Algebraic and geometric characterizations of double-cross matrices of polylines

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Abstract

We study the double-cross matrix descriptions of polylines in the two-dimensional plane. The double-cross matrix is a qualitative description of polylines in which exact, quantitative information is given up in favour of directional information. First, we give an algebraic characterization of the double-cross matrix of a polyline and derive some properties of double-cross matrices from this characterisation. Next, we give a geometric characterization of double-cross similarity of two polylines, using the technique of local carrier orders of polylines. To end, we identify the transformations of the plane that leave the double-cross matrix of all polylines in the two-dimensional plane invariant.

Keywords: Spatial reasoning; Double-cross matrix; Qualitative spatial models; Polylines; Trajectory and Moving Object Data.

1 Introduction and summary of results

Polylines arise in Geographical Information Science (GIS) in a multitude of ways. One recent example comes from the collection of moving object data, where trajectories of moving persons (or animals), that carry GPS-equipped devices, are collected in the form of time-space points that are registered at certain (ir)regular moments in time. The spatial trace of this movement is a collection of points in two-dimensional space. There are several methods to extend the trajectory in between the measured sample points, of which linear interpolation is a popular method (Güting & Schneider, 2005). The resulting curve in the two-dimensional geographical space is a *polyline*.

Another example comes from shape recognition and retrieval, which arises in domains, such as computer vision, image analysis and GIS, in general. Here, closed polylines (where the starting point coincides with the end point) or polygons, often occur as the boundary of two-dimensional shapes or regions. Shape recognition and retrieval are central problems in this context.

In examples, such as the above, there are, roughly speaking, two very distinct approaches to deal with polygonal curves and shapes. On the one hand, there are the *quantitative* approaches and on the other hand there are the *qualitative* approaches. Initially, most research efforts have dealt with the quantitative methods (Bookstein, 1986; Dryden & Mardia, 1998; Kent & Mardia, 1986; Mokhtarian & Mackworth, 1992). Only afterwards, the qualitative approaches have gained more attention, mainly supported by research in cognitive science that provides evidence that qualitative models of shape representation are much more expressive

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than their quantitative counterpart and reflect better the way in which humans reason about their environment (Gero, 1999). Polygonal shapes and polygonal curves are very complex spatial phenomena and it is commonly agreed that a qualitative representation of space is more suitable to deal with them (Meathrel, 2001).

Within the qualitative approaches to describe two-dimensional movement or shapes, two major approaches can be distinguished: the region-based and the boundary-based approach. The *region-based* approach, using concepts such as circularity, orientation with respect to the coordinate axis, can only distinguish between shapes with large dissimilarities (Schlieder, 1996). The *boundary-based*, using concepts such as extremes in and changes of curvature of the polyline representing the shape, gives more precise tools to distinguish polylines and polygons. Examples of the boundary-based approaches are found in (Gottfried, 2003; Jungert, 1993; Kulik & Egenhofer, 2003; Latecki & Lakämper, 2000; Leyton, 2000; Meathrel, 2001; Schlieder, 1996).

The principles behind qualitative approaches to deal with polylines are also related to the field of spatial reasoning. *Spatial reasoning* has as one of its main objectives to present geographic information in a qualitative way to be able to reason about it (see, for example, Chapter 12 in (Giannotti & Pedreschi, 2008), also for spatio-temporal reasoning) and it can be seen as the processing of information about an spatial environment that is immediately available to humans (or animals) through direct observation. The reason for using a *qualitative representation* is that the available information is often imprecise, partial and subjective (Freksa, 1992). If we return to the example of trajectory data, we can see that if a person orients her- or himself at a certain location in a city and then moves away from this location, she or he remembers her or his current location by using a mental map that takes the relative turns into account with respect to the original starting point, rather than the precise metric information about her or his trajectory. For such navigational problems, a person will for instance remember: "I left the station and went straight; passing a church to my right; then taking two left turns; ..."

One of the formalisms to qualitatively describe polylines in the plane is given by the *double-cross calculus*. In this method, a *double-cross matrix* captures the relative position of any two line segments in a polyline by describing it with respect to a double cross based on the starting points of these line segments. The double-cross calculus was introduced as a formalism to qualitatively represent a configuration of vectors in the plane (Freksa, 1992; Zimmermann & Freksa, 1996). For an overview of the use of constraint calculi in qualitative spatial reasoning, we refer to (Renz & Nebel, 2007). In the double-cross formalism, the relative position (or orientation) of two (located) vectors is encoded by means of a 4-tuple, whose entries come from the set $\{0, +, -\}$. Such a 4-tuple expresses the relative orientation of two vectors with respect to each other. The double-cross formalism is used, for instance, in the *qualitative trajectory calculus*, which, in turn, has been used to test polyline similarity with applications to query-by-sketch, indexing and classification (Kuijpers, Moelans, & Van de Weghe, 2006).

Two polylines are called *double-cross similar* if their double-cross matrices are identical. Two polylines, that are quite different from a quantitative or metric perspective, may have the same double-cross matrices, as we illustrate below. The idea is that they follow a similar pattern of relative turns, which reflects how humans visualize and remember movement patterns.

In this paper, we provide an extensive algebraic and geometric interpretation of the double-cross matrix of a polyline and of double-cross similarity of polylines. To start with, we give a collection of polynomial constraints (polynomial equalities and inequalities) on the coordinates of the measured points of a polyline (its vertices) that express the information contained in the double-cross matrix of a polyline. This algebraic characterisation can be used to effectively verify double-cross similarity of polylines and to generate double-cross similar polylines by means of tools from algebraic geometry, implemented, for instance, in software packages like MATHEMATICA (Wolfram Research, 2015). This algebraic characterization of the double-cross matrix also allows us to prove a number of properties of double-cross matrices. As an example, we mention a high degree of symmetry in the double-cross matrix along its main diagonal.

Next, we turn to a geometrical interpretation of double-cross similarity of two polylines. This geometrical interpretation is based on local geometric information of the polyline in its vertices. This information is called the *local carrier order* and it uses (local) compass directions in the vertices of a polyline to locate the relative position of the other vertices. Our main result in this context says that two polylines are double-cross similar if and only if they have the same local carrier order structure.

From the definition of the doubtle-cross matrix of a polyline it is clear that this matrix remains the same if, for instance, we translate or rotate the polyline in the two-dimensional plane. In a final part of this paper, we identify the exact set of transformations of the two-dimensional plane that leave double-cross matrices invariant. Our main (and rather technical) result states that the largest group of transformations of the plane, that is double-cross invariant consist of the *similaritiy* transformations of the plane onto itself. *Grosso modo*, the similarities of the plane are the translations, rotations and homotecies (scalings) of the plane. This result allows us, for instance, to prove any statement about double-cross matrices of a polyline, only for polylines start in the origin of the two-dimensional plane and have a unit length first line segment.

Organization. This paper is organized as follows. Section 2 gives the definition of a polyline, the double-cross matrix of a polyline and double-cross similarity of two polylines. Section 3 gives our algebraic characterization of the double-cross matrix of a polyline. In Section 4, we derive a number of properties of double-cross matrices from the algebraic characterisation. In Section 5, we give a geometric characterization of the double-cross similarity of two polylines in terms of the local carrier order. And finally, in Section 6, we characterize the double-cross invariant transformations of the plane. In this section, we identify the transformations of the plane that leave the double-cross matrix of all polylines invariant.

2 Definitions and preliminaries

In this section, we give the definitions of a polyline, of the double-cross matrix of a polyline and of double-cross similarity of two polylines.

2.1 Polylines in the plane

Let $\mathbf{R}$ denote the sets of the real numbers, and let $\mathbf{R}^2$ denote the two-dimensional real plane. To stress that some real values are constants, we use sans serif characters: $\mathsf{x}, \mathsf{y}, \mathsf{x}_0, \mathsf{y}_0, \mathsf{x}_1, \mathsf{y}_1, \ldots$. Real variables are denoted in normal characters. For constant points of $\mathbf{R}^2$, we use the sans serif characters $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1, \ldots$

The following definition specifies what we mean by *polylines*. We define polylines as a finite sequences of points in $\mathbf{R}^2$ (which is often used as their finite representation). When we add the line segments between consecutive points we obtain what we call the *semantics* of the polyline. We also introduce some terminology about polylines.

Definition 1 A *polyline* (in $\mathbf{R}^2$) is an ordered list $P = \langle(\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N)\rangle$ of points in $\mathbf{R}^2$. We call the points $(\mathsf{x}_i, \mathsf{y}_i)$, $0 \leq i \leq N$, the *vertices* of the polyline. We assume that no two consecutive vertices are identical, that is: $(\mathsf{x}_i, \mathsf{y}_i) \neq (\mathsf{x}_{i+1}, \mathsf{y}_{i+1})$, for $0 \leq i < N$.

We call N is the *size* of the polyline P. The vertices $(\mathsf{x}_0, \mathsf{y}_0)$ and $(\mathsf{x}_N, \mathsf{y}_N)$ are respectively called the *start* and *end vertex* of P. The line segments connecting the points $(\mathsf{x}_i, \mathsf{y}_i)$ and $(\mathsf{x}_{i+1}, \mathsf{y}_{i+1})$, for $0 \leq i < N$, are called the *(line) segments* of the polyline P. The *semantics* of P, denoted $\mathsf{sem}(P)$, is the union of the line segments of P. □

So, the semantics, $\mathsf{sem}(P)$, is the following union of line segments:

$$\bigcup_{i=0}^{N-1} \left\{(x, y) \in \mathbf{R}^2 \mid \exists\lambda \in [0, 1] : (x, y) = \lambda \cdot (\mathsf{x}_i, \mathsf{y}_i) + (1 - \lambda) \cdot (\mathsf{x}_{i+1}, \mathsf{y}_{i+1})\right\},$$

which is a polygonal curve in $\mathbf{R}^2$. Further on, we will loosely use the term polyline also to refer to the semantics of a polyline, although, *stricto sensu*, a polyline is a ordered list of points in $\mathbf{R}^2$.

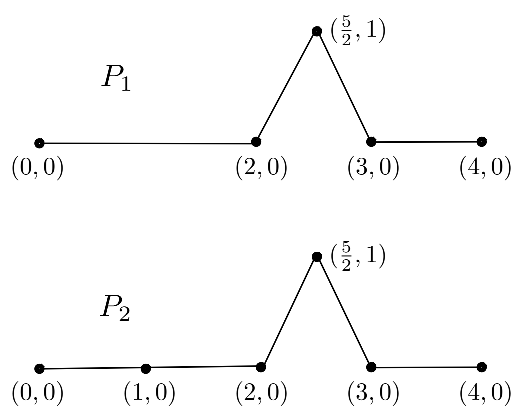

Figure 1: An example of a polyline $P_1 = \langle(0, 0), (2, 0), (\frac{5}{2}, 1), (3, 0), (4, 0)\rangle$ and a polyline $P_2 = \langle(0, 0), (1, 0), (2, 0), (\frac{5}{2}, 1), (3, 0), (4, 0)\rangle$. Although they have a different vertex set and a different size, still $\mathsf{sem}(P_1) = \mathsf{sem}(P_2)$.

We remark that two polylines with a different number of vertices, may have the same semantics. Figure 1 gives an example of such polylines. We also remark that the line segments, appearing in the semantics, may intersect in points which may or may be not vertices, as is illustrated by the polyline shown in Figure 2. A polyline where the start and end vertex coincide and which has no other self-intersections in its semantics is a *polygon*. Finally, we remark that it is reasonable to assume that polylines coming from GIS applications have vertices with rational coordinates.

Below, we stick to the notation introduced in the above definitions. Furthermore, as a standard, we abbreviate $(\mathsf{x}_i, \mathsf{y}_i)$ by p_i. We also use the following notational conventions. The *(located) vector*[1] from p_i to p_j is denoted by $\overrightarrow{\mathsf{p}_i\mathsf{p}_j}$. The counter-clockwise angle (expressed in degrees) measured from $\overrightarrow{\mathsf{p}_i\mathsf{p}_j}$ to $\overrightarrow{\mathsf{p}_i\mathsf{p}_k}$ is denoted by $\angle(\overrightarrow{\mathsf{p}_i\mathsf{p}_j}, \overrightarrow{\mathsf{p}_i\mathsf{p}_k})$, as illustrated in Figure 3.

[1]By the *located vector* from p to q, we mean an ordered pair (p, q) of points of $\mathbf{R}^2$, which we denote $\overrightarrow{\mathsf{pq}}$. We use this concept to represent the oriented line segment between p and q.

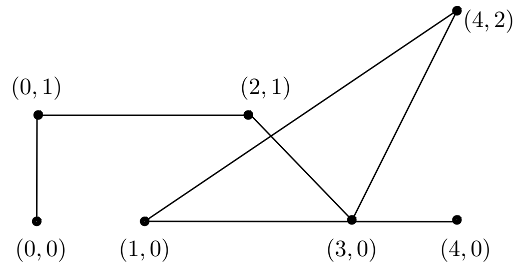

Figure 2: An example of a polyline $P = \langle(0,0),(0,1),(2,1),(3,0),(4,2),(1,0),(4,0)\rangle$ and its semantics $\mathsf{sem}(P)$. We see that two of the line segments of its semantics intersect in a point that is not a vertex. The last line segment of the polyline intersects two other line setments in a vertex.

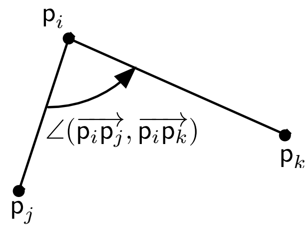

Figure 3: The counter-clockwise angle $\angle(\overrightarrow{\mathsf{p}_i\mathsf{p}_j}, \overrightarrow{\mathsf{p}_i\mathsf{p}_k})$ from $\overrightarrow{\mathsf{p}_i\mathsf{p}_j}$ to $\overrightarrow{\mathsf{p}_i\mathsf{p}_k}$.

2.2 The double-cross matrix of a polyline

In this section, we define the *double-cross matrix* of a polyline.

2.2.1 The double-cross value of two (located) vectors

The double-cross calculus was introduced as a formalism to qualitatively represent a configuration of vectors in the plane $\mathbf{R}^2$ (Freksa, 1992; Zimmermann & Freksa, 1996). In this formalism, the relative position (or orientation) of two (located) vectors is encoded by means of a 4-tuple, whose entries come from the set $\{0, +, -\}$. Such a 4-tuple expresses the relative orientation of two vectors with respect to each other.

We associate to a polyline $P = \langle(\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N)\rangle$, with $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$, the (located) vectors $\overrightarrow{\mathsf{p}_0\mathsf{p}_1}, \overrightarrow{\mathsf{p}_1\mathsf{p}_2}, \ldots, \overrightarrow{\mathsf{p}_{N-1}\mathsf{p}_N}$, representing the oriented line segments between the consecutive vertices of P. Because of the assumption in Definition 1, the vectors $\overrightarrow{\mathsf{p}_0\mathsf{p}_1}$, $\overrightarrow{\mathsf{p}_1\mathsf{p}_2}, \ldots, \overrightarrow{\mathsf{p}_{N-1}\mathsf{p}_N}$ all have a strictly positive length. In the double-cross formalism, the relative orientation between $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$ is given by means of a 4-tuple

$$(C_1\ C_2\ C_3\ C_4) \in \{-, 0, +\}^4.$$

We follow the traditional notation of this 4-tuple *without commas*. To determine C_1, C_2, C_3 and C_4, for $\mathsf{p}_i \neq \mathsf{p}_j$, first of all, a *double cross* is defined for the vectors $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$, determined by the following three lines:

- the line L_{ij} through p_i and p_j;
- the line P_{iji} through p_i, perpendicular on L_{ij}; and

- the line P_{ijj} through p_j, perpendicular on L_{ij}.

These three lines are illustrated in Figure 4. These three lines determine a cross at p_i and a cross at p_j. Hence the name "double cross." The entries C_1, C_2, C_3 and C_4 express in which quadrants or on which half lines p_{i+1} and p_{j+1} are located with respect to the double cross.

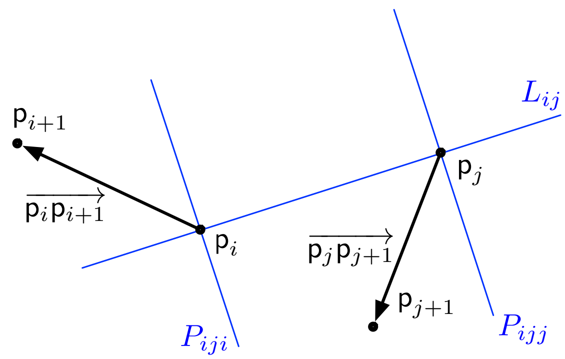

Figure 4: The double cross (in blue): the lines L_{ij}, P_{iji} and P_{ijj}.

We now define this more formally and follow the historical use of the double cross (see, for instance, (Freksa, 1992; Zimmermann & Freksa, 1996). In this definition, an interval (a, b) of angles, represents the open interval between a and b on the counter-clockwise oriented circle.

Definition 2 Let $P = \langle(\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N)\rangle$ be a polyline, with $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$, for $0 \leq i \leq N$, and with associated vectors $\overrightarrow{\mathsf{p}_0\mathsf{p}_1}, \overrightarrow{\mathsf{p}_1\mathsf{p}_2}, \ldots, \overrightarrow{\mathsf{p}_{N-1}\mathsf{p}_N}$. For $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$ with $0 \leq i, j < N$, $i \neq j$ and $\mathsf{p}_i \neq \mathsf{p}_j$, we define

$$\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (C_1\ C_2\ C_3\ C_4)$$

as follows:

$$C_1 = \begin{cases} - & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_i\mathsf{p}_j}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) \in (-90°, 90°) \\ 0 & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_i\mathsf{p}_j}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) \in \{-90°, 90°\} \\ + & \text{else} \end{cases}$$

$$C_2 = \begin{cases} - & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_j\mathsf{p}_i}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) \in (-90°, 90°) \\ 0 & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_j\mathsf{p}_i}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) \in \{-90°, 90°\} \\ + & \text{else} \end{cases}$$

$$C_3 = \begin{cases} - & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_i\mathsf{p}_j}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) \in (0°, 180°) \\ 0 & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_i\mathsf{p}_j}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) \in \{0°, 180°\} \\ + & \text{else} \end{cases}$$

$$C_4 = \begin{cases} - & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_j\mathsf{p}_i}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) \in (0°, 180°) \\ 0 & \text{if} \quad \angle(\overrightarrow{\mathsf{p}_j\mathsf{p}_i}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) \in \{0°, 180°\} \\ + & \text{else.} \end{cases}$$

For $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$, with $\mathsf{p}_i = \mathsf{p}_j$, we define, for reasons of continuity,[2]

$$\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (0\ 0\ 0\ 0).$$

[2]This argumentation is given in (Forbus, 1990).

□

So, in particular, when $i = j$, we have $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (0\ 0\ 0\ 0)$.

We remark that the values C_1 and C_3 describe the location of the point p_{i+1} or, equivalently, the orientation of the vector $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ with respect to the cross at p_i (formed by the lines L_{ij} and P_{iji}). We see that each of the four quadrants and four half lines determined by the cross at p_i are determined by a unique combination of C_1 and C_3 values. Similarly, the values C_2 and C_4 describe the location of the point p_{j+1} or, equivalently, the orientation of the vector $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$ with respect to the cross at p_j (formed by the lines L_{ij} and P_{ijj}).

The quadrants and half lines where C_1, C_2, C_3 and C_4 take different values are graphically illustrated in Figure 5. For example, the 4-tuple $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}})$ for the vectors $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$, shown in Figure 4, is $(+\ -\ -\ -)$.

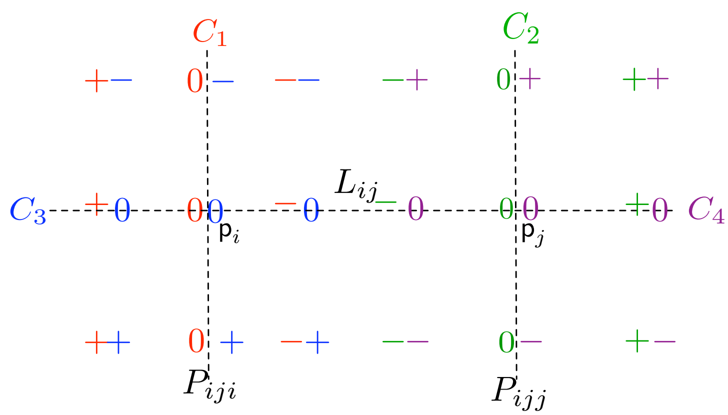

Figure 5: The quadrants and half lines where C_1, C_2, C_3 and C_4 take different values.

Further on, we will sometimes use the notation $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}})[k]$ to indicate C_k, for $k = 1, 2, 3, 4$. Obviously, this notation does not hide the dependence on i and j.

Remark. Since C_1, C_2, C_3 and C_4 take values from the set $\{-, 0, +\}$, it may seem that there are $3^4 = 81$ possible values for the tuples $(C_1\ C_2\ C_3\ C_4)$.

However, some combinations are not possible because of the assumption in Definition 1, that says that two consecutive vertices of a polyline have to be different. This means that C_1 and C_3 cannot be both 0 and that C_2 and C_4 cannot be both 0, in each case with the exception of C_1, C_2, C_3 and C_4 all being 0, that is $(C_1\ C_2\ C_3\ C_4) = (0\ 0\ 0\ 0)$. So, we have $81 - 8 - 8 = 65$ possible values for $(C_1\ C_2\ C_3\ C_4)$.

This number of 65 possible values for the tuples $(C_1\ C_2\ C_3\ C_4)$ can also be reached in another way. The point p_{i+1} can be in one of four quadrants around p_i or on one of four half lines starting in p_i. These are 8 possible locations for p_{i+1}. Similarly, we have 8 possible locations for p_{j+1} in the quadrants and half lines starting in p_j. This gives $8 \times 8 = 64$ possible combinations. Together with the case $(C_1\ C_2\ C_3\ C_4) = (0\ 0\ 0\ 0)$, we reach a total number of 65 possibilities. □

2.2.2 The double-cross matrix of a polyline

Based on the definition of $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}})$, we now define the double-cross matrix of a polyline.

	$\overrightarrow{\mathsf{p}_0\mathsf{p}_1}$	$\overrightarrow{\mathsf{p}_1\mathsf{p}_2}$	$\overrightarrow{\mathsf{p}_2\mathsf{p}_3}$	$\overrightarrow{\mathsf{p}_3\mathsf{p}_4}$	$\overrightarrow{\mathsf{p}_4\mathsf{p}_5}$
$\overrightarrow{\mathsf{p}_0\mathsf{p}_1}$	$(0\ 0\ 0\ 0)$	$(-\ -\ 0\ +)$	$(-\ +\ +\ -)$	$(-\ -\ +\ -)$	$(-\ +\ -\ +)$
$\overrightarrow{\mathsf{p}_1\mathsf{p}_2}$	$(-\ -\ +\ 0)$	$(0\ 0\ 0\ 0)$	$(-\ -\ 0\ +)$	$(-\ +\ +\ +)$	$(-\ -\ +\ +)$
$\overrightarrow{\mathsf{p}_2\mathsf{p}_3}$	$(-\ -\ -\ +)$	$(-\ -\ +\ 0)$	$(0\ 0\ 0\ 0)$	$(-\ +\ 0\ -)$	$(-\ -\ -\ +)$
$\overrightarrow{\mathsf{p}_3\mathsf{p}_4}$	$(-\ -\ -\ +)$	$(+\ -\ +\ +)$	$(+\ -\ -\ 0)$	$(0\ 0\ 0\ 0)$	$(-\ -\ 0\ +)$
$\overrightarrow{\mathsf{p}_4\mathsf{p}_5}$	$(+\ -\ +\ -)$	$(-\ -\ +\ +)$	$(-\ -\ +\ -)$	$(-\ -\ +\ 0)$	$(0\ 0\ 0\ 0)$

Table 1: The entries of the double-cross matrix of the polyline of Figure 6.

Definition 3 Let $P = \langle(\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \dots, (\mathsf{x}_N, \mathsf{y}_N)\rangle$ be a polyline, with $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$, for $0 \leq i \leq N$, and with associated vectors $\overrightarrow{\mathsf{p}_0\mathsf{p}_1}, \overrightarrow{\mathsf{p}_1\mathsf{p}_2}, \dots, \overrightarrow{\mathsf{p}_{N-1}\mathsf{p}_N}$. The *double-cross matrix* of P, denoted $\mathsf{DCM}(P)$, is the $N \times N$ matrix with the entries $\mathsf{DCM}(P)[i, j] = \mathsf{DC}(\overrightarrow{p_ip_{i+1}}, \overrightarrow{p_jp_{j+1}})$, for $0 \leq i, j < N$. □

For example, the entries of the double-cross matrix of the polyline of Figure 6 are given in Table 1. This first example can be used to illustrate some properties of this matrix that are proven in Section 4. First, we observe that on the diagonal always $(0\ \ 0\ \ 0\ \ 0)$ appears. We also see that there is a certain degree of symmetry along the diagonal. If $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (C_1\ C_2\ C_3\ C_4)$, then we have $\mathsf{DC}(\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) = (C_2\ C_1\ C_4\ C_3)$. These two observations imply that it suffices to know a double-cross matrix *above its diagonal*.

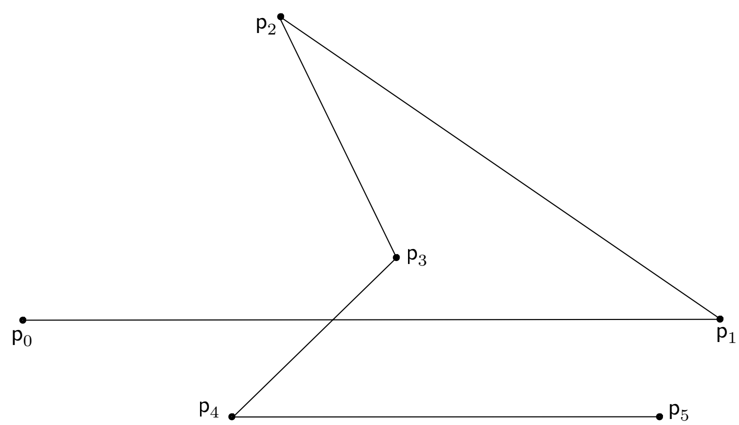

Figure 6: An example of a polyline.

2.3 Double-cross similarity of polylines

We now define *double-cross similarity* of two polylines of equal size.

Definition 4 Let P and Q be polylines of the same size. We say that P and Q are *double-cross similar* if $\mathsf{DCM}(P) = \mathsf{DCM}(Q)$. □

We stress that Definition 4 requires that the two polygons have to be of the same size before we can speak of their double-cross similarity.

Figure 7 depicts two polylines, P and Q, which are double-cross similar. The entries of their double-cross matrices are given in Table 2. In polyline P of Figure 7, at each vertex, the polyline bends around 10 degrees to the left. In polyline Q, this is only around 2 degrees.

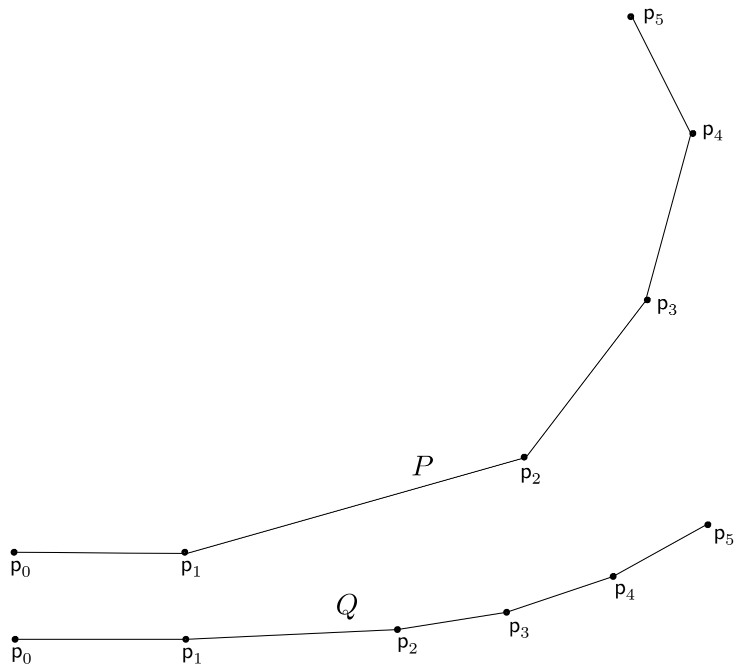

Figure 7: The polylines P and Q are double-cross similar.

	$\overrightarrow{p_0p_1}$	$\overrightarrow{p_1p_2}$	$\overrightarrow{p_2p_3}$	$\overrightarrow{p_3p_4}$	$\overrightarrow{p_4p_5}$
$\overrightarrow{p_0p_1}$	$(0\ 0\ 0\ 0)$	$(-\ +\ 0\ +)$	$(-\ +\ +\ +)$	$(-\ +\ +\ +)$	$(-\ +\ +\ +)$
$\overrightarrow{p_1p_2}$	$(+\ -\ +\ 0)$	$(0\ 0\ 0\ 0)$	$(-\ +\ 0\ +)$	$(-\ +\ +\ +)$	$(-\ +\ +\ +)$
$\overrightarrow{p_2p_3}$	$(+\ -\ +\ +)$	$(+\ -\ +\ 0)$	$(0\ 0\ 0\ 0)$	$(-\ +\ 0\ +)$	$(-\ +\ +\ +)$
$\overrightarrow{p_3p_4}$	$(+\ -\ +\ +)$	$(+\ -\ +\ +)$	$(+\ -\ +\ 0)$	$(0\ 0\ 0\ 0)$	$(-\ +\ 0\ +)$
$\overrightarrow{p_4p_5}$	$(+\ -\ +\ +)$	$(+\ -\ +\ +)$	$(+\ -\ +\ +)$	$(+\ -\ +\ 0)$	$(0\ 0\ 0\ 0)$

Table 2: The entries of the double-cross matrix of the polylines of Figure 7.

Nevertheless, all relative positions of oriented line segments remain the same. As the most extreme example, if we compare $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_4p_5}$ in both polylines, we see that $\overrightarrow{p_4p_5}$ almost makes a 90° left angle with the central line of the double cross in the polyline P, whereas, this is only some 10° in the polyline Q. Still, both P and Q have the same double-cross entry for $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_4p_5}$.

3 An algebraic characterization of the double-cross matrix of a polyline

In this section, we give an algebraic characterization of the entries in the double-cross matrix of a polyline. This algebraic characterisation can be used to effectively verify double-cross similarity of polylines.

Let $P = \langle(x_0, y_0), (x_1, y_1), , \ldots, (x_N, y_N)\rangle$ be a polyline and let $p_i = (x_i, y_i)$, for $0 \leq i \leq N$. Theorem 1 gives algebraic expressions to calculate the entries $DC(\overrightarrow{p_ip_{i+1}}, \overrightarrow{p_jp_{j+1}})$ of a double-cross matrix in terms of the x- and y-coordinates of the points p_i, p_{i+1}, p_j and p_{j+1}. Further on, we use this theorem extensively to prove properties of double-cross matrices.

Before stating and proving this theorem, we recall some elementary notations from algebra and some formula's in the following remark.

Remark. The well-known formula to calculate the (counter-clockwise) angle θ between two vectors[3] $\vec{a}$ and $\vec{b}$ in $\mathbf{R}^2$ (and also, in general, in $\mathbf{R}^n$) is

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}.$$

Here, the $\cdot$ in the numerator denotes the *inner product*[4] of two vectors and $|\vec{a}|$ is the norm or length of $\vec{a}$ (and the $\cdot$ in the denominator is the product of real numbers).

The above formula implies that we have $\cos\theta = 0$ if and only if $\vec{a}\cdot\vec{b} = 0$ if and only if $\theta \in \{90°, -90°\}$. So, $\vec{a}\cdot\vec{b} = 0$ means that $\vec{a}$ is perpendicular to $\vec{b}$. On the other hand, we have $\cos\theta > 0$ and thus $\vec{a}\cdot\vec{b} > 0$, when $\theta \in (-90°, 90°)$. And finally $\vec{a}\cdot\vec{b} < 0$ is equivalent to $\theta \in (90°, 270°)$.

If $\vec{a} = (a, b) \in \mathbf{R}^2$, then $\vec{a}^\perp = (-b, a)$ is the unique vector, perpendicular to $\vec{a}$ and of the same length of $\vec{a}$, such that the (counter-clockwise) angle from $\vec{a}$ to $\vec{a}^\perp$ is $90°$. □

In the following theorem, we use the function

$$\mathrm{sign} : \mathbf{R} \to \{-, 0, +\} : x \mapsto \mathrm{sign}(x) = \begin{cases} - & \text{if} \quad x < 0; \\ 0 & \text{if} \quad x = 0; \text{ and} \\ + & \text{if} \quad x > 0. \end{cases}$$

We also work with the following convention concerning signs: $--$ is $+$; -0 is 0; and $-+$ is $-$.

Theorem 1 *Let* $P = \langle(\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N)\rangle$ *be a polyline and let* $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$*, for* $0 \le i \le N$*. Then,* $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (C_1\ C_2\ C_3\ C_4)$ *with*

$$\begin{array}{lcll} C_1 & = & - & \mathrm{sign}((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)); \\ C_2 & = & & \mathrm{sign}((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{j+1} - \mathsf{x}_j) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{j+1} - \mathsf{y}_j)); \\ C_3 & = & - & \mathrm{sign}((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)); \textit{ and} \\ C_4 & = & & \mathrm{sign}((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{j+1} - \mathsf{y}_j) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{j+1} - \mathsf{x}_j)). \end{array}$$

Proof. We have $\mathsf{p}_i = \mathsf{p}_j$ if and only if $\mathsf{x}_j - \mathsf{x}_i = 0$ and $\mathsf{y}_j - \mathsf{y}_i = 0$ and in this case the four (instantiated) polynomials in the statement of the theorem evaluate to zero.

Next, we assume $\mathsf{p}_i \neq \mathsf{p}_j$. We consider the following vectors in $\mathbf{R}^2$:

- $\overrightarrow{u_{ij}} = (\mathsf{x}_j - \mathsf{x}_i, \mathsf{y}_j - \mathsf{y}_i)$;
- $\overrightarrow{u_{ji}} = (\mathsf{x}_i - \mathsf{x}_j, \mathsf{y}_i - \mathsf{y}_j)$;
- $\overrightarrow{v_i} = (\mathsf{x}_{i+1} - \mathsf{x}_i, \mathsf{y}_{i+1} - \mathsf{y}_i)$; and
- $\overrightarrow{v_j} = (\mathsf{x}_{j+1} - \mathsf{x}_j, \mathsf{y}_{j+1} - \mathsf{y}_j)$.

[3] Now, we are not talking about *located* vectors like before, but vectors in the common sense.
[4] The inner product is also called scalar product.

We remark that $\overrightarrow{u_{ij}} = -\overrightarrow{u_{ji}}$ and that the vectors $\overrightarrow{u_{ij}}$, $\overrightarrow{v_i}$ and $\overrightarrow{v_j}$ (in the common sense of the word vector) are the (located) vectors $\overrightarrow{\mathsf{p}_i\mathsf{p}_j}$, $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$ translated to the origin of $\mathbf{R}^2$.

• C_1: Now, we apply the above cosine-formula to $\overrightarrow{a} = \overrightarrow{u_{ij}}$ and $\overrightarrow{b} = \overrightarrow{v_i}$ to obtain an expression for C_1. Because C_1 is negative towards p_j, we get the minus-sign in the following expression for C_1:

$$\begin{aligned} C_1 &= -\text{sign}(\overrightarrow{u_{ij}} \cdot \overrightarrow{v_i}) \\ &= -\text{sign}((\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)). \end{aligned}$$

• C_2: Next, we apply the cosine-formula to $\overrightarrow{a} = \overrightarrow{u_{ji}}$ and $\overrightarrow{b} = \overrightarrow{v_j}$ to obtain an expression for C_2. Again, because C_2 is negative towards p_i, we get the minus-sign in $C_2 = -\text{sign}(\overrightarrow{u_{ji}} \cdot \overrightarrow{v_j})$. This means that

$$\begin{aligned} C_2 &= \text{sign}(\overrightarrow{u_{ij}} \cdot \overrightarrow{v_j}) \\ &= \text{sign}((\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{x}_{j+1} - \mathsf{x}_j) + (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{y}_{j+1} - \mathsf{y}_j)). \end{aligned}$$

• C_3: Here, we apply the cosine-formula to $\overrightarrow{a} = \overrightarrow{u_{ij}}^{\perp}$ and $\overrightarrow{b} = \overrightarrow{v_i}$ and get $C_3 = -\text{sign}(\overrightarrow{u_{ij}}^{\perp} \cdot \overrightarrow{v_i})$. We have a minus-sign here, because $C_3 = -$ in the direction of $\overrightarrow{u_{ij}}^{\perp}$. Since $\overrightarrow{u_{ij}}^{\perp} = (-(\mathsf{y}_j - \mathsf{y}_i), \mathsf{x}_j - \mathsf{x}_i)$, we get

$$\begin{aligned} C_3 &= -\text{sign}(\overrightarrow{u_{ij}}^{\perp} \cdot \overrightarrow{v_i}) \\ &= \text{sign}((\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) - (\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)). \end{aligned}$$

• C_4: Finally, we apply the cosine-formula to $\overrightarrow{a} = \overrightarrow{u_{ji}}^{\perp}$ and $\overrightarrow{b} = \overrightarrow{v_j}$. Since $C_4 = -$ in the direction of $\overrightarrow{u_{ji}}^{\perp}$, we have $C_4 = -\text{sign}(\overrightarrow{u_{ji}}^{\perp} \cdot \overrightarrow{v_j})$. Since $\overrightarrow{u_{ji}}^{\perp} = (\mathsf{y}_j - \mathsf{y}_i, -(\mathsf{x}_j - \mathsf{x}_i))$, we get

$$\begin{aligned} C_4 &= -\text{sign}(\overrightarrow{u_{ji}}^{\perp} \cdot \overrightarrow{v_j}) \\ &= \text{sign}(-(\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{x}_{j+1} - \mathsf{x}_j) + (\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)). \end{aligned}$$

This concludes the proof. □

In the following property, we show that the double-cross value (0 0 0 0), which, for reasons of continuity, is the value in the case $\mathsf{p}_i = \mathsf{p}_j$ (see Definition 2), can only occur in that exceptional case.

Proposition 1 *Let* $P = \langle(\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), \ldots, (\mathsf{x}_N, \mathsf{y}_N)\rangle$ *be a polyline and let* $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$. *Then,* $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (0\ 0\ 0\ 0)$ *if and only if* $\mathsf{p}_i = \mathsf{p}_j$.

Proof. As already observed in the proof of Theorem 1, $\mathsf{p}_i = \mathsf{p}_j$ implies $\mathsf{x}_j - \mathsf{x}_i = 0$ and $\mathsf{y}_j - \mathsf{y}_i = 0$ and in this case the four (instantiated) polynomials of Theorem 1 evaluate to zero. This implies that $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (0\ 0\ 0\ 0)$.

For the converse, we have to show that if the four polynomials evaluate to zero, then $\mathsf{p}_i = \mathsf{p}_j$. We prove this by assuming $\mathsf{p}_i \neq \mathsf{p}_j$ and deriving a contradiction. If $\mathsf{p}_i \neq \mathsf{p}_j$, then $\mathsf{x}_j - \mathsf{x}_i \neq 0$ or $\mathsf{y}_j - \mathsf{y}_i \neq 0$. First, we consider the case $\mathsf{x}_j - \mathsf{x}_i \neq 0$.

As a first subcase, we consider the case $\mathsf{y}_j - \mathsf{y}_i = 0$. Then we get from the equations $C_1 = 0$ and $C_3 = 0$ that $(\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) = 0$ and $(\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i) = 0$. Since $\mathsf{x}_j - \mathsf{x}_i \neq 0$ is assumed, this implies that $\mathsf{x}_{i+1} - \mathsf{x}_i = 0$ and $\mathsf{y}_{i+1} - \mathsf{y}_i = 0$. This contradicts the assumption in Definition 1, which says that no two consecutive vertices of a polyline are identical.

As a second subcase, we consider the case $\mathsf{y}_j - \mathsf{y}_i \neq 0$. Then we get from $C_1 = 0$ that

$$\mathsf{x}_{i+1} - \mathsf{x}_i = \frac{-(\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)}{\mathsf{x}_j - \mathsf{x}_i} = \frac{(\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)}{\mathsf{y}_j - \mathsf{y}_i}.$$

From $C_3 = 0$, we get $(\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i) = (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i)$.

Combined, these two equalities imply $((\mathsf{x}_j - \mathsf{x}_i)^2 + (\mathsf{y}_j - \mathsf{y}_i)^2) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i) = 0$. Since in this case $(\mathsf{x}_j - \mathsf{x}_i)^2 + (\mathsf{y}_j - \mathsf{y}_i)^2 > 0$, we conclude $\mathsf{y}_{i+1} - \mathsf{y}_i = 0$. But then, again using the equation for C_1, we get $(\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) = 0$, or $\mathsf{x}_{i+1} - \mathsf{x}_i = 0$. So, we have both $\mathsf{x}_{i+1} - \mathsf{x}_i = 0$ and $\mathsf{y}_{i+1} - \mathsf{y}_i = 0$, which again contradicts the assumption in Definition 1. We have contradiction in all cases and this concludes the proof of the first case. The case $\mathsf{y}_j - \mathsf{y}_i \neq 0$ has a completely analogous proof, now using $C_2 = 0$ and $C_4 = 0$ instead of $C_1 = 0$ and $C_3 = 0$. This concludes the proof. □

We end this section by remarking that all the factors appearing in the algebraic expressions, given by the theorem, that is $\mathsf{x}_j - \mathsf{x}_i$, $\mathsf{x}_{i+1} - \mathsf{x}_i$, $\mathsf{y}_j - \mathsf{y}_i$, $\mathsf{y}_{i+1} - \mathsf{y}_i$, $\mathsf{x}_{j+1} - \mathsf{x}_j$ and $\mathsf{y}_{j+1} - \mathsf{y}_j$ are differences in x-coordinate or differences in y-coordinate values.

4 Some properties of double-cross matrices that can be derived from their algebraic characterisation

In this section, we give some basic properties of double-cross matrices of polylines. In most cases, these properties can be derived from the algebraic characterization of the entries of a double-cross matrix, that we presented in previous section.

4.1 Symmetry in the double-cross matrix of a polyline

In Section 2.2.2, we have already announced by the example polyline given in Figure 6 with its double-cross matrix given in Table 1, that a double-cross matrix exhibits symmetry properties. We prove these properties in this section. The first property is by definition, the second needs some inspection of polynomials. The conclusion is that it is enough to know a double-cross matrix *above its diagonal*.

Proposition 2 *If* $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N \rangle$ *is a polyline, then* $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) = (0\ 0\ 0\ 0)$, *for* $0 \leq i < N$. □

The following property says how $\mathsf{DC}(\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}})$ can be derived from $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}})$ in a straightforward way.

Proposition 3 *Let* $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N \rangle$ *be a polyline. If* $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (C_1\ C_2\ C_3\ C_4)$, *then* $\mathsf{DC}(\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) = (C_2\ C_1\ C_4\ C_3)$.

Proof. Let $P = \langle (\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N) \rangle$ be a polyline and let $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$. We use the polynomials given in Theorem 1 to prove this result. Essentially, what we do is to interchange the role of i and j. If $i = j$, nothing has to be shown. So, we assume $i \neq j$. If we interchange in

$$
\begin{array}{lcll}
C_1 & = & - & \text{sign}((\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)); \\
C_2 & = & & \text{sign}((\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{x}_{j+1} - \mathsf{x}_j) + (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{y}_{j+1} - \mathsf{y}_j)); \\
C_3 & = & - & \text{sign}((\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i)); \text{ and} \\
C_4 & = & & \text{sign}((\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{y}_{j+1} - \mathsf{y}_j) - (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{x}_{j+1} - \mathsf{x}_j)).
\end{array}
$$

the role of i and j, we get $\mathsf{DC}(\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) = (C_1'\ C_2'\ C_3'\ C_4')$, with

$$
\begin{array}{lcll}
C_1' & = & - & \mathrm{sign}((\mathsf{x}_i - \mathsf{x}_j)\cdot(\mathsf{x}_{j+1} - \mathsf{x}_j) + (\mathsf{y}_i - \mathsf{y}_j)\cdot(\mathsf{y}_{j+1} - \mathsf{y}_j));\\
C_2' & = & & \mathrm{sign}((\mathsf{x}_i - \mathsf{x}_j)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_i - \mathsf{y}_j)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i));\\
C_3' & = & - & \mathrm{sign}((\mathsf{x}_i - \mathsf{x}_j)\cdot(\mathsf{y}_{j+1} - \mathsf{y}_j) - (\mathsf{y}_i - \mathsf{y}_j)\cdot(\mathsf{x}_{j+1} - \mathsf{x}_j)); \text{ and}\\
C_4' & = & & \mathrm{sign}((\mathsf{x}_i - \mathsf{x}_j)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_i - \mathsf{y}_j)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)).
\end{array}
$$

It is easy to see that $C_1' = C_2$, $C_2' = C_1$, $C_3' = C_4$ and $C_4' = C_3$. □

These two properties imply that only the $\frac{N\cdot(N-1)}{2}$ entries above the diagonal of the double-cross matrix of a polyline are significant.

4.2 The double-cross value of consecutive line segments

The following property says what the entries in the double-cross matrix of two successive line segments $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}}$ in a polyline $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N\rangle$ look like. These values correspond to entries in the double-cross matrix just above (or below) its diagonal.

Proposition 4 *Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N\rangle$ be a polyline. Of $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}})$ the entries C_1 and C_3 are fixed to $-$ and 0. That is,*

$$\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}}) = (-\ C_2\ 0\ C_4),$$

for any $0 \le i < N-1$.

Proof. Let $0 \le i < N-1$. We start with the entry $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}})[1] = -\mathrm{sign}((\mathsf{x}_{i+1} - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_{i+1} - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) = -\mathrm{sign}((\mathsf{x}_{i+1} - \mathsf{x}_i)^2 + (\mathsf{y}_{i+1} - \mathsf{y}_i)^2)$. Because of the assumption in Definition 1, we have $(\mathsf{x}_{i+1} - \mathsf{x}_i)^2 + (\mathsf{y}_{i+1} - \mathsf{y}_i)^2 > 0$ and we can conclude that $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}})[1] = -$.

For the third entry we have $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}})[3] = -\mathrm{sign}((\mathsf{x}_{i+1} - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_{i+1} - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) = -\mathrm{sign}(0) = 0$. This concludes the proof. □

The following property shows that more values depend on one another.

Proposition 5 *Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N\rangle$ be a polyline and let $1 \le i < N-1$. If $\mathsf{DC}(\overrightarrow{\mathsf{p}_{i-1}\mathsf{p}_i}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}) = (-\ C_2\ 0\ C_4)$, with $C_2 = +$ or 0, then $\mathsf{DC}(\overrightarrow{\mathsf{p}_{i-1}\mathsf{p}_i}, \overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}}) = (-\ C_2'\ C_4\ C_4')$, for some $C_2', C_4' \in \{-, 0, +\}$.*

Proof. Let $\mathsf{DC}(\overrightarrow{\mathsf{p}_{i-1}\mathsf{p}_i}, \overrightarrow{\mathsf{p}_{i+1}\mathsf{p}_{i+2}})$ be $(C_1'\ C_2'\ C_3'\ C_4')$. We have the following expressions:

$$
\begin{array}{lcl}
C_2 & = & \mathrm{sign}((\mathsf{x}_i - \mathsf{x}_{i-1})\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_i - \mathsf{y}_{i-1})\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i))\\
C_4 & = & \mathrm{sign}((\mathsf{x}_i - \mathsf{x}_{i-1})\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_i - \mathsf{y}_{i-1})\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i))\\
C_1' & = & -\mathrm{sign}((\mathsf{x}_{i+1} - \mathsf{x}_{i-1})\cdot(\mathsf{x}_i - \mathsf{x}_{i-1}) + (\mathsf{y}_{i+1} - \mathsf{y}_{i-1})\cdot(\mathsf{y}_i - \mathsf{y}_{i-1}))\\
C_3' & = & -\mathrm{sign}((\mathsf{x}_{i+1} - \mathsf{x}_{i-1})\cdot(\mathsf{y}_i - \mathsf{y}_{i-1}) - (\mathsf{y}_{i+1} - \mathsf{y}_{i-1})\cdot(\mathsf{x}_i - \mathsf{x}_{i-1}))
\end{array}
$$

Let us abbreviate the first two expression as $C_2 = \mathrm{sign}(c_2)$ and $C_4 = \mathrm{sign}(c_4)$ and the latter two as $C_1' = -\mathrm{sign}(c_1')$ and $C_3' = -\mathrm{sign}(c_3')$.

Then we have $c_1' = ((\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{x}_i - \mathsf{x}_{i-1}))\cdot(\mathsf{x}_i - \mathsf{x}_{i-1}) + ((\mathsf{y}_{i+1} - \mathsf{y}_i) + (\mathsf{y}_i - \mathsf{y}_{i-1}))\cdot(\mathsf{y}_i - \mathsf{y}_{i-1}) = (\mathsf{x}_i - \mathsf{x}_{i-1})^2 + (\mathsf{y}_i - \mathsf{y}_{i-1})^2 + c_2$. Since, by assumption $c_2 \ge 0$, it follows from the assumption in Definition 1 that $c_1' > 0$ and thus $C_1' = -\mathrm{sign}(c_1') = -$.

Further, we have $c_3' = ((\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{x}_i - \mathsf{x}_{i-1}))\cdot(\mathsf{y}_i - \mathsf{y}_{i-1}) - ((\mathsf{y}_{i+1} - \mathsf{y}_i) + (\mathsf{y}_i - \mathsf{y}_{i-1}))\cdot(\mathsf{x}_i - \mathsf{x}_{i-1}) = -c_4$. So, $C_3' = -\mathrm{sign}(c_3') = -\mathrm{sign}(-c_4) = C_4$. This concludes the proof. □

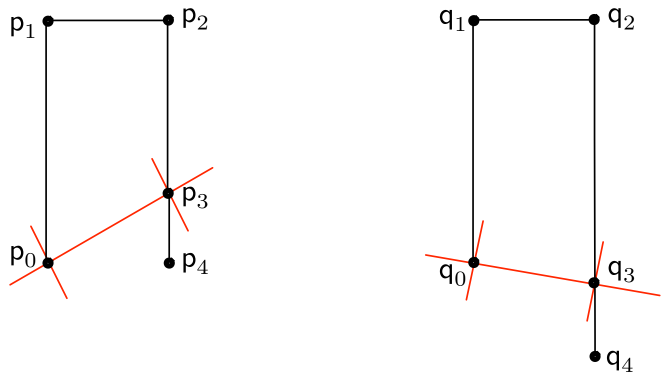

Figure 8: Two polylines that differ in the length of their third segment.

4.3 On the length of line segments of a polyline

The following properties shows that changing the length of segments in a polyline may or may not influence certain entries in its double-cross matrix.

Proposition 6 *Let $P = \langle(\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N)\rangle$ be a polyline and let $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$, for $0 \leq i \leq N$. Changing the length of $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$ does not influence the value of $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}})$.*

Proof. If we take $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}) = (C_1\ C_2\ C_3\ C_4)$ and $\mathsf{DC}(\overrightarrow{\mathsf{p}_i\mathsf{p}'_{i+1}}, \overrightarrow{\mathsf{p}_j\mathsf{p}'_{j+1}}) = (C'_1\ C'_2\ C'_3\ C'_4)$, where $\overrightarrow{\mathsf{p}_i\mathsf{p}'_{i+1}}$ is $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$ scaled by a factor c, with $c > 0$ and $\overrightarrow{\mathsf{p}_j\mathsf{p}'_{j+1}}$ is $\overrightarrow{\mathsf{p}_j\mathsf{p}_{j+1}}$ scaled by a factor d, with $d > 0$, then we first observe that $\mathsf{p}'_{i+1} = (x_i + c\cdot(x_{i+1} - x_i), y_i + c\cdot(y_{i+1} - y_i)$ and $\mathsf{p}'_{j+1} = (x_j + c\cdot(x_{j+1} - x_j), y_j + c\cdot(y_{j+1} - y_j)$. If we abbreviate It is then easily verified that $C'_1 = -\text{sign}(c\cdot c_1) = -\text{sign}(c_1) = C_1$, since $c > 0$. Similarly, we get $C'_2 = \text{sign}(d\cdot c_2) = \text{sign}(c_2) = C_2$, $C'_3 = -\text{sign}(c\cdot c_3) = -\text{sign}(c_3) = C_3$ and $C'_4 = \text{sign}(d\cdot c_4) = \text{sign}(c_4) = C_4$, since also $d > 0$. This concludes the proof. □

The length of the last segment of a polyline does not influence the double-cross matrix. Only its direction matters. This follows straightforwardly from the definition.

Proposition 7 *Let $P = \langle\mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N\rangle$ be a polyline. Changing the length of $\overrightarrow{\mathsf{p}_{N-1}\mathsf{p}_N}$ does not change $\mathsf{DCM}(P)$.* □

For segments, that differ from the last, this is not the case, as the following property shows.

Proposition 8 *Let $P = \langle\mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N\rangle$ be a polyline. Changing the length of $\overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}}$, for $0 \leq i < N - 1$, may change $\mathsf{DCM}(P)$.* □

Proof. Consider the polylines $P = \langle\mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4\rangle$ and $Q = \langle\mathsf{q}_0, \mathsf{q}_1, \mathsf{q}_2, \mathsf{q}_3, \mathsf{q}_4\rangle$ of Figure 8. They only differ in the length of their third segment. For P, we have $\mathsf{DC}(\overrightarrow{\mathsf{p}_0\mathsf{p}_1}, \overrightarrow{\mathsf{p}_3\mathsf{p}_4}) = (-\ -\ -\ -)$, whereas for Q, we have $\mathsf{DC}(\overrightarrow{\mathsf{q}_0\mathsf{q}_1}, \overrightarrow{\mathsf{q}_3\mathsf{q}_4}) = (+\ +\ -\ -)$. □

4.4 The quadrant of points of a polyline

In Section 6, we will see that we can apply a similarity transformation to a polyline without changing its double-cross matrix. Without loss of generality, we may therefore assume that the first line segment of the polyline is the unit interval of the x-axis, that is, $\mathsf{p}_0 = (0,0)$ and $\mathsf{p}_1 = (1,0)$.

The following property states that we can derive the quadrants in which all the other points are located from the double-cross matrix.

Proposition 9 *Let* $P = \langle \mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_N \rangle$ *be a polyline and assume that* $\mathsf{p}_0 = (0,0)$ *and* $\mathsf{p}_1 = (1,0)$. *Let* $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$ *for* $2 \leq i \leq N$. *From* $\mathsf{DC}(\overrightarrow{\mathsf{p}_0\mathsf{p}_1}, \overrightarrow{\mathsf{p}_i\mathsf{p}_{i+1}})$, *we can determine* $\mathrm{sign}(\mathsf{x}_i)$ *and* $\mathrm{sign}(\mathsf{y}_i)$.

Proof. It is clear that $C_1 = -\mathrm{sign}((\mathsf{x}_i - 0)\cdot 1 + \mathsf{y}_i \cdot 0) = -\mathrm{sign}(\mathsf{x}_i)$ and that $C_3 = -\mathrm{sign}(\mathsf{x}_i \cdot 0 + \mathsf{y}_i \cdot 1) = -\mathrm{sign}(-\mathsf{y}_i)$. □

5 A geometric characterization of the double-cross similarity of two polylines

In this section, we define the *local carrier order* of a polyline. This is a geometric concept and the main result of this section is a characterization of double-cross similarity of two polylines in terms of their local carrier orders.

5.1 The local carrier order of a polyline

Here, we give the definition of the local carrier order of a polyline. First, we introduce some notation for half-lines.

Definition 5 Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_N \rangle$ be a polyline in $\mathbf{R}^2$ and let $0 \leq i < N$. If $\mathsf{p}_i \neq \mathsf{p}_j$, the (directed) half-line starting in p_i through p_j will be denoted by $\overline{\mathsf{p}_i\mathsf{p}_j}$. The half-line, also starting in p_i, but in the opposite direction is denoted $-\overline{\mathsf{p}_i\mathsf{p}_j}$. The half-lines $\overline{\mathsf{p}_i\mathsf{p}_j}$ and $-\overline{\mathsf{p}_i\mathsf{p}_j}$, for $0 \leq j \leq N$ with $j \neq i$ and $\mathsf{p}_j \neq \mathsf{p}_i$, are called the *carriers at* p_i.

The perpendicular half-line on $\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$ starting in p_i directing to the right of $\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$ (that is, making a 90° clockwise angle with $\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$) as $\overline{\mathsf{p}_i}^{\perp_r}$ and the opposite perpendicular half-line starting in p_i as $\overline{\mathsf{p}_i}^{\perp_\ell}$. The half-lines $\overline{\mathsf{p}_i}^{\perp_r}$ and $\overline{\mathsf{p}_i}^{\perp_\ell}$ are called the *perpendiculars at* p_i. □

For $0 \leq i < N$, the vertex p_i has $2N$ carriers and 2 perpendiculars. For an illustration of the half-lines of and of this single cross between p_i and p_{i+1}, we refer to Figure 9.

Now, we define the local carrier order of a vertex p_i of a polyline P, for $0 \leq i < N$. This local carrier order consists of nine sets. One keeps track which p_j's are equal to p_i and the other eight are corresponding to eight directions of a compass card. We use the image of a 8-point compass with the northern cardinal direction in the direction of the half-line $\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$ to name these sets.

In the following definition, we say that a half-line $\overline{\ell}$ is *strictly between* the two perpendicular half-lines $\overline{\ell_1}$ and $\overline{\ell_2}$, if they all have the same starting point and $\overline{\ell}$ is in the quadrant determined by $\overline{\ell_1}$ and $\overline{\ell_2}$ (following the counter-clockwise direction).

Definition 6 Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_N \rangle$ be a polyline in $\mathbf{R}^2$. For $0 \leq i < N$, we define the following nine sets for the vertex p_i:

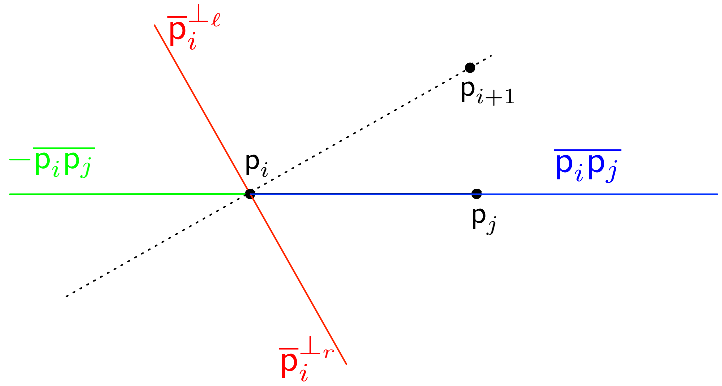

Figure 9: An example the half-lines $\overline{\mathsf{p}_i\mathsf{p}_j}$ (in blue), $-\overline{\mathsf{p}_i\mathsf{p}_j}$ (in green) and the two perpendicular half-lines $\overline{\mathsf{p}_i}^{\perp_r}$ and $\overline{\mathsf{p}_i}^{\perp_\ell}$ (in red).

- $\mathsf{N}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ equal to $\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$;
- $\mathsf{NE}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ strictly between $\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$ and $\overline{\mathsf{p}_i}^{\perp_r}$;
- $\mathsf{E}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ equal to $\overline{\mathsf{p}_i}^{\perp_r}$;
- $\mathsf{SE}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ strictly between $\overline{\mathsf{p}_i}^{\perp_r}$ and $-\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$;
- $\mathsf{S}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ equal to $-\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$;
- $\mathsf{SW}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ strictly between $-\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$ and$\overline{\mathsf{p}_i}^{\perp_\ell}$;
- $\mathsf{W}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ equal to $\overline{\mathsf{p}_i}^{\perp_\ell}$; and
- $\mathsf{NW}(\mathsf{p}_i)$ is the set of $\overline{\mathsf{p}_i\mathsf{p}_j}$ strictly between $\overline{\mathsf{p}_i}^{\perp_\ell}$ and $\overline{\mathsf{p}_i\mathsf{p}_{i+1}}$,

with $0 \le j < i$ or $i < j < N$. Finally, $\mathsf{Eq}(\mathsf{p}_i)$ is the set of p_j that are equal to p_i. The *local carrier order of P in its vertex* p_i, for $0 \le i < N$, denoted as $\mathsf{LCO}(P, \mathsf{p}_i)$, is the list of sets

$$\langle \mathsf{Eq}(\mathsf{p}_i), \mathsf{N}(\mathsf{p}_i), \mathsf{NE}(\mathsf{p}_i), \mathsf{E}(\mathsf{p}_i), \mathsf{SE}(\mathsf{p}_i), \mathsf{S}(\mathsf{p}_i), \mathsf{SW}(\mathsf{p}_i), \mathsf{W}(\mathsf{p}_i), \mathsf{NW}(\mathsf{p}_i)\rangle$$

and the *local carrier order of P* is the the list

$$\langle \mathsf{LCO}(P, \mathsf{p}_0), \mathsf{LCO}(P, \mathsf{p}_1), ..., \mathsf{LCO}(P, \mathsf{p}_{N-1})\rangle.$$

□

We remark that if $\mathsf{p}_j \in \mathsf{Eq}(\mathsf{p}_i)$, then the half-line $\overline{\mathsf{p}_i\mathsf{p}_j}$ makes no sense and therefor does not appear in any of the sets $\mathsf{N}(\mathsf{p}_i)$, ..., $\mathsf{NW}(\mathsf{p}_i)$.

As an illustration we use the polyline P depicted in Figure 10. Here, the local carrier orders in the vertices are given by:

- $\mathsf{LCO}(P, \mathsf{p}_0) = \langle \{\}, \{\overline{\mathsf{p}_0\mathsf{p}_1}\}, \{\overline{\mathsf{p}_0\mathsf{p}_2}, \overline{\mathsf{p}_0\mathsf{p}_4}\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\overline{\mathsf{p}_0\mathsf{p}_3}\}\rangle$
- $\mathsf{LCO}(P, \mathsf{p}_1) = \langle \{\}, \{\overline{\mathsf{p}_1\mathsf{p}_2}\}, \{\}, \{\}, \{\overline{\mathsf{p}_1\mathsf{p}_0}\}, \{\}, \{\}, \{\}, \{\overline{\mathsf{p}_1\mathsf{p}_3}, \overline{\mathsf{p}_1\mathsf{p}_4}\}\rangle$
- $\mathsf{LCO}(P, \mathsf{p}_2) = \langle \{\}, \{\}, \{\overline{\mathsf{p}_2\mathsf{p}_4}\}, \{\}, \{\}, \{\}, \{\overline{\mathsf{p}_2\mathsf{p}_0}\}, \{\}, \{\overline{\mathsf{p}_2\mathsf{p}_3}\}\rangle$

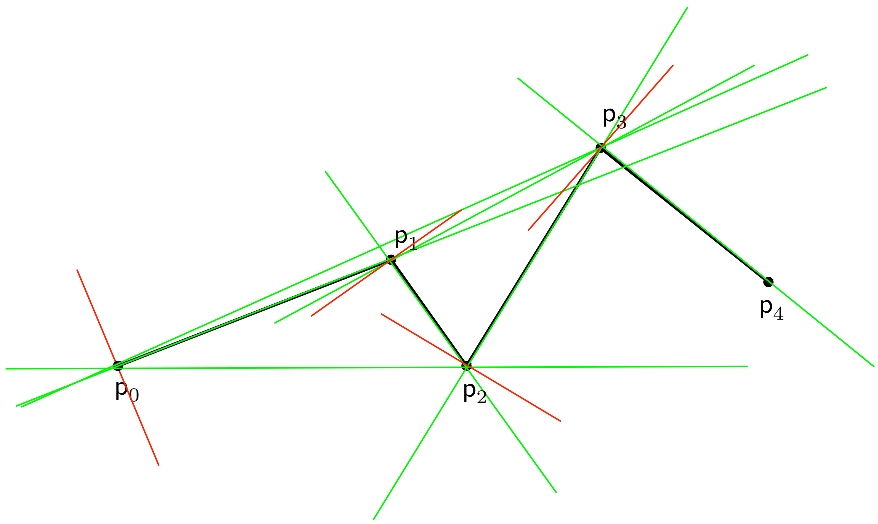

Figure 10: A polyline $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \mathsf{p}_2, , \mathsf{p}_3, \mathsf{p}_4 \rangle$ with its carriers (in green) and its perpendiculars (in red).

- $\mathsf{LCO}(P, \mathsf{p}_3) = \langle \{\}, \{\overline{\mathsf{p}_3\mathsf{p}_4}\}, \{\overline{\mathsf{p}_3\mathsf{p}_2}\}, \{\}, \{\overline{\mathsf{p}_3\mathsf{p}_1}, \overline{\mathsf{p}_3\mathsf{p}_0}\}, \{\}, \{\}, \{\}, \{\} \rangle$

We now define the notion of *local-carrier-order similarity* of two polylines.

Definition 7 Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_N \rangle$ and $Q = \langle \mathsf{q}_0, \mathsf{q}_1, , \ldots, \mathsf{q}_N \rangle$ be polylines of equal size. We say that P and Q are *local-carrier-order similar* if $\mathsf{LCO}(P, \mathsf{p}_i) = \mathsf{LCO}(Q, \mathsf{q}_i)$ for all $i = 0, 1, ..., N-1$, that is, if $\mathsf{LCO}(P) = \mathsf{LCO}(Q)$ (always, modulo changing p_i in q_i). □

5.2 An algebraic characterization of the local carrier order of a polyline

In this section, we give algebraic conditions to express the local carrier order of a polyline. Hereto, it suffices to give for each vertex p_i, with $0 \leq i < N$, in the polyline $P = \langle \mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_N \rangle$ characterizations of the sets in the list

$$\langle \mathsf{Eq}(\mathsf{p}_i), \mathsf{N}(\mathsf{p}_i), \mathsf{NE}(\mathsf{p}_i), \mathsf{E}(\mathsf{p}_i), \mathsf{SE}(\mathsf{p}_i), \mathsf{S}(\mathsf{p}_i), \mathsf{SW}(\mathsf{p}_i), \mathsf{W}(\mathsf{p}_i), \mathsf{NW}(\mathsf{p}_i) \rangle.$$

The following property gives this characterization. The proof of this property uses the same algebraic tools as the proof of Theorem 1 and we will skip the (straightforward) details.

We remark that, obviously, the algebraic characterisation of $\mathsf{Eq}(\mathsf{p}_i)$ is given by equalities on the coordinates.

Proposition 10 *Let $P = \langle (\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N) \rangle$ be a polyline and let $\mathsf{p}_i = (\mathsf{x}_i, \mathsf{y}_i)$, for $0 \leq i \leq N$. For $0 \leq j < i$ or $i < j < N$, the following table gives algebraic conditions for the halfline $\overline{\mathsf{p}_i\mathsf{p}_j}$ to belong to $\mathsf{X}(\mathsf{p}_i)$ with $\mathsf{X} \in \{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}\}$:*

$\mathsf{X} =$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{X}(\mathsf{p}_i)$ *is equivalent to*
N	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) < 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) = 0$
NE	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) < 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) < 0$
E	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) = 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) < 0$
SE	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) > 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) < 0$
S	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) > 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) = 0$
SW	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) > 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) > 0$
W	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) = 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) > 0$
NW	$-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i)) < 0$ *and* $-((\mathsf{x}_j - \mathsf{x}_i)\cdot(\mathsf{y}_{i+1} - \mathsf{y}_i) - (\mathsf{y}_j - \mathsf{y}_i)\cdot(\mathsf{x}_{i+1} - \mathsf{x}_i)) > 0$

□

5.3 A characterization of double-cross similarity of polylines in terms of their local carrier order

In this section, we give a geometric characterization of double-cross similarity of polylines in terms of their local carrier orders. The main theorem that we prove in this section is the following.

Theorem 2 *Let P and Q be polylines of equal size. Then, P and Q are double-cross similar if and only if they are local-carrier-order similar. That is*

$$\mathsf{DCM}(P) = \mathsf{DCM}(Q) \quad \textit{if and only if} \quad \mathsf{LCO}(P) = \mathsf{LCO}(Q).$$

The two directions of this theorem are proven in Lemma 1 and Lemma 2 (or their immediate Corollaries 1 and 2).

Lemma 1 *Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N \rangle$ be a polyline. For i, j with $0 \le i \le j < N$, we can derive the value of the 4-tuple $\mathsf{DCM}(P)[i, j] = (C_1\ C_2\ C_3\ C_4)$ from $\mathsf{LCO}(P, \mathsf{p}_i)$ and $\mathsf{LCO}(P, \mathsf{p}_j)$.*

Proof. Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N \rangle$ be a polyline of size N. If $\mathsf{p}_j \in \mathsf{Eq}(\mathsf{p}_i)$, then $\mathsf{DCM}(P)[i, j] = (0\ 0\ 0\ 0)$. This is in particular true if $i = j$.

If $\mathsf{p}_j \notin \mathsf{Eq}(\mathsf{p}_i)$, then the following twelve easily observable facts show how to determine C_1, C_2, C_3 and C_4 (for instance, by a detailed inspection of Figure 5).

C_1	is equivalent to
0	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{W}(\mathsf{p}_i) \cup \mathsf{E}(\mathsf{p}_i)$
$+$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{SE}(\mathsf{p}_i) \cup \mathsf{S}(\mathsf{p}_i) \cup \mathsf{SW}(\mathsf{p}_i)$
$-$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{NW}(\mathsf{p}_i) \cup \mathsf{N}(\mathsf{p}_i) \cup \mathsf{NE}(\mathsf{p}_i)$

C_2	is equivalent to
0	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{W}(\mathsf{p}_j) \cup \mathsf{E}(\mathsf{p}_j)$
$+$	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{SE}(\mathsf{p}_j) \cup \mathsf{S}(\mathsf{p}_j) \cup \mathsf{SW}(\mathsf{p}_j)$
$-$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{NW}(\mathsf{p}_i) \cup \mathsf{N}(\mathsf{p}_i) \cup \mathsf{NE}(\mathsf{p}_i)$

C_3	is equivalent to
0	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{N}(\mathsf{p}_i) \cup \mathsf{S}(\mathsf{p}_i)$
$+$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{SW}(\mathsf{p}_i) \cup \mathsf{W}(\mathsf{p}_i) \cup \mathsf{NW}(\mathsf{p}_i)$
$-$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{NE}(\mathsf{p}_i) \cup \mathsf{E}(\mathsf{p}_i) \cup \mathsf{SE}(\mathsf{p}_i)$

C_4	is equivalent to
0	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{N}(\mathsf{p}_j) \cup \mathsf{S}(\mathsf{p}_j)$
$+$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{SW}(\mathsf{p}_j) \cup \mathsf{W}(\mathsf{p}_j) \cup \mathsf{NW}(\mathsf{p}_j)$
$-$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{NE}(\mathsf{p}_i) \cup \mathsf{E}(\mathsf{p}_i) \cup \mathsf{SE}(\mathsf{p}_i)$

This concludes the proof. □

This lemma has the following immediate corollary.

Corollary 1 *Let P be a polyline in $\mathbf{R}^2$. Then, the matrix $\mathsf{DCM}(P)$ can be reconstructed from the local carrier order $\mathsf{LCO}(P)$.*

Proof. Properties 2 and 3 show that it is sufficient to know a double-cross matrix of a polyline on and above its diagonal in order to complete it below its diagonal. And Lemma 1 shows how the local carrier order gives the double-cross matrix on and above its diagonal. This concludes the proof. □

Now, we turn to the other implication of Theorem 2.

Lemma 2 *Let $P = \langle \mathsf{p}_0, \mathsf{p}_1, , \ldots, \mathsf{p}_N \rangle$ be a polyline in $\mathbf{R}^2$ of size N. If $0 \leq i < j < N$, then $\mathsf{DCM}(P)[i,j]$ contains enough information to derived to which set of $\mathsf{LCO}(P, \mathsf{p}_i)$ the half-lines $\overline{\mathsf{p}_i\mathsf{p}_j}$ belong and to which set of $\mathsf{LCO}(P, \mathsf{p}_j)$ the half-lines $\overline{\mathsf{p}_j\mathsf{p}_i}$ belong.*

Proof. Let $\mathsf{DCM}(P)[i,j] = (C_1\ C_2\ C_3\ C_4)$, for $0 \leq i < j < N$. Again, the following facts are easily observable (for instance, by a detailed inspection of Figure 5).

C_1	C_3	is equivalent to
$-$	$-$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{NE}(\mathsf{p}_i)$
$-$	0	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{N}(\mathsf{p}_i)$
$-$	$+$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{NW}(\mathsf{p}_i)$
0	$-$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{E}(\mathsf{p}_i)$
0	0	$\mathsf{p}_i = \mathsf{p}_{i+1}$ is excluded
0	$+$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{W}(\mathsf{p}_i)$
$+$	$-$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{SE}(\mathsf{p}_i)$
$+$	0	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{S}(\mathsf{p}_i)$
$+$	$+$	$\overline{\mathsf{p}_i\mathsf{p}_j} \in \mathsf{SW}(\mathsf{p}_i)$

C_2	C_4	is equivalent to
$-$	$-$	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{NE}(\mathsf{p}_j)$
$-$	0	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{N}(\mathsf{p}_j)$
$-$	$+$	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{NW}(\mathsf{p}_j)$
0	$-$	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{E}(\mathsf{p}_j)$
0	0	$\mathsf{p}_j = \mathsf{p}_{j+1}$ is excluded
0	$+$	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{W}(\mathsf{p}_j)$
$+$	$-$	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{SE}(\mathsf{p}_j)$
$+$	0	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{S}(\mathsf{p}_j)$
$+$	$+$	$\overline{\mathsf{p}_j\mathsf{p}_i} \in \mathsf{SW}(\mathsf{p}_j)$

This concludes the proof. □

This lemma has the following immediate corollary.

Corollary 2 *Given* $\mathsf{DCM}(P)$*,* $\mathsf{LCO}(P, \mathsf{p}_i)$ *can be constructed for all* $0 \leq i < N$*.* □

Combined, Corollaries 1 and 2 prove Theorem 2.

6 A characterization of the double-cross invariant transformations of the plane

In this section, we identify the transformations[5] of the plane $\mathbf{R}^2$ that leave the double-cross matrix of all polylines invariant.

If $\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ is a transformation and if p and q are points in $\mathbf{R}^2$, then we write $\alpha(\overrightarrow{\mathsf{pq}})$ for $\overrightarrow{\alpha(\mathsf{p})\alpha(\mathsf{q})}$.

What do we mean by applying a transformation of the plane to a polyline? The following definition says that we mean it to be the polyline formed by the transformed vertices.

Definition 8 Let $\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ be a transformation. Let $P = \langle (\mathsf{x}_0, \mathsf{y}_0), (\mathsf{x}_1, \mathsf{y}_1), , \ldots, (\mathsf{x}_N, \mathsf{y}_N) \rangle$ be a polyline. We define $\alpha(P)$ to be the polyline $\langle \alpha(\mathsf{x}_0, \mathsf{y}_0), \alpha(\mathsf{x}_1, \mathsf{y}_1), , \ldots, \alpha(\mathsf{x}_N, \mathsf{y}_N) \rangle$. □

We remark that since a transformation α is a bijective function, the assumption in Definition 1, which says that no two consecutive vertices of a polyline are identical, will hold for $\alpha(P)$ if it holds for the polyline P.

We now define the notion of double-cross invariant transformation of the plane.

Definition 9 Let $\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ be a transformation. Let P be a polyline. We say that α *leaves* P *invariant* if P and $\alpha(P)$ are double-cross similar, that is, if $\mathsf{DCM}(P) = \mathsf{DCM}(\alpha(P))$.

We say that α is a *double-cross invariant* transformation if it leaves all polylines invariant. A group of transformations of $\mathbf{R}^2$ is *double-cross invariant* if all its members are double-cross invariant transformations. □

The main aim of this section is to prove the following theorem, which says that the largest group of transformations that is double-cross invariant consists of the translations, rotations and homotecies (or scalings)[6] of the plane. The elements of this group are called the *similarities* of $\mathbf{R}^2$.

Theorem 3 *The largest group of transformations of* $\mathbf{R}^2$*, that is double-cross invariant consist of the* similarity transformations *of the plane onto itself, that is, transformations of the form*

$$\alpha : \mathbf{R}^2 \to \mathbf{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \mathsf{c} \cdot \begin{pmatrix} \mathsf{a} & -\mathsf{b} \\ \mathsf{b} & \mathsf{a} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \mathsf{d} \\ \mathsf{e} \end{pmatrix},$$

where $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e} \in \mathbf{R}$*,* $\mathsf{c} \neq 0$ *and* $\mathsf{a}^2 + \mathsf{b}^2 = 1$*.* □

[5] A transformation is a continuous, bijective mapping of the plane $\mathbf{R}^2$ onto itself.

[6] A homotecy of the plane is a transformation of the form $\alpha_{\mathsf{c}} : \mathbf{R}^2 \to \mathbf{R}^2 : (x, y) \mapsto \mathsf{c} \cdot (x, y)$, where $c \in \mathbf{R}$ and $\mathsf{c} \neq 0$.

We remark that the condition $\mathsf{a}^2 + \mathsf{b}^2 = 1$ implies that a and b cannot be both zero. In fact, we can see a as $\cos\varphi$ and b as $\sin\varphi$, where φ is the angle of the rotation expressed by the matrix.

We prove this theorem by proving three lemma's. Lemma 3 proves *soundness* and Lemma 5 proves *completeness*. Lemma 4 is a purely technical lemma.

Lemma 3 *The translations, rotations and homotecies of the plane (that is, the transformations given in Theorem 1) are double-cross invariant transformations.*

Proof. We consider the three types of transformations separately, since we can apply them one after the other. In all cases, we use the algebraic characterization, given by Theorem 1.

1. Translations. We have already remarked that all the factors appearing in the algebraic expressions, given by given by Theorem 1, that is $(\mathsf{x}_j - \mathsf{x}_i)$, $(\mathsf{x}_{i+1} - \mathsf{x}_i)$, $(\mathsf{y}_j - \mathsf{y}_i)$, $(\mathsf{y}_{i+1} - \mathsf{y}_i)$, $(\mathsf{x}_{j+1} - \mathsf{x}_j)$ and $(\mathsf{y}_{j+1} - \mathsf{y}_j)$ are differences in x-coordinates or differences in y-coordinates. A translation $\tau_{(\mathsf{d},\mathsf{e})} : \mathbf{R}^2 \to \mathbf{R}^2 : (x, y) \mapsto (x + \mathsf{d}, y + \mathsf{e})$, therefore leaves these differences unaltered. For instance, $(\mathsf{x}_j - \mathsf{x}_i)$ is transformed to $(\mathsf{x}_j + \mathsf{d} - (\mathsf{x}_i + \mathsf{d}))$, which is, of course, the original value $(\mathsf{x}_j - \mathsf{x}_i)$. None of the expressions given by Theorem 1 are therefore changed and the double-cross condition remain the same.

2. Rotations. Let

$$\rho_{(\mathsf{a},\mathsf{b})} : \mathbf{R}^2 \to \mathbf{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{a} & -\mathsf{b} \\ \mathsf{b} & \mathsf{a} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

with $\mathsf{a}^2 + \mathsf{b}^2 = 1$, be a rotation (that fixes the origin).

The expression for C_1 is transformed to

$$(\mathsf{a} \cdot (\mathsf{x}_j - \mathsf{x}_i) - \mathsf{b} \cdot (\mathsf{y}_j - \mathsf{y}_i)) \cdot (\mathsf{a} \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) - \mathsf{b} \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)) + \\ (\mathsf{b} \cdot (\mathsf{x}_j - \mathsf{x}_i) + \mathsf{a} \cdot (\mathsf{y}_j - \mathsf{y}_i))) \cdot (\mathsf{b} \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) + \mathsf{a} \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i)),$$

which simplifies to $(\mathsf{a}^2 + \mathsf{b}^2) \cdot ((\mathsf{x}_j - \mathsf{x}_i) \cdot (\mathsf{x}_{i+1} - \mathsf{x}_i) + (\mathsf{y}_j - \mathsf{y}_i) \cdot (\mathsf{y}_{i+1} - \mathsf{y}_i))$, which is the original polynomial since $\mathsf{a}^2 + \mathsf{b}^2 = 1$. For C_2, C_3 and C_4, a similar straightforward computation shows that the polynomials remain the same.

3. Homotecies. A homotecy $\alpha_{\mathsf{c}} : \mathbf{R}^2 \to \mathbf{R}^2 : (x, y) \mapsto \mathsf{c} \cdot (x, y)$, transforms the differences $(\mathsf{x}_j - \mathsf{x}_i)$, $(\mathsf{x}_{i+1} - \mathsf{x}_i)$, $(\mathsf{y}_j - \mathsf{y}_i)$, $(\mathsf{y}_{i+1} - \mathsf{y}_i)$, $(\mathsf{x}_{j+1} - \mathsf{x}_j)$ and $(\mathsf{y}_{j+1} - \mathsf{y}_j)$ to $(\mathsf{c} \cdot \mathsf{x}_j - \mathsf{c} \cdot \mathsf{x}_i)$, $(\mathsf{c} \cdot \mathsf{x}_{i+1} - \mathsf{c} \cdot \mathsf{x}_i)$, $(\mathsf{c} \cdot \mathsf{y}_j - \mathsf{c} \cdot \mathsf{y}_i)$, $(\mathsf{c} \cdot \mathsf{y}_{i+1} - \mathsf{c} \cdot \mathsf{y}_i)$, $(\mathsf{c} \cdot \mathsf{x}_{j+1} - \mathsf{c} \cdot \mathsf{x}_j)$ and $(\mathsf{c} \cdot \mathsf{y}_{j+1} - \mathsf{c} \cdot \mathsf{y}_j)$. This means that the polynomials given by Theorem 1 are multiplied by c^2, which is strictly larger than zero, for $\mathsf{c} \neq 0$. The signs of these polynomials are therefore unaltered. And the double-cross value of the scaled polyline is the same as the original one. □

Before we can turn to completeness, we need the following technical lemma.

Lemma 4 *Let $f : \mathbf{R} \to \mathbf{R} : t \mapsto f(t)$ be a strictly monotone increasing function. If*

$$f(\frac{s+t}{2}) = \frac{f(s) + f(t)}{2}$$

for any $s, t \in \mathbf{R}$, then $f(t) = (f(1) - f(0)) \cdot t + f(0)$.

Proof. Suppose that f is a function as described and suppose that there is a $t_0 \in \mathbf{R}$ such that $f(t_0) \neq (f(1) - f(0)) \cdot t_0 + f(0)$. We remark that therefore t_0 cannot be 0 or 1.

If $f(-t_0) = (f(1) - f(0)) \cdot (-t_0) + f(0)$, then it follows from $2 \cdot f(0) = 2 \cdot f(\frac{t_0 - t_0}{2}) = f(t_0) + f(-t_0)$ that also $f(t_0) = (f(1) - f(0)) \cdot t_0 + f(0)$. We may therefore assume $0 < t_0$.

If $f(\frac{t_0}{2}) = (f(1) - f(0)) \cdot (\frac{t_0}{2}) + f(0)$, then it follows from $2 \cdot f(\frac{0+t_0}{2}) = f(0) + f(t_0)$ that also $f(t_0) = (f(1) - f(0)) \cdot t_0 + f(0)$. We may therefore assume $0 < t_0 < 1$.

Claim. For any $n \in \mathbf{N}$ and any k, with $0 \leq k \leq 2^n$, we have that

$$f(\frac{k}{2^n}) = (f(1) - f(0)) \cdot \frac{k}{2^n} + f(0).$$

We first prove this claim (by induction on n). For $n = 0$, and $k = 0, 1$, we respectively have $f(0) = (f(1) - f(0)) \cdot 0 + f(0)$ and $f(1) = (f(1) - f(0)) \cdot 1 + f(0)$.

Assume now that the claim is true for n. We prove it holds for $n + 1$. We consider $\frac{k}{2^{n+1}}$ and distinguish between the cases, $0 \leq k \leq 2^n$ and $k = k' + 2^n$ with $0 < k' \leq 2^n$. If $0 \leq k \leq 2^n$, then $f(\frac{k}{2^{n+1}}) = f(\frac{1}{2}(0 + \frac{k}{2^n})) = \frac{1}{2}(f(0) + f(\frac{k}{2^n}))$, which by the induction hypothesis equals $\frac{1}{2}(f(0) + (f(1) - f(0)) \cdot \frac{k}{2^n} + f(0))$ or $(f(1) - f(0)) \cdot \frac{k}{2^{n+1}} + f(0)$.

If $k = k' + 2^n$ with $0 < k' \leq 2^n$, then $f(\frac{2^n + k'}{2^{n+1}}) = f(\frac{1}{2}(1 + \frac{k'}{2^n})) = \frac{1}{2}(f(1) + f(\frac{k'}{2^n}))$, which by the induction hypothesis equals $\frac{1}{2}(f(1) + (f(1) - f(0)) \cdot \frac{k'}{2^n} + f(0))$ which equals $\frac{1}{2}(f(1) - f(0)) + (f(1) - f(0)) \cdot \frac{k'}{2^{n+1}} + f(0)$ or $(f(1) - f(0)) \cdot \frac{k' + 2^n}{2^{n+1}} + f(0)$ which is $(f(1) - f(0)) \cdot \frac{k}{2^{n+1}} + f(0)$. This concludes the proof of the claim. □

To conclude the proof, let $0 < t_0 < 1$ and assume first that $f(t_0) > (f(1) - f(0)) \cdot t_0 + f(0)$. This means that $t_0 < \frac{f(t_0) - f(0)}{f(1) - f(0)}$. We remark that since f is assumed to be strictly monotone, $f(1) - f(0) \neq 0$ and therefore the division is allowed. Choose k and n such that

$$\frac{k}{2^n} \leq t_0 < \frac{k+1}{2^n} < \frac{f(t_0) - f(0)}{f(1) - f(0)},$$

with $0 \leq k \leq 2^n$. Then $f(\frac{k+1}{2^n}) = (f(1) - f(0)) \cdot \frac{k+1}{2^n} + f(0) < f(t_0)$, although $t_0 < \frac{k+1}{2^n}$, which contradicts the fact that f is strictly monotone increasing.

If we assume $f(t_0) < (f(1) - f(0)) \cdot t_0 + f(0)$ on the other hand, we have $\frac{f(t_0) - f(0)}{f(1) - f(0)} < t_0$. Choose k and n such that

$$\frac{f(t_0) - f(0)}{f(1) - f(0)} < \frac{k}{2^n} < t_0 \leq \frac{k+1}{2^n},$$

with $0 \leq k \leq 2^n$. Then $f(\frac{k}{2^n}) = (f(1) - f(0)) \cdot \frac{k}{2^n} + f(0) > f(t_0)$, although $\frac{k}{2^n} < t_0$, which contradicts the fact that f is strictly monotone increasing. In both cases, we obtain a contradiction and this concludes the proof. □

The following lemma proves completeness.

Lemma 5 *The similarity transformations of the plane (given in Theorem 1) are the only double-cross invariant transformations.*

Proof. Let $\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ be a double-cross invariant transformation.

(1) We consider polylines $P = \langle \mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2 \rangle$, where p_0, p_1 and p_2 are collinear points with p_1 between p_0 and p_2. By the assumption in Definition 1, p_1 should be *strictly* between p_0 and p_2 The only relevant entry in the double-cross matrix of this polyline is $\mathsf{DC}(\overrightarrow{\mathsf{p}_0\mathsf{p}_1}, \overrightarrow{\mathsf{p}_1\mathsf{p}_2})$

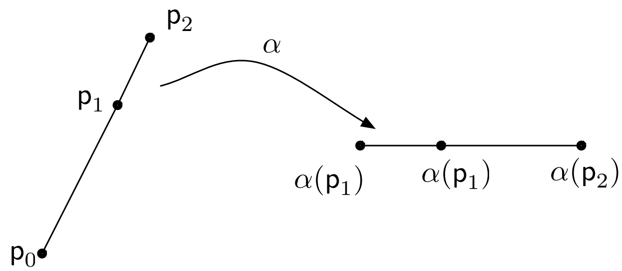

Figure 11: A collinearity and betweenness-preserving transformation of the plane.

which is $(-\ +\ 0\ 0)$. In $\alpha(P)$, $\mathsf{DC}(\alpha(\overrightarrow{\mathsf{p}_0\mathsf{p}_1}), \alpha(\overrightarrow{\mathsf{p}_1\mathsf{p}_2}))$ should also be $(-\ +\ 0\ 0)$. This implies that $\alpha(\mathsf{p}_0)$, $\alpha(\mathsf{p}_1)$ and $\alpha(\mathsf{p}_2)$ should also be collinear, with $\alpha(\mathsf{p}_1)$ (strictly) between $\alpha(\mathsf{p}_0)$ and $\alpha(\mathsf{p}_2)$. This means that α preserves *collinearity* and *betweenness*.

(2) We consider polylines $P = \langle \mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2 \rangle$, where $\angle(\overrightarrow{\mathsf{p}_1\mathsf{p}_0}, \overrightarrow{\mathsf{p}_1\mathsf{p}_2}) = 90°$, that is, the polyline takes a right turn at p_1. The only relevant entry in the double-cross matrix of this polyline is again $\mathsf{DC}(\overrightarrow{\mathsf{p}_0\mathsf{p}_1}, \overrightarrow{\mathsf{p}_1\mathsf{p}_2})$ which is now $(-\ 0\ 0\ -)$. In $\alpha(P)$, $\mathsf{DC}(\alpha(\overrightarrow{\mathsf{p}_0\mathsf{p}_1}), \alpha(\overrightarrow{\mathsf{p}_1\mathsf{p}_2}))$ should also be $(-\ 0\ 0\ -)$. This means that α is a *right-turn-preserving* transformation. This is illustrated in Figure 12. Similarly, α is a *left-turn-preserving* transformation.

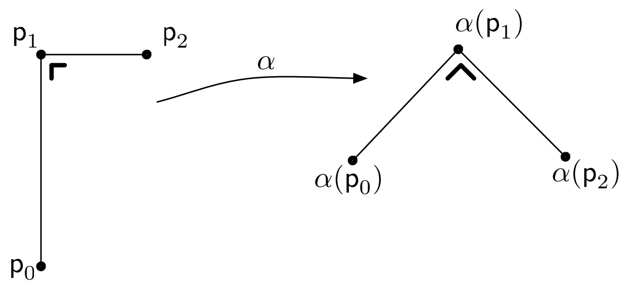

Figure 12: A right-turn-preserving transformation of the plane.

(3) We consider the polyline $P = \langle \mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_7, \mathsf{p}_8, \mathsf{p}_9, \mathsf{p}_{10}, \mathsf{p}_{11}, , \mathsf{p}_{12} \rangle$, with $\mathsf{p}_0 = \mathsf{p}_4 = \mathsf{p}_7 =, \mathsf{p}_{12}(0,0)$, $\mathsf{p}_1 = \mathsf{p}_5 = (0,1)$, $\mathsf{p}_2 = \mathsf{p}_{10} = (1,1)$, $\mathsf{p}_3 = \mathsf{p}_8 = (1,0)$ and $\mathsf{p}_6 = \mathsf{p}_9 = (\frac{1}{2}, \frac{1}{2})$, depicted in Figure 13. This polyline forms a square with its two diagonals after making six 90° right-turns and two 90° left-turns . It is also closed in the sense that its start and end vertex are equal. The transformation α, which according to (2) preserves right and left turns, therefore has to map P again to a square with its diagonals. This means α is a *square-preserving* transformation. In particular, α preserves parallel line segments. Also, p_6, which is the midpoint between p_0 and p_2 is mapped to $\alpha(\mathsf{p}_6)$, which should be the midpoint between $\alpha(\mathsf{p}_0)$ and $\alpha(\mathsf{p}_2)$. This means α is a *midpoint-preserving* transformation.

Suppose $\alpha(0,0) = (a,b)$. If $\tau_{(-a,-b)}$ is the translation $(x,y) \mapsto (x-a, y-b)$, then $\tau_{(-a,-b)} \circ \alpha(0,0) = (0,0)$. Suppose $\tau_{(-a,-b)} \circ \alpha(1,0) = (c,d)$. Let $\rho_{(c,d)}$ be the rotation with $(0,0)$ as center that brings (c,d) to the positive x-axis, that is, to $(\sqrt{c^2+d^2}, 0)$. We remark that (c,d) cannot be the origin since α is assumed to be a bijective function. So, also $\tau_{(-a,-b)} \circ \alpha$ is bijective. Furthermore, let $\sigma_{\sqrt{c^2+d^2}}$ be the scaling $(x,y) \mapsto \frac{1}{\sqrt{c^2+d^2}}(x,y)$ and let

$$\beta = \sigma_{\sqrt{c^2+d^2}} \circ \rho_{(c,d)} \circ \tau_{(-a,-b)} \circ \alpha.$$

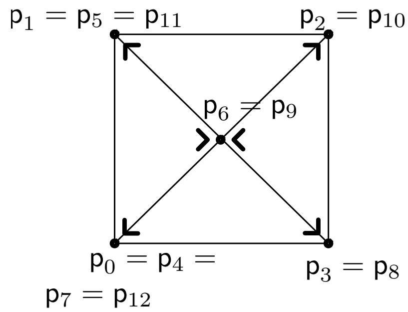

Figure 13: A polyline that is a square with its two diagonals. The six 90° right-turns are indicated in bold.

Then we have that $\beta(0,0) = (0,0)$ and $\beta(1,0) = (1,0)$.

Since α is a double-cross invariant transformation, by assumption, and since $\sigma_{\sqrt{c^2+d^2}}$, $\rho_{(c,d)}$ and $\tau_{(-a,-b)}$ are double-cross invariant transformations by Lemma 3, also β is a double-cross invariant transformation. And β also inherits from α the properties of preserving betweenness, collinearity, right- and left turns, squares, parallel line segments and midpoints. Because β preserves squares, we also have $\beta(0,1) = (0,1)$.

We now claim the following.

Claim: The transformation β is the identity.

The proof of this claim finishes the proof. Indeed, then we have

$$\alpha = \sigma^{-1}_{\sqrt{c^2+d^2}} \circ \rho^{-1}_{(c,d)} \circ \tau^{-1}_{(-a,-b)},$$

which is of the required form.

Proof of the claim: First, we show that β is the identity on the x-axis and next we do the same for all lines perpendicular to the x-axis. Hereto, we consider the function

$$\beta_x : \mathbf{R} \to \mathbf{R} : x \mapsto \beta_x(x) := \pi_x(\beta(x,0)),$$

where π_x is the projection on the first component, that is, $\pi_x(x,y) := x$. Since $\beta(0,0) = (0,0)$ and $\beta(1,0) = (1,0)$ and β preserves collinearity, β maps the x-axis onto the x-axis and we have $\beta_x(0) = 0$ and $\beta_x(1) = 1$. Furthermore, since β and hence β_x preserves betweenness, β_x is strictly monotone increasing. Indeed, let $s, t \in \mathbf{R}$ with $s < t$. With respect to 0 and 1, we can consider the twelve possible locations of s and t: $s < t < 0$; $s < t = 0 < 1$; $s < 0 < t < 1$; $s < 0 < t = 1$; $s < 0 < 1 < t$; $s = 0 < t < 1$; $s = 0 < t = 1$; $s = 0 < 1 < t$; $0 < s < t = 1$; $0 < s < 1 < t$; $0 < s = 1 < t$; and $0 < 1 < s < t$. In all cases, except $s = 0 < t = 1$, we have three points. So, here we can use the fact that β preserves betweenness to show that $\beta_x(s) < \beta_x(t)$. In the case $s = 0 < t = 1$, we have $\beta_x(s) = \beta_x(0) = 0 < 1 = \beta_x(1) = \beta_x(t)$. Finally, since β preserves midpoints, also for β_x, we have

$$\beta_x(\frac{s+t}{2}) = \frac{\beta_x(s) + \beta_x(t)}{2},$$

for all $s, t \in \mathbf{R}$. All the conditions to apply Lemma 4 are therefore fulfilled. And we get $\beta_x(x) = (\beta_x(1) - \beta_x(0)) \cdot x + \beta_x(0) = (1-0) \cdot x + 0 = x$.

Now, we fix some $\mathsf{x}_0 \in \mathbf{R}$ and consider the function

$$\beta_{\mathsf{x}_0,y} : \mathbf{R} \to \mathbf{R} : y \mapsto \beta_{\mathsf{x}_0,y}(y) := \pi_y(\beta(\mathsf{x}_0, y)),$$

where $\pi_y(x, y) := y$. Since β preserves parallel line segments, $\beta_{\mathsf{x}_0,y}$ maps the line with equation $x = \mathsf{x}_0$ onto itself (since it maps the y-axis to itself). Since β also preserves the rectangle given by the polyline

$$P = \langle(0,0),(0,1),(1,1),(\mathsf{x}_0,1),(\mathsf{x}_0,0),(1,0),(0,0),(0,1)\rangle$$

(for $\mathsf{x}_0 = 1$, we can omit $(\mathsf{x}_0, 1)$ and $(\mathsf{x}_0, 0)$ from the list) onto itself, we have again have $\beta_{\mathsf{x}_0,y}(0) = 0$ and $\beta_{\mathsf{x}_0,y}(1) = 1$. The function $\beta_{\mathsf{x}_0,y}$ also inherits from β the property of preserving midpoints and is strictly monotonic increasing on the line $x = \mathsf{x}_0$. So, again we can apply Lemma 4 to show that $\beta_{\mathsf{x}_0,y}$ is the identity.

Since $\beta(x, y) = (\beta_x(x), \beta_{x,y}(y))$, we obtain that β is the identity transformation. This finishes the proof of the claim and also of the lemma. □

7 Conclusion

We have studied the double-cross matrix descriptions of polylines in the two-dimensional plane from an algebraic and geometrical point of view. We have first given an algebraic characterization of the double-cross matrix of a polyline. This algebraic characterisation allowed us to prove some basic properties of double-cross matrices. We have give a geometric characterization of double-cross similarity of two polylines by means of the notion of the local carrier orders of polylines. To end, we identify the transformations of the plane that leave the double-cross matrix of all polylines in the two-dimensional plane invariant.

Research on double-cross matrices gives rise to many questions of which we name a few here. Firstly, variants of double crosses can be imagined, for instance, to include the temporal dimension of moving object data. Another variant would be to rotate the double cross by 45°. This would make notions of straight ahead, back, right and left more relative. We can also envisage double crosses with a finer structure. Thay may have 8, 16 or more lines determining the “crosses”.

Our algebraic characterization of the double-cross matrix of a polyline, also raises the question of the realisability of double-cross matrices. This question adds up to the following: given an $N \times N$ matrix of 4-tuples over the set $\{-, 0, +\}$, decide if this is the double-cross matrix of a polyline. A matrix where (0 0 0 0) doesn't appear on the diagonal, for instance, cannot be the double-cross matrix of a polyline. This decision problem leads to non-trivial problems in computational algebraic geometry.

8 Acknowledgments

A part of Section 3 has already appeared, as work of the present authors, in the form of a section in a conference extended abstract (Kuijpers et al., 2006). However, the current version of this section contains more technical details, full proofs and additional results. Most results in Section 5 have also appeared in the form of part of a conference extended abstract by the same authors (Kuijpers & Moelans, 2008), but for the special case of polylines on a grid. Here, these results are presented for arbitrary polylines and with detailed proofs.

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