,l} u^k \otimes u^l$$

for all $u^k \in A^k$ and $u^l \in A^l$. Note that $1^k \cdot u^k = u^k$. We have

$$\begin{aligned}
& 1_1 \cdot u^k \otimes 1_2 \cdot u^l \\
= & \sum_{i,s} \sum_{p+q=s} X_i^i(p) \cdot u^k \otimes X_{i+p}^{i+p}(q) \cdot u^l \\
= & \sum_{i,s} \sum_{p+q=s} (X_i^i(p) 1^k) \cdot u^k \otimes (X_{i+p}^{i+p}(q) 1^l) \cdot u^l \\
= & \sum_{s} \sum_{p+q=s} X_k^k(p) \cdot u^k \otimes (X_l^l(q) \cdot u^l = \delta_{k,l} u^k \otimes u^l,
\end{aligned}$$

where Lemma 5.3.11 was applied to the last equality. Similarly, $A \otimes_{t\ R} H = \bigoplus_i (A^i \otimes H^i)$.

Now the map $\gamma|_{A^i \otimes A^i}$ is well-defined. For all $a \otimes b \in A^i \otimes A^i$, we have

$$ab_{(0)} \otimes b_{(1)} \in A^i \otimes H^i.$$

Thus $\gamma(A^i \otimes A^i) \subseteq A^i \otimes H^i$. Since A is a braided Galois object, the canonical map is bijective. Therefore, the map $\gamma|_{A^i \otimes A^i}$ is bijective. □

Lemma 5.3.22. *If A is a quantum commutative Galois object, then A^i is an H^i-Galois object over H^i for any $i \in \mathbb{V}$. Moreover, $A^i \simeq A^j$ as (H^i, H^j)-comodule algebras, for all $i, j \in \mathbb{V}$.*

Proof. The proof follows from Corollary 5.3.16, Example 5.3.19, Lemma 3.3.20 and Corollary 5.3.21. □

Corollary 5.3.23. *If A is a quantum commutative Galois object, then A is the direct sum $\bigoplus_{i\in\mathbb{V}} A^i$, where every A^i is an H^i-Galois object and $A^i \simeq A^j$ as (H^i, H^j)-comodule algebras for any $i, j \in \mathbb{V}$.*

Proof. The proof follows from Lemma 5.3.7 and 5.3.22. □

Now we state one of main results in the section.

Theorem 5.3.24. *Let A be a $\mathbb{C}$-algebra with unity. Then A is a quantum commutative Galois object if and only if A is the direct sum $\bigoplus_{i\in\mathbb{V}} A^i$, where every A^i is an H^i-Galois object and $A^i \simeq A^j$ as (H^i, H^j)-comodule algebras for any $i, j \in \mathbb{V}$.*

Proof. It remains to be shown that the converse of Corollary 5.3.23 is true. Now assume that A is the direct sum $\bigoplus_i A^i$, where every A^i is an H^i-Galois object and $A^i \simeq A^j$ for any $i, j \in V$. We need to construct a left H-module structure on A such that A becomes a braided Galois object.

First of all, fix some $i \in \mathbb{V}$. By assuption, there is a family of isomorphisms:

$$\begin{aligned}
&\lambda_1 : A^i \longrightarrow A^{i+1}\\
&\lambda_2 : A^{i+1} \longrightarrow A^{i+2}\\
&\ldots \quad \ldots \quad \ldots \quad \ldots\\
&\lambda_N : A^{i+N-1} \longrightarrow A^{i+N}.
\end{aligned}$$

We may choose λ_i such that $\lambda_n \ldots \lambda_2\lambda_1 = 1$ because of $A^{i+N} = A^i$. For example, take $\lambda_N = \lambda_1^{-1}\lambda_2^{-1}\ldots\lambda_{N-1}^{-1}$.

Given $k, l \in \mathbb{V}$, we define an isomorphism from A^k to A^l as compositions of some of $\lambda_1, \ldots, \lambda_n$, which we denote by f_k^l, and the inverse by f_l^k. For example, from A^{i+4} to A^{i+6} :

$$f_{i+6}^{i+4} = \lambda_6\lambda_5 : A^{i+4} \longrightarrow A^{i+5} \longrightarrow A^{i+6};$$

from A^{i+6} to A^{i+4}:

$$f_{i+4}^{i+6} = \lambda_4 \ldots \lambda_1 \lambda_n \ldots \lambda_7 : A^{i+6} \longrightarrow A^{i+7} \ldots \longrightarrow A^{i+n} = A^i \longrightarrow A^{i+1} \longrightarrow \ldots \longrightarrow A^{i+4}.$$

It is clear that the following equations hold:

$$f_k^k = 1,\ f_k^l = f_u^l f_k^u,\ \ f_k^l = f_w^l \cdots f_u^v f_k^u.$$

Next we come to construct a left H-action on A. For a fixed i, we first choose a basis $\{u_0^i, u_1^i, ..., u_{n-1}^i\}$ of A^i such that u_0^i is the identity element of A^i . So $\{f_i^j(u_0^i), f_i^j(u_1^i), ..., f_i^j(u_{n-1}^i)\}$ is also a basis of A^j for any $j \in \mathbb{V}$. It is obvious that $f_i^j(u_0^i)$ is also the identity element of A^j. Moreover, since every f_i^j is an (H^i, H^j)-comodule algebra isomorphism, we have

$$f_i^j(u_p^i)_{(0)} \otimes f_i^j(u_p^i)_{(1)} = f_i^j(u^i_{p(0)}) \otimes \iota_i^j(u^i_{p(1)}). \tag{5.2}$$

Now write u_p^j for $f_i^j(u_p^i)$. We obtain

$$f_j^k(u_p^j) = f_j^k f_i^j(u_p^i) = f_i^k(u_p^i) = u_p^k,$$

for any $k \in \mathbb{V}$. In particular, $f_j^j(u_p^j) = u_p^j$. Define a $\mathbb{C}$-linear map $\varphi : H \otimes A \longrightarrow A$ by:

$$\varphi(X_n^m(p) \otimes u_q^k) = \delta_{p,0}\delta_{n,k} f_k^m(u_q^k),$$

for any $m, n, p, k, q \in \mathbb{V}$. It is easy to see that φ is well-defined. In fact,

$$\varphi(X_n^m(p) \otimes u_q^k) = \delta_{p,0}\delta_{n,k} u_q^m.$$

Now we claim that (A, φ) is a left H-module algebra. On one hand,

$$\begin{aligned} 1_H \cdot u_q^k &= \sum_{i,p} X_i^i(p) \cdot u_q^k = \sum_p X_k^k(p) \cdot u_q^k \\ &= X_k^k(0) \cdot u_q^k = u_q^k. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} X_v^u(w) \cdot (X_n^m(p) \cdot u_q^k) &= \delta_{p,0}\delta_{n,k} X_v^u(w) \cdot u_p^m = \delta_{p,0}\delta_{n,k}\delta_{w,0}\delta_{v,m} u_p^u \\ &= \delta_{w,p}\delta_{v,m}\delta_{w,0}\delta_{n,k} u_p^u = \delta_{w,p}\delta_{v,m} X_n^u(w) \cdot u_q^k \\ &= (X_v^u(w) X_n^m(p)) \cdot u_q^k. \end{aligned}$$

So (A, φ) is a left H-module. Moreover, we also have

$$\begin{aligned} X_v^u(w) \cdot 1_A &= X_v^u(w) \cdot (\sum_i u_0^i) = \sum_i \delta_{v,i}\delta_{w,0}u_0^u \\ &= \delta_{w,0}u_0^u = \delta_{w,0}\sum_p \sum_i X_u^u(p) \cdot u_0^i \\ &= \delta_{w,0}(\sum_p X_u^u(p)) \cdot (\sum_i u_0^i) \\ &= \delta_{w,0}(\sum_p X_u^u(p)) \cdot 1_A \\ &= \varepsilon_t(X_v^u(w)) \cdot 1_A. \end{aligned}$$

Since $A^i A^j = 0 = A^j A^i$, we have for $w \neq 0$,

$$\begin{aligned} &\sum_{s+t=w} (X_v^u(s) \cdot u_p^k)(X_{v+s}^{u+s}(t) \cdot u_q^k) \\ &= \sum_{s+t=w} [\delta_{s,0}\delta_{v,k} f_k^u(u_p^k)][\delta_{t,0}\delta_{v+s,k} f_k^{u+s}(u_p^k)] \\ &= \delta_{w,0}\delta_{v,k}[f_k^u(u_p^k)][f_k^u(u_p^k)] \\ &= \delta_{w,0}\delta_{v,k} f_k^u(u_p^k u_q^k) = X_v^u(w) \cdot (u_p^k u_q^k). \end{aligned}$$

Thus (A, φ) is also a left H-module algebra.

We also need to show that the right coaction ρ is left H-linear. On one hand,

$$\begin{aligned} \rho(X_n^m(p) \cdot u_q^k) &= \delta_{p,0}\delta_{n,k}\rho(u_q^m) = \delta_{p,0}\delta_{n,k} u_{q_{(0)}}^m \otimes u_{q_{(1)}}^m \\ &\overset{(5.2)}{=} \delta_{p,0}\delta_{n,k} f_i^m(u_{p(0)}^i) \otimes \iota_i^m(u_{p(1)}^i). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\sum_{s+t=p} X_n^m(s) \cdot u_{q_{(0)}}^k \otimes X_{n+s}^{m+s}(t) \cdot u_{q_{(1)}}^k \\ =\;& \delta_{p,0} X_n^m(0) \cdot u_{q_{(0)}}^k \otimes X_n^m(0) \cdot u_{q_{(1)}}^k \\ =\;& \delta_{p,0}\delta_{n,k} f_k^m(u_{q_{(0)}}^k) \otimes \iota_k^m(u_{q_{(1)}}^k) \\ \overset{(5.2)}{=}\;& \delta_{p,0}\delta_{n,k} f_k^m(f_i^k(u_{p(0)}^i)) \otimes \iota_k^m(\iota_i^k(u_{p(0)}^i)) \\ =\;& \delta_{p,0}\delta_{n,k} f_i^m(u_{p(0)}^i) \otimes \iota_i^m(u_{p(0)}^i). \end{aligned}$$

So $\rho(X_n^m(p)\cdot u_q^k)=\sum_{s+t=p}X_n^m(s)\cdot u^k_{q_{(0)}}\otimes X_{n+s}^{m+s}(t)\cdot u^k_{q_{(1)}}$. Thus ρ is left H-linear.

Finally, we verify that (A,φ,ρ) is a right ${}_RH$-comodule algebra. Note that $A^iA^j=0=A^jA^i$ for any $i\neq j$. We have $\rho(\delta_{k,l}u^ku^l)=\rho(u^ku^l)$ and so

$$\begin{aligned}
&\sum_{i,j,p}u^k_{(0)}(X^j_{j+p}(i-j)\cdot u^l_{(0)})\otimes(X^i_j(p)\cdot u^k_{(1)})u^l_{(1)}\\
=&\sum_{i,j}u^k_{(0)}(X^j_j(i-j)\cdot u^l_{(0)})\otimes(X^i_j(0)\cdot u^k_{(1)})u^l_{(1)}\\
=&\sum_{i}u^k_{(0)}(X^k_k(i-k)\cdot u^l_{(0)})\otimes(X^i_k(0)\cdot u^k_{(1)})u^l_{(1)}\\
=&\ u^k_{(0)}(X^k_k(l-k)\cdot u^l_{(0)})\otimes(X^l_k(0)\cdot u^k_{(1)})u^l_{(1)}\\
=&\ \delta_{k,l}u^k_{(0)}u^l_{(0)}\otimes u^k_{(1)}u^l_{(1)}=\rho(\delta_{k,l}u^ku^l),
\end{aligned}$$

for any $u^k\in A^k$ and $u^l\in A^l$. It is not hard to see that A is a quantum commutative Galois object. □

Corollary 5.3.25. *If A' is a Galois object over Hopf algebra H^i for some $i\in\mathbb{V}$, then there exists a quantum commutative ${}_RH$-Galois object such that its i-th direct summand is A'.*

Proof. Consider the direct sum $\bigoplus_{i\in\mathbb{V}}A'^i$, where every algebra A'^i is a copy of A'. Since ι_i^j is an isomorphism from H^i to H^j for any $j\in\mathbb{V}$, we can equip A'^j with a right H^j-comodule structure defined by

$$(1\otimes\iota_i^j)\rho: A'\longrightarrow A'\otimes H^i\longrightarrow A'\otimes H^j,\ a'\longmapsto a'_{(0)}\otimes\iota_i^j(a'_{(1)}).$$

It is clear that A'^j is a right H^j-Galois object. Moreover, we have an isomorphism (id,ι_i^j) from A'^i to A'^j, i.e., $A'^i\simeq A'^j$. By Theorem 5.3.23, $\bigoplus_{i\in\mathbb{V}}A'^i$ is a quantum commutative Galois object, and the i-th direct summand of $\bigoplus_{i\in\mathbb{V}}A'^i$ is A'. □

Lemma 5.3.26. *Let A and B be two quantum commutative ${}_RH$-Galois objects. Then $A\simeq B$ as braided Galois objects if and only if there exists some $i\in\mathbb{V}$ such that $A^i\simeq B^i$ as H^i-Galois objects.*

Proof. Assume that there exists an isomorphism h from A^i to B^i for some $i\in\mathbb{V}$. For any $j\in\mathbb{V}$, we have an (H^j,H^i)-comodule isomorphism $f:A^j\longrightarrow A^i$ and an (H^i,H^j)-comodule isomorphism $g:B^i\longrightarrow B^j$.

Now we check that the composition $g\circ(h\circ f)$ is an isomorphism from Galois object A^j to Galois object B^j. It is enough to show that $g\circ(h\circ f)$ is right-colinear.For any $a\in A^j$, we have

$$\begin{aligned}\rho[g((h\circ f)(a))] &= g((h\circ f)(a))_{(0)}\otimes g((h\circ f)(a))_{(1)}\\ &= g((h\circ f)(a)_{(0)})\otimes \iota_i^j((h\circ f)(a)_{(1)})\\ &= (g\circ h)(f(a)_{(0)})\otimes \iota_i^j(f(a)_{(1)})\\ &= (g\circ h)(f(a_{(0)}))\otimes \iota_i^j(\iota_j^i(a_{(1)}))\\ &= (g\circ h)(f(a_{(0)}))\otimes a_{(1)},\end{aligned}$$

where the third equality stems from the fact that h is a comodule map. So $A\simeq B$. The proof of the converse is clear. □

Following Theorem 5.3.24 and Lemma 5.3.26, we can see that a quantum commutative Galois object over ${}_RH$ is uniquely determined by a Galois object over a Hopf algebra H^i for some $i\in V$. Therefore, we can derive an isomorphism between the group $Gal^{qc}({}_RH)$ of quantum commutative Galois objets and the group $Gal(H^i)$ of isomorphism classes of Galois objets over H^i.

Theorem 5.3.27. *Let H be the dual of $A_{N,t}$. Then there exists a group isomorphism*

$$\Omega: Gal^{qc}({}_RH)\longrightarrow Gal(H^i),\quad A\longmapsto A^i,$$

for any fixed $i\in\mathbb{V}$.

Proof. By Theorem 5.3.22, Ω is well-defined. Following Theorem 5.3.21, we have

$$A\Box_{{}_RH}B=\bigoplus_{i\in\mathbb{V}}(A^i\Box_{H^i}B^i).$$

Then

$$\Omega(A\Box_{{}_RH}B)=A^i\Box_{H^i}B^i=\Omega(A)\Box_{H^i}\Omega(B).$$

Clearly, $\Omega({}_RH))=H^i$. Using Corollary 5.3.24, we define Ω' by

$$\Omega': Gal(H^i)\longrightarrow Gal^{qc}({}_RH),\quad A'\longmapsto\bigoplus_{i\in\mathbb{V}}A'^i,$$

where $\bigoplus_{i\in\mathbb{V}}A'^i$ is equipped with the same structures as in Corollary 5.3.24. It is easy to see that $\Omega\Omega' = 1$. It follows from Corollary 5.3.25 that $\Omega'\Omega = 1$. □

Remark 5.3.28. (1) Following Theorem 5.3.27, the computation of the Brauer group of a modular category obtained from $SU(N)_L$-SOS models can be transferred to characterizing the group of Galois objects over some corresponding Hopf algebra H^i given in Proposition 5.3.6. The Hopf algebra H^i is finite dimensional and commutative. So the Galois group is given by the second Sweedler's cohomology group.

(2) From Corollary 5.3.23, a quantum commutative Galois object is the direct sum of Galois objects over some Hopf algebras. Note that the direct sum of Hopf algebras can be viewed as a weak Hopf algebra. So a quantum commutative Galois object can be also regarded as a Galois object over a weak Hopf algebra. This motivates us to consider the group of bi-Galois objects over a weak Hopf algebra in the next chapter.

Chapter 6

The group of bi-Galois objects over a weak Hopf algebra

The study of bi-Galois extensions was initiated by Van Oystaeyen and Zhang in [82]. They introduced bi-Galois extensions over a commutative Hopf algebra in order to establish a Galois-type correspondence. The theory was generalized by Schauenburg to any non-commutative Hopf algebra, see [66], and was further developed. In the same paper, Schauenburg constructed the groupoid of all bi-Galois objects under cotensor product since a bi-Galois object A induces an equivalence $A\Box-$ between the categories of comodules. If one fixes a Hopf algebra H, then the two-sided H-bi-Galois objects form a subgroup $Gal(H)$, which was independently discovered by Van Oystaeyen and Zhang in an unpublished paper [81]. Moreover, it was shown in [68] that a Hopf-(bi)Galois object is the same as a faithfully flat quantum torsor in [35].

The theory of bi-Galois objects has been generalized to the case of a bialgebroid (or a coring), see [6]. For example, it was shown that there exists a bijective correspondence between faithfully flat A-B torsors and bi-Galois objects over bialgebroids, and that a bi-Galois object also induces an equivalence between the categories of comodules over bialgebroids, see [6]. So one can ask whether we can form the group of bi-Galois objects in the case of a Hopf algebroid. In this chapter, we will explicitly work out this group in the case of a weak Hopf algebra, for the reason that we explained in Remark 5.3.28 (2).

6.1 Cotensor products

Let H be a weak Hopf algebra. Let A and B be a right H-comodule algebra and a left H-comodule algebra. We first have the cotensor product $A \boxdot_H B$. By Example 1.6.21, there exists a Hopf algebroid $\mathcal{H}$ consisting of a left bialgebroid H_l and a right bialgebroid H_r:

$$H_l = (H, H_t, id_{H_t}, S^{-1}|_{H_t}, \Delta, \varepsilon_t), \quad H_r = (H, H_s, id_{H_s}, S^{-1}|_{H_s}, \Delta, \varepsilon_s).$$

So we can work with the cotensor products: $A\square_{H_l}B$ (over left bialgebroid H_l) and $A\square_{H_r}B$ (over right bialgebroid H_r). In this section, we will mainly discuss the relation between the three cotensor products so that we can use them freely in the next section.

Let M be an ${H_t}^e$-bimodule. Then M is an (H_t, H_s)-bimodule with the following bimodule structure:

$$xm = (x \otimes \overline{1})m, \qquad ux = u(x \otimes \overline{1}); \tag{6.1}$$

$$ym = (1 \otimes \overline{S(y)})m, \quad my = u(1 \otimes \overline{S(y)}), \tag{6.2}$$

for all $m \in M, x \in H_t$ and $y \in H_s$.

Let M and N be two ${H_t}^e$-bimodules. Since $S : H_t \longrightarrow H_s$ is anti-isomorphic, the product $\times_{H_t}$ in the sense of Takeuchi can be written as

$$M \otimes_{H_t} N = \int_x {}_{S^{-1}(x)}M \otimes {}_xN, \quad \forall\ x \in H_t;$$

$$M \times_{H_t} N = \int^t \int_x {}_{S^{-1}(x)}M_{S^{-1}(t)} \otimes {}_xN_t, \quad \forall x, t \in H_t.$$

Now define a subspace $M \odot N$ of $M \otimes N$ as

$$\{m \otimes n \in M \otimes N | m \otimes n = 1_1 \cdot m \cdot 1_1 \otimes 1_2 \cdot m \cdot 1_2\}.$$

Let us first work out the relation between $M \times_{H_t} N$ and $M \odot N$. Note that H_t is Frobenius-separable. By [7, 20] we get a map

$$P : M \otimes_{H_t} N \longrightarrow \Delta(1) \cdot (M \otimes N), \quad m \otimes_{H_t} n \longmapsto 1_1 \cdot m \otimes 1_2 \cdot n,$$

with the inverse given by

$$P'' : \Delta(1) \cdot (M \otimes N) \longrightarrow M \otimes_{H_t} N,\ 1_1 \cdot m \otimes 1_2 \cdot n \ \longmapsto m \otimes_{H_t} n.$$

Moreover, $M \times_{H_t} N$ is an ${H_t}^e$-bimodule with the following structure:

$$\begin{aligned}(x \otimes \overline{x'}) \cdot (m \otimes_{H_t} n) &= xm \otimes_{H_t} S^{-1}(x')n,\\ (m \otimes_{H_t} n) \cdot (x \otimes \overline{x'}) &= mx \otimes_{H_t} nS^{-1}(x'),\end{aligned} \tag{6.3}$$

for all $x, x' \in H_t$, $m \otimes_{H_t} n \in M \times_{H_t} N$. Similarly, $M \odot N$ is also an ${H_t}^e$-bimodule with the following structure:

$$\begin{aligned}(x \otimes \overline{x'}) \cdot [(1_1 m' 1'_1) \otimes (1_2 n' 1'_2)] &= x 1_1 m' 1'_1 \otimes S^{-1}(x') 1_2 n' 1'_2,\\ [(1_1 m' 1'_1) \otimes (1_2 n' 1'_2)] \cdot (x \otimes \overline{x'}) &= 1_1 m' 1'_1 x \otimes 1_2 n' 1'_2 S^{-1}(x'),\end{aligned} \tag{6.4}$$

for all $x, x' \in H_t$ and $m' \in M, n' \in N$.

Lemma 6.1.1. *Let M and N be two ${H_t}^e$-bimodules. Then $M \times_{H_t} N = M1_1 \otimes_{H_t} N1_2$.*

Proof. Take $m \in M, n \in N$ and $x \in H_t$. We have

$$m1_1 S^{-1}(x) \otimes_{H_t} n1_2 = S(1_1) \otimes_{H_t} n1_2 x.$$

Thus $M1_1 \otimes_{H_t} N1_2 \subseteq M \times_{H_t} N$.

Conversely, we need to show that $m'1'_1 \otimes_{H_t} n'1'_2 = m' \otimes_{H_t} n'$ for all $m' \otimes_{H_t} n' \in M \times_{H_t} N$. Note that $m' \otimes_{H_t} n' = 1_1 m' \otimes_{H_t} 1_2 n'$ and

$$1_1 m' S^{-1}(x) \otimes_{H_t} 1_2 n' = m' S^{-1}(x) \otimes_{H_t} n' = m' \otimes_{H_t} n' x,$$

for all $x, x' \in H_t$. Using the map P we get

$$1_1 m' S^{-1}(x) \otimes 1_2 n' = 1_1 m' \otimes 1_2 n' x,$$

which implies $1_1 m' S^{-1}(x) \otimes (1_2 n') x' = 1_1 m' \otimes (1_2 n' x) x'$. In particular,

$$\begin{aligned}1_1 m' 1'_1 \otimes 1_2 n' 1'_2 &= 1_1 m' S^{-1}[S(1'_1)] \otimes (1_2 n') 1'_2\\ &= 1_1 m' \otimes (1_2 n' S(1'_1)) 1'_2 = 1_1 m' \otimes 1_2 n'.\end{aligned}$$

Applying the map P'' to both sides of the above, we obtain

$$1_1m'1'_1 \otimes_{H_t} 1_2n'1'_2 = 1_1m' \otimes_{H_t} 1_2n'.$$

Since $1_1m' \otimes_{H_t} 1_2n' = m' \otimes_{H_t} n'$, we have $m'1'_1 \otimes_{H_t} n'1'_2 = m' \otimes_{H_t} n'$. Then $m' \otimes_{H_t} n' \in M1_1 \otimes_{H_t} N1_2$. Therefore, $M \times_{H_t} N = M1_1 \otimes_{H_t} N1_2$. □

Lemma 6.1.2. *Let M and N be two ${H_t}^e$-bimodules. Then, with the structures (6.3) and (6.4), $M \times_{H_t} N \simeq M \odot N$ as ${H_t}^e$-bimodules.*

Proof. Note that $M \times_{H_t} N = M1_1 \otimes_{H_t} N1_2$. It is easy to verify that $P : M1_1 \otimes_{H_t} N1_2 \longrightarrow M \odot N$ is isomorphic. Now it remains to be checked that P is ${H_t}^e$-bilinear. Indeed, the left linearity of P follows from

$$\begin{aligned}
&P[(x \otimes \overline{x'}) \cdot (m1'_1 \otimes_{H_t} n1'_2)] \\
&= P(xm1'_1 \otimes_{H_t} S^{-1}(x')n1'_2) = x1_1m1'_1 \otimes S^{-1}(x')1_2n1'_2 \\
&= (x \otimes \overline{x'}) \cdot (1_1m1'_1 \otimes_{H_t} 1_2n1'_2) = (x \otimes \overline{x'}) \cdot P(m1_1 \otimes_{H_t} n1_2),
\end{aligned}$$

for all $m \in M, n \in N$ and $x, x' \in H_t$. Similarly, P is right ${H_t}^e$-linear. □

Example 6.1.3. Let M be an H-bicomodule. By [59] M is an ${H_t}^e$-bimodule with the following induced structure:

$$\begin{aligned}
(x \otimes \overline{x'}) \cdot m &= \varepsilon(xm_{[0]_{[-1]}})m_{[0]_{[0]}}\varepsilon(S^{-1}(x')m_{[1]}), \\
m \cdot (x \otimes \overline{x'}) &= \varepsilon(m_{[0]_{[-1]}}x)m_{[0]_{[0]}}\varepsilon(m_{[1]}S^{-1}(x')),
\end{aligned}$$

for any $x, x' \in H_t$ and $m \in M$, where $\rho^L(m) = m_{[-1]} \otimes m_{[0]}$ and $\rho^R(m) = m_{[0]} \otimes m_{[1]}$. So M is an (H_t, H_s)-bimodule with the structures (6.1) and (6.2). If M and N are two H-bicomodules, then $M \times_{H_t} N \simeq M \odot N$ as ${H_t}^e$-bimodules.

Now let (U, i) be an ${H_t}^e$-algebra. U is a natural ${H_t}^e$-bimodue, which restricts to an (H_t, H_s)-bimodule structure:

$$xu = i(x \otimes \overline{1})u, \quad ux = ui(x \otimes \overline{1}); \; yu = i(1 \otimes \overline{S(y)})u, \quad uy = ui(1 \otimes \overline{S(y)}),$$

for all $u \in U, x \in H_t$ and $y \in H_s$.

Lemma 6.1.4. *Let (U, i) and (V, j) be two ${H_t}^e$-algebras. Then $U \odot V$ is an ${H_t}^e$-algebra with unity $1_1 1_U \otimes 1_2 1_V$:*

$$[\Delta(1)(u \otimes v)\Delta(1)][\Delta(1)(u' \otimes v')\Delta(1)] = \Delta(1)(u 1_1 u' \otimes v 1_2 v')\Delta(1),$$
$$g : A^e \longrightarrow \Delta(1)(U \otimes V)\Delta(1), x \otimes \overline{y} \longmapsto \Delta(1)(i(x) \otimes j(\overline{x'}))\Delta(1),$$

for all $u \in U$, $v \in V$ and $x, x' \in H_t$.

Proof. Straightforward. □

Let (U, i) and (V, j) be two ${H_t}^e$-algebras. Then $U \times_{H_t} V$ is an ${H_t}^e$-algebra with unity $1 \otimes_{H_t} 1$ and the following structure:

$$(u \otimes_{H_t} v)(u' \otimes_{H_t} v') = uu' \otimes_{H_t} vv', \quad \forall u \otimes_{H_t} v, u' \otimes_{H_t} v' \in U \times_{H_t} V,$$
$$h : {H_t}^e \longrightarrow U \times_{H_t} V, x \otimes \overline{y} \longmapsto i(x) \otimes_{H_t} j(\overline{x'}), \ \forall x, x' \in H_t.$$

Proposition 6.1.5. *Let (U, i) and (V, j) be two ${H_t}^e$-algebras. Then $U \times_{H_t} V$ and $U \odot V$ are isomorphic as ${H_t}^e$-algebras.*

Proof. Note that $u \otimes_{H_t} v = u\overline{S(1_1)} \otimes_{H_t} v 1_2$ for all $u \otimes_{H_t} v \in U \times_{H_t} V$. We have

$$\begin{aligned} P[(u \otimes_{H_t} v)(u' \otimes_{H_t} v')] &= P[(u\overline{S(1_1)} \otimes_{H_t} v 1_2)(u' \otimes_{H_t} v')] \\ &= P[(u\overline{S(1_1)}u' \otimes_{H_t} v 1_2 v')] \\ &= \Delta(1)(u\overline{S(1_1)}u' \otimes v 1_2 v')\Delta(1) \\ &= [\Delta(1)(u \otimes v)\Delta(1)][\Delta(1)(u' \otimes v')\Delta(1)] \\ &= P[(u \otimes_{H_t} v)]\pi[(u' \otimes_{H_t} v')]. \end{aligned}$$

The rest of the proof is easy. □

Let A be a left H-comodule algebra. By [11], the left comodule structure on A induces an algebra map

$$i : H_t \longrightarrow A, \quad x \longmapsto \varepsilon(1_{[-1]}x)1_{[0]},$$

for any $x \in H_t$, which makes A into an H_t-algebra. Similarly, a right H-comodule algebra B is an H_s-algebra with the structure map given by,

$$j : H_s \longrightarrow H, \ y \longmapsto \varepsilon(1_{[1]}y)1_{[0]}, \ \forall y \in H_s.$$

Lemma 6.1.6. *Let A be an H-bicomodule algebra. Then (A, k) is an ${H_t}^e$-algebra, where the map k is given by*

$$k : H_t \otimes {H_t}^{op} \longrightarrow A, \ x \otimes \overline{x'} \longmapsto \varepsilon(x1_{[-1]})1_{[0]}1'_{[0]}\varepsilon(S^{-1}(x')1'_{[1]}).$$

Proof. Note that

$$\begin{aligned}\varepsilon(1_{[-1]}x)1_{[0]}1'_{[0]}\varepsilon(1'_{[1]}y) &= \varepsilon(1_{[-1]}x)1_{[0]_{[0]}}\varepsilon(y\varepsilon_t(1_{[0]_{[1]}})) = \varepsilon(1_{[0]_{[-1]}}x)1_{[0]_{[0]}}\varepsilon(y1_{[1]}) \\ &= \varepsilon(\varepsilon_s(1_{[0]_{[-1]}})x)1_{[0]_{[0]}}\varepsilon(y1_{[1]}) = \varepsilon(1_{[-1]}x)1'_{[0]}1_{[0]}\varepsilon(y1'_{[1]}),\end{aligned}$$

for all $x \in H_t$ and $y \in H_s$. We have a well-defined algebra map

$$k' : H_t \otimes H_s \longrightarrow A, \ x \otimes y \longmapsto \varepsilon(x1_{[-1]})1_{[0]}1'_{[0]}\varepsilon(y1'_{[1]}).$$

The map k follows from k' since H_s is isomorphic to H_t^{op}. □

Since an H-bicomodule algebra A is an H_t^e-algebra, A has a natural H_t^e-bimodule structure:

$$\begin{aligned}(x \otimes \overline{x'}) \cdot a &= \varepsilon(x1_{[-1]})1_{[0]}1'_{[0]}a\varepsilon(S^{-1}(x')1'_{[1]}), \\ (x \otimes \overline{x'}) \cdot a &= \varepsilon(x1_{[-1]})a1_{[0]}1'_{[0]}\varepsilon(S^{-1}(x')1'_{[1]}),\end{aligned}$$

for any $x, x' \in H_t$ and $a \in A$.

Following Proposition 6.1.5 and Lemma 6.1.6, we obtain the following consequence:

Corollary 6.1.7. *Let A and B be two bicomodule algebras. Then $A \times_{H_t} B$ and $A \odot B$ are isomorphic as ${H_t}^e$-algebras.*

Observe that both $A \odot B$ and $A \times_{H_t} B$ are H-bicomodule algebras with the same H-bicomodule structure ρ^L and ρ^R,

$$\rho^L(a \otimes b) = a_{[-1]} \otimes (a_{[0]} \otimes b), \quad \rho^R(a \otimes b) = (a \otimes b_{[0]}) \otimes b_{[1]},$$

for all $a \otimes b \in A \odot B$ or $a \otimes b \in A \times_{H_t} B$.

Following Corollary 6.1.7, we get the following:

Proposition 6.1.8. *Let A and B be two bicomodule algebras. Then $A \times_{H_t} B$ and $A \odot B$ are isomorphic as H-bicomodule algebras.*

Remark 6.1.9. Let A and B be two H-bicomodule algebras. One can define $A\times^{H_s} B$ (see [40]) similarly. Moreover, $A\times^{H_s} B$ and $A\odot B$ are isomorphic as H-bicomodule algebras.

Let M and N be a right H-comodule and a left H-comodule respectively. The cotensor product over H is as follow:

$$M\,\square_H\,N=\{m\otimes n\in M\otimes N|\rho^R(m)\otimes n=m\otimes\rho^L(n)\}.$$

Lemma 6.1.10. *Let A and B be a right H-comodule algebra and a left H-comodule algebra respectively. Then*

$$A\,\square_H\,B\subset A\odot B.$$

Proof. Note that $a_{[0]}\otimes a_{[1]}\otimes b=a\otimes b_{[-1]}\otimes b_{[0]}$ for all $a\otimes b\in A\,\square_H\,B$. Applying $id\otimes\varepsilon_t\otimes id$ to both sides, we get $a_{[0]}\otimes\varepsilon_t(a_{[1]})\otimes b=a\otimes\varepsilon_t(b_{[-1]})\otimes b_{[0]}$. So

$$\begin{aligned}1_1\cdot a\otimes 1_2\cdot b &= 1_{[0]}\cdot a\otimes 1_{[1]}\cdot b=a_{[0]}\otimes\varepsilon_t(a_{[1]})\cdot b\\ &= a\otimes\varepsilon_t(b_{[-1]})\cdot b_{[0]}=a\otimes\varepsilon_t(b_{[-1]})\cdot b_{[0]}\\ &= a\otimes\varepsilon(\varepsilon_t(b_{[-1]})b_{[0]_{[-1]}})b_{[0]_{[0]}}\\ &= a\otimes b.\end{aligned}$$

Similarly, $a\cdot 1_1\otimes b\cdot 1_2=a\otimes b$. Thus $a\otimes b\in A\odot B$. □

Now let us recall the relation between H-comodule algebras and $H_l(H_r)$-comodule algebras. Let A be a left H-comodule algebra. By [4, 11] A is a left H_l-comodule algebra as well as a left H_r-comodule algebra. Here, A is a left H_l-comodule algebra with the following H_t-bimodule structure:

$$x\cdot a=a_{[0]}\varepsilon(xa_{[-1]})\quad a\cdot x=a_{[0]}\varepsilon(a_{[-1]}x), \tag{6.5}$$

for all $x\in H_t$ and $a\in A$. The above actions induce the following actions:

$$y\triangleright a=a\cdot S^{-1}(y),\quad a\triangleleft y=S^{-1}(y)\cdot a,$$

for any $y\in H_s$ and $a\in A$. Then $(A,\triangleright,\triangleleft)$ is an H_R-comodule algebra over an H_s-coring H_r (or a right bialgebroid H_r).

Similarly, if B is a right H-comodule algebra, then B is a right H_r-comodule algebra with the following H_s-bimodule structure:

$$y \cdot b = b_{[0]}\varepsilon(xb_{[1]}) \quad b \cdot y = b_{[0]}\varepsilon(b_{[1]}y), \tag{6.6}$$

for all $y \in H_s$ and $b \in B$, and right H_l-comodule algebra with H_t-bimodule structure:

$$x \blacktriangleright b = b \cdot S^{-1}(x), \quad b \blacktriangleleft x = S^{-1}(x) \cdot b.$$

In fact, the action $\triangleright, \triangleleft, \blacktriangleright$ and $\blacktriangleleft$ are induced by Example 6.1.3.

Now let A be an H-bicomodule algebra. We have

$$y \cdot a = a_{[0]}\varepsilon(ya_{[1]}) = a_{[0]}\varepsilon(y\varepsilon_t(a_{[1]})) = 1_{[0]}a\varepsilon(y1_{[1]}) = (y \cdot 1)a,$$

for all $y \in H_s$ and $a \in A$. Take $x \in H_t$. Then $x \cdot 1 \in A^{coH}$ follows from

$$\begin{aligned} 1_{[0]_{[0]}}\varepsilon(x1_{[-1]}) \otimes 1_{[0]_{[1]}} &= 1_{[0]_{[0]}}\varepsilon(x1_{[0]_{[-1]}}) \otimes 1_{[1]} \\ &= x \cdot 1_{[0]} \otimes \otimes 1_{[1]} = (x \cdot 1)1_{[0]} \otimes \otimes 1_{[1]}. \end{aligned}$$

Now we can define an algebra map

$$\mu : H_t \longrightarrow A^{coH}, \; x \longmapsto x \cdot 1.$$

Similarly, there is also an algebra map

$$\nu : H_s \longrightarrow {}^{coH}A, \; y \longmapsto y \cdot 1.$$

Let (A, ρ^R) and (B, ρ^L) be a right H-comodule algebra and a left H-comodule algebra respectively. We view (A, ρ^R) and (B, ρ^L) as a right H_l-comodule algebra and a left H_l-comodule algebra respectively. So we have the cotensor product:

$$A \square_{H_l} B = \{a \otimes b \in A \otimes_{H_t} B | \rho^R(a) \otimes b = a \otimes \rho^L(b)\}.$$

Now define $A \overline{\square}^t_H B$ as a subspace of $A \times_{H_t} B$:

$$A \overline{\square}^t_H B = \{a \otimes b \in A \times_{H_t} B | \rho^R(a) \otimes b = a \otimes \rho^L(b)\}.$$

Lemma 6.1.11. *Let H be a weak Hopf algebra. If (A, ρ^R) is a right H-comodule* *algebra and (B,ρ^L) is a left H-comodule algebra, then as vector spaces*

$$A\Box_{H_l}B = A\overline{\Box}^t_H B \text{ and } A\Box_{H_l}B \simeq A \boxdot_H B.$$

Proof. In order to prove $A\Box_{H_l}B = A\overline{\Box}^t_H B$, it is sufficient to show that $A\Box_{H_l}B \subseteq A \times_{H_t} B$ since the definition of $\times_{H_t}$ implies that $A\Box_{H_l}B \supset A\overline{\Box}^t_H B$.

Take $a\otimes b \in A\Box_{H_l}B$. We have that $a_{[0]}\otimes a_{[1]}\otimes b = a\otimes b_{[-1]}\otimes b_{[0]}$. Since $a_{[0]}\otimes a_{[1]}\otimes b \in (A\times_{H_t} H)\otimes_t B$ and $a\otimes b_{[-1]}\otimes b_{[0]} \in A\otimes_t (H\times_{H_t} B)$, we get that $a_{[0]}\otimes a_{[1]}\overline{x}\otimes b = a\otimes b_{[-1]}\overline{x}\otimes b_{[0]}$ for all $x\in H_t$. Observe that

$$(a\overline{x})_{[0]}\otimes (a\overline{x})_{[1]}\otimes b = a_{[0]}\otimes a_{[1]}\overline{x}\otimes b = a\otimes b_{[-1]}\overline{x}\otimes b_{[0]} = a\otimes b_{[-1]}\otimes b_{[0]}x.$$

It follows from the counit axioms that $a\overline{x}\otimes b = a\otimes bx$. Thus $a\otimes b\in A\times_{H_t} B$.

The proof of the second statement is similar to the proof of Lemma 6.1.2. □

Similarly, we can also think (A,ρ^R) and (B,ρ^L) as a right H_r-comodule algebra and a left H_r-comodule algebra respectively, and define the cotensor product $A\Box_{H_r}B$.

Lemma 6.1.12. *Let H be a weak hopf algebra. If A is a right H-comodule algebra and B is a left H-comodule algebra, then as vector spaces $A\Box_{H_r}B \simeq A\boxdot_H B$.*

Proof. Similar to the proof of Lemma 6.1.11. □

Now we can state the relation between these three cotensor products.

Proposition 6.1.13. *Let H be a weak Hopf algebra. If A and B are two H-bicomodule algebras, then as H-bicomodule algebras*

$$A\Box_{H_r}B \simeq A\boxdot_H B \simeq A\Box_{H_l}B.$$

Proof. We only show that $A\Box_{H_l}B \simeq A\boxdot_H B$ as H-bicomodule algebras. Following Proposition 6.1.8, $A\odot B \simeq A\times_{H_t} B$ as H-comodule algebras. Given $a'\otimes b'\in A\Box_{H_l}B$, we have $a'_{[0]}\otimes a'_{[1]}\otimes b' = a'\otimes b'_{[-1]}\otimes b'_{[0]}$ and

$$a'_{[0]}\otimes a'_{[1]}\otimes b'_{[0]}\otimes b'_{[1]} = a'\otimes b'_{[-1]}\otimes b'_{[0]_{[0]}}\otimes b'_{[0]_{[1]}} = a'\otimes b'_{[0]_{[-1]}}\otimes b'_{[0]_{[0]}}\otimes b'_{[1]}.$$

So $a\otimes b_{[0]}\otimes b_{[1]}\in A\Box_{H_l}B\otimes H$. Thus $A\Box_{H_l}B$ is a right H-subcomodule. Moreover, we also have

$$a_{[0]}a'_{[0]} \otimes a_{[1]}a'_{[1]} \otimes bb' = (a_{[0]} \otimes a_{[1]} \otimes b)(a'_{[0]} \otimes a'_{[1]} \otimes b') = aa' \otimes b_{[-1]}b'_{[-1]} \otimes b_{[0]}b'_{[0]},$$

for any $a \otimes b, a' \otimes b' \in A\Box^t_H B$. Hence, $A\Box_{H_l} B$ is a subalgebra of $A \times_{H_t} B$.

Using Lemma 6.1.7 and 6.1.11 we get an isomorphism $P : A \odot B \simeq A \times_{H_t} B$ as ${H_t}^e$-algebras, and so $A \boxdot_H B \simeq A\Box_{H_l} B$ as ${H_t}^e$-subalgebras. It remains to be checked that P is right colinear. Indeed,

$$\begin{aligned} \rho^R[\Delta(1)(a \otimes b)\Delta(1)] &= 1_1 \cdot a \cdot 1_1 \otimes (1_2 \cdot b \cdot 1_2)_{[0]} \otimes (1_2 \cdot b \cdot 1_2)_{[1]} \\ &= 1_1 \cdot a \cdot 1_1 \otimes 1_2 \cdot b_{[0]} \cdot 1_2 \otimes b_{[1]} = P(a \otimes b_{[0]}) \otimes b_{[1]}. \end{aligned}$$

So $A \boxdot_H B \simeq A\Box_{H_l} B$ as right comodule algebras. Similarly, $A \boxdot_H B \simeq A\Box_{H_l} B$ as left comodule algebras. Therefore, $A \boxdot_H B \simeq A\Box_{H_l} B$ as bicomodule algebras. $\Box$

From Proposition 6.1.13, we can choose freely these cotensor products for H-bicomodule algebras if we need.

6.2 The group of bi-Galois objects

Here we discuss how to form the group of Galois objects over a weak Hopf algebra.

Definition 6.2.1. A weak Hopf algebra H is called faithfully flat if H is faithfully flat as (H_t, H_s)-bimodule.

Definition 6.2.2. Let H be a faithfully flat weak Hopf algebra. A weak H-bicomodule algebra A is called an H-bi-Galois object if the following conditions are satisfied:

- A/A^{coH} is a right weak H-Galois extension with $A^{coH} \simeq H_t$,
- $A/{}^{coH}A$ is a left weak H-Galois extension with ${}^{coH}A \simeq H_s$,
- A is faithfully flat as an (H_t, H_s)-bimodule under the actions (6.5) and (6.6) .

Remark 6.2.3. By [4] the definition of a left H_l (or a right H_r)-Galois extension is equivalent to the one of a left (right) weak H-Galois extension. So Definition 6.2.5 is a special case of [6]. Moreover, one-sided H-Galois object was also defined in [6]. In particular, H is a trivial bi-Galois object.

Lemma 6.2.4. *Let A and B be two H-bi-Galois objects. Then the following hold:*

1. $A\Box_{H_r}B/H_t$ is a right weak H-Galois extension;

2. $A\Box_{H_l}B/H_s$ is a left weak H-Galois extension;

3. $A\boxdot_H B$ is a faithfully flat bi-Galois object.

Proof. Since A and B are H-bi-Galois objects, it is not hard to see that

$$(A\Box_{H_r}B)^{coH}\simeq B^{coH}\simeq H_t.$$

By the proof of Theorem 5.6 in [6], we have an isomorphism

$$(A\Box_{H_r}B)\otimes_{H_t}(A\Box_{H_r}B)\simeq A\Box_{H_r}(B\otimes_{H_s{}^{op}}B)\simeq A\Box_{H_r}(B\otimes_{H_t}B).$$

The canonical map

$$\begin{aligned}\gamma:(A\Box^t_H B)\otimes_{H_t}(A\Box^t_H B)&\longrightarrow(A\Box^t_H B)\otimes_{H_s}H\\(a\otimes b)\otimes(a'\otimes b')&\longmapsto(aa'\otimes bb'_{[0]})\otimes b'_{[1]}\end{aligned}$$

is bijective since

$$(A\Box_{H_r}B)\otimes_{H_t}(A\Box_{H_r}B)\simeq A\Box_{H_r}(B\otimes_{H_t}B)\simeq A\Box_{H_r}(B\otimes_{H_s}H_r)\simeq(A\Box_{H_r}B)\otimes_{H_s}H_r.$$

So $A\Box_{H_r}B/H_t$ is a right H_r-Galois extension. By Remark 6.2.3, $A\Box_{H_r}B/H_t$ is a right weak H-Galois extension. Similarly, the second statement is true. The last one follows from Lemma 6.1.13 and the former two statements. □

Lemma 6.2.5. *Let A be an H-bi-Galois object. So is A^{op}.*

Proof. It is not hard to check that the opposite algebra A^{op} is a left H-comodule algebra with the coaction: $A^{op}\longrightarrow H\otimes A^{op},\ a\longmapsto S^{-1}(a_{[1]})\otimes a_{[0]}$. Moreover, the coinvaiant subalgebra ${}^{coH}A^{op}$ is just the opposite of A^{coH}. So ${}^{coH}A^{op}\simeq H_t^{op}\simeq H_s$. That A^{op} is a left H-Galois extension follows from

$$A^{op}\otimes_{H_s}A^{op}\simeq A\otimes_{H_t}A\simeq A1_{[0]}\otimes H1_{[1]}\simeq 1_{[-1]}H\otimes 1_{[0]}A^{op}.$$ □

Now assume that A is a right H-comodule algebra. Then $A\otimes_{H_s}A$ is a Hopf module with the following structure:

$$(a \otimes b)c = a \otimes bc, \ \ \rho(a \otimes b) = a_{[0]} \otimes b_{[0]} \otimes a_{[1]}b_{[1]}.$$

Lemma 6.2.6. *Let H be a weak Hopf algebra and A a right H-Galois object. Then*

$$Hom^H(A, V \otimes_{H_t} A) \simeq Hom_{-H_t}((A \otimes_{H_s} A)^{coH}, V).$$

Proof. By the structure theorem of Hopf modules from [16], we have

$$B : Hom^H_{-A}(A \otimes_{H_s} A, V \otimes_{H_t} A) \simeq Hom_{-H_t}((A \otimes_{H_s} A)^{coH}, V).$$

So we only need to show that $Hom^H(A, V \otimes_{H_t} A) \simeq Hom^H_{-A}(A \otimes_{H_s} A, V \otimes_{H_t} A)$. Take $\alpha \in Hom^H(A, V \otimes_{H_t} A)$. It follows from the H-colinerity that the map α is H_s-linear. We have a well-defined map

$$T : Hom^H(A, V \otimes_{H_t} A) \longrightarrow Hom^H_{-A}(A \otimes_{H_s} A, V \otimes_{H_t} A), \ \ T(\alpha)(a \otimes b) = \alpha(a)b.$$

Consider another map

$$T^{-1} : Hom^H_{-A}(A \otimes_{H_s} A, V \otimes_{H_t} A) \longrightarrow Hom^H(A, V \otimes_{H_t} A), \ T^{-1}(\beta)(a) = \beta(a \otimes_{H_s} 1).$$

Then the map T^{-1} is the inverse of T since

$$TT'(\beta)(a \otimes b) = T^{-1}(\beta)(a)b = \beta(a \otimes_{H_s} 1)b = \beta(a \otimes b);$$
$$T^{-1}T(\alpha)(a) = T(\alpha)(a \otimes_{H_s} 1) = \alpha(a).$$

Thus the map T is bijective. □

It was proved that there exists a bijective correspondence between faithfully flat A-B torsors and bi-Galois objects over $\times_A$-Hopf algebras, see [6, Thm.5.2] or [40, Thm.5.2.10]. Let A be a right H-Galois object. Then A is a faithfully flat H_s-H_t torsor with the following structure:

$$\delta(a) = a_{[0]} \otimes a^{[1]}_{[1]} \otimes a^{[2]}_{[1]} \in A \otimes_{H_s} A \otimes_{H_t} A, \ \forall \ \in A.$$

The following construction of a $\times_A$-Hopf algebra from a Galois object is a special case of Theorem 5.2.10 in [40]:

Let H be a faithfully flat weak Hopf algebra and A a right H-Galois object. Then $L := (A \otimes_{H_s} A)^{coH}$ is a $\times_{H_t}$-Hopf algebra with the following Hopf structure:

$$\begin{aligned}
&(a \otimes b)(a' \otimes b') = aa' \otimes b'b, \\
&\overline{\Delta}(a \otimes b) = a_{[0]} \otimes a_{[1]}^{[1]} \otimes a_{[1]}^{[2]} \otimes b, \\
&\overline{\varepsilon_t}(a \otimes b) = ab \in A^{coH} = H_t,
\end{aligned}$$

Moreover, A is also a left L-Galois object.

The following proposition is a generalization of Lemma 3.2 in [66]:

Proposition 6.2.7. *Let H be a faithfully flat weak Hopf algebra and A a right H-Galois objec. Set $L := (A \otimes_{H_s} A)^{coH}$. Then the following statements hold:*

1. *There is a right H-colinear map $\delta : A \longrightarrow L \otimes_{H_t} A$ given by $\delta(a) = a_{[0]} \otimes a_{[1]}^{[1]} \otimes a_{[1]}^{[2]}$, which has the following universal property:*
   Given a right H_t-module V and an H-colinear map $\phi : A \longrightarrow V \otimes_{H_t} A$, there is a unique right H_t-linear map $f : L \longrightarrow V$ with $\phi = (f \otimes_{H_t} 1)\delta$.
2. *The map $\delta : A \longrightarrow L \times_{H_t} A$ is an algebra map;*
3. *If V is an H_t-algebra and $\phi : A \longrightarrow V \times_{H_t} A$ is an H_t-algebra map, so is the induced map $f : L \longrightarrow V$;*
4. *If A is an H-bi-Galois object, then L is isomorphic to H_L.*

Proof. Although the proof is a bit similar to the one of Lemma 3.2 in [66], we write it down for the completeness. Since A is a right Galois object, $(A \otimes_{H_s} A)^{coH} \otimes_{H_t} A \simeq A \otimes_{H_s} A$ as Doi-Hopf modules. This isomorphism υ is given by

$$\upsilon(a' \otimes b' \otimes c') = (a' \otimes b')c', \forall a' \otimes b' \otimes c' \in (A \otimes_{H_s} A)^{coH} \otimes_{H_t} A.$$

with the inverse υ' given by

$$\upsilon'(a \otimes b) = a_{[0]} \otimes a_{[1]}^{[1]} \otimes a_{[1]}^{[2]} b, \ \forall\ a \otimes b \in A \otimes_{H_s} A.$$

Using Lemma 6.2.6 we get an isomorphism

$$B \circ T : Hom^H(A, V \otimes_{H_t} A) \simeq Hom_{-H_t}((A \otimes_{H_s} A)^{coH}, V),$$

where B and T are the same as in Lemma 6.2.6. Let $\delta := T^{-1} \circ B^{-1}(id_L)$. In fact, the map $B^{-1}(id_L)$ is the isomorphism υ'. We have

$$\delta(a) = a_{[0]} \otimes a_{[1]}^{[1]} \otimes a_{[1]}^{[2]}, \ \forall a \in A.$$

Given a right H_t-module V and an H-colinear map $\phi : A \longrightarrow V \otimes_{H_t} A$, there is a unique right H_t-linear map $f : L \longrightarrow V$ such that $f = B \circ T(\phi)$. That $\phi = (f \otimes_{H_t} 1)\delta$ follows from $f(a \otimes_{H_s} b) \otimes_{H_t} 1 = B^{-1}(f)(a \otimes_{H_s} b) = T\phi(a \otimes_{H_s} b) = \phi(a)b$. Thus the first statement holds.

The second statement follows from the definition of an H_s-H_t torsor, see Definition 5.2.1 in [40].

Now let us look at the third statement. Note that $V \times_{H_t} A \subseteq V \otimes_{H_t} A$. For any $\phi : A \longrightarrow V \times_{H_t} A$, by the first statement, there is a unique right H_t-linear map $f : L \longrightarrow V$ with $\phi = (f \otimes_{H_t} 1)\delta$. We have

$$f(aa' \otimes b'b) \otimes 1 = \phi(aa')b'b = \phi(a)\phi(a')b'b = f(a \otimes b)f(a' \otimes b') \otimes 1,$$

for all $a \otimes b, a' \otimes b' \in (A \otimes_{H_s} A)^{coH}$. So f is an H_t-algebra map.

Now assume that A is an H-bi-Galois object. Then A is also a left H_L-Galois object. Note that L is a $\times_{H_t}$-Hopf algebra and A is a left L-Galois object. By the thrid statement there is a unique algebra map $f : L \longrightarrow H$ such that ${}^A\rho = (f \otimes 1)^L\rho$, where ${}^A\rho$ and ${}^L\rho$ denote the left H-coaction and L-coaction respectively. By the universal property of L, the map f is an H_t-coring map and so f is a morphism from L to H_L. Denote by γ and γ' the canonical maps of left L-Galois and left H_L-Galois A respectively. Then $\gamma = (f \otimes 1)\gamma'$. Note that γ and γ' are bijective, and that A is faithfully flat. Thus the map f is bijective. □

Using Proposition 6.2.7 and [67], we obtain

Corollary 6.2.8. *Let H be a faithfully flat weak Hopf algebra. If A is an H- bi-Galois object, then $(A \otimes_{H_s} A)^{coH}$ is isomorphic to H as a weak Hopf algebra.*

Proposition 6.2.9. *Let H be a faithfully flat weak Hopf algebra. If A is an H-bi-Galois object, then $A \boxdot_H A^{op} \simeq (A \otimes_{H_s} A)^{coH}$ and so $A^{op} \boxdot_H A \simeq H$ as bi-Galois objects.*

Proof. Let $A \overline{\otimes} A = \{a \otimes b \in A \otimes A | a \otimes b = a_{[1]} \otimes b_{[1]}\varepsilon(a_{[1]}b_{[1]})\}$. Then $A \overline{\otimes} A$ is a Hopf module with the following structure:

$$(a \otimes b)c = a \otimes bc,\ \ \rho(a \otimes b) = a_{[0]} \otimes b_{[0]} \otimes a_{[1]}b_{[1]}.$$

Similar to the proof of Lemma 6.1.2, we can get that $A \otimes_{H_s} A$ is isomorphic to $A \overline{\otimes} A$ as Hopf modules. By the structure theorem of Hopf modules, $(A \overline{\otimes} A)^{coH} \simeq (A \otimes_{H_s} A)^{coH}$ as right H_t-modules.

Now we verify that $(A \otimes_{H_s} A)^{coH} \simeq A \boxtimes_H A^{op}$. Note that the H-comodule structure on A induces the following H_t-bimodule on A^{op} and H_s-bimodule on A:

$$\begin{aligned} \overline{b} \cdot x = \overline{1_{[0]} b}\varepsilon(S^{-1}(1_{[1]})x), \quad a \cdot y = a1_{[0]}\varepsilon(y1_{[1]}) \\ x \cdot \overline{b} = \overline{b1_{[0]}}\varepsilon(S^{-1}(1_{[1]})x), \quad y \cdot a = 1_{[0]}a\varepsilon(y1_{[1]}), \end{aligned}$$

for any $a, b \in A, x \in H_t$ and $y \in H_s$. We have

$$\begin{aligned} a \cdot 1_1 \otimes \overline{b} \cdot 1_2 &= a1_{[0]}\varepsilon(1_1 1_{[1]}) \otimes \overline{1'_{[0]} b}\varepsilon(S^{-1}(1'_{[1]})1_2) = a1_{[0]}\varepsilon(1_1 1_{[1]}) \otimes \overline{1'_{[0]} b}\varepsilon(S(1_2)1'_{[1]}) \\ &= a1_{[0]}\varepsilon(1_{[1]}1_1)\varepsilon(1_2 1'_{[1]}) \otimes \overline{1'_{[0]} b} = a1_{[0]}\varepsilon(1_{[1]}1'_{[1]}) \otimes \overline{1'_{[0]} b}. \end{aligned}$$

Similarly, $1_1 \cdot a \otimes 1_2 \cdot \overline{b} = 1_{[0]}a\varepsilon(1_{[1]}1'_{[1]}) \otimes \overline{b1'_{[0]}}$. On one hand,

$$\begin{aligned} a \cdot 1_1 \otimes [\overline{b} \cdot 1_2]_{[-1]} \otimes [\overline{b} \cdot 1_2]_{[0]} &= a \cdot 1_1 \otimes S^{-1}(b_{[1]})1_2 \otimes \overline{b_{[0]}} \\ &= a1_{[0]}\varepsilon(1_1 1_{[1]}) \otimes S^{-1}(b_{[1]})1_2 \otimes \overline{b_{[0]}} \\ &= a1_{[0]} \otimes S^{-1}(b_{[1]})1_{[1]} \otimes \overline{b_{[0]}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [1_1 \cdot a]_{[0]} \otimes [1_1 \cdot a]_{[1]} \otimes 1_2 \cdot \overline{b} &= a_{[0]} \otimes 1_1 a_{[1]} \otimes 1_2 \cdot \overline{b} \\ &= a_{[0]} \otimes 1_1 a_{[1]} \otimes \overline{b1_{[0]}}\varepsilon(S^{-1}(1_{[1]})1_2) \\ &= a_{[0]} \otimes S^{-1}(1_{[1]})a_{[1]} \otimes \overline{b1_{[0]}}. \end{aligned}$$

Take $a \otimes b \in (A \otimes_{H_s} A)^{coH}$. We have $a_{[0]} \otimes b_{[0]} \otimes a_{[1]}b_{[1]} = a \otimes b1_{[0]} \otimes 1_{[1]}$ and

$$\begin{aligned} a1_{[0]} \otimes S^{-1}(b_{[1]})1_{[1]} \otimes b_{[0]} &= a_{[0]} \otimes S^{-1}(b_{[1]})S^{-1}(a_{[2]})a_{[1]} \otimes b_{[0]} \\ &= a_{[0]} \otimes S^{-1}(a_{[2]}b_{[1]})a_{[1]} \otimes b_{[0]} \\ &= a_{[0]} \otimes S^{-1}(1_{[1]})a_{[1]} \otimes b1_{[0]}. \end{aligned}$$

So the following equation holds:

$$a1_{[0]} \otimes S^{-1}(b_{[1]})1_{[1]} \otimes \overline{b_{[0]}} = a_{[0]} \otimes S^{-1}(1_{[1]})a_{[1]} \otimes \overline{b1_{[0]}},$$

which means that $a \cdot 1_1 \otimes [\overline{b} \cdot 1_2]_{[-1]} \otimes [\overline{b} \cdot 1_2]_{[0]} = [1_1 \cdot a]_{[0]} \otimes [1_1 \cdot a]_{[1]} \otimes 1_2 \cdot \overline{b}$. Then $1_1 \cdot a \otimes 1_2 \cdot \overline{b} = a \otimes \overline{b} = a \cdot 1_1 \otimes \overline{b} \cdot 1_2$. Thus $(A\overline{\otimes}A)^{coH} \subset A \boxdot_H A^{op}$. Now define a k-linear map

$$\theta : (A\overline{\otimes}A)^{coH} \longrightarrow A \boxdot_H A^{op}, \quad a \otimes b \longmapsto a \otimes \overline{b}.$$

We find the inverse of θ. Let $a\otimes\overline{b} \in A\boxdot_H A^{op}$. Then $a_{[0]}\otimes a_{[1]}\otimes b = a\otimes S^{-1}(b_{[1]})\otimes b_{[0]}$. We have

$$\begin{aligned} a_{[0]} \otimes b_{[0]} \otimes a_{[1]}b_{[1]} &= a \otimes b_{[0]_{[0]}} \otimes S^{-1}(b_{[1]})b_{[0]_{[1]}} \\ &= a \otimes b_{[0]} \otimes S^{-1}(b_{[2]})b_{[1]} \\ &= a \otimes b_{[0]} \otimes S^{-1}[S(b_{[1]})b_{[2]}] \\ &= a \otimes b_{[0]} \otimes S^{-1}[\varepsilon_s(b_{[1]})] \\ &= a \otimes b1_{[0]} \otimes S^{-1}[S(1_{[1]})] = a \otimes b1_{[0]} \otimes 1_{[1]}. \end{aligned}$$

So $a \otimes b \in (A\overline{\otimes}A)^{coH}$. Now we can define another k-linear map

$$\theta' : A \boxdot_H A^{op} \longrightarrow (A\overline{\otimes}A)^{coH}, \quad a \otimes \overline{b} \longmapsto a \otimes b.$$

It is easy to see that θ' is the inverse of θ. Thus, $A \boxdot_H A^{op} \simeq (A\overline{\otimes}A)^{coH}$ as vector spaces.

Finally, by Lemma 6.1.13, $A \boxdot A^{op} \simeq A\Box_{H_r}A^{op}$. So $A\Box_{H_r}A^{op} \simeq (A \otimes_{H_s} A)^{coH}$. Similar to [66], $A\Box_{H_r}A^{op} \simeq (A \otimes_{H_s} A)^{coH}$ as left L-comodule algebras. Hence, $A \boxdot_H A^{op} \simeq H$. Similarly, $A^{op} \boxdot_H A \simeq H$. □

Now we can form the main result of this chapter.

Theorem 6.2.10. *Let H be a faithfully flat weak Hopf algebra. Let $Gal(H, H_t)$ be the set of isomorphism classes of H- bi-Galois objects. Then $Gal(H, H_t)$ forms a group under the cotensor product $\boxdot_H$.*

Proof. Follows from Proposition 6.2.13. □

By [5] we can easily define a bi-cleft object and form the group of isomorphism classes of H-bi-cleft objects, a subgroup of the group $G(H, H_t)$.

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