2014 | Faculty of Sciences

DOCTORAL DISSERTATION

Green rings of finite dimensional pointed Hopf algebras of rank one

Doctoral dissertation submitted to obtain the degree of doctor of Science: Mathematics, to be defended by:

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D/2014/2451/34

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Acknowledgements

I would like to take this opportunity to thank the people who helped me, directly or indirectly, to complete this thesis.

First of all, I am grateful to my promoter, Professor Yinhuo Zhang for his continued guidance, discussions and support during my stay in Belgium. I want also to thank my promoter at Yangzhou University, Professor Libin Li, for his advice, encouragement and patience. Special thanks to Professor Huixiang Chen for numerous helpful discussions and for sharing good ideas with me.

Thanks also to my current and former colleagues at University of Hasselt. Without Dr. Xiaowang Zhang's generous help, I would have spent much more time to get used to the life in Belgium. I am so grateful to Jeroen Dello for his help to translate the abstract of the thesis; to Dr. Haixing Zhu for his kindness and many helpful discussion; to Dr. Xiaolan Yu for her kindness and patience. Special thanks to the secretaries who have been helping me to organize the defense of this thesis.

Lastly, but most importantly, I am forever indebted to my family. Without their continuous encourage and ultimate support, I could not have completed this thesis within 3 years.

Samenvatting

In deze thesis bestuderen we Green ringen van eindig dimensionale pointed Hopf algebras van rang één over een algebraïsch gesloten veld $\Bbbk$ met karakteristiek 0. Verder proberen we de verworven eigenschappen van Green ringen van pointed Hopf algebras van rang één te veralgemenen voor eindig dimensionale Hopf algebras.

In Hoofdstuk 2 en Hoofdstuk 3 werken we met eindig dimensionale pointed Hopf algebras van rang één (van het nilpotente type en het niet-nilpotente type, respectievelijk). We vertonen de Green ring van een dergelijke Hopf algebra in termen van voortbrengers en relaties, waaruit we kunnen besluiten dat de Green ring commutatief en symmetrisch is met een duale basis geassocieerd aan bepaalde bijna split rijen. Het blijkt dat het Jacobson radicaal van de Green ring een hoofd-ideaal is, voortgebracht door een speciaal element voorgesteld door een lineaire combinatie van projectieve modulen, en dat de idempotenten van de Green ring triviaal zijn. Bovendien zijn de niet-triviale idempotenten van de gecomplexifieerde Green ring van een pointed Hopf algebra van rang één van het nilpotente type volledig bepaald.

In Hoofdstuk 4 bestuderen we de stabiele Green ring (i.e., de Green ring van de stabiele categorie) van een eindig dimensionale pointed Hopf algebra van rang één. We tonen aan dat de stabiele Green ring overeenkomt met het quotient van de Green ring van H modulo het ideaal voortgebracht door alle projectieve modulen. Daarenboven, de gecomplexifieerd stabiele Green ring is een groepachtige algebra, en bijgevolg een bi-Frobenius algebra.

In Hoofdstuk 5 bestuderen we de Green ring van een willekeurige eindig dimensionale Hopf algebra H. Ten eerste onderzoeken we kwantum dimensies van indecomposabele H-modulen. Hierdoor kunnen we een antwoord geven op de vraag van Cibils.

Vervolgens bestuderen we enkele ringtheoretische eigenschappen van de Green ring $r(H)$ van H, waaronder de beschrijvingen van zekere belangrijke eenzijdige idealen, de nilpotente idealen en de centrale primitieve idempotenten van $r(H)$. Bovendien bewijzen we dat de stabiele Green ring van H een associatieve non-degenerate bi-lineaire vorm bezit die geïnduceerd wordt door de bi-lineaire vorm op $r(H)$. Ten slotte, indien H een sferische Hopf algebra is, vormt de quotiënt categorie van H-modulen modulo alle verwaarloosbare morfismen een addititieve semisimpele sferische monoïdale categorie. We tonen aan dat de Green ring van de quotiënt categorie isomorf is met de quotiënt ring van $r(H)$ modulo alle indecomposabele modulen met kwamtum dimensie nul. In het bijzonder, indien H van het eindige representatie type is, dan is de gecomplexifieerd quotiënt ring een groepachtige algebra, en bijgevolg een bi-Frobenius algebra.

Introduction

Let H-mod be the category of finite dimensional representations of a Hopf algebra H over a field $\Bbbk$. In the study of the monoidal structure of H-mod one has to consider the decompositions of the tensor product of representations in H-mod. In particular, the decomposition of the tensor product of any two indecomposable representations in H-mod. However, in general, very little is known about how a tensor product of two indecomposable representations decomposes into a direct sum of indecomposable representations. One method of addressing this problem is to consider the tensor product as the multiplication of the Green ring (or the representation ring) $r(H)$ of H, and to study the ring-theoretical properties of the Green ring.

A lot of work have been done in this direction. Firstly, Green [39, 40], Benson and Carlson, etc., considered the semi-simplicity of the representation ring $r(\Bbbk G)$ for modular representations of a finite group G. One of the interesting results they obtained is that $\Bbbk G$ is of finite representation type if and only if there are no nilpotent elements in $r(\Bbbk G)$. In general, it is difficult to determine all nilpotent elements of $r(\Bbbk G)$ if $\Bbbk G$ is of infinite representation type (see [4, 46, 63]). For the Green rings of Hopf algebras, if H is a finite dimensional semi-simple Hopf algebra, then the Green ring $r(H)$ is equal to the Grothendieck ring and is semi-simple (see, e.g. [57, 78]). When H is the enveloping algebra of a complex semi-simple Lie algebra, the Green ring has been studied by Cartan and Weyl (see [37]). Here we would like to mention the recent work by Sergeev and Veselov for basic classical Lie superalgebras (see [68]). Darpö and Herschend have presented a general description of the Green ring of the polynomial Hopf algebra $\Bbbk[x]$ in [26] in case the ground field $\Bbbk$ is perfect. For Green rings of quantum algebras, we refer to the work by Domokos and Lenagan (see [27]). In [70], Wakui computed the Green rings of all non-semisimple Hopf algebras of dimension 8 in terms of generators and relations over an algebraically closed field

$\Bbbk$ of characteristic 0. Cibils in [21] determined all the graded Hopf algebras on a cycle path coalgebra (which are just equal to the generalized Taft algebras (see [62])), and considered the decomposition of two indecomposable representations (see also [41]). Moreover, Cibils also computed the Green ring of the Sweedler 4-dimensional Hopf algebra in terms of generators and relations.

Recently, Chen, Van oystaeyen and Zhang, Li and Zhang, successfully computed the Green rings of the Taft algebras and the Green rings of the generalized Taft algebras respectively in [24] and [56]. They have made use of Cibils' decomposition formulas of tensor products [21]. The nilpotent elements of those Green rings were completely determined in terms of the linear combinations of projective indecomposable modules. In [47], the Green rings of pointed tensor categories of finite type were investigated where the quiver techniques were applied to study the comodules over a graded coquasi-Hopf algebra. For Hopf algebras of infinite type, the Green rings are usually not finitely generated. In this case, the Green rings are difficult to study. For example, the Green ring of the Drinfeld double of the Sweedler Hopf algebra [19] and the Green rings of the rank 2 Taft algebras [53] are not finitely generated although they can be computable.

Motivated by the aforementioned work, we study the Green rings of a family of finite dimensional pointed Hopf algebras, called finite dimensional pointed Hopf algebras of rank one. This family contains the (generalized) Taft algebras, the half quantum group [41] and the Radford Hopf algebras [62]. The classification of this family of Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic 0 has been given respectively in [20] and [50]. In the case of characteristic $p > 0$, the classification was given by Scherotzke in [64]. The authors in [75] constructed this family of Hopf algebras in the point view of Hopf-Ore extensions.

In this thesis, we compute the Green ring of a pointed Hopf algebra of rank one in terms of generators and relations. We show that the Green ring of such a Hopf algebra is commutative and symmetric with a dual basis (to the canonical basis) associated to certain almost split sequences. We are able to describe the Jacobson radical and the idempotents of the Green ring. It turns out that the Jacobson radical of the Green ring is principal generated by a special element, and the idempotents of the Green ring are trivial. We then study the stable Green ring (i.e., the Green ring of the stable category) of the aforementioned Hopf algebra. We show that the stable Green ring is isomorphic to the quotient of the Green ring of the Hopf algebra modulo all projective modules. Moreover, the complexified stable Green ring admits a group-like algebra structure, and hence is a bi-Frobenius algebra. Some properties of the Green ring can be extended to the Green ring of an arbitrary finite dimensional Hopf algebra. So in the second part of the thesis, we investigate the ring-theoretical properties of the Green ring of a finite dimensional Hopf algebra. This includes the descriptions of some import one-sided ideals, the nilpotent ideals and the idempotents of the Green ring. In particular, in the case where the Hopf algebra is spherical, the quotient of the Green ring modulo the objects of quantum dimension zero is semisimple. If, in addition, H is of finite representation type, then the complexified quotient ring admits a group-like algebra structure, and hence becomes a bi-Frobenius algebra.

Now let us formulate the main results with more details. Throughout, we work over a fixed algebraically closed field $\Bbbk$ of characteristic 0. Let H be a Hopf algebra, and $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ the coradical filtration of H. Suppose that the coradical H_0 is a Hopf subalgebra of H. Then each H_i is a free H_0-module. Consider $\Bbbk$ as the trivial right H_0-module. If H is generated as an algebra by H_1 and $\dim_{\Bbbk}(\Bbbk \otimes_{H_0} H_1) = n+1$, then H is called a Hopf algebra of rank n [50]. Every finite dimensional pointed Hopf algebra of rank one comes from a group datum $\mathcal{D} = (G, \chi, g, \mu)$, which is either of nilpotent type or of non-nilpotent type (see Definition 1.4.1). Denote by $H_{\mathcal{D}}$ the Hopf algebra associated to a group datum $\mathcal{D}$. $H_{\mathcal{D}}$ is said to be of nilpotent type (resp. non-nilpotent type) if the associated group datum $\mathcal{D}$ is nilpotent (resp. non-nilpotent). The thesis is organized as follows.

In Chapter 1, we recall the notions of a bi-Frobenius algebra, a group-like algebra, as well as a Nakayama algebra. As almost split sequences will play the central role in this thesis, we will recall the basic of the Auslander-Reiten theory. Roughly speaking, the Green ring (or the representation ring) of a Hopf algebra is the free abelian group generated by the isomorphism classes $[X]$ of finite dimensional representations X with the addition induced by the direct sum and the multiplication induced by the tensor product. Some existing properties of Green rings will be presented in Section 1.3. One of the fundamental properties of the Green ring of a finite dimensional Hopf algebra H is that the Green ring can be endowed with an associative non-degenerate $\mathbb{Z}$-bilinear form. More precisely, denote by $\delta_{[Z]}$ the element $[X] - [Y] + [Z]$ in the Green ring if $0 \to X \to Y \to Z \to 0$ is an almost split sequence. If Z is projective, then $\delta_{[Z]} = [Z] - [\mathrm{rad} Z]$. For H-modules X and Y, we define

$$([X], [Y]) = \dim_{\Bbbk} \mathrm{Hom}_H(X, Y^*).$$

This leads to an associative non-degenerate $\mathbb{Z}$-bilinear form on $r(H)$. Moreover, the bilinear form is symmetric if the square of the antipode of H is inner. In view of this, we obtain the following important property of the Green ring of a Hopf algebra H:

Theorem 1. [Theorem 1.3.4] *If H is a finite dimensional Hopf algebra of finite representation type, then the Green ring $r(H)$ of H is Frobenius with the Frobenius homomorphism $\phi(x) = (x, 1)$, for any $x \in r(H)$.*

In Chapter 2, we study the Green ring of a finite dimensional pointed rank one Hopf algebra $H_{\mathcal{D}}$ of nilpotent type. We refer to [71] for the published version of a part of work of this chapter. The Hopf algebra $H_{\mathcal{D}}$ of nilpotent type is a Nakayama algebra. We shall determine all finite dimensional indecomposable $H_{\mathcal{D}}$-modules up to isomorphism (Theorem 2.1.4). The almost split sequence ending at a non-projective indecomposable $H_{\mathcal{D}}$-module M is obtained by tensoring M over $\Bbbk$ on the right (or on the left) with the almost split sequence ending at the trivial module $\Bbbk$. This enables us to deduce the Clebsch-Gordan formulas for the decompositions of tensor products of indecomposable modules (Proposition 2.2.6). According to the decomposition formulas, we deliver the structure of the Green ring $r(H_{\mathcal{D}})$ of $H_{\mathcal{D}}$ in terms of generators and relations. It turns out that the Green ring $r(H_{\mathcal{D}})$ is isomorphic to a polynomial ring with one variable over the Green ring $r(\Bbbk G)$ (i.e., the Grothendieck ring of $\Bbbk G$) modulo a relation given by a Dickson polynomial $F_n(a, z)$ (see Equation (2.7) or (2.8)) multiplied with $\delta_{[\Bbbk]}$, the almost split sequence ending at the trivial module $\Bbbk$:

Theorem 2. [Theorem 2.3.4] *Let $H_{\mathcal{D}}$ be the Hopf algebra associated to a group datum $\mathcal{D} = (G, \chi, g, \mu)$ of nilpotent type. Let $r(\Bbbk G)$ be the Green ring of the group algebra $\Bbbk G$ and $r(\Bbbk G)[z]$ the polynomial ring in variable z over $r(\Bbbk G)$. Then the Green ring $r(H_{\mathcal{D}})$ is isomorphic to $r(\Bbbk G)[z]/I$, where I is the ideal of $r(\Bbbk G)[z]$ generated by the element $(1+a-z)F_n(a, z)$ and a represents the isomorphism class of the 1-dimensional simple module with the character χ^{-1}.*

It is well-known that two group algebras which are not gauge equivalent may possess the same character ring (or the Grothendieck ring). This happens to Hopf algebras as well. As a consequence of the aforementioned structure theorem, we characterize the group data, and give a sufficient condition for two pointed rank one Hopf algebras of nilpotent type which are not gauge equivalent to share the same Green ring (Proposition 2.3.6). Taft algebras give such examples (Example 2.3.7).

As mentioned above, the Green ring $r(H_{\mathcal{D}})$ possesses an associative non-degenerate

$\mathbb{Z}$-bilinear form $(-,-)$. This form is symmetric since the square of the antipode of $H_{\mathcal{D}}$ is inner. It follows that $r(H_{\mathcal{D}})$ is symmetric with a dual basis associated to certain almost split sequences. Let $\mathcal{P}$ be the free abelian group generated by the isomorphism classes of indecomposable projective $H_{\mathcal{D}}$-modules. Then $\mathcal{P}$ forms an ideal of $r(H_{\mathcal{D}})$. Denote by $\mathcal{P}^{\perp}$ the ideal of $r(H_{\mathcal{D}})$ orthogonal to $\mathcal{P}$ with respect to the form $(-,-)$. We show that $\mathcal{P}^{\perp}$ is a principal ideal generated by $\delta_{[\Bbbk]}$ (Proposition 2.4.1). The quotient ring $r(H_{\mathcal{D}})/\mathcal{P}^{\perp}$ is precisely the Grothendieck ring $G_0(H_{\mathcal{D}})$ of $H_{\mathcal{D}}$ (Remark 2.5.3). The following property of the Jacobson radical of the Green ring can be deduced from the Frobenius property of $r(H_{\mathcal{D}})$:

Theorem 3. [Theorem 2.5.2] *The Jacobson radical $J(r(H_{\mathcal{D}}))$ of $r(H_{\mathcal{D}})$ is exactly the intersection $\mathcal{P} \cap \mathcal{P}^{\perp}$.*

As a direct consequence, the square of the Jacobson radical $J(r(H_{\mathcal{D}}))$ of $r(H_{\mathcal{D}})$ is equal to zero. In order to describe the Jacobson radical $J(r(H_{\mathcal{D}}))$ of $r(H_{\mathcal{D}})$ in terms of generators, we consider the complexified Green algebra $R(H_{\mathcal{D}}) := \mathbb{C} \otimes_{\mathbb{Z}} r(H_{\mathcal{D}})$ over the field $\mathbb{C}$. We determine the dimension of the Jacobson radical of $R(H_{\mathcal{D}})$ by calculating the number of simple modules over $R(H_{\mathcal{D}})$. The rank of the Jacobson radical of $r(H_{\mathcal{D}})$ is in fact equal to the dimension of the Jacobson radical of $R(H_{\mathcal{D}})$. It turns out that the Jacobson radical of $r(H_{\mathcal{D}})$ is a principal ideal generated by a special element represented by a linear combination of projective modules:

Theorem 4. [Theorem 2.5.7] *The Jacobson radical of $r(H_{\mathcal{D}})$ is a principal ideal generated by the element $M[0,n]\theta$.*

The element $M[0,n]$ is the isomorphism class of the projective cover of the trivial module $\Bbbk$ and θ is a polynomial of the element a, where a is the isomorphism class of 1-dimensional simple $H_{\mathcal{D}}$-module with the character χ^{-1}. This explains the reason why those nilpotent elements of the Green rings of the generalized Taft algebras are of a special linear combination form of projective modules [56]. Another result that can be deduced from the Frobenius property of $r(H_{\mathcal{D}})$ is described as follows.

Theorem 5. [Theorem 2.6.2] *The Green ring $r(H_{\mathcal{D}})$ has only trivial idempotents.*

In view of this, we turn to study the idempotents of the complexified Green algebra $R(H_{\mathcal{D}}) = \mathbb{C} \otimes_{\mathbb{Z}} r(H_{\mathcal{D}})$. By lifting all primitive idempotents of the commutative semisimple algebra $R(H_{\mathcal{D}})/J(R(H_{\mathcal{D}}))$, we are able to determine the idempotents of the complexified Green algebra $R(H_{\mathcal{D}})$ completely (Theorem 2.6.5). As an application, we compute all primitive idempotents of complexified Green algebra $R(T_3)$ of the Taft algebra T_3.

In the final part of this chapter, we shall apply the obtained results to compute the Green ring of the Hopf algebra $H_{\mathcal{D}}$ of nilpotent type such that the group G in the group datum $\mathcal{D}$ is a dihedral group. In this case, the Green ring of $H_{\mathcal{D}}$ is generated over $\mathbb{Z}$ by four generators with five relations (Theorem 2.8.5).

In Chapter 3, we shall deal with a finite dimensional pointed rank one Hopf algebra $H_{\mathcal{D}}$ of non-nilpotent type. We first show that the quotient $\overline{H_{\mathcal{D}}} = H_{\mathcal{D}}/H_{\mathcal{D}}(1-e)$ is a finite dimensional pointed rank one Hopf algebra of nilpotent type, and $H_{\mathcal{D}}(1-e)$ is a semisimple subalgebra of $H_{\mathcal{D}}$, where the element e is a central idempotent of $H_{\mathcal{D}}$. Accordingly, we determine all finite dimensional indecomposable $H_{\mathcal{D}}$-modules up to isomorphism (Theorem 3.1.10). Thanks to the results obtained in Chapter 2, we are able to establish the Clebsch-Gordan formulas for the decompositions of tensor products of indecomposable $H_{\mathcal{D}}$-modules. It turns out that the decompositions depend mainly on those decompositions of simple $\Bbbk G$-modules (Proposition 3.2.3, Proposition 3.2.4 and Proposition 3.2.5).

Let $r(\overline{H_{\mathcal{D}}})$ be the Green ring of pointed rank one Hopf algebra $\overline{H_{\mathcal{D}}}$ of nilpotent type. Then $r(\overline{H_{\mathcal{D}}})$ is a subring of $r(H_{\mathcal{D}})$ (deduced from Proposition 3.1.3). Denote by $\mathcal{P}$ the free abelian group generated by the isomorphism classes of indecomposable projective $H_{\mathcal{D}}$-modules. Then $\mathcal{P}$ is an ideal of $r(H_{\mathcal{D}})$. We form a direct sum $r(\overline{H_{\mathcal{D}}}) \bigoplus \mathcal{P}$ as free $\mathbb{Z}$-modules. Then $r(\overline{H_{\mathcal{D}}}) \bigoplus \mathcal{P}$ is endowed with a commutative ring structure with multiplication given by

$$(b_1, c_1)(b_2, c_2) = (b_1 b_2, b_1 c_2 + c_1 b_2 + c_1 c_2),$$

for any $b_1, b_2 \in r(\overline{H_{\mathcal{D}}})$ and $c_1, c_2 \in \mathcal{P}$. The ring $r(\overline{H_{\mathcal{D}}}) \bigoplus \mathcal{P}$ can be regarded as a certain trivial extension of $r(\overline{H_{\mathcal{D}}})$ with respect to $\mathcal{P}$. Moreover, the Green ring $r(H_{\mathcal{D}})$ is isomorphic to a quotient ring of this trivial extension:

Theorem 6. [Theorem 3.3.1] *Let $\mathcal{I}$ be the submodule of the $\mathbb{Z}$-module $r(\overline{H_{\mathcal{D}}}) \bigoplus \mathcal{P}$ generated by the elements $(-M[i,n], M[i,n])$, for certain isomorphism classes of indecomposable projective modules $M[i,n]$. Then $\mathcal{I}$ is an ideal of the ring $r(\overline{H_{\mathcal{D}}}) \bigoplus \mathcal{P}$ and the quotient ring $(r(\overline{H_{\mathcal{D}}}) \bigoplus \mathcal{P})/\mathcal{I}$ is isomorphic to $r(H_{\mathcal{D}})$.*

The generators and relations of the Green ring $r(H_{\mathcal{D}})$ of $H_{\mathcal{D}}$ of non-nilpotent type are more complicated than those in the case of nilpotent type (Theorem 3.3.4). The main reason is that a Hopf algebra $H_{\mathcal{D}}$ of non-nilpotent type does not possess the Chevalley property whereas a Hopf algebra of nilpotent type does have. Hence the free abelian group generated by all simple $H_{\mathcal{D}}$-modules is no longer a subring of $r(H_{\mathcal{D}})$. Nevertheless, the Green ring $r(H_{\mathcal{D}})$ of $H_{\mathcal{D}}$ of non-nilpotent type has similar ring-theoretical properties to those in the case of nilpotent type. For example, the Jacobson radical of $r(H_{\mathcal{D}})$ is also equal to the intersection $\mathcal{P} \cap \mathcal{P}^{\perp}$ (Theorem 3.4.2), which is a principal ideal (Theorem 3.4.3); the Green ring $r(H_{\mathcal{D}})$ has only trivial idempotents (Theorem 3.4.4), etc.. Finally, as an example, we present explicitly the Green ring of a Radford Hopf algebra in terms of generators and relations (Theorem 3.5.2).

In Chapter 4, we study the Green ring of the stable category of a finite dimensional pointed Hopf algebra of rank one. Let $H_{\mathcal{D}}$ be such a Hopf algebra associated to a group datum $\mathcal{D}$ (of nilpotent or non-nilpotent type). Recall that the stable category $H_{\mathcal{D}}$-$\underline{\text{mod}}$, the quotient category of $H_{\mathcal{D}}$-mod modulo the morphisms factoring through projective modules, is triangulated [43] with the monoidal structure derived from that of $H_{\mathcal{D}}$-mod. The Green ring of the stable category $H_{\mathcal{D}}$-$\underline{\text{mod}}$ is called the stable Green ring of $H_{\mathcal{D}}$, and denoted $r_{st}(H_{\mathcal{D}})$. The structure of the stable Green ring $r_{st}(H_{\mathcal{D}})$ is given as follows.

Theorem 7. [Theorem 4.1.1] *The stable Green ring $r_{st}(H_{\mathcal{D}})$ is isomorphic to the quotient ring $r(H_{\mathcal{D}})/\mathcal{P}$, where $\mathcal{P}$ is the ideal of $r(H_{\mathcal{D}})$ generated by the isomorphism classes of indecomposable projective $H_{\mathcal{D}}$-modules.*

The stable Green ring $r_{st}(H_{\mathcal{D}})$ is semisimple since the Jacobson radical of $r(H_{\mathcal{D}})$ is contained in $\mathcal{P}$. One of the most interesting properties of the complexified stable Green algebra $R_{st}(H_{\mathcal{D}}) := \mathbb{C} \otimes_{\mathbb{Z}} r_{st}(H_{\mathcal{D}})$ is that it possesses a group-like algebra structure (Proposition 4.1.2). Thus, the stable Green algebra $R_{st}(H_{\mathcal{D}})$ is a bi-Frobenius algebra (Remark 1.1.8). As a consequence, many interesting properties of group-like algebras and bi-Frobenius algebras (see [28, 29, 30, 31]) can be applied to the stable Green algebra $R_{st}(H_{\mathcal{D}})$.

In the final part of this chapter, we study the Nakayama functor $\mathcal{N}$ and the syzygy functor Ω of $H_{\mathcal{D}}$-mod. We give a necessary and sufficient condition for the stable category $H_{\mathcal{D}}$-$\underline{\text{mod}}$ to be Calabi-Yau. It turns out that $H_{\mathcal{D}}$-$\underline{\text{mod}}$ is Calabi-Yau if and only if the order of $\chi(g)$ is 2, where χ and g are factors in the group datum $\mathcal{D} = (G, \chi, g, \mu)$ (Proposition 4.3.5). If the stable category $H_{\mathcal{D}}$-$\underline{\text{mod}}$ is not Calabi-Yau, we use the results of Cibils and Zhang [25] to determine the minimal, consequently all the d-th Calabi-Yau objects of $H_{\mathcal{D}}$-$\underline{\text{mod}}$ (Theorem 4.3.6). This raises naturally the following question: what is the role of the Calabi-Yau objects in the stable Green ring $r_{st}(H_{\mathcal{D}})$?

After the study of Green rings of finite dimensional pointed Hopf algebras of rank one, we turn to investigate the Green ring $r(H)$ of a finite dimensional Hopf algebra H in the general case. The first step we need to do is to describe $\delta_{[X]}$ for any indecomposable H-module X, since they not only form a basis of $r(H)$ (if H is of finite representation type), but also play a key role in the structure of $r(H)$ (e.g., Theorem 2.3.4, Proposition 2.4.1, Theorem 2.5.2, etc.). In the case where H is a finite dimensional pointed Hopf algebra of rank one, the element $\delta_{[X]}$ satisfies the following relations (Section 2.4):

$$[X]\delta_{[\Bbbk]} = \delta_{[X]} = \delta_{[\Bbbk]}[X] \text{ if } X \text{ is not projective,} \tag{I}$$

$$[X]\delta_{[\Bbbk]} = 0 = \delta_{[\Bbbk]}[X] \text{ if } X \text{ is projective.} \tag{II}$$

The relation (II) holds in general since the short exact sequence obtained by tensoring a projective module with an almost split sequence is split, whereas the relation (I) does not work in general. However, as we shall see that there are still some techniques to characterize whether or not an H-module satisfies the relation (I).

In Chapter 5, let H be an arbitrary finite dimensional Hopf algebra over the field $\Bbbk$. We begin with the study of quantum dimensions of H-modules using the techniques from [42, 80]. We first determine when an H-module is of quantum dimension zero or non-zero. In particular, we answer the question raised by Cibils in [21, Remark 5.8]: when does the trivial module $\Bbbk$ appear as a direct summand of the tensor product $M \otimes N$ ($\Bbbk \mid M \otimes N$ for short) for any two indecomposable modules M and N?

Theorem 8. [Theorem 5.1.7] *Let X and Y be two indecomposable H-modules.*

(1) $\Bbbk \mid Y \otimes X^*$ *if and only if there are isomorphisms* $f : X \to Y$ *and* $g : Y \to X^{**}$ *such that* $Tr^L_X(g \circ f) = id_{\Bbbk}$.

(2) $\Bbbk \mid X^* \otimes Y$ *if and only if there are isomorphisms* $f : X^{**} \to Y$ *and* $g : Y \to X$ *such that* $Tr^R_X(g \circ f) = id_{\Bbbk}$.

Here $\mathrm{Tr}_X^L(g\circ f)$ (resp. $\mathrm{Tr}_X^R(g\circ f)$) is the left (resp. right) quantum dimension of the map $g\circ f$ (see Section 5.1).

If H is not semisimple, we have an almost split sequence $0\to\tau(\Bbbk)\to E\xrightarrow{\sigma}\Bbbk\to 0$ ending at the trivial module $\Bbbk$. By tensoring X with the sequence, we obtain the following two exact sequences:

$$0\to\tau(\Bbbk)\otimes X\to E\otimes X\xrightarrow{\sigma\otimes id_X}X\to 0,$$

$$0\to X\otimes\tau(\Bbbk)\to X\otimes E\xrightarrow{id_X\otimes\sigma}X\to 0.$$

We show that the map $\sigma\otimes id_X$ (resp. $id_X\otimes\sigma$) is either a right almost split morphism or a split epimorphism depending on $\Bbbk\mid X\otimes X^*$ (resp. $\Bbbk\mid X^*\otimes X$) or not. If the map $\sigma\otimes id_X$ (resp. $id_X\otimes\sigma$) is a split epimorphism, it is obvious that $\delta_{[\Bbbk]}[X]=0$ (resp. $[X]\delta_{[\Bbbk]}=0$). If the map $\sigma\otimes id_X$ (resp. $id_X\otimes\sigma$) is right almost split, we show that $\delta_{[\Bbbk]}[X]=\delta_{[X]}$ (resp. $[X]\delta_{[\Bbbk]}=[X]$) (see Proposition 5.2.1 for the general case). Thus, the relations (I) and (II) can be generalized to more general form as follows.

Theorem 9. [Theorem 5.2.2] *Let X be an indecomposable H-module.*

(1) *If $\Bbbk\nmid X^*\otimes X$, then $[X]\delta_{[\Bbbk]}=0$.*

(2) *If $\Bbbk\nmid X\otimes X^*$, then $\delta_{[\Bbbk]}[X]=0$.*

(3) *If $\Bbbk\mid X^*\otimes X$, then $[X]\delta_{[\Bbbk]}=\delta_{[X]}$.*

(4) *If $\Bbbk\mid X\otimes X^*$, then $\delta_{[\Bbbk]}[X]=\delta_{[X]}$.*

Now let $\mathcal{P}_+$ (resp. $\mathcal{P}_-$) be the free abelian group generated by all indecomposable H-modules X with $\Bbbk\nmid X\otimes X^*$ (resp. $\Bbbk\nmid X^*\otimes X$). Then $\mathcal{P}_+$ (resp. $\mathcal{P}_-$) is a right (resp. left) ideal of $r(H)$ (deduced from Corollary 5.1.9). The nilpotent ideals of $r(H)$ are contained in $\mathcal{P}_+\cap\mathcal{P}_-$ (Proposition 5.2.7). For every central primitive idempotent E of $r(H)$, it is either $E\in\mathcal{P}_+\cap\mathcal{P}_-$ or $1-E\in\mathcal{P}_+\cap\mathcal{P}_-$ (Proposition 5.2.8). Let $\mathcal{J}_+$ (resp. $\mathcal{J}_-$) denote the free abelian group generated by $\delta_{[X]}$ with $\Bbbk\mid X\otimes X^*$ (resp. $\Bbbk\mid X^*\otimes X$). Then $\mathcal{J}_+$ (resp. $\mathcal{J}_-$) is a right (resp. left) ideal of $r(H)$ generated by $\delta_{[\Bbbk]}$ (deduced from Theorem 5.2.2). If H is of finite representation type, then the relations between these one-sided ideals are described as follows (Proposition 5.2.5):

$$\mathcal{J}_+=\mathcal{P}_-^{\perp}=(\mathcal{P}_+^{\perp})^*\quad\text{and}\quad\mathcal{J}_-=\mathcal{P}_+^{\perp}=(\mathcal{P}_-^{\perp})^*. \tag{III}$$

In case H is a finite dimensional pointed Hopf algebra of rank one, we have $\mathcal{P}_+ = \mathcal{P}_-$, denoted $\mathcal{P}$; and $\mathcal{J}_+ = \mathcal{J}_-$, denoted $\mathcal{J}$. In this case, the relation (III) becomes $\mathcal{J} = \mathcal{P}^\perp = (\mathcal{P}^\perp)^*$, yielding Proposition 2.4.1, see also Equation (3.13).

Similar to the bilinear form $(-,-)$ defined on the Green ring $r(H)$, we define a bilinear form $(-,-)_{st}$ on the stable Green ring $r_{st}(H)$ of H. This bilinear form is also associative and non-degenerate. Moreover, it is symmetric if S^2 is inner (Proposition 5.3.5). Thus, $r_{st}(H)$ is Frobenius if H is of finite representation type.

In the last part of this chapter, we devote ourselves to the study of the Green ring of a spherical Hopf algebra H, where the finite dimensional H-module category H-mod forms a spherical category [14]. The quantum dimension $\mathbf{d}(X)$ of H-module X defined by the pivotal structure of H-mod satisfies

$$\mathbf{d}(X) = \mathbf{d}(X^*) \text{ and } \mathbf{d}(X \otimes Y) = \mathbf{d}(X)\mathbf{d}(Y).$$

In this case, $\mathcal{P} := \mathcal{P}_+ = \mathcal{P}_-$ is a two-sided ideal of $r(H)$ generated by isomorphism classes of indecomposable modules of quantum dimension zero. Let $\mathbf{B} = \{[X_i] \mid i \in \mathbb{I}\}$ consisting of the isomorphism classes of indecomposable modules $[X_i]$ with $\mathbf{d}(X_i) \neq 0$ and $\mathbb{Z}\mathbf{B}$ the free abelian group generated by $\mathbf{B}$. Define the map T from $\mathbb{Z}\mathbf{B}$ to $\mathbb{Z}$ by letting $T(x)$ be the coefficient of $[\Bbbk]$ in the linear expression of $x \in \mathbb{Z}\mathbf{B}$. We take a similar approach to the one in [16] and establish a ring structure on $\mathbb{Z}\mathbf{B}$ as follows.

Theorem 10. [Theorem 5.4.2] *The free abelian group $\mathbb{Z}\mathbf{B}$ admits a ring structure as follows:*

(1) *The multiplication law is given by $x \cdot y = \sum_{i \in \mathbb{I}} T(xy[X_{i^*}])[X_i]$ for $x, y \in \mathbb{Z}\mathbf{B}$.*

(2) *The $\mathbb{Z}$-bilinear form $[-,-]$ on $\mathbb{Z}\mathbf{B}$ given by $[x,y] = T(xy)$ is associative, symmetric, non-degenerate and $*$-invariant.*

(3) *The quantum dimension map $\mathbf{d}$ from $\mathbb{Z}\mathbf{B}$ to $\Bbbk$ is a ring homomorphism.*

(4) *$\mathbb{Z}\mathbf{B}$ is isomorphic to the quotient ring $r(H)/\mathcal{P}$.*

Because of the isomorphism $\mathbb{Z}\mathbf{B} \cong r(H)/\mathcal{P}$, the ring $\mathbb{Z}\mathbf{B}$ can be regarded as the Green ring of certain factor category of H-mod. More precisely, since the H-module category H-mod is spherical, there is a bilinear pairing given by

$$\Theta : \mathrm{Hom}_H(X,Y) \times \mathrm{Hom}_H(Y,X) \to \Bbbk, \ \ \Theta(f,g) = \mathrm{Tr}_X^L(\theta_X \circ g \circ f).$$

A morphism f from X to Y is called negligible if $\Theta(f,g) = 0$ for any morphism g from Y to X. Let $\mathcal{J}(X,Y)$ be the set consisting of all negligible morphisms from X to Y. Then the negligible morphisms form a monoidal ideal, i.e., composing or tensoring a negligible morphism with any morphism yields a negligible morphism [60, P.118]. This leads to a factor category H-$\underline{\mathrm{mod}}$, where the objects are those of H-mod while the morphism spaces are given by the quotient:

$$\underline{\mathrm{Hom}}_H(X,Y) := \mathrm{Hom}_H(X,Y)/\mathcal{J}(X,Y).$$

The factor category H-$\underline{\mathrm{mod}}$ is an additive semisimple $\Bbbk$-linear spherical category [15] with the monoidal structure derived from that of H-mod.

Theorem 11. [Theorem 5.4.3] *The Green ring of the factor category H-$\underline{\mathrm{mod}}$ is isomorphic to the quotient ring $r(H)/\mathcal{P}$, where $\mathcal{P}$ is the ideal of $r(H)$ generated by all indecomposable modules of quantum dimension zero.*

The set of isomorphism classes of simple objects of the category H-$\underline{\mathrm{mod}}$ is not finite in general. However, the finiteness property is necessary if one wants to construct a manifold invariant from this category [15, Theorem 5.1]. It is easy to see that the category H-$\underline{\mathrm{mod}}$ possesses the finiteness property if and only if the complexified algebra $\Bbbk\mathbf{B} := \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}\mathbf{B}$ is a finite dimensional algebra. We show that the algebra $\Bbbk\mathbf{B}$ is finite dimensional if and only if it possesses a non-zero left or right integral with respect to $\mathbf{d}$ (Proposition 5.4.7). Moreover, if the algebra $\Bbbk\mathbf{B}$ is finite dimensional, it is a group-like algebra (Proposition 5.4.8), and hence a bi-Frobenius algebra.

Let $H_{\mathcal{D}}$ be a finite dimensional pointed Hopf algebra of rank one associated to the group datum $\mathcal{D} = (G, \chi, g, \mu)$. Then $H_{\mathcal{D}}$ is spherical if and only if the order of $\chi(g)$ is 2 (Remark 5.4.4). In this case, the factor category $H_{\mathcal{D}}$-$\underline{\mathrm{mod}}$ of $H_{\mathcal{D}}$-mod modulo all negligible morphisms is equivalent to $\Bbbk\widetilde{G}$-mod and the Green ring $\mathbb{Z}\mathbf{B}$ of $H_{\mathcal{D}}$-$\underline{\mathrm{mod}}$ is isomorphic to the Grothendieck ring $G_0(\Bbbk\widetilde{G})$, where the group $\widetilde{G}$ is equal to G or a quotient of G depending on whether the group datum $\mathcal{D}$ is of nilpotent type or of non-nilpotent type.

Notations and conventions

Throughout, $\Bbbk$ is a fixed algebraically closed field with characteristic 0. All vector spaces, algebras (associative with identities) and modules are assumed to be over $\Bbbk$ unless otherwise stated. The unfinished dim and tensor $\otimes$ mean $\dim_{\Bbbk}$ and $\otimes_{\Bbbk}$ respectively. The letters $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ stand for the sets of natural numbers, integers and complex numbers respectively. Given a positive integer n, we let $\mathbb{Z}_n$ be $\mathbb{Z}/(n)$. For a finite set B, we denote by $|B|$ the cardinality of B, and by sp$B$ the vector space spanned over $\Bbbk$ by the set B.

We define the q-binomial coefficients for $0 \leq k \leq n$ by

$$\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!},$$

where $(n)_q = 1 + q + \cdots + q^{n-1}$ and $(n)_q! = (1)_q(2)_q \cdots (n)_q$.

Without specifically stated, a Hopf algebra H means a finite dimensional Hopf algebra with an antipode S. The group of group-like elements of H is denoted by $G(H)$. The trivial H-module will be denoted by $\Bbbk$. For an element $a \in H$, we use the Sweedler's notation $\triangle(a) = \sum a_1 \otimes a_2$ for the comultiplication of H.

For a Hopf algebra H and a finite dimensional H-module M, we write M^*, rad$M$, soc$M$ and $\mathrm{P}(M)$ for the dual, the radical, the socle and the projective cover of M respectively. The dual space $M^* = \mathrm{Hom}(M, \Bbbk)$ is an H-module with the H-module structure given by $(hf)(v) = f(S(h)v)$ for $h \in H$, $f \in M^*$ and $v \in M$. Given any two H-modules M and N, the notation $M \mid N$ (resp. $M \nmid N$) means that M is (resp. is not) a direct summand of N.

Chapter 1

Preliminaries

In this chapter, we first recall the definitions of a Frobenius algebra, a bi-Frobenius algebra and a group-like algebra. After that, we shall collect concepts and results from the Auslander-Reiten theory. The main theme of this thesis is about Green rings of Hopf algebras. So we spend a bit more time not only on recalling definitions but also on working out some basic properties of Green rings which will be used in other chapters. In the final part, we recall the construction and classification of finite dimensional pointed Hopf algebras of rank one by means of group data.

1.1 Bi-Frobenius algebras

1.1.1 Frobenius algebras

Frobenius algebras occur in many different fields of mathematics, such as topological quantum field theory [2], Hopf algebras and quantum Yang-Baxter equations [18, 55]. In the following, the notion of a Frobenius algebra is defined directly over a field $\Bbbk$, although it can also be defined over a commutative ring (e.g., [51, 58]).

Let A be a finite dimensional $\Bbbk$-algebra. We denote by the dual $A^* = \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$. Then A^* has a natural A-A-bimodule structure given by

$$(a \rightharpoonup f \leftharpoonup b)(c) = f(bca), \text{ for } a, b, c \in A, \ f \in A^*.$$

Definition 1.1.1. (cf. [22, 23]) The pair (A, ϕ) is called a *Frobenius algebra* provided that $\phi \in A^*$ such that the right A-module morphism $\theta_A : A \to A^*,\ a \mapsto \phi \leftharpoonup a$ is bijective; or equivalently, the left A-module morphism ${}_A\theta : A \to A^*,\ a \mapsto a \rightharpoonup \phi$ is bijective.

The linear form ϕ is called a *Frobenius homomorphism*. Moreover, A is a *symmetric algebra* provided that A is isomorphic to A^* as A-A-bimodules.

Remark 1.1.2. If (A, ϕ) is a Frobenius algebra, then $\langle a, b\rangle := \phi(ab)$ for $a, b \in A$, is a non-degenerate associative bilinear form over A. Conversely, if A is equipped with a non-degenerate associative bilinear form $\langle -, -\rangle$, then $\phi := \langle 1, -\rangle$ is a Frobenius homomorphism of A [2, Proposition 1]. Accordingly, one of the equivalent definitions of a Frobenius algebra is that A is Frobenius if and only if A is equipped with a non-degenerate bilinear form $\langle -, -\rangle : A \times A \to \Bbbk$ satisfying the associative $\langle ab, c\rangle = \langle a, bc\rangle$, for any $a, b, c \in A$. Moreover, if the bilinear form is symmetric $\langle a, b\rangle = \langle b, a\rangle$ for $a, b \in A$, then A is a symmetric algebra.

We refer to [29, 51, 58] for the following basic properties of Frobenius algebras. The $\Bbbk$-linear map θ_A given in Definition 1.1.1 induces the $\Bbbk$-linear isomorphism

$$\Theta : A \otimes A \xrightarrow{id \otimes \theta_A} A \otimes A^* \cong \mathrm{End}_{\Bbbk}(A).$$

Hence there exists a unique element $\sum_{i=1}^n a_i \otimes b_i \in A \otimes A$ such that $\Theta(\sum_{i=1}^n a_i \otimes b_i) = id_A$. The pair $\{a_i, b_i \mid 1 \le i \le n\}$ is called a *dual basis* of (A, ϕ) and $\sum_{i=1}^n a_i \otimes b_i$ is the *Casimir element* of (A, ϕ). Moreover, (A, ϕ) is symmetric if and only if

$$\sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^n b_i \otimes a_i.$$

According to the map Θ given above, we have the following:

$$x = \sum_{i=1}^n a_i \phi(b_i x) = \sum_{i=1}^n a_i \langle b_i, x\rangle, \text{ for } x \in A, \tag{1.1}$$

or equivalently,

$$x = \sum_{i=1}^n \phi(x a_i) b_i = \sum_{i=1}^n \langle x, a_i\rangle b_i, \text{ for } x \in A. \tag{1.2}$$

In fact, both of them is equivalent to

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, a_i\rangle\langle b_i, y\rangle \tag{1.3}$$

for any $x, y \in A$ (cf. [58]).

The following are basic examples of Frobenius algebras over the field $\Bbbk$ (cf. [22]).

Example 1.1.3. Let H be a finite dimensional Hopf algebra over the field $\Bbbk$. Let $\lambda \in H^*$ be a non-zero left integral and $\Lambda \in H$ such that $\lambda(\Lambda) = 1$. Then (H, λ) is a Frobenius algebra with the dual basis $\{S(\Lambda_1), \Lambda_2\}$, where $\triangle(\Lambda) = \sum \Lambda_1 \otimes \Lambda_2$. In a similar fashion, one can see that if $\gamma \in H^*$ is a non-zero right integral, then there exists a left integral $\Gamma \in H$ such that $\gamma(\Gamma) = 1$. Then (H, γ) is a Frobenius algebra with the dual basis $\{\Gamma_1, S(\Gamma_2)\}$, where $\triangle(\Gamma) = \sum \Gamma_1 \otimes \Gamma_2$. As shown in [57] that H is symmetric if and only if H is unimodular and the square of antipode is inner.

Let (A, ϕ) be a Frobenius algebra with the dual basis $\{a_i, b_i \mid 1 \leq i \leq n\}$. Any algebra morphism $\varepsilon : A \to \Bbbk$ is called an *augmentation* of A. Suppose that (A, ϕ) has an augmentation ε, then one of the elements in the following set

$$\int_A^l = \{t \in A \mid at = \varepsilon(a)t, \text{ for } a \in A\}$$

is called a *left integral* of A. Similarly, one of the elements in the following set

$$\int_A^r = \{t \in A \mid ta = \varepsilon(a)t, \text{ for } a \in A\}$$

is called a *right integral* of A.

For an augmentation ε of A, there is a unique $\Lambda_\varepsilon \in A$ such that $\langle \Lambda_\varepsilon, -\rangle = \varepsilon$. By the equations (1.1) and (1.2), one can show that a right integral of A with respect to ε is $\Lambda_\varepsilon = \sum_{i=1}^n \varepsilon(a_i)b_i$ and $\int_A^r = \Bbbk\Lambda_\varepsilon \cong \Bbbk$. Similarly, a left integral of A with respect to ε is ${}_\varepsilon\Lambda = \sum_{i=1}^n \varepsilon(b_i)a_i$ and $\int_A^l = \Bbbk_\varepsilon\Lambda \cong \Bbbk$.

1.1.2 Bi-Frobenius algebras

Let C be a coalgebra over the field $\Bbbk$. Then C has a natural structure of left and right C^*-module under the left action $f \rightharpoonup c = \sum c_1 f(c_2)$, and the right action

$c \leftharpoonup f = \sum f(c_1)c_2$, for any $f \in C^*$ and $c \in C$ with $\triangle(c) = \sum c_1 \otimes c_2$. Moreover, for any $c \in C$, the induced maps $c \leftharpoonup : C^* \to C$ and $\rightharpoonup c : C^* \to C$ are morphisms of right and left C^*-modules respectively.

Definition 1.1.4. (cf. [29, 31]) A *Frobenius coalgebra* is a pair (C, t) where C is a finite dimensional coalgebra and $t \in C$ such that the morphism $t \leftharpoonup : C^* \to C,\ f \mapsto t \leftharpoonup f$ is bijective; or equivalently, the morphism $\rightharpoonup t : C^* \to C,\ f \mapsto f \rightharpoonup t$ is bijective.

The notion of a Frobenius coalgebra has a nice characterization that is analogue to the characterizations of a Frobenius algebras [28, 30].

The concept of a bi-Frobenius algebra was introduced by Doi and Takeuchi in [31] and further investigated in [30, 28] as a natural generalized of finite dimensional Hopf algebras.

Definition 1.1.5. (cf. [30]) Let H be a finite dimensional algebra and coalgebra over the field $\Bbbk$, $\phi \in H^*$, $t \in H$. Define a map S by

$$S : H \to H,\ S(x) = t \leftharpoonup (x \rightharpoonup \phi) = \phi(t_1 x)t_2.$$

The quadruple (H, ϕ, t, S) is called a *bi-Frobenius algebra* if the following hold:

(BF1) The counit ε of the coalgebra H is an algebra morphism.

(BF2) The unity 1 is a group-like element of H.

(BF3) (H, ϕ) is a Frobenius algebra.

(BF4) (H, t) is a Frobenius coalgebra.

(BF5) S is an anti-algebra and anti-coalgebra morphism, i.e., $S(ab) = S(b)S(a)$, $S(1) = 1$ and $\Delta(S(a)) = \sum S(a_2) \otimes S(a_1)$, $\varepsilon(S(a)) = \varepsilon(a)$.

The map S given above is necessarily bijective [31], it is called the *antipode* of the bi-Frobenius algebra H. It does not mean a convolution inverse of identity. This is true in the particular situation of Hopf algebras. A dual basis of (H, ϕ, t, S) is given by $\{S^{-1}(t_2), t_1\}$ [29]. Since H is necessary finite dimensional, the $\Bbbk$-linear dual H^* is also an algebra and coalgebra. The comultiplication in H^* is given by

$$\triangle(f)(a \otimes b) = f(ab),$$

for $f \in H^*$ and $a, b \in H$. It is easy to see that (H^*, t, ϕ, S^*) becomes a bi-Frobenius algebra. We call it the *dual bi-Frobenius algebra* of H.

Example 1.1.6. Let H be a finite dimensional Hopf algebra. Choose the right integral $\gamma \in H^*$ and the left integral $\Gamma \in H$ such that $\gamma(\Gamma) = 1$. Then (H, γ, Γ, S) becomes a bi-Frobenius algebra.

It is interesting to construct bi-Frobenius algebras that are not Hopf algebras. Using known results on the existence of large Hadamard matrices, the author in [44] constructs a class of bi-Frobenius algebras of arbitrarily large dimension satisfying the additional condition

$$S * id = id * S = \varepsilon \tag{1.4}$$

and that are not Hopf algebras. This family of bi-Frobenius algebras satisfying the condition (1.4) is also studied in [67]. There are many other approaches to construct bi-Frobenius algebras that are not Hopf algebras, e.g., [76, 77]. As we shall see that one of main results of this thesis is that the stable Green algebras of certain finite dimensional Hopf algebras are bi-Frobenius algebras that are not Hopf algebras.

1.1.3 Group-like algebras

The notion of a group-like algebra was introduced by Doi in [28] generalizing the group algebra of a finite group and a scheme ring (Bose-Mesner algebra) of a non-commutative association scheme.

Definition 1.1.7. (cf.[28]) Let $(A, \varepsilon, \mathbf{b}, *)$ be a quadruple, where A is a finite dimensional algebra over a field $\Bbbk$ with unit 1, ε is an algebra morphism from A to $\Bbbk$, the set $\mathbf{b} = \{b_i \mid i \in I\}$ is a $\Bbbk$-basis of A such that $0 \in I$ and $b_0 = 1$, and $*$ is an involution of the index set I. Then $(A, \varepsilon, \mathbf{b}, *)$ is called a group-like algebra if the following hold:

(G1) $\varepsilon(b_i) = \varepsilon(b_{i^*}) \neq 0$ for all $i \in I$.

(G2) $p_{ij}^k = p_{j^*i^*}^{k^*}$ for all $i, j, k \in I$, where p_{ij}^k is the structure constant for $\mathbf{b}$ defined by $b_i b_j = \sum_{k \in I} p_{ij}^k b_k$.

(G3) $p_{ij}^0 = \delta_{i,j^*}\varepsilon(b_i)$ for all $i, j \in I$.

Group-like algebras have some special properties (e.g., [28]). Group-like algebras of dimension 2 and 3 have been determined in [28]. For group-like algebras of dimension 4, we refer to [29].

Remark 1.1.8. [28, Remark 3.2] Let $(A, \varepsilon, \mathbf{b}, *)$ be a group-like algebra. Then A becomes a coalgebra by defining $\triangle(b_i) = \frac{1}{\varepsilon(b_i)} b_i \otimes b_i$. Let $\phi \in A^*$ such that $\phi(b_i) = \delta_{0,i}$ and $t = \sum_{i \in I} b_i$. Define the $\Bbbk$-linear map S from A to itself given by $S(b_i) = b_{i^*}$ for any $i \in I$. Then (A, ϕ, t, S) becomes a bi-Frobenius algebra with the dual basis $\{b_i, \frac{b_{i^*}}{\varepsilon(b_i)} \mid i \in I\}$.

If a group-like algebra is also a Hopf algebra, then it needs to be a group algebra [44, Corollary 2]. Because of this, a bi-Frobenius algebra coming from a group-like algebra is not a Hopf algebra if the algebra itself is not a group algebra.

1.2 Auslander-Reiten Theory

The aim of this section is to collect several results about Auslander-Reiten theory which are needed in this thesis. For these concepts, we refer to books [5, 6].

1.2.1 Auslander-Reiten translate

Let A be a finite dimensional algebra over $\Bbbk$ and A-mod (resp. mod-A) the finite dimensional left (resp. right) module category of A. There are several ingredients that go into the topic of Auslander-Reiten translate of A-mod. One is the functor $D : A\text{-mod} \to \text{mod-}A$ which is defined as $DX = \mathrm{Hom}_{\Bbbk}(X, \Bbbk)$, for $X \in A$-mod. We also want to use another functor $\mathrm{Hom}_A(-, A) : A\text{-mod} \to \text{mod-}A$. If M is a left A-module, then $\mathrm{Hom}_A(M, A)$ is a right A-module given by $(fa)(u) = f(u)a$ for $a \in A$, $u \in M$ and $f \in \mathrm{Hom}_A(M, A)$.

Let M be in A-mod and $P_0 \xrightarrow{p_0} M \to 0$ the projective cover of M. We denote by $P_1 \xrightarrow{p_1} \ker p_0$ the projective cover of $\ker p_0$. Then the sequence $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ is called a *minimal projective presentation* of M. One can continue the process forever and get what is called a *minimal projective resolution*, but we are only interested in the P_1 and P_0 terms.

Applying the functor $\mathrm{Hom}_A(-, A)$ to $P_1 \xrightarrow{p_1} P_0$, one obtains a right A-module map $p_1^* : \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A)$. The *transpose* of M is defined to be $\mathrm{Tr}(M) := \mathrm{coker}(p_1^*)$ and the *Auslander-Reiten translate* of M is $\mathrm{DTr}(M)$, the dual of transpose of left A-module M.

1.2.2 Almost split sequences

In this subsection we give an introduction to almost split sequences, a special type of short exact sequences of modules which play a central role in the representation theory of artin algebras.

Let X and Y be two A-modules. The morphism $f : M \to N$ is a *split monomorphism* if there exists $g : N \to M$ such that $g \circ f = id_M$, and $f : M \to N$ is a *split epimorphism* if there exists $g : N \to M$ such that $f \circ g = id_N$.

In the following, we introduce some special morphisms, called left and right almost split morphisms, which gives rise in a natural way to the notion of an almost split sequence.

Definition 1.2.1. The map $f : M \to N$ is called *left almost split* if f is not a split monomorphism and if there is $g : M \to X$ with g not a split monomorphism, then there is $h : N \to X$ such that $h \circ f = g$. Dually, $f : M \to N$ is called *right almost split* if f is not split epimorphism and if there is $g : Y \to N$ with g not split epimorphism, then there is $h : Y \to M$ such that $f \circ h = g$.

We also need the notion of minimality.

Definition 1.2.2. The map $f : M \to N$ is called *left minimal* if for all $h : N \to N$ with $h \circ f = f$, then h is an isomorphism. Dually, $f : M \to N$ is called *right minimal* if for all $h : M \to M$ with $f \circ h = f$, then h is an isomorphism.

Finally, we say that $f : M \to N$ is *left minimal almost split* if f is both left minimal and left almost split. Similarly, we have the notion of *right minimal almost split*.

Definition 1.2.3. A short exact sequence $0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$ is called *almost split* if f is left minimal almost split and g is right minimal almost split.

The following proposition [5, Proposition 1.14, ChV] gives many equivalent conditions for a short exact sequence to be almost split.

Proposition 1.2.4. *The following are equivalent for a short exact sequence* $0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$.

(1) *The sequence is an almost split sequence.*

(2) *The morphism f is left minimal almost split.*

(3) *The morphism g is right minimal almost split.*

(4) *X is indecomposable and g is right almost split.*

(5) *Y is indecomposable and f is left almost split.*

(6) *X is isomorphic to $DTrY$ and g is right almost split.*

(7) *Y is isomorphic to $TrDX$ and f is left almost split.*

We end with an introduction to the existence and uniqueness of almost split sequence.

Theorem 1.2.5. *[5, Theorem1.15, ChV] We have the following existence of almost split sequence:*

(1) *If Y is an indecomposable non-projective module, then there is an almost split sequence $0 \to X \to M \to Y \to 0$.*

(2) *If X is an indecomposable non-injective module, then there is an almost split sequence $0 \to X \to M \to Y \to 0$.*

An almost split sequence is determined uniquely by either of its end terms in the following sense (cf. [5, Theorem1.16, ChV]).

Theorem 1.2.6. *The following are equivalent for two almost split sequences $0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$ and $0 \to X' \xrightarrow{f'} M' \xrightarrow{g'} Y' \to 0$.*

(1) $X \cong X'$.

(2) $Y \cong Y'$.

(3) *The two sequences are isomorphic (i.e., there is a commutative diagram of the following form with the vertical morphisms isomorphisms)*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X' \arrow[r, "f'"] & M' \arrow[r, "g'"] & Y' \arrow[r] & 0. \end{tikzcd}$$

As we see in the following, it is easy to determine whether a special short exact sequence is almost split or not.

Proposition 1.2.7. *[5, Corollary 2.4, ChV] Let Y be indecomposable with $\underline{End}_A(Y)$ a division ring, where $\underline{End}_A(Y)$ is the quotient of $End_A(Y)$ modulo all endomorphisms of Y which factor through a projective module. Then the following are equivalent for a short exact sequence $\delta : 0 \to DTr(Y) \to M \to Y \to 0$.*

(1) *δ is almost split.*

(2) *δ does not split.*

(3) *M is not isomorphic to $Y \bigoplus DTr(Y)$.*

The dual version of Proposition 1.2.7 is given as follows.

Proposition 1.2.8. *Let X be indecomposable with $\overline{End}_A(X)$ a division ring, where $\overline{End}_A(X)$ is the quotient of $End_A(X)$ modulo all endomorphisms of X which factor through an injective module. Then the following are equivalent for a short exact sequence $\delta : 0 \to X \to M \to TrD(X) \to 0$.*

(1) *δ is almost split.*

(2) *δ does not split.*

(3) *M is not isomorphic to $X \bigoplus TrD(X)$.*

1.2.3 Nakayama algebras

Nakayama algebras are of considerable interest because next to semisimple algebras they are the best understood artin algebras. Since Nakayama algebras are defined in terms of uniserial modules, we start this subsection with a discussion of these modules.

For a finite dimensional A-module M, the sequence

$$M \supseteq \mathrm{rad}M \supseteq \mathrm{rad}^2 M \supseteq \cdots \supseteq 0$$

is called the *radical series* of M. Because M has finite dimension as a $\Bbbk$-vector space, the series has finite composition length, and the least positive integer m such that $\mathrm{rad}^m M = 0$ is called the *radical length* of M and is denoted by $\mathrm{rl}(M)$.

The dual notion is that of the socle series of M. Recall that the *socle* of M is the sum of all simple submodules of M. Let $\mathrm{soc}^0 M = 0$. If $\mathrm{soc}^i M$ is already defined, then $\mathrm{soc}^{i+1} M = \pi^{-1}(\mathrm{soc}(M/\mathrm{soc}^i M))$, where $\pi : M \to M/\mathrm{soc}^i M$ denoted the canonical epimorphism. Thus, one obtains an sequence

$$0 = \mathrm{soc}^0 M \subseteq \mathrm{soc}M \subseteq \mathrm{soc}^2 M \subseteq \cdots \subseteq M.$$

Because M has finite composition length, there is a least positive integer m such that $\mathrm{soc}^m M = M$. The number m is called *socle length* of M and is denoted by $\mathrm{sl}(M)$.

In general, the radical and socle series of M do not coincide. However, we always have $\mathrm{sl}(M) = \mathrm{rl}(M)$ (cf. [6, Proposition 1.3]), which is called the *Loewy length* of M. The Loewy length of M is not more then $\mathrm{l}(M)$, the length of the composition series of M.

A left A-module M is called *uniserial* if it has a unique composition series. Obviously, if M is uniserial, so is a submodule of M and a quotient of M. If M is uniserial, then it is indecomposable since it has only one simple socle. We have the following useful characterizations of uniserial modules [5, Proposition 2.1, ChIV].

Proposition 1.2.9. *The following are equivalent for a left A-module M.*

(1) *M is uniserial.*

(2) *There is only one composition series of M.*

(3) *The radical series of M is a composition series of M.*

(4) *The socle series of M is a composition series of M.*

(5) *The Loewy length of M is exactly the length of M, namely, $l(M) = rl(M)$.*

If both indecomposable projective and indecomposable injective A-modules are uniserial, then A is a *Nakayama algebra*.

The following description of Auslander-Reiten translate for uniserial modules over Nakayama algebra is used in this thesis.

Proposition 1.2.10. *[5, Proposition 2.6, ChIV] Suppose M is a uniserial non-projective module of length n over a Nakayama algebra.*

(1) *TrM and $DTrM$ are uniserial.*

(2) *$l(M) = l(DTrM)$.*

(3) *If $P \to M$ is a projective cover of M, then $DTrM \cong radP/rad^{n+1}P$.*

1.3 Bilinear forms on Green rings

Let H be an arbitrary finite dimensional Hopf algebra over the field $\Bbbk$ (see [49, 61] for standard facts about Hopf algebras) and $F(H)$ the free abelian group generated by the isomorphism classes of finite dimensional left H-modules. The abelian group $F(H)$ becomes a ring if we endow $F(H)$ with a multiplication given by the tensor product $[M][N] = [M \otimes N]$. The *Green ring* (or *representation ring*) $r(H)$ of the Hopf algebra H is defined to be the quotient ring of $F(H)$ modulo the relations $[M \oplus N] = [M] + [N]$, for any two H-modules M and N. The identity of the associative ring $r(H)$ is represented by the trivial H-module $[\Bbbk]$. Note that $r(H)$ has a $\mathbb{Z}$-basis $\mathrm{ind}(H)$ consisting of the isomorphism classes of finite dimensional indecomposable H-modules, see [27, 70, 78].

The *Grothendieck ring* $G_0(H)$ of the Hopf algebra H is the quotient ring of $F(H)$ modulo exact sequences of H-modules $0 \to X \to Y \to Z \to 0$, i.e., $[Y] = [X] + [Z]$. The Grothendieck ring $G_0(H)$ possesses a basis given by the isomorphism classes of simple H-modules, see [5, P.5]. Both $r(H)$ and $G_0(H)$ are augmented $\mathbb{Z}$-algebras with the dimension augmentation. Moreover, there is a natural ring epimorphism from $r(H)$ to $G_0(H)$ given by $[M] \mapsto [M]$ for any finite dimensional H-module M. If H is semisimple, then the ring epimorphism is the identity map.

For any indecomposable H-module Z, if Z is not projective, there is a unique almost split sequence of H-modules $0 \to X \to Y \to Z \to 0$ ending at Z, we follow the notation given in [5, Section 4, ChVI] and denote by $\delta_{[Z]}$ the following element in $r(H)$:

$$\delta_{[Z]} = [X] - [Y] + [Z].$$

If Z is indecomposable projective, then we write $\delta_{[Z]} = [Z] - [\mathrm{rad} Z]$.

We define

$$\langle [X], [Y] \rangle = \dim \mathrm{Hom}_H(X, Y),$$

for any two H-modules X and Y. Then $\langle -, - \rangle$ induces a $\mathbb{Z}$-bilinear form on $r(H)$. The following result can be found from [5, Proposition 4.1, ChVI].

Lemma 1.3.1. *For any indecomposable H-module X and $x \in r(H)$, the following hold in $r(H)$:*

(1) $\langle [X], \delta_{[M]} \rangle = 1$ *if $X \cong M$, and 0 otherwise.*

(2) $x = \sum_{[M]\in ind(H)} \langle x, \delta_{[M]} \rangle [M]$.

(3) $\{\delta_{[M]} \mid [M] \in ind(H)\}$ *is linearly independent.*

(4) *H is of finite representation type if and only if $\{\delta_{[M]} \mid [M] \in ind(H)\}$ is a basis of $r(H)$.*

(5) *H is of finite representation type if and only if $\{\delta_{[M]} \mid [M] \in ind(H)$ and M not projective $\}$ is a basis of the kernel of the natural ring epimorphism from $r(H)$ to $G_0(H)$.*

Let $*$ be a $\mathbb{Z}$-linear map from $r(H)$ to itself given by the dual: $[M]^* = [M^*]$. Then $*$ is an anti-automorphism of $r(H)$. The inverse of $*$ under composition is denoted by $\star$, namely, $\star * = * \star = id$. Obviously, if S^2 is an inner automorphism of H, then $*$ is an involution of $r(H)$. The $\mathbb{Z}$-bilinear form $\langle -, - \rangle$ defined above is neither associative nor symmetric. However, we may modify it as follows:

$$([X], [Y]) := \langle [X], [Y]^* \rangle = \dim \mathrm{Hom}_H(X, Y^*),$$

for any two H-modules X and Y. Then $(-, -)$ extends to a $\mathbb{Z}$-bilinear form on $r(H)$.

Lemma 1.3.2. *Let X, Y and Z be H-modules. The bilinear form $(-, -)$ satisfies the following properties:*

(1) $([X][Y], [Z]) = ([X], [Y][Z])$.

(2) $([X], [Y]) = ([Y]^{**}, [X])$. *Thus,* $([X], [Y]) = ([Y], [X])$ *if S^2 of H is inner.*

(3) $([X]^*, [Y]) = \langle [Y], [X] \rangle$.

Proof. (1) The associativity follows from the following:

$$\begin{aligned}
([X][Y], [Z]) &= \dim \mathrm{Hom}_H(X \otimes Y, Z^*) \\
&= \dim(Z^* \otimes (X \otimes Y)^*)^H \\
&= \dim((Y \otimes Z)^* \otimes X^*)^H \\
&= \dim \mathrm{Hom}_H(X, (Y \otimes Z)^*) \\
&= ([X], [Y][Z]).
\end{aligned}$$

(2) The $\Bbbk$-linear isomorphism $\mathrm{Hom}_H(X, Y^*) \cong \mathrm{Hom}_H(Y^{**}, X^*)$ (cf. [57]) implies that $([X], [Y]) = ([Y]^{**}, [X])$. If S^2 is inner, then the anti-automorphism $*$ of $r(H)$ is an involution. Hence $([X], [Y]) = ([Y], [X])$.

(3) Since $\star$ is the inverse of $*$, we have $([X]^*, [Y]) = ([X]^{\star\star\star}, [Y])$. By Part (2), $([X]^{\star\star\star}, [Y]) = ([Y], [X]^\star)$. According to the definition of $(-,-)$, we have $([Y], [X]^\star) = \langle [Y], [X]^{\star\star} \rangle = \langle [Y], [X] \rangle$. □

Denote by $\delta^*_{[M]}$ (resp. $\delta^\star_{[M]}$) the image of $\delta_{[M]}$ under the anti-automorphism $*$ (resp. $\star$) of $r(H)$ for any indecomposable H-module M.

Lemma 1.3.3. *For any indecomposable H-module X and $x \in r(H)$, the following hold in $r(H)$:*

(1) $(\delta^*_{[M]}, [X]) = 1$ *if $X \cong M$, and 0 otherwise.*

(2) $x = \sum_{[M] \in ind(H)} (\delta^*_{[M]}, x)[M]$.

(3) *The form $(-,-)$ (resp. $\langle -,- \rangle$) is non-degenerate.*

Proof. Part (1) and Part (2) follow from Lemma 1.3.1 and Lemma 1.3.2. To check Part (3), we have that if $(x, [M]) = 0$ for any $[M] \in \mathrm{ind}(H)$, then $(x, \delta^\star_{[M]}) = 0$ for any $[M] \in \mathrm{ind}(H)$. This implies that $(\delta^*_{[M]}, x) = 0$ by Lemma 1.3.2 (2). It follows from Part (2) that $x = 0$. If $([M], x) = 0$ for any $[M] \in \mathrm{ind}(H)$, then it is also $x = 0$ by Lemma 1.3.2 (2). □

Now we are ready to state main result of this subsection.

Theorem 1.3.4. *If H is a finite dimensional Hopf algebra of finite representation type, then the Green ring $r(H)$ of H is Frobenius with the Frobenius homomorphism $\phi(x) = (x, 1)$, for any $x \in r(H)$. The Nakayama automorphism of $r(H)$ is $\mathcal{N} = \star^2$. Moreover, $\mathcal{N} = id$ and $r(H)$ is symmetric if S^2 is inner.*

Proof. It follows from Lemma 1.3.2 (1) and Lemma 1.3.3 that the Green ring $r(H)$ is endowed with an associative and non-degenerate $\mathbb{Z}$-bilinear form $(-,-)$. Hence the Green ring $r(H)$ is Frobenius with the Frobenius homomorphism $\phi(x) = (x, 1)$, for $x \in r(H)$. For any $x, y \in r(H)$, the Nakayama automorphism $\mathcal{N}$ of $r(H)$ is defined by $\phi(xy) = \phi(y\mathcal{N}(x))$. By Lemma 1.3.2, we have

$$\phi(yx^{\star\star}) = (y, x^{\star\star}) = (x^{\star\star\star\star}, y) = (x, y) = \phi(xy).$$

This implies that $\mathcal{N}(x) = x^{\star\star}$. Consequently, $\mathcal{N} = \star\star$. If S^2 is inner, then $\mathcal{N} = id$. In this case, the bilinear form $(-,-)$ is symmetric and $r(H)$ is a symmetric $\mathbb{Z}$-algebra. □

Remark 1.3.5. For any finite dimensional Hopf algebra H of finite representation type, by Theorem 1.3.4, the Green ring $r(H)$ is Frobenius with the dual basis $\{\delta^*_{[M]}, [M] \mid [M] \in \mathrm{ind}(H)\}$ with respect to the bilinear form $(-,-)$. One of properties of dual basis we will use later is that Lemma 1.3.3 (2) is equivalent to

$$x = \sum_{[M]\in\mathrm{ind}(H)} (x, [M])\delta^*_{[M]}, \tag{1.5}$$

for this we refer to [58]. Moreover, if H is semisimple, then S^2 is inner (cf. [54]) and

$$\delta^*_{[M]} = ([M] - [\mathrm{rad}M])^* = [M]^* = [M^*].$$

In this case, $r(H) = G_0(H)$, which is symmetric (cf. Theorem 1.3.4) and semisimple (cf. [81, Lemma 2]) with the dual basis $\{[M]^*, [M] \mid [M] \in \mathrm{ind}(H)\}$.

1.4 Classification of pointed Hopf algebras of rank one

A Hopf algebra is *pointed*, if all its simple left or right comodules are 1-dimensional. That is, the coradical of the Hopf algebra is a group algebra [61].

Let H_0 be the coradical of Hopf algebra H. We define

$$H_i = \Delta^{-1}(H \otimes H_{i-1} + H_0 \otimes H),$$

for $i \geq 1$. Then $\{H_i \mid i \geq 0\}$ is called the *coradical filtration* of Hopf algebra H. If H is pointed, then its coradical filtration is a Hopf algebra filtration (cf. [61, Lemma 5.2.8]). Coradical filtration is important in the classification of pointed Hopf algebras, see e.g., [7, 8].

Let $\{H_i \mid i \geq 0\}$ be the coradical filtration of H. Assume that the coradical H_0 is a Hopf subalgebra of H. Then each H_i is a free H_0-module. Consider $\Bbbk$ as the trivial right H_0-module. If H is generated as an algebra by H_1 and $\dim(\Bbbk \otimes_{H_0} H_1) = n + 1$, then H is called a *Hopf algebra of rank* n (cf. [50, 75]) .

Krop and Radford defined the notion of rank so as to give a measure of complexity for Hopf algebras. One of the simplest pointed Hopf algebras mentioned here is so-called finite dimensional pointed Hopf algebras of rank one. The (generalized) Taft algebras and the half quantum group [41] are typical examples of such Hopf algebras. Every finite dimensional pointed Hopf algebra of rank one can be obtained from a group datum stated as follows.

Definition 1.4.1. (cf. [50, 20]) A quadruple $\mathcal{D} = (G, \chi, g, \mu)$ is called a *group datum* if G is a finite group, g an element in the center of G, χ a $\Bbbk$-linear character of G, and $\mu \in \Bbbk$ subject to $\chi^n = 1$ or $\mu(g^n - 1) = 0$, where n is the order of $q := \chi(g)$. If $\mu(g^n - 1) = 0$, then the group datum $\mathcal{D}$ is said to be of *nilpotent type*. Otherwise, it is of *non-nilpotent type* .

Given a group datum $\mathcal{D} = (G, \chi, g, \mu)$. Let $H_{\mathcal{D}}$ be an associative algebra generated by y and all h in G such that $\Bbbk G$ is a subalgebra of $H_{\mathcal{D}}$ and

$$y^n = \mu(g^n - 1), \ yh = \chi(h)hy, \tag{1.6}$$

for any $h \in G$. Then the algebra $H_{\mathcal{D}}$ is finite dimensional with a canonical $\Bbbk$-basis

$\{y^i h \mid h \in G,\ 0 \leq i \leq n-1\}$. Thus, $\dim H_{\mathcal{D}} = n|G|$.

Remark 1.4.2. If the order of $\chi(g)$ is $n = 1$, then $H_{\mathcal{D}}$ is nothing but $\Bbbk G$. To avoid this, we always assume that $n \geq 2$ throughout this thesis. In this case, $\chi(g) \neq 1$. This implies that $g \neq 1$ and $\chi \neq \varepsilon$.

The algebra $H_{\mathcal{D}}$ is endowed with a Hopf algebra structure. The comultiplication $\triangle$, the counit ε, and the antipode S are given respectively by

$$\triangle(y) = y \otimes g + 1 \otimes y,\ \varepsilon(y) = 0,\ S(y) = -yg^{-1},$$

$$\triangle(h) = h \otimes h,\ \varepsilon(h) = 1,\ S(h) = h^{-1},$$

for all $h \in G$.

It is easy to see that $H_{\mathcal{D}}$ is a pointed Hopf algebra of rank one, the group of group-like elements of $H_{\mathcal{D}}$ is G. If the group datum $\mathcal{D}$ is of nilpotent type, then the Hopf algebra $H_{\mathcal{D}}$ is said to be of *nilpotent type*. Otherwise, it is of *non-nilpotent type*.

Example 1.4.3. Let G be a cyclic group of order m with a generator g and χ a $\Bbbk$-linear character of G such that the order of $\chi(g)$ is m.

- The group datum $\mathcal{D} = (G, \chi, g, \mu)$ is of nilpotent type, and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is nothing but a Taft algebra [24].
- Suppose $d > 1$ is a divisor of m. Then the group datum $\mathcal{D} = (G, \chi, g^d, \mu)$ is of nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is a generalized Taft algebra [56].
- Suppose $d > 1$ is a divisor of m. Then the group datum $\mathcal{D} = (G, \chi^d, g, \mu)$ ($\mu \neq 0$) is of non-nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is a Radford Hopf algebra [62].

The $\Bbbk$-linear character χ induces an automorphism σ of $\Bbbk G$ as follows:

$$\sigma(a) = \Sigma\chi(a_1)a_2, \tag{1.7}$$

for any $a \in \Bbbk G$ with the comultiplication $\triangle(a) = \sum a_1 \otimes a_2$. In view of this, we have

$$y^j a = \sigma^j(a)y^j,\ \text{for } j \geq 0. \tag{1.8}$$

The spaces of left and right integrals of $H_{\mathcal{D}}$ are described respectively as follows.

Lemma 1.4.4. *Let $E = \frac{1}{|G|}\sum_{h\in G} h$. Then the spaces of left and right integrals of $H_{\mathcal{D}}$ are spanned respectively by Ey^{n-1} and $y^{n-1}E$.*

Proof. If $H_{\mathcal{D}}$ is of nilpotent type, then $hEy^{n-1} = Ey^{n-1} = \varepsilon(h)Ey^{n-1}$ for any $h \in G$, and $yEy^{n-1} = 0 = \varepsilon(y)Ey^{n-1}$. Thus, Ey^{n-1} is a non-zero left integral of $H_{\mathcal{D}}$. It is similar that $y^{n-1}E$ is a non-zero right integral of $H_{\mathcal{D}}$.

If $H_{\mathcal{D}}$ is of non-nilpotent type, then $hEy^{n-1} = Ey^{n-1} = \varepsilon(h)Ey^{n-1}$ for any $h \in G$. It follows from $\chi(g^n) = 1$ that

$$\begin{aligned}
yEy^{n-1} &= \frac{1}{|G|}\sum_{h\in G} yhy^{n-1}\\
&= \frac{1}{|G|}\sum_{h\in G} \chi(h)hy^{n}\\
&= \frac{1}{|G|}\sum_{h\in G} \chi(h)h\mu(g^n-1)\\
&= \frac{\mu}{|G|}\sum_{h\in G} (\chi(hg^n)hg^n-\chi(h)h)\\
&= 0.
\end{aligned}$$

Hence $yEy^{n-1} = \varepsilon(y)Ey^{n-1}$ and Ey^{n-1} is a left integral of $H_{\mathcal{D}}$. Similarly, $y^{n-1}E$ is a right integral of $H_{\mathcal{D}}$. □

It follows from Lemma 1.4.4 that the space of left integrals is not equal to the right one, hence $H_{\mathcal{D}}$ is neither unimodular nor symmetric (cf. [57]).

The family of finite dimensional pointed Hopf algebras of rank one coincides with the family of non-semisimple monomial Hopf algebras discussed in [20]. The classification of such Hopf algebras over an algebraically closed field of characteristic 0 has been given respectively in [20, 50]. Following Krop and Radford [50, Theorem 1], we present the classification of finite dimensional pointed Hopf algebras of rank one as follows.

Proposition 1.4.5. *We have the following classification result:*

(1) *Every finite dimensional pointed Hopf algebra of rank one over the field $\Bbbk$ is isomorphic to $H_{\mathcal{D}}$ for some group datum $\mathcal{D}$.*

(2) *The Hopf algebra $H_{\mathcal{D}}$ associated to any group datum $\mathcal{D}$ is a finite dimensional pointed Hopf algebra of rank one.*

(3) *Let $\mathcal{D} = (G, \chi, g, \mu)$ and $\mathcal{D}' = (G', \chi', g', \mu')$ be two group data. Then $H_{\mathcal{D}}$ and $H_{\mathcal{D}'}$ are isomorphic as Hopf algebras if and only if there is a group isomorphism $f : G \to G'$ such that $f(g) = g'$, $\chi = \chi' \circ f$ and $\beta\mu'(g'^n - 1) = \mu(g'^n - 1)$ for some non-zero $\beta \in \Bbbk$, where n is the order of $\chi(g)$.*

In the case of characteristic $p > 0$, the classification of finite dimensional pointed Hopf algebras of rank one was given by Scherotzke in [64]. The classification of infinite or finite dimensional pointed Hopf algebras of rank one over an arbitrary field $\Bbbk$ was obtained in [75].

Remark 1.4.6. If the group datum $\mathcal{D} = (G, \chi, g, \mu)$ is of nilpotent type, namely, $\mu(g^n - 1) = 0$, where n is the order of $\chi(g)$, then it is either $\mu = 0$ or $g^n - 1 = 0$. In both cases, Proposition 1.4.5 (3) implies that the Hopf algebras associated to (G, χ, g, μ) and $(G, \chi, g, 0)$ respectively are isomorphic. Because of this fact, we always assume that $\mu = 0$ for any group datum $\mathcal{D} = (G, \chi, g, \mu)$ of nilpotent type. In case the group datum $\mathcal{D} = (G, \chi, g, \mu)$ is of non-nilpotent type, namely, $\mu(g^n - 1) \neq 0$ and $\chi^n = 1$, where n is the order of $\chi(g)$, without loss of generality we may assume that $\mu = 1$ for the group datum $\mathcal{D} = (G, \chi, g, \mu)$ of non-nilpotent type (for this, see [50, Corollary 1]).

The Green rings of Taft algebras and the generalized Taft algebras have been studied respectively in [24] and [56]. In both cases, the Green rings are commutative and are generated by two elements subject to certain relations defined recursively. The nilpotent elements of the aforementioned Green rings have been completely determined in [56]. Stimulated by the above works, the main goal of this thesis is to compute the Green rings of finite dimensional pointed Hopf algebras of rank one, and attempt to extend those obtained properties of Green rings from the pointed Hopf algebras of rank one to finite dimensional Hopf algebras in general.

Chapter 2

Pointed Hopf algebras of rank one: nilpotent type

In this chapter, H is always a finite dimensional pointed rank one Hopf algebra of nilpotent type. H is a Nakayama algebra, this enables us to determine all finite dimensional indecomposable H-modules up to isomorphism. In order to obtain the decompositions of tensor products of indecomposable H-modules, we study almost split sequences of H-modules. From the uniqueness of an almost split sequence, we are able to deduce the Clebsch-Gordan formulas for the decompositions of tensor product of two indecomposable modules. Using the obtained Clebsch-Gordan formulas, we present the Green ring $r(H)$ of H in terms of generators and relations. The Jacobson radical of $r(H)$ is a principal ideal generated by a special element. The idempotents of both the Green ring $r(H)$ and the complexified Green algebra $R(H)$ are completely determined.

2.1 Indecomposable representations

Throughout this chapter, H is a finite dimensional pointed Hopf algebra of rank one associated to the group datum $\mathcal{D} = (G, \chi, g, 0)$ of nilpotent type. In this case, H is called a pointed rank one Hopf algebra of nilpotent type.

Note that the Jacobson radical J of H is generated by y and $H/J \cong \Bbbk G$. An H-

module V is simple if and only if $yV = 0$ and V restricts to a simple $\Bbbk G$-module. Thus a complete set of non-isomorphic simple $\Bbbk G$-modules forms a complete set of non-isomorphic simple H-modules. In the sequel, we fix such a complete set $\{V_i \mid i \in \Omega_0\}$ of non-isomorphic simple $\Bbbk G$-modules. In particular, $0 \in \Omega_0$ since we assume that $V_0 = \Bbbk$, the trivial $\Bbbk G$-module.

Remark 2.1.1. The fact that each simple H-module V_i restricts to a simple $\Bbbk G$-module implies that there exists a primitive idempotent e_i of $\Bbbk G$ such that $V_i \cong \Bbbk G e_i$. Since e_i is also a primitive idempotent of H and $He_i/\mathrm{rad}(He_i) \cong V_i$, we see that He_i is the projective cover of the simple H-module V_i, for $i \in \Omega_0$.

Now let V_χ and $V_{\chi^{-1}}$ be two (1-dimensional) simple $\Bbbk G$-modules corresponding to the $\Bbbk$-linear characters χ and χ^{-1} respectively. For any simple $\Bbbk G$-module V_i, $i \in \Omega_0$, the tensor product $V_{\chi^{-1}} \otimes V_i \cong V_i \otimes V_{\chi^{-1}}$ is simple as well. Hence there is a unique permutation τ of the index set Ω_0 such that

$$V_{\chi^{-1}} \otimes V_i \cong V_i \otimes V_{\chi^{-1}} \cong V_{\tau(i)}.$$

The inverse of τ is determined by

$$V_\chi \otimes V_i \cong V_i \otimes V_\chi \cong V_{\tau^{-1}(i)}.$$

Lemma 2.1.2. *For any $i \in \Omega_0$ and $t \in \mathbb{Z}$, there is a bijective map $\widetilde{\sigma}_{i,t}$ from V_i to $V_{\tau^t(i)}$ such that $\widetilde{\sigma}_{i,t}(av) = \sigma^t(a)\widetilde{\sigma}_{i,t}(v)$, for any $a \in \Bbbk G$ and $v \in V_i$.*

Proof. For a fixed non-zero element $u \in V_{\chi^{-t}}$, the map

$$V_i \to V_i \otimes V_{\chi^{-t}}, \ v \mapsto v \otimes u$$

composed with the isomorphism $V_i \otimes V_{\chi^{-t}} \cong V_{\tau^t(i)}$ gives the desired bijective map. □

Let x be a variable and V a $\Bbbk G$-module. For any $k \in \mathbb{N}$, consider $x^k V$ as a vector space defined by $x^k u + x^k v = x^k(u+v)$ and $\lambda(x^k u) = x^k(\lambda u)$, for $u, v \in V, \lambda \in \Bbbk$. Then $x^k V$ becomes a $\Bbbk G$-module defined by

$$h(x^k v) = \chi^{-k}(h) x^k hv, \tag{2.1}$$

for any $h \in G$ and $v \in V$.

Lemma 2.1.3. *We have* $\Bbbk G$*-module isomorphisms* $x^k V_i \cong V_{\chi^{-k}} \otimes V_i \cong V_{\tau^k(i)}$*, for any* $i \in \Omega_0$ *and* $k \in \mathbb{N}$*,*

For any $i \in \Omega_0$ and $1 \leq j \leq n$, the direct sum

$$M(i,j) := V_i \oplus xV_i \oplus \cdots \oplus x^{j-1}V_i$$

is a $\Bbbk G$-module, where each summand is a simple $\Bbbk G$-module defined by (2.1). We give an action of y on $M(i,j)$ as follows:

$$y(x^k v) = \begin{cases} x^{k+1}v, & 0 \leq k \leq j-2, \\ 0, & k = j-1, \end{cases} \tag{2.2}$$

for any $v \in V_i$. Then $M(i,j)$ becomes an H-module with $\dim M(i,j) = j \dim V_i$. Moreover, it is easy to see that $M(i,1) \cong V_i$ and $M(i,n) \cong He_i$. Let B_i be a set consisting of a basis of V_i. Then $\{x^k v \mid 0 \leq k \leq j-1, v \in B_i\}$ forms a basis of $M(i,j)$. In particular, $\{1, x, \cdots, x^{j-1}\}$ forms a basis of $M(0,j)$, where we identify $x^k 1$ with x^k, for $1 \in \Bbbk$.

Theorem 2.1.4. *For any* $i \in \Omega_0$*,* $1 \leq j \leq n$*, we have the following:*

(1) *rad*$M(i,1) = 0$ *and rad*$M(i,j) \cong M(\tau(i), j-1)$*, for* $2 \leq j \leq n$*.*

(2) *soc*$M(i,j) \cong V_{\tau^{j-1}(i)}$*,* $M(i,j)/$*rad*$M(i,j) \cong V_i$ *and the projective cover of* $M(i,j)$ *is* $P(M(i,j)) \cong M(i,n)$*.*

(3) $M(i,j)$ *is indecomposable and uniserial.* H *is a Nakayama algebra, and therefore it is of finite representation type.*

(4) $M(i,j) \cong M(k,l)$ *if and only if* $i = k$ *and* $j = l$*. Moreover, the set* $\{M(i,j) \mid i \in \Omega_0,\ 1 \leq j \leq n\}$ *forms a complete set of finite dimensional indecomposable* H*-modules up to isomorphism.*

Proof. (1) Since $M(i,1) \cong V_i$ is simple, we have $\mathrm{rad}M(i,1) = 0$. For $2 \leq j \leq n$, note that $J = (y)$, we have $\mathrm{rad}M(i,j) = \bigoplus_{k=1}^{j-1} x^k V_i$ as a vector space. Define a $\Bbbk$-linear map

$$\mathrm{rad}M(i,j) \to M(\tau(i), j-1),\ x^k v \mapsto x^{k-1}\widetilde{\sigma}_{i,1}(v),$$

for any $v \in V_i$ and $1 \leq k \leq j-1$. By Lemma 2.1.2, it is straightforward to check that the map above is an H-module isomorphism.

(2) It follows from $J = (y)$ that

$$\mathrm{soc}M(i,j) = \{u \in M(i,j) \mid yu = 0\} = x^{j-1}V_i \cong V_{\tau^{j-1}(i)}$$

as H-modules. Since $\mathrm{rad}M(i,j) = \bigoplus_{k=1}^{j-1} x^k V_i$, we have $M(i,j)/\mathrm{rad}M(i,j) \cong V_i$. The following isomorphism

$$P(M(i,j)) \cong P(M(i,j)/\mathrm{rad}M(i,j)) \cong P(V_i) \cong M(i,n)$$

follows from Remark 2.1.1.

(3) That $M(i,j)$ is indecomposable follows from the fact that $\mathrm{soc}M(i,j)$ is simple. The proof of $M(i,j)$ to be uniserial is similar to the one of [50, Proposition 3]. Denote by $N_{i,l}$ the submodule of $M(i,j)$:

$$N_{i,l} = x^l V_i \oplus \cdots \oplus x^{j-1} V_i, \text{ for } 0 \le l \le j-1.$$

Suppose that N is a non-zero submodule of $M(i,j)$. Then there exists a largest l such that $N \subseteq N_{i,l}$. Since $N_{i,l+1}$ is a maximal submodule of $N_{i,l}$, we conclude that $N + N_{i,l+1} = N_{i,l}$. However, $N_{i,l+1} = \mathrm{rad}N_{i,l}$, whence $N = N_{i,l}$ by Nakayama's lemma. Thus $M(i,j)$ is uniserial. Since H is Frobenius (and hence self-injective), the indecomposable projective modules $M(i,n)$ for $i \in \Omega_0$, are also injective modules. Hence they are uniserial. It follows that H is a Nakayama algebra.

(4) If $M(i,j) \cong M(k,l)$, by Part (2),

$$V_i \cong M(i,j)/\mathrm{rad}M(i,j) \cong M(k,l)/\mathrm{rad}M(k,l) \cong V_k,$$

for $i,k \in \Omega_0$. This implies that $i = k$. Comparing the dimensions of the vector spaces, we obtain that $j = l$. Since H is a Nakayama algebra, every indecomposable H-module M is a quotient of an indecomposable projective module $M(i,n)$ for some $i \in \Omega_0$. Thus M is of the form $M(i,j)$, for some $1 \le j \le n$. □

2.2 Almost split sequences and Clebsch-Gordan formulas

Almost split sequences over Nakayama algebras have been much studied, see e.g., [5]. In this section, we first point out that the Auslander-Reiten translate of an H-module $M(i,j)$ is nothing but $M(\tau(i),j)$. We then show that the almost split sequence ending at a non-projective module $M(i,j)$ can be obtained by tensoring $M(i,j)$ over $\Bbbk$ on the right (or on the left) with the almost split sequence ending at the trivial H-module $\Bbbk$. This approach using almost split sequences works also for the $\mathscr{H}$-modules, where $\mathscr{H}$ is a Hopf algebra associated to the quiver A_∞^∞, see [21]. But it does not work for more general Hopf algebras, see e.g. [42]. After that we use the uniqueness of an almost split sequence to determine the decompositions of the tensor products of indecomposable H-modules. To begin with, we need the following results.

Proposition 2.2.1. *For any $i \in \Omega_0$ and $1 \le j \le n$, we have*

$$V_i \otimes M(0,j) \cong M(0,j) \otimes V_i \cong M(i,j).$$

In particular,

$$V_{\chi^{-1}} \otimes M(0,j) \cong M(0,j) \otimes V_{\chi^{-1}} \cong M(\tau(0),j).$$

Proof. The map

$$V_i \otimes M(0,j) \to M(i,j),\ v \otimes x^k \mapsto x^k v,$$

for $0 \le k \le j-1$ and $v \in V_i$ is an H-module isomorphism. Similarly, let ω_i be the scalar such that $gv = \omega_i v$, for $v \in V_i$. The map

$$M(0,j) \otimes V_i \to M(i,j),\ x^k \otimes v \mapsto \omega_i^{-k} x^k v,$$

for $0 \le k \le j-1$ and $v \in V_i$ is an H-module isomorphism. □

Lemma 2.2.2. *For any $i \in \Omega_0$ and $1 < j < n$, we have the following:*

(1) *There is an injective morphism from $radM(i,j)$ to $M(0,2) \otimes M(i,j)$.*

(2) *There is an injective morphism from $radM(i,j)$ to $M(i,j) \otimes M(0,2)$.*

Proof. We only prove Part (1) and the proof of Part (2) is similar. Let ω_i be a scalar such that $gv = \omega_i v$, for any $v \in V_i$. Denote by N the subspace of $M(0,2) \otimes M(i,j)$

spanned by the following elements:

$$1 \otimes x^k v - \zeta_k x \otimes x^{k-1} v, \text{ for } v \in V_i, \ 1 \leq k \leq j-1,$$

where $\zeta_k = \frac{\omega_i}{1-q}(\frac{1}{q^{j-1}} - \frac{1}{q^{k-1}})$ and $q = \chi(g)$. Then the action of H on N is stable:

$$h(1 \otimes x^k v - \zeta_k x \otimes x^{k-1} v) = \chi^{-k}(h)(1 \otimes x^k hv - \zeta_k x \otimes x^{k-1} hv) \in N,$$

for $h \in G$, $1 \leq k \leq j-1$,

$$y(1 \otimes x^k v - \zeta_k x \otimes x^{k-1} v) = 1 \otimes x^{k+1} v - \zeta_{k+1} x \otimes x^k v \in N,$$

for $1 \leq k \leq j-2$ and $y(1 \otimes x^{j-1} v - \zeta_{j-1} x \otimes x^{j-2} v) = 0$. Now we define a $\Bbbk$-linear map ι from $\mathrm{rad}M(i,j) = \bigoplus_{k=1}^{j-1} x^k V_i$ to N as follows:

$$\iota(x^k v) = 1 \otimes x^k v - \zeta_k x \otimes x^{k-1} v, \text{ for } 1 \leq k \leq j-1. \tag{2.3}$$

It is obvious that the map ι is an H-module isomorphism. Hence ι is injective from $\mathrm{rad}M(i,j)$ to $M(0,2) \otimes M(i,j)$. □

The permutation τ of the index set Ω_0 is related to the Auslander-Reiten translate of H-modules as shown in the following.

Proposition 2.2.3. *The Auslander-Reiten translate DTr$(M(i,j))$ of $M(i,j)$ is isomorphic to $M(\tau(i),j)$, for any non-projective indecomposable H-module $M(i,j)$.*

Proof. By Theorem 2.1.4, $M(i,j)$ is uniserial and consequently the length of $M(i,j)$ coincides with the radical length of $M(i,j)$, which is exactly the number j, see Proposition 1.2.9 (5). It also follows from Theorem 2.1.4 that the projective cover of $M(i,j)$ is $M(i,n)$. Thus, by Proposition 1.2.10 (3),

$$\mathrm{DTr}(M(i,j)) \cong \mathrm{rad}M(i,n)/\mathrm{rad}^{j+1}M(i,n) \cong M(\tau(i),j),$$

as desired. □

Recall that $M(0,2) = V_0 \oplus xV_0$, and the summand xV_0 is isomorphic to $V_{\chi^{-1}}$ as $\Bbbk G$-modules, where the isomorphism is given by $\rho : V_{\chi^{-1}} \to xV_0$, $\rho(u) = x$, for a fixed non-zero element $u \in V_{\chi^{-1}}$ and $x \in xV_0$.

Lemma 2.2.4. *The sequence*

$$0 \to V_{\chi^{-1}} \xrightarrow{\alpha} M(0,2) \xrightarrow{\beta} V_0 \to 0 \tag{2.4}$$

is an almost split sequence of H-modules, where $\alpha = \binom{0}{\rho}$ and $\beta = (id, 0)$.

Proof. It is obvious that the maps α and β are both H-module morphisms. The short sequence (2.4) is exact but not split since $M(0,2) \cong V_0 \oplus V_{\chi^{-1}}$ holds just as $\Bbbk G$-modules but not as H-modules. Note that V_0 is a non-projective brick (that is, $\mathrm{End}(V_0) = \Bbbk$), by Proposition 2.2.3,

$$\mathrm{DTr}(V_0) = \mathrm{DTr}(M(0,1)) \cong M(\tau(0), 1) \cong V_{\chi^{-1}}.$$

Hence, the sequence (2.4) is almost split according to Proposition 1.2.7. □

For any indecomposable H-module $M(i,j)$, we consider the following two short exact sequences obtained by tensoring $M(i,j)$ over $\Bbbk$ with the almost split sequence (2.4) on the right and on the left respectively:

$$0 \to M(\tau(i), j) \to M(0,2) \otimes M(i,j) \to M(i,j) \to 0, \tag{2.5}$$

$$0 \to M(\tau(i), j) \to M(i,j) \otimes M(0,2) \to M(i,j) \to 0. \tag{2.6}$$

Proposition 2.2.5. *If $M(i,j)$ is non-projective, that is, $j \neq n$, then the short exact sequences of (2.5) and (2.6) ending at $M(i,j)$ are both almost split.*

Proof. We only show that the sequence (2.5) is almost split and the same argument also works for the sequence (2.6). If $j = 1$, then the ending term of the sequence (2.5) is the simple module V_i. In this case, the same argument in the proof of Lemma 2.2.4 shows that the sequence (2.5) is almost split. We assume now that $2 \leq j \leq n-1$. Since $M(\tau(i), j) \cong \mathrm{DTr}M(i,j)$, by [5, Proposition 2.2, ChV], we only need to verify that each non-isomorphism $f : M(i,j) \to M(i,j)$ factors through the following map:

$$(id, 0) \otimes id_{M(i,j)} : M(0,2) \otimes M(i,j) \to M(i,j).$$

Note that $M(i,j)$ is indecomposable and uniserial, and f is not an isomorphism. Thus, the image of f is contained in $\mathrm{rad}M(i,j)$, which is the unique maximal submodule of

$M(i,j)$. Hence the left triangle in the following diagram is commutative:

$$
\begin{tikzcd}
M(i,j) \arrow[rr, "f"] \arrow[dr, "f"'] & & M(i,j) & \\
& \mathrm{rad}M(i,j) \arrow[ur, hook] \arrow[rr, "\iota"'] & & M(0,2)\otimes M(i,j). \arrow[ul, "(id,0)\otimes id_{M(i,j)}"']
\end{tikzcd}
$$

By Lemma 2.2.2, there exists an injective map ι from $\mathrm{rad}M(i,j)$ to $M(0,2)\otimes M(i,j)$ given by $\iota(x^k v) = 1\otimes x^k v - \zeta_k x\otimes x^{k-1}v$ (see (2.3)), and

$$((id,0)\otimes id_{M(i,j)})(1\otimes x^k v - \zeta_k x\otimes x^{k-1}v) = 1\otimes x^k v = x^k v,$$

for $1\le k\le j-1$ and $v\in V_i$. This implies that the right triangle in the above diagram is also commutative. As a consequence, the non-isomorphism f factors through the map $(id,0)\otimes id_{M(i,j)}$. $\square$

Proposition 2.2.6. *For $i\in\Omega_0$ and $1\le j\le n$, we have the following:*

(1) *If $2\le j\le n-1$, then $M(0,2)\otimes M(i,j)\cong M(i,j)\otimes M(0,2)\cong M(i,j+1)\oplus M(\tau(i),j-1)$.*

(2) *If $j=n$, then $M(0,2)\otimes M(i,n)\cong M(i,n)\otimes M(0,2)\cong M(i,n)\oplus M(\tau(i),n)$.*

Proof. (1) For each non-projective indecomposable module $M(i,j)$, on the one hand, the sequences (2.5) and (2.6) ending at $M(i,j)$ are almost split. On the other hand, by the proof of [5, Theorem 2.1, ChVI] that the following sequence is also almost split (we omit the maps of the sequence):

$$0\to M(\tau(i),j)\to M(\tau(i),j-1)\oplus M(i,j+1)\to M(i,j)\to 0.$$

Since an almost split sequence is uniquely determined by its beginning and ending terms, we obtain that

$$M(0,2)\otimes M(i,j)\cong M(i,j)\otimes M(0,2)\cong M(\tau(i),j-1)\oplus M(i,j+1).$$

Part (2) is obvious as the short exact sequences (2.5) and (2.6) with $j=n$ are split because $M(i,n)$ is projective. $\square$

It is possible to give the decomposition of the tensor product $M(i,k)\otimes M(j,l)$ by virtue of Proposition 2.2.6. However, we do not continue this process since it is more tedious. Instead, we leave it to the next section and express it as a multiplication rule in the Green ring of H.

To end this section, we describe the dual of the indecomposable H-modules, which will be used later. Let M be a finite dimensional H-module. Recall that the dual space M^* is an H-module given by $(hf)(v) = f(S(h)v)$, for $h\in H,\ f\in M^*$ and $v\in M$.

Proposition 2.2.7. *For any $i\in\Omega_0$ and $1\le j\le n$, we have the following:*

(1) $M(i,j)^*\cong M(\tau^{1-j}(i^*),j)$, *where $i^*\in\Omega_0$ determined by $(V_i)^*\cong V_{i^*}$.*

(2) $M(i,j)^{**}\cong M(i,j)$.

Proof. (1) We first study the dual $M(0,j)^*$ of the indecomposable module $M(0,j)$. Let $\{1,x,\cdots,x^{j-1}\}$ be the basis of $M(0,j)$ with the dual $\{1^*,x^*,\cdots,(x^{j-1})^*\}$. The actions of $h\in G$ and y on the dual basis are confirmed to be

$$h(x^k)^*=\chi^k(h)(x^k)^*,\quad y(x^k)^*=-q^{k-1}(x^{k-1})^*,$$

for $0\le k\le j-1$. Here we point out that $y(1^*)=0$. Thus $M(0,j)^*$ is isomorphic to $V_{\chi^{j-1}}\otimes M(0,j)$ with the isomorphism given by

$$(x^k)^*\mapsto(-1)^kq^{\frac{k(k-1)}{2}}(u\otimes x^{j-1-k}),$$

for $0\le k\le j-1$ and $0\ne u\in V_{\chi^{j-1}}$. It follows from Proposition 2.2.1 that

$$\begin{aligned}M(i,j)^*&\cong(M(0,j)\otimes V_i)^*\cong(V_i)^*\otimes M(0,j)^*\\&\cong(V_i)^*\otimes V_{\chi^{j-1}}\otimes M(0,j)\cong V_{i^*}\otimes V_{\chi^{j-1}}\otimes M(0,j)\\&\cong V_{\tau^{1-j}(i^*)}\otimes M(0,j)\cong M(\tau^{1-j}(i^*),j).\end{aligned}$$

(2) Follows from the fact that the square of the antipode of H is inner. □

2.3 Generators and relations of Green rings

In this section, we study the Green ring $r(H)$ of pointed rank one Hopf algebra H of nilpotent type and present the Green ring $r(H)$ in terms of generators and relations. We denote by $M[i,j]$ the isomorphism class of indecomposable H-module $M(i,j)$ in $r(H)$. In particular, we set $1 = [V_0]$ and $a = [V_{\chi^{-1}}] = [V_{\tau(0)}]$.

Proposition 2.3.1. *The following hold in the Green ring $r(H)$:*

(1) $M[i,j] = [V_i]M[0,j] = M[0,j][V_i]$, *for $i \in \Omega_0$ and $1 \leq j \leq n$.*

(2) $M[0,2]M[0,j] = M[0,j]M[0,2] = M[0,j+1] + aM[0,j-1]$, *for $2 \leq j \leq n-1$.*

(3) $M[0,2]M[0,n] = M[0,n]M[0,2] = (1+a)M[0,n]$.

(4) $r(H)$ *is commutative and generated as a ring by $[V_i]$ for $i \in \Omega_0$ and $M[0,2]$ over $\mathbb{Z}$.*

Proof. Part (1) follows from Proposition 2.2.1. Part (2) and Part (3) follow from Proposition 2.2.6. Part (4) is a consequence of Part (1), Part (2) and the fact that $\{M[i,j] \mid i \in \Omega_0, 1 \leq j \leq n\}$ forms a $\mathbb{Z}$-basis of $r(H)$. □

Now we give the multiplication of $M[0,k]M[0,l]$ in $r(H)$ as follows.

Proposition 2.3.2. *For $1 \leq k, l \leq n$, we have the following in $r(H)$:*

(1) *If $k+l-1 \leq n$, then*

$$M[0,k]M[0,l] = \sum_{t=0}^{\min\{k,l\}-1} M[\tau^t(0), k+l-1-2t].$$

(2) *If $k+l-1 \geq n$, let $r = k+l-1-n$. Then*

$$M[0,k]M[0,l] = \sum_{t=0}^{r} M[\tau^t(0), n] + \sum_{t=r+1}^{\min\{k,l\}-1} M[\tau^t(0), k+l-1-2t].$$

In particular,

$$M[0,k]M[0,n] = \sum_{t=0}^{k-1} M[\tau^t(0), n] = (1 + a + \cdots + a^{k-1})M[0,n].$$

Proof. (1) We proceed by induction on $k+l-1$ for $1 \leq k+l-1 \leq n$. It is obvious that the identity holds for $k+l-1=1$. For a fixed $1 < p \leq n-1$, suppose that the identity holds for $1 < k+l-1 \leq p$. We show that it holds for the case $k+l-1 = p+1$. We may now assume that $k \geq 2$ without loss of generality. Since $k+l-1 = p+1$ implies that $(k-1)+l-1 \leq p$ and $(k-2)+l-1 \leq p$, we may apply the induction hypothesis on $(k-1)+l-1 \leq p$ and $(k-2)+l-1 \leq p$. We obtain the following two equalities:

$$M[0,k-1]M[0,l] = \sum_{t=0}^{\min\{k-1,l\}-1} M[\tau^t(0), k-1+l-1-2t],$$

and

$$M[0,k-2]M[0,l] = \sum_{t=0}^{\min\{k-2,l\}-1} M[\tau^t(0), k-2+l-1-2t].$$

Now consider the product $M[0,2]M[0,k-1]M[0,l]$. On the one hand, we apply the induction assumption to get that

$$\begin{aligned}
& M[0,2](M[0,k-1]M[0,l]) \\
= & M[0,2] \sum_{t=0}^{\min\{k-1,l\}-1} M[\tau^t(0), k-1+l-1-2t] \\
= & \sum_{t=0}^{\min\{k-1,l\}-1} M[0,2]M[\tau^t(0), k-1+l-1-2t] \\
= & \sum_{t=0}^{\min\{k-1,l\}-1} (M[\tau^t(0), k-1+l-2t] + M[\tau^{t+1}(0), k-1+l-2-2t]).
\end{aligned}$$

On the other hand, if we apply Proposition 2.3.1 (2) on the product, we obtain that

$$\begin{aligned}
& (M[0,2]M[0,k-1])M[0,l] \\
= & (M[0,k] + M[\tau(0), k-2])M[0,l] \\
= & M[0,k]M[0,l] + aM[0,k-2]M[0,l] \\
= & M[0,k]M[0,l] + \sum_{t=0}^{\min\{k-2,l\}-1} M[\tau^{t+1}(0), k-2+l-1-2t].
\end{aligned}$$

Together with these equations, we obtain that

$$\begin{aligned} & M[0,k]M[0,l] + \sum_{t=0}^{\min\{k-2,l\}-1} M[\tau^{t+1}(0), k-2+l-1-2t] \\ = & \sum_{t=0}^{\min\{k-1,l\}-1} (M[\tau^t(0), k-1+l-2t] + M[\tau^{t+1}(0), k-1+l-2-2t]). \end{aligned}$$

By discussing the cases $k-1<l$, $k-1=l$ and $k-1>l$ respectively, we always obtain that

$$M[0,k]M[0,l] = \sum_{t=0}^{\min\{k,l\}-1} M[\tau^t(0), k+l-1-2t].$$

Thus we have proved Part (1) for the case $k+l-1=p+1$.

(2) The proof of Part (2) is similar to the proof of Part (1) by induction on $k+l-1$ for the case $n \leq k+l-1 \leq 2n-1$. □

The multiplication of $M[i,k]$ with $M[j,l]$ in $r(H)$ is deduced as follows, which is correspondence to the decomposition of the tensor product $M(i,k) \otimes M(j,l)$ of H-modules.

Corollary 2.3.3. *For any $i,j \in \Omega_0$, let $V_i \otimes V_j \cong \bigoplus_s V_s$.*

(1) *If $k+l-1 \leq n$, then*

$$M[i,k]M[j,l] = \sum_s \sum_{t=0}^{\min\{k,l\}-1} M[\tau^t(s), k+l-1-2t].$$

Moreover, $aM[j,l] = M[\tau(j), l]$.

(2) *If $k+l-1 \geq n$, then*

$$M[i,k]M[j,l] = \sum_s (\sum_{t=0}^{r} M[\tau^t(s), n] + \sum_{t=r+1}^{\min\{k,l\}-1} M[\tau^t(s), k+l-1-2t]),$$

where $r = k+l-1-n$.

Now we are ready to describe the structure of the Green ring $r(H)$ of H. Let $\Bbbk[y,z]$ be the polynomial ring with variables y and z over $\Bbbk$ and $F_k(y,z)$ the polynomials in

$\Bbbk[y,z]$ defined recursively as follows:

$$F_1(y,z)=1,\ F_2(y,z)=z,\ F_k(y,z)=zF_{k-1}(y,z)-yF_{k-2}(y,z),\ k\geq 3. \tag{2.7}$$

Define the matrix $\mathbf{A}$ as follows:

$$\mathbf{A}=\begin{pmatrix} z & -y \\ 1 & 0 \end{pmatrix}.$$

Then the recursive relations of (2.7) can be written as

$$\mathbf{A}^k\begin{pmatrix} F_2(y,z) \\ F_1(y,z) \end{pmatrix}=\begin{pmatrix} F_{k+2}(y,z) \\ F_{k+1}(y,z) \end{pmatrix},$$

for $k\geq 0$. In addition, for any $k\geq 2$, the polynomial $F_k(y,z)$ can be expressed explicitly as follows (cf. [24, Lemma 3.11]):

$$F_k(y,z)=\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}(-1)^i\binom{k-1-i}{i}y^iz^{k-1-2i}, \tag{2.8}$$

where $\lfloor\frac{k-1}{2}\rfloor$ stands for the biggest integer that not more than $\frac{k-1}{2}$.

The polynomials $F_k(y,z)$ for $k\geq 1$ are called the generalized Fibonacci polynomial in [24] and [56] since if $y=-1$ the polynomials $F_k(-1,z)\in\Bbbk[z]$ are the well-known Fibonacci polynomials [45]. These polynomials are also referred to be the Dickson polynomials (of the second type), see [59, 48]. As we shall see that the Dickson polynomials are fundamental factors in the structure of the Green ring $r(H)$.

Theorem 2.3.4. *Let $r(\Bbbk G)$ be the Green ring of the group algebra $\Bbbk G$ and $r(\Bbbk G)[z]$ the polynomial ring with variable z over $r(\Bbbk G)$. Then the Green ring $r(H)$ is isomorphic to $r(\Bbbk G)[z]/I$, where I is the ideal of $r(\Bbbk G)[z]$ generated by the element $(1+a-z)F_n(a,z)$.*

Proof. According to Proposition 2.3.1, $r(H)$ is generated as a ring by $M[0,2]$ over $r(\Bbbk G)$. Hence there is a unique ring epimorphism Φ from $r(\Bbbk G)[z]$ to $r(H)$ such that

$$\Phi: r(\Bbbk G)[z]\to r(H),\ g(z)\mapsto g(M[0,2]),$$

for any polynomial $g(z)\in r(\Bbbk G)[z]$. It is easy to check by induction on j that

$\Phi(F_j(a,z)) = M[0,j]$, for $1 \leq j \leq n$. Now let I be the ideal of $r(\Bbbk G)[z]$ generated by the element $(1+a-z)F_n(a,z)$. By Proposition 2.3.1, we have

$$\Phi((1+a-z)F_n(a,z)) = (1+a-M[0,2])M[0,n] = 0.$$

This leads to a natural ring epimorphism $\overline{\Phi}$ from $r(\Bbbk G)[z]/I$ to $r(H)$ such that $\overline{\Phi}(\overline{g(z)}) = \Phi(g(z))$ for any $g(z) \in r(\Bbbk G)[z]$, where $\overline{g(z)}$ stands for the coset $g(z)+I$ in $r(\Bbbk G)[z]/I$. Observe that as a $\mathbb{Z}$-module, $r(\Bbbk G)[z]/I$ has a $\mathbb{Z}$-basis $\{\overline{[V_i]z^j} \mid i \in \Omega_0,\ 0 \leq j \leq n-1\}$. Thus, $r(\Bbbk G)[z]/I$ and $r(H)$ both have the same rank as free $\mathbb{Z}$-modules, and therefore the map $\overline{\Phi}$ is an isomorphism. $\square$

Remark 2.3.5. (1) If H is a Taft algebra or a generalized Taft algebra, then Theorem 2.3.4 recovers the main results of [24] and [56].

(2) Since $\Bbbk G$ is semisimple, the Green ring $r(\Bbbk G)$ of $\Bbbk G$ is exactly the Grothendieck ring $G_0(\Bbbk G)$ of $\Bbbk G$. Note that the Jacobson radical $J = (y)$ of H is a Hopf ideal and $H/J \cong \Bbbk G$. The Grothendieck ring $G_0(H)$ of H is isomorphic to $r(\Bbbk G)$, see [57].

(3) Consider $F_j(a,z)$, $1 \leq j \leq n$ the Dickson polynomials in $r(\Bbbk G)[z]$. Note that $\Phi(F_j(a,z)) = M[0,j]$, see the proof of Theorem 2.3.4. By Proposition 2.3.2, we have the following expressions for the multiplication of the Dickson polynomials of the second type.

- If $k+l-1 \leq n$, then

$$F_k(a,z)F_l(a,z) \equiv \sum_{t=0}^{\min\{k,l\}-1} a^t F_{k+l-1-2t}(a,z)\ (mod\ I).$$

- If $k+l-1 \geq n$, let $r = k+l-1-n$. Then

$$F_k(a,z)F_l(a,z) \equiv \sum_{t=0}^{r} a^t F_n(a,z) + \sum_{t=r+1}^{\min\{k,l\}-1} a^t F_{k+l-1-2t}(a,z)\ (mod\ I),$$

where I is the ideal given in Theorem 2.3.4.

Two Hopf algebras are said to be *gauge equivalent* if their representation categories are tensor equivalent. It is obvious that two gauge equivalent Hopf algebras possess the same Green ring. However, the converse is not true in general. In fact, the representation category of a Hopf algebra can not be solely determined by its Green ring, see e.g. [70, Remark 1.8]. In the following, we give a sufficient condition for two non-gauge equivalent Hopf algebras $H_{\mathcal{D}}$ to share the same Green ring.

Proposition 2.3.6. *Let $\mathcal{D} = (G, \chi, g, 0)$ and $\mathcal{D}' = (G, \chi', g, 0)$ be two group data of nilpotent type. Denote by the set $G_k = \{h \in G \mid h^2 = g^{-k}\}$ and the element*

$$\vartheta = \sum_{k=0}^{n-1} \sum_{h \in G_k} (-1)^k h^{-k} g^{-\frac{k(1+k)}{2}}$$

in $\Bbbk G$. Suppose that $\chi(\vartheta) \neq \chi'(\vartheta)$ and there is an automorphism δ of $r(\Bbbk G)$ such that $\delta(a) = a'$, where $a = [V_{\chi^{-1}}]$ and $a' = [V_{\chi'^{-1}}]$. Then the representation categories of Hopf algebras $H_{\mathcal{D}}$ and $H_{\mathcal{D}'}$ are not tensor equivalent, but the Green rings $r(H_{\mathcal{D}})$ and $r(H_{\mathcal{D}'})$ are isomorphic.

Proof. It is straightforward to check that $\chi(\vartheta)$ (resp. $\chi'(\vartheta)$) is the trace of the antipode of Hopf algebra $H_{\mathcal{D}}$ (resp. $H_{\mathcal{D}'}$). The condition $\chi(\vartheta) \neq \chi'(\vartheta)$ implies that the representation categories of Hopf algebras $H_{\mathcal{D}}$ and $H_{\mathcal{D}'}$ are not tensor equivalent since the trace of antipode of a Hopf algebra is a gauge invariant (i.e., invariant under tensor equivalence) (see [52]). By Theorem 2.3.4, the Green ring $r(H_{\mathcal{D}})$ (resp. $r(H_{\mathcal{D}'})$) is isomorphic to $r(\Bbbk G)[z]/I$, where I is the ideal of $r(\Bbbk G)[z]$ generated by the element $(1 + a - z)F_n(a, z)$ (resp. $(1 + a' - z)F_n(a', z)$). Hence the automorphism δ of $r(\Bbbk G)$ such that $\delta(a) = a'$ induces an isomorphism from $r(H_{\mathcal{D}})$ to $r(H_{\mathcal{D}'})$. □

Example 2.3.7. Let G be a cyclic group of order n generated by g, χ a linear character of G such that the order of χ is n and $a = [V_{\chi^{-1}}]$. Given two group data $\mathcal{D} = (G, \chi, g, 0)$ and $\mathcal{D}' = (G, \chi^i, g, 0)$ such that $\gcd(i, n) = 1$. The Hopf algebras $H_{\mathcal{D}}$ and $H_{\mathcal{D}'}$ are nothing but two Taft Hopf algebras. It is easy to see that the Green ring $r(\Bbbk G)$ is isomorphic to $\mathbb{Z}C_n$, where $C_n = \langle a \rangle$ is a cyclic group of order n. Let δ be the endomorphism of $\mathbb{Z}C_n$ defined by $\delta(a) = a^i$. Then δ is an automorphism of $\mathbb{Z}C_n$ since $\gcd(i, n) = 1$. By Proposition 2.3.6 (or [65]) the representation categories of the Taft Hopf algebras $H_{\mathcal{D}}$ and $H_{\mathcal{D}'}$ are not gauge equivalent since $\chi(g) \neq \chi^i(g)$. But the Green rings $r(H_{\mathcal{D}})$ and $r(H_{\mathcal{D}'})$ are isomorphic.

2.4 Frobenius properties of Green rings

In this section, we study the Frobenius property of the Green ring $r(H)$ of H of nilpotent type. The Green ring $r(H)$ is a symmetric ring (because the square of antipode is inner) with an associative symmetric and non-degenerate $\mathbb{Z}$-bilinear form $(-,-)$. We give a dual basis of $r(H)$ explicitly with respect to the form $(-,-)$. We show that the principal ideals of $r(H)$ generated respectively by the projective cover of the trivial module and by the almost split sequence ending at the trivial module are orthogonal with respect to the form $(-,-)$.

Recall that $\delta_{[Z]} = [X] - [Y] + [Z]$ if $0 \to X \to Y \to Z \to 0$ is an almost split sequence, and $\delta_{[Z]} = [Z] - [\mathrm{rad} Z]$ if Z is indecomposable projective. Note that the sequence (2.5) ending at the non-projective indecomposable module $M(i,j)$ is almost split. In view of this, for $i \in \Omega_0$ and $1 \le j \le n-1$, we have that

$$\begin{aligned}
\delta_{M[i,j]} &= M[\tau(i), j] - [M(0,2) \otimes M(i,j)] + M[i,j] \\
&= aM[i,j] - M[0,2]M[i,j] + M[i,j] \\
&= (1 + a - M[0,2])M[i,j] \\
&= \delta_{[\Bbbk]} M[i,j].
\end{aligned}$$

For $i \in \Omega_0$, we have that

$$\begin{aligned}
\delta_{M[i,n]} &= M[i,n] - [\mathrm{rad} M(i,n)] \\
&= M[i,n] - M[\tau(i), n-1] \\
&= M[i,n] - aM[i,n-1].
\end{aligned}$$

For any $i \in \Omega_0$, by Proposition 2.3.1 (3), we have that

$$\delta_{[\Bbbk]} M[i,n] = 0. \tag{2.9}$$

According to $\delta_{M[i,j]}$ given above, we obtain an interesting equality as follows:

$$[V_i] = \delta_{M[i,1]} + \delta_{M[i,2]} + \cdots + \delta_{M[i,n]}. \tag{2.10}$$

Note that the Green ring $r(H)$ is commutative. The duality functor of H-module category given in Proposition 2.2.7 induces an automorphism $*$ of $r(H)$ as follows:

$$M[i,j]^* = M[\tau^{1-j}(i^*), j] = a^{1-j}[V_i^*]M[0,j]. \tag{2.11}$$

This automorphism of $r(H)$ is an involution, namely, $M[i,j]^{**} = M[i,j]$, for $i \in \Omega_0$ and $1 \leq j \leq n$. We denote by $\delta^*_{M[i,j]}$ the image of $\delta_{M[i,j]}$ under the dual automorphism. By (2.11), for any $i \in \Omega_0$ and $1 \leq j \leq n-1$, we have

$$\begin{aligned} \delta^*_{M[i,j]} &= (1 + a - M[0,2])^* M[i,j]^* \\ &= (1 + a^{-1} - a^{-1}M[0,2])a^{1-j}[V_{i^*}]M[0,j] \\ &= \delta_{[\Bbbk]} a^{-j}[V_{i^*}]M[0,j] \\ &= \delta_{M[\tau^{-j}(i^*),j]}, \end{aligned}$$

$$\delta^*_{M[i,n]} = (M[i,n] - aM[i,n-1])^* = a^{1-n}[V_i^*](M[0,n] - M[0,n-1]).$$

In particular, we have the following result:

$$\begin{aligned} &\delta^*_{M[i,n]}\delta_{M[j,n]} \\ &= a^{1-n}[V_i]^*(M[0,n] - M[0,n-1])(M[j,n] - aM[j,n-1]) \\ &= a^{1-n}[V_i]^*[V_j](M[0,n] - M[0,n-1])(M[0,n] - aM[0,n-1]) \\ &= a^{1-n}[V_i]^*[V_j]a^{n-1} \\ &= [V_i]^*[V_j]. \end{aligned} \tag{2.12}$$

It follows from Theorem 1.3.4 that $r(H)$ possesses an associative symmetric and non-degenerate $\mathbb{Z}$-bilinear form $(-,-)$ with the dual basis $\{\delta^*_{M[i,j]}, M[i,j] \mid i \in \Omega_0, 1 \leq j \leq n\}$ with respect to the form $(-,-)$. For any $x \in r(H)$, by Lemma 1.3.3, we have

$$x = \sum_{i \in \Omega_0} \sum_{j=1}^{n} (x, \delta^*_{M[i,j]}) M[i,j], \tag{2.13}$$

or equivalently (see (1.5)),

$$x = \sum_{i \in \Omega_0} \sum_{j=1}^{n} (M[i,j], x) \delta^*_{M[i,j]}. \tag{2.14}$$

Note that $M[i,j] = [V_i](1 + a + \cdots + a^{j-1})$ holds in the Grothendieck ring $G_0(H)$

of H for $i \in \Omega_0$ and $1 \leq j \leq n$. The natural ring epimorphism from $r(H)$ to $G_0(H)$ is given by

$$M[i,j] \mapsto [V_i](1 + a + \cdots + a^{j-1}), \text{ for } i \in \Omega_0,\ 1 \leq j \leq n, \tag{2.15}$$

whose kernel is precisely spanned by $\delta_{M[i,j]}$ (or $\delta^*_{M[i,j]}$) for $i \in \Omega_0$ and $1 \leq j \leq n-1$.

For any simple $\Bbbk G$-module V_i, $i \in \Omega_0$, the character of V_i is denoted by χ_i. It is well known that the free abelian group $\mathrm{Irr}(\Bbbk G) := \sum_{i \in \Omega_0} \mathbb{Z}\chi_i$ with the convolution product forms a ring, which is called a *character ring* of $\Bbbk G$. The character ring $\mathrm{Irr}(\Bbbk G)$ is isomorphic to the Green ring $r(\Bbbk G)$ via the map $\chi_i \mapsto [V_i]$ for $i \in \Omega_0$ since $\Bbbk G$ is semisimple. The equality

$$[V_i][V_j] = \delta_{i,j^*}[V_0] + \sum_{0 \neq k \in \Omega_0} \gamma_k [V_k] \tag{2.16}$$

follows from the fact that $\chi_i\chi_j = \delta_{i,j^*}\chi_0 + \sum_{0 \neq k \in \Omega_0} \gamma_k \chi_k$, for some non-negative integers γ_k. The equality (2.16) could also be interpreted by the equation (2.13) if we consider V_i, $i \in \Omega_0$ the simple H-modules. In fact,

$$\begin{aligned}
[V_i][V_j] &= \sum_{k \in \Omega_0} \sum_{l=1}^{n} ([V_i][V_j], \delta^*_{M[k,l]}) M[k,l] \\
&= \sum_{k \in \Omega_0} ([V_i][V_j], \delta^*_{M[k,1]}) M[k,1] \\
&= ([V_i][V_j], \delta^*_{M[0,1]}) M[0,1] + \sum_{0 \neq k \in \Omega_0} ([V_i][V_j], \delta^*_{M[k,1]}) M[k,1] \\
&= ([V_i], \delta^*_{M[j^*,1]}) M[0,1] + \sum_{0 \neq k \in \Omega_0} ([V_i][V_j], \delta^*_{M[k,1]}) M[k,1] \\
&= \delta_{i,j^*}[V_0] + \sum_{0 \neq k \in \Omega_0} ([V_i][V_j], \delta^*_{M[k,1]}) [V_k].
\end{aligned}$$

Let $\mathcal{P}$ be the free abelian group generated by the isomorphism classes of indecomposable projective H-modules. Then $\mathcal{P}$ is an ideal of $r(H)$, which is in fact a principal ideal generated by $M[0,n]$. Denote by

$$\mathcal{P}^{\perp} = \{x \in r(H) \mid (x,y) = 0, \text{for } y \in \mathcal{P}\}.$$

Then $\mathcal{P}^{\perp}$ is an ideal of $r(H)$ since $\mathcal{P}$ itself is. We denote by $\mathcal{J}$ the free abelian group generated by all almost split sequences, namely,

$$\mathcal{J} = \mathbb{Z}\{\delta_{M[i,k]} \mid i \in \Omega_0, 1 \leq k \leq n-1\}.$$

It follows from Lemma 1.3.1 (5) that the free abelian group $\mathcal{J}$ is exactly the kernel of the natural ring epimorphism from $r(H)$ to $G_0(H)$ given by (2.15). In view of this, $\mathcal{J}$ is an ideal of $r(H)$. Moreover, it is a principal ideal generated by $\delta_{[\Bbbk]}$ since $\delta_{[\Bbbk]} \in \mathcal{J}$ and $\delta_{M[i,k]} = \delta_{[\Bbbk]}M[i,k]$ for $i \in \Omega_0$ and $1 \leq k \leq n-1$.

The relation between the ideals $\mathcal{P}$ generated by $M[0,n]$ and $\mathcal{J}$ generated by $\delta_{[\Bbbk]}$ is shown in the following that they are orthogonal with respect to the form $(-,-)$.

Proposition 2.4.1. *We have* $\mathcal{J} = \mathcal{P}^{\perp} = (\mathcal{P}^{\perp})^*$.

Proof. Note that the bilinear form $(-,-)$ is symmetric and $r(H)$ is commutative. For any $M[j,n] \in \mathcal{P}$, by (2.9), we have

$$\begin{aligned}(\delta_{M[i,k]}, M[j,n]) &= (\delta_{[\Bbbk]}M[i,k], M[j,n]) \\ &= (\delta_{[\Bbbk]}M[j,n], M[i,k]) \\ &= (0, M[i,k]) \\ &= 0.\end{aligned}$$

This implies that $\delta_{M[i,k]} \in \mathcal{P}^{\perp}$, and therefore $\mathcal{J} \subseteq \mathcal{P}^{\perp}$. For any $x \in \mathcal{P}^{\perp}$, by (2.14),

$$x = \sum_{i\in\Omega_0}\sum_{j=1}^{n}(M[i,j],x)\delta^*_{M[i,j]} = \sum_{i\in\Omega_0}\sum_{j=1}^{n-1}(M[i,j],x)\delta^*_{M[i,j]}.$$

Then $x^* = \sum_{i\in\Omega_0}\sum_{j=1}^{n-1}(M[i,j],x)\delta_{M[i,j]}$ and

$$\begin{aligned}(x^*, M[k,n]) &= \sum_{i\in\Omega_0}\sum_{j=1}^{n-1}(M[i,j],x)(\delta_{M[i,j]}, M[k,n]) \\ &= \sum_{i\in\Omega_0}\sum_{j=1}^{n-1}(M[i,j],x)(\delta_{[\Bbbk]}M[k,n], M[i,j]) \\ &= \sum_{i\in\Omega_0}\sum_{j=1}^{n-1}(M[i,j],x)(0, M[i,j]) \\ &= 0.\end{aligned}$$

Thus, $x^* \in \mathcal{P}^\perp$, and therefore $x = x^{**} \in (\mathcal{P}^\perp)^*$. This implies that $\mathcal{P}^\perp \subseteq (\mathcal{P}^\perp)^*$. For any $x \in \mathcal{P}^\perp$, by (2.14) as well,

$$x = \sum_{i\in\Omega_0}\sum_{j=1}^{n}(M[i,j],x)\delta^*_{M[i,j]} = \sum_{i\in\Omega_0}\sum_{j=1}^{n-1}(M[i,j],x)\delta^*_{M[i,j]}.$$

Then $x^* = \sum_{i\in\Omega_0}\sum_{j=1}^{n-1}(M[i,j],x)\delta_{M[i,j]} \in \mathcal{J}$, and hence $(\mathcal{P}^\perp)^* \subseteq \mathcal{J}$. We obtain that $\mathcal{J} = \mathcal{P}^\perp = (\mathcal{P}^\perp)^*$. $\square$

2.5 Jacobson radicals of Green rings

In this section, we study the Jacobson radical of the Green ring $r(H)$ of H of nilpotent type. We need the following useful lemma, which was done by Zhu for semisimple Hopf algebras [81, Lemma 1].

Lemma 2.5.1. *Let X and Y be any two indecomposable H-modules and $x \in r(H)$.*

(1) *The coefficient of $[\Bbbk]$ in the linear expression of $[Y][X]^*$ is 1 if $X \cong Y$ and not projective, and 0 otherwise.*

(2) *If $xx^* = 0$, then $x \in \mathcal{P}$.*

Proof. (1) If X is projective, then the coefficient of the identity $[\Bbbk]$ in $[Y][X]^*$ is 0 since $\Bbbk$ is not projective. If X is not projective, by Lemma 1.3.3, the coefficient of the identity $[\Bbbk]$ in $[Y][X]^*$ is equal to $(\delta^*_{[\Bbbk]}, [Y][X]^*) = (\delta^*_{[X]}, [Y])$, which is 1 if $X \cong Y$, and 0 otherwise.

(2) Suppose $x = \sum_{i\in\Omega_0, 1\leq j\leq n-1} \alpha_{ij} M[i,j] + x_0$, for $\alpha_{ij} \in \mathbb{Z}$ and $x_0 \in \mathcal{P}$. By Part (1), the coefficient of $[\Bbbk]$ in xx^* is $\sum_{i\in\Omega_0, 1\leq j\leq n-1} \alpha_{ij}^2$. Thus, if $\alpha_{ij} \neq 0$ for some $i \in \Omega_0$ and $1 \leq j \leq n-1$, then $xx^* \neq 0$. Otherwise, if $xx^* = 0$, then $\alpha_{ij} = 0$ for any $i \in \Omega_0$ and $1 \leq j \leq n-1$. This implies that $x = x_0 \in \mathcal{P}$. □

Theorem 2.5.2. *The Jacobson radical $J(r(H))$ of $r(H)$ is the intersection $\mathcal{P} \cap \mathcal{P}^\perp$.*

Proof. For any $x \in \mathcal{P} \cap \mathcal{P}^\perp$ and $y \in r(H)$, we have $xy \in \mathcal{P} \cap \mathcal{P}^\perp$ and hence $(x^2, y) = (x, xy) = 0$. It follows that $x^2 = 0$ since the bilinear form $(-,-)$ is non-degenerate. As a consequence, $x \in J(r(H))$. Conversely, for any $x \in J(r(H))$, we denote by

$$x = \sum_{i\in\Omega_0} \sum_{1\leq j\leq n-1} \alpha_{ij} M[i,j] + x_0,$$

for $\alpha_{ij} \in \mathbb{Z}$ and $x_0 \in \mathcal{P}$. Then

$$xx^* = \sum_{i,k\in\Omega_0} \sum_{1\leq j,l\leq n-1} \alpha_{ij}\alpha_{kl} M[i,j]M[k,l]^* + x_1$$

for some $x_1 \in \mathcal{P}$ since $\mathcal{P}$ is an ideal of $r(H)$ satisfying $\mathcal{P} = \mathcal{P}^*$. If we denote by

$$y := xx^* = \sum_{i\in\Omega_0} \sum_{1\leq j\leq n-1} \beta_{ij} M[i,j] + x_2,$$

for some $x_2 \in \mathcal{P}$, then the coefficient of $[\Bbbk]$ in y is $\beta_{01} := \sum_{i\in\Omega_0}\sum_{1\leq j\leq n-1}\alpha_{ij}^2$ by Lemma 2.5.1 (1). Consider

$$y^2 = yy^* = \sum_{i,k\in\Omega_0}\sum_{1\leq j,l\leq n-1}\beta_{ij}\beta_{kl}M[i,j]M[k,l]^* + x_3,$$

for some $x_3 \in \mathcal{P}$. According to Lemma 2.5.1 (1) as well, we have the coefficient of $[\Bbbk]$ in y^2 is $\sum_{i\in\Omega_0}\sum_{1\leq j\leq n-1}\beta_{ij}^2$. If $\beta_{01} \neq 0$, then $\sum_{i\in\Omega_0}\sum_{1\leq j\leq n-1}\beta_{ij}^2 \neq 0$, and hence $y^2 \neq 0$. By repeating this process, we obtain that if $\beta_{01} \neq 0$, then $y^{2^n} \neq 0$, for any $n > 0$, this contradicts to the fact that $y \in J(r(H))$. Therefore, $\beta_{01} = 0$ and $x = x_0 \in \mathcal{P}$.

For any $x \in J(r(H))$, to verify $x \in \mathcal{P}^{\perp}$, we may write x as follows:

$$x = \sum_{i\in\Omega_0}\alpha_i\delta_{M[i,n]} + x_0,$$

for $\alpha_i \in \mathbb{Z}$ and $x_0 \in \mathcal{J}$ since $\{\delta_{M[i,j]} \mid i \in \Omega_0, 1 \leq j \leq n-1\}$ is a basis of $\mathcal{J}$ and such a basis of $\mathcal{J}$ together with $\{\delta_{M[i,n]} \mid i \in \Omega_0\}$ forms a basis of $r(H)$. Note that $\mathcal{J}$ is an ideal of $r(H)$ and $\mathcal{J} = \mathcal{J}^*$ (see Proposition 2.4.1). It follows that

$$\begin{aligned} x^*x &= \sum_{i,j\in\Omega_0}\alpha_i\alpha_j[V_i]^*[V_j] + x_1 \quad \text{by (2.12)} \\ &= \sum_{i\in\Omega_0}\beta_i[V_i] + x_1 \\ &= \sum_{i\in\Omega_0}\beta_i\delta_{M[i,n]} + x_2, \quad \text{by (2.10)} \end{aligned}$$

where $\beta_0 = \sum_{i\in\Omega_0}\alpha_i^2$ and $x_1, x_2 \in \mathcal{J}$. We denote by $y := x^*x = \sum_{i\in\Omega_0}\beta_i\delta_{M[i,n]} + x_2$. It follows that

$$\begin{aligned} y^2 &= y^*y \\ &= \sum_{i,j\in\Omega_0}\beta_i\beta_j[V_i]^*[V_j] + x_3 \quad \text{by (2.12)} \\ &= \sum_{i\in\Omega_0}\gamma_i[V_i] + x_3 \\ &= \sum_{i\in\Omega_0}\gamma_i\delta_{M[i,n]} + x_4, \quad \text{by (2.10)} \end{aligned}$$

where $\gamma_0 = \sum_{i\in\Omega_0} \beta_i^2$ and $x_3, x_4 \in \mathcal{J}$. If $\beta_0 \neq 0$, then $\gamma_0 = \sum_{i\in\Omega_0} \beta_i^2 \neq 0$, and therefore $y^2 \neq 0$. By repeating this process, we obtain that if $\beta_0 \neq 0$, then the power of y can not be zero, a contradiction to the fact that $y \in J(r(H))$. Hence $\beta_0 = 0$ and $x = x_0 \in \mathcal{J} = \mathcal{P}^\perp$. As a result, $J(r(H)) = \mathcal{P} \cap \mathcal{P}^\perp$. $\square$

Remark 2.5.3. (1) Observe that $\mathcal{P}^* = \mathcal{P}$. It follows from Proposition 2.4.1 that $(\mathcal{P}^*)^\perp = (\mathcal{P}^\perp)^*$.

(2) By Proposition 2.4.1, the kernel of the natural ring epimorphism from $r(H)$ to $G_0(H)$ is precisely $\mathcal{P}^\perp$. We obtain that $r(H)/\mathcal{P}^\perp \cong G_0(H)$.

(3) For any $x, y \in \mathcal{P} \cap \mathcal{P}^\perp$ and $z \in r(H)$, we have $(xy, z) = (x, yz) = 0$. It follows that $xy = 0$ since the form $(-,-)$ is non-degenerate. Hence the square of $J(r(H))$ is zero.

We have known that the Jacobson radical of $r(H)$ is the intersection $\mathcal{P} \cap \mathcal{P}^\perp$. In the following, we assume that the pointed rank one Hopf algebra H of nilpotent type is defined over $\Bbbk = \mathbb{C}$. We study the Jacobson radical of the Green ring $r(H)$ in terms of generators.

Suppose that the order of χ in the group datum $\mathcal{D} = (G, \chi, g, 0)$ is l, so is the order of $a = [V_{\chi^{-1}}]$. Note that $q = \chi(g)$ is a primitive n-th root of unity, and $q^l = (\chi(g))^l = \chi^l(g) = 1$. We obtain that l is divisible by n.

Since $\mathbb{C}G$ is semisimple, the complexified Green algebra $R(\mathbb{C}G) := \mathbb{C} \otimes_{\mathbb{Z}} r(\mathbb{C}G)$ is a commutative semisimple algebra, see [78, 81]. As a consequence, all the simple modules over $R(\mathbb{C}G)$ is one dimensional and the number of non-isomorphic simple modules is equal to the rank of $r(\mathbb{C}G)$, which is equal to the number of non-isomorphic simple $\mathbb{C}G$-modules, namely, $|\Omega_0|$.

Let $\{W_j \mid j \in \Omega_0\}$ be a complete set of non-isomorphic simple $R(\mathbb{C}G)$-modules. The action of a on W_j is a scalar multiple by ω^{t_j}, where $\omega = \cos\frac{2\pi}{l} + i\sin\frac{2\pi}{l}$ is a primitive l-th root of unity and $0 \leq t_j \leq l-1$. We divide the index set Ω_0 into three parts:

$$\Omega_0^1 = \{j \mid j \in \Omega_0, t_j = 0\},$$

$$\Omega_0^2 = \{j \mid j \in \Omega_0, t_j \neq 0 \text{ and } \frac{l}{n} \nmid t_j\},$$

$$\Omega_0^3 = \{j \mid j \in \Omega_0, t_j \neq 0 \text{ and } \frac{l}{n} \mid t_j\}.$$

The cardinalities of the sets Ω_0^1, Ω_0^2 and Ω_0^3 are denoted by d_1, d_2 and d_3 respectively. Obviously, $d_1 + d_2 + d_3 = |\Omega_0|$.

Let $\alpha = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$ and N_j the number of distinct roots of the equation

$$(z - \omega^{t_j} - 1)F_n(\omega^{t_j}, z) = 0, \tag{2.17}$$

for any $j \in \Omega_0$. We need the following lemma:

Lemma 2.5.4. *The distinct roots of the equation (2.17) are described as follows:*

(1) *If $j \in \Omega_0^3$, then the equation (2.17) has $N_j = n - 1$ distinct roots:*

$$\alpha_{j,k} = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k}), \text{ for } 1 \le k \le n-1.$$

(2) *If $j \in \Omega_0^1 \cup \Omega_0^2$, then the equation (2.17) has $N_j = n$ distinct roots:*

$$\alpha_{j,k} = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k}), \text{ for } 1 \le k \le n-1 \text{ and } \alpha_{j,n} = \omega^{t_j} + 1.$$

Proof. Let $b_j = \cos(\frac{t_j\pi}{l} + \frac{3\pi}{2}) + i\sin(\frac{t_j\pi}{l} + \frac{3\pi}{2})$. Then $b_j^2 = -\omega^{t_j}$. The relations between the polynomials $F_k(\omega^{t_j}, z)$ and the Fibonacci polynomials $F_k(-1, z)$ are established by induction on k as follows:

$$F_k(\omega^{t_j}, z) = b_j^{k-1}F_k(-1, b_j^{-1}z), \text{ for } k \ge 1.$$

In particular, $F_n(\omega^{t_j}, z) = b_j^{n-1}F_n(-1, b_j^{-1}z)$. Since the distinct roots of $F_n(-1, z) = 0$ are $\alpha_k = 2i\cos\frac{k\pi}{n} = i(\alpha^k + \alpha^{-k})$, for $1 \le k \le n-1$ (here $i^2 = -1$), see [45]. It follows that the distinct roots of $F_n(\omega^{t_j}, z) = 0$ are

$$\begin{aligned}
\alpha_{j,k} &= 2b_j i\cos\frac{k\pi}{n} \\
&= (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(\cos(\frac{t_j\pi}{l} + \frac{3\pi}{2}) + i\sin(\frac{t_j\pi}{l} + \frac{3\pi}{2}))(\alpha^k + \alpha^{-k}) \\
&= (\cos\frac{t_j\pi}{l} + i\sin\frac{t_j\pi}{l})(\alpha^k + \alpha^{-k}) \\
&= \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k}),
\end{aligned}$$

for $1 \le k \le n-1$. Here $\sqrt{\omega^{t_j}}$ always means $\cos\frac{t_j\pi}{l} + i\sin\frac{t_j\pi}{l}$. This implies that the equation (2.17) has roots $\omega^{t_j} + 1$ and $\sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})$, for $1 \le k \le n-1$. Now

$\omega^{t_j} + 1 = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})$ if and only if $\cos\frac{t_j\pi}{l} = \cos\frac{k\pi}{n}$ if and only if $k = s$ and $t_j = \frac{ls}{n}$, for a unique $1 \le s \le n-1$, as desired. □

Proposition 2.5.5. *Let $R(H) := \mathbb{C} \otimes_{\mathbb{Z}} r(H)$ be the complexified Green algebra.*

(1) *$R(H)$ has exactly $n|\Omega_0| - d_3$ simple modules and each of them is of dimension one;*

(2) *The dimension of the Jacobson radical $J(R(H))$ of $R(H)$ is d_3.*

Proof. (1) The fact that $R(H)$ is commutative and the ground field is $\mathbb{C}$ implies that each simple $R(H)$-module is of dimension one. According to Theorem 2.3.4, the Green ring $r(H)$ is isomorphic to $r(\mathbb{C}G)[z]/I$. This means that each one dimensional $R(H)$-module is a lift from a one dimensional $R(\mathbb{C}G)[z]$-module. Since the action of a on each simple $R(\mathbb{C}G)$-module W_j is a scalar multiple by some ω^{t_j}, the $R(\mathbb{C}G)$-module W_j becomes a simple $R(H)$-module if and only if the action of z on W_j is a scalar multiple by a root of the equation (2.17). This equation has N_j distinct roots. We conclude that the number of non-isomorphic simple $R(H)$-modules lifted from W_j is N_j. By Lemma 2.5.4, the number of non-isomorphic simple $R(H)$-modules is

$$\sum_{j\in\Omega_0} N_j = (|\Omega_0| - d_3)n + d_3(n-1) = |\Omega_0|n - d_3,$$

as desired.

(2) Note that the dimension of $R(H)$ is $|\Omega_0|n$ and the dimension of the quotient algebra $R(H)/J(R(H))$ is equal to the number of non-isomorphic simple $R(H)$-modules, namely, $|\Omega_0|n - d_3$. Thus, the dimension of the Jacobson radical $J(R(H))$ of $R(H)$ is d_3. □

Now let θ be the following element in $r(\mathbb{C}G)$:

$$\theta = (1-a)(1 + a^n + a^{2n} + \cdots + a^{(\frac{l}{n}-1)n}),$$

and (θ) the principal ideal of $r(\mathbb{C}G)$ generated by θ. Then $\mathbb{C} \otimes_{\mathbb{Z}} (\theta)$ is an ideal of $R(\mathbb{C}G) = \mathbb{C} \otimes_{\mathbb{Z}} r(\mathbb{C}G)$. Multiplying (θ) by the element $M[0,n]$, we get a $\mathbb{Z}$-submodule $M[0,n](\theta)$ of $r(H)$. Since $r(H)$ is freely over $\mathbb{Z}$, the submodule $M[0,n](\theta)$ of $r(H)$ is freely as well.

Lemma 2.5.6. *We have the following:*

(1) *The rank of* (θ) *is* d_3.

(2) *The rank of* $M[0,n](\theta)$ *is equal to the rank of* (θ).

Proof. (1) Note that the quotient algebra $R(\mathbb{C}G)/(\mathbb{C}\otimes_{\mathbb{Z}}(\theta))$ is commutative semisimple. We first determine the dimension of $R(\mathbb{C}G)/(\mathbb{C}\otimes_{\mathbb{Z}}(\theta))$ by calculating the number of non-isomorphic simple $R(\mathbb{C}G)/(\mathbb{C}\otimes_{\mathbb{Z}}(\theta))$-modules. Observe that each simple $R(\mathbb{C}G)/(\mathbb{C}\otimes_{\mathbb{Z}}(\theta))$-module is precisely a simple $R(\mathbb{C}G)$-module W_j such that $\theta W_j = 0$, whereas, $\theta W_j = 0$ if and only if

$$(1-\omega^{t_j})(1+\omega^{nt_j}+\omega^{2nt_j}\cdots+\omega^{(\frac{l}{n}-1)nt_j})=0,$$

if and only if $j\in\Omega_0^1\cup\Omega_0^2$. As a consequence, there exist exactly d_1+d_2 distinct simple $R(\mathbb{C}G)/(\mathbb{C}\otimes_{\mathbb{Z}}(\theta))$-modules and hence the dimension of $R(\mathbb{C}G)/(\mathbb{C}\otimes_{\mathbb{Z}}(\theta))$ is d_1+d_2. It follows that the dimension of $\mathbb{C}\otimes_{\mathbb{Z}}(\theta)$ is $|\Omega_0|-(d_1+d_2)=d_3$. Therefore, as a free $\mathbb{Z}$-module, the rank of (θ) is d_3.

(2) We prove the general case: if I is an ideal of $r(\mathbb{C}G)$, then both I and $M[0,n]I$ have the same rank as free $\mathbb{Z}$-modules. Let $\theta_1,\theta_2,\cdots,\theta_k$ be a $\mathbb{Z}$-basis of I. It is obvious that $M[0,n]I$ is generated as a $\mathbb{Z}$-module by $M[0,n]\theta_1, M[0,n]\theta_2,\cdots,M[0,n]\theta_k$. We claim that the foregoing generators form a $\mathbb{Z}$-basis of $M[0,n]I$. Indeed, if

$$\alpha_1M[0,n]\theta_1+\alpha_2M[0,n]\theta_2+\cdots+\alpha_kM[0,n]\theta_k=0,$$

where each $\alpha_i\in\mathbb{Z}$, then

$$M[0,n](\alpha_1\theta_1+\alpha_2\theta_2+\cdots+\alpha_k\theta_k)=0. \tag{2.18}$$

Denote by $K_0(H)$ the Grothendieck group of the category of finite dimensional projective left H-modules. That is, the abelian group generated by the isomorphism classes $M[i,n]$ of projective H-modules $M(i,n)$ modulo the relations

$$[M(i,n)\oplus M(j,n)]=M[i,n]+M[j,n].$$

Then $K_0(H)$ is a free abelian group with a basis $\{M[i,n]\mid i\in\Omega_0\}$. It is obvious that $K_0(H)$ admits a right action from $G_0(H)=r(\mathbb{C}G)$:

$$M[i,n]\cdot[V_j]=[M(i,n)\otimes V_j].$$

Thus $K_0(H)$ is a right module over $G_0(H)$. In fact, $K_0(H)$ is freely of rank 1 with the generator $M[0,n]$ as a right $G_0(H)$-module. Now the equation (2.18) implies that

$$K_0(H)(\alpha_1\theta_1 + \alpha_2\theta_2 + \cdots + \alpha_k\theta_k) = 0.$$

Hence $\alpha_1\theta_1 + \alpha_2\theta_2 + \cdots + \alpha_k\theta_k = 0$ since the right $G_0(H)$-module $K_0(H)$ is faithful, see [57]. We obtain that $\alpha_i = 0$, for $1 \leq i \leq k$. Therefore, the rank of I is equal to the rank of $M[0,n]I$. $\square$

Theorem 2.5.7. *The Jacobson radical of $r(H)$ is a principal ideal generated by the element $M[0,n]\theta$.*

Proof. Note that

$$(1 + a + \cdots + a^{n-1})\theta = (1 - a^n)(1 + a^n + a^{2n} \cdots + a^{(\frac{l}{n}-1)n}) = 0.$$

By Proposition 2.3.2, we obtain $M[0,n]^2 = (1 + a + \cdots + a^{n-1})M[0,n]$. This yields that $(M[0,n]\theta)^2 = 0$. Hence $M[0,n](\theta) \subseteq J(r(H))$. On the one hand, the rank of $M[0,n](\theta)$ is d_3 by Lemma 2.5.6. Consequently, the rank of $J(r(H))$ is equal or greater than d_3. On the other hand, since $\mathbb{C} \otimes_{\mathbb{Z}} J(r(H)) \subseteq J(R(H))$, the dimension of $J(R(H))$ is d_3 by Proposition 2.5.5. It follows that the rank of $J(r(H))$ is equal or less than d_3. We conclude that the rank of $J(r(H))$ is equal to d_3. Therefore,

$$J(r(H)) = M[0,n](\theta).$$

Let $(M[0,n]\theta)$ be the principal ideal of $r(H)$ generated by $M[0,n]\theta$. We have the following inclusions:

$$J(r(H)) = M[0,n](\theta) \subseteq (M[0,n]\theta) \subseteq J(r(H)).$$

It follows that $J(r(H)) = (M[0,n]\theta)$, as desired. $\square$

Remark 2.5.8. (1) In case H is a generalized Taft algebra, the Jacobson radical of the Green ring $r(H)$ has been calculated in [56], where each nilpotent element is represented by a linear combination of certain projective indecomposables. Now we understand the form from the result of Theorem 2.5.7.

(2) We have already checked that the Jacobson radical $J(r(H))$ of $r(H)$ is precisely $\mathcal{P} \cap \mathcal{P}^{\perp}$ (Theorem 2.5.2). To see that $M[0,n]\theta \in \mathcal{P} \cap \mathcal{P}^{\perp}$, by induction on k

we have that $\delta_{[\Bbbk]}(M[0,1]+\cdots+M[0,k]) = 1+aM[0,k]-M[0,k+1]$ for each $1 \leq k \leq n-1$. This implies that

$$M[0,n] = (1+a+\cdots+a^{n-1}) - \delta_{[\Bbbk]}\sum_{k=1}^{n-1}(M[0,1]+\cdots+M[0,k])a^{n-1-k}.$$

Thus,

$$M[0,n]\theta = -\delta_{[\Bbbk]}\sum_{k=1}^{n-1}(M[0,1]+\cdots+M[0,k])a^{n-1-k}\theta,$$

which belongs to $\mathcal{P}$ as well as $\mathcal{P}^{\perp}$ since $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are principal ideals of $r(H)$ generated by $M[0,n]$ and $\delta_{[\Bbbk]}$ respectively.

2.6 Idempotents of (complexified) Green rings

In this section, we first prove that the Green ring $r(H)$ of H possesses only trivial idempotents. However, the complexified Green algebra $R(H)$ has many idempotents. We shall determine all idempotents of $R(H)$ through construction. To this aim, we first need to determine the idempotents of the Green ring of a group algebra, which might be found in other literature. For the sake of completeness, we describe them as follows.

Proposition 2.6.1. *For any finite group G, the Green ring (i.e., Grothendieck ring) $r(\Bbbk G)$ of group algebra $\Bbbk G$ has only trivial idempotents.*

Proof. Suppose that $\{V_i \mid i \in \Omega_0\}$ is a complete set of all simple $\Bbbk G$-modules up to isomorphism. For any $x \in r(\Bbbk G)$, suppose $x = \sum_{i\in\Omega_0} \alpha_i[V_i]$ for each $\alpha_i \in \mathbb{Z}$. Note that the multiplicity of the trivial module $\Bbbk$ as a direct summand of $V_i \otimes V_j^*$ is 1 if $V_i \cong V_j$, and 0 otherwise (cf. [81, Lemma 1]). In view of this, the coefficient of the identity $[\Bbbk]$ in xx^* is $\sum_{i\in\Omega_0} \alpha_i^2$. Thus $x = 0$ if and only if $xx^* = 0$. If E be a primitive idempotent of $r(\Bbbk G)$, so is E^* since the dual $*$ is an anti-automorphism of $r(\Bbbk G)$. Then $E = E^*$ or $EE^* = 0$. If $EE^* = 0$, then $E = 0$. If $E = E^*$, comparing the coefficient of the identity $[\Bbbk]$ in both sides of the equation $EE^* = E$, we obtain that $E = 0$ or $E = 1$, as desired. $\square$

Theorem 2.6.2. *The Green ring $r(H)$ of H has only trivial idempotents.*

Proof. Let E be a primitive idempotent of $r(H)$. We first prove that $E \in \mathcal{P}$ or $1 - E \in \mathcal{P}$. Note that E is primitive, so is E^* since the dual $*$ is an automorphism of $r(H)$. Then $E = E^*$ or $EE^* = 0$. If $EE^* = 0$, then $E \in \mathcal{P}$ by Lemma 2.5.1 (2). If $E = E^*$, let $E = \sum_{i\in\Omega_0, 1\le j\le n-1} \alpha_{ij} M[i,j] + E_0$, for $\alpha_{ij} \in \mathbb{Z}$ and $E_0 \in \mathcal{P}$. Comparing the coefficient of the identity $[\Bbbk]$ in both sides of the equation $EE^* = E$, we obtain that $\sum_{i\in\Omega_0, 1\le j\le n-1} \alpha_{ij}^2 = \alpha_{01}$. This implies that $\alpha_{01} = 0$ or 1, and $\alpha_{ij} = 0$ for any other α_{ij}. Hence E has the form $\alpha_{01} + E_0$, and therefore $E \in \mathcal{P}$ or $1 - E \in \mathcal{P}$.

If $E \in \mathcal{P}$, we write

$$E = \sum_{i\in\Omega_0} \beta_i M[i,n] = (\sum_{i\in\Omega_0} \beta_i [V_i]) M[0,n].$$

The equality $E^2 = E$ implies that

$$(\sum_{i\in\Omega_0} \beta_i[V_i])^2 M[0,n]^2 = (\sum_{i\in\Omega_0} \beta_i[V_i])M[0,n].$$

Note that $M[0,n]^2 = (1+a+\cdots+a^{n-1})M[0,n]$. Then

$$(\sum_{i\in\Omega_0} \beta_i[V_i])^2(1+a+\cdots+a^{n-1})M[0,n] = \sum_{i\in\Omega_0} \beta_i[V_i]M[0,n].$$

As a result,

$$(\sum_{i\in\Omega_0} \beta_i[V_i])^2(1+a+\cdots+a^{n-1}) = \sum_{i\in\Omega_0} \beta_i[V_i], \tag{2.19}$$

for this, we refer to the proof of Lemma 2.5.6 (2). The equation (2.19) means that $(\sum_{i\in\Omega_0} \beta_i[V_i])(1+a+\cdots+a^{n-1})$ is an idempotent of $r(\Bbbk G)$. It follows from Proposition 2.6.1 that $(\sum_{i\in\Omega_0} \beta_i[V_i])(1+a+\cdots+a^{n-1})$ is equal to 0 or 1. However, it could not be 1 since $(1+a+\cdots+a^{n-1})\theta = 0$. Thus, $(\sum_{i\in\Omega_0} \beta_i[V_i])(1+a+\cdots+a^{n-1}) = 0$. By taking it into (2.19), we have that $\sum_{i\in\Omega_0} \beta_i[V_i] = 0$. It follows that $E = 0$.

If $1-E \in \mathcal{P}$, it is similar that $1-E = 0$, and hence $E = 1$, as desired. □

In the following, we assume that H is defined over the field $\mathbb{C}$. We shall study the idempotents of $R(H) = \mathbb{C} \otimes_{\mathbb{Z}} r(H)$ by the study of simple modules over $R(H)$.

The Green algebra $R(\mathbb{C}G) = \mathbb{C} \otimes_{\mathbb{Z}} r(\Bbbk G)$ is commutative semisimple with the basis $\{e_i \mid i \in \Omega_0\}$ consisting of all primitive orthogonal idempotents of $R(\mathbb{C}G)$ such that $e_i W_j = \delta_{i,j} W_j$, for $i, j \in \Omega_0$, where $\{W_i \mid i \in \Omega_0\}$ is the set of all non-isomorphic (one dimensional) simple $R(\mathbb{C}G)$-modules. Note that the action of a on W_j is a scalar multiple by ω^{t_j}, where $\omega = \cos\frac{2\pi}{l} + i\sin\frac{2\pi}{l}$ and $0 \le t_j \le l-1$. We have $a = \sum_{j\in\Omega_0} \omega^{t_j} e_j$.

Let $W_{j,k}$ be a simple $R(H)$-module lifted by W_j. That is, $W_{j,k}$ is the same as W_j as an $R(\Bbbk G)$-module, while the generator $M[0,2]$ of $R(H)$ acts on W_j is the scalar multiple by $\alpha_{j,k}$ determined in Lemma 2.5.4. It follows from Lemma 2.5.4 and Proposition 2.5.5 that

$$\{W_{j,k} \mid j \in \Omega_0^1 \cup \Omega_0^2, 1 \le k \le n\} \cup \{W_{j,k} \mid j \in \Omega_0^3, 1 \le k \le n-1\}$$

forms a complete set of simple $R(H)$-modules up to isomorphism. Obviously, the set above is also a complete set of simple $R(H)/J(R(H))$-modules since every simple

$R(H)$-module is annihilated by the Jacobson radical $J(R(H))$ of $R(H)$. For any simple $R(H)/J(R(H))$-module $W_{j,k}$, there exists a unique algebra morphism $\Phi_{j,k}$ from $R(H)/J(R(H))$ to $\mathbb{C}$ such that

$$\Phi_{j,k}(\overline{e_i}) = \delta_{i,j},\ \Phi_{j,k}(\overline{a}) = \omega^{t_j} \text{ and } \Phi_{j,k}(\overline{M[0,2]}) = \alpha_{j,k}.$$

Conversely, every algebra morphism from $R(H)/J(R(H))$ to $\mathbb{C}$ is determined in this way since $R(H)/J(R(H))$ is commutative semisimple over $\mathbb{C}$. Hence there is a one to one correspondence between the set of non-isomorphic simple $R(H)/J(R(H))$-modules and the set of distinct algebra morphisms from $R(H)/J(R(H))$ to $\mathbb{C}$.

Lemma 2.6.3. *For the algebra morphism $\Phi_{j,k}$ defined above, we have the following:*

(1) *If $j \in \Omega_0^1 \cup \Omega_0^2 \cup \Omega_0^3$ and $1 \leq k \leq n-1$, then*

$$\Phi_{j,k}(\overline{M[0,s]}) = (\sqrt{\omega^{t_j}})^{s-1}\frac{\alpha^{ks} - \alpha^{-ks}}{\alpha^k - \alpha^{-k}},$$

for $1 \leq s \leq n$. Moreover, $\Phi_{j,k}(\overline{M[0,n]}) = 0$.

(2) *If $j \in \Omega_0^1 \cup \Omega_0^2$ and $k = n$, then*

$$\Phi_{j,n}(\overline{M[0,s]}) = \begin{cases} \frac{1-\omega^{st_j}}{1-\omega^{t_j}}, & j \in \Omega_2, \\ s, & j \in \Omega_1, \end{cases}$$

for $1 \leq s \leq n$.

Proof. (1) By induction on s. If $s = 1$, it is trivial since $\overline{M[0,1]}$ is the identity of $R(H)/J(R(H))$. If $s = 2$, then $\Phi_{j,k}(\overline{M[0,2]}) = \alpha_{j,k} = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})$. Suppose it holds for $s \leq i$. To prove $s = i+1$, by induction assumption, we have

$$\begin{aligned}
&\Phi_{j,k}(\overline{M[0,i+1]}) \\
&= \Phi_{j,k}(\overline{M[0,2]})\Phi_{j,k}(\overline{M[0,i]}) - \Phi_{j,k}(\overline{a})\Phi_{j,k}(\overline{M[0,i-1]}) \\
&= \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})(\sqrt{\omega^{t_j}})^{i-1}((\alpha^k)^{i-1} + (\alpha^k)^{i-3} + \cdots + (\alpha^k)^{1-i}) \\
&- \omega^{t_j}(\sqrt{\omega^{t_j}})^{i-2}((\alpha^k)^{i-2} + (\alpha^k)^{i-4} + \cdots + (\alpha^k)^{2-i}) \\
&= (\sqrt{\omega^{t_j}})^i((\alpha^k)^i + (\alpha^k)^{i-2} + \cdots + (\alpha^k)^{-i}) \\
&= (\sqrt{\omega^{t_j}})^i\frac{\alpha^{k(i+1)} - \alpha^{-k(i+1)}}{\alpha^k - \alpha^{-k}}.
\end{aligned}$$

Moreover,

$$\Phi_{j,k}(\overline{M[0,n]}) = (\sqrt{\omega^{t_j}})^{n-1}\frac{\alpha^{kn}-\alpha^{-kn}}{\alpha^k-\alpha^{-k}} = 0$$

since $\alpha = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$.

(2) If $j \in \Omega_0^1 \cup \Omega_0^2$ and $k = n$, by Lemma 2.5.4, we have

$$\Phi_{j,n}(\overline{M[0,2]}) = \alpha_{j,n} = \omega^{t_j} + 1.$$

Now the result follows by the induction on s. $\square$

Let $E_{j,k}$ be an element of $R(H)$ such that $\overline{E_{j,k}}$ is a primitive idempotent of $R(H)/J(R(H))$ and $\Phi_{i,s}(\overline{E_{j,k}}) = \delta_{i,j}\delta_{s,k}$. Then

$$\{\overline{E_{j,k}} \mid j \in \Omega_0^1 \cup \Omega_0^2, 1 \le k \le n\} \cup \{\overline{E_{j,k}} \mid j \in \Omega_0^3, 1 \le k \le n-1\}$$

forms an orthogonal basis of $R(H)/J(R(H))$.

Lemma 2.6.4. *We have the follows:*

(1) *The set $\{e_jM[0,k] \mid j \in \Omega_0, 1 \le k \le n\}$ forms a basis of $R(H)$.*

(2) *The set $\{e_jM[0,n] \mid j \in \Omega_0^3\}$ forms a basis of $J(R(H))$.*

(3) *The set $\{\overline{e_jM[0,k]} \mid j \in \Omega_0^1 \cup \Omega_0^2, 1 \le k \le n\} \cup \{\overline{e_jM[0,k]} \mid j \in \Omega_0^3, 1 \le k \le n-1\}$ forms a basis of $R(H)/J(R(H))$.*

Proof. (1) Observe from (2.8) that the polynomial $F_k(a,z)$ is of degree $k-1$ with the leading coefficient 1 in the polynomial algebra $R(\Bbbk G)[z]$. Let I be the ideal of $R(\Bbbk G)[z]$ generated by the element $(z-a-1)F_n(a,z)$. Then the quotient algebra $R(\Bbbk G)[z]/I$ has a $\mathbb{C}$-basis $\overline{e_jF_k(a,z)}$, for $j \in \Omega_0$ and $1 \le k \le n$. By Theorem 2.3.4, the Green algebra $R(H)$ is isomorphic to $R(\Bbbk G)[z]/I$. Moreover, the image of $\overline{e_jF_k(a,z)}$ under the isomorphism is $e_jM[0,k]$. We conclude that $e_jM[0,k]$ for $j \in \Omega_0$ and $1 \le k \le n$, is a basis of the Green algebra $R(H)$.

(2) If $j \in \Omega_0^3$, then

$$1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-1)t_j} = \frac{1-\omega^{nt_j}}{1-\omega^{t_j}} = 0.$$

Together with $a = \sum_{j\in\Omega_0} \omega^{t_j} e_j$, we have that

$$\begin{aligned}(e_jM[0,n])^2 &= e_jM[0,n]^2 \\ &= e_j(1 + a + a^2 + \cdots + a^{n-1})M[0,n] \\ &= e_j(1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-1)t_j})M[0,n] \\ &= 0.\end{aligned}$$

This implies that $e_jM[0,n] \in J(R(H))$ for $j \in \Omega_0^3$. Moreover, it forms a basis of $J(R(H))$ since $e_jM[0,n]$ for $j \in \Omega_0^3$ is linear independent by Part (1), and the dimension of $J(R(H))$ is equal to the cardinality of Ω_0^3 by Proposition 2.5.5.

(3) This follows immediately from Part (1) and Part (2). $\square$

In the following, we shall write the primitive orthogonal idempotents $\overline{E_{j,k}}$ as a linear combination of a basis of $R(H)/J(R(H))$ given in Lemma 2.6.4 (3).

Let $\overline{\Lambda} = \sum_{i,k} \beta_{i,k}\overline{E_{i,k}}$ be an arbitrary element of $R(H)/J(R(H))$ for $\beta_{i,k} \in \mathbb{C}$. The equality $\Phi_{i,k}(\overline{E_{j,s}}) = \delta_{i,j}\delta_{k,s}$ implies that $\Phi_{i,k}(\overline{\Lambda}) = \beta_{i,k}$. It follows that

$$\overline{\Lambda} = \sum_{i,k} \Phi_{i,k}(\overline{\Lambda})\overline{E_{i,k}}.$$

As a consequence,

$$\overline{e_jM[0,s]} = \sum_{i,k} \Phi_{i,k}(\overline{e_jM[0,s]})\overline{E_{i,k}} = \sum_{k} \Phi_{j,k}(\overline{M[0,s]})\overline{E_{j,k}}, \tag{2.20}$$

where if $j \in \Omega_0^1 \cup \Omega_0^2$, then the sum $\sum_k$ runs from 1 to n; if $j \in \Omega_0^3$, then the sum $\sum_k$ runs from 1 to $n-1$.

For any $j \in \Omega_0^1 \cup \Omega_0^2 \cup \Omega_0^3$, we consider the following matrix:

$$\mathbf{A_j} = \begin{pmatrix} \Phi_{j,1}(\overline{M[0,1]}) & \Phi_{j,2}(\overline{M[0,1]}) & \cdots & \Phi_{j,n-1}(\overline{M[0,1]}) \\ \Phi_{j,1}(\overline{M[0,2]}) & \Phi_{j,2}(\overline{M[0,2]}) & \cdots & \Phi_{j,n-1}(\overline{M[0,2]}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{j,1}(\overline{M[0,n-1]}) & \Phi_{j,2}(\overline{M[0,n-1]}) & \cdots & \Phi_{j,n-1}(\overline{M[0,n-1]}) \end{pmatrix}.$$

By Lemma 2.6.3, the (s,k)-entry of the matrix $\mathbf{A_j}$ is

$$\Phi_{j,k}(\overline{M[0,s]}) = (\sqrt{\omega^{t_j}})^{s-1}\frac{\alpha^{ks}-\alpha^{-ks}}{\alpha^k-\alpha^{-k}},$$

where $1 \leq k \leq n-1$ and $\alpha^k - \alpha^{-k} \neq 0$. Let $\mathbf{B}$ be the matrix with (k,s)-entry $\alpha^{ks}-\alpha^{-ks}$ for $1 \leq k,s \leq n-1$. Let $\mathbf{C_j}$ and $\mathbf{D}$ be two diagonal matrices given by

$$\mathbf{C_j} = \begin{pmatrix} 1 & & & & \\ & \sqrt{\omega^{t_j}} & & & \\ & & (\sqrt{\omega^{t_j}})^2 & & \\ & & & \ddots & \\ & & & & (\sqrt{\omega^{t_j}})^{n-2} \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} \alpha-\alpha^{-1} & & & \\ & \alpha^2-\alpha^{-2} & & \\ & & \ddots & \\ & & & \alpha^{n-1}-\alpha^{-(n-1)} \end{pmatrix}.$$

It is clear that $\mathbf{B}$ is symmetric, and $\mathbf{A_j} = \mathbf{C_j B D^{-1}}$ with the inverse $\mathbf{A_j^{-1}} = \mathbf{D B^{-1} C_j^{-1}}$ (as we shall see that $\mathbf{A_j}$ is invertible). Suppose the (k,s)-entry of the matrix $\mathbf{B^{-1}}$ is $\theta_{k,s}$, for $1 \leq k,s \leq n-1$. Then the (k,s)-entry of the matrix $\mathbf{A_j^{-1}}$ is

$$\beta_{k,s}^{(j)} = (\sqrt{\omega^{t_j}})^{1-s}(\alpha^k-\alpha^{-k})\theta_{k,s},$$

for $1 \leq k,s \leq n-1$.

If $j \in \Omega_0^3$, then the linear relations of (2.20) can be written as follows:

$$\begin{pmatrix} \overline{e_jM[0,1]} \\ \overline{e_jM[0,2]} \\ \vdots \\ \overline{e_jM[0,n-1]} \end{pmatrix} = \mathbf{A_j} \begin{pmatrix} \overline{E_{j,1}} \\ \overline{E_{j,2}} \\ \vdots \\ \overline{E_{j,n-1}} \end{pmatrix}. \tag{2.21}$$

Observe that the matrix $\mathbf{A_j}$ given above is invertible since $\overline{e_jM[0,k]}$ (resp. $\overline{E_{j,k}}$) are linear independent for $1 \leq k \leq n-1$. By (2.21), the idempotents $\overline{E_{j,k}}$ could be expressed as a linear combination as follows:

$$\overline{E_{j,k}} = \sum_{s=1}^{n-1} \beta_{k,s}^{(j)} \overline{e_j M[0,s]} = (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} \overline{e_j M[0,s]}, \tag{2.22}$$

for $j \in \Omega_3$ and $1 \leq k \leq n-1$.

If $j \in \Omega_0^1 \cup \Omega_0^2$, then the linear relations of (2.20) can be written as follows:

$$\begin{pmatrix} \overline{e_j M[0,1]} \\ \overline{e_j M[0,2]} \\ \vdots \\ \overline{e_j M[0,n]} \end{pmatrix} = \begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix} \begin{pmatrix} \overline{E_{j,1}} \\ \overline{E_{j,2}} \\ \vdots \\ \overline{E_{j,n}} \end{pmatrix}, \tag{2.23}$$

where $\begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}$ is a block matrix with the entries determined by Lemma 2.6.3. More explicitly, the column vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 + \omega^{t_j} \\ \vdots \\ 1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-2)t_j} \end{pmatrix},$$

the row vector $\mathbf{0}$ is a zero vector and the scalar $\delta = 1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-1)t_j} \neq 0$. Similarly, the matrix $\begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}$ is invertible with the inverse given by

$$\begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}^{\mathbf{-1}} = \begin{pmatrix} \mathbf{A_j^{-1}} & -\delta^{-1}\mathbf{A_j^{-1}b} \\ \mathbf{0} & \delta^{-1} \end{pmatrix},$$

where $-\delta^{-1}\mathbf{A_j^{-1}b}$ is a column vector with the k-th entry

$$-\delta^{-1} \sum_{s=1}^{n-1} (1 + \omega^{t_j} + \cdots + \omega^{(s-1)t_j}) \beta_{k,s}^{(j)},$$

for $1 \leq k \leq n-1$. Now the idempotents $\overline{E_{j,k}}$ could be expressed as follows:

$$\begin{aligned}
\overline{E_{j,k}} &= \sum_{s=1}^{n-1} \beta_{k,s}^{(j)} \overline{e_j M[0,s]} - \delta^{-1} \sum_{s=1}^{n-1} (1+\omega^{t_j} + \cdots + \omega^{(s-1)t_j}) \beta_{k,s}^{(j)} \overline{e_j M[0,n]} \\
&= \sum_{s=1}^{n-1} \beta_{k,s}^{(j)} (\overline{e_j M[0,s]} - \frac{1+\omega^{t_j} + \cdots + \omega^{(s-1)t_j}}{1+\omega^{t_j} + \cdots + \omega^{(n-1)t_j}} \overline{e_j M[0,n]}) \qquad (2.24) \\
&= (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} (\overline{e_j M[0,s]} - \frac{1+\omega^{t_j} + \cdots + \omega^{(s-1)t_j}}{1+\omega^{t_j} + \cdots + \omega^{(n-1)t_j}} \overline{e_j M[0,n]})
\end{aligned}$$

for $j \in \Omega_0^1 \cup \Omega_0^2$ and $1 \leq k \leq n-1$. If $j \in \Omega_0^1 \cup \Omega_0^2$ and $k = n$, then

$$\overline{E_{j,n}} = \delta^{-1} \overline{e_j M[0,n]} = \frac{1}{1+\omega^{t_j} + \cdots + \omega^{(n-1)t_j}} \overline{e_j M[0,n]}. \qquad (2.25)$$

We have obtained the primitive orthogonal idempotents $\overline{E_{j,k}}$ as a linear combination of a basis of $R(H)/J(R(H))$ as shown in (2.22), (2.24) and (2.25) for each case. In the following, we delete the upper bar in the equations (2.22), (2.24) and (2.25) and obtain the elements $E_{j,k}$ in $R(H)$ as follows:

- $E_{j,k} := (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} e_j M[0,s]$, for $j \in \Omega_0^3$ and $1 \leq k \leq n-1$
- $E_{j,k} := (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} e_j (M[0,s] - \frac{1+\omega^{t_j}+\cdots+\omega^{(s-1)t_j}}{1+\omega^{t_j}+\cdots+\omega^{(n-1)t_j}} M[0,n])$, for $j \in \Omega_0^1 \cup \Omega_0^2$ and $1 \leq k \leq n-1$.
- $E_{j,k} := \frac{1}{1+\omega^{t_j}+\cdots+\omega^{(n-1)t_j}} e_j M[0,n]$, for $j \in \Omega_0^1 \cup \Omega_0^2$ and $k = n$.

Now the idempotents of $R(H)$ can be described explicitly as follows.

Theorem 2.6.5. *Let $e_{j,k}$ be the idempotent of $R(H)$ such that $\overline{e_{j,k}} = \overline{E_{j,k}}$.*

(1) *If $j \in \Omega_0^1 \cup \Omega_0^2$, then $e_{j,k} = E_{j,k}$, for $1 \leq k \leq n$.*

(2) *If $j \in \Omega_0^3$, then $e_{j,k} = E_{j,k} + \gamma_{j,k} e_j M[0,n]$, for $1 \leq k \leq n-1$, where*

$$\gamma_{j,k} = (1 - 2\delta_{k,\frac{nt_j}{l}}) \frac{\alpha^{\frac{nt_j}{l}} (\alpha^k - \alpha^{-k})^2}{\alpha^{\frac{nt_j}{l}} - \alpha^{-\frac{nt_j}{l}}} \sum_{s+t-1 \geq n} \theta_{k,s} \theta_{k,t} (\alpha^{\frac{(s+t)nt_j}{l}} - \alpha^{-\frac{(s+t)nt_j}{l}}).$$

Proof. (1) Note that $e_{j,k}$ is the idempotent of $R(H)$ such that $\overline{e_{j,k}} = \overline{E_{j,k}}$. It follows that $e_{j,k} - E_{j,k} \in J(R(H))$. For any $i \neq j$, we obtain that $e_i e_{j,k} \in e_i J(R(H)) \subseteq$

$J(R(H))$ since $e_i(e_{j,k}-E_{j,k})\in e_iJ(R(H))$ and $e_iE_{j,k}=0$. It follows that $e_ie_{j,k}=0$ because $e_ie_{j,k}$ is idempotent. Hence $e_{j,k}$ belongs to $e_jR(H)$. By Lemma 2.6.4 (2), we have that

$$\begin{aligned}e_{j,k}-E_{j,k}&\in e_jR(H)\cap J(R(H))\\&=e_jJ(R(H))\\&=\begin{cases}\mathrm{sp}\{e_jM[0,n]\}, & j\in\Omega_0^3;\\ 0, & j\in\Omega_0^1\cup\Omega_0^2.\end{cases}\end{aligned}\tag{2.26}$$

Therefore, Part (1) is proved.

(2) By (2.26), we denote by $e_{j,k}=E_{j,k}+\gamma_{j,k}e_jM[0,n]$ for $j\in\Omega_0^3$ and $\gamma_{j,k}\in\mathbb{C}$. According to Theorem 2.5.7, $(J(R(H)))^2=0$, we have

$$E_{j,k}+\gamma_{j,k}e_jM[0,n]=(E_{j,k}+\gamma_{j,k}e_jM[0,n])^2=E_{j,k}^2+2\gamma_{j,k}e_jM[0,n]E_{j,k}.$$

This implies that

$$E_{j,k}^2-E_{j,k}=\gamma_{j,k}(e_jM[0,n]-2e_jM[0,n]E_{j,k}).$$

Note that $a=\sum_{j\in\Omega_0}\omega^{t_j}e_j$ and $e_ja=\omega^{t_j}e_j$. We have

$$\begin{aligned}&e_jM[0,n]E_{j,k}\\&=(\alpha^k-\alpha^{-k})\sum_{s=1}^{n-1}(\sqrt{\omega^{t_j}})^{1-s}\theta_{k,s}e_jM[0,n]M[0,s]\\&=(\alpha^k-\alpha^{-k})\sum_{s=1}^{n-1}(\sqrt{\omega^{t_j}})^{1-s}\theta_{k,s}e_j(1+a+\cdots+a^{s-1})M[0,n]\\&=(\alpha^k-\alpha^{-k})\sum_{s=1}^{n-1}(\sqrt{\omega^{t_j}})^{1-s}\theta_{k,s}(1+\omega^{t_j}+\cdots+\omega^{(s-1)t_j})e_jM[0,n]\\&=(\alpha^k-\alpha^{-k})\sum_{s=1}^{n-1}(\sqrt{\omega^{t_j}})^{1-s}\theta_{k,s}\frac{\omega^{st_j}-1}{\omega^{t_j}-1}e_jM[0,n].\end{aligned}\tag{2.27}$$

Note that $j\in\Omega_0^3$ implies that $\frac{l}{n}\mid t_j$. Suppose $t_j=\frac{l}{n}p$ for some integer p, then

$$\sqrt{\omega^{t_j}}=\cos\frac{t_j\pi}{l}+i\sin\frac{t_j\pi}{l}=\cos\frac{p\pi}{n}+i\sin\frac{p\pi}{n}=\alpha^p=\alpha^{\frac{nt_j}{l}}.$$

Now (2.27) can be written as

$$\begin{aligned}
e_jM[0,n]E_{j,k} &= (\alpha^k - \alpha^{-k})\sum_{s=1}^{n-1}\alpha^{\frac{(1-s)nt_j}{l}}\theta_{k,s}\frac{\alpha^{\frac{2snt_j}{l}}-1}{\alpha^{\frac{2nt_j}{l}}-1}e_jM[0,n] \\
&= \frac{\alpha^k-\alpha^{-k}}{\alpha^{\frac{nt_j}{l}}-\alpha^{-\frac{nt_j}{l}}}e_jM[0,n]\sum_{s=1}^{n-1}\theta_{k,s}(\alpha^{\frac{snt_j}{l}}-\alpha^{-\frac{snt_j}{l}}) \\
&= \frac{\alpha^k-\alpha^{-k}}{\alpha^{\frac{nt_j}{l}}-\alpha^{-\frac{nt_j}{l}}}e_jM[0,n]\delta_{k,\frac{nt_j}{l}} \\
&= \delta_{k,\frac{nt_j}{l}}e_jM[0,n],
\end{aligned}$$

here $\sum_{s=1}^{n-1}\theta_{k,s}(\alpha^{\frac{snt_j}{l}}-\alpha^{-\frac{snt_j}{l}}) = \delta_{k,\frac{nt_j}{l}}$ since $\mathbf{B^{-1}B} = \mathbf{E}$. Hence,

$$E_{j,k}^2 - E_{j,k} = \gamma_{j,k}(1-2\delta_{k,\frac{nt_j}{l}})e_jM[0,n]. \tag{2.28}$$

The rest is to determine the coefficient of the term $e_jM[0,n]$ in $E_{j,k}^2 - E_{j,k}$. Note that $E_{j,k}$ has no the term $e_jM[0,n]$. It suffices to consider the coefficient of the term $e_jM[0,n]$ in $E_{j,k}^2$. Note that

$$\begin{aligned}
E_{j,k}^2 &= ((\alpha^k-\alpha^{-k})\sum_{s=1}^{n-1}(\sqrt{\omega^{t_j}})^{1-s}\theta_{k,s}e_jM[0,s])^2 \\
&= (\alpha^k-\alpha^{-k})^2\sum_{s,t=1}^{n-1}(\sqrt{\omega^{t_j}})^{2-s-t}\theta_{k,s}\theta_{k,t}e_jM[0,s]M[0,t].
\end{aligned}$$

By Proposition 2.3.2, we have that the term $e_jM[0,n]$ appears in $e_jM[0,s]M[0,t]$ if and only if $s+t-1 \geq n$. In this case, it is straightforward to check that the term $e_jM[0,n]$ in $e_jM[0,s]M[0,t]$ is

$$\begin{aligned}
&\sum_{s+t-1\geq n} e_j(\sum_{q=0}^{s+t-1-n} M[\tau^q(0),n]) = \sum_{s+t-1\geq n} e_j(\sum_{q=0}^{s+t-1-n} a^q)M[0,n] \\
&= \sum_{s+t-1\geq n}(\sum_{q=0}^{s+t-1-n}\omega^{qt_j})e_jM[0,n] = \sum_{s+t-1\geq n}\frac{1-\omega^{(s+t-n)t_j}}{1-\omega^{t_j}}e_jM[0,n] \\
&= \sum_{s+t-1\geq n}\frac{1-\omega^{(s+t)t_j}}{1-\omega^{t_j}}e_jM[0,n].
\end{aligned}$$

We conclude that the coefficient of the term $e_jM[0,n]$ in $E^2_{j,k}$ is

$$\begin{aligned}&(\alpha^k-\alpha^{-k})^2\sum_{s+t-1\geq n}(\sqrt{\omega^{t_j}})^{2-s-t}\theta_{k,s}\theta_{k,t}\frac{1-\omega^{(s+t)t_j}}{1-\omega^{t_j}}\\ =&\frac{\alpha^{\frac{nt_j}{l}}(\alpha^k-\alpha^{-k})^2}{\alpha^{\frac{nt_j}{l}}-\alpha^{-\frac{nt_j}{l}}}\sum_{s+t-1\geq n}\theta_{k,s}\theta_{k,t}(\alpha^{\frac{(s+t)nt_j}{l}}-\alpha^{-\frac{(s+t)nt_j}{l}})\end{aligned}$$

since $\sqrt{\omega^{t_j}}=\alpha^{\frac{nt_j}{l}}$. Comparing the scalars of the equation (2.28), we conclude that

$$\frac{\alpha^{\frac{nt_j}{l}}(\alpha^k-\alpha^{-k})^2}{\alpha^{\frac{nt_j}{l}}-\alpha^{-\frac{nt_j}{l}}}\sum_{s+t-1\geq n}\theta_{k,s}\theta_{k,t}(\alpha^{\frac{(s+t)nt_j}{l}}-\alpha^{-\frac{(s+t)nt_j}{l}})=\gamma_{j,k}(1-2\delta_{k,\frac{nt_j}{l}}).$$

Therefore,

$$\gamma_{j,k}=(1-2\delta_{k,\frac{nt_j}{l}})\frac{\alpha^{\frac{nt_j}{l}}(\alpha^k-\alpha^{-k})^2}{\alpha^{\frac{nt_j}{l}}-\alpha^{-\frac{nt_j}{l}}}\sum_{s+t-1\geq n}\theta_{k,s}\theta_{k,t}(\alpha^{\frac{(s+t)nt_j}{l}}-\alpha^{-\frac{(s+t)nt_j}{l}}).$$

We complete the proof. $\square$

2.7 Idempotents of the Green algebra of a Taft algebra

In this section, as an example, we will determine all primitive idempotents of the Green algebra of Taft algebra T_3.

Let $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and $\omega = \alpha^2$. Then ω is a primitive 3-th root of unity. The Taft algebra T_3 is generated over the ground field $\mathbb{C}$ by two elements g and y subject to the relations (cf. [24, 69])

$$g^3 = 1,\ y^3 = 0,\ yg = \omega gy.$$

T_3 is a Hopf algebra with comultiplication $\triangle$, counit ε, and the antipode S given respectively by

$$\triangle(y) = y \otimes g + 1 \otimes y,\ \varepsilon(y) = 0,\ S(y) = -yg^{-1},$$

$$\triangle(g) = g \otimes g,\ \varepsilon(g) = 1,\ S(g) = g^{-1}.$$

Note that $\dim T_3 = 9$ and $\{g^i y^j \mid 0 \leq i, j \leq 2\}$ forms a $\mathbb{C}$-basis for T_3. Let G be the cyclic group generated by g and χ a $\mathbb{C}$-linear character of G such that $\chi(g) = \omega$. Then T_3 is the pointed rank one Hopf algebra associated to the group datum $\mathcal{D} = (G, \chi, g, 0)$ of nilpotent type. Thus, by the result of Theorem 2.1.4, $\{M(i,j) \mid i \in \Omega_0, 1 \leq j \leq 3\}$ forms a complete set of indecomposable T_3-modules up to isomorphism, where $\Omega_0 = \{0, 1, 2\}$.

The Green ring $r(T_3)$ of T_3 is commutative with a $\mathbb{Z}$-basis $M[i,j]$ for $0 \leq i \leq 2$ and $1 \leq j \leq 3$. Denote by a one of $M[i,1]$ for $i \in \Omega_0$ such that the character of $M(i,1)$ as a simple $\mathbb{C}G$-module is χ^{-1}. The multiplication formulas of Green ring $r(T_3)$ is stated as follows: $M[0,1] = 1$, the identity of $r(T_3)$, $a^3 = 1$, $M[i,j] = a^i M[0,j]$ and

$$M[0,2]M[0,2] = a + M[0,3],$$

$$M[0,2]M[0,3] = (1+a)M[0,3],$$

$$M[0,3]M[0,3] = (1+a+a^2)M[0,3].$$

By Theorem 2.3.4, see also [24, Theorem 3.10], the Green ring $r(T_3)$ is isomorphic to the quotient ring

$$\mathbb{Z}[a,z]/(a^3-1,(1+a-z)F_3(a,z)),$$

which admits only trivial idempotents. Let $R(T_3)$ be the complexified Green algebra. That is, $R(T_3)$ is isomorphic to the algebra $\mathbb{C}[a,z]/(a^3-1,(1+a-z)F_3(a,z))$. In the following, we follow the notations given in Section 2.6 and determine all primitive idempotents of $R(T_3)$.

Let $R(\mathbb{C}G)$ be the complexified Green algebra of the group algebra $\mathbb{C}G$. Then $R(\mathbb{C}G)$ is isomorphic to $\mathbb{C}[a]/(a^3-1)$, which is a subalgebra of $R(T_3)$. It is obvious that the primitive idempotents of $\mathbb{C}[a]/(a^3-1)$ are

$$e_j = \frac{1}{3}(1+\omega^{-j}a+\omega^{-2j}a^2),$$

for $0 \leq j \leq 2$, see e.g., [70, Equation (2.1)]. It follows that $a = e_0 + \omega e_1 + \omega^2 e_2$. Let W_j for $0 \leq j \leq 2$ be all (one dimensional) simple modules over $R(\mathbb{C}G)$ such that the generator a acts on W_j is a scalar multiple by ω^j (i.e., $t_j = j$ in this case). Then the subsets Ω_0^1, Ω_0^2 and Ω_0^3 of Ω_0 given respectively in section 2.5 are $\Omega_0^1 = \{0\}$, $\Omega_0^2 = \emptyset$ and $\Omega_0^3 = \{1,2\}$. Let $W_{i,j}$ be the same as W_j as a simple $R(\mathbb{C}G)$-module while the generator z acts on it as the scalar multiple by $\alpha^i(\alpha^j+\alpha^{-j})$ for $0 \leq i \leq 2$ and $1 \leq j \leq 2$. Also, let $W_{0,3}$ be W_0 as an $R(\mathbb{C}G)$-module and z acts on $W_{0,3}$ as the scalar multiple by 2. Then $\{W_{i,j} \mid 0 \leq i \leq 2, 1 \leq j \leq 2\} \cup \{W_{0,3}\}$ forms all simple $R(T_3)$-modules up to isomorphism (Proposition 2.5.5 (1)).

Now the matrices $\mathbf{B}$, $\mathbf{C_j}$ for $0 \leq j \leq 2$ and $\mathbf{D}$ given in Section 2.6 can be written as follows:

$$\mathbf{B} = (\alpha-\alpha^{-1})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{C_j} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^j \end{pmatrix}, \text{ and } \mathbf{D} = \begin{pmatrix} \alpha-\alpha^{-1} & 0 \\ 0 & \alpha^2-\alpha^{-2} \end{pmatrix}.$$

It follows that

$$\mathbf{A_j} = \mathbf{C_j}\mathbf{B}\mathbf{D^{-1}} = \begin{pmatrix} 1 & 1 \\ \alpha^j & -\alpha^j \end{pmatrix},$$

Let $E_{i,j} \in R(T_3)$ such that $\overline{E_{ij}}$ is an idempotent of $R(T_3)/J(R(T_3))$ determined by simple module $W_{i,j}$. Namely, $\overline{E_{i,j}} \cdot W_{k,l} = \delta_{i,k}\delta_{j,l}W_{k,l}$. Then the equations (2.21) and (2.23) become the following

$$\begin{pmatrix} \overline{e_j M[0,1]} \\ \overline{e_j M[0,2]} \end{pmatrix} = \mathbf{A_j} \begin{pmatrix} \overline{E_{j,1}} \\ \overline{E_{j,2}} \end{pmatrix}, \tag{2.29}$$

for $1 \leq j \leq 2$, and

$$\begin{pmatrix} \overline{e_0 M[0,1]} \\ \overline{e_0 M[0,2]} \\ \overline{e_0 M[0,3]} \end{pmatrix} = \begin{pmatrix} \mathbf{A_0} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix} \begin{pmatrix} \overline{E_{0,1}} \\ \overline{E_{0,2}} \\ \overline{E_{0,3}} \end{pmatrix}, \tag{2.30}$$

where $\mathbf{b}$ is the column vector $\binom{1}{2}$ and $\delta = 3$. Since $\mathbf{A}_j$ and $\begin{pmatrix} \mathbf{A_0} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}$ are both invertible with the inverse matrices given respectively by

$$\mathbf{A_j^{-1}} = \frac{1}{2}\begin{pmatrix} 1 & \alpha^{-j} \\ 1 & -\alpha^{-j} \end{pmatrix},$$

for $0 \leq j \leq 2$, and

$$\begin{pmatrix} \mathbf{A_0} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A_0^{-1}} & -\frac{1}{3}\mathbf{A_0^{-1}b} \\ \mathbf{0} & \frac{1}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 & 3 & -3 \\ 3 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

In view of this, all primitive idempotents of $R(T_3)/J(R(T_3))$ are determined by (2.29) and (2.30) and can be stated explicitly as follows:

- $\overline{E_{j,1}} = \frac{1}{2}\overline{e_j M[0,1]} + \frac{\alpha^{-j}}{2}\overline{e_j M[0,2]}$ for $1 \leq j \leq 2$,
- $\overline{E_{j,2}} = \frac{1}{2}\overline{e_j M[0,1]} - \frac{\alpha^{-j}}{2}\overline{e_j M[0,2]}$ for $1 \leq j \leq 2$,
- $\overline{E_{0,1}} = \frac{1}{2}\overline{e_0 M[0,1]} + \frac{1}{2}\overline{e_0 M[0,2]} - \frac{1}{2}\overline{e_0 M[0,3]}$,
- $\overline{E_{0,2}} = \frac{1}{2}\overline{e_0 M[0,1]} - \frac{1}{2}\overline{e_0 M[0,2]} + \frac{1}{6}\overline{e_0 M[0,3]}$,
- $\overline{E_{0,3}} = \frac{1}{3}\overline{e_0 M[0,3]}$.

In the following, we shall lift all the idempotents $\overline{E_{i,j}}$ of the quotient algebra $R(T_3)/J(R(T_3))$ to the Green algebra $R(T_3)$. We first delete the upper bar in the above equations and obtain the element $E_{j,k}$ in $R(H)$ as follows:

$$E_{j,1} := e_j(\frac{1}{2}M[0,1] + \frac{\alpha^{-j}}{2}M[0,2]) \text{ for } 1 \le j \le 2,$$

$$E_{j,2} := e_j(\frac{1}{2}M[0,1] - \frac{\alpha^{-j}}{2}M[0,2]) \text{ for } 1 \le j \le 2,$$

$$E_{0,1} := e_0(\frac{1}{2}M[0,1] + \frac{1}{2}M[0,2] - \frac{1}{2}M[0,3]),$$

$$E_{0,2} := e_0(\frac{1}{2}M[0,1] - \frac{1}{2}M[0,2] + \frac{1}{6}M[0,3]),$$

$$E_{0,3} := \frac{1}{3}e_0M[0,3].$$

We need to compute the scalar $\gamma_{j,k}$ described in Theorem 2.6.5. Note that the (k,s)-entry of the matrix $\mathbf{B^{-1}}$ is $\theta_{k,s}$. Then

$$\begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} = \mathbf{B^{-1}} = \frac{1}{2(\alpha - \alpha^{-1})}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now by Theorem 2.6.5, for $1 \le j,k \le 2$, we have

$$\begin{aligned}
\gamma_{j,k} &= (1-2\delta_{k,j})\frac{\alpha^j(\alpha^k - \alpha^{-k})^2}{\alpha^j - \alpha^{-j}} \sum_{s+t-1\ge 3} \theta_{k,s}\theta_{k,t}(\alpha^{(s+t)j} - \alpha^{-(s+t)j}) \\
&= (1-2\delta_{k,j})\frac{\alpha^j(\alpha^k - \alpha^{-k})^2}{\alpha^j - \alpha^{-j}} \theta_{k,2}\theta_{k,2}(\alpha^{4j} - \alpha^{-4j}) \\
&= (1-2\delta_{k,j})\frac{\alpha^j(\alpha^k - \alpha^{-k})^2}{\alpha^j - \alpha^{-j}} (\frac{1}{2(\alpha - \alpha^{-1})})^2(\alpha^{4j} - \alpha^{-4j}) \\
&= \begin{cases} \frac{\alpha}{4}, & (j,k) = (1,1) \\ -\frac{\alpha}{4}, & (j,k) = (1,2) \\ \frac{\alpha^2}{4}, & (j,k) = (2,1) \\ -\frac{\alpha^2}{4}, & (j,k) = (2,2) \end{cases} \\
&= (-1)^{k-1}\frac{\alpha^j}{4}.
\end{aligned}$$

It follows from Theorem 2.6.5 that all primitive idempotents $e_{i,j}$ of $R(T_3)$ are exactly as follows:

- $e_{j,1} = E_{j,1} + \gamma_{j,1}e_jM[0,3] = e_j(\frac{1}{2}M[0,1] + \frac{\alpha^{-j}}{2}M[0,2] + \frac{\alpha^j}{4}M[0,3])$ for $1 \le j \le 2$,

- $e_{j,2} = E_{j,2} + \gamma_{j,2} e_j M[0,3] = e_j(\frac{1}{2}M[0,1] - \frac{\alpha^{-j}}{2}M[0,2] - \frac{\alpha^j}{4}M[0,3])$ for $1 \le j \le 2$,
- $e_{0,1} = E_{0,1} = e_0(\frac{1}{2}M[0,1] + \frac{1}{2}M[0,2] - \frac{1}{2}M[0,3])$,
- $e_{0,2} = E_{0,2} = e_0(\frac{1}{2}M[0,1] - \frac{1}{2}M[0,2] + \frac{1}{6}M[0,3])$,
- $e_{0,3} = E_{0,3} = \frac{1}{3}e_0 M[0,3]$.

For instance, to see that $e_{j,2}^2 = e_{j,2}$, by using the equalities $e_j a = \omega^j e_j = \alpha^{2j} e_j$ and $1 + \alpha^{2j} + \alpha^{4j} = 0$ for $1 \le j \le 2$, we have that

$$\begin{aligned}
e_{j,2}^2 &= e_j^2(\frac{1}{2}M[0,1] - \frac{\alpha^{-j}}{2}M[0,2] - \frac{\alpha^j}{4}M[0,3])^2 \\
&= e_j(\frac{1+\alpha^{-2j}a}{4}M[0,1] - \frac{\alpha^{-j}}{2}M[0,2]) \\
&+ e_j((\frac{\alpha^{-2j} - \alpha^j + 1 + a}{4} + \frac{\alpha^{2j}(1+a+a^2)}{16})M[0,3]) \\
&= e_j(\frac{1}{2}M[0,1] - \frac{\alpha^{-j}}{2}M[0,2]) \\
&+ e_j((\frac{\alpha^{-2j} - \alpha^j + 1 + \alpha^{2j}}{4} + \frac{\alpha^{2j}(1+\alpha^{2j}+\alpha^{4j})}{16})M[0,3]) \\
&= e_j(\frac{1}{2}M[0,1] - \frac{\alpha^{-j}}{2}M[0,2] - \frac{\alpha^j}{4}M[0,3]) \\
&= e_{j,2}.
\end{aligned}$$

2.8 An example

In this section, we compute the Green ring of a pointed rank one Hopf algebra H of nilpotent type such that the group $G(H)$ is a dihedral group.

Assume that $s > 0$ is a fixed odd integer. The dihedral group of order $2s$ is defined as follows:

$$D_{2s} = \langle b, c \mid b^2 = c^{2s} = (cb)^2 = 1\rangle,$$

where c^s is the unique non-trivial central element of D_{2s}. The simple modules over the group algebra $\mathbb{C}D_{2s}$ are given as follows [66]:

- four simple modules of dimension one :

$$F(i,j) : c \mapsto (-1)^i,\ b \mapsto (-1)^j,\ i,j \in \mathbb{Z}_2,$$

- $s-1$ simple modules of dimension 2:

$$V(l) : c \mapsto \begin{pmatrix} \theta^l & 0 \\ 0 & \theta^{-l} \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ 1 \le l \le s-1,$$

  where θ is a $2s$-th primitive root of unity.

For the sake of completeness, we include the Grothendieck ring $G_0(\mathbb{C}D_{2s})$ of the dihedral group $\mathbb{C}D_{2s}$ described by generators and relations. The result might be found in other literature. We know that the number of simple $\mathbb{C}D_{2s}$-modules is $4 + s - 1 = s + 3$, and so is the rank of $G_0(\mathbb{C}D_{2s})$. The following decompositions of the tensor products of simple modules over $\mathbb{C}D_{2s}$ is straightforward and hence we omit the proof.

Proposition 2.8.1. (1) $F(i,j) \otimes F(k,t) \cong F(i+k, j+t),\ i,j,k,t \in \mathbb{Z}_2.$

(2) $F(0,j) \otimes V(l) \cong V(l),\ j \in \mathbb{Z}_2,\ 1 \le l \le s-1.$

(3) $F(1,j) \otimes V(l) \cong V(s-l),\ j \in \mathbb{Z}_2,\ 1 \le l \le s-1.$

(4) $V(l) \otimes V(h) \cong V(l-h) \oplus V(l+h),\ 1 \le l,h \le \frac{s-1}{2}$ *and* $l > h.$

(5) $V(l) \otimes V(l) \cong V(2l) \oplus F(0,0) \oplus F(0,1),\ 1 \le l \le \frac{s-1}{2}.$

We need the following lemma.

Lemma 2.8.2. *For* $2 \leq i \leq s-1$, $[V(i)] = [V(i-1)][V(1)] - [V(i-2)]$ *holds in* $G_0(\mathbb{C}D_{2s})$, *where* $[V(0)] := [F(0,0)] + [F(0,1)]$.

Proof. By Proposition 2.8.1 (4), we have

$$[V(i)] = [V(i-1)][V(1)] - [V(i-2)], \text{ for } 3 \leq i \leq \frac{s+1}{2}. \tag{2.31}$$

To show that it also holds for $\frac{s+3}{2} \leq i \leq s-1$, we multiply by $[F(1,0)]$ to both sides of the equality (2.31) and by Proposition 2.8.1 (3), we have that

$$[V(s-i)] = [V(s-i+1)][V(1)] - [V(s-i+2)].$$

Let $j = s-i+2$. Then $\frac{s+3}{2} \leq j \leq s-1$ and the above equality becomes

$$[V(j)] = [V(j-1)][V(1)] - [V(j-2)].$$

Therefore, $[V(i)] = [V(i-1)][V(1)] - [V(i-2)]$ holds for $3 \leq i \leq s-1$. For the case $i = 2$, the equality follows from Proposition 2.8.1 (5). □

Let $\mathbb{Z}[x_1, x_2, x_3]$ be the polynomial ring over $\mathbb{Z}$ with variables x_1, x_2 and x_3. We define a sequence of polynomials $f_i(x_2, x_3)$ in the subring $\mathbb{Z}[x_2, x_3]$ of $\mathbb{Z}[x_1, x_2, x_3]$ recursively as follows:

$$f_0(x_2, x_3) = 1 + x_2, \ f_1(x_2, x_3) = x_3,$$
$$f_i(x_2, x_3) = x_3 f_{i-1}(x_2, x_3) - f_{i-2}(x_2, x_3), \text{ for } i \geq 2.$$

We have the following.

Lemma 2.8.3. *Let* $(x_2x_3 - x_3)$ *be the ideal of* $\mathbb{Z}[x_2, x_3]$ *generated by* $x_2x_3 - x_3$. *There exist some polynomials* $h_i(x_3) \in \mathbb{Z}[x_3]$ *with degree* $i-2$ *and* $p_i(x_2) \in \mathbb{Z}[x_2]$ *such that* $f_i(x_2, x_3) \equiv x_3^i + h_i(x_3) + p_i(x_2)$ *modulo* $(x_2x_3 - x_3)$, *for* $i \geq 3$.

Proof. The proof can be completed by induction on the index i. □

Theorem 2.8.4. *Let* I *be the ideal of* $\mathbb{Z}[x_1, x_2, x_3]$ *generated by the polynomials* $x_1^2 - 1, x_2^2 - 1, x_2x_3 - x_3, f_{\frac{s+1}{2}}(x_2, x_3) - x_1 f_{\frac{s-1}{2}}(x_2, x_3)$. *The Grothendieck ring* $G_0(\mathbb{C}D_{2s})$ *of* $\mathbb{C}D_{2s}$ *is isomorphic to the quotient ring* $\mathbb{Z}[x_1, x_2, x_3]/I$.

Proof. By Proposition 2.8.1 (1) and Lemma 2.8.2, the Grothendieck ring $G_0(\mathbb{C}D_{2s})$ of $\mathbb{C}D_{2s}$ is generated as a ring by generators $[F(1,0)], [F(0,1)]$ and $[V(1)]$. Hence, there is a unique ring epimorphism φ from $\mathbb{Z}[x_1, x_2, x_3]$ to $G_0(\mathbb{C}D_{2s})$ such that

$$\varphi(x_1) = [F(1,0)],\ \varphi(x_2) = [F(0,1)] \text{ and } \varphi(x_3) = [V(1)].$$

Moreover, it is easy to check by induction on i that $\varphi(f_i(x_2, x_3)) = [V(i)]$, for $1 \leq i \leq s-1$. According to Proposition 2.8.1, the ideal I of $\mathbb{Z}[x_1, x_2, x_3]$ is contained in the kernel of φ, namely,

$$\begin{gathered}
\varphi(x_1^2 - 1) = [F(1,0)]^2 - [F(0,0)] = 0,\\
\varphi(x_2^2 - 1) = [F(0,1)]^2 - [F(0,0)] = 0,\\
\varphi(x_2 x_3 - x_3) = [F(0,1)][V(1)] - [V(1)] = 0,\\
\varphi(f_{\frac{s+1}{2}}(x_2, x_3) - x_1 f_{\frac{s-1}{2}}(x_2, x_3)) = [V(\frac{s+1}{2})] - [F(1,0)][V(\frac{s-1}{2})] = 0.
\end{gathered}$$

Thus, φ induces a ring epimorphism $\overline{\varphi}$ from $\mathbb{Z}[x_1, x_2, x_3]/I$ to $G_0(\mathbb{C}D_{2s})$ such that $\overline{\varphi}(\overline{f}) = \varphi(f)$ for any $f \in \mathbb{Z}[x_1, x_2, x_3]$.

If $s = 1$, then $0 = f_{\frac{s+1}{2}}(x_2, x_3) - x_1 f_{\frac{s-1}{2}}(x_2, x_3) = f_1(x_2, x_3) - x_1 f_0(x_2, x_3) = x_3 - x_1(1 + x_2)$ holds in $\mathbb{Z}[x_1, x_2, x_3]/I$. Hence as a $\mathbb{Z}$-module, $\mathbb{Z}[x_1, x_2, x_3]/I$ has a $\mathbb{Z}$-basis $\{\overline{1}, \overline{x_1}, \overline{x_2}, \overline{x_1 x_2}\}$ and the rank of $\mathbb{Z}[x_1, x_2, x_3]/I$ is 4, which is equal to the rank of $G_0(\mathbb{C}D_{2s})$. It follows that $\overline{\varphi}$ is an isomorphism.

If $s = 3$, then $0 = f_{\frac{s+1}{2}}(x_2, x_3) - x_1 f_{\frac{s-1}{2}}(x_2, x_3) = f_2(x_2, x_3) - x_1 f_1(x_2, x_3) = x_3^2 - x_1 x_3 - x_2 - 1$ holds in $\mathbb{Z}[x_1, x_2, x_3]/I$. Hence, as a $\mathbb{Z}$-module, $\mathbb{Z}[x_1, x_2, x_3]/I$ has a $\mathbb{Z}$-basis $\{\overline{1}, \overline{x_1}, \overline{x_2}, \overline{x_1 x_2}, \overline{x_3}, \overline{x_1 x_3}\}$ and the rank of $\mathbb{Z}[x_1, x_2, x_3]/I$ is 6, the same as the rank of $G_0(\mathbb{C}D_{2s})$. Therefore, $\overline{\varphi}$ is an isomorphism.

If $s = 5$, then we have $0 = f_{\frac{s+1}{2}}(x_2, x_3) - x_1 f_{\frac{s-1}{2}}(x_2, x_3) = f_3(x_2, x_3) - x_1 f_2(x_2, x_3) = x_3^3 - x_1 x_3^2 - (2 + x_2) x_3 + x_1(1 + x_2)$ in $\mathbb{Z}[x_1, x_2, x_3]/I$. Thus, as a $\mathbb{Z}$-module, $\mathbb{Z}[x_1, x_2, x_3]/I$ has a $\mathbb{Z}$-basis $\{\overline{1}, \overline{x_1}, \overline{x_2}, \overline{x_1 x_2}, \overline{x_3}, \overline{x_3^2}, \overline{x_1 x_3}, \overline{x_1 x_3^2}\}$ and the rank of $\mathbb{Z}[x_1, x_2, x_3]/I$ is 8, the same as the rank of $G_0(\mathbb{C}D_{2s})$. So $\overline{\varphi}$ is an isomorphism.

If $s \geq 7$, by Lemma 2.8.3, the following hold in $\mathbb{Z}[x_1, x_2, x_3]/I$,

$$\begin{aligned}
0 &= f_{\frac{s+1}{2}}(x_2, x_3) - x_1 f_{\frac{s-1}{2}}(x_2, x_3)\\
&= x_3^{\frac{s+1}{2}} - x_1 x_3^{\frac{s-1}{2}} + h_{\frac{s+1}{2}}(x_3) - x_1 h_{\frac{s-1}{2}}(x_3) + p_{\frac{s+1}{2}}(x_2) - x_1 p_{\frac{s-1}{2}}(x_2),
\end{aligned}$$

where the degrees of $h_{\frac{s+1}{2}}(x_3)$ and $h_{\frac{s-1}{2}}(x_3)$ are $\frac{s-3}{2}$ and $\frac{s-5}{2}$ respectively. Hence, as a $\mathbb{Z}$-module, $\mathbb{Z}[x_1, x_2, x_3]/I$ has a $\mathbb{Z}$-basis

$$\{\overline{1}, \overline{x_1}, \overline{x_2}, \overline{x_1x_2}, \overline{x_3^i}, \overline{x_1x_3^i} \mid 1 \leq i \leq \frac{s-1}{2}\}.$$

We conclude that the rank of $\mathbb{Z}[x_1, x_2, x_3]/I$ is $4 + 2 \times \frac{s-1}{2} = s + 3$, which is the same as the rank of $G_0(\mathbb{C}D_{2s})$. Hence $\overline{\varphi}$ is an isomorphism. $\square$

Denote by χ the character of the simple $\mathbb{C}D_{2s}$-module $F(1, 0)$. It is obvious that the order of χ is 2. The Hopf algebra $H_{\mathcal{D}}$ stemming from the group datum $\mathcal{D} = (D_{2s}, \chi, c^s, 0)$ is of nilpotent type. Note that $q := \chi(c^s) = (-1)^s = -1$, which is of order 2. By Theorem 2.1.4, the set of indecomposable $H_{\mathcal{D}}$-modules consists of simple modules as well as their projective covers. Let $a = [V_{\chi^{-1}}] = [V_\chi] = [F(1, 0)]$. Then $a^2 = 1$. By Theorem 2.3.4, the Green ring $r(H_{\mathcal{D}})$ of $H_{\mathcal{D}}$ is isomorphic to $r(\mathbb{C}D_{2s})[z]/I$, where I is the ideal generated by $(z - a - 1)z$. Since the order of χ in the group datum $\mathcal{D} = (D_{2s}, \chi, c^s, 0)$ is equal to the order of $\chi(c^s)$, by Theorem 2.5.7, the Jacobson radical of $r(H_{\mathcal{D}})$ is the principal ideal generated by the element $[P(F(0, 0))](1 - a)$. Thanks to Theorem 2.8.4, we obtain the following.

Theorem 2.8.5. *Let I be an ideal of $\mathbb{Z}[x_1, x_2, x_3, x_4]$ generated by the polynomials $x_1^2 - 1, x_2^2 - 1, x_2x_3 - x_3, f_{\frac{s+1}{2}}(x_2, x_3) - x_1 f_{\frac{s-1}{2}}(x_2, x_3)$ and $x_4^2 - x_1x_4 - x_4$. The Green ring $r(H_{\mathcal{D}})$ of $H_{\mathcal{D}}$ is isomorphic to the quotient ring $\mathbb{Z}[x_1, x_2, x_3, x_4]/I$. The Jacobson radical of $r(H_{\mathcal{D}})$ (under the isomorphism) is a principal ideal of $\mathbb{Z}[x_1, x_2, x_3, x_4]/I$ generated by the element $\overline{x_4} - \overline{x_1x_4}$.*

Chapter 3

Pointed Hopf algebras of rank one: non-nilpotent type

In this chapter, we continue our study of Green rings of finite dimensional pointed rank one Hopf algebras, but concentrate ourselves on those Hopf algebras of non-nilpotent type. If H is finite dimensional pointed rank one Hopf algebra of non-nilpotent type, then the quotient Hopf algebra $\overline{H}$ of H modulo a Hopf ideal is isomorphic to a finite dimensional pointed rank one Hopf algebra of nilpotent type. This leads to the fact that the Green ring $r(\overline{H})$ is a subring of the Green ring $r(H)$. We then determine all finite dimensional indecomposable H-modules and describe the Clebsch-Gordan formulas of the tensor products of indecomposable modules. We reconstruct $r(H)$ from the subring $r(\overline{H})$ together with the ideal of $r(H)$ generated by isomorphism classes of indecomposable projective modules. Moreover, the Jacobson radical of $r(H)$ turns out to be exactly the Jacobson radical of $r(\overline{H})$, a principal ideal generated by a special element. Finally, as an example, we present the Green ring of a Radford Hopf algebra explicitly.

3.1 Indecomposable representations

Throughout this chapter, $\mathcal{D} = (G, \chi, g, 1)$ is a fixed group datum of non-nilpotent type. That is, $g^n - 1 \neq 0$ and $\chi^n = 1$, where n is the order of $q := \chi(g)$. It is obvious that n is in fact the order of χ. Denote by H the Hopf algebra $H_{\mathcal{D}}$ associated to the group datum $\mathcal{D}$. Then H is a finite dimensional pointed rank one Hopf algebra of non-nilpotent type. The relations of (1.6) now become

$$y^n = g^n - 1,\ yh = \chi(h)hy,\ \text{for } h \in G.$$

In this section, we construct an idempotent e of H and show that the quotient algebra $\overline{H} := H/H(1-e)$ is isomorphic to a pointed rank one Hopf algebra of nilpotent type. Thus, we obtain all finite dimensional indecomposable $\overline{H}$-modules by Theorem 2.1.4. We determine all finite dimensional simple modules over $H(1-e)$ and it turns out that all of them are projective. Thus, $H(1-e)$ is semisimple. As a consequence, we obtain all finite dimensional indecomposable H-modules up to isomorphism.

Note that the order of $q = \chi(g)$ is n. It follows that the order of g in group G is nr for some integer r. Denote by

$$N = \{1, g^n, g^{2n}, \cdots, g^{(r-1)n}\}$$

and

$$e = \frac{1}{r}(1 + g^n + g^{2n} + \cdots + g^{(r-1)n}).$$

It is not difficult to see that the element e is a central idempotent of H and N is a normal subgroup of G. Denote by $\overline{G} = G/N$ and $\overline{h} = \pi(h)$, where π is the natural epimorphism from G to $\overline{G}$. Then the character χ of G induces the character $\overline{\chi}$ of $\overline{G}$ such that $\overline{\chi} \circ \pi = \chi$. Let $\overline{\mathcal{D}} = (\overline{G}, \overline{\chi}, \overline{g}, 0)$. Then $\overline{\mathcal{D}}$ is a group datum of nilpotent type since $\overline{g}^n - 1 = 0$, where n is the order of $\chi(g) = \overline{\chi}(\overline{g})$. Let $\overline{H}$ be the Hopf algebra $H_{\overline{\mathcal{D}}}$ associated to the group datum $\overline{\mathcal{D}}$. That is, $\overline{H}$ is generated as an algebra by z and all $\overline{h} \in \overline{G}$ such that $\Bbbk\overline{G}$ is a subalgebra of $\overline{H}$ and

$$z^n = 0,\ z\overline{h} = \overline{\chi}(\overline{h})\overline{h}z = \chi(h)\overline{h}z,\ \text{for } \overline{h} \in \overline{G}.$$

The comultiplication $\triangle$, the counit ε, and antipode S of $\overline{H}$ are given respectively by

$$\triangle(z) = z \otimes \overline{g} + 1 \otimes z,\ \varepsilon(z) = 0,\ S(z) = -z\overline{g}^{-1},$$

$$\triangle(\overline{h}) = \overline{h} \otimes \overline{h},\ \varepsilon(\overline{h}) = 1,\ S(\overline{h}) = \overline{h}^{-1},$$

for all $\overline{h} \in \overline{G}$. Accordingly, $\overline{H}$ is a finite dimensional pointed rank one Hopf algebra of nilpotent type with a $\Bbbk$-basis $\{\overline{h}z^i \mid \overline{h} \in \overline{G}, 0 \leq i \leq n-1\}$.

Lemma 3.1.1. *Let $\Bbbk G(g^n - 1)$ be the ideal of $\Bbbk G$ generated by $g^n - 1$. We have that $\Bbbk G/\Bbbk G(g^n - 1) \cong \Bbbk\overline{G}$.*

Proof. The natural epimorphism π from G to $\overline{G}$ induces the group algebra epimorphism $\pi : \Bbbk G \to \Bbbk\overline{G}$. We claim that $\ker\pi = \Bbbk G(g^n - 1)$. The inclusion $\Bbbk G(g^n - 1) \subseteq \ker\pi$ is obvious. Conversely, let $G = Nh_1 \cup Nh_2 \cup \cdots \cup Nh_s$ be the right coset decomposition of G respect to the normal subgroup N. For any $a \in \Bbbk G$, a can be written as $a = \sum_{i=1}^{s} a_i$, where each term a_i is a linear combination of the elements from the set Nh_i. More explicitly, $a_i = (\sum_{j=0}^{r-1} \lambda_{ij} g^{jn})h_i$, where each constant $\lambda_{ij} \in \Bbbk$. If $\pi(a) = 0$, then

$$0 = \sum_{i=1}^{s} \overline{a_i} = \sum_{i=1}^{s}(\sum_{j=0}^{r-1} \lambda_{ij}\overline{g^{jn}})\overline{h_i} = \sum_{i=1}^{s}(\sum_{j=0}^{r-1} \lambda_{ij})\overline{h_i}. \tag{3.1}$$

Note that $\{\overline{h_i} \mid 1 \leq i \leq s\}$ forms a basis of $\Bbbk\overline{G}$. The equality (3.1) implies that $\sum_{j=0}^{r-1} \lambda_{ij} = 0$ for each i. We obtain that

$$a_i = (\sum_{j=0}^{r-1} \lambda_{ij} g^{jn})h_i = \sum_{j=1}^{r-1} \lambda_{ij}(g^{jn} - 1)h_i \in \Bbbk G(g^n - 1),$$

as desired. □

Remark 3.1.2. Observe that $1 - e = (1 - g^n)\frac{1}{r}\sum_{k=0}^{r-1}(1 + g^n + \cdots + g^{(k-1)n})$ while $1 - g^n = (1 - e)(1 - g^n)$, and both of them are central elements of H.

(1) The ideals $\Bbbk G(1 - e)$ and $\Bbbk G(1 - g^n)$ coincide, hence $\Bbbk\overline{G} \cong \Bbbk Ge$.

(2) The ideals $H(1 - e)$ and $H(1 - g^n)$ coincide, hence $H(1 - e)$ is a Hopf ideal of H since $H(1 - g^n)$ is.

The relation between the Hopf algebra H of non-nilpotent type and $\overline{H}$ of nilpotent type is described as follows.

Proposition 3.1.3. *For the Hopf algebras H and $\overline{H}$ as above, we have the following:*

(1) *$\overline{H}$ is isomorphic to the quotient Hopf algebra $H/H(1 - e)$ of H.*

(2) *$\overline{H}$ is isomorphic to the subalgebra He of H.*

Proof. (1) It is easy to check that the algebra epimorphism $\rho : H \to \overline{H}$ given by $\rho(y) = z$ and $\rho(h) = \overline{h}$, for all $h \in G$ respects the Hopf algebra structure. The inclusion $H(1-e) \subseteq \ker \rho$ is obvious since $\rho(e) = 1$. To verify that $\ker \rho = H(1-e)$, we only need to show that the restriction of ρ to the summand He of H is injective. In fact, if $\rho(\sum_{i=0}^{n-1} a_i y^i e) = 0$, for each $a_i \in \Bbbk G$, then $\sum_{i=0}^{n-1} \overline{a_i e} z^i = 0$. It follows that $\overline{a_i e} = 0$. By Lemma 3.1.1, $\overline{a_i e} = 0$ if and only if $a_i e \in (g^n - 1)$. It follows that $a_i e = (g^n - 1) b_i$, for some $b_i \in \Bbbk G$. As a result, $a_i e = a_i e^2 = e(g^n - 1) b_i = 0$, as desired.

(2) By Part (1), it is immediate that $He \cong H/H(1-e) \cong \overline{H}$ as algebras. $\square$

For the group G in the datum $\mathcal{D}$, let V be a $\Bbbk G$-module, $k \in \mathbb{N}$ and x a variable. It is similar to Chapter 2 that $x^k V$ is a $\Bbbk G$-module given by

$$h(x^k v) = \chi^{-k}(h) x^k h v,$$

for any $h \in G$ and $v \in V$.

Let $\{V_i \mid i \in \Omega\}$ be a complete set of non-isomorphic simple $\Bbbk G$-modules. Since g is a central element of G, the action of g^n on each V_i is a scalar multiple by a non-zero element λ_i. Let $\Omega_0 = \{i \in \Omega \mid \lambda_i = 1\}$ and $\Omega_1 = \{i \in \Omega \mid \lambda_i \neq 1\}$. In particular, $0 \in \Omega_0$ since we denote by V_0 the trivial H-module $\Bbbk$. It follows from Lemma 3.1.1 that $\{V_i \mid i \in \Omega_0\}$ is a complete set of non-isomorphic simple modules over $\Bbbk\overline{G}$.

For any $i \in \Omega_0$ and $1 \leq j \leq n$, let $M(i,j) := V_i \oplus x V_i \oplus \cdots \oplus x^{j-1} V_i$. Then $M(i,j)$ is an H-module given by

$$h(x^k v) = \chi^{-k}(h) x^k h v, \ 0 \leq k \leq j-1,$$

and

$$y(x^k v) = \begin{cases} x^{k+1} v, & 0 \leq k \leq j-2, \\ 0, & k = j-1, \end{cases}$$

for all $h \in G$ and $v \in V_i$. Note that $\lambda_i = 1$ for $i \in \Omega_0$. For any $x^k v \in M(i,j)$,

$$e(x^k v) = \frac{1}{r} \sum_{s=0}^{r-1} g^{sn}(x^k v) = x^k (\frac{1}{r} \sum_{s=0}^{r-1} g^{sn} v) = x^k (\frac{1}{r} \sum_{s=0}^{r-1} \lambda_i^s v) = x^k v. \qquad (3.2)$$

Thus, the action of the idempotent e on $M(i,j)$ is the identity. It follows from Proposition 3.1.3 (1) that each $M(i,j)$ is exactly an $\overline{H}$-module. Moreover, by Theorem 2.1.4,

$\{M(i,j) \mid i \in \Omega_0, 1 \leq j \leq n\}$ is a complete set of non-isomorphic indecomposable $\overline{H}$-modules.

We have already described all indecomposable modules over $\overline{H} \cong He$. In the following, we shall study the representations of $H(1-e)$.

For any $j \in \Omega_1$, let $P_j := V_j \oplus xV_j \oplus \cdots \oplus x^{n-1}V_j$. Then P_j is an H-module with the actions given by

$$h(x^k v) = \chi^{-k}(h)x^k hv, \ 0 \leq k \leq n-1,$$

and

$$y(x^k v) = \begin{cases} x^{k+1}v, & 0 \leq k \leq n-2, \\ (g^n - 1)v, & k = n-1, \end{cases}$$

for all $h \in G$ and $v \in V_j$. Note that $\lambda_j \neq 1$ and $\lambda_j^r = 1$, for $j \in \Omega_1$. For any $x^k v \in P_j$,

$$e(x^k v) = \frac{1}{r}\sum_{s=0}^{r-1} g^{sn}(x^k v) = x^k(\frac{1}{r}\sum_{s=0}^{r-1} g^{sn}v) = x^k(\frac{1}{r}\sum_{s=0}^{r-1} \lambda_j^s v) = 0, \tag{3.3}$$

hence each P_j is exactly an $H(1-e)$-module.

For any H-module V, the subspace $V_y = \{v \in V \mid yv = 0\}$ is a submodule of V. If $V_y = V$, then V is called *y-torsion*. If $V_y = 0$, then V is called *y-torsionfree*.

Obviously, if V is simple, then V is either y-torsion or y-torsionfree. To investigate the representations of the subalgebra $H(1-e)$ of H, we begin with the study of simple y-torsion and y-torsionfree H-modules respectively.

Proposition 3.1.4. *Let V be an H-module. Then V is simple y-torsion if and only if V is simple over $\Bbbk\overline{G}$. Therefore, $\{V_i \mid i \in \Omega_0\}$ is a complete set of non-isomorphic simple y-torsion H-modules.*

Proof. Suppose V is simple y-torsion, then $V = V_y$. That is, for any $v \in V$, $yv = 0$. In particular, $(g^n - 1)v = y^n v = 0$. Hence V is simple over the quotient algebra $H/(y, g^n - 1)$, where $(y, g^n - 1)$ is the ideal of H generated by y and $g^n - 1$. By Lemma 3.1.1, we have the following isomorphisms:

$$H/(y, g^n - 1) \cong \Bbbk G/(g^n - 1) \cong \Bbbk\overline{G}.$$

Therefore, V is simple over $\Bbbk\overline{G}$. Conversely, if V is simple over $\Bbbk\overline{G}$, by Lemma 3.1.1, V is simple over $\Bbbk G$ satisfying $(g^n-1)v=0$, for any $v\in V$. Define $yv=0$ for any $v\in V$. Then V becomes a simple y-torsion H-module. $\square$

Proposition 3.1.5. *Let V be an H-module. Then V is simple y-torsionfree if and only if $V\cong P_j$, for some $j\in\Omega_1$. Moreover, $\{P_j\mid j\in\Omega_1\}$ forms a complete set of simple y-torsionfree H-modules (not necessary mutually non-isomorphism).*

Proof. We assume that V is a simple y-torsionfree H-module. Then V is semisimple as a $\Bbbk G$-module since $\Bbbk G$ is a semisimple subalgebra of H. Hence there is some $j\in\Omega$ such that V_j is a direct summand of V. Since $V_j+yV_j+\cdots+y^{n-1}V_j$ is a submodule of H-module V, by the simplicity of V, we have that $V=V_j+yV_j+\cdots+y^{n-1}V_j$. We claim that $j\in\Omega_1$ (i.e., $\lambda_j\neq 1$) and the sum $V_j+yV_j+\cdots+y^{n-1}V_j$ is a direct sum. In fact, if $\lambda_j=1$, then for any $v\in V_j$, $y^nv=(g^n-1)v=(\lambda_j-1)v=0$. It follows that $v=0$ since V is y-torsionfree, a contradiction. Note that g is a central element of G, there is a scalar ω_j satisfying $gv=\omega_jv$, for any $v\in V_j$. If $v_0+yv_1+\cdots+y^{n-1}v_{n-1}=0$, where each $v_i\in V_j$, we have

$$\begin{aligned}0&=g^i(v_0+yv_1+\cdots+y^{n-1}v_{n-1})\\&=g^iv_0+q^{-i}yg^iv_1+\cdots+q^{-(n-1)i}y^{n-1}g^iv_{n-1}\\&=\omega_j^i(v_0+q^{-i}yv_1+\cdots+q^{-(n-1)i}y^{n-1}v_{n-1}).\end{aligned}$$

This implies that

$$v_0+q^{-i}yv_1+\cdots+q^{-(n-1)i}y^{n-1}v_{n-1}=0,\ \text{for } 0\le i\le n-1. \tag{3.4}$$

Therefore, $v_0=yv_1=\cdots=y^{n-1}v_{n-1}=0$ since the order of q is n and the coefficient matrix determined by the equations (3.4) is a Vandermonde matrix which is invertible. Now we have $V=V_j\oplus yV_j\oplus\cdots\oplus y^{n-1}V_j$ and $j\in\Omega_1$. Moreover, V is isomorphic to P_j as follows:

$$V\to P_j,\ \sum_{i=0}^{n-1}y^iv_i\mapsto\sum_{i=0}^{n-1}x^iv_i,$$

for each $v_i\in V_j$, as asserted the first part of the proposition.

To prove the rest of the proposition, we first claim that $P_j=V_j\oplus xV_j\oplus\cdots\oplus x^{n-1}V_j$ is y-torsionfree, for $j\in\Omega_1$. In fact, if $y(v_0+xv_1+\cdots+x^{n-1}v_{n-1})=0$, then $(g^n-1)(v_0+xv_1+\cdots+x^{n-1}v_{n-1})=y^n(v_0+xv_1+\cdots+x^{n-1}v_{n-1})=0$. That is,

$(\lambda_j - 1)(v_0 + xv_1 + \cdots + x^{n-1}v_{n-1}) = 0$. We obtain that $v_0 + xv_1 + \cdots + x^{n-1}v_{n-1} = 0$ since $\lambda_j \neq 1$. P_j is simple. To obtain this, suppose V is a non-zero simple submodule of P_j, then V is also simple y-torsionfree. By the first part of the proposition, $V \cong P_{j'}$ for some $j' \in \Omega_1$. Moreover, $V_{j'} \subseteq P_{j'} \cong V \subseteq P_j$. Since $V_{j'}$ is a simple $\Bbbk G$-module and P_j is a direct sum with the summands of simple $\Bbbk G$-modules, there exists some k such that $V_{j'} \cong x^k V_j$. Hence, $\dim V_{j'} = \dim x^k V_j = \dim V_j$, it implies that $\dim V = \dim P_{j'} = \dim P_j$. As a result, $V = P_j$. $\square$

In order to determine all non-isomorphic simple y-torsionfree H-modules, we need to define a permutation on the index set Ω of simple $\Bbbk G$-modules.

Let V_χ and $V_{\chi^{-1}}$ be two 1-dimensional simple $\Bbbk G$-modules with respect to the $\Bbbk$-linear character χ and χ^{-1} respectively. For any simple $\Bbbk G$-module V_s, $s \in \Omega$, the tensor product $V_{\chi^{-1}} \otimes V_s \cong V_s \otimes V_{\chi^{-1}}$ is also simple, it is similar to Chapter 2 that there is a unique permutation τ of the index set Ω determined by

$$V_{\chi^{-1}} \otimes V_s \cong V_s \otimes V_{\chi^{-1}} \cong V_{\tau(s)},$$

for some $\tau(s) \in \Omega$. Moreover, it is easy to see that $s \in \Omega_0$ (resp. $s \in \Omega_1$) if and only if $\tau(s) \in \Omega_0$ (resp. $\tau(s) \in \Omega_1$). That is, τ permutes the index set Ω_0 and Ω_1 respectively.

Lemma 3.1.6. *For any $s \in \Omega$ and $t \in \mathbb{Z}$, the following hold as $\Bbbk G$-modules:*

(1) $V_s \otimes V_\chi \cong V_\chi \otimes V_s \cong V_{\tau^{-1}(s)}$.

(2) $V_s \otimes V_{\chi^{-t}} \cong V_{\tau^t(s)}$.

(3) *There is an bijective $\widetilde{\sigma}_{s,t}$ from V_s to $V_{\tau^t(s)}$ such that $\widetilde{\sigma}_{s,t}(hv) = \chi^t(h)h\widetilde{\sigma}_{s,t}(v)$, for any $h \in G$ and $v \in V_s$.*

(4) *$xV_s \cong V_{\tau(s)}$. Moreover, $V_i \cong V_j$ if and only if $xV_i \cong xV_j$, for $i, j \in \Omega$.*

(5) *The order of the permutation τ is n. Moreover, $x^t V_s \cong V_s$ if and only if t is divisible by n, for any $s \in \Omega$.*

Proof. Part (1) and Part (2) are obvious. Part (3) is the same as Lemma 2.1.2.

(4) The $\Bbbk$-linear map $xv \mapsto u \otimes v$ gives an isomorphism from xV_s to $V_{\chi^{-1}} \otimes V_s$, where $0 \neq u \in V_{\chi^{-1}}$. Moreover, $V_i \cong V_j$ if and only if $V_{\tau(i)} \cong V_{\tau(j)}$ if and only if $xV_i \cong xV_j$, for $i, j \in \Omega$.

(5) Note that the order of χ is n. So is the order of τ by Part (2). Suppose the action of g on V_s is a scalar multiple by the element ω_s, then the action of g on $x^t V_s$ is a scalar multiple by $\omega_s q^{-t}$. If $x^t V_s \cong V_s$, then $\omega_s = \omega_s q^{-t}$ and hence t is divisible by n since the order of q is n. Conversely, if t is divisible by n, it is obvious that $x^t V_s \cong V_s$ since the order of τ is confirmed to be n. $\square$

Let $\langle\tau\rangle$ be a group generated by the permutation τ. Then $\langle\tau\rangle$ acts on the index set Ω_1. By Lemma 3.1.6 (5), each $\langle\tau\rangle$-orbit has exactly n distinct element, hence the cardinality of Ω_1 is divisible by n. Let $\sim$ be the equivalence relation on Ω_1 defined by $i \sim j$ if and only if i and j belong to the same $\langle\tau\rangle$-orbit, for any $i, j \in \Omega_1$. The equivalence class of i is denoted by $[i]$. Denote by $\overline{\Omega}_1$ the set consisting of all distinct equivalence classes. Let $P_{[i]}$ stand for one of P_j, for $j \in \Omega_1$ such that $j \sim i$. $P_{[i]}$ is well-defined as shown in the following.

Proposition 3.1.7. *The set $\{P_{[i]} \mid [i] \in \overline{\Omega}_1\}$ is a complete set of non-isomorphic simple y-torsionfree H-modules.*

Proof. Suppose $P_i \cong P_j$ as H-modules, so is it as $\Bbbk G$-modules. By Krull-Schmidt theorem, the direct summand V_i of P_i is isomorphic as $\Bbbk G$-modules to a direct summand $x^k V_j$ of P_j. By Lemma 3.1.6 (4), we have $i = \tau^k(j)$. Thus, i and j belong to the same $\langle\tau\rangle$-orbit. Conversely, if $i = \tau^k(j)$, for some k, then $V_i \cong V_{\tau^k(j)} \cong x^k V_j$ as $\Bbbk G$-modules. We have $\dim P_i = \dim P_j$ since $\dim V_i = \dim x^k V_j = \dim V_j$. Let ς denote the isomorphism from V_i to $x^k V_j$. The $\Bbbk$-linear map

$$P_i \to P_j, \ \sum_{s=0}^{n-1} x^s v_s \mapsto \sum_{s=0}^{n-1} y^s \varsigma(v_s),$$

where each $v_s \in V_i$ and the term $y^s \varsigma(v_s)$ stands for the action of y^s on $\varsigma(v_s)$, is an injective H-module morphism. Comparing the dimension, we conclude that $P_i \cong P_j$. By virtue of Proposition 3.1.5, we complete the proof. $\square$

Proposition 3.1.8. *The set $\{P_{[i]} \mid [i] \in \overline{\Omega}_1\}$ is a complete set of non-isomorphic simple modules over $H(1-e)$.*

Proof. It follows from (3.3) that the simple y-torsionfree H-module P_j is simple over the quotient algebra $H/He \cong H(1-e)$. Conversely, if V is a simple $H(1-e)$-module, then the natural projective from H to the summand $H(1-e)$ of H makes V an H-module, namely, $ev = 0$, for any $v \in V$. V is simple over H. In fact, if V^0 is a non-zero submodule of V, for any $0 \neq v_0 \in V^0$, by the simplicity of V as an $H(1-e)$-module, we have

$$V = H(1-e)v_0 = \{a(1-e)v_0 \mid a \in H\} = \{av_0 \mid a \in H\} \subseteq V^0.$$

We conclude that $V = V^0$. V is y-torsionfree. To prove this, we suppose $yv = 0$, for some $v \in V$. Then $(g^n - 1)v = y^n v = 0$, and hence $g^n v = v$. It follows that $0 = ev = \frac{1}{r}\sum_{k=0}^{r-1} g^{kn} v = v$. We conclude that V is a simple y-torsionfree H-module. It follows from Proposition 3.1.5 that $V \cong P_j$, for some $j \in \Omega_1$. By (3.3) as well, each P_j is annihilated by the idempotent e. Then $P_i \cong P_j$ as $H(1-e)$-modules if and only if $P_i \cong P_j$ as H-modules. According to Proposition 3.1.7, this is precisely i and j belong to the same $\langle\tau\rangle$-orbit. □

Corollary 3.1.9. *We have the following:*

(1) *For any $j \in \Omega_1$, P_j is projective both as an H-module and as an $H(1-e)$-module.*

(2) *The subalgebra $H(1-e)$ of H is semisimple.*

(3) *The Jacobson radical of H is a principal ideal of H generated by ye.*

Proof. (1) For each $j \in \Omega_1$, let e_j be the primitive idempotent of $\Bbbk G$ such that $V_j \cong \Bbbk G e_j$ as $\Bbbk G$-modules. Then e_j is also an idempotent of H. Let He_j be the left ideal of H generated by e_j. Obviously,

$$He_j = \Bbbk G e_j \oplus y\Bbbk G e_j \oplus \cdots \oplus y^{n-1}\Bbbk G e_j$$

with the direct summands of simple $\Bbbk G$-modules. Denote by ζ_j the isomorphism from V_j to $\Bbbk G e_j$ and consider the following $\Bbbk$-linear map:

$$P_j \to He_j,\ \sum_{k=0}^{n-1} x^k v_k \mapsto \sum_{k=0}^{n-1} y^k \zeta_j(v_k),$$

where each $v_k \in V_j$ and $y^k\zeta_j(v_k)$ stands for the multiplication of y^k with $\zeta_j(v_k)$ in H. It is easy to see that the map given above is an H-module isomorphism. Hence for each $j \in \Omega_1$, P_j is projective over H. To see that each P_j is also projective over $H(1-e)$, note that each P_j is annihilated by the idempotent e and P_j is isomorphic to He_j as H-modules, we conclude that $ee_j = 0$. Thus, e_j is an idempotent of the subalgebra $H(1-e)$ and $H(1-e)e_j = He_j$. Now the H-module isomorphism

$P_j \cong He_j = H(1-e)e_j$ is also an $H(1-e)$-module isomorphism. Hence P_j is projective over $H(1-e)$.

(2) It follows from Part (1) and Proposition 3.1.8 that all simple $H(1-e)$-modules are projective. As a result, $H(1-e)$ is semisimple.

(3) Observe that the ideal (ye) of H generated by ye is nilpotent and the quotient algebra $H/(ye) \cong H(1-e) \bigoplus \Bbbk Ge$ is semisimple. The Jacobson radical of H is a principal ideal generated by ye. $\square$

Recall that a finite dimensional Hopf algebra over $\Bbbk$ is said to have the *Chevalley property* if the tensor product of any two simple modules is semisimple [38, Definition 7.2.1]. One of the equivalent conditions is that the radical of the Hopf algebra is a Hopf ideal [38, Proposition 7.2.2]. In view of this, we obtain that the pointed rank one Hopf algebra H of non-nilpotent type has no the Chevalley property since the Jacobson radical of H is not a Hopf ideal. As we shall see that the tensor product of any two simple H-modules is not necessary semisimple.

We summarize the main results of this section as follows.

Theorem 3.1.10. *Let H and $\overline{H}$ be two Hopf algebras associated to the group data $\mathcal{D} = (G, \chi, g, 1)$ of non-nilpotent type and $\overline{\mathcal{D}} = (\overline{G}, \overline{\chi}, \overline{g}, 0)$ of nilpotent type.*

(1) *$\overline{H} \cong H/H(1-e)$ as Hopf algebras and $\overline{H} \cong He$ as algebras.*

(2) *The subalgebra $H(1-e)$ of H is semisimple.*

(3) *The set $\{M(i,k), P_{[j]} \mid i \in \Omega_0, 1 \leq k \leq n, [j] \in \overline{\Omega}_1\}$ forms a complete set of indecomposable H-modules up to isomorphism.*

3.2 Clebsch-Gordan formulas

In this section, we investigate the decomposition formulaes of tensor product of indecomposable H-modules. It turns out that the decomposition of the tensor product of two indecomposable H-modules depends mainly on the decomposition of the tensor products of simple $\Bbbk G$-modules. For the sake of simplicity, we denote by π_s the projection from $V_i \otimes V_j$ to the summand V_s if not confused.

Lemma 3.2.1. *Let V_s be a direct summand of $V_i \otimes V_j$ as $\Bbbk G$-modules.*

(1) *If $i, j \in \Omega_0$, then $s \in \Omega_0$.*

(2) *If $i \in \Omega_0$, $j \in \Omega_1$ or $i \in \Omega_1$, $j \in \Omega_0$, then $s \in \Omega_1$.*

(3) *If $i, j \in \Omega_1$, and $\lambda_i\lambda_j = 1$, then $s \in \Omega_0$.*

(4) *If $i, j \in \Omega_1$, and $\lambda_i\lambda_j \neq 1$, then $s \in \Omega_1$.*

Proof. The projective $\pi_s : V_i \otimes V_j \to V_s$ shows that $\lambda_i\lambda_j = \lambda_s$, as desired. □

For any $i \in \Omega$, let B_i be a set consisting of a basis of V_i. Then $\mathrm{sp}B_i = V_i$. Obviously, the following set

$$\Gamma = \{x^t u \otimes x^s v \mid 0 \leq t, s \leq n-1, u \in B_i, v \in B_j\}$$

forms a basis of $P_i \otimes P_j$, where $i, j \in \Omega_1$. Similarly,

$$\Pi = \{x^t u \otimes x^s v \mid 0 \leq t \leq k-1, 0 \leq s \leq n-1, u \in B_i, v \in B_j\}$$

and

$$\Lambda = \{x^s v \otimes x^t u \mid 0 \leq t \leq k-1, 0 \leq s \leq n-1, u \in B_i, v \in B_j\}$$

form bases of $M(i,k) \otimes P_j$ and $P_j \otimes M(i,k)$ respectively, where $i \in \Omega_0$, $j \in \Omega_1$ and $1 \leq k \leq n$. We need another bases of $P_i \otimes P_j$, $M(i,k) \otimes P_j$ and $P_j \otimes M(i,k)$ stated respectively as follows.

Lemma 3.2.2. *We have the following:*

(1) *For any $i, j \in \Omega_1$, the set*

$$\widehat{\Gamma} = \{y^s(x^t u \otimes v) \mid 0 \leq s, t \leq n-1, u \in B_i, v \in B_j\}$$

forms a basis of $P_i \otimes P_j$.

(2) *For any* $i \in \Omega_0$, $1 \le k \le n$ *and* $j \in \Omega_1$,

$$\widehat{\Pi} = \{y^s(x^t u \otimes v) \mid 0 \le s \le n-1, 0 \le t \le k-1, u \in B_i, v \in B_j\},$$

$$\widehat{\Lambda} = \{y^s(v \otimes x^t u) \mid 0 \le s \le n-1, 0 \le t \le k-1, u \in B_i, v \in B_j\}$$

form bases of $M(i,k) \otimes P_j$ *and* $P_j \otimes M(i,k)$ *respectively.*

Proof. (1) Since the cardinality of the set Γ is equal to the cardinality of the set $\widehat{\Gamma}$, we only need to verify that $x^t u \otimes x^s v \in \mathrm{sp}\widehat{\Gamma}$, for any $0 \le s, t \le n-1$ and $u \in B_i, v \in B_j$. The proof is proceeded by induction on s, for all $0 \le t \le n-1$ and $u \in B_i, v \in B_j$. It is obvious that the result holds for $s = 0$. Suppose for all $0 \le t \le n-1$, $u \in B_i, v \in B_j$ and $1 \le s \le d$, where $1 \le d \le n-2$, we have $x^t u \otimes x^s v \in \mathrm{sp}\widehat{\Gamma}$. To consider the case $s = d+1$, note that

$$\Delta(y^{d+1}) = \sum_{p=0}^{d+1} \binom{d+1}{p}_q y^{d+1-p} \otimes g^{d+1-p} y^p,$$

see e.g., [50, Eq.(1)], we have the following:

$$\begin{aligned}
y^{d+1}(x^t u \otimes v) &= \sum_{p=0}^{d+1} \binom{d+1}{p}_q (y^{d+1-p} \otimes g^{d+1-p} y^p)(x^t u \otimes v) \\
&= \sum_{p=0}^{d} \binom{d+1}{p}_q (y^{d+1-p} \otimes g^{d+1-p} y^p)(x^t u \otimes v) + (1 \otimes y^{d+1})(x^t u \otimes v) \\
&= \sum_{p=0}^{d} \mu_p (x^{n_p} u \otimes x^p v) + (x^t u \otimes x^{d+1} v),
\end{aligned}$$

where n_p is the remainder resulting from dividing $d+1-p+t$ by n and $\mu_p \in \Bbbk$. By induction assumption, we have $\sum_{p=0}^{d} \mu_p (x^{n_p} u \otimes x^p v) \in \mathrm{sp}\widehat{\Gamma}$. It follows that

$$x^t u \otimes x^{d+1} v = y^{d+1}(x^t u \otimes v) - \sum_{p=0}^{d} \mu_p (x^{n_p} u \otimes x^p v) \in \mathrm{sp}\widehat{\Gamma},$$

for any $0 \le t \le n-1$ and $u \in B_i$, $v \in B_j$, as desired.

(2) The proof of Part (2) is similar to Part (1). $\square$

For any $0 \leq t \leq n-1$, we write

$$\widehat{\Gamma}_t = \{x^t u \otimes v, y(x^t u \otimes v), \cdots, y^{n-1}(x^t u \otimes v) \mid \text{for all } u \in B_i, v \in B_j\}.$$

It is obvious that $\widehat{\Gamma}_t$ is a subset of $\widehat{\Gamma}$ with the cardinality $n \dim V_i \dim V_j$ and $\widehat{\Gamma} = \bigcup_{t=0}^{n-1} \widehat{\Gamma}_t$, which is a disjoint union. Moreover, the subspace $\mathrm{sp}\widehat{\Gamma}_t$ of $P_i \otimes P_j$ is a submodule of $P_i \otimes P_j$.

Proposition 3.2.3. *For any $i, j \in \Omega_1$, we have the following decompositions:*

(1) *If $\lambda_i \lambda_j = 1$, then $V_i \otimes V_j \cong \bigoplus_s V_s$ for some $s \in \Omega_0$. In this case,*

$$P_i \otimes P_j \cong \bigoplus_{t=0}^{n-1} \bigoplus_s M(\tau^t(s), n).$$

(2) *If $\lambda_i \lambda_j \neq 1$, then $V_i \otimes V_j \cong \bigoplus_s V_s$ for some $s \in \Omega_1$. In this case,*

$$P_i \otimes P_j \cong \bigoplus_{t=0}^{n-1} \bigoplus_s P_{\tau^t(s)}.$$

Proof. (1) If $\lambda_i \lambda_j = 1$, by Lemma 3.2.1, $V_i \otimes V_j \cong \bigoplus_s V_s$, for some $s \in \Omega_0$. For any $0 \leq t \leq n-1$, we consider the following $\Bbbk$-linear map:

$$\varphi_t : \mathrm{sp}\widehat{\Gamma}_t \to \bigoplus_s M(\tau^t(s), n), \ y^k(x^t u \otimes v) \mapsto \sum_s x^k \widetilde{\sigma}_{s,t} \pi_s(u \otimes v),$$

for $0 \leq k \leq n-1$ and $u \in B_i, v \in B_j$. Here the map $\widetilde{\sigma}_{s,t}$ is given in Lemma 3.1.6 (3). We first verify that φ_t respects module structure. In fact, for $0 \leq k \leq n-2$,

$$\varphi_t(y^{k+1}(x^t u \otimes v)) = \sum_s x^{k+1} \widetilde{\sigma}_{s,t} \pi_s(u \otimes v) = y\varphi_t(y^k(x^t u \otimes v)).$$

For $k = n-1$, on the one hand,

$$\varphi_t(y^n(x^t u \otimes v)) = \varphi_t((g^n - 1)(x^t u \otimes v)) = (\lambda_i \lambda_j - 1)\varphi_t(x^t u \otimes v) = 0.$$

On the other hand, by the observation of H-module structure of $M(\tau^t(s), n)$, we obtain that

$$y\varphi_t(y^{n-1}(x^t u \otimes v)) = y(\sum_s x^{n-1} \widetilde{\sigma}_{s,t} \pi_s(u \otimes v)) = 0.$$

By Lemma 3.1.6 (3), we have $\widetilde{\sigma}_{s,t}(hv) = \chi^t(h)h\widetilde{\sigma}_{s,t}(v)$ for $h \in G$ and $v \in V_s$. Then

$$\begin{aligned}
\varphi_t(hy^k(x^tu \otimes v)) &= \chi^{-k}(h)\varphi_t(y^k(hx^tu \otimes hv)) \\
&= \chi^{-k-t}(h)\varphi_t(y^k(x^thu \otimes hv)) \\
&= \chi^{-k-t}(h)\sum_s x^k\widetilde{\sigma}_{s,t}\pi_s(hu \otimes hv) \\
&= \chi^{-k-t}(h)\sum_s x^k\widetilde{\sigma}_{s,t}(h\pi_s(u \otimes v)) \\
&= \chi^{-k-t}(h)\sum_s x^k\chi^t(h)h\widetilde{\sigma}_{s,t}(\pi_s(u \otimes v)) \\
&= h\sum_s x^k\widetilde{\sigma}_{s,t}\pi_s(u \otimes v) \\
&= h\varphi_t(y^k(x^tu \otimes v)).
\end{aligned}$$

Hence, φ_t is an H-module morphism. To prove φ_t is injective, we suppose that

$$\varphi_t(\sum_{k=0}^{n-1}\sum_{u\in B_i}\sum_{v\in B_j}\beta_{k,u,v}y^k(x^tu \otimes v)) = 0,$$

for $\beta_{k,u,v} \in \Bbbk$. Then

$$0 = \sum_s\sum_{k=0}^{n-1}\sum_{u\in B_i}\sum_{v\in B_j}\beta_{k,u,v}x^k\widetilde{\sigma}_{s,t}\pi_s(u \otimes v) \in \bigoplus_s M(\tau^t(s), n).$$

It follows that $\sum_{u\in B_i}\sum_{v\in B_j}\beta_{k,u,v}\pi_s(u\otimes v) = 0$ since $\sum_s$ and $\sum_{k=0}^{n-1}$ are direct sums and the map $\widetilde{\sigma}_{s,t}$ is bijective. Note that the map π_s is projective. We obtain

$$\sum_{u\in B_i}\sum_{v\in B_j}\beta_{k,u,v}(u \otimes v) = 0.$$

Therefore, $\beta_{k,u,v} = 0$, for each k, u, v since $u \otimes v,\ u \in B_i,\ v \in B_j$ is a basis of $V_i \otimes V_j$. In view of the dimension,

$$\begin{aligned}
\dim \mathrm{sp}\widehat{\Gamma}_t = n \dim V_i \dim V_j &= n(\sum_s \dim V_s) \\
&= n(\sum_s \dim V_{\tau^t(s)}) = \dim(\bigoplus_s M(\tau^t(s), n)).
\end{aligned}$$

We obtain that φ_t is an H-module isomorphism. Now by Lemma 3.2.2 (1), we obtain the desired result:

$$P_i \otimes P_j = \bigoplus_{t=0}^{n-1} \mathrm{sp}\widehat{\Gamma}_t \cong \bigoplus_{t=0}^{n-1} \bigoplus_s M(\tau^t(s), n).$$

(2) If $\lambda_i \lambda_j \neq 1$, by Lemma 3.2.1, $V_i \otimes V_j \cong \bigoplus_s V_s$, for some $s \in \Omega_1$. For any $0 \leq t \leq n-1$, we define the following $\Bbbk$-linear map:

$$\psi_t : \mathrm{sp}\widehat{\Gamma}_t \to \bigoplus_s P_{\tau^t(s)},\ y^k(x^t u \otimes v) \mapsto \sum_s x^k \widetilde{\sigma}_{s,t} \pi_s(u \otimes v),$$

for $0 \leq k \leq n-1$ and $u \in B_i, v \in B_j$. It is analogous to the proof of Part (1) that ψ_t is an H-module isomorphism. By Lemma 3.2.2 (1),

$$P_i \otimes P_j = \bigoplus_{t=0}^{n-1} \mathrm{sp}\widehat{\Gamma}_t \cong \bigoplus_{t=0}^{n-1} \bigoplus_s P_{\tau^t(s)}.$$

We complete the proof. □

For any $0 \leq t \leq k-1$, let

$$\widehat{\Pi}_t = \{x^t u \otimes v, y(x^t u \otimes v), \cdots, y^{n-1}(x^t u \otimes v) \mid \text{for all } u \in B_i, v \in B_j\}.$$

It is obvious that $\widehat{\Pi} = \overset{k-1}{\underset{t=0}{\cup}} \widehat{\Pi}_t$ is a disjoint union and each subset $\widehat{\Pi}_t$ of $\widehat{\Pi}$ has exactly the same cardinality $n \dim V_i \dim V_j$. Moreover, the subspace $\mathrm{sp}\widehat{\Pi}_t$ of $M(i,k) \otimes P_j$ is a submodule of $M(i,k) \otimes P_j$.

Proposition 3.2.4. *If $i \in \Omega_0$ and $j \in \Omega_1$, then $V_i \otimes V_j \cong \bigoplus_s V_s$ for some $s \in \Omega_1$. In this case,*

$$M(i,k) \otimes P_j \cong P_j \otimes M(i,k) \cong \bigoplus_{t=0}^{k-1} \bigoplus_s P_{\tau^t(s)}.$$

Proof. For $i \in \Omega_0$, $j \in \Omega_1$ and $1 \leq k \leq n$, by Lemma 3.2.1, $V_i \otimes V_j \cong V_j \otimes V_i \cong \bigoplus_s V_s$, for some $s \in \Omega_1$. Consider the following $\Bbbk$-linear map

$$\phi_t : \mathrm{sp}\widehat{\Pi}_t \to \bigoplus_s P_{\tau^t(s)},\ y^k(x^t u \otimes v) \mapsto \sum_s x^k \widetilde{\sigma}_{s,t} \pi_s(u \otimes v),$$

for $0 \leq k \leq n-1$ and $u \in B_i, v \in B_j$. It is analogous to the proof of Proposition 3.2.3 that ϕ_t is an H-module isomorphism. Consequently,

$$M(i,k) \otimes P_j = \bigoplus_{t=0}^{k-1} \mathrm{sp}\widehat{\Pi}_t \cong \bigoplus_{t=0}^{k-1} \bigoplus_{s} P_{\tau^t(s)}.$$

The same argument as above shows that

$$P_j \otimes M(i,k) \cong \bigoplus_{t=0}^{k-1} \bigoplus_{s} P_{\tau^t(s)}$$

since $V_j \otimes V_i \cong V_i \otimes V_j$ and hence $V_j \otimes V_i$ has the same decomposition $\bigoplus_s V_s$. □

Note that $M(i,k)$ and $M(j,l)$ are also $\overline{H}$-modules. It follows from Proposition 3.1.3 (1) that the left $\overline{H}$-module category is a monoidal full subcategory of left H-module category. Thus the decomposition of the tensor product $M(i,k) \otimes M(j,l)$ as H-modules is the same as the decomposition of $M(i,k) \otimes M(j,l)$ as $\overline{H}$-modules. We present the decomposition as follows, which can be deduced from Corollary 2.3.3.

Proposition 3.2.5. *For any $i, j \in \Omega_0$, let $V_i \otimes V_j \cong \bigoplus_s V_s$.*

(1) *If $k + l - 1 \leq n$, then*

$$M(i,k) \otimes M(j,l) \cong \bigoplus_{s} \bigoplus_{t=0}^{\min\{k,l\}-1} M(\tau^t(s), k+l-1-2t).$$

(2) *If $k + l - 1 \geq n$, then*

$$M(i,k) \otimes M(j,l) \cong \bigoplus_{s} (\bigoplus_{t=0}^{r} M(\tau^t(s), n) \bigoplus \bigoplus_{t=r+1}^{\min\{k,l\}-1} M(\tau^t(s), k+l-1-2t)),$$

where $r = k + l - 1 - n$.

As an immediate consequence of Proposition 3.2.3, Proposition 3.2.4 and Proposition 3.2.5, we have the following corollary.

Corollary 3.2.6. *For any two H-modules M and N, we have that $M \otimes N \cong N \otimes M$.*

3.3 The structure of Green rings

In this section, we shall describe the structure of the Green ring $r(H)$. Denote by $M[i,k]$ and by $P_{[j]}$ the isomorphism classes of indecomposable H-modules $M(i,k)$ and P_j respectively. We write 1 for $[\Bbbk]$ and a for $[V_{\chi^{-1}}]$. Then $a^n = 1$ since the order of χ is n.

Let $r(\overline{H})$ be the Green ring of $\overline{H}$. Then $r(\overline{H})$ is a subring of $r(H)$ deduced from Proposition 3.1.3. The generators and relations of $r(\overline{H})$ have been described in Theorem 2.3.4. Let $\mathcal{P}$ be the free abelian group generated by all isomorphism classes of indecomposable projective H-modules. That is,

$$\mathcal{P} = \mathbb{Z}\{M[i,n], P_{[j]} \mid i \in \Omega_0, [j] \in \overline{\Omega}_1\}.$$

Then $\mathcal{P}$ is a two-sided ideal of $r(H)$. Let $r(\overline{H}) \bigoplus \mathcal{P}$ be the direct sum as free $\mathbb{Z}$-modules. Then $r(\overline{H}) \bigoplus \mathcal{P}$ is a commutative ring with the multiplication given by

$$(b_1, c_1)(b_2, c_2) = (b_1 b_2, b_1 c_2 + c_1 b_2 + c_1 c_2),$$

for any $b_1, b_2 \in r(\overline{H})$ and $c_1, c_2 \in \mathcal{P}$. Obviously, the identity of $r(\overline{H}) \bigoplus \mathcal{P}$ is $(1, 0)$.

Theorem 3.3.1. *Let $\mathcal{I}$ be the submodule of $\mathbb{Z}$-module $r(\overline{H}) \bigoplus \mathcal{P}$ generated by the elements $(-M[i,n], M[i,n])$ for $i \in \Omega_0$. Then $\mathcal{I}$ is a two-sided ideal of the ring $r(\overline{H}) \bigoplus \mathcal{P}$ and the quotient ring $(r(\overline{H}) \bigoplus \mathcal{P})/\mathcal{I}$ is isomorphic to $r(H)$.*

Proof. For any $b \in r(\overline{H})$ and $c \in \mathcal{P}$, note that $bM[i,n]$ is a $\mathbb{Z}$-linear combination of elements of the form $M[j,n]$ for $j \in \Omega_0$. Then

$$(b,c)(-M[i,n], M[i,n]) = (-bM[i,n], bM[i,n]) \in \mathcal{I}.$$

Thus, $\mathcal{I}$ is a two-sided ideal of $r(\overline{H}) \bigoplus \mathcal{P}$ since $r(\overline{H}) \bigoplus \mathcal{P}$ is commutative. Define the $\mathbb{Z}$-linear map φ from $r(\overline{H}) \bigoplus \mathcal{P}$ to $r(H)$ as follows:

$$\varphi((b,c)) = b + c$$

for any $b \in r(\overline{H})$ and $c \in \mathcal{P}$. It is straightforward to check that φ is a ring epimorphism with $\ker \varphi = \mathcal{I}$. We conclude that $(r(\overline{H}) \bigoplus \mathcal{P})/\mathcal{I} \cong r(H)$. $\square$

In the following, we will present the Green ring $r(H)$ in terms of generators and relations. The following relations are deduced from the Clebsch-Gordan formulas of H-modules.

Proposition 3.3.2. *For any $i, j \in \Omega_1$, the following hold in $r(H)$:*

(1) *If $\lambda_i\lambda_j = 1$, then $V_i \otimes V_j \cong \bigoplus_s V_s$ for some $s \in \Omega_0$. In this case,*

$$P_{[i]}P_{[j]} = (1 + a + \cdots + a^{n-1}) \sum_s M[s, n]. \tag{3.5}$$

(2) *If $\lambda_i\lambda_j \neq 1$, then $V_i \otimes V_j \cong \bigoplus_s V_s$ for some $s \in \Omega_1$. In this case,*

$$P_{[i]}P_{[j]} = n \sum_s P_{[s]}. \tag{3.6}$$

Proof. (1) It follows from Proposition 3.2.3 (1) that

$$\begin{aligned} P_{[i]}P_{[j]} &= \sum_{t=0}^{n-1} \sum_s M[\tau^t(s), n] \\ &= \sum_{t=0}^{n-1} \sum_s a^t M[s, n] \\ &= (1 + a + \cdots + a^{n-1}) \sum_s M[s, n]. \end{aligned}$$

In addition, the expression (3.5) is well-defined. Suppose $i_1 = \tau^k(i)$ and $j_1 = \tau^p(j)$, for some integers k and p. Then $V_{i_1} \cong V_i \otimes V_{\chi^{-k}}$ and $V_{j_1} \cong V_j \otimes V_{\chi^{-p}}$. This implies that

$$\lambda_{i_1}\lambda_{j_1} = \lambda_i\chi^{-k}(g^n)\lambda_j\chi^{-p}(g^n) = \lambda_i\lambda_j = 1.$$

If $V_i \otimes V_j \cong \bigoplus_s V_s$, for some $s \in \Omega_0$, by Lemma 3.1.6, we have

$$\begin{aligned} V_{i_1} \otimes V_{j_1} &\cong V_i \otimes V_{\chi^{-k}} \otimes V_j \otimes V_{\chi^{-p}} \\ &\cong V_i \otimes V_j \otimes V_{\chi^{-(k+p)}} \\ &\cong (\bigoplus_s V_s) \otimes V_{\chi^{-(k+p)}} \\ &\cong \bigoplus_s V_{\tau^{k+p}(s)}. \end{aligned}$$

Note that $a^n = 1$. By Proposition 3.2.3 (1), we obtain

$$\begin{aligned} P_{[i_1]}P_{[j_1]} &= \sum_{t=0}^{n-1}\sum_s M[\tau^t(\tau^{k+p}(s)), n] \\ &= \sum_{t=0}^{n-1}\sum_s a^{t+k+p}M[s,n] \\ &= (1 + a + \cdots + a^{n-1})\sum_s M[s,n] \\ &= P_{[i]}P_{[j]}. \end{aligned}$$

(2) Note that $P_{[\tau^t(s)]} = P_{[s]}$ since $\tau^t(s) \sim s$. By Proposition 3.2.3 (2), we obtain

$$P_{[i]}P_{[j]} = \sum_{t=0}^{n-1}\sum_s P_{[\tau^t(s)]} = \sum_{t=0}^{n-1}\sum_s P_{[s]} = n\sum_s P_{[s]}.$$

To show that the expression (3.6) is well-defined, we suppose $i_1 = \tau^k(i)$ and $j_1 = \tau^p(j)$, for some integers k and p. Then $\lambda_{i_1}\lambda_{j_1} \neq 1$ since $\lambda_i\lambda_j \neq 1$. If $V_i \otimes V_j \cong \bigoplus_s V_s$, for some $s \in \Omega_1$, then $V_{i_1} \otimes V_{j_1} \cong \bigoplus_s V_{\tau^{k+p}(s)}$ as shown above. By Proposition 3.2.3 (2), we obtain that

$$P_{[i_1]}P_{[j_1]} = \sum_{t=0}^{n-1}\sum_s P_{[\tau^t(\tau^{k+p}(s))]} = \sum_{t=0}^{n-1}\sum_s P_{[s]} = n\sum_s P_{[s]} = P_{[i]}P_{[j]},$$

as desired. □

Proposition 3.3.3. *If $i \in \Omega_0$ and $j \in \Omega_1$, then $V_i \otimes V_j \cong \bigoplus_s V_s$ for some $s \in \Omega_1$. In this case,*

$$M[i,k]P_{[j]} = k\sum_s P_{[s]}. \tag{3.7}$$

Moreover, $[V_i]P_{[j]} = \sum_s P_{[s]}$ and $M[0,2]P_{[j]} = 2P_{[j]}$.

Proof. By Proposition 3.2.4, we have

$$M[i,k]P_{[j]} = \sum_{t=0}^{k-1}\sum_s P_{[\tau^t(s)]} = \sum_{t=0}^{k-1}\sum_s P_{[s]} = k\sum_s P_{[s]}.$$

It is similar that the expression (3.7) is well-defined. □

Let $X = \{X_{[i]} \mid [i] \in \overline{\Omega}_1\}$ and $r(\overline{H})[X]$ be the polynomial ring over $r(\overline{H})$ in the variables $X_{[i]}$ for $[i] \in \overline{\Omega}_1$. Let I be the ideal of $r(\overline{H})[X]$ generated by the following four families of elements:

$$X_{[i]}X_{[j]} - (1 + a + \cdots + a^{n-1})\sum_s M[s, n], \tag{3.8}$$

where $i, j \in \Omega_1,\ \lambda_i\lambda_j = 1$ and $V_i \otimes V_j \cong \bigoplus_s V_s$, for some $s \in \Omega_0$;

$$X_{[i]}X_{[j]} - n\sum_s X_{[s]}, \tag{3.9}$$

where $i, j \in \Omega_1$, $\lambda_i\lambda_j \neq 1$ and $V_i \otimes V_j \cong \bigoplus_s V_s$, for some $s \in \Omega_1$;

$$[V_i]X_{[j]} - \sum_s X_{[s]}, \tag{3.10}$$

where $i \in \Omega_0$, $j \in \Omega_1$ and $V_i \otimes V_j \cong \bigoplus_s V_s$, for some $s \in \Omega_1$;

$$M[0, 2]X_{[j]} - 2X_{[j]}, \tag{3.11}$$

where $j \in \Omega_1$.

Theorem 3.3.4. *The Green ring $r(H)$ of H is isomorphic to the quotient ring $r(\overline{H})[X]/I$.*

Proof. It follows from Corollary 3.2.6 that $r(H)$ is commutative and generated as a ring by $P_{[j]}$, for $[j] \in \overline{\Omega}_1$, over the subring $r(\overline{H})$. Hence there is a unique ring epimorphism Φ from $r(\overline{H})[X]$ to $r(H)$ such that the restriction of Φ to $r(\overline{H})$ is the identity and $\Phi(X_{[j]}) = P_{[j]}$ for $[j] \in \overline{\Omega}_1$. By Proposition 3.3.2 and Proposition 3.3.3, it is easy to check that the map Φ vanishes the generators of the ideal I given from (3.8) to (3.11). Hence Φ induces a unique ring epimorphism $\overline{\Phi}$ from $r(\overline{H})[X]/I$ to $r(H)$ such that $\overline{\Phi}(\overline{z}) = \Phi(z)$ for any $z \in r(\overline{H})[X]$, where $\overline{z}$ means the image of z under the natural epimorphism from $r(\overline{H})[X]$ to $r(\overline{H})[X]/I$. Observe that as a free $\mathbb{Z}$-module, $r(\overline{H})[X]/I$ has a $\mathbb{Z}$-basis $\{\overline{M[i, k]}, \overline{X_{[j]}} \mid i \in \Omega_0, 1 \leq k \leq n, [j] \in \overline{\Omega}_1\}$. We conclude that $r(\overline{H})[X]/I$ and $r(H)$ both have the same rank as free $\mathbb{Z}$-modules. As a result, the map $\overline{\Phi}$ is an isomorphism. $\square$

3.4 Jacobson radicals and idempotents of Green rings

In this section, we use the Frobenius properties of Green rings to show that the Green ring $r(H)$ of a pointed rank one Hopf algebra H of non-nilpotent type has the analogous ring-theoretical properties with those of the Green ring of a pointed rank one Hopf algebra of nilpotent type.

Similar to the notations in Section 2.4, we make use of the following notations:

$$\delta_{M[i,j]} = (1 + a - M[0,2])M[i,j] = \delta_{[\Bbbk]}M[i,j]$$

for any indecomposable non-projective H-module $M(i,j)$, and

$$\delta_{M[i,n]} = M[i,n] - [\mathrm{rad}M(i,n)] = M[i,n] - aM[i,n-1]$$

for any indecomposable projective H-module $M(i,n)$. Note that P_k is simple projective for $k \in \Omega_1$. Then

$$\delta_{[P_k]} = [P_k] - [\mathrm{rad}P_k] = P_{[k]}.$$

Now the set

$$\{\delta_{M[i,j]}, P_{[k]} \mid i \in \Omega_0, 1 \leq j \leq n, [k] \in \overline{\Omega}_1\} \tag{3.12}$$

forms a basis of $r(H)$.

Let

$$\mathcal{P}^{\perp} = \{x \in r(H) \mid (x,y) = 0, \text{for } y \in \mathcal{P}\}.$$

Then $\mathcal{P}^{\perp}$ is an ideal of $r(H)$ since $\mathcal{P}$ is. Denote by $\mathcal{J}$ the free abelian group generated by all almost split sequences of H-modules. That is,

$$\mathcal{J} = \mathbb{Z}\{\delta_{M[i,k]} \mid i \in \Omega_0, 1 \leq k \leq n-1\}.$$

Then $\mathcal{J}$ is exactly the kernel of the natural ring epimorphism from $r(H)$ to $G_0(H)$ given by

$$M[i,j] \mapsto [V_i](1 + a + \cdots + a^{j-1}),\ P_{[k]} \mapsto P_{[k]} \text{ for } i \in \Omega_0,\ 1 \leq j \leq n,\ [k] \in \overline{\Omega}_1.$$

In view of this, $\mathcal{J}$ is an ideal of $r(H)$ generated by $\delta_{[\Bbbk]}$. We have the following relation

$$\mathcal{J} = \mathcal{P}^{\perp} = (\mathcal{P}^{\perp})^*, \tag{3.13}$$

which is similar to Proposition 2.4.1.

In the following, we use (3.13) to describe the Jacobson radical of $r(H)$. We first introduce the following lemma.

Lemma 3.4.1. *If $P^*_{[k]}P_{[l]}$ is expressed as a linear combination of the basis of (3.12), then the coefficient of $\delta_{M[0,n]}$ in $P^*_{[k]}P_{[l]}$ is n if $[k]=[l]$ and 0 otherwise.*

Proof. By the same argument of Proposition 2.2.7, we obtain that the dual $P^*_{[k]} \cong V_{\chi^{-1}} \otimes P_{[k^*]}$, hence $P^*_{[k]} = aP_{[k^*]}$, where $k^* \in \Omega_1$ satisfying $(V_k)^* \cong V_{k^*}$. Thus, if $\lambda_{k^*}\lambda_l \neq 1$ (then $[k] \neq [l]$), by Proposition 3.3.2, $P^*_{[k]}P_{[l]}$ can be written as a linear combination of elements of the form $P_{[s]}$ for some $[s] \in \overline{\Omega}_1$. In this case, the coefficient of $\delta_{M[0,n]}$ in $P^*_{[k]}P_{[l]}$ is 0.

If $\lambda_{k^*}\lambda_l = 1$, by Proposition 3.3.2, we have that

$$\begin{aligned} P^*_{[k]}P_{[l]} &= aP_{[k^*]}P_{[l]} \\ &= a(1+a+\cdots+a^{n-1})[V_k]^*[V_l]M[0,n] \\ &= (1+a+\cdots+a^{n-1})[V_k]^*[V_l]M[0,n]. \end{aligned}$$

It follows from Remark 2.5.8 (2) that

$$M[0,n] = (1+a+\cdots+a^{n-1}) - \sum_{k=1}^{n-1}(\delta_{M[0,1]}+\cdots+\delta_{M[0,k]})a^{n-1-k}.$$

In view of this,

$$\begin{aligned} P^*_{[k]}P_{[l]} &= (1+a+\cdots+a^{n-1})^2[V_k]^*[V_l] \\ &\quad - (1+a+\cdots+a^{n-1})[V_k]^*[V_l]\sum_{k=1}^{n-1}(\delta_{M[0,1]}+\cdots+\delta_{M[0,k]})a^{n-1-k} \\ &= n(1+a+\cdots+a^{n-1})[V_k]^*[V_l] \\ &\quad - (1+a+\cdots+a^{n-1})[V_k]^*[V_l]\sum_{k=1}^{n-1}(\delta_{M[0,1]}+\cdots+\delta_{M[0,k]}). \end{aligned}$$

By (2.10), the coefficient of $\delta_{M[0,n]}$ in $P^*_{[k]}P_{[l]}$ is exactly the coefficient of $[\Bbbk]$ in $n(1+a+\cdots+a^{n-1})[V_k]^*[V_l]$, which is n if k and l belong to the same $\langle\tau\rangle$-orbit (i.e. $[k]=[l]$) and 0 otherwise. ∎

Theorem 3.4.2. *The Jacobson radical $J(r(H))$ of $r(H)$ is the intersection $\mathcal{P} \cap \mathcal{P}^{\perp}$.*

Proof. We only need to check that $J(r(H)) \subseteq \mathcal{P}^{\perp}$, since the other inclusions are similar to that of Theorem 2.5.2. For any $x \in J(r(H))$, we suppose

$$x = \sum_{i \in \Omega_0} \alpha_i \delta_{M[i,n]} + \sum_{[k] \in \overline{\Omega}_1} \beta_{[k]} P_{[k]} + x_0,$$

for $\alpha_i, \beta_{[k]} \in \mathbb{Z}$ and $x_0 \in \mathcal{J}$. It follows from Proposition 3.3.3 that

$$x^* x = (\sum_{i,j \in \Omega_0} \alpha_i \alpha_j \delta^*_{M[i,n]} \delta_{M[j,n]}) + (\sum_{[k],[l] \in \overline{\Omega}_1} \beta_{[k]} \beta_{[l]} P^*_{[k]} P_{[l]}) + \sum_{[k] \in \overline{\Omega}_1} \xi_{[k]} P_{[k]} + x_1,$$

for some $\xi_{[k]} \in \mathbb{Z}$ and $x_1 \in \mathcal{J}$. Note that $\delta^*_{M[i,n]} \delta_{M[j,n]} = [V_i]^*[V_j]$ by (2.12), the coefficient of $\delta_{M[0,n]}$ in $[V_i]^*[V_j]$ is 1 if $i = j$ and 0 otherwise by (2.10), and the coefficient of $\delta_{M[0,n]}$ in $P^*_{[k]} P_{[l]}$ is n if $[k] = [l]$ and 0 otherwise by Lemma 3.4.1. Thus, if we write

$$y := x^* x = \sum_{i \in \Omega_0} \mu_i \delta_{M[i,n]} + \sum_{[k] \in \overline{\Omega}_1} \zeta_{[k]} P_{[k]} + x_2,$$

for some $x_2 \in \mathcal{J}$, then $\mu_0 = \sum_{i \in \Omega_0} \alpha_i^2 + n \sum_{[k] \in \overline{\Omega}_1} \beta^2_{[k]}$. It is similar that if

$$y^2 = y^* y = \sum_{i \in \Omega_0} \gamma_i \delta_{M[i,n]} + \sum_{[k] \in \overline{\Omega}_1} \eta_{[k]} P_{[k]} + x_3,$$

for some $x_3 \in \mathcal{J}$, then $\gamma_0 = \sum_{i \in \Omega_0} \mu_i^2 + n \sum_{[k] \in \overline{\Omega}_1} \zeta^2_{[k]}$. If $\mu_0 \neq 0$, then $\gamma_0 \neq 0$, and therefore $y^2 \neq 0$. By repeating this process, we obtain that the power of y can not be zero if $\mu_0 \neq 0$, a contradiction to the fact that $y \in J(r(H))$. Hence $\mu_0 = 0$ and $x = x_0 \in \mathcal{J} = \mathcal{P}^{\perp}$. □

In the following, we study the Jacobson radical of the Green ring $r(H)$ in terms of generators.

Theorem 3.4.3. *The Jacobson radical of $r(H)$ is a principal ideal generated by the element $(a-1)M[0,n]$.*

Proof. Consider the $\mathbb{Z}$-linear map ϕ from $r(H)$ to $r(\Bbbk G)$ as follows:

$$\phi([V_i]) = [V_i], \ \phi(M[s,k]) = (1 + a + \cdots + a^{k-1})[V_s],$$

and

$$\phi(P_{[j]}) = (1 + a + \cdots + a^{n-1})[V_j],$$

for any $i, s \in \Omega_0$, $1 \leq k \leq n$ and $[j] \in \overline{\Omega}_1$. The map ϕ is well-defined. In fact, if $P_{[j]} = P_{[k]}$, then $j = \tau^l(k)$, for some integer l. In this case,

$$\begin{aligned}(1 + a + \cdots + a^{n-1})[V_j] &= (1 + a + \cdots + a^{n-1})[V_{\tau^l(k)}] \\ &= (1 + a + \cdots + a^{n-1})a^l[V_k] \\ &= (1 + a + \cdots + a^{n-1})[V_k].\end{aligned}$$

It is straightforward to verify that the map ϕ is a ring homomorphism. We claim that $\ker \phi \subseteq r(\overline{H})$. In fact, if the element

$$\sum_{s\in\Omega_0}\sum_{k=1}^{n}\gamma_{s,k}M[s,k] + \sum_{[j]\in\overline{\Omega}_0}\gamma_{[j]}P_{[j]}$$

belongs to $\ker \phi$, where each $\gamma_{s,k}, \gamma_{[j]} \in \mathbb{Z}$, then the image of this element under ϕ is zero. Namely,

$$\sum_{s\in\Omega_0}\sum_{k=1}^{n}\gamma_{s,k}(1 + a + \cdots + a^{k-1})[V_s] + \sum_{[j]\in\overline{\Omega}_0}\gamma_{[j]}(1 + a + \cdots + a^{n-1})[V_j] = 0.$$

Note that the sum $\sum_{s\in\Omega_0}\sum_{k=1}^{n}\gamma_{s,k}(1+a+\cdots+a^{k-1})[V_s]$ is a linear combination of elements of $\{[V_i] \mid i \in \Omega_0\}$, while the sum $\sum_{[j]\in\overline{\Omega}_0}\gamma_{[j]}(1+a+\cdots+a^{n-1})[V_j]$ is a linear combination of elements of $\{[V_i] \mid i \in \Omega_1\}$. We obtain that both of them are equal to zero. However, $\sum_{[j]\in\overline{\Omega}_0}\gamma_{[j]}(1 + a + \cdots + a^{n-1})[V_j] = 0$ implies that each $\gamma_{[j]} = 0$ since the sum $[V_j] + a[V_j] + \cdots + a^{n-1}[V_j]$ is a $\langle\tau\rangle$-orbit sum (the sum of elements in the same $\langle\tau\rangle$-orbit). We conclude that $\ker \phi \subseteq r(\overline{H})$.

Note that the map ϕ maps nilpotent elements to nilpotent elements and $r(\Bbbk G)$ is semisimple [81]. Thus, the Jacobson radical $J(r(H))$ of $r(H)$ is contained in $\ker \phi$. It follows from $\ker \phi \subseteq r(\overline{H})$ that $J(r(H)) \subseteq r(\overline{H})$. Consequently, $J(r(H)) \subseteq J(r(\overline{H}))$ since the Jacobson radical of $r(\overline{H})$ is the unique maximum nilpotent ideal of $r(\overline{H})$. It is obvious that $J(r(\overline{H})) \subseteq J(r(H))$ since $r(\overline{H})$ is a subring of $r(H)$. It follows that $J(r(H)) = J(r(\overline{H}))$. As mentioned in Theorem 2.5.7 that $J(r(\overline{H}))$ is a principal ideal of $r(\overline{H})$ generated by $M[0,n]\theta$, where $\theta = 1 - a$ since the order of a is n. Hence $J(r(H))$ is a principal ideal of $r(H)$ generated by $(a - 1)M[0, n]$. $\square$

In the final part of this section, we show that the Green ring $r(H)$ has only trivial idempotents.

Theorem 3.4.4. *The Green ring $r(H)$ of H has only trivial idempotents.*

Proof. By the same argument with the proof of Theorem 2.6.2, we conclude that $E \in \mathcal{P}$ or $1 - E \in \mathcal{P}$ for each idempotent E of $r(H)$. We suppose $E \in \mathcal{P}$. Then $E = \sum_{i \in \Omega_0} \beta_i M[i,n] + \sum_{[j] \in \overline{\Omega}_1} \alpha_{[j]} P_{[j]}$. Note that the image of E under the ring epimorphism ϕ from $r(H)$ to $r(\Bbbk G)$ given in the proof of Theorem 3.4.3 is

$$\phi(E) = (\sum_{i \in \Omega_0} \beta_i [V_i] + \sum_{[j] \in \overline{\Omega}_1} \alpha_{[j]} [V_j])(1 + a + \cdots + a^{n-1}).$$

It follows that $\phi(E)$ is an idempotent of $r(\Bbbk G)$. However, $r(\Bbbk G)$ has only trivial idempotents (Proposition 2.6.1). Hence $\phi(E) = 0$ or 1. But $(1+a+\cdots+a^{n-1})(1-a) = 0$, this implies that $\phi(E) = 0$, namely,

$$(\sum_{i \in \Omega_0} \beta_i [V_i] + \sum_{[j] \in \overline{\Omega}_1} \alpha_{[j]} [V_j])(1 + a + \cdots + a^{n-1}) = 0.$$

Hence, $\alpha_{[j]} = 0$ for each $[j] \in \overline{\Omega}_1$ since the sum $[V_j] + a[V_j] + \cdots + a^{n-1}[V_j]$ is a $\langle \tau \rangle$-orbit sum. As a result,

$$E = \sum_{i \in \Omega_0} \beta_i M[i,n].$$

Now by the same argument with the proof of Theorem 2.6.2, we obtain that $E = 0$.

If $1 - E \in \mathcal{P}$, it is similar that $1 - E = 0$, and therefore $E = 1$. $\square$

3.5 Green rings of Radford Hopf algebras

In this section, we apply the results obtained in previous sections to a family of Hopf algebras, known as Radford Hopf algebras. This family of Hopf algebras was introduced by Radford in [62] in order to give an example of a Hopf algebra whose Jacobson radical is not a Hopf ideal. As a matter of fact, Radford Hopf algebras are examples of pointed rank one Hopf algebras of non-nilpotent type.

Let G be a cyclic group of order mn with generator g. Suppose V_i is a one dimensional vector space such that the action of g on V_i is the scalar multiple by ω^i, where ω is a primitive mn-th root of unity. Then $\{V_i \mid i \in \mathbb{Z}_{mn}\}$ forms a complete set of non-isomorphic simple $\Bbbk G$-modules. Let χ be the $\Bbbk$-linear character of $V_{m(n-1)}$. Namely, $\chi(g) = \omega^{m(n-1)} = \omega^{-m}$, where ω^{-m} is a primitive n-th root of unity. Then the order of χ is n and the $\Bbbk$-linear character of V_m is χ^{-1}.

Let $\mathcal{D} = (G, \chi, g, 1)$. Then the group datum $\mathcal{D}$ is of non-nilpotent type since $g^n - 1 \neq 0$ and $\chi^n = 1$. Let H be the Hopf algebra associated to the group datum $\mathcal{D}$. That is, H is generated as an algebra by g and y subject to the relations

$$g^{mn} = 1, \ yg = \chi(g)gy = \omega^{-m}gy, \ y^n = g^n - 1.$$

The comultiplication $\triangle$, counit ε, and antipode S are given respectively by

$$\triangle(y) = y \otimes g + 1 \otimes y, \ \varepsilon(y) = 0, \ S(y) = -yg^{-1},$$

$$\triangle(g) = g \otimes g, \ \varepsilon(g) = 1, \ S(g) = g^{-1}.$$

H is a finite dimensional pointed Hopf algebra of rank one with a $\Bbbk$-basis $\{g^i y^j \mid 0 \leq i \leq mn-1, \ 0 \leq j \leq n-1\}$ and $\dim H = mn^2$. It is obvious that H is indeed a Radford Hopf algebra.

Let $N = \{1, g^n, g^{2n}, \cdots, g^{(m-1)n}\}$, $\overline{G} = G/N$ and $\overline{\chi}$ the $\Bbbk$-linear character of G/N such that $\overline{\chi}(\overline{g^i}) = \chi(g^i)$, for $0 \leq i \leq mn-1$. The Hopf algebra associated to the group datum $\overline{\mathcal{D}} = (\overline{G}, \overline{\chi}, \overline{g}, 0)$ of nilpotent type is nothing but the Taft algebra T_n. Let $e = \frac{1}{m}\sum_{k=0}^{m-1} g^{kn}$. Then e is a central idempotent of H and H has the decomposition $H \cong He \oplus H(1-e)$. By Theorem 3.1.10, the subalgebra He is isomorphic to T_n as algebras and $H(1-e)$ is semisimple. Moreover, $H(1-e)$ can be decomposed as $m-1$ copies of matrix algebras $M_n(\Bbbk)$.

Let Ω_0 be a subset of $\mathbb{Z}_{mn}$ consisting of all elements divided by m and Ω_1 a complementary subset of Ω_0. Let τ be the permutation of $\mathbb{Z}_{mn}$ determined by $V_{\chi^{-1}} \otimes V_i \cong V_{\tau(i)}$, where $V_{\chi^{-1}}$ is exactly the simple $\Bbbk G$-module V_m with the character χ^{-1}. It is easy to see that $\tau(i) = m + i$, for any $i \in \mathbb{Z}_{mn}$. Denote by $\langle\tau\rangle$ the subgroup of the symmetry group $\mathbb{S}_{mn}$ generated by the permutation τ. Then $\langle\tau\rangle$ acts on the index set $\mathbb{Z}_{mn}$. We obtain m distinct $\langle\tau\rangle$-orbits $[0], [1], [2], \cdots, [m-1]$, where $[i] = \{i, m+i, 2m+i, \cdots, (n-1)m+i\}$, for $0 \le i \le m-1$. Moreover, $\Omega_0 = [0]$ and $\Omega_1 = [1] \cup [2] \cup \cdots \cup [m-1]$.

It follows from Theorem 3.1.10 that

$$\{M(i,j), P_{[k]} \mid i \in \Omega_0, 1 \le j \le n, 1 \le k \le m-1\}$$

is a complete set of non-isomorphic indecomposable H-modules. Observe that $V_i \otimes V_j \cong V_{i+j}$ and $(\omega^i\omega^j)^n = 1$ if and only if $m \mid i+j$ for any $i, j \in \mathbb{Z}_{mn}$. The Proposition 3.3.2 and Proposition 3.3.3 turn out to be the following.

Proposition 3.5.1. *Let $i, j \in \Omega_1$, $s \in \Omega_0$ and $1 \le k \le n$.*

(1) *If $m \mid i+j$, then $P_{[i]}P_{[j]} = (1 + a + \cdots + a^{n-1})M[0,n]$, where the element $a = [V_{\chi^{-1}}] = [V_m]$.*

(2) *If $m \nmid i+j$, then $P_{[i]}P_{[j]} = nP_{[i+j]}$.*

(3) *$M[s,k]P_{[j]} = kP_{[j]}$. Moreover $[V_s]P_{[j]} = P_{[j]}$ and $M[0,2]P_{[j]} = 2P_{[j]}$.*

We denote by $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]$ the polynomial ring over $\mathbb{Z}$ in the variables $Y, Z, X_1, X_2, \cdots, X_{m-1}$. Denote by $F_n(Y, Z)$ a Dickson polynomial of the second type (see (2.8)) and I the ideal of $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]$ generated by the following elements:

$$Y^n - 1,\ (1 + Y - Z)F_n(Y, Z),\ YX_1 - X_1,\ ZX_1 - 2X_1, \tag{3.14}$$

$$X_1^k - n^{k-1}X_k, \text{ for } 1 \le k \le m-1, \tag{3.15}$$

$$X_1^m - n^{m-2}(1 + Y + \cdots + Y^{n-1})F_n(Y, Z). \tag{3.16}$$

Theorem 3.5.2. *The Green ring $r(H)$ of Radford Hopf algebra H is isomorphic to the quotient ring $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]/I$.*

Proof. Let $r(T_n)$ be the Green ring of the Taft algebra T_n. Then $r(T_n)$ is generated as a ring by the simple module $a = [V_m]$ and the indecomposable module $M[0, 2]$ subject to the relations $a^n = 1$ and $(1+a-M[0,2])F_n(a, M[0, 2]) = 0$, see [24, Theorem 3.10] or Theorem 2.3.4. It follows from Theorem 3.3.4 that $r(H)$ is commutative and generated as a ring by $P_{[k]}$, for $1 \leq k \leq m-1$ over the subring $r(T_n)$. In view of this, there is a unique ring epimorphism Φ from $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]$ to $r(H)$ such that

$$\Phi(Y) = a,\ \Phi(Z) = M[0, 2],\ \Phi(X_k) = P_{[k]},\ \text{for } 1 \leq k \leq m-1.$$

It follows from Proposition 3.5.1 that the map Φ vanishes the generators of the ideal I given in (3.14), (3.15) and (3.16). Hence Φ induces a unique ring epimorphism $\overline{\Phi}$ from $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]/I$ to $r(H)$ such that $\overline{\Phi}(\overline{z}) = \Phi(z)$ for any z in $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]$, where $\overline{z}$ means the image of z under the natural morphism from $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]$ to $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]/I$. Observe that $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]/I$ as a $\mathbb{Z}$-module has a $\mathbb{Z}$-basis $\{\overline{Y^i Z^j}, \overline{X_k} \mid 0 \leq i, j \leq n-1, 1 \leq k \leq m-1\}$. Thus, as free $\mathbb{Z}$-modules, $\mathbb{Z}[Y, Z, X_1, X_2, \cdots, X_{m-1}]/I$ and $r(H)$ both have the same rank $n^2 + m - 1$. It follows that the map $\overline{\Phi}$ is an isomorphism. □

Recall that the Grothendieck ring $G_0(H)$ of Radford Hopf algebra H is isomorphic to the quotient of $r(H)$ modulo the principal ideal generated by $\delta_{[\Bbbk]}$ (see Section 3.4). By Theorem 3.5.2, we have the following description of $G_0(H)$ in terms of generators and relations.

Corollary 3.5.3. *The Grothendieck ring $G_0(H)$ of Radford Hopf algebra H is isomorphic to the quotient ring $\mathbb{Z}[Y, X_1, X_2, \cdots, X_{m-1}]/I_0$, where I_0 is the ideal of $\mathbb{Z}[Y, X_1, X_2, \cdots, X_{m-1}]$ generated by $Y^n - 1, YX_1 - X_1, X_1^j - n^{j-1}X_j$ for $1 \leq j \leq m-1$ and $X_1^m - n^{m-1}(1 + Y + \cdots + Y^{n-1})$.*

Chapter 4

Stable categories of pointed Hopf algebras of rank one

Let H be an arbitrary finite dimensional pointed Hopf algebra of rank one. In this chapter, we show that the stable Green ring of H, i.e., the Green ring of the stable category of H, is isomorphic to the quotient ring of $r(H)$ modulo the ideal generated by indecomposable projective H-modules. Then we show that the complexified stable Green algebra possesses a group-like algebra structure, hence is a bi-Frobenius algebra. Finally, we study Calabi-Yau objects of the stable category of H. In particular, we use the results of Cibils and Zhang to determine the minimal, consequently all Calabi-Yau objects in the stable category of H.

4.1 Stable Green rings

Let H be a finite dimensional pointed Hopf algebra of rank one. In this section, we describe the stable Green ring of H and show that the complexified stable Green algebra is indeed a group-like algebra, a notation due to Doi [28].

Denote by G the group of group-like elements of H. For the sake of convenience, we assume that

$$\widetilde{G} = \begin{cases} G, & \text{if } H \text{ is of nilpotent,} \\ \overline{G}, & \text{if } H \text{ is of non-nilpotent,} \end{cases}$$

where $\overline{G} = G/N$ (see Section 3.1). Then $\{V_i \mid i \in \Omega_0\}$ forms a complete set of non-isomorphic simple $\Bbbk\widetilde{G}$-modules.

Recall that the *stable category* H-$\underline{\text{mod}}$ is the quotient category of H-mod modulo the morphisms factoring through the projective modules. This category is triangulated [43] with the monoidal structure derived from that of H-mod. The Green ring of the stable category H-$\underline{\text{mod}}$, denoted $r_{st}(H)$, is called the *stable Green ring* of H.

Theorem 4.1.1. *The stable Green ring $r_{st}(H)$ is isomorphic to the quotient ring $r(H)/\mathcal{P}$, where $\mathcal{P}$ is the ideal of $r(H)$ generated by the isomorphism classes of projective H-modules. Precisely, $r_{st}(H)$ is isomorphic to $r(\Bbbk\widetilde{G})[z]/(F_n(a,z))$.*

Proof. The functor F from H-mod to H-$\underline{\text{mod}}$ given by $F(M) = M$ for any H-module M, and $F(\phi) = \underline{\phi}$ for $\phi \in \text{Hom}_H(M,N)$ with the canonical image $\underline{\phi} \in \underline{\text{Hom}}(M,N)$ is a full, dense tensor functor. Such a functor defines a ring epimorphism f from $r(H)$ to $r_{st}(H)$ such that $f(\mathcal{P}) = 0$. Hence there is a unique ring epimorphism $\overline{f}$ from $r(H)/\mathcal{P}$ to $r_{st}(H)$ such that $\overline{f}(\overline{x}) = f(x)$, for any $x \in r(H)$ with the canonical image $\overline{x} \in r(H)/\mathcal{P}$. The rank of $r_{st}(H)$ is the same as the rank of $r(H)/\mathcal{P}$ since there is one to one correspondence between the indecomposable objects in H-$\underline{\text{mod}}$ and the non-projective indecomposable objects in H-mod. We conclude that $r_{st}(H)$ is isomorphic to $r(H)/\mathcal{P}$. It follows from Theorem 2.3.4 and Theorem 3.3.4 that the quotient $r(H)/\mathcal{P}$ is always isomorphic to $r(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ no matter whether H is of nilpotent type or of non-nilpotent type, as desired. □

The stable Green ring $r_{st}(H)$ is semisimple since $J(r(H)) \subseteq \mathcal{P}$. The set

$$\{\overline{M[i,j]} \mid i \in \Omega_0, 1 \le j \le n-1\}$$

forms a $\mathbb{Z}$-basis of $r_{st}(H)$. Note that $\mathcal{P}^* = \mathcal{P}$. The dual $*$ induces an automorphism of the stable Green ring $r_{st}(H)$. Consider the complexified stable Green algebra $R_{st}(H) := \mathbb{C} \otimes_{\mathbb{Z}} r_{st}(H)$, we shall show that this algebra is a group-like algebra.

According to the definition of a group-like algebra given in Definition 1.1.7, we know that the complexified Green algebra $R(\Bbbk\widetilde{G}) = \mathbb{C}\otimes_{\mathbb{Z}} r(\Bbbk\widetilde{G})$ is a group-like algebra, where the algebra morphism ε from $R(\Bbbk\widetilde{G})$ to $\mathbb{C}$ is given by $\varepsilon([V_i]) = \dim(V_i)$, for

$i \in \Omega_0$, $\mathbf{b} = \{\dim(V_i)[V_i] \mid i \in \Omega_0\}$ and the involution $*$ is induced by the dual map $[V_{i^*}] = [V_i]^*$, for $i \in \Omega_0$.

Let ε be the algebra morphism from $R(\Bbbk\widetilde{G})[z]$ to $\mathbb{C}$ such that

$$\varepsilon([V_i]) = \dim(V_i), \text{ for } i \in \Omega_0, \ \varepsilon(z) = 2\cos\frac{\pi}{n}.$$

It follows from the proof of Lemma 2.5.4 that $\varepsilon(F_j(a,z)) = F_j(1, 2\cos\frac{\pi}{n}) \neq 0$ for $1 \leq j \leq n-1$ and $\varepsilon(F_n(a,z)) = F_n(1, 2\cos\frac{\pi}{n}) = 0$. Since $R_{st}(H)$ is isomorphic to $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ (see Theorem 4.1.1), ε is exactly the algebra morphism from $R_{st}(H)$ to $\mathbb{C}$ such that

$$\varepsilon(\overline{M[i,j]}) = \dim(V_i) F_j(1, 2\cos\frac{\pi}{n}).$$

We denote by $I = \{(i,j) \mid i \in \Omega_0, \ 1 \leq j \leq n-1\}$ and $b_{(i,j)} = \varepsilon(\overline{M[i,j]})\overline{M[i,j]}$, for $(i,j) \in I$. Then the set $\mathbf{b} = \{b_{(i,j)} \mid (i,j) \in I\}$ forms a basis of $R_{st}(H)$ with $b_{(0,1)} = 1$. Since the automorphism $*$ of $r(H)$ preserves the ideal $\mathcal{P}$, the automorphism $*$ of $r(H)$ induces an automorphism over the quotient ring $r(H)/\mathcal{P} \cong r_{st}(H)$, namely, for any $(i,j) \in I$,

$$(\overline{M[i,j]})^* = \overline{M[i,j]^*}. \tag{4.1}$$

Moreover, by (2.11), we have that

$$\begin{aligned}
\varepsilon(\overline{M[i,j]^*}) &= \varepsilon(\overline{M[\tau^{1-j}(i^*), j]}) = \varepsilon(\overline{a^{1-j}})\varepsilon(\overline{[V_{i^*}]})\varepsilon(\overline{M[0,j]}) \\
&= \dim(V_{i^*})\varepsilon(\overline{M[0,j]}) = \dim(V_i)\varepsilon(\overline{M[0,j]}) \\
&= \varepsilon(\overline{M[i,j]}).
\end{aligned} \tag{4.2}$$

The automorphism $*$ of $r(H)$ induces an involution on I, namely,

$$(i,j)^* = (\tau^{1-j}(i^*), j), \tag{4.3}$$

for any $(i,j) \in I$. With the above notations, we have the following:

Proposition 4.1.2. *The quadruple $(R_{st}(H), \varepsilon, \mathbf{b}, *)$ is a group-like algebra.*

Proof. We check the conditions (G1)-(G3) given in Definition 1.1.7. The condition (G1) follows from (4.2) and (4.3). For the condition (G2), we have

$$(b_{(i,j)})^* = \varepsilon(\overline{M[i,j]})(\overline{M[i,j]})^* = \varepsilon(\overline{M[i,j]^*})\overline{M[i,j]^*} = b_{(i,j)^*}, \tag{4.4}$$

for any $(i,j) \in I$, because of (4.2), (4.1) and (4.3). For any $(i,k),(j,l) \in I$, suppose

$$b_{(i,k)}b_{(j,l)} = \sum_{(s,t)\in I} p^{(s,t)}_{(i,k)(j,l)} b_{(s,t)}, \tag{4.5}$$

where $p^{(s,t)}_{(i,k)(j,l)} \in \mathbb{C}$. Applying the dual automorphism $*$ and (4.4) to the equality (4.5), we obtain that $b_{(j,l)^*}b_{(i,k)^*} = \sum_{(s,t)\in I} p^{(s,t)}_{(i,k)(j,l)} b_{(s,t)^*}$. It follows from (4.5) that $b_{(j,l)^*}b_{(i,k)^*} = \sum_{(r,q)\in I} p^{(r,q)}_{(j,l)^*(i,k)^*} b_{(r,q)}$. We conclude that $p^{(s,t)}_{(i,k)(j,l)} = p^{(s,t)^*}_{(j,l)^*(i,k)^*}$ for any $(i,k),(j,l),(s,t) \in I$. Now we verify the condition (G3). For any $(i,k),(j,l) \in I$ such that $k \neq l$, note that $(j,l)^* = (\tau^{1-l}(j^*), l)$. By Corollary 2.3.3, the multiplication $b_{(i,k)}b_{(j,l)}$ does not contain the term $b_{(0,1)}$. In this case, $p^{(0,1)}_{(i,k)(j,l)} = 0 = \delta_{(i,k),(j,l)^*}\varepsilon(b_{(i,k)})$ and the condition (G3) holds. If $k = l$, by Proposition 2.3.2, we have

$$\begin{aligned} b_{(i,k)}b_{(j,k)} &= \varepsilon(\overline{M[i,k]})\varepsilon(\overline{M[j,k]})\overline{M[i,k]M[j,k]} \\ &= \varepsilon(\overline{M[i,k]})\varepsilon(\overline{M[j,k]})\overline{[V_i][V_j]M[0,k]^2} \\ &= \varepsilon(\overline{M[i,k]})\varepsilon(\overline{M[j,k]})\overline{[V_i][V_j]}(\overline{a}^{k-1} + \sum_{s=2}^{r_k} \overline{a^{k-s}M[0,2s-1]}), \end{aligned}$$

where $r_k = k$ if $2k-1 \leq n$, and $n-k$ otherwise. It follows from equation (2.16) that

$$\overline{[V_i][V_j]a^{k-1}} = \overline{[V_i][V_{\tau^{k-1}(j)}]} = \delta_{i,\tau^{1-k}(j^*)}\overline{[V_0]} + \sum_{t=2}^{m} \gamma_t \overline{[V_t]},$$

for some integers γ_t. Hence the coefficient $p^{(0,1)}_{(i,k)(j,k)}$ of $b_{(0,1)}$ in $b_{(i,k)}b_{(j,k)}$ is

$$\begin{aligned} p^{(0,1)}_{(i,k)(j,k)} &= \varepsilon(\overline{M[i,k]})\varepsilon(\overline{M[j,k]})\delta_{i,\tau^{1-k}(j^*)} \\ &= \varepsilon(\overline{M[i,k]})\varepsilon(\overline{M[j,k]})\delta_{(i,k),(j,k)^*} \\ &= \varepsilon(\overline{M[i,k]})\varepsilon(\overline{M[i,k]^*})\delta_{(i,k),(j,k)^*} \\ &= \varepsilon(b_{(i,k)})\delta_{(i,k),(j,k)^*}. \end{aligned}$$

Therefore, the condition (G3) is satisfied. □

4.2 Bi-Frobenius algebra structure

The notion of a bi-Frobenius algebra was introduced by Doi and Takeuchi in [31]. It generalizes the notion of a finite dimensional Hopf algebra. A group-like algebra can be viewed as a bi-Frobenius algebra in a natural way, see Remark 1.1.8. Following the similar approach, we define on $(R_{st}(H), \varepsilon, \mathbf{b}, *)$ a bi-Frobenius algebra structure as follows.

$(R_{st}(H), \phi)$ is a Frobenius algebra with the Frobenius homomorphism ϕ given by

$$\phi(b_{(i,j)}) = \delta_{(0,1),(i,j)},$$

for $(i,j) \in I$. The pair $\{b_{(i,j)}, \frac{b_{(i,j)^*}}{\varepsilon(b_{(i,j)})} \mid (i,j) \in I\}$ forms a dual basis of $(R_{st}(H), \phi)$. $R_{st}(H)$ is a coalgebra with the counit ε above, and comultiplication $\triangle$ defined by

$$\triangle(b_{(i,j)}) = \frac{1}{\varepsilon(b_{(i,j)})} b_{(i,j)} \otimes b_{(i,j)},$$

for $(i,j) \in I$. Now let $t = \sum_{(i,j)\in I} b_{(i,j)}$. Then $(R_{st}(H), t)$ is a Frobenius coalgebra. Define a map

$$S : R_{st}(H) \to R_{st}(H),\ b_{(i,j)} \mapsto b_{(i,j)^*},$$

for $(i,j) \in I$. It is easy to see that the map S is an anti-algebra and anti-coalgebra map, so is an antipode of $R_{st}(H)$. The quadruple $(R_{st}(H), \phi, t, S)$ forms a bi-Frobenius algebra, see Definition 1.1.5. As a consequence, various properties of group-like algebras and bi-Frobenius algebras (see [28, 29, 30, 31]) hold for $R_{st}(H)$.

We have given $R_{st}(H)$ a bi-Frobenius structure, where the coalgebra structure of $R_{st}(H)$ is defined directly on indecomposable H-modules. Since the stable Green algebra $R_{st}(H)$ is isomorphic to $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$. In the final, we will identity $R_{st}(H)$ with $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ and translate the bi-Frobenius structure of $R_{st}(H)$ to the algebra $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ directly.

Recall that the Dickson polynomial $F_s(y,z)$ of the second type is defined recursively by (2.7). It follows from the general form of $F_s(y,z)$ given by (2.8) that $\{\overline{[V_i]z^j} \mid i \in \Omega_0, 0 \le j \le n-2\}$ forms a basis of $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$.

Lemma 4.2.1. *We have the following reverse version of the Dickson polynomials:*

$$z^s = \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{i} \frac{s+1-2i}{s+1-i} y^i F_{s+1-2i}(y,z), \ \textit{for } s \geq 0.$$

Proof. We only give the proof of the case where s is even, and the same argument works for the case where s is odd. For the sake of simplicity, we write F_k for $F_k(y,z)$ for $k \geq 1$. It is obvious that the equality holds for $s = 0$. Suppose that the equality holds for $s = 2m$, we show that it also holds for $s = 2m+2$. By the recursive relations of (2.7), we have $z^2 F_1 = F_3 + yF_1$ and $z^2 F_{2m+1-2i} = F_{2m+3-2i} + 2yF_{2m+1-2i} + y^2 F_{2m-1-2i}$. By induction hypothesis, we have

$$\begin{aligned}
z^{2m+2} &= z^2 \sum_{i=0}^{m} \binom{2m}{i} \frac{2m+1-2i}{2m+1-i} y^i F_{2m+1-2i} \\
&= \sum_{i=0}^{m-1} \binom{2m}{i} \frac{2m+1-2i}{2m+1-i} y^i z^2 F_{2m+1-2i} + \binom{2m}{m} \frac{1}{m+1} y^m z^2 F_1 \\
&= \sum_{i=0}^{m-1} \binom{2m}{i} \frac{2m+1-2i}{2m+1-i} y^i (F_{2m+3-2i} + 2yF_{2m+1-2i} + y^2 F_{2m-1-2i}) \\
&+ \binom{2m}{m} \frac{1}{m+1} y^m (F_3 + yF_1) \\
&= \sum_{i=0}^{m-1} \binom{2m}{i} \frac{2m+1-2i}{2m+1-i} y^i F_{2m+3-2i} + 2\sum_{i=1}^{m} \binom{2m}{i-1} \frac{2m+3-2i}{2m+2-i} y^i F_{2m+3-2i} \\
&+ \sum_{i=2}^{m+1} \binom{2m}{i-2} \frac{2m+5-2i}{2m+3-i} y^i F_{2m+3-2i} + \binom{2m}{m} \frac{1}{m+1} y^m (F_3 + yF_1) \\
&= \sum_{i=2}^{m-1} (\binom{2m}{i} \frac{2m+1-2i}{2m+1-i} + 2\binom{2m}{i-1} \frac{2m+3-2i}{2m+2-i} + \binom{2m}{i-2} \frac{2m+5-2i}{2m+3-i}) \\
&\cdot y^i F_{2m+3-2i} + F_{2m+3} + (2m+1)yF_{2m+1} \\
&+ \binom{2m+2}{m} \frac{3}{m+3} y^m F_3 + \binom{2m+2}{m+1} \frac{1}{m+2} y^{m+1} F_1 \\
&= \sum_{i=0}^{m+1} \binom{2m+2}{i} \frac{2m+3-2i}{2m+3-i} y^i F_{2m+3-2i}.
\end{aligned}$$

We complete the proof for the case s is even. $\square$

If we identify $R_{st}(H)$ with $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ under the isomorphism given in

Theorem 4.1.1, then the basis $\overline{M[i,j]}$ in $R_{st}(H)$, for $i \in \Omega_0$ and $1 \leq j \leq n-1$ is regarded as $\overline{[V_i]F_j(a,z)}$ in $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$. Conversely, by Lemma 4.2.1, the basis $\overline{[V_i]z^j}$ in $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ is regarded as an element as follows:

$$\sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{k} \frac{j+1-2k}{j+1-k} \overline{[V_i]a^k F_{j+1-2k}(a,z)}.$$

While

$$\sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{k} \frac{j+1-2k}{j+1-k} \overline{[V_i]a^k F_{j+1-2k}(a,z)} = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{k} \frac{j+1-2k}{j+1-k} \overline{M[\tau^k(i), j+1-2k]}.$$

With this identification, the bi-Frobenius algebra structure of $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ induced from $R_{st}(H)$ is given as a proposition as follows.

Proposition 4.2.2. *The quadruple $(R(\Bbbk\widetilde{G})[z]/(F_n(a,z)), \phi, t, S)$ forms a bi-Frobenius algebra. The Frobenius homomorphism ϕ is given by*

$$\phi(\overline{[V_i]z^j}) = \begin{cases} \binom{j}{\frac{j}{2}} \frac{2}{j+2}, & j \text{ is even and } [V_i] = a^{-\frac{j}{2}}, \\ 0, & \text{otherwise.} \end{cases}$$

The comultiplication $\triangle$ of $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ is given by

$$\triangle(\overline{[V_i]z^j}) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{k} \frac{j+1-2k}{j+1-k} \frac{1}{\dim(V_i) F_{j+1-2k}(1, 2\cos\frac{\pi}{n})}$$

$$\cdot \overline{[V_{\tau^k(i)}]F_{j+1-2k}(a,z)} \otimes \overline{[V_{\tau^k(i)}]F_{j+1-2k}(a,z)}.$$

The element t is given by

$$t = \sum_{(i,j)\in I} b_{(i,j)} = \sum_{(i,j)\in I} \dim(V_i) F_j(1, 2\cos\frac{\pi}{n}) \overline{[V_i]F_j(a,z)}.$$

The anti-algebra and anti-coalgebra morphism S is given by

$$S(\overline{[V_i]z^j}) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{k} \frac{j+1-2k}{j+1-k} \overline{[V_{i^*}]a^{k-j} F_{j+1-2k}(a,z)}.$$

From the discussions above, we know that the bi-Frobenius algebra structure of the quotient algebra $R(\Bbbk\widetilde{G})[z]/(F_n(a,z))$ defined on the basis $\overline{[V_j]z^j}$ (polynomials) is more complicated then that defined on $\overline{M[i,j]}$ (indecomposable H-modules). From this point of view, it is an effective way to construct a bi-Frobenius algebra from the stable Green algebra of a Hopf algebra.

4.3 Calabi-Yau objects

In this section, we use the results of Cibils and Zhang in [25] to determine the minimal, consequently all Calabi-Yau objects in stable category of H. Since H is a self-injective algebra, the Nakayama functor $\mathcal{N} := H^* \otimes_H -$, Heller's syzygy functor Ω, and the Auslander-Reiten translate $\mathrm{DTr} \cong \Omega^2 \circ \mathcal{N} \cong \mathcal{N} \circ \Omega^2$ [5, P.126] are endo-equivalences of H-$\underline{\mathrm{mod}}$ [5, Ch IV]. Note that H-$\underline{\mathrm{mod}}$ is a Hom-finite Krull-Schmidt triangulated $\Bbbk$-category with the shift functor $[1] = \Omega^{-1}$ [43, P.16], one gets the Serre functor $F := [1] \circ \mathrm{DTr} \cong \Omega \circ \mathcal{N}$ of H-$\underline{\mathrm{mod}}$ [25].

Note that H is Frobenius. There are $\lambda \in \int_{H^*}^l$ and $\Lambda \in \int_H^r$ such that $\lambda(\Lambda) = 1$ and λ is a Frobenius homomorphism with the dual basis $\{S(\Lambda_1), \Lambda_2\}$. Since $S(\Lambda) \in \int_H^l$, there is a group-like element α of H^* such that $S(\Lambda)b = \alpha(b)S(\Lambda)$, for $b \in H$. By Lemma 1.4.4, we have $Ey^{n-1} \in \int_H^l$, and $Ey^{n-1}h = \chi^{n-1}(h)Ey^{n-1}$, $Ey^{n-1}y = 0$. It follows that

$$\alpha(h) = \chi^{n-1}(h), \text{ for } h \in G, \text{ and } \alpha(y) = 0.$$

For the non-degenerate associative bilinear $\langle b, c\rangle := \lambda(bc)$, there is a Nakayama automorphism $\mu : H \to H$ such that $\langle b, c\rangle = \langle \mu(c), b\rangle$, for $b, c \in H$. For any H-module M, we denote by $M^{(\mu)}$ the H-module with underlying $\Bbbk$-space M and action $b \cdot u := \mu(b)u$, for $b \in H$ and $u \in M$. Since twisting M by an inner automorphism reproduces M, the Nakayama automorphisms induce naturally equivalent automorphisms on H-mod.

Lemma 4.3.1. *With the notions above, the Nakayama automorphism μ of H is given by $\mu(h) = \chi^{n-1}(h)h$, $\mu(y) = y$, for $h \in G$.*

Proof. Using the dual basis $\{S(\Lambda_1), \Lambda_2\}$ with respect to the Frobenius homomorphism λ, we have $\mu(b) = \sum \lambda(\mu(b)S(\Lambda_1))\Lambda_2 = \sum \lambda(S(\Lambda_1)b)\Lambda_2$. Applying S^2 to the equality, and recalling that λ is a left integral in H^*, we obtain that

$$\begin{aligned} S^2(\mu(b)) &= \sum S^2(\Lambda_2)\lambda(S(\Lambda_1)b) \\ &= \sum S^2(\Lambda_3)S(\Lambda_2)b_1\lambda(S(\Lambda_1)b_2) \\ &= \sum b_1\lambda(S(\Lambda)b_2) \\ &= \sum b_1\lambda(\alpha(b_2)S(\Lambda)) \\ &= \sum b_1\alpha(b_2). \end{aligned}$$

Then $\mu(b) = \sum \overline{S}^2(b_1)\alpha(b_2)$, where $b \in H$ and $\overline{S}$ is the inverse of the antipode S under composition. Since $\overline{S}^2(h) = h$, $\overline{S}^2(y) = qy$ and $\alpha(h) = \chi^{n-1}(h)$, $\alpha(y) = 0$, we have $\mu(h) = \chi^{n-1}(h)h$, $\mu(y) = y$, as desired. □

To describe the Calabi-Yau objects in H-mod, we first need to determine the Nakayama functor $\mathcal{N}$ of H-modules.

Lemma 4.3.2. *For any non-projective indecomposable module $M(i,j)$, $i \in \Omega_0$ and $1 \leq j \leq n-1$, we have the following:*

(1) $\mathcal{N}(M(i,j)) \cong M(\tau^{1-n}(i), j)$.

(2) $\Omega(M(i,j)) \cong M(\tau^j(i), n-j)$ *and* $\Omega^{-1}(M(i,j)) \cong M(\tau^{j-n}(i), n-j)$.

Proof. (1) Let $\psi := \mu \otimes id$ be the automorphism of the algebra $H \otimes H^{\mathrm{op}}$. Consider $H^{(\psi)}$ the same as H as vector space while it is an $H \otimes H^{\mathrm{op}}$-module given by $(b \otimes c) \cdot x = \mu(b)xc$, for $b, c, x \in H$. Then the map

$$\Psi : H^{(\psi)} \to H^*,$$

given by $\Psi(b)(c) = \langle b, c\rangle$ is bijective since the form $\langle -, -\rangle$ is non-degenerate. Moreover, the bijective above is an $H \otimes H^{\mathrm{op}}$-module isomorphism, where H^* is $H \otimes H^{\mathrm{op}}$-modules given by $((b \otimes c) \cdot g)(x) = g(cxb)$, for $b, c, x \in H$ and $g \in H^*$. In fact,

$$\begin{aligned}
\Psi((b \otimes c) \cdot x)(z) &= \Psi(\mu(b)xc)(z) = \langle \mu(b)xc, z\rangle \\
&= \langle x, czb\rangle = \Psi(x)(czb) \\
&= ((b \otimes c) \cdot \Psi(x))(z),
\end{aligned}$$

for any $b, c, x, z \in H$. As a result,

$$\mathcal{N}(M) \cong H^* \otimes_H M \cong H^{(\psi)} \otimes_H M \cong M^{(\mu)},$$

for any H-module M. For any non-projective indecomposable module $M(i,j)$, $i \in \Omega_0$ and $1 \leq j \leq n-1$, by Lemma 4.3.1, the following linear map

$$M(i,j)^{(\mu)} \to V_{\chi^{n-1}} \otimes M(i,j), \; x^k v \mapsto u \otimes x^k v,$$

for any $v \in V_i$ and a fixed $0 \neq u \in V_{\chi^{n-1}}$ is an H-module isomorphism. We conclude that $\mathcal{N}(M(i,j)) \cong V_{\chi^{n-1}} \otimes M(i,j) \cong M(\tau^{1-n}(i), j)$.

(2) Note that the epimorphism $p : M(i,n) \to M(i,j)$ given by

$$p(\sum_{k=0}^{n-1} x^k v_k) = \sum_{k=0}^{j-1} x^k v_k, \text{ for } v_k \in V_i,$$

induces an isomorphism $M(i,n)/\mathrm{rad}M(i,n) \cong M(i,j)/\mathrm{rad}M(i,j)$. The map p is a projective cover (cf. [5, Proposition 4.3, ChI]), and therefore $\Omega(M(i,j)) = \ker p$. It is straightforward to check that the map from $\Omega(M(i,j))$ to $M(\tau^j(i), n-j)$ given by

$$\sum_{k=j}^{n-1} x^k v_k \mapsto \sum_{k=j}^{n-1} x^{k-j} \widetilde{\sigma}_{i,j}(v_k)$$

is an H-module isomorphism. We conclude that $\Omega(M(i,j)) \cong M(\tau^j(i), n-j)$. The isomorphism $\Omega^{-1}(M(i,j)) \cong M(\tau^{j-n}(i), n-j)$ follows from that $\Omega(M(i,j)) \cong M(\tau^j(i), n-j)$. $\square$

Remark 4.3.3. As mentioned previously, the Auslander-Reiten translate $\mathrm{DTr} \cong \Omega^2 \circ \mathcal{N} \cong \mathcal{N} \circ \Omega^2$, by Lemma 4.3.2, we obtain that $\mathrm{DTr}M(i,j) \cong M(\tau(i), j)$, which is exactly the result of Proposition 2.2.3.

By the observation of Lemma 4.3.2, we have the following corollary:

Corollary 4.3.4. *For any non-projective indecomposable module $M(i,j)$ and $m \in \mathbb{Z}$,*

$$\Omega^m(M(i,j)) \cong \begin{cases} M(\tau^{\frac{mn}{2}}(i), j), & 2 \mid m, \\ M(\tau^{j+\frac{(m-1)n}{2}}(i), n-j), & 2 \nmid m. \end{cases}$$

Recall that H-$\underline{\mathrm{mod}}$ is *Calabi-Yau* if and only if $\mathcal{N} \cong \Omega^{-(d+1)}$ of functors for some integer d. Denote by $\circ([1])$ the order of $[1]$. If $\circ([1]) = \infty$, then the integer d above is unique and is called the *Calabi-Yau dimension* of H. If $\circ([1])$ is finite, then the minimal non-negative integer d such that $\mathcal{N} \cong \Omega^{-(d+1)}$ is called the *Calabi-Yau dimension* of H (see e.g., [34, 25]).

Proposition 4.3.5. *The stable category H-$\underline{\mathrm{mod}}$ is Calabi-Yau if and only if $n = 2$. In this case, the Calabi-Yau dimension of H is zero.*

Proof. We assume that H-$\underline{\mathrm{mod}}$ is Calabi-Yau, there is some integer d such that $\mathcal{N} \cong \Omega^{-(d+1)}$ and this isomorphism can be taken as H-modules (cf. [25]). For any non-projective indecomposable module $M(i,j)$, by Lemma 4.3.2 and Corollary 4.3.4, $\mathcal{N}(M(i,j)) \cong \Omega^{-(d+1)}(M(i,j))$ if and only if

$$M(\tau^{1-n}(i),j) \cong \begin{cases} M(\tau^{-\frac{(d+1)n}{2}}(i),j), & 2 \mid d+1, \\ M(\tau^{j-\frac{(d+2)n}{2}}(i),n-j), & 2 \mid d. \end{cases}$$

In the case $2 \mid d+1$, the isomorphism $M(\tau^{1-n}(i),j) \cong M(\tau^{-\frac{(d+1)n}{2}}(i),j)$ implies that the order of τ divides $\frac{(d-1)n}{2}+1$. However, the order of τ is divisible by n. This yields that $n=1$, a contradiction to Remark 1.4.2.

In the case $2 \mid d$, the isomorphism $M(\tau^{1-n}(i),j) \cong M(\tau^{j-\frac{(d+2)n}{2}}(i),n-j)$ implies that n is even, $j=\frac{n}{2}$ and the order of τ divides $\frac{(d-1)n}{2}+1$. Note that the order of τ is divisible by n. It follows that $n=2$.

Conversely, if $n=2$, then H is a Nakayama algebra of Loewy length 2. Hence H-$\underline{\mathrm{mod}}$ is Calabi-Yau with the Calabi-Yau dimension zero (cf. [34, Proposition 2.1]). $\square$

In the following, we shall determine all Calabi-Yau objects of H-$\underline{\mathrm{mod}}$ for the case $n>2$. A d-th Calabi-Yau object M of H-$\underline{\mathrm{mod}}$ is said to be *minimal* if any proper direct summand of M is not a d-th Calabi-Yau object. Since every d-th Calabi-Yau object is a direct sum of finitely many minimal d-th Calabi-Yau objects [25, Theorem 4.2], we only need to describe all minimal Calabi-Yau objects of H-$\underline{\mathrm{mod}}$. By [25, Corollary 4.3], every minimal d-th Calabi-Yau object of H-$\underline{\mathrm{mod}}$ is of the form

$$\bigoplus_{0\leq k\leq r_{ij}-1} (\Omega^{d+1}\circ \mathcal{N})^k(M(i,j)),$$

where r_{ij} is the relative order of $\Omega^{d+1}\circ\mathcal{N}$ with respect to non-projective indecomposable module $M(i,j)$. That is, r_{ij} is the minimal positive integer such that $(\Omega^{d+1}\circ\mathcal{N})^{r_{ij}}(M(i,j)) \cong M(i,j)$. It follows from Corollary 4.3.4 that

$$(\Omega^{d+1}\circ\mathcal{N})(M(i,j)) \cong \begin{cases} M(\tau^{1+\frac{(d-1)n}{2}}(i),j), & 2 \mid d+1, \\ M(\tau^{j+1+\frac{(d-2)n}{2}}(i),n-j), & 2 \mid d. \end{cases}$$

If $2 \mid d+1$, then for any $m \in \mathbb{Z}$, we have

$$(\Omega^{d+1}\circ\mathcal{N})^m(M(i,j)) \cong M(\tau^{m+\frac{(d-1)mn}{2}}(i),j).$$

If $2 \mid d$, then

$$(\Omega^{d+1} \circ \mathcal{N})^m(M(i,j)) \cong \begin{cases} M(\tau^{m+\frac{(d-1)mn}{2}}(i), j), & 2 \mid m, \\ M(\tau^{j+m+\frac{(d-1)mn-n}{2}}(i), n-j), & 2 \nmid m. \end{cases}$$

Consequently, together with Lemma 4.3.2, all minimal d-th Calabi-Yau objects of H-<u>mod</u> are described completely as follows.

Theorem 4.3.6. *Let $n > 2$ and M be a minimal d-th Calabi-Yau object of H-<u>mod</u>.*

(1) *If d is odd, then M is isomorphic to one of the following:*

$$\bigoplus_{0 \le m \le m_i - 1} M(\tau^{m+\frac{(d-1)mn}{2}}(i), j),$$

where $i \in \Omega_0$, $1 \le j \le n-1$ and m_i is the least positive integer satisfying $\tau^{m_i+\frac{(d-1)m_i n}{2}}(i) = i$.

(2) *If d is even, then M is isomorphic to one of the following:*

$$\bigoplus_{\substack{0 \le m \le m_i - 1, \\ 2 \mid m}} M(\tau^{m+\frac{(d-1)mn}{2}}(i), j) \bigoplus \bigoplus_{\substack{1 \le m \le m_i, \\ 2 \nmid m}} M(\tau^{j+m+\frac{(d-1)mn-n}{2}}(i), n-j),$$

where $i \in \Omega_0$, $1 \le j \le n-1$ and m_i is the least positive integer satisfying $\tau^{m_i+\frac{(d-1)m_i n}{2}}(i) = i$.

If we restrict to the case that H is of non-nilpotent type, then the order of τ is n. Note that n is the least positive integer satisfying $\tau^n(i) = i$ for any $i \in \Omega_0$ (Lemma 3.1.6 (5)). According to Theorem 4.3.6, the minimal d-th Calabi-Yau objects of H-<u>mod</u> can be described explicitly as follows.

Corollary 4.3.7. *Let H be a finite dimensional pointed rank one Hopf algebra of non-nilpotent type, $n > 2$ and M a minimal d-th Calabi-Yau object of H-<u>mod</u>. If d is odd, then M is isomorphic to one of the following:*

$$\bigoplus_{0 \le k \le n-1} M(\tau^k(i), j),$$

where $i \in \Omega_0$, $1 \le j \le n-1$.

If d is even, then M is isomorphic to one of the following:

(1)
$$\bigoplus_{0\le k\le \frac{n}{2}-1}(M(\tau^{2k}(i),j)\bigoplus M(\tau^{2k}(i),n-j)),$$

where n is even, $i\in\Omega_0$, $1\le j\le n-1$, j is odd and $j\neq\frac{n}{2}$.

(2)
$$\bigoplus_{0\le k\le \frac{n}{2}-1}M(\tau^{2k}(i),j),$$

where n is even, $i\in\Omega_0$, $j=\frac{n}{2}$ is odd.

(3)
$$\bigoplus_{0\le k\le \frac{n}{2}-1}(M(\tau^{2k}(i),j)\bigoplus M(\tau^{2k+1}(i),n-j)),$$

where n is even, $i\in\Omega_0$, $1\le j\le n-1$ and j is even.

(4)
$$\bigoplus_{0\le k\le n-1}(M(\tau^{k}(i),j)\bigoplus M(\tau^{k}(i),n-j)),$$

where n is odd, $i\in\Omega_0$, $1\le j\le n-1$.

Chapter 5

Green rings of finite dimensional Hopf algebras

In this chapter, we attempt to extend those ring-theoretical properties of the Green ring of a finite dimensional pointed Hopf algebra of rank one to the Green ring of an arbitrary finite dimensional Hopf algebra H. We first study the quantum dimensions of H-modules. We determine when an H-module is of quantum dimension zero or non-zero. In particular, we answer the question raised by Cibils: when does the trivial module appear as a direct summand of the tensor product $M \otimes N$ for any two indecomposable modules M and N?

We then study some properties of the Green ring $r(H)$ of H by means of the bilinear form $(-,-)$. This involves the descriptions of some one-sided ideals, the nilpotent ideals and central primitive idempotents of $r(H)$. In addition, we show that the stable Green ring of H possesses an associative non-degenerate bilinear form induced by the bilinear form $(-,-)$ on $r(H)$.

If H is a finite dimensional spherical Hopf algebra, then the quotient ring of $r(H)$ modulo all objects of quantum dimension zero is isomorphic to the Green ring of the quotient category of H-module category modulo all negligible morphisms. In particular, if H is of finite representation type, then the complexified quotient ring is a group-like algebra, and hence a bi-Frobenius algebra.

5.1 Quantum dimensions

In this section, we shall use the techniques given in [42], [80] and [5, Section 4, ChV] to characterize when the trivial module appears as a direct summand of the tensor product $M\otimes N$, a question raised by Cibils. As we shall see, these equivalent assertions are also principal in the study of the Green ring of a Hopf algebra.

Let H be a finite dimensional Hopf algebra over the field $\Bbbk$. For any two finite dimensional H-modules X and Y, the $\Bbbk$-linear space $\mathrm{Hom}_{\Bbbk}(X,Y)$ is an H-module defined by $(hf)(x)=\sum h_1 f(S(h_2)x)$, for $h\in H$, $x\in X$ and $f\in \mathrm{Hom}_{\Bbbk}(X,Y)$. In the special case where $Y=\Bbbk$, the trivial H-module, then $X^*=\mathrm{Hom}_{\Bbbk}(X,\Bbbk)$ is an H-module given by $(hf)(x)=f(S(h)x)$, for $h\in H$, $x\in X$ and $f\in X^*$.

The *evaluation* of H-module X is the morphism $\mathrm{ev}_X : X^*\otimes X\to\Bbbk$ given by $\mathrm{ev}_X(f\otimes x)=f(x)$. The *coevaluation* of X is the morphism $\mathrm{coev}_X:\Bbbk\to X\otimes X^*$ given by $\mathrm{coev}_X(1)=\sum_i x_i\otimes x_i^*$, where $\{x_i\}$ is a basis of X and $\{x_i^*\}$ its dual basis in X^*.

For any H-module X, the *left quantum dimension* of $\theta\in\mathrm{Hom}_H(X,X^{**})$ is defined by the following composition

$$\mathrm{Tr}_X^L(\theta):\Bbbk\xrightarrow{\mathrm{coev}_X}X\otimes X^*\xrightarrow{\theta\otimes id_{X^*}}X^{**}\otimes X^*\xrightarrow{\mathrm{ev}_{X^*}}\Bbbk. \tag{5.1}$$

Similarly, the *right quantum dimension* of $\theta\in\mathrm{Hom}_H(X^{**},X)$ is defined by

$$\mathrm{Tr}_X^R(\theta):\Bbbk\xrightarrow{\mathrm{coev}_{X^*}}X^*\otimes X^{**}\xrightarrow{id_{X^*}\otimes\theta}X^*\otimes X\xrightarrow{\mathrm{ev}_X}\Bbbk. \tag{5.2}$$

Because of $\mathrm{End}_H(\Bbbk)\cong\Bbbk$, $\mathrm{Tr}_X^L(\theta)$ and $\mathrm{Tr}_X^R(\theta)$ can be regarded as elements of $\Bbbk$.

Remark 5.1.1. Applying the duality functor $*$ to (5.1) and (5.2) respectively, one obtains that $\mathrm{Tr}_X^L(\theta)=\mathrm{Tr}_{X^*}^R(\theta^*)$ and $\mathrm{Tr}_X^R(\theta)=\mathrm{Tr}_{X^*}^L(\theta^*)$. We refer the reader to [36, Proposition 1.37.1] for the similar results in the categorical setting.

Remark 5.1.2. Let P be a projective H-module. If H is not semisimple, then $\mathrm{Tr}_P^L(\theta)=0$ for any $\theta\in\mathrm{Hom}_H(P,P^{**})$. Otherwise, the morphism coev_P is a split monomorphism by (5.1), and $\Bbbk$ is a direct summand of the projective module $P\otimes P^*$. Hence $\Bbbk$ is projective, implying that H is semisimple. Similarly, if H is not semisimple, then $\mathrm{Tr}_P^R(\theta)=0$ for any $\theta\in\mathrm{Hom}_H(P^{**},P)$. If H is involutory, that is, $S^2=id_H$, then the map $\theta:P\to P^{**}$ given by $\theta(x)(f)=f(x)$ for $x\in P$ and $f\in P^*$ is an

H-module isomorphism. In particular, $\mathrm{Tr}_P^L(\theta) = \mathrm{Tr}_P^R(\theta^{-1}) = \dim P$. This implies that an involutory Hopf algebra over the field $\Bbbk$ of characteristic 0 is semisimple (the converse is always true [54]).

Lemma 5.1.3. *For H-modules X, Y and Z, we have the following canonical isomorphisms functorial in X, Y and Z:*

(1) $\Phi_{X,Y,Z} : Hom_H(X \otimes Y, Z) \to Hom_H(X, Z \otimes Y^*)$.

(2) $\Psi_{X,Y,Z} : Hom_H(X, Y \otimes Z) \to Hom_H(Y^* \otimes X, Z)$.

Proof. These isomorphisms come from [13, Lemma 2.1.6]. More explicitly, the isomorphism $\Phi_{X,Y,Z}$ is given by $\Phi_{X,Y,Z}(\alpha) = (\alpha \otimes id_{Y^*}) \circ (id_X \otimes \mathrm{coev}_Y)$ with the inverse map $\Phi^{-1}_{X,Y,Z}(\beta) = (id_Z \otimes \mathrm{ev}_Y) \circ (\beta \otimes id_Y)$ for $\alpha \in \mathrm{Hom}_H(X \otimes Y, Z)$ and $\beta \in \mathrm{Hom}_H(X, Z \otimes Y^*)$. Similarly, $\Psi_{X,Y,Z}(\gamma) = (\mathrm{ev}_Y \otimes id_Z) \circ (id_{Y^*} \otimes \gamma)$ with the inverse map $\Psi^{-1}_{X,Y,Z}(\delta) = (id_Y \otimes \delta) \circ (\mathrm{coev}_Y \otimes id_X)$ for $\gamma \in \mathrm{Hom}_H(X, Y \otimes Z)$ and $\delta \in \mathrm{Hom}_H(Y^* \otimes X, Z)$. $\square$

The canonical isomorphisms given in Lemma 5.1.3 have the following properties.

Proposition 5.1.4. *Let X be an indecomposable H-module and $\theta : X \to X^{**}$ an H-module isomorphism. For any H-module Y, we have the following:*

(1) *The canonical isomorphism $\Phi_{Y,X^*,\Bbbk} : Hom_H(Y \otimes X^*, \Bbbk) \to Hom_H(Y, X^{**})$ preserves split epimorphisms.*

(2) *The canonical isomorphism $\Psi_{Y,X,\Bbbk} : Hom_H(Y, X) \to Hom_H(X^* \otimes Y, \Bbbk)$ reflects split epimorphisms.*

Proof. (1) If the map $\alpha \in \mathrm{Hom}_H(Y \otimes X^*, \Bbbk)$ is a split epimorphism, then there is some $\beta \in \mathrm{Hom}_H(\Bbbk, Y \otimes X^*)$ such that $\alpha \circ \beta = id_\Bbbk$. For the map β, there is $\gamma \in \mathrm{Hom}_H(X, Y)$ such that $\beta = \Phi_{\Bbbk,X,Y}(\gamma)$. Note that the composition $\Phi_{Y,X^*,\Bbbk}(\alpha) \circ \gamma \circ \theta^{-1} \in \mathrm{End}_H(X^{**})$. If $\Phi_{Y,X^*,\Bbbk}(\alpha) \circ \gamma \circ \theta^{-1} \in \mathrm{radEnd}_H(X^{**})$, so is $(\Phi_{Y,X^*,\Bbbk}(\alpha) \circ \gamma \circ \theta^{-1}) \otimes id_{X^*} \in \mathrm{radEnd}_H(X^{**} \otimes X^*)$. Hence the following endomorphism of $\Bbbk$

$$ev_{X^*} \circ ((\Phi_{Y,X^*,\Bbbk}(\alpha) \circ \gamma \circ \theta^{-1} \circ \theta) \otimes id_{X^*}) \circ coev_X,$$

factoring through $(\Phi_{Y,X^*,\Bbbk}(\alpha)\circ\gamma\circ\theta^{-1})\otimes id_{X^*}$, is zero. However,

$$\begin{aligned}
&ev_{X^*}\circ((\Phi_{Y,X^*,\Bbbk}(\alpha)\circ\gamma\circ\theta^{-1}\circ\theta)\otimes id_{X^*})\circ coev_X\\
&=\Phi^{-1}_{Y,X^*,\Bbbk}(\Phi_{Y,X^*,\Bbbk}(\alpha))\circ\Phi_{\Bbbk,X,Y}(\gamma)\\
&=\alpha\circ\beta\\
&=id_{\Bbbk},
\end{aligned}$$

a contradiction. This means that $\Phi_{Y,X^*,\Bbbk}(\alpha)\circ\gamma\circ\theta^{-1}$ is an automorphism of X^{**} since $\mathrm{End}_H(X^{**})$ is local. Thus, the map $\Phi_{Y,X^*,\Bbbk}(\alpha)$ is a split epimorphism.

(2) Given $\alpha\in\mathrm{Hom}_H(Y,X)$ such that $\Psi_{Y,X,\Bbbk}(\alpha)$ is a split epimorphism. There is some $\beta\in\mathrm{Hom}_H(\Bbbk,X^*\otimes Y)$ such that $\Psi_{Y,X,\Bbbk}(\alpha)\circ\beta=id_{\Bbbk}$. For the map β, there is $\gamma\in\mathrm{Hom}_H(X^{**},Y)$ such that $\beta=\Psi^{-1}_{\Bbbk,X^*,Y}(\gamma)$. Note that the composition $\alpha\circ\gamma\circ\theta\in\mathrm{End}_H(X)$. If $\alpha\circ\gamma\circ\theta\in\mathrm{radEnd}_H(X)$, so is $id_{X^*}\otimes(\alpha\circ\gamma\circ\theta)\in\mathrm{radEnd}_H(X^*\otimes X)$. Hence the following endomorphism of $\Bbbk$

$$ev_X\circ(id_{X^*}\otimes(\alpha\circ\gamma\circ\theta\circ\theta^{-1}))\circ coev_{X^*},$$

factoring through $id_{X^*}\otimes(\alpha\circ\gamma\circ\theta)$, is zero. However,

$$\begin{aligned}
&ev_X\circ(id_{X^*}\otimes(\alpha\circ\gamma\circ\theta\circ\theta^{-1}))\circ coev_{X^*}\\
&=\Psi_{Y,X,\Bbbk}(\alpha)\circ\Psi^{-1}_{\Bbbk,X^*,Y}(\gamma)\\
&=\Psi_{Y,X,\Bbbk}(\alpha)\circ\beta\\
&=id_{\Bbbk},
\end{aligned}$$

a contradiction. This means that $\alpha\circ\gamma\circ\theta$ is an automorphism of X since $\mathrm{End}_H(X)$ is local. Thus, the map α is a split epimorphism. $\square$

As an immediate consequence of Proposition 5.1.4, we have the following corollary.

Corollary 5.1.5. *Let X and Y be indecomposable H-modules and $X\cong X^{**}$.*

(1) *If $\Bbbk\mid Y\otimes X^*$, then $Y\cong X^{**}$.*

(2) *If $\Bbbk\mid X^*\otimes Y$, then $X\cong Y$.*

To pursue Corollary 5.1.5 even further, we need some preparations. For any integer $m>0$, the m-th power of dual $*$ on X is denoted by X^{*m}. If $\{x_i\}$ is a basis of X, we denote by $\{x_i^{*m}\}$ the basis of X^{*m} which is dual to the basis $\{x_i^{*m-1}\}$ of X^{*m-1}. That is, $\langle x_i^{*m}, x_j^{*m-1}\rangle = \delta_{i,j}$. With these notations, we have the following.

Lemma 5.1.6. *Let X be an indecomposable H-module.*

(1) *For any $\theta \in Hom_H(X, X^{**})$, if $Tr_X^L(\theta) \neq 0$, then the map θ is an isomorphism.*

(2) *For any $\theta \in Hom_H(X^{**}, X)$, if $Tr_X^R(\theta) \neq 0$, then the map θ is an isomorphism.*

Proof. We only prove Part (1) and the proof of Part (2) is similar. Denote by $\mathbf{A}$ the transformation matrix of $\theta \in \mathrm{Hom}_H(X, X^{**})$ with respect to the bases $\{x_i\}$ of X and $\{x_i^{**}\}$ of X^{**}. The left quantum dimension of θ is $\mathrm{Tr}_X^L(\theta) = \mathrm{tr}(\mathbf{A})$, the usual trace of the matrix $\mathbf{A}$. Suppose that $S^{2n} = id_H$. Then the map

$$\gamma : X^{*2n} \to X,\ \sum_i \lambda_i x_i^{*2n} \mapsto \sum_i \lambda_i x_i$$

is an H-module isomorphism. Moreover, the matrix of the map γ with respect to the basis $\{x_i^{*2n}\}$ of X^{*2n} and the basis $\{x_i\}$ of X is the identity matrix. Consider the following composition:

$$\Theta : X \xrightarrow{\theta} X^{**} \xrightarrow{\theta^{**}} X^{****} \to \cdots \to X^{*2n-2} \xrightarrow{\theta^{*2n-2}} X^{*2n} \xrightarrow{\gamma} X.$$

The matrix of the map Θ from X to itself under the basis $\{x_i\}$ of X is $\mathbf{A}^n$. Note that $\mathrm{End}_H(X)$ is local, the map Θ is either nilpotent or isomorphic. If Θ is nilpotent, then $\mathbf{A}^n$, and hence $\mathbf{A}$, is nilpotent. This implies that $\mathrm{Tr}_X^L(\theta) = \mathrm{tr}(\mathbf{A}) = 0$, a contradiction. Thus, Θ is an isomorphism, and therefore the map θ is an isomorphisms. $\square$

Cibils in [21, Remark 5.8] mentioned the following question: when is the trivial module a direct summand of tensor product $M \otimes N$ for a finite dimensional Hopf algebra with antipode of order bigger than 2? Now we are ready to answer this question as follows.

Theorem 5.1.7. *Let X and Y be two indecomposable H-modules.*

(1) $\Bbbk \mid Y \otimes X^*$ *if and only if there are isomorphisms $f : X \to Y$ and $g : Y \to X^{**}$ such that $Tr_X^L(g \circ f) = id_\Bbbk$.*

(2) $\Bbbk \mid X^* \otimes Y$ *if and only if there are isomorphisms $f : X^{**} \to Y$ and $g : Y \to X$ such that $Tr_X^R(g \circ f) = id_\Bbbk$.*

Proof. We only prove Part (1), the same argument works for Part (2). If there are isomorphisms $X \xrightarrow{f} Y \xrightarrow{g} X^{**}$ such that $\mathrm{Tr}_X^L(g \circ f) = id_{\Bbbk}$, by (5.1), we have

$$id_{\Bbbk} = \mathrm{Tr}_X^L(g \circ f) = \mathrm{ev}_{X^*} \circ (g \otimes id_{X^*}) \circ (f \otimes id_{X^*}) \circ \mathrm{coev}_X.$$

This implies that the map $(f \otimes id_{X^*}) \circ \mathrm{coev}_X : \Bbbk \to Y \otimes X^*$ is a split monomorphism, and hence $\Bbbk \mid Y \otimes X^*$. Conversely, if $\Bbbk \mid Y \otimes X^*$, then there is a map $\alpha : \Bbbk \to Y \otimes X^*$ and a map $\beta : Y \otimes X^* \to \Bbbk$ such that $\beta \circ \alpha = id_{\Bbbk}$. For the map α, by Lemma 5.1.3, there is $f : X \to Y$ such that

$$\alpha = \Phi_{\Bbbk,X,Y}(f) = (f \otimes id_{X^*}) \circ (id_{\Bbbk} \otimes \mathrm{coev}_X).$$

Similarly, for the map β, there is $g : Y \to X^{**}$ such that

$$\beta = \Phi^{-1}_{Y,X^*,\Bbbk}(g) = (id_{\Bbbk} \otimes \mathrm{ev}_{X^*}) \circ (g \otimes id_{X^*}).$$

This yields that

$$\begin{aligned}
\mathrm{Tr}_X^L(g \circ f) &= \mathrm{ev}_{X^*} \circ (g \otimes id_{X^*}) \circ (f \otimes id_{X^*}) \circ \mathrm{coev}_X \\
&= (id_{\Bbbk} \otimes \mathrm{ev}_{X^*}) \circ (g \otimes id_{X^*}) \circ (f \otimes id_{X^*}) \circ (id_{\Bbbk} \otimes \mathrm{coev}_X) \\
&= \beta \circ \alpha \\
&= id_{\Bbbk}.
\end{aligned}$$

The fact that the composition $g \circ f$ is an isomorphism follows from Lemma 5.1.6. We obtain that f and g are both isomorphisms. □

For any two indecomposable modules X and Y, one knows little about how to decompose the tensor product $X \otimes Y$ into a direct sum of indecomposable modules. However, there are still some rules that the decomposition should follow as shown in the following.

Proposition 5.1.8. *Let X, Y and M be H-modules with X and M indecomposable.*

(1) *If $M \mid X \otimes Y$ and $\Bbbk \mid M \otimes M^*$, then $\Bbbk \mid X \otimes X^*$ and $X \mid M \otimes Y^*$.*

(2) *If $M \mid Y \otimes X$ and $\Bbbk \mid M^* \otimes M$, then $\Bbbk \mid X^* \otimes X$ and $X \mid Y^* \otimes M$.*

Proof. (1) We only prove Part (1) and the proof of Part (2) is similar. The conditions $\Bbbk \mid M \otimes M^*$ and $M \mid X \otimes Y$ imply that $\Bbbk \mid X \otimes Y \otimes M^*$. Suppose $Y \otimes M^* \cong \bigoplus_i N_i^*$

for some indecomposable modules N_i. There is an indecomposable module N_i such that $\Bbbk \mid X \otimes N_i^*$. By Theorem 5.1.7 (1), we obtain $X \cong N_i \cong N_i^{**}$. It follows that $\Bbbk \mid X \otimes N_i^* \cong X \otimes X^*$. Note that $\Bbbk \mid M \otimes M^*$ implies that $M \cong M^{**}$. Then $X \cong N_i^{**}$ implies that $X \mid (Y \otimes M^*)^* \cong M \otimes Y^*$, as desired. □

As an immediate consequence of Proposition 5.1.8, we obtain the following corollary, which will be used in the next section.

Corollary 5.1.9. *Let X and M be two indecomposable H-modules.*

(1) *If $\Bbbk \nmid X \otimes X^*$ and $M \mid X \otimes Y$, then $\Bbbk \nmid M \otimes M^*$.*

(2) *If $\Bbbk \nmid X^* \otimes X$ and $M \mid Y \otimes X$, then $\Bbbk \nmid M^* \otimes M$.*

In the following of this section, H is always a non-semisimple Hopf algebra. We have the almost split sequence

$$0 \to \tau(\Bbbk) \to E \xrightarrow{\sigma} \Bbbk \to 0 \tag{5.3}$$

ending at the trivial module $\Bbbk$. By tensoring (over $\Bbbk$) the sequence (5.3) with an indecomposable module X, we obtain the following two short exact sequences:

$$0 \to \tau(\Bbbk) \otimes X \to E \otimes X \xrightarrow{\sigma \otimes id_X} X \to 0 \tag{5.4}$$

$$0 \to X \otimes \tau(\Bbbk) \to X \otimes E \xrightarrow{id_X \otimes \sigma} X \to 0. \tag{5.5}$$

Lemma 5.1.10. *For any two H-modules X and Y, we have the commutative diagrams:*

$$\begin{array}{ccc} Hom_H(Y, X \otimes E) & \xrightarrow{(id_X \otimes \sigma)_*} & Hom_H(Y, X) \\ {\scriptstyle \Psi_{Y,X,E}}\downarrow & & {\scriptstyle \Psi_{Y,X,\Bbbk}}\downarrow \\ Hom_H(X^* \otimes Y, E) & \xrightarrow{\sigma_*} & Hom_H(X^* \otimes Y, \Bbbk), \end{array} \tag{5.6}$$

$$\begin{array}{ccc} Hom_H(Y \otimes X, E) & \xrightarrow{\sigma_*} & Hom_H(Y \otimes X, \Bbbk) \\ {\scriptstyle \Phi_{Y,X,E}}\downarrow & & {\scriptstyle \Phi_{Y,X,\Bbbk}}\downarrow \\ Hom_H(Y, E \otimes X^*) & \xrightarrow{(\sigma \otimes id_{X^*})_*} & Hom_H(Y, X^*). \end{array} \tag{5.7}$$

Proof. The verification of the commutative diagrams follows from Lemma 5.1.3 and is straightforward. □

In the following, we shall give some equivalent characterizations on whether or not the trivial module $\Bbbk$ appears in the tensor product $X^* \otimes X$ (resp. $X \otimes X^*$). For the case where the square of antipode is inner, we refer to [42, 80].

Proposition 5.1.11. *Let X be an indecomposable H-module. The following are equivalent:*

(1) $\Bbbk \nmid X^* \otimes X$.

(2) *The map $Hom_H(X^* \otimes X, E) \xrightarrow{\sigma_*} Hom_H(X^* \otimes X, \Bbbk)$ is surjective.*

(3) *The map $Hom_H(X, X \otimes E) \xrightarrow{(id_X \otimes \sigma)_*} Hom_H(X, X)$ is surjective.*

(4) *The map $X \otimes E \xrightarrow{id_X \otimes \sigma} X$ is a split epimorphism.*

(5) *The map $E \otimes X^* \xrightarrow{\sigma \otimes id_{X^*}} X^*$ is a split epimorphism.*

Proof. (1) $\Leftrightarrow$ (2). If $\Bbbk \nmid X^* \otimes X$, then for any $\alpha \in \mathrm{Hom}_H(X^* \otimes X, \Bbbk)$, the map α is not a split epimorphism. Since σ is right almost split from E to $\Bbbk$, there is a map β from $X^* \otimes X$ to E such that $\sigma \circ \beta = \alpha$. This implies that the map σ_* is surjective. Conversely, if the map σ_* is surjective, then $\Bbbk \nmid X^* \otimes X$. Otherwise, by Theorem 5.1.7 (2), there is an isomorphism $\theta : X^{**} \to X$ such that $\mathrm{Tr}_X^R(\theta) = id_{\Bbbk}$. For the map $\mathrm{ev}_X : X^* \otimes X \to \Bbbk$, there is some $\beta \in \mathrm{Hom}_H(X^* \otimes X, E)$ such that $\sigma \circ \beta = \mathrm{ev}_X$ since the map σ_* is surjective. It follows that

$$id_{\Bbbk} = \mathrm{Tr}_X^R(\theta) = \mathrm{ev}_X \circ (id_{X^*} \otimes \theta) \circ \mathrm{coev}_{X^*} = \sigma \circ \beta \circ (id_{X^*} \otimes \theta) \circ \mathrm{coev}_{X^*}.$$

We obtain that the map σ is a split epimorphism, a contradiction to the fact that σ is right almost split.

(2) $\Leftrightarrow$ (3). It follows from (5.6) that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_H(X, X \otimes E) & \xrightarrow{(id_X \otimes \sigma)_*} & \mathrm{Hom}_H(X, X) \\ {\scriptstyle \Psi_{X,X,E}} \downarrow & & {\scriptstyle \Psi_{X,X,\Bbbk}} \downarrow \\ \mathrm{Hom}_H(X^* \otimes X, E) & \xrightarrow{\sigma_*} & \mathrm{Hom}_H(X^* \otimes X, \Bbbk). \end{array}$$

This implies that σ_* is surjective if and only if $(id_X \otimes \sigma)_*$ is surjective.

(3) $\Leftrightarrow$ (4). If $(id_X \otimes \sigma)_*$ is surjective, then for the map $id_X \in \mathrm{Hom}_H(X, X)$, there is a map $\alpha \in \mathrm{Hom}_H(X, X \otimes E)$ such that $(id_X \otimes \sigma)_*(\alpha) = id_X$. This means that

$(id_X \otimes \sigma) \circ \alpha = id_X$, and hence $id_X \otimes \sigma$ is a split epimorphism. Conversely, if $id_X \otimes \sigma$ is a split epimorphism, there is $\alpha \in \mathrm{Hom}_H(X, X \otimes E)$ such that $(id_X \otimes \sigma) \circ \alpha = id_X$. For any $\beta \in \mathrm{Hom}_H(X, X)$, we have $(id_X \otimes \sigma)_*(\alpha \circ \beta) = \beta$. It follows that the map $(id_X \otimes \sigma)_*$ is surjective.

(2) $\Leftrightarrow$ (5). It follows from (5.7) that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_H(X^* \otimes X, E) & \xrightarrow{\sigma_*} & \mathrm{Hom}_H(X^* \otimes X, \Bbbk) \\ {\scriptstyle \Phi_{X^*,X,E}}\downarrow & & {\scriptstyle \Phi_{X^*,X,\Bbbk}}\downarrow \\ \mathrm{Hom}_H(X^*, E \otimes X^*) & \xrightarrow{(\sigma \otimes id_{X^*})_*} & \mathrm{Hom}_H(X^*, X^*). \end{array}$$

is commutative. We obtain that σ_* is surjective if and only if $(\sigma \otimes id_{X^*})_*$ is surjective. If $(\sigma \otimes id_{X^*})_*$ is surjective, for $id_{X^*} \in \mathrm{Hom}_H(X^*, X^*)$, there is $\alpha \in \mathrm{Hom}_H(X^*, E \otimes X^*)$ such that $id_{X^*} = (\sigma \otimes id_{X^*})_*(\alpha) = (\sigma \otimes id_{X^*}) \circ \alpha$. This implies that the map $\sigma \otimes id_{X^*}$ is a split epimorphism. Conversely, if $\sigma \otimes id_{X^*}$ is a split epimorphism, there is $\alpha \in \mathrm{Hom}_H(X^*, E \otimes X^*)$ such that $(\sigma \otimes id_{X^*}) \circ \alpha = id_{X^*}$. For any $\beta \in \mathrm{Hom}_H(X^*, X^*)$, we obtain that $(\sigma \otimes id_{X^*})_*(\alpha \circ \beta) = \beta$. It follows that the map $(\sigma \otimes id_{X^*})_*$ is surjective. $\square$

Similarly, there are some equivalent characterizations of the property that $\Bbbk \nmid X \otimes X^*$. However, we only need the following.

Proposition 5.1.12. *Let X be an indecomposable H-module. The following are equivalent:*

(1) $\Bbbk \nmid X \otimes X^*$.

(2) *The map $E \otimes X \xrightarrow{\sigma \otimes id_X} X$ is a split epimorphism.*

Proof. Let Y be an indecomposable module such that $Y^* \cong X$ (such a Y exists as the order of S^2 is finite). Then $\Bbbk \nmid X \otimes X^*$ if and only if $\Bbbk \nmid (Y^* \otimes Y)^*$ if and only if $\Bbbk \nmid Y^* \otimes Y$. By Proposition 5.1.11, this is precisely that $\sigma \otimes id_{Y^*}$ is a split epimorphism. $\square$

Proposition 5.1.13. *Let X be an indecomposable H-module. The following are equivalent:*

(1) $\Bbbk \mid X^* \otimes X$.

(2) *The map* $X \otimes E \xrightarrow{id_X \otimes \sigma} X$ *is right almost split.*

Proof. If $id_X \otimes \sigma$ is right almost split, then it is not a split epimorphism. By Proposition 5.1.11, we have that $\Bbbk \mid X^* \otimes X$. Conversely, if $\Bbbk \mid X^* \otimes X$, by Proposition 5.1.11, the map $id_X \otimes \sigma$ is not a split epimorphism. The condition $\Bbbk \mid X^* \otimes X$ also implies that $X \cong X^{**}$. Now by Proposition 5.1.4 (2), for any non-split epimorphism $\alpha \in \mathrm{Hom}_H(Y, X)$, the map $\Psi_{Y,X,\Bbbk}(\alpha) \in \mathrm{Hom}_H(X^* \otimes Y, \Bbbk)$ is also non-split epimorphism. For the map $\Psi_{Y,X,\Bbbk}(\alpha)$, there is a map $\beta \in \mathrm{Hom}_H(X^* \otimes Y, E)$ such that

$$\sigma \circ \beta = \Psi_{Y,X,\Bbbk}(\alpha)$$

since σ is right almost split. For the map β, we obtain the map $\Psi^{-1}_{Y,X,E}(\beta) \in \mathrm{Hom}_H(Y, X \otimes E)$. We claim that the map $\Psi^{-1}_{Y,X,E}(\beta)$ satisfies the relation $(id_X \otimes \sigma) \circ \Psi^{-1}_{Y,X,E}(\beta) = \alpha$, and therefore $id_X \otimes \sigma$ is right almost. Note that the commutative diagram (5.6) implies that

$$\Psi_{Y,X,\Bbbk} \circ (id_X \otimes \sigma)_* = \sigma_* \circ \Psi_{Y,X,E}.$$

It follows that

$$\begin{aligned}
\alpha &= \Psi^{-1}_{Y,X,\Bbbk}(\sigma \circ \beta) \\
&= (\Psi^{-1}_{Y,X,\Bbbk} \circ \sigma_*)(\beta) \\
&= ((id_X \otimes \sigma)_* \circ \Psi^{-1}_{Y,X,E})(\beta) \\
&= (id_X \otimes \sigma) \circ \Psi^{-1}_{Y,X,E}(\beta).
\end{aligned}$$

This completes the proof. □

Proposition 5.1.14. *Let X be an indecomposable H-module. The following are equivalent:*

(1) $\Bbbk \mid X \otimes X^*$.

(2) *The map* $E \otimes X \xrightarrow{\sigma \otimes id_X} X$ *is right almost split.*

Proof. If the map $\sigma \otimes id_X$ is right almost split, then it is not a split epimorphism. By Proposition 5.1.12, we obtain $\Bbbk \mid X \otimes X^*$. Conversely, to show that $\sigma \otimes id_X$ is right almost split, it is equivalent to showing that $\sigma \otimes id_{X^{**}}$ is right almost split because $\Bbbk \mid X \otimes X^*$ implies that $X \cong X^{**}$ (Theorem 5.1.7 (1)). Note that $\Bbbk \mid X^{**} \otimes X^{***}$.

It follows from Proposition 5.1.12 that the map $\sigma \otimes id_{X^{**}}$ is not a split epimorphism. For any $\alpha \in \mathrm{Hom}_H(Y, X^{**})$ which is not a split epimorphism, by Proposition 5.1.4 (1), $\Phi^{-1}_{Y,X^*,\Bbbk}(\alpha) \in \mathrm{Hom}_H(Y \otimes X^*, \Bbbk)$ is also not split epimorphism. We obtain a map $\beta \in \mathrm{Hom}_H(Y \otimes X^*, E)$ such that

$$\sigma \circ \beta = \Phi^{-1}_{Y,X^*,\Bbbk}(\alpha)$$

since the map σ is right almost split. In the following, we will verify that the map $\Phi_{Y,X^*,E}(\beta) \in \mathrm{Hom}_H(Y, E \otimes X^{**})$ satisfies $(\sigma \otimes id_{X^{**}}) \circ \Phi_{Y,X^*,E}(\beta) = \alpha$, and therefore the map $\sigma \otimes id_{X^{**}}$ is right almost split. To this end, by replacing X with X^* in commutative diagram (5.7), we obtain that

$$\Phi_{Y,X^*,\Bbbk} \circ \sigma_* = (\sigma \otimes id_{X^{**}})_* \circ \Phi_{Y,X^*,E}.$$

Then

$$\begin{aligned}
\alpha &= \Phi_{Y,X^*,\Bbbk}(\sigma \circ \beta) \\
&= (\Phi_{Y,X^*,\Bbbk} \circ \sigma_*)(\beta) \\
&= ((\sigma \otimes id_{X^{**}})_* \circ \Phi_{Y,X^*,E})(\beta) \\
&= (\sigma \otimes id_{X^{**}}) \circ \Phi_{Y,X^*,E}(\beta).
\end{aligned}$$

We complete the proof. □

An indecomposable module satisfying one of the equivalent conditions in Proposition 5.1.14 is called a *splitting trace module* (cf. [3, 32, 42, 80]).

5.2 Some ring-theoretical properties of Green rings

In this section, we study some ring-theoretical properties of the Green ring $r(H)$ of a finite dimensional Hopf algebra H. We use the bilinear form $(-,-)$ to investigate relations of some one-sided ideals of $r(H)$. We describe nilpotent ideals and central primitive idempotents of $r(H)$.

Recall that $\delta_{[Z]} = [X] - [Y] + [Z]$ if $0 \to X \to Y \to Z \to 0$ is almost split. The existence condition of $\delta_{[Z]}$ can be simplified as follows.

Proposition 5.2.1. *Let $0 \to X \to Y \xrightarrow{\alpha} Z \to 0$ be a short exact sequence of H-modules ending at an indecomposable non-projective module Z. If the map α is right almost split, then $\delta_{[Z]} = [X] - [Y] + [Z]$.*

Proof. Note that the sequence

$$0 \to X \to Y \xrightarrow{\alpha} Z \to 0 \tag{5.8}$$

is exact and the map α is right almost split. It follows from [5, Theorem 2.2, ChI] that the middle term Y has a decomposition $Y = Y_1 \bigoplus Y_2$ such that the restriction of α to the summand Y_1, denoted by $\alpha|_{Y_1}$, is right minimal, and the restriction to the summand Y_2 is zero. We obtain that $\alpha|_{Y_1}$ is both right minimal and right almost split. According to [5, Proposition 1.12, ChV], the sequence

$$0 \to \ker(\alpha|_{Y_1}) \xrightarrow{\iota} Y_1 \xrightarrow{\alpha|_{Y_1}} Z \to 0$$

is almost split, where ι is the inclusion map. Thus, $\delta_{[Z]} = [\ker(\alpha|_{Y_1})] - [Y_1] + [Z]$. Meanwhile, it is easy to see that the sequence

$$0 \to \ker(\alpha|_{Y_1}) \bigoplus Y_2 \xrightarrow{\iota \coprod id_{Y_2}} Y_1 \bigoplus Y_2 \xrightarrow{\alpha} Z \to 0 \tag{5.9}$$

is exact. Applying the short five lemma to the sequences (5.8) and (5.9), we obtain that $X \cong \ker(\alpha|_{Y_1}) \bigoplus Y_2$. It follows that

$$\begin{aligned}\delta_{[Z]} &= [\ker(\alpha|_{Y_1})] - [Y_1] + [Z] \\ &= [\ker(\alpha|_{Y_1}) \bigoplus Y_2] - [Y_1 \bigoplus Y_2] + [Z] \\ &= [X] - [Y] + [Z].\end{aligned}$$

We complete the proof. □

For any indecomposable module X, we have the following description of $\delta_{[X]}$, which is fundamental in the structure of $r(H)$.

Theorem 5.2.2. *Let X be an indecomposable H-module.*

(1) *If $\Bbbk \nmid X^* \otimes X$, then $[X]\delta_{[\Bbbk]} = 0$.*

(2) *If $\Bbbk \nmid X \otimes X^*$, then $\delta_{[\Bbbk]}[X] = 0$.*

(3) *If $\Bbbk \mid X^* \otimes X$, then $[X]\delta_{[\Bbbk]} = \delta_{[X]}$.*

(4) *If $\Bbbk \mid X \otimes X^*$, then $\delta_{[\Bbbk]}[X] = \delta_{[X]}$.*

Proof. If H is semisimple, then $\Bbbk \mid X^* \otimes X$ and $\Bbbk \mid X \otimes X^*$. In this case, Part (3) and Part (4) are trivial because $\delta_{[\Bbbk]} = [\Bbbk]$ and $\delta_{[X]} = [X]$. In the following, we assume that H is not semisimple. If X is projective, then $\Bbbk \nmid X \otimes X^*$ and $\Bbbk \nmid X^* \otimes X$. Note that the sequences (5.4) and (5.5) ending at the projective module X are split. Thus,

$$\delta_{[\Bbbk]}[X] = ([\tau(\Bbbk)] - [E] + [\Bbbk])[X] = [\tau(\Bbbk) \otimes X] - [E \otimes X] + [X] = 0$$

and

$$[X]\delta_{[\Bbbk]} = [X]([\tau(\Bbbk)] - [E] + [\Bbbk]) = [X \otimes \tau(\Bbbk)] - [X \otimes E] + [X] = 0.$$

If X is not projective, we only prove Part (1) and Part (3). The proofs of Part (2) and Part (4) are similar. If $\Bbbk \nmid X^* \otimes X$, by Proposition 5.1.11, the map $id_X \otimes \sigma$ is a split epimorphism. It follows that $[X \otimes E] = [X \otimes \tau(\Bbbk)] + [X]$, and therefore $[X]\delta_{[\Bbbk]} = 0$. If $\Bbbk \mid X^* \otimes X$, then the map $id_X \otimes \sigma$ is right almost split by Proposition 5.1.13. It follows from Proposition 5.2.1 that $\delta_{[X]} = [X \otimes \tau(\Bbbk)] - [X \otimes E] + [X] = [X]\delta_{[\Bbbk]}$, as desired. □

As an application, we shall determine the multiplicity of the trivial module $\Bbbk$ in the decompositions of the tensor product $X \otimes X^*$ and $X^* \otimes X$ respectively. For the case where H is semisimple over the field $\Bbbk$ of characteristic 0, this was done by Zhu [81, Lemma 1], see also [79, Proposition 2.1].

Corollary 5.2.3. *Let X be an indecomposable H-module.*

(1) *If $\Bbbk \mid X^* \otimes X$, then the multiplicity of $\Bbbk$ in $X^* \otimes X$ is 1.*

(2) *If $\Bbbk \mid X \otimes X^*$, then the multiplicity of $\Bbbk$ in $X \otimes X^*$ is 1.*

Proof. (1) By Lemma 1.3.3 (2), the multiplicity of the trivial module $\Bbbk$ in $X^* \otimes X$ is $(\delta^*_{[\Bbbk]}, [X^*][X])$. By Theorem 5.2.2, we have that $(\delta^*_{[\Bbbk]}, [X^*][X]) = (([X]\delta_{[\Bbbk]})^*, [X]) = (\delta^*_{[X]}, [X]) = 1$.

(2) Note that $\Bbbk \mid X \otimes X^*$ if and only if $\Bbbk \mid X^{**} \otimes X^*$. The multiplicity of the trivial module $\Bbbk$ in $X^{**} \otimes X^*$ is $(\delta^*_{[\Bbbk]}, [X^{**}][X^*]) = (([X^*]\delta_{[\Bbbk]})^*, [X^*]) = (\delta^*_{[X^*]}, [X^*]) = 1$, as desired. $\square$

Proposition 5.2.4. *Let $0 \to X \to Y \to Z \to 0$ be an almost split sequence of H-modules.*

(1) $\Bbbk \mid Z \otimes Z^*$ *if and only if* $\Bbbk \mid X \otimes X^*$.

(2) $\Bbbk \mid Z^* \otimes Z$ *if and only if* $\Bbbk \mid X^* \otimes X$.

Proof. We only prove Part (1) because Part (2) follows from Part (1). Applying the duality functor $*$ to the almost split sequence $0 \to X \to Y \to Z \to 0$, we get the almost split sequence $0 \to Z^* \to Y^* \to X^* \to 0$ (cf. [5, P.144]). This implies that $\delta^*_{[Z]} = \delta_{[X^*]}$. If $\Bbbk \mid Z \otimes Z^*$, then $\delta_{[\Bbbk]}[Z] = \delta_{[Z]}$ by Theorem 5.2.2. We claim that $\Bbbk \mid X \otimes X^*$. Otherwise, $\Bbbk \nmid X^{**} \otimes X^*$, then $[X^*]\delta_{[\Bbbk]} = 0$ by Theorem 5.2.2. However,

$$1 = (\delta^*_{[X^*]}, [X^*]) = (\delta^{**}_{[Z]}, [X^*]) = ([X^*], \delta_{[Z]}) = ([X^*]\delta_{[\Bbbk]}, [Z]) = 0,$$

a contradiction. Conversely, if $\Bbbk \mid X \otimes X^*$, then $\Bbbk \mid X^{**} \otimes X^*$. This yields that $[X^*]\delta_{[\Bbbk]} = \delta_{[X^*]}$. We claim that $\Bbbk \mid Z \otimes Z^*$. Otherwise, $\delta_{[\Bbbk]}[Z] = 0$ by Theorem 5.2.2. It follows that

$$1 = (\delta^*_{[Z]}, [Z]) = (\delta_{[X^*]}, [Z]) = ([X^*], \delta_{[\Bbbk]}[Z]) = 0,$$

a contradiction. $\square$

Denote by $\mathcal{J}_+$ and $\mathcal{J}_-$ the free abelian groups as follows:

$$\mathcal{J}_+ = \mathbb{Z}\{\delta_{[M]} \mid [M] \in \operatorname{ind}(H) \text{ and } \Bbbk \mid M \otimes M^*\},$$

$$\mathcal{J}_- = \mathbb{Z}\{\delta_{[M]} \mid [M] \in \operatorname{ind}(H) \text{ and } \Bbbk \mid M^* \otimes M\}.$$

It follows from Theorem 5.2.2 that $\mathcal{J}_+$ and $\mathcal{J}_-$ are right and left ideals of $r(H)$ generated respectively by $\delta_{[\Bbbk]}$. Moreover, by Proposition 5.2.4, we have $\mathcal{J}_+^* = \mathcal{J}_-$ and $\mathcal{J}_-^* = \mathcal{J}_+$.

Let $\mathcal{P}_+$ and $\mathcal{P}_-$ denote the free abelian groups respectively as follows:

$$\mathcal{P}_+ = \mathbb{Z}\{[M] \in \operatorname{ind}(H) \mid \Bbbk \nmid M \otimes M^*\},$$

$$\mathcal{P}_- = \mathbb{Z}\{[M] \in \operatorname{ind}(H) \mid \Bbbk \nmid M^* \otimes M\}.$$

It follows from Corollary 5.1.9 that $\mathcal{P}_+$ is a right ideal of $r(H)$ and $\mathcal{P}_-$ is a left ideal of $r(H)$. Obviously, $\mathcal{P}_+^* = \mathcal{P}_-$ and $\mathcal{P}_-^* = \mathcal{P}_+$.

Let $\mathcal{P}_+^\perp$ and $\mathcal{P}_-^\perp$ be the subgroups of $r(H)$ orthogonal to $\mathcal{P}_+$ and $\mathcal{P}_-$ with respect to the form $(-,-)$ respectively. Namely,

$$\mathcal{P}_+^\perp = \{x \in r(H) \mid (y, x) = 0 \text{ for } y \in \mathcal{P}_+\},$$

$$\mathcal{P}_-^\perp = \{x \in r(H) \mid (x, y) = 0 \text{ for } y \in \mathcal{P}_-\}.$$

Then $\mathcal{P}_+^\perp$ is a left ideal of $r(H)$ since $\mathcal{P}_+$ is a right ideal of $r(H)$. Similarly, $\mathcal{P}_-^\perp$ is a right ideal of $r(H)$.

The relations between these one-sided ideals of $r(H)$ are described as follows.

Proposition 5.2.5. *Suppose H is of finite representation type.*

(1) $\mathcal{J}_+ = \mathcal{P}_-^\perp = (\mathcal{P}_+^\perp)^*$.

(2) $\mathcal{J}_- = \mathcal{P}_+^\perp = (\mathcal{P}_-^\perp)^*$.

Proof. It is sufficient to show Part (1) since the proof of Part (2) is similar. For any two indecomposable H-modules X and Y satisfying $\Bbbk \mid X \otimes X^*$ and $\Bbbk \nmid Y^* \otimes Y$, by Theorem 5.2.2, we have

$$(\delta_{[X]}, [Y]) = (\delta_{[\Bbbk]}[X], [Y]) = ([Y^{**}]\delta_{[\Bbbk]}, [X]) = (0, [X]) = 0.$$

This implies that $\mathcal{J}_+ \subseteq \mathcal{P}_-^\perp$. For any $x \in \mathcal{P}_-^\perp$, then

$$\begin{aligned} x &= \sum_{[M]\in\operatorname{ind}(H)} (x, [M])\delta^*_{[M]} \quad \text{by (1.5)} \\ &= \sum_{\Bbbk \mid M^* \otimes M} (x, [M])\delta^*_{[M]} \quad (\text{as } x \in \mathcal{P}_-^\perp). \end{aligned}$$

If $\Bbbk \mid M^* \otimes M$ and $\Bbbk \nmid Y \otimes Y^*$ for indecomposable modules M and Y, by Theorem

5.2.2, we have that

$$([Y],\delta_{[M]})=(\delta^{**}_{[M]},[Y])=(\delta_{[M^{**}]},[Y])=([M^{**}],\delta_{[\Bbbk]}[Y])=([M^{**}],0)=0.$$

This implies that $\delta_{[M]}\in\mathcal{P}_+^\perp$, and hence $x=\sum_{\Bbbk|M^*\otimes M}(x,[M])\delta^*_{[M]}\in(\mathcal{P}_+^\perp)^*$. We obtain $\mathcal{P}_-^\perp\subseteq(\mathcal{P}_+^\perp)^*$. For any $x\in\mathcal{P}_+^\perp$, we have

$$\begin{aligned}x&=\sum_{[M]\in\mathrm{ind}(H)}(x,[M])\delta^*_{[M]}\quad\text{by (1.5)}\\&=\sum_{[M]\in\mathrm{ind}(H)}([M^{**}],x)\delta^*_{[M]}\\&=\sum_{\Bbbk|M\otimes M^*}([M^{**}],x)\delta^*_{[M]}\quad(\text{as } x\in\mathcal{P}_+^\perp).\end{aligned}$$

Thus,

$$x^*=\sum_{\Bbbk|M\otimes M^*}([M^{**}],x)\delta^{**}_{[M]}\in\mathcal{J}_+^{**}=\mathcal{J}_+,$$

implying that $(\mathcal{P}_+^\perp)^*\subseteq\mathcal{J}_+$. We obtain the desired result. □

For a finite dimensional semisimple Hopf algebra over the field $\Bbbk$ of characteristic 0, its Green ring (i.e., Grothendieck ring) is semisimple (cf. [81, Lemma 2] or [79, Proposition 2.2]). In the following, we give a description of nilpotent ideals and central primitive idempotents of $r(H)$ for any finite dimensional Hopf algebra H over the field $\Bbbk$ of characteristic 0. We first need the following useful lemma.

Lemma 5.2.6. *For any $x\in r(H)$, we have the following:*

(1) *If $xx^*=0$, then $x\in\mathcal{P}_+$.*

(2) *If $x^*x=0$, then $x\in\mathcal{P}_-$.*

Proof. It suffices to prove Part (1) as the same argument works for Part (2). We write

$$x=\sum_{\Bbbk|M\otimes M^*}\lambda_{[M]}[M]+\sum_{\Bbbk\nmid M\otimes M^*}\lambda_{[M]}[M],$$

where $\lambda_{[M]}\in\mathbb{Z}$ and the sum $\sum_{\Bbbk|M\otimes M^*}$ (resp. $\sum_{\Bbbk\nmid M\otimes M^*}$) runs over all elements $[M]\in\mathrm{ind}(H)$ such that $\Bbbk\mid M\otimes M^*$ (resp. $\Bbbk\nmid M\otimes M^*$). By Theorem 5.1.7 (1) and Corollary 5.2.3, the coefficient of the identity $[\Bbbk]$ in the linear expression of xx^*

(with the basis ind(H)) is $\sum_{\Bbbk|M\otimes M^*}\lambda_{[M]}^2$. Thus, if $xx^*=0$, then $\lambda_{[M]}=0$, for any indecomposable module M satisfying $\Bbbk \mid M\otimes M^*$. Hence $x=\sum_{\Bbbk\nmid M\otimes M^*}\lambda_{[M]}[M]\in \mathcal{P}_+$. $\square$

Recall that a two-sided ideal I of a ring R is call *nilpotent* if I^m=0 for some natural number m. This means that $a_1\cdots a_m=0$ for any $a_1\cdots,a_m\in I$.

Proposition 5.2.7. *If I is a nilpotent ideal of $r(H)$, then $I\subseteq \mathcal{P}_+\cap\mathcal{P}_-$.*

Proof. Let I be a nilpotent ideal of $r(H)$ such that $I^m=0$. For any $x\in I$, let $x_0=x$ and $x_{i+1}=x_ix_i^*$ for $i\geq 0$. For instance, $x_1=xx^*$, $x_2=xx^*x^{**}x^*$, $x_3=xx^*x^{**}x^*x^{**}x^{***}x^{**}x^*$, etc.. Note that the order of the duality operator $*$ is finite and $I^m=0$. There exists some k such that $x_k=0$. We write

$$x=\sum_{\Bbbk|M\otimes M^*}\lambda_{[M]}[M]+\sum_{\Bbbk\nmid M\otimes M^*}\lambda_{[M]}[M]$$

and

$$x_1=xx^*=\sum_{\Bbbk|M\otimes M^*}\mu_{[M]}[M]+\sum_{\Bbbk\nmid M\otimes M^*}\mu_{[M]}[M],$$

for $\lambda_{[M]}$ and $\mu_{[M]}$ in $\mathbb{Z}$. As shown in the proof of Lemma 5.2.6, the coefficient of $[\Bbbk]$ in $x_1=xx^*$ is $\mu_{[\Bbbk]}=\sum_{\Bbbk|M\otimes M^*}\lambda_{[M]}^2$ and the coefficient of $[\Bbbk]$ in $x_2=x_1x_1^*$ is $\sum_{\Bbbk|M\otimes M^*}\mu_{[M]}^2$. If $\mu_{[\Bbbk]}\neq 0$, then $\sum_{\Bbbk|M\otimes M^*}\mu_{[M]}^2\neq 0$, and hence $x_2\neq 0$. By repeating this process, we obtain that $x_i\neq 0$ for any $i\geq 0$ if $\mu_{[\Bbbk]}\neq 0$. This contradicts to the fact that $x_k=0$. Thus, $\mu_{[\Bbbk]}=0$ and $x=\sum_{\Bbbk\nmid M\otimes M^*}\lambda_{[M]}[M]\in\mathcal{P}_+$. Similarly, if $x\in I$, then $x\in\mathcal{P}_-$. We obtain that $I\subseteq\mathcal{P}_+\cap\mathcal{P}_-$. $\square$

If H is of finite representation type, then the Green ring $r(H)$ is Frobenius (and hence artinian). The Jacobson radical $J(r(H))$ of $r(H)$ is the largest nilpotent ideal of $r(H)$. According to Proposition 5.2.7, we have $J(r(H))\subseteq\mathcal{P}_+\cap\mathcal{P}_-$.

Proposition 5.2.8. *Let E be a central primitive idempotent of $r(H)$. Then $E\in\mathcal{P}_+\cap\mathcal{P}_-$ or $1-E\in\mathcal{P}_+\cap\mathcal{P}_-$.*

Proof. If E is a central primitive idempotent of $r(H)$, so is E^* since the duality operator $*$ is an anti-automorphism of $r(H)$. It follows that $E=E^*$ or $EE^*=E^*E=0$. If $EE^*=E^*E=0$, by Lemma 5.2.6, $E\in\mathcal{P}_+$ as well as $E\in\mathcal{P}_-$. If

$E = E^*$, denote by

$$E = \sum_{\Bbbk \mid M \otimes M^*} \lambda_{[M]}[M] + \sum_{\Bbbk \nmid M \otimes M^*} \lambda_{[M]}[M].$$

Comparing the coefficients of $[\Bbbk]$ in both sides of the equation $EE^* = E$, we obtain that $\sum_{\Bbbk \mid M \otimes M^*} \lambda_{[M]}^2 = \lambda_{[\Bbbk]}$. This implies that $\lambda_{[\Bbbk]} = 0$ or 1, and $\lambda_{[M]} = 0$ for any $[M]$ satisfying $\Bbbk \mid M \otimes M^*$ and $[M] \neq [\Bbbk]$. Hence E is reduced to be

$$E = \lambda_{[\Bbbk]}[\Bbbk] + \sum_{\Bbbk \nmid M \otimes M^*} \lambda_{[M]}[M].$$

Meanwhile, we denote by

$$E = \sum_{\Bbbk \mid M^* \otimes M} \mu_{[M]}[M] + \sum_{\Bbbk \nmid M^* \otimes M} \mu_{[M]}[M].$$

Similarly, by comparing the coefficients of $[\Bbbk]$ in both sides of the equation $E^*E = E$, we obtain that

$$E = \mu_{[\Bbbk]}[\Bbbk] + \sum_{\Bbbk \nmid M^* \otimes M} \mu_{[M]}[M].$$

Thus, $\mu_{[\Bbbk]} = \lambda_{[\Bbbk]}$ which is equal to 0 or 1. Therefore, $E \in \mathcal{P}_+ \cap \mathcal{P}_-$ if $\mu_{[\Bbbk]} = \lambda_{[\Bbbk]} = 0$, and $1 - E \in \mathcal{P}_+ \cap \mathcal{P}_-$ if $\mu_{[\Bbbk]} = \lambda_{[\Bbbk]} = 1$, as desired. $\square$

5.3 Bilinear forms on stable Green rings

In this section, we study the stable Green ring (i.e., Green ring of a stable category) of a finite dimensional Hopf algebra H. We show that the stable Green ring admits an associative non-degenerate $\mathbb{Z}$-bilinear form induced by the form $(-,-)$ on $r(H)$.

Let H be a finite dimensional Hopf algebra over the field $\Bbbk$. Denote by H-mod the category of finite dimensional left H-module. Recall that the stable category H-$\underline{\text{mod}}$ has the same objects as H-mod does, and the space of morphisms from X to Y in H-$\underline{\text{mod}}$ is the quotient space

$$\underline{\mathrm{Hom}}_H(X,Y) := \mathrm{Hom}_H(X,Y)/\mathcal{P}(X,Y)$$

where $\mathcal{P}(X,Y)$ is the subspace of $\mathrm{Hom}_H(X,Y)$ consisting of morphisms factoring through projective modules.

The stable category H-$\underline{\text{mod}}$ is a triangulated [43] monoidal category with the monoidal structure stemming from that of H-mod. The Green ring $r_{st}(H)$ of the stable category H-$\underline{\text{mod}}$ is called the stable Green ring of H. The following proposition is similar to Theorem 4.1.1.

Proposition 5.3.1. *The stable Green ring $r_{st}(H)$ of H is isomorphic to the quotient ring $r(H)/\mathcal{P}$, where $\mathcal{P}$ is the ideal of $r(H)$ generated by the isomorphism classes of indecomposable projective H-modules.*

Note that the form $(-,-)$ on $r(H)$ given by $([X],[Y]) = \dim \mathrm{Hom}_H(X,Y^*)$ is associative and non-degenerate. In the following, we show that this form induces a $\mathbb{Z}$-bilinear form on the stable Green ring $r_{st}(H)$. We first need the following lemmas.

Lemma 5.3.2. *For H-modules X, Y and Z, the canonical isomorphisms $\Phi_{X,Y,Z}$ and $\Psi_{X,Y,Z}$ given in Lemma 5.1.3 induce respectively the following $\Bbbk$-linear isomorphisms:*

(1) $\underline{\Phi}_{X,Y,Z} : \underline{Hom}_H(X \otimes Y, Z) \to \underline{Hom}_H(X, Z \otimes Y^*)$.

(2) $\underline{\Psi}_{X,Y,Z} : \underline{Hom}_H(X, Y \otimes Z) \to \underline{Hom}_H(Y^* \otimes X, Z)$.

Proof. We only prove Part (1) and the proof of Part (2) is similar. If $\alpha \in \mathrm{Hom}_H(X \otimes Y, Z)$ factors through a projective module P, then $\Phi_{X,Y,Z}(\alpha)$ factors through the projective module $P \otimes Y^*$ by Lemma 5.1.3 (1). Thus, $\Phi_{X,Y,Z}(\mathcal{P}(X \otimes Y, Z)) \subseteq \mathcal{P}(X, Z \otimes$

Y^*). Conversely, for any $\beta \in \mathcal{P}(X, Z \otimes Y^*)$ which factors through a projective module P, by Lemma 5.1.3 (1), the map $\Phi^{-1}_{X,Y,Z}(\beta)$ factors through the projective module $P \otimes Y$. We obtain that

$$\Phi_{X,Y,Z}(\mathcal{P}(X \otimes Y, Z)) = \mathcal{P}(X, Z \otimes Y^*).$$

This induces a $\Bbbk$-linear isomorphism $\underline{\Phi}_{X,Y,Z}$ from the quotient space $\underline{\mathrm{Hom}}_H(X \otimes Y, Z)$ to $\underline{\mathrm{Hom}}_H(X, Z \otimes Y^*)$, as desired. □

Recall that the form $\langle -, - \rangle$ on $r(H)$ is defined by $\langle [X], [Y] \rangle = \dim \mathrm{Hom}_H(X, Y)$ for any two H-modules X and Y. Similarly, we define a $\mathbb{Z}$-linear $\langle -, - \rangle_{st}$ on $r_{st}(H)$ by

$$\langle [X], [Y] \rangle_{st} := \dim \underline{\mathrm{Hom}}_H(X, Y)$$

for objects X and Y in H-$\underline{\mathrm{mod}}$.

Lemma 5.3.3. *Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be an almost split sequence of H-modules ending at the indecomposable non-projective module Z. For any indecomposable non-projective H-module M, we have*

$$\langle [M], \delta_{[Z]} \rangle_{st} = \begin{cases} 0, & M \ncong Z \text{ and } \Omega^{-1}M \ncong Z; \\ 1, & M \cong Z \text{ and } \Omega^{-1}M \ncong Z; \\ 2, & M \cong Z \text{ and } \Omega^{-1}M \cong Z. \end{cases}$$

Proof. Note that the functor $\underline{\mathrm{Hom}}_H(M, -)$ is naturally isomorphic to the functor $\mathrm{Ext}^1_H(\Omega^{-1}M, -)$. Applying the functor $\mathrm{Hom}_H(\Omega^{-1}M, -)$ to the given almost split sequence, we obtain the following long exact sequence:

$$0 \to \mathrm{Hom}_H(\Omega^{-1}M, X) \to \mathrm{Hom}_H(\Omega^{-1}M, Y) \to \mathrm{Hom}_H(\Omega^{-1}M, Z)$$

$$\to \underline{\mathrm{Hom}}_H(M, X) \xrightarrow{\overline{\alpha}} \underline{\mathrm{Hom}}_H(M, Y) \xrightarrow{\overline{\beta}} \underline{\mathrm{Hom}}_H(M, Z).$$

Denote by $\mathrm{rad}_H(M, Z)$ the space of morphisms f from M to Z such that $id_M - g \circ f$ is isomorphic for any $g \in \mathrm{Hom}_H(Z, M)$. Then the image of the map $\overline{\beta}$ is $\underline{\mathrm{rad}}_H(M, Z)$ since the map $\beta : Y \to Z$ is right almost split. Thus, we obtain the sequence:

$$0 \to \mathrm{Hom}_H(\Omega^{-1}M, X) \to \mathrm{Hom}_H(\Omega^{-1}M, Y) \to \mathrm{Hom}_H(\Omega^{-1}M, Z)$$

$$\to \underline{\mathrm{Hom}}_H(M, X) \xrightarrow{\overline{\alpha}} \underline{\mathrm{Hom}}_H(M, Y) \xrightarrow{\overline{\beta}} \underline{\mathrm{rad}}_H(M, Z) \to 0.$$

This implies that

$$\langle [M], \delta_{[Z]} \rangle_{st} = \langle [\Omega^{-1}M], \delta_{[Z]} \rangle + \dim \underline{\mathrm{Hom}}_H(M, Z)/\underline{\mathrm{rad}}_H(M, Z).$$

Recall that $\dim \underline{\mathrm{Hom}}_H(M, Z)/\underline{\mathrm{rad}}_H(M, Z) = 1$ if $M \cong Z$ and 0 otherwise; and $\langle [\Omega^{-1}M], \delta_{[Z]} \rangle = 1$ if $\Omega^{-1}M \cong Z$ and 0 otherwise. So the statement follows. $\square$

Remark 5.3.4. The proof of Lemma 5.3.3 is motivated by the proof of [33, Lemma 1.4]. If $M \ncong Z$ and $\Omega^{-1}M \ncong Z$, then $\langle [M], \delta_{[Z]} \rangle_{st} = 0$. This result was proved in [35, Lemma 3.2].

Define a $\mathbb{Z}$-linear form on $r_{st}(H)$ by

$$([X], [Y])_{st} := \dim \underline{\mathrm{Hom}}_H(X, Y^*).$$

The form $(-, -)_{st}$ has the following properties.

Proposition 5.3.5. *For objects X, Y and Z in H-$\underline{\mathrm{mod}}$, the following hold:*

(1) $([X][Y], [Z])_{st} = ([X], [Y][Z])_{st}$.

(2) $([X], [Y])_{st} = ([Y^{**}], [X])_{st}$. *If S^2 is inner, then* $([X], [Y])_{st} = ([Y], [X])_{st}$.

(3) *The form $(-, -)_{st}$ is non-degenerate.*

Proof. Part (1) and Part (2) follow from Lemma 5.3.2, Part (3) follows from Lemma 5.3.3. $\square$

Corollary 5.3.6. *If H is of finite representation type, then the stable Green ring $r_{st}(H)$ is a Frobenius ring. Moreover, $r_{st}(H)$ is symmetric if S^2 is inner.*

5.4 Green rings of spherical Hopf algebras

In this section, we will devote ourselves to the study of the Green ring $r(H)$ of a spherical Hopf algebra H. In this case, the free abelian group generated by all indecomposable modules of quantum dimension zero forms an ideal $\mathcal{P}$ of $r(H)$. The quotient ring of $r(H)$ modulo $\mathcal{P}$ can be regarded as the Green ring of a factor category of H-mod. If H is of finite representation type, the complexified quotient ring is a group-like algebra and a bi-Frobenius algebra.

Let H be a finite dimensional non-semisimple *spherical Hopf algebra* over the field $\Bbbk := \mathbb{C}$. That is, there is a $\omega \in G(H)$ such that

$$S^2(h) = \omega h \omega^{-1} \text{ for } h \in H, \tag{5.10}$$

$$\mathrm{Tr}_X^L(\theta_X \vartheta) = \mathrm{Tr}_X^R(\vartheta \theta_X^{-1}) \text{ for } \vartheta \in \mathrm{End}_H(X). \tag{5.11}$$

Here X is a finite dimensional H-module and the isomorphism $\theta_X : X \to X^{**}$ given by $\theta_X(x)(f) = f(\omega x)$ is a *pivotal structure* of H-mod. That is, this map is natural in X and satisfies $\theta_{X\otimes Y} \cong \theta_X \otimes \theta_Y$ for any two H-modules X and Y.

The condition (5.11) is only required for any simple H-module [1, Proposition 2.1]. Applying $\vartheta = id_X$ to (5.11), we obtain that $\mathbf{d}(X) := \mathrm{Tr}_X^L(\theta_X) = \mathrm{Tr}_X^R(\theta_X^{-1})$, which is the quantum dimension of H-module X. Observe that

$$\mathbf{d}(X) = \mathbf{d}(X^*) \text{ and } \mathbf{d}(X \otimes Y) = \mathbf{d}(X)\mathbf{d}(Y).$$

The quantum dimensions of H-modules defines a ring homomorphism from $r(H)$ to $\Bbbk$ preserving the dual.

By Theorem 5.1.7, $\Bbbk \nmid X \otimes X^*$ if and only if $\Bbbk \nmid X^* \otimes X$ if and only if $\mathbf{d}(X) = \mathbf{d}(X^*) = 0$. Then $\mathcal{P} := \mathcal{P}_+ = \mathcal{P}_-$, which is the two-sided ideal of $r(H)$ generated by indecomposable modules of quantum dimension zero.

Let $\mathbf{B} = \{[X_i] \mid i \in \mathbb{I}\}$ denote the set consisting of all $[X] \in \mathrm{ind}(H)$ with $\mathbf{d}(X) \neq 0$. Then $0 \in \mathbb{I}$ since $[X_0] := [\Bbbk] \in \mathbf{B}$. Note that $\mathbf{d}(X) = \mathbf{d}(X^*)$. Then $[X_i] \in \mathbf{B}$ if and only if $[X_i^*] \in \mathbf{B}$. Hence the duality functor $*$ of H-mod induces an involution on the index set $\mathbb{I}$ defined by $[X_{i^*}] := [X_i^*]$, for any $i \in \mathbb{I}$. Moreover, $\mathcal{P}^* = \mathcal{P}$. Thus, the dual $*$ induces an involution over $r(H)/\mathcal{P}$, namely, $(\overline{[X_i]})^* = \overline{[X_{i^*}]}$, for any $i \in \mathbb{I}$.

Let $\mathbb{Z}\mathbf{B}$ be the free abelian group generated by the set $\mathbf{B}$ and T the $\mathbb{Z}$-linear map from $\mathbb{Z}\mathbf{B}$ to $\mathbb{Z}$ given by $T(x) = (x, \delta^*_{[\Bbbk]})$. Namely, $T(x)$ stands for the coefficient of $[\Bbbk]$ in the linear expression of x. The map T is in fact a trace map as shown in the following.

Lemma 5.4.1. *For any $x, y \in \mathbb{Z}\mathbf{B}$, the map T satisfies the following:*

(1) $T(x) = T(x^*)$.

(2) $T(xy) = T(yx)$.

(3) $T(xy) = \sum_{i\in\mathbb{I}} T(x[X_{i^*}])T([X_i]y)$.

Proof. Part (1) is obvious. Part (2) follows from the fact that $\delta_{[\Bbbk]}$ (and hence $\delta^*_{[\Bbbk]}$) is a central element of $r(H)$ (see Theorem 5.2.2) if H is spherical. To verify Part (3), for any $x \in \mathbb{Z}\mathbf{B}$, we obtain that the coefficient of $[X_i]$ in x is $(x, \delta^*_{[X_i]}) = (x[X_{i^*}], \delta^*_{[\Bbbk]}) = T(x[X_{i^*}])$. This implies that

$$x = \sum_{i\in\mathbb{I}} T(x[X_{i^*}])[X_i]. \tag{5.12}$$

Hence $\sum_{i\in\mathbb{I}} T(x[X_{i^*}])T([X_i]y) = T(xy)$, as desired. $\square$

It was mentioned in [16] that if a monoid generated by a finite set admits a fusion rule, then it gives rise to a fusion ring structure on the monoid. We follow a similar approach and give $\mathbb{Z}\mathbf{B}$ a ring structure as follows.

Theorem 5.4.2. *The free abelian group $\mathbb{Z}\mathbf{B}$ admits a ring structure as follows:*

(1) *The multiplication law is given by $x\cdot y = \sum_{i\in\mathbb{I}} T(xy[X_{i^*}])[X_i]$ for any $x, y \in \mathbb{Z}\mathbf{B}$.*

(2) *The $\mathbb{Z}$-bilinear form $[-,-]$ on $\mathbb{Z}\mathbf{B}$ given by $[x, y] = T(xy)$ is associative symmetric non-degenerate and $*$-invariant.*

(3) *The map $\mathbf{d} : \mathbb{Z}\mathbf{B} \to \Bbbk$ given by the quantum dimension is a ring homomorphism.*

(4) *$\mathbb{Z}\mathbf{B}$ is isomorphic to the quotient ring $r(H)/\mathcal{P}$.*

Proof. (1) The sum $\sum_{i\in\mathbb{I}} T(xy[X_{i^*}])[X_i]$ is finite, since

$$\sum_{i\in\mathbb{I}} T(xy[X_{i^*}])[X_i] = \sum_{i\in\mathbb{I}} (xy[X_{i^*}], \delta^*_{[\Bbbk]})[X_i] = \sum_{i\in\mathbb{I}} (xy, \delta^*_{[X_i]})[X_i],$$

which is a part of the linear expression of xy by (1.5). To verify the associativity of the law, for any $x, y, z \in \mathbf{B}$, one has

$$\begin{aligned}(x \cdot y) \cdot z &= \sum_{j\in\mathbb{I}} T(xy[X_{j^*}])[X_j] \cdot z \\ &= \sum_{i,j\in\mathbb{I}} T(xy[X_{j^*}])T([X_j]z[X_{i^*}])[X_i] \\ &= \sum_{i\in\mathbb{I}} T(xyz[X_{i^*}])[X_i] \quad \text{by Lemma 5.4.1 (3).}\end{aligned}$$

On the other hand,

$$\begin{aligned}x \cdot (y \cdot z) &= x \cdot \sum_{j\in\mathbb{I}} T(yz[X_{j^*}])[X_j] \\ &= \sum_{i,j\in\mathbb{I}} T(yz[X_{j^*}])T(x[X_j][X_{i^*}])[X_i] \\ &= \sum_{i,j\in\mathbb{I}} T(yz[X_{j^*}])T([X_j][X_{i^*}]x)[X_i] \quad \text{by Lemma 5.4.1 (2)} \\ &= \sum_{i\in\mathbb{I}} T(yz[X_{i^*}]x)[X_i] \quad \text{by Lemma 5.4.1 (3)} \\ &= \sum_{i\in\mathbb{I}} T(xyz[X_{i^*}])[X_i].\end{aligned}$$

Thus, the multiplication is associative. One can repeat this process and obtain

$$x_1 \cdot x_2 \cdots x_m = \sum_{i\in\mathbb{I}} T(x_1 x_2 \cdots x_m[X_{i^*}])[X_i].$$

The identity of $\mathbb{Z}\mathbf{B}$ is $[\Bbbk]$, which follows from (5.12). The dual operator $*$ is an anti-automorphism of $\mathbb{Z}\mathbf{B}$. Indeed, one has

$$\begin{aligned}(x \cdot y)^* &= \sum_{i\in\mathbb{I}} T(xy[X_{i^*}])[X_{i^*}] \\ &= \sum_{i\in\mathbb{I}} T(xy[X_i])[X_i] \\ &= \sum_{i\in\mathbb{I}} T([X_{i^*}]y^*x^*)[X_i] \quad \text{by Lemma 5.4.1 (1)} \\ &= \sum_{i\in\mathbb{I}} T(y^*x^*[X_{i^*}])[X_i] = y^* \cdot x^*.\end{aligned}$$

(2) Note that

$$T(x \cdot y) = \sum_{i \in \mathbb{I}} T(xy[X_{i^*}])T([X_i]) = T(xy).$$

This yields the associativity of the form. The symmetry stems from $T(xy) = T(yx)$ and the non-degeneracy of the form follows from the orthogonality: $T([X_i][X_{j^*}]) = \delta_{i,j}$ for $i, j \in \mathbb{I}$. Moreover, the form is $*$-invariant: $[x^*, y^*] = T(x^*y^*) = T(yx) = T(xy) = [x, y]$, for any $x, y \in \mathbb{Z}\mathbf{B}$.

(3) Note that for any finite dimensional indecomposable H-module X, $\mathbf{d}([X]) = 0$ if $[X]$ is not in $\mathbf{B}$. Now for any $x, y \in \mathbb{Z}\mathbf{B}$, one has

$$\begin{aligned}
\mathbf{d}(x \cdot y) &= \sum_{i \in \mathbb{I}} T(xy[X_{i^*}])\mathbf{d}([X_i]) \\
&= \sum_{i \in \mathbb{I}} (xy[X_{i^*}], \delta^*_{[\Bbbk]})\mathbf{d}([X_i]) \\
&= \sum_{i \in \mathbb{I}} (xy, \delta^*_{[X_i]})\mathbf{d}([X_i]) \\
&= \sum_{[X] \in \mathrm{ind}(H)} (xy, \delta^*_{[X]})\mathbf{d}([X]) \\
&= \mathbf{d}\Big(\sum_{[X] \in \mathrm{ind}(H)} (xy, \delta^*_{[X]})[X]\Big) \\
&= \mathbf{d}(xy) = \mathbf{d}(x)\mathbf{d}(y).
\end{aligned}$$

(4) The map φ from $\mathbb{Z}\mathbf{B}$ to $r(H)/\mathcal{P}$ given by $\varphi(x) = \overline{x}$ is obviously $\mathbb{Z}$-linear and bijective. This map also preserves the ring structure:

$$\begin{aligned}
\varphi(x \cdot y) &= \sum_{i \in \mathbb{I}} T(xy[X_{i^*}])\overline{[X_i]} \\
&= \sum_{i \in \mathbb{I}} (xy[X_{i^*}], \delta^*_{[\Bbbk]})\overline{[X_i]} \\
&= \sum_{i \in \mathbb{I}} (xy, \delta^*_{[X_i]})\overline{[X_i]} \\
&= \sum_{[X] \in \mathrm{ind}(H)} (xy, \delta^*_{[X]})\overline{[X]} \\
&= \overline{xy} = \varphi(x)\varphi(y),
\end{aligned}$$

for any $x, y \in \mathbb{Z}\mathbf{B}$, as desired. $\square$

In the following, we give an interpretation of $\mathbb{Z}\mathbf{B}$ from a categorical point of view. We show that $\mathbb{Z}\mathbf{B}$ is the Green ring of a factor category of H-mod.

Note that H-mod is a spherical category. For any two H-modules X and Y, there is a bilinear pairing given by

$$\Theta : \mathrm{Hom}_H(X, Y) \times \mathrm{Hom}_H(Y, X) \to \Bbbk, \ \Theta(f, g) = \mathrm{Tr}_X^L(\theta_X \circ g \circ f).$$

A morphism f from X to Y is called *negligible* if $\Theta(f, g) = 0$ for any morphism g from Y to X. Let $\mathcal{J}(X, Y)$ be the set consisting of all negligible morphisms from X to Y. It follows from Theorem 5.1.7 that

$$\mathcal{J}(X, Y) = \mathrm{Hom}_H(X, Y) \text{ if } \mathbf{d}(X) = 0 \text{ or } \mathbf{d}(Y) = 0.$$

The negligible morphisms form a monoidal ideal, i.e. composing or tensoring a negligible morphism with any morphism yields a negligible morphism [60, P.118]. This leads to a factor category H-$\underline{\mathrm{mod}}$, where the objects are those of H-mod while the morphism spaces are the quotient:

$$\underline{\mathrm{Hom}}_H(X, Y) := \mathrm{Hom}_H(X, Y)/\mathcal{J}(X, Y).$$

The factor category H-$\underline{\mathrm{mod}}$ is an additive semisimple $\Bbbk$-linear spherical category [15] with the monoidal structure derived from that of H-mod.

Theorem 5.4.3. *The Green ring of the factor category H-$\underline{\mathrm{mod}}$ is isomorphic to the quotient ring $r(H)/\mathcal{P}$.*

Proof. The canonical functor F from H-mod to H-$\underline{\mathrm{mod}}$ given by $F(M) = M$ for any H-module M, and $F(g) = \overline{g}$ for $g \in \mathrm{Hom}_H(M, N)$ with the canonical image $\overline{g} \in \underline{\mathrm{Hom}}_H(M, N)$ is a full and dense tensor functor. Such a functor defines a ring epimorphism ψ from $r(H)$ to the Green ring of H-$\underline{\mathrm{mod}}$ such that $\psi(\mathcal{P}) = 0$. Hence there is a unique ring epimorphism $\overline{\psi}$ from $r(H)/\mathcal{P}$ to the Green ring of H-$\underline{\mathrm{mod}}$ such that $\overline{\psi}(\overline{x}) = \psi(x)$, for any $x \in r(H)$ with the canonical image $\overline{x} \in r(H)/\mathcal{P}$. The rank of the Green ring of H-$\underline{\mathrm{mod}}$ is the same as that of $r(H)/\mathcal{P}$ since there is one to one correspondence between the set of simple objects in H-$\underline{\mathrm{mod}}$ and the set of isomorphism classes of indecomposable finite dimensional H-modules with non-zero quantum dimension [1, Theorem 2.7]. We conclude that the Green ring of H-$\underline{\mathrm{mod}}$ is isomorphic to $r(H)/\mathcal{P}$. $\square$

Remark 5.4.4. For any finite dimensional pointed Hopf algebra $H_{\mathcal{D}}$ of rank one associated to the group datum $\mathcal{D} = (G, \chi, g, \mu)$, by (5.11), we obtain that $H_{\mathcal{D}}$ is spherical if and only if the order of $\chi(g)$ is $n = 2$. In this case, all indecomposable $H_{\mathcal{D}}$-modules are simple $\Bbbk\widetilde{G}$-modules (see Section 4.1 for this notion) together with some projective modules. Thus, if V is projective, then $\mathbf{d}([V]) = 0$. If V is simple, then $\mathbf{d}([V]) = \chi_V(g)\dim(V)$, where χ_V is the character of V. Obviously, the factor category $H_{\mathcal{D}}$-$\underline{\text{mod}}$ is equivalent to $\Bbbk\widetilde{G}$-mod and the Green ring $\mathbb{Z}\mathbf{B}$ is isomorphic to the Grothendieck ring $G_0(\Bbbk\widetilde{G})$.

Recall that the form $(-,-)$ on $r(H)$ is given by $([X],[Y]) = \dim \mathrm{Hom}_H(X, Y^*)$. In the following, we show that the form $[-,-]$ on $\mathbb{Z}\mathbf{B}$ defined by the map T is essentially induced by the form $(-,-)$ on $r(H)$. That is, $[[X],[Y]] = \dim \underline{\mathrm{Hom}}_H(X, Y^*)$.

Lemma 5.4.5. *For H-modules X, Y and Z, the canonical isomorphisms given in Lemma 5.1.3 induce respectively the following canonical isomorphisms:*

(1) $\underline{\Phi}_{X,Y,Z} : \underline{Hom}_H(X \otimes Y, Z) \to \underline{Hom}_H(X, Z \otimes Y^*)$.

(2) $\underline{\Psi}_{X,Y,Z} : \underline{Hom}_H(X, Y \otimes Z) \to \underline{Hom}_H(Y^* \otimes X, Z)$.

Proof. We only prove Part (1) and the same argument works for Part (2). Note that $\mathrm{Tr}^L_X(\theta_X\vartheta) = \mathrm{tr}_X(\omega\vartheta)$, where $\mathrm{tr}_X(\omega\vartheta)$ is the usual trace of the map $\omega\vartheta$ of X.

Claim 1. $\Phi_{X,Y,Z}(\mathcal{J}(X \otimes Y, Z)) \subseteq \mathcal{J}(X, Z \otimes Y^*)$.

Indeed, If $f \in \mathcal{J}(X \otimes Y, Z)$, then for any $g \in \mathrm{Hom}_H(Z \otimes Y^*, X)$, the morphism $(id_X \otimes \theta_Y^{-1}) \circ \Phi_{Z,Y^*,X}(g)$ is in $\mathrm{Hom}_H(Z, X \otimes Y)$ and hence

$$\mathrm{tr}_Z(f \circ (id_X \otimes \theta_Y^{-1}) \circ \Phi_{Z,Y^*,X}(g)\omega) = 0.$$

In the following, we shall use this to prove that $\Phi_{X,Y,Z}(f) \in \mathcal{J}(X, Z \otimes Y^*)$, namely, $\mathrm{tr}_{Z\otimes Y^*}(\Phi_{X,Y,Z}(f) \circ g\omega) = 0$. For any H-module M, we denote by $\{m_i\}$ the basis of M, and by $\{m_i^*\}$ the dual basis of M^*. We have

$$\sum_i \omega m_i \otimes \omega m_i^* = \sum_i m_i \otimes m_i^*,$$

or equivalently,

$$\sum_i m_i \otimes \omega m_i^* = \sum_i \omega^{-1} m_i \otimes m_i^*.$$

With these equalities, it is straightforward to check that the image of the basis $\{z_i\}$ of Z under the morphism $\omega f \circ (id_X \otimes \theta_Y^{-1}) \circ \Phi_{Z,Y^*,X}(g)$ is

$$(\omega f \circ (id_X \otimes \theta_Y^{-1}) \circ \Phi_{Z,Y^*,X}(g))(z_i) = \sum_j f(\omega g(z_i \otimes y_j^*) \otimes y_j).$$

It follows that

$$\begin{aligned} 0 &= \mathrm{tr}_Z(f \circ (id_X \otimes \theta_Y^{-1}) \circ \Phi_{Z,Y^*,X}(g)\omega) \\ &= \mathrm{tr}_Z(\omega f \circ (id_X \otimes \theta_Y^{-1}) \circ \Phi_{Z,Y^*,X}(g)) \\ &= \sum_{i,j} \langle z_i^*, f(\omega g(z_i \otimes y_j^*) \otimes y_j)\rangle. \end{aligned} \tag{5.13}$$

On the other hand, the image of the basis $\{z_i \otimes y_k^*\}$ of $Z \otimes Y^*$ under the morphism $\omega\Phi_{X,Y,Z}(f) \circ g$ is

$$(\omega\Phi_{X,Y,Z}(f) \circ g)(z_i \otimes y_k^*) = \sum_j f(\omega g(z_i \otimes y_k^*) \otimes y_j) \otimes y_j^*.$$

Therefore,

$$\begin{aligned} \mathrm{tr}_{Z\otimes Y^*}(\Phi_{X,Y,Z}(f) \circ g\omega) &= \mathrm{tr}_{Z\otimes Y^*}(\omega\Phi_{X,Y,Z}(f) \circ g) \\ &= \sum_{i,k,j} \langle z_i^* \otimes y_k^{**}, f(\omega g(z_i \otimes y_k^*) \otimes y_j) \otimes y_j^*\rangle \\ &= \sum_{i,j} \langle z_i^*, f(\omega g(z_i \otimes y_j^*) \otimes y_j)\rangle. \\ &= 0. \text{ by (5.13)} \end{aligned}$$

Claim 2. $\Phi_{X,Y,Z}(\mathcal{J}(X \otimes Y, Z)) = \mathcal{J}(X, Z \otimes Y^*)$.

For any morphism $\alpha \in \mathcal{J}(X, Z \otimes Y^*) \subseteq \mathrm{Hom}_H(X, Z \otimes Y^*) = \Phi_{X,Y,Z}(\mathrm{Hom}_H(X \otimes Y, Z))$, there is some $f \in \mathrm{Hom}_H(X \otimes Y, Z)$ such that $\alpha = \Phi_{X,Y,Z}(f)$. Now we check that $f \in \mathcal{J}(X \otimes Y, Z)$. Namely, for any $g \in \mathrm{Hom}_H(Z, X \otimes Y)$, the trace $\mathrm{tr}_Z(f \circ g\omega)$ needs to be zero. Since the morphism $\Phi^{-1}_{Z,Y^*,X}((id_X \otimes \theta_Y) \circ g)$ is in $\mathrm{Hom}_H(Z \otimes Y^*, X)$, we have

$$\mathrm{tr}_{Z\otimes Y^*}(\alpha\circ\Phi^{-1}_{Z,Y^*,X}((id_X\otimes\theta_Y)\circ g)\omega)=0.$$

On the one hand, if we write $g(z_i)=\sum_j x_{ij}\otimes y_j\in X\otimes Y$, then the image of the basis $\{z_i\otimes y_k^*\}$ of $Z\otimes Y^*$ under the morphism $\omega\alpha\circ\Phi^{-1}_{Z,Y^*,X}((id_X\otimes\theta_Y)\circ g)$ is given as follows:

$$(\omega\alpha\circ\Phi^{-1}_{Z,Y^*,X}((id_X\otimes\theta_Y)\circ g))(z_i\otimes y_k^*)=\sum_{j,s}\langle y_k^*,\omega y_j\rangle\omega f(x_{ij}\otimes y_s)\otimes\omega y_s^*.$$

It follows that

$$\begin{aligned}
0&=\mathrm{tr}_{Z\otimes Y^*}(\alpha\circ\Phi^{-1}_{Z,Y^*,X}((id_X\otimes\theta_Y)\circ g)\omega)\\
&=\mathrm{tr}_{Z\otimes Y^*}(\omega\alpha\circ\Phi^{-1}_{Z,Y^*,X}((id_X\otimes\theta_Y)\circ g))\\
&=\sum_{i,k,j,s}\langle y_k^*,\omega y_j\rangle\langle z_i^*\otimes y_k^{**},\omega f(x_{ij}\otimes y_s)\otimes\omega y_s^*)\rangle\\
&=\sum_{i,k,j,s}\langle y_k^*,\omega y_j\rangle\langle z_i^*\otimes y_k^{**},\omega f(x_{ij}\otimes\omega^{-1}y_s)\otimes y_s^*)\rangle\\
&=\sum_{i,k,j}\langle y_k^*,\omega y_j\rangle\langle z_i^*,\omega f(x_{ij}\otimes\omega^{-1}y_k)\rangle \qquad (5.14)\\
&=\sum_{i,k,j}\langle \omega y_k^*,\omega y_j\rangle\langle z_i^*,\omega f(x_{ij}\otimes y_k)\rangle\\
&=\sum_{i,k,j}\langle y_k^*,y_j\rangle\langle z_i^*,\omega f(x_{ij}\otimes y_k)\rangle\\
&=\sum_{i,j}\langle z_i^*,\omega f(x_{ij}\otimes y_j)\rangle.
\end{aligned}$$

Now the image of the basis $\{z_i\}$ of Z under the morphism $\omega f\circ g$ is given by $(\omega f\circ g)(z_i)=\sum_j\omega f(x_{ij}\otimes y_j)$ and by (5.14), we have $\mathrm{tr}_Z(f\circ g\omega)=\mathrm{tr}_Z(\omega f\circ g)=\sum_{i,j}\langle z_i^*,\omega f(x_{ij}\otimes y_j)\rangle=0$. We complete the proof. □

Proposition 5.4.6. *The form $[-,-]$ on the Green ring $\mathbb{Z}\mathbf{B}$ of the factor category H-$\underline{mod}$ can be defined by $[[X_i],[X_j]]=\dim\underline{Hom}_H(X_i,X_j^*)$, for any $i,j\in\mathbb{I}$.*

Proof. The associativity of the form follows from Lemma 5.4.5 (1). The symmetry and non-degeneracy of the form stem from the fact that $\underline{\mathrm{Hom}}_H(X_i,X_j^*)$ is isomorphic to $\Bbbk$ if $X_i\cong X_j^*$, and 0 otherwise. □

In general the number of the isomorphism classes of simple objects in the additive semisimple spherical category H-$\underline{\text{mod}}$ is not necessary finite. However, the finiteness is necessary if one wants to construct a manifold invariant from this category [15]. In the following, we give a characterization of the finiteness of H-$\underline{\text{mod}}$ by means of the Green ring $\mathbb{Z}\mathbf{B}$.

Let $\Bbbk\mathbf{B} := \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}\mathbf{B}$ be the $\Bbbk$-algebra obtained by the scalar extension from $\mathbb{Z}$ to $\Bbbk$. The ring homomorphism $\mathbf{d}$ from $\mathbb{Z}\mathbf{B}$ to $\Bbbk$ can be regarded as an algebra morphism from $\Bbbk\mathbf{B}$ to $\Bbbk$ in a natural way. A *left integral* of $\Bbbk\mathbf{B}$ with respect to $\mathbf{d}$ is one of the following elements:

$$\int_{\Bbbk\mathbf{B}}^{l} = \{t \in \Bbbk\mathbf{B} \mid x \cdot t = \mathbf{d}(x)t, \text{ for } x \in \Bbbk\mathbf{B}\}.$$

Similarly, one can define the set of *right integrals* of $\Bbbk\mathbf{B}$ with respect to $\mathbf{d}$:

$$\int_{\Bbbk\mathbf{B}}^{r} = \{t \in \Bbbk\mathbf{B} \mid t \cdot x = \mathbf{d}(x)t, \text{ for } x \in \Bbbk\mathbf{B}\}.$$

If the spaces of left integral and right integral coincide, then $\Bbbk\mathbf{B}$ is called *unimodular*.

Proposition 5.4.7. *The algebra* $\Bbbk\mathbf{B}$ *is finite dimensional if and only if* $\int_{\Bbbk\mathbf{B}}^{l}$ *or* $\int_{\Bbbk\mathbf{B}}^{r}$ *of* $\Bbbk\mathbf{B}$ *is not zero.*

Proof. If the $\Bbbk$-algebra $\Bbbk\mathbf{B}$ is finite dimensional, then it is Frobenius (and also symmetric) with the dual basis $\{[X_i], [X_{i^*}] \mid i \in \mathbb{I}\}$. In this case, the element

$$\sum_{i\in\mathbb{I}} \mathbf{d}([X_i])[X_{i^*}]$$

is a non-zero left (and also right) integral of $\Bbbk\mathbf{B}$ [58]. Conversely, if $\Bbbk\mathbf{B}$ has a non-zero left integral t, we write $t = \sum_{i\in\mathbb{I}} \lambda_i[X_i]$, a linear combination with $\lambda_i \neq 0$ for only finitely many $i \in \mathbb{I}$. By comparing the coefficient of $[\Bbbk]$ in the equality $t^* \cdot t = \mathbf{d}(t^*)t$, we obtain that $\sum_{i\in\mathbb{I}} \lambda_i^2 = \mathbf{d}(t^*)\lambda_0$. This implies that $\lambda_0 \neq 0$. To verify that the set $\mathbf{B} = \{[X_i] \mid i \in \mathbb{I}\}$ is finite, we need to show that any $[X_i]$ with $i \in \mathbb{I}$ is a non-zero summand of t. Indeed, the coefficient of $[X_i]$ in t is

$$\lambda_i = [[X_{i^*}], t] = [[\Bbbk], [X_{i^*}] \cdot t] = \mathbf{d}([X_{i^*}])[[\Bbbk], t] = \mathbf{d}([X_{i^*}])\lambda_0 \neq 0,$$

as desired. $\square$

In the sequel, we assume that H is of finite representation type. In this case, the algebra $\Bbbk\mathbf{B}$ is finite dimensional symmetric and semisimple (see Proposition 5.2.7) with the dual basis $\{[X_i], [X_{i^*}] \mid i \in \mathbb{I}\}$ with respect to the form $[-,-]$. Moreover, it is unimodular with $\int_{\Bbbk\mathbf{B}}^{l}$ as well as $\int_{\Bbbk\mathbf{B}}^{r}$ spanned by $t = \sum_{i\in\mathbb{I}} \mathbf{d}([X_i])[X_{i^*}]$. Denote by $x_i := \mathbf{d}([X_i])[X_i]$, for any $i \in \mathbb{I}$. Then the set $\mathbf{b} = \{x_i \mid i \in \mathbb{I}\}$ forms a basis of $\Bbbk\mathbf{B}$.

Proposition 5.4.8. *The quadruple* $(\Bbbk\mathbf{B}, \mathbf{d}, \mathbf{b}, *)$ *is a group-like algebra.*

Proof. We verify the conditions (G1)-(G3) given in Definition 1.1.7. The condition (G1) is obvious. To verify the condition (G2), we have

$$x_i^* = \mathbf{d}([X_i])([X_i])^* = \mathbf{d}([X_{i^*}])[X_{i^*}] = x_{i^*}, \tag{5.15}$$

for any $i \in \mathbb{I}$. Now for any $i, j \in \mathbb{I}$, we suppose that

$$x_i \cdot x_j = \sum_{k\in\mathbb{I}} p_{ij}^k x_k, \tag{5.16}$$

where $p_{ij}^k \in \Bbbk$. On the one hand, applying the dual operator $*$ to the equality (5.16) and using (5.15), we obtain that $x_{j^*} \cdot x_{i^*} = \sum_{k\in\mathbb{I}} p_{ij}^k x_{k^*}$. On the other hand, we have $x_{j^*} \cdot x_{i^*} = \sum_{l\in\mathbb{I}} p_{j^*i^*}^l x_l$ by (5.15). Thus, $p_{ij}^k = p_{j^*i^*}^{k^*}$ for any $i, j, k \in \mathbb{I}$. Now we verify the condition (G3). Extending the map T from $\Bbbk\mathbf{B}$ to $\Bbbk$ by linearity, one has for any $i, j \in \mathbb{I}$ that

$$\begin{aligned}
p_{ij}^0 &= T(x_i \cdot x_j) \\
&= \mathbf{d}([X_i])\mathbf{d}([X_j])T([X_i] \cdot [X_j]) \\
&= \mathbf{d}([X_i])\mathbf{d}([X_j])T([X_i][X_j]) \\
&= \mathbf{d}([X_i])\mathbf{d}([X_j])([X_i][X_j], \delta_{[\Bbbk]}^*) \\
&= \mathbf{d}([X_i])\mathbf{d}([X_j])([X_i], \delta_{[X_{j^*}]}^*) \\
&= \mathbf{d}([X_i])\mathbf{d}([X_j])\delta_{i,j^*} \\
&= \delta_{i,j^*}\mathbf{d}(x_i).
\end{aligned}$$

Therefore, the condition (G3) is satisfied. $\square$

A group-like algebra can be viewed as a bi-Frobenius algebra in a natural way, see [29, Example 3.2]. Following this approach, we define on $(\Bbbk\mathbf{B}, \mathbf{d}, \mathbf{b}, *)$ a bi-Frobenius algebra structure as follows.

$(\Bbbk\mathbf{B}, \phi)$ is a Frobenius algebra with the Frobenius homomorphism ϕ given by

$\phi(x_i) = \delta_{0,i}$, for $i \in \mathbb{I}$. Equivalently,

$$\phi([X_i]) = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

The set $\{x_i, \frac{x_i^*}{\mathbf{d}(x_i)} \mid i \in \mathbb{I}\}$ forms a dual basis of $(\Bbbk\mathbf{B}, \phi)$. This is equivalent to saying that $\{[X_i], [X_{i^*}] \mid i \in \mathbb{I}\}$ is a dual basis of $\Bbbk\mathbf{B}$ with respect to the Frobenius homomorphism ϕ. From the observation above, we conclude that the Frobenius homomorphism ϕ of $\Bbbk\mathbf{B}$ is nothing but the map determined by the form $[-,-]$, namely, $\phi(x) = [x, 1]$ for any $x \in \Bbbk\mathbf{B}$.

$\Bbbk\mathbf{B}$ is a coalgebra with the counit given by $\mathbf{d}$, and the comultiplication $\triangle$ defined by $\triangle(x_i) = \frac{1}{\mathbf{d}(x_i)} x_i \otimes x_i$, or equivalently,

$$\triangle([X_i]) = \frac{1}{\mathbf{d}([X_i])}[X_i] \otimes [X_i],$$

for $i \in \mathbb{I}$. Let $t = \sum_{i \in \mathbb{I}} x_i = \sum_{i \in \mathbb{I}} \mathbf{d}([X_i])[X_i]$ (t is nothing but an integral of $\Bbbk\mathbf{B}$ associated to the counit $\mathbf{d}$). Then $(\Bbbk\mathbf{B}, t)$ is a Frobenius coalgebra. Define a map $S : \Bbbk\mathbf{B} \to \Bbbk\mathbf{B}$ by $S(x_i) = x_{i^*}$, that is, $S([X_i]) = [X_{i^*}]$ for $i \in \mathbb{I}$. It is easy to see that the map S is an anti-algebra and anti-coalgebra morphism, so is an antipode of $\Bbbk\mathbf{B}$. Then the quadruple $(\Bbbk\mathbf{B}, \phi, t, S)$ forms a bi-Frobenius algebra. Thus, various properties of group-like algebras and bi-Frobenius algebras (see [28, 29, 30, 31]) hold for $\Bbbk\mathbf{B}$.

Bibliography

[1] N. Andruskiewitsch, I. Angiono, A. G. Iglesias, et al, From Hopf algebras to tensor categories, Conformal field theories and tensor categories, Springer Berlin Heidelberg, 1-31 (2014).

[2] L. Abrams, Two-dimensional topological quantum field theories and Frobenius algebras, J. Knot Theory Ramifications **5** (5), 569-587 (1996).

[3] M. Auslander, J. F. Carlson, Almost-split sequences and group rings, J. Algebra **103**, 122-140 (1986).

[4] L. Archer, On certain quotients of the Green rings of dihedral 2-groups, J. Pure Appl. Algebra **212**, 1888-1897 (2008).

[5] M. Auslander, I. Reiten, S. Smalϕ, Representation theory of Artin algebras, Cambridge Studies in Advanced Mathematics, Vol. **36**, Cambridge University Press (1994).

[6] I. Assem, D. Simson, A. Skowronski, Elements of the representation theory of associative algebras: Techniques of representation theory, Cambridge University Press (2006).

[7] N. Andruskiewitsch, H.-J. Schneider, Pointed Hopf algebras, New Directions in Hopf Algebras, 1-68, Cambridge University Press (2002).

[8] N. Andruskiewitsch, H.-J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, Ann. Math. **171**, 375-417 (2010).

[9] N. Andruskiewitsch, D. Radford, H.-J. Schneider, Complete reducibility theorems for modules over pointed Hopf algebras, J. Algebra **324** (11), 2932-2970 (2010).

[10] M. Auslander, Representation theory of artin algebras I, Commun. Algebra **1**, 177-268 (1974).

[11] M. Auslander, Representation theory of artin algebras II, Commun. Algebra **1**, 269-310 (1974).

[12] M. Auslander, I. Reiten, Representation theory of artin algebras III: Almost split sequences, Commun. Algebra **3**, 239-294 (1975).

[13] B. Bakalov, A. A. Kirillov, Lectures on tensor categories and modular functors, Amer. Math. Soc., Providence (2001).

[14] J. W. Barrett, B. W. Westbury, Spherical categories, Adv. Math. **143**, 357-375 (1999).

[15] J. W. Barrett, B. W. Westbury, Invariants of piecewise-linear 3-manifolds, Trans. Amer. Math. Soc. **348** (10), 3997-4022 (1996).

[16] A. Beauville, Conformal blocks, fusion rules and the Verlinde formula, Proceedings of the Hirzebruch **65** Conference on Algebraic Geometry, Israel Math. Conf. Proc. **9** (1996).

[17] D. J. Benson, The Green ring of a finite group, J. Algebra **87**, 290-331 (1984).

[18] K. Beidar, Y. Fong, A. Stolin, On Frobenius algebras and the quantum Yang-Baxter equation, Trans. Amer. Math. Soc. **349** (9), 3823-3836 (1997).

[19] H. Chen, The Green ring of Drinfeld double $D(H_4)$, arXiv:1209.3471v1, online Algebras and Representation Theory.

[20] X. Chen, H. Huang, Y. Ye, and P. Zhang, Monomial Hopf algebras, J. Algebra **275**, 212-232 (2004).

[21] C. Cibils, A quiver quantum group, Commun. Math. Phys. **157**, 459-477 (1993).

[22] M. Cohen, S. Westreich, Structure constants related to symmetric Hopf algebras, J. Algebra **324**, 3219-3240 (2010).

[23] C. W. Curtis, I. Reiner, Methods of representation theory, vol. **1**, Wiley-Interscience, New York (1981).

[24] H. Chen, F. V. Oystaeyen, Y. Zhang, The Green rings of Taft algebras, Proc. Amer. Math. Soc. **142**, 765-775 (2014).

[25] C. Cibils, P. Zhang, Calabi-Yau objects in triangulated categories, Trans. Amer. Math. Soc. **361**, 6501-6519 (2009).

[26] E. Darpö, M. Herschend, On the representation ring of the polynomial algebra over perfect field, Math. Z **265**, 605-615 (2011).

[27] M. Domokos, T. H. Lenagan, Representation rings of quantum groups, J. Algebra **282**, 103-128 (2004).

[28] Y. Doi, Bi-Frobenius algebras and group-like algebras, Lecture notes in pure and applied Mathematics, 143-156 (2004).

[29] Y. Doi, Group-like algebras and their representations, Commun. Algebra **38** (7), 2635-2655 (2010).

[30] Y. Doi, Substructures of bi-Frobenius algebras, J. Algebra **256**, 568-582 (2002).

[31] Y. Doi, M. Takeuchi, BiFrobenius algebras, Contemp. Math. **267**, 67-98 (2000).

[32] K. Erdmann, E. L. Green, N. Snashall, R. Taillefer, Representation theory of the Drinfeld doubles of a family of Hopf algebras, J. Pure Appl. Algebra **204** (2), 413-454 (2006).

[33] K. Erdmann, O. Kerner, On the stable module category of a self-injective algebra, Trans. Amer. Math. Soc. **352** (5), 2389-2405 (2000).

[34] K. Erdmann, A. Skowroński, The stable Calabi-Yau dimension of tame symmetric algebras, J. Math. Soc. Japan **58** (1), 97-128 (2006).

[35] K. Erdmann, A. Skowroński, On Auslander-Reiten components of blocks and self-injective biserial algebras, Trans. Amer. Math. Soc. **330** (1), 165-189 (1992).

[36] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories, lecture note for the MIT course 18.769, (2009). available at: www-math.mit.edu/ etingof/tenscat. pdf.

[37] W. Fulton, J. Harris, Representation theory, Grad. Texts Math, **129**, Spring-Verlag, New York (1991).

[38] S. Gelaki, On the classification of finite-dimensional triangular Hopf algebras, New Directions in Hopf Algebras, 69-116, Cambridge University Press (2002).

[39] J. A. Green, The modular representation algebra of a finite group III, J. Math. **6** (4), 607-619 (1962).

[40] J. A. Green, A transfer theorem for modular representations, J. Algebra **1**, 71-84 (1964).

[41] E. Gunnlaugsdöttir, Monoidal structure of the category of u_q^+-modules, Linear Algebra Appl. **365**, 183-199 (2003).

[42] E. L. Green, E. N. Marcos, and Ø. Solberg, Representations and almost split sequences for Hopf algebras, Representation theory of algebras, 237-245 (1996).

[43] D. Happel, Triangulated categories in the representation of finite dimensional algebras, Cambridge University Press (1988).

[44] M. Haim, Group-like algebras and Hadamard matrices, J. Algebra **308**, 215-235 (2007).

[45] V. E. Hoggart, J. M. Bicknell, Roots of Fibonacci polynomials, Fibonacci Quarterly **11** (3), 271-274 (1973).

[46] H. Heldner, Nilpotent elemnets of order 3 in the Green ring of the dihedral 2-group, Arch. Math(Basel) **63**, 193-196 (1994).

[47] H. Huang, F. V. Oystaeyen, Y. Yang, and Y. Zhang, The Green rings of pointed tensor categories of finite type, J. Pure Appl. Algebra **218**, 333-342 (2014).

[48] X. D. Hou, G. L. Mullen, J. A. Sellers, J. L. Yucas, Reversed Dickson polynomials over finite fields, Finite Fields Appl. **15** (6), 748-773 (2009).

[49] C. Kassel, Quantum groups, Springer-Verlag, New York (1995).

[50] L. Kropa, D. E. Radford, Finite-dimensional Hopf algebras of rank one in characteristic zero, J. Algebra **302**, 214-230 (2006).

[51] L. Kadison, A. A. Stolin, Separability and Hopf algebras, Contemp. Math. **259**, 279-298 (1999).

[52] Y. Kashina, S. Montgomery, S. H. Ng, On the trace of the antipode and higher indicators, Israel J. Math. **188** (1), 57-89 (2012).

[53] Y. Li, N. Hu, The Green rings of the 2-rank Taft algebra and its two relatives twisted, J. Algebra **410**, 1-35 (2014).

[54] R. G. Larson, D. E. Radford, Semisimple cosemisimple Hopf algebras, Amer. J. Math. **109**, 187-195 (1987).

[55] R. G. Larson, M. Sweedler, An associative orthogonal bilinear form for Hopf algebras, Amer. J. Math. **91**, 75-93 (1969).

[56] L. Li, Y. Zhang, The Green rings of the generalized Taft Hopf algebras, Contemp. Math. **585**, 275-288 (2013).

[57] M. Lorenz, Representations of finite-dimensional Hopf algebras, J. Algebra **188**, 476-505 (1997).

[58] M. Lorenz, Some applications of Frobenius algebras to Hopf algebras, Contemp. Math. **537**, 269-289 (2011).

[59] R. Lidl, G. L. Mullen, G. Turnwald, Dickson polynomials, Harlow: Longman Scientific & Technical **65** (1993).

[60] M. Müger, Tensor categories: a selective guided tour, Rev. Un. Mat. Argentina **51** (1), 95-163 (2010).

[61] S. Montgomery, Hopf Algebras and their actions on rings, In: CBMS Series in Math, vol.**82**. Amer. Math. Soc., Providence (1993)

[62] D. E. Radford, On the coradical of a finite-dimensional Hopf algebra, Proc. Amer. Math. Soc. **53** (1), 9-15 (1975).

[63] I. Reiner, Nilpotent elements in rings of integral representations, Proc. Amer. Math. Soc. **17**, 270-274 (1966).

[64] S. Scherotzke, Classification of pointed rank one Hopf algebras, J. Algebra **319**, 2889-2912 (2008).

[65] P. Schauenburg, Bi-Galois objects over the Taft algebras, Israel J Math. **115** (1), 101-123 (2000).

[66] J. P. Serre, Linear representations of finite groups, Graduate Texts in Mathematics, Vol. **42**, New York, Springer-Verlag (1977).

[67] W. F. Santos, M. Haim, Radford's formula for biFrobenius algebras and applications, Commun. Algebra **36** (4), 1301-1310 (2008).

[68] A. N. Sergeev, A. P. Veselov, Grothendieck rings of basic classical Lie superalgebras, Ann. Math **173**, 663-703 (2011).

[69] E. J. Taft, The order of the antipode of finite-dimensional Hopf algebra, Proc. Nat. Acad. Sci. USA **68**, 2631-2633 (1971).

[70] M. Wakui, Various structures associated to the representation categories of eight-dimensional nonsemisimple Hopf algebras, Algebras and Representation Theory **7**, 491-515 (2004).

[71] Z. Wang, L. Li, Y. Zhang, Green rings of pointed rank one Hopf algebras of nilpotent type, arXiv: math/1304.3611v3, online Algebras and Representation Theory.

[72] Z. Wang, L. Li, Y. Zhang, Green rings of pointed rank one Hopf algebras of non-nilpotent type, preprint.

[73] Z. Wang, L. Li, Y. Zhang, Bilinear forms on Green rings of finite dimensional Hopf algebras, in preparation.

[74] Z. Wang, L. Li, Y. Zhang, On Green rings of spherical Hopf algebras, in preparation.

[75] Z. Wang, L. You and H. Chen, Hopf-Ore extensions and pointed Hopf algebras of rank one, arXiv: math/1206.3943v1.

[76] Y. Wang, P. Zhang, Construct bi-Frobenius algebras via quivers, Tsukuba J. Math. **28** (1), 215-227 (2004).

[77] Y. Wang, X. Chen, Construct non-graded bi-Frobenius algebras via quivers, Science in China Series A: Mathematics **50** (3), 450-456 (2007).

[78] S. J. Witherspoon, The representation ring of the quantum double of a finite group, J. Algebra **179**, 305-329 (1996).

[79] S. J. Witherspoon, The representation ring and the centre of a Hopf algebra, Canad. J. Math. **51** (4), 881-896 (1999).

[80] S. Yang, Finite dimensional representations of u-Hopf algebras, Commun. Algebra **29** (12), (2001): 5359-5370.

[81] Y. Zhu, Hopf algebras of prime dimension, Internat. Math. Res. Not. **1**, 53-59 (1994).

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