A REMARK ON A THEOREM BY DELIGNE

M. VAN DEN BERGH

Abstract. We give a proof avoiding spectral sequences of Deligne's decomposition theorem for objects in a triangulated category admitting a Lefschetz homomorphism.

Below $\mathcal{A}$ is a triangulated category equipped with a bounded t-structure. In addition $\mathcal{A}$ will be equipped with an auto-equivalence $A \mapsto A(1)$ compatible with the t-structure.

In [1, 2] Deligne proves the following result:

Theorem 1. *Let A be an object of $\mathcal{A}$ equipped with an endomorphism $\phi : A[-1] \to A[1](1)$ such that its iterates $\phi^n : A[-n] \to A[n](n)$ induce isomorphisms $H^{-n}(A) \to H^n(A)(n)$. Then there exists an isomorphism $A \cong \oplus_k H^{-k}(A)[k]$.*

Deligne's proof is slightly indirect. He first shows that a decomposition as asserted in Theorem 1 exists if and only if for every cohomological functor $T : \mathcal{A} \to \mathcal{C}$ to an abelian category the resulting spectral sequence

$$E_2^{pq} : T^p(H^q(A)) \Rightarrow T^{p+q}(A) \tag{1}$$

degenerates. He then proceeds to show that (1) does indeed degenerate.

The aim of this note is to give a proof of Theorem 1 which avoids the use of spectral sequences.

Let A, ϕ be as in the statement of Theorem 1. We start with the following statement:

(Hyp_n) $A \cong A_n \oplus \left(\oplus_{|k|>n} H^{-k}(A)[k]\right)$ with $A_n \in \mathcal{A}^{[-n,n]}$.

By the boundedness of the t-structure (Hyp_n) is true for $n \gg 0$. We need to show that it is true for $n = 0$, so we use descending induction on n.

Assume (Hyp_n) is true for $n \geq 1$. We will show that (Hyp_{n-1}) is also true. Without loss of generality we may assume that the isomorphism in the statement of (Hyp_n) is an equality. Let $i : A_n \to A$, $p : A \to A_n$ be respectively the inclusion and the projection map. They induce identifications $H^q(A) = H^q(A_n)$ for $|q| \leq n$. Let α be the composition of the following maps

$$H^{-n}(A) \to A_n[-n] \xrightarrow{i} A[-n] \xrightarrow{\phi^n} A[n](n) \xrightarrow{p} A_n[n](n) \to H^n(A)(n)$$

where the first and last map are obtained from the canonical maps $H^{-n}(A_n)[n] \to A_n \to H^n(A_n)[-n]$ which exist because $A_n \in \mathcal{A}^{[-n,n]}$.

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The author is a senior researcher at the FWO.

Applying $H^0(-)$ and the hypotheses we find that $\alpha = H^0(\phi^n)$ and hence it is an isomorphism. Composing arrows we find that α is also the composition of maps

$$H^{-n}(A) \to A_n[-n] \to H^n(A)(n) \tag{2}$$

$$H^{-n}(A) \to A_n[n] \to H^n(A)(n) \tag{3}$$

inducing isomorphisms on H^0.

From (2) it follows that

$$A_n[-n] \cong H^{-n}(A) \oplus C \tag{4}$$

for some $C \in \mathcal{A}^{[1,2n]}$. Shifting and substituting in (3) we deduce that α is a composition

$$H^{-n}(A) \to H^{-n}(A)[2n] \oplus C[2n] \to H^n(A)(n)$$

Since $\mathrm{Hom}_{\mathcal{A}}(H^{-n}(A)[2n], H^n(A)(n)) = 0$ we see that α is actually a composition

$$H^{-n}(A) \to C[2n] \to H^n(A)(n)$$

and these maps still induce isomorphisms in degree zero. Thus $C[2n] \cong H^n(A)(n) \oplus D$ for $D \in \mathcal{A}^{[-2n+1,-1]}$. Shifting and substituting in (4) yields a decomposition

$$A_n \cong H^{-n}(A)[n] \oplus H^n(A)(n)[-n] \oplus D[-n]$$

Putting $A_{n-1} = D[-n]$ finishes the induction step and the proof.

Remark 2. It follows from the above proof that the decomposition asserted in Theorem 1 still exists if we have maps $\phi_n : A[-n] \to A[n](n)$ inducing isomorphisms $H^{-n}(A) \to H^n(A)(n)$ which are not necessarily powers of a fixed $\phi : A[-1] \to A[1](1)$. However I have no example where this extra generality applies.

Remark 3. In [2] Deligne constructs several canonical isomorphisms $A \cong \oplus_k H^{-k}(A)[k]$. We have not tried to duplicate these constructions with our approach.

References

[1] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 35, 259–278.

[2] P. Deligne, *Décompositions dans la catégorie dérivée*, Motives (Seattle, WA, 1991) (Providence, RI), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 115–128.

Departement WNI, Limburgs Universitair Centrum, 3590 Diepenbeek, Belgium.

E-mail address: vdbergh@luc.ac.be