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Samenvatting

Voor een commutatieve algebra A die eindig voortgebracht is over een lichaam k, werd het affiene schema $\operatorname{Spec} A$ gedefinieerd. Met een eindig voortgebrachte commutatieve gegradeerde algebra $A = k + A_1 + A_2 + \ldots$, associeert men een projectief schema $\operatorname{Proj} A$. In beide gevallen, werd ringtheorie gebruikt om de meetkundige structuur van deze schemas te beschrijven en omgekeerd, door gebruik te maken van onder andere het fundamentele artikel van Serre [33], kunnen we de meetkundige structuur gebruiken om iets te zeggen over de algebras. Men kan dit beschouwen als een reden om te proberen een meetkundig object te associëren met een niet-commutatieve gegradeerde algebra, dat ons meer kan vertellen over de algebra zelf en vice versa.

In dit werk, beschouwen we de niet-commutatieve gegradeerde algebras die regulier zijn in de zin van Artin en Schelter. In het bijzonder diegene die globale dimensie 3 hebben. Zij k een algebraïsch gesloten lichaam en $A = k + A_1 + A_2 + \ldots$ zo een drie dimensionale Artin-Schelter reguliere algebra, voortgebracht in graad één.

Artin, Tate en Van den Bergh associeerden in hun artikel [6] de volgende meetkundige data met A: een projectief schema E, een automorfisme σ van E en een inverteerbare schoof $\mathcal{L}$ op E. Zij zochten ook naar een weg terug en definieerden een algebra B, die isomorf is met A in het geval A lineair is, en met A/gA als A elliptisch is, waar g een regulier normaliserend element is van graad 3 of 4. Sinds het artikel [8], noemen we B de getwiste homogene coördinatenring van A en in dat artikel voorzagen Artin en Van den Bergh ons van een weg terug van de meetkundige gegevens naar de algebra zelf.

Dit mooie samenspel tussen ringtheoretische aspecten en niet-commutatieve meetkunde is weergegeven in Hoofdstuk 1. We gebruiken het in Hoofdstuk 2 om een beschrijving te geven van de categorie van de eindig voortgebrachte gegradeerde A-modulen van Gelfand-Kirillov dimensie één (modulo de eindig dimensionale over k). We vinden deze beschrijving door het belangrijkste resultaat van Hoofdstuk 2 te gebruiken in het geval $\mathcal{A}$ de categorie is van de gegradeerde rechtse A-modulen (modulo de torsiemodulen).

Voor het bewijs van Stelling 2.0.1, gebruiken we een resultaat van Gabriel (zie Stelling 2.1.1) en dit impliceert dat we enkele algemeenheden over pseudocompacte ringen nodig hebben. Omdat sommige lezers van dit werk, bang zouden kunnen worden bij de confrontatie met topologische ringen, hebben we Sectie 2.1-2.3 aan hen gewijd. Het blijkt dat voor een pseudocompact moduul M over een pseudocompacte ring A, behoorlijk wat eigenschappen geldig zijn in de categorie $\mathrm{PC}(A)$ van pseudocompacte modulen over A, als en slechts als ze geldig zijn in $\mathrm{Mod}(A)$, met of zonder bijkomende hypothesen. Bijvoorbeeld, $M \in \mathrm{PC}(A)$ is simpel, resp. noethers in $\mathrm{PC}(A)$ als en slechts als het simpel, resp. noethers is in $\mathrm{Mod}(A)$. Voor verdere informatie aangaande $\mathrm{PC}(A)$ en de connectie met $\mathrm{Mod}(A)$, verwijzen we naar Sectie 2.1. Om zich nog beter te voelen bij het werken met pseudocompacte ringen A, voorzien we de lezer van een matrixvoorstelling voor zo een ringen. Als $(e_i)_{i\in I}$ een sommeerbare verzameling is van orthogonale idempotenten in A zodat $\sum e_i = 1$ en we stellen dat $A_{ij} = e_iAe_j$, dan geldt, met een kleine hypothese, dat $A = \prod_{i,j} A_{ij}$ als topologische ruimten. We kunnen hetzelfde doen voor een pseudocompact A-moduul M, i.e. stel $M_i = e_iM$, dan is $M = \prod_i M_i$ als topologische ruimten. Dit werk werd gedaan in Sectie 2.2.

Nu, zonder het resultaat van Gabriel, kunnen we de categorie $\mathcal{C}_f$ die we willen beschrijven, reeds schrijven als $\oplus_{z\in E/<\tau>}\mathcal{C}_{f,z}$, met $\mathcal{C}_{f,z}$ een zekere volle deelcategorie van $\mathcal{C}_f$. Gabriel zorgt ervoor dat we precies weten wat $\mathcal{C}_{f,z}$ is. Meer bepaald, er volgt uit Stelling 2.1.1 dat de duale categorie van $\mathcal{C}_{f,z}$ equivalent is met de categorie van linkse pseudocompacte modulen met eindige lengte over een ring C_z. Dus als we de vorm van C_z kennen, zijn we klaar.

Van de ring C_z kunnen we bewijzen (zie Sectie 2.6) dat hij voldoet aan de hypothesen (A) en (B) van Sectie 2.4. Het is om die reden dat we in die laatste sectie de pseudocompacte ringen classificeren die voldoen aan (A) en (B). Hieruit volgt dan dat C_z kan voorgesteld worden door een matrix, waarvan de vorm afhankelijk is van de kardinaliteit van de τ-orbit $O_\tau(z)$ van z. In het geval dat $|O_\tau(z)| = \infty$, wordt C_z gegeven door de $\mathbb{Z}\times\mathbb{Z}$ beneden-driehoeksmatrices met coëfficiënten in een lokale pseudocompacte ring R. We vonden dat R isomorf is met $k[[x]]$. Als $|O_\tau(z)| < \infty$, dan maakt de matrixvoorstelling van C_z ook gebruik van een lokale pseudocompacte ring R maar hier was het moeilijker om de vorm van R te bepalen. Dit probleem werd opgelost in Sectie 2.5. Dus zijn we klaar met de beschrijving van $\mathcal{C}_f$, de categorie waarin we geinteresseerd waren en dit beeindigt ook het tweede hoofdstuk.

In Hoofdstuk 3, gaan we verder met de studie van drie dimensionale Artin-Schelter reguliere algebras, in het bijzonder diegene die 3 generatoren hebben van graad één en eindig zijn over hun centrum. Deze keer gebruiken we het samenspel tussen ringtheoretische aspecten en niet-commutatieve meetkunde, in tegenstelling tot het voorgaande hoofdstuk, om iets te besluiten over de meetkunde, meer bepaald, over het centrum van het projectieve schema Proj A.

Zij A dus een drie dimensionale 3 generator Artin-Schelter reguliere algebra die eindig is over zijn centrum R. Zij X de klassieke Proj van R en stel $\mathcal{O}_\Delta = \tilde{A}$. $\mathcal{O}_\Delta$ is een schoof van $\mathcal{O}_X$-algebras, waarvan we het centrum noteren door $\mathcal{Z}$. Grothendieck voorzag ons van een constructie van een projectief schema Z, waarvan de structuurschoof precies $\mathcal{Z}$ is. Dit projectief schema noemen we het centrum van Proj A en we vermoeden dat $Z \cong \mathbb{P}^2$.

Dit was reeds bewezen door Artin in het geval dat E glad is [3] en door Izusu Mori [25] in het geval dat E bestaat uit drie verschillende lijnen. Alhoewel we dit vermoeden niet in het algemeen kunnen bewijzen, zullen we aantonen dat het waar is als k een algebraïsch gesloten lichaam is van karakteristiek 0.

Onze manier om het vermoeden te bewijzen in dit geval, is zeer verschillend van de benaderingen van Artin en Mori. Artin maakt gebruik van de meetkunde van lijn en “vette” puntmodulen over A, terwijl Mori expliciete berekeningen gebruikt. Onze strategie start met het ontwikkelen van de eerste beginselen van een intersectietheorie voor $\mathcal{O}_\Delta$.

We moeten erop wijzen dat in [21], [26] alternatieve intersectietheorieën ontwikkeld werden voor niet-commutatieve ringen. Al die theorieën zijn vanzelfsprekend equivalent (op de doorsnede van hun definitiedomeinen) maar vertalingen tussen hen zijn soms moeilijk. Daarom prefereren we een eigen definitie te hanteren die sterker verbonden is met de theorie van orders.

We willen een intersectietheorie voor $\mathcal{O}_\Delta$ ontwikkelen omdat we de zelf-intersectie van het dualiserende moduul ω_Z van $\mathcal{Z}$ willen berekenen, in het geval dat A elliptisch is. Merk op dat, als A lineair is, $Z = E$ zodat we klaar zijn in dit geval. Stel $\mathcal{J} := (gA)\tilde{}$. Door een resultaat van Yekutieli [41], volgt dat het dualiserende moduul ω_Δ van $\mathcal{O}_\Delta$ gelijk is aan $\mathcal{J}$. Omdat we zullen bewijzen dat $\mathcal{O}_\Delta$ een maximaal order is, volgt hieruit dat $\mathcal{J}^s = \mathcal{O}_\Delta \otimes_{\mathcal{O}_Z} \omega_Z$, waar s de PI-graad is van $\mathcal{O}_\Delta$. Door gebruik te maken van onze intersectietheorie, volgt dat de zelf-intersectie van ω_Z gelijk is aan 9.

Aan de andere kant, omdat $\mathcal{J}^{-1}$ ampel is en in karakteristiek 0, $\mathcal{O}_Z$ een directe sommant is van $\mathcal{O}_\Delta$, hebben we dat ω_Z^{-1} ampel is op Z. Samen met het feit dat Z glad is, bepaalt dit de mogelijke vormen van het oppervlak Z. We beeindigen dit hoofstuk door te kijken naar de zelf-intersecties van deze mogelijkheden waaruit we kunnen besluiten dat $Z \cong \mathbb{P}^2$.

In het laatste hoofdstuk van dit werk, keren we terug naar de ringen R die we nodig hadden voor de beschrijving van de categorie $\mathcal{C}_f$ van Hoofdstuk 2. Meer bepaald beschouwen we ringen van de vorm $C = k\langle\langle x, y\rangle\rangle/(\psi)$, waar ψ alleen termen heeft van totale graad ≥ 2 en k een lichaam is van karakteristiek p. We vermoeden dat hun centrum $Z(C)$ ofwel triviaal is, ofwel een ring van formele machtreeksen in twee variabelen z en w. We geloven ook dat als het kwadratische deel van ψ van de vorm $yx - xy$ is en $p > 0$, de generatoren van $Z(C)$ de volgende vorm hebben: $z = x^{p^n} + \varphi(x)$, $w = y^{p^n} + \theta(x, y)$ voor een zekere $n > 0$ en waar φ en θ ofwel triviaal zijn, ofwel alleen termen bevatten in x, y van totale graad $> p^n$.

We bewijzen dit vermoeden in het geval C een Ore extensie $B[[y; \sigma, \delta]]$ is, met $B = k[[x]]$, σ een k-lineair automorfisme van B en δ een k-lineaire σ-derivatie van B. De gevallen waarin het centrum van C niet-triviaal is, zijn:

- $\sigma = \mathrm{id}$, $p > 0$.
- $\delta = 0$, $\sigma \neq \mathrm{id}$, $\mathrm{orde}(\sigma) < \infty$.
- $\delta \neq 0$, $\sigma \neq \mathrm{id}$, $\mathrm{orde}(\sigma) < \infty$.

In het eerste geval, kunnen we $z = x^p$ nemen en $w = y^p - c_p(x)y$, waar $c_p(x)$ het speciale element van B is gedefinieerd in Sectie 4.1. Als de x-adic valuatie $v(\delta(x))$ van $\delta(x) \geq 3$ is, dan is $v(c_p(x)) > p - 1$ waaruit het tweede deel van het vermoeden volgt in dit geval.

Als $\delta = 0$, σ niet-triviaal is en eindige orde n heeft, bekomen we dat $z = x\sigma(x)\ldots\sigma^{n-1}(x) \in B$ en $w = y^n$. Onder de hypothesen van het tweede deel van het vermoeden, bewijzen we dat z van de vorm x^n+ hogere orde termen is en $n = p^m$, voor een zekere m.

In het laatste geval, duikt hetzelfde element z van B op als in het voorgaande geval. Van w kunnen we enkel bewijzen dat het de vorm heeft, vereist door het vermoeden, maar we kunnen er geen mooie uitdrukking voor geven.

In de laatste sectie van dit werk, voorzien we de lezer van een nieuw bewijs van een resultaat van G. Baron en A. Schinzel. We hebben dit gevonden omdat we dit resultaat oorspronkelijk gebruikten toen we bewezen in het geval dat σ triviaal is en $p > 0$, dat $w = y^p - c_p(x)y \in Z(C)$. Later gebruikten we hiervoor een andere manier en die leidde ook tot het nieuwe bewijs. Waar het bewijs van [12] nogal technisch is, is het onze rechttoe rechtaan en het is gebasseerd op algemene berekeningen met derivaties.

Contents

Introduction

For a commutative algebra A which is finitely generated over a field k, the affine scheme $\operatorname{Spec} A$ was defined. To a finitely generated commutative graded algebra $A = k + A_1 + A_2 + \ldots$, one associates a projective scheme $\operatorname{Proj} A$. In both cases, ringtheory was used te describe the geometric structure of these schemes and vice versa, using amongst other things the fundamental paper of Serre [33], we can use the geometric structure to say something about the algebras. One could consider this as a reason to try to associate to a non-commutative graded algebra, a geometrical object, which can tell us more about the algebra itself and vice versa.

In this work, we consider the non-commutative graded algebras which are regular in the sense of Artin and Schelter. In particular those who have global dimension 3. Assume that k is an algebraically closed field. Let $A = k + A_1 + A_2 + \ldots$, be such a three dimensional Artin-Schelter regular algebra, generated in degree one.

Artin, Tate and Van den Bergh associated in their paper [6] the following geometric data to A: a projective scheme E, an automorphism σ of E and an invertible sheaf $\mathcal{L}$ on E. They were also looking for a way back and defined an algebra B, which is isomorphic to A in the case A is linear, and to A/gA when A is elliptic, where g is a regular normalizing element of degree 3 or 4. Since the paper [8], we call B the twisted homogeneous coordinate ring of A and in this paper Artin and Van den Bergh provided us with the way back from the geometric data to the algebra itself.

This nice interplay between ringtheoretical aspects and non-commutative geometry is presented in Chapter 1. We use it in Chapter 2 to give a description of the category of finitely generated graded A-modules of Gelfand-Kirillov dimension one (modulo those of finite dimension over k). We find this description, using the main result of Chapter 2 in the case $\mathcal{A}$ is the category of graded right A-modules (modulo torsion modules).

For the proof of Theorem 2.0.1, we use a result of Gabriel (see Theorem 2.1.1) and this involves generalities on pseudocompact rings. Since some of the readers of this work, might feel frightened when confronted with topological rings, we dedicated Section 2.1-2.3 to them. It turns out that for a pseudocompact module M over a pseudocompact ring A, quite some properties hold in the category $\mathrm{PC}(A)$ of pseudocompact modules over A, if and only if they hold in $\mathrm{Mod}(A)$, with or without some additional hypotheses. For instance, $M \in \mathrm{PC}(A)$ is simple, resp. noetherian in $\mathrm{PC}(A)$ if and only if it is simple, resp. noetherian in $\mathrm{Mod}(A)$. For further information on $\mathrm{PC}(A)$ and its connection with $\mathrm{Mod}(A)$, we refer to Section 2.1. To even feel more comfortable when working with pseudocompact rings A, we provide the reader with a matrix representation for such rings. If we assume that $(e_i)_{i\in I}$ is a summable set of orthogonal idempotents in A such that $\sum e_i = 1$ and put $A_{ij} = e_i A e_j$, then under some minor hypotheses, $A = \prod_{i,j} A_{ij}$ as topological spaces. The same can be done for a pseudocompact A-module M, i.e. put $M_i = e_i M$, then $M = \prod_i M_i$ as topological spaces. This work is done in Section 2.2.

Now, without the result of Gabriel, we are able already to write the category $\mathcal{C}_f$, which we want to describe, as $\oplus_{z\in E/<\tau>}\mathcal{C}_{f,z}$, where $\mathcal{C}_{f,z}$ is some full subcategory of $\mathcal{C}_f$. Further Gabriel sees to it that we know exactly what $\mathcal{C}_{f,z}$ is. That is, from Theorem 2.1.1 it follows that the dual category of $\mathcal{C}_{f,z}$ is equivalent to the category of left pseudocompact modules of finite length over a ring C_z. So if we know the form of C_z, we are through.

For the ring C_z, we can prove (see Section 2.6) that it satisfies the hypotheses (A) and (B) of Section 2.4. For this reason, we classify in this last section, pseudocompact rings satisfying (A) and (B). It follows that C_z can be represented by a matrix, whose form depends on the cardinality of the τ-orbit $O_\tau(z)$ of z. In the case that $|O_\tau(z)| = \infty$, C_z is given by the $\mathbb{Z} \times \mathbb{Z}$ lower triangular matrices with entries in a local pseudocompact ring R. We found that R is isomorphic to $k[[x]]$. If $|O_\tau(z)| < \infty$, the matrix representation of C_z also involves a local pseudocompact ring R but here it was harder to deduce the form of R. This problem is solved in Section 2.5. So we are finished with the description of $\mathcal{C}_f$, the category we were interested in and also with our second chapter.

In Chapter 3, we continue with the study of three dimensional Artin-Schelter regular algebras, in particular those who have 3 generators of degree one and are finite over their center. This time, we use the interplay between ringtheoretical aspects and non-commutative geometry, in contrast with the previous chapter, to decide something on the geometry, to be more precise, on the center of the projective scheme Proj A.

So let A be a three dimensional three generator Artin-Schelter regular algebra which is finite over its center R. Let X be the classical Proj of R and put $\mathcal{O}_\Delta = \tilde{A}$. $\mathcal{O}_\Delta$ is a sheaf of $\mathcal{O}_X$-algebras, whose center we denote by $\mathcal{Z}$. Grothendieck provided us with the construction of a projective scheme Z, whose structure sheaf is precisely $\mathcal{Z}$. This projective scheme is called the center of Proj A and we conjecture that $Z \cong \mathbb{P}^2$.

This was already proved by Artin in the case that E is smooth [3] and by Izusu Mori [25] in the case that E consists of three distinct lines. Although we are not able to prove the conjecture in general, we will show that it holds if k is an algebraically closed field of characteristic 0.

Our method for proving the conjecture in this case, is very different from the approaches by Artin and Mori. Artin uses the geometry of line and "fat" point modules over A, whereas Mori uses explicit computation. Our strategy starts with the development of the rudiments of an intersection theory for $\mathcal{O}_\Delta$.

We should point out that in [21], [26], alternative intersection theories for non-commutative rings were introduced. All these theories are of course equivalent (on the intersections of their domains of definition) but translating between them sometimes takes some effort. That is why we have preferred to use our own definition which is more directly tied to the theory of orders.

The reason for developping an intersection theory for $\mathcal{O}_\Delta$, is that we want to compute the self-intersection of the dualizing module ω_Z of $\mathcal{Z}$, in the case A is elliptic. Note that, when A is linear, $Z = E$ so we are through. Put $\mathcal{J} := (gA)\tilde{}$. By a result of Yekutieli [41], it follows that the dualizing module ω_Δ of $\mathcal{O}_\Delta$ is equal to $\mathcal{J}$. Since we will prove that $\mathcal{O}_\Delta$ is a maximal order, this yields $\mathcal{J}^s = \mathcal{O}_\Delta \otimes_{\mathcal{O}_Z} \omega_Z$, where s is the PI-degree of $\mathcal{O}_\Delta$. Using our intersection theory, it follows that ω_Z has self-intersection 9.

On the other hand, since $\mathcal{J}^{-1}$ is ample and in characteristic 0, $\mathcal{O}_Z$ is a direct summand of $\mathcal{O}_\Delta$, we deduce that ω_Z^{-1} is ample on Z. Together with the fact that Z is smooth, this determines the possible shapes of the surface Z. We finish this chapter with looking at the self-intersections of this possibilities for Z, from which we obtain that $Z \cong \mathbb{P}^2$.

In the final chapter of this work, we return to the rings R needed for the description of the category $\mathcal{C}_f$ of Chapter 2. To be more precise, we consider rings of the form $C = k\langle\langle x, y\rangle\rangle/(\psi)$, where ψ only has terms of total degree ≥ 2 and k is a field of characteristic p. We conjecture that their center $Z(C)$

is either trivial, or else a formal power series ring in two variables z and w. We also believe that if the quadratic part of ψ is of the form $yx - xy$ and $p > 0$, the generators of $Z(C)$ have the form $z = x^{p^n} + \varphi(x)$, $w = y^{p^n} + \theta(x, y)$ for some $n > 0$ and where φ and θ are either trivial, or else contain only terms in x, y of total degree $> p^n$.

We prove this conjecture in the case C is an Ore extension $B[[y; \sigma, \delta]]$, where $B = k[[x]]$, σ is a k-linear automorphism of B and δ is a k-linear σ-derivation of B. The cases in which the center of C is non-trivial are:

- $\sigma = \mathrm{id}$, $p > 0$.
- $\delta = 0$, $\sigma \neq \mathrm{id}$, $\mathrm{order}(\sigma) < \infty$.
- $\delta \neq 0$, $\sigma \neq \mathrm{id}$, $\mathrm{order}(\sigma) < \infty$.

In the first case, we can take $z = x^p$ and $w = y^p - c_p(x)y$, where $c_p(x)$ is the special element of B defined in Section 4.1. If the x-adic valuation $v(\delta(x))$ of $\delta(x)$ is ≥ 3, then $v(c_p(x)) > p-1$ which yields the second part of the conjecture in this case.

When $\delta = 0$, σ is non-trivial and has finite order n, we obtain that $z = x\sigma(x)\ldots\sigma^{n-1}(x) \in B$ and $w = y^n$. Under the hypotheses of the second part of the conjecture, we prove that z is of the form $x^n +$ higher order terms and $n = p^m$, for some m.

In the last case, the same element z of B appears as in the previous case. For w, we can only prove that it has the form required in the conjecture, but we can't give a nice expression for it.

In the final section of this work, we provide the reader with a new proof of a result by G. Baron and A. Schinzel. We found this since we originally relied on the result when proving in the case that σ is trivial and $p > 0$, that $w = y^p - c_p(x)y \in Z(C)$. Afterwards, we used another approach which also lead to the new proof. Whereas the proof in [12] is rather technical, ours is straightforward and relies on general computations with derivations.

Chapter 1

Preliminaries

Although it is strictly not necessary, we assume that k is an algebraically closed field.

In the Chapters 2 and 3 of this work, we consider graded algebras $A = k + A_1 + A_2 + \dots$ that are regular in the sense of Artin-Schelter. In this chapter we work out some generalities about these algebras which will be used in the following ones.

Throughout, let $A = k + A_1 + A_2 + \dots$ be a graded algebra that is generated by finitely many elements of degree one. We have the following definition.

Definition 1.0.1. *A is Artin-Schelter regular of dimension d if it satisfies the following conditions*

(i) A has finite global dimension d.

(ii) A has polynomial growth
i.e. $\dim_k A_n \leq cn^\delta$, for some positive real numbers c, δ.

(iii) A is Gorenstein, meaning that

$$\operatorname{Ext}^n_A(k, A) = \begin{cases} k & \text{if } n = d \\ 0 & \text{otherwise} \end{cases}$$

The Gelfand-Kirillov dimension of A is 1 more than the minimal δ in (ii). Note that for every regular algebra we know, the minimal such δ is equal to $d - 1$.

1.1 Three dimensional Artin-Schelter regular algebras

The three dimensional regular algebras generated in degree one, were classified in [5] and later in [6], [7]. As was shown in [5], there are two possibilities for such an algebra A.

- A has three generators of degree one and three defining relations of degree two.
- A has two generators of degree one and two defining relations of degree three.

The number of generators will be denoted by r, and the degrees of the defining relations by s. So the possible values are

$$(r,s) = \begin{cases} (3,2) \\ (2,3) \end{cases} \quad \text{and } r+s=5$$

Throughout this chapter, we assume that an algebra A can be presented by r generators of degree one and r defining relations of degree s, with (r,s) as above.

Definition 1.1.1. *1. Let A be an algebra which can be presented by r generators x_j of degree one and r relations $f_i = \sum_{j=1}^{r} m_{ij}x_j$ of degree s, such that, $(r,s)=(3,2)$ or $(2,3)$ as before. Let $M=(m_{ij})_{i,j}$ and write for the defining relations $f = Mx$. The algebra A is called standard if there is an invertible matrix $Q \in \mathrm{GL}_r(k)$ such that $x^tM = (Qf)^t$.*

2. A standard algebra A is nondegenerate if the rank of the matrix $M(p)$ is at least $r-1$ for all points $p \in \mathbb{P}^2$ if $(r,s)=(3,2)$, or for all points $p \in \mathbb{P}^1 \times \mathbb{P}^1$ if $(r,s)=(2,3)$.

We recall from [6]

Theorem 1.1.2. *1. The regular algebras of global dimension 3 generated in degree one are exactly the nondegenerate standard algebras.*

2. They are left and right noetherian.

Let A be a graded algebra as before. Write $\tilde{f}_i$ for the multilinearizations of the defining relations of A and let Γ denote the locus of common zeros of the $\tilde{f}_i$. Thus, $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ if $r = 3$, and $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ if $r = 2$. Define the projections

$$\begin{aligned} \mathrm{pr}_s(p^{(1)} \times \ldots \times p^{(s)}) \ &= (p^{(1)} \times \ldots \times p^{(s-1)}) \\ &\quad \text{(drop the last component)} \\ \mathrm{pr}_1(p^{(1)} \times \ldots \times p^{(s)}) \ &= (p^{(2)} \times \ldots \times p^{(s)}) \\ &\quad \text{(drop the first component)} \end{aligned}$$

If A is standard, then the images of Γ under these two projections are the same (see [6], §4). Let E be the image of Γ under these projections. Since $f = Mx$, it is clear that E is the locus of zeros of the multihomogenized polynomial $\det(\widetilde{M})$.

If $\det(\widetilde{M})$ is not identically zero, then E is a divisor of degree 3 in $\mathbb{P}^2$ in case $r = 3$, and a divisor of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ in the case $r = 2$. We then say that A is *elliptic*.
Otherwise, E is all of $\mathbb{P}^2$ if $r = 3$, and all of $\mathbb{P}^1 \times \mathbb{P}^1$ when $r = 2$. We call A *linear* in this case.

Since $E = \mathrm{pr}_s(\Gamma) = \mathrm{pr}_1(\Gamma)$, we may view Γ as the graph of a correspondence $\mathrm{pr}_s(p) \longrightarrow \mathrm{pr}_1(p)$ from E to itself. Observe that A is nondegenerate if and only if one of the projections $\mathrm{pr}_s, \mathrm{pr}_1 : \Gamma \longrightarrow E$ is an isomorphism ([6], Lemma 4.4). That is, if and only if Γ is the graph of an automorphism $\sigma : E \longrightarrow E$.

So to every three dimensional Artin-Schelter regular algebra A, generated in degree one, we can associate a triple $\mathcal{T}(A) = (E, \sigma, \mathcal{L})$, where E is the scheme $\mathrm{pr}_s(\Gamma)$, σ is the automorphism of E defined by Γ and $\mathcal{L}$ is the invertible $\mathcal{O}_E$-module $\pi^*\mathcal{O}(1)$, with π the inclusion of E in $\mathbb{P}^2$ if $r = 3$, or the projection of E on the first factor $\mathbb{P}^1$ if $r = 2$. In both cases the map $\pi : E \longrightarrow \mathbb{P}^{r-1}$ is the morphism defined by the sections of $\mathcal{L}$. Furthermore, in the case that $r = 2$, σ is of the form $\sigma(p, q) = (q, f(p, q))$, if A is elliptic and if A is linear, then σ has the form $\sigma(p, q) = (q, \tau(p))$, where τ is an isomorphism of $\mathbb{P}^1$.

1.2 Algebras associated to a triple

In this section we start with an abstract triple $\mathcal{T}$ and define from it two algebras $A = \mathcal{A}(\mathcal{T})$ and $B = \mathcal{B}(\mathcal{T})$, and a homomorphism $\beta = \beta(\mathcal{T}) : \mathcal{A}(\mathcal{T}) \longrightarrow \mathcal{B}(\mathcal{T})$.

Definition 1.2.1. *A triple $\mathcal{T}$ is a set $(E, \sigma, \mathcal{L})$, where E is a scheme, σ is an automorphism of E, and $\mathcal{L}$ is an invertible sheaf on E whose global sections define a morphism $\pi : E \longrightarrow \mathbb{P}^{r-1}$, and where one of the following holds:*

Case $r = 3$

(a) *E is a divisor of degree 3 in $\mathbb{P}^2$, and $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$.*

(b) *$E = \mathbb{P}^2$, and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$.*

Case $r = 2$

(a) *E is a divisor of bidegree $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, σ has the form $\sigma(p,q) = (q, f(p,q))$, and $\mathcal{L} = \mathrm{pr}^* \mathcal{O}_{\mathbb{P}^1}(1)$ where pr is the projection of E on the first factor $\mathbb{P}^1$.*

(b) *$E = \mathbb{P}^1 \times \mathbb{P}^1$, σ has the form $\sigma(p,q) = (q, \tau(p))$, where τ is an automorphism of $\mathbb{P}^1$, and $\mathcal{L} = \mathrm{pr}^* \mathcal{O}_{\mathbb{P}^1}(1)$ with pr as in (a).*

We will say that a triple of type (a) is *elliptic* and triples of type (b) will be called *linear*.

Definition 1.2.2. *A triple $\mathcal{T} = (E, \sigma, \mathcal{L})$ is said to be regular if it satisfies*

$$\begin{cases} (\sigma - 1)^2 \lambda = 0 & \text{if } r = 3 \\ (\sigma - 1)(\sigma^2 - 1)\lambda = 0 & \text{if } r = 2 \end{cases}$$

where λ denotes the class of $\mathcal{L}$ in $\mathrm{Pic}(E)$.

So let $\mathcal{T} = (E, \sigma, \mathcal{L})$ be a triple and $\pi : E \longrightarrow \mathbb{P}^{r-1}$ be the morphism determined by the global sections of $\mathcal{L}$.

Set $B_0 = \mathrm{H}^0(E, \mathcal{O}_E)$ and for each integer $n > 0$,

$$B_n = \mathrm{H}^0(E, \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}})$$

where $\mathcal{L}^{\sigma}$ is the pullback $\sigma^* \mathcal{L}$ and tensor products are taken over $\mathcal{O}_E$. We define

$$\mathcal{B}(\mathcal{T}) = B = \oplus_{n \geq 0} B_n$$

Multiplication on B is defined by the rule that if $a \in B_m$ and $b \in B_n$, then

$$a \cdot b = a \otimes b^{\sigma^m}$$

where $b^{\sigma^m} = b \circ \sigma^m$.

To define $\mathcal{A}(\mathcal{T})$, let $T = \sum T_n$ be the tensor algebra over k on $T_1 = \mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(1))$. The isomorphism

$$\pi^* : T_1 \longrightarrow B_1$$

induces a homomorphism $T \longrightarrow B$. Let $J = \sum J_n$ be its kernel, and let I be the two-sided ideal of T generated by J_s. We define

$$\mathcal{A}(\mathcal{T}) = A = T/I$$

The composition of the natural homomorphisms

$$T/I \longrightarrow T/J \longrightarrow B$$

yields a canonical homomorphism $\beta(\mathcal{T}) = \beta : A \longrightarrow B$ which is bijective in degree 1.

Recall from Theorem 6.8 in [6].

Theorem 1.2.3. *Let $\mathcal{T}$ be a triple. Let $A = \mathcal{A}(\mathcal{T})$ and $B = \mathcal{B}(\mathcal{T})$. Then*

1. *β is always surjective.*
2. *If $\mathcal{T}$ is linear, then β is an isomorphism.*
3. *If $\mathcal{T}$ is elliptic and regular, the kernel of β has the form $gA = Ag$, where g is a non-zero normalizing element of degree $s+1$.*
4. *If $\mathcal{T}$ is regular, then A is a regular algebra of global dimension 3, and in the elliptic case, the element g of (3.) is left and right regular.*

Now let A be a three dimensional regular algebra generated in degree one and $\mathcal{T}(A) = (E, \sigma, \mathcal{L})$ the triple associated to A. Let A' and B be the two algebras associated to the triple $\mathcal{T}(A)$. From [6] it follows that

Proposition 1.2.4.

1. *The algebras A' and A are canonically isomorphic.*
2. *If A is linear, then $A \cong B$.*
3. *If A is elliptic, then $B \cong A/gA$, where g is a regular normalizing element of degree $s+1$.*

Note that since the algebra $\mathcal{A}(\mathcal{T})$ defined above, depends only on $\mathcal{B}(\mathcal{T})$, we can look here at A via B and this approach was found very rewarding.

1.3 Modules over regular algebras

Let A be a regular algebra of dimension 3, generated in degree one. The graded A-modules we are interested in, are the so-called point modules, since they are in one-to-one correspondence with the points of the associated scheme E (see [6]).

Definition 1.3.1. *A point module M is a graded right A-module which satisfies the following properties*

(i) $M_0 = k$.

(ii) M_0 *generates* M, *and*

(iii) $\dim_k M_n = 1$, *for all* $n \geq 0$.

For a graded left or right module M which is *locally finite*, i.e. each graded piece M_n is a finite dimensional k-vector space, one defines the *Hilbert function* by

$$n \mapsto \dim_k M_n$$

and the *Hilbert series* by

$$h_M(t) = \sum_n (\dim_k M_n) t^n$$

Note that, since A itself is locally finite as an A-module, it has a Hilbert series.

Definition 1.3.2. *Let M be a graded locally finite A-module. The leading coefficient $e(M)$ of the series expansion of h_M is powers of $1-t$, is called the multiplicity of M.*

The multiplicity $e(M)$ of M is positive and an integer multiple of the multiplicity $e(A)$ of A. Furthermore, we have

$$e(A) = \begin{cases} 1 & \text{if } r = 3 \\ 1/2 & \text{if } r = 2 \end{cases}$$

In particular, we have for a point module M that

$$h_M(t) = \frac{1}{1-t} \quad \text{and} \quad e(M) = 1$$

Furthermore, the Gelfand-Kirillov dimension of M is equal to 1, M is also a B-module and critical [7], that is

Definition 1.3.3. *A graded A-module M is critical, if for every graded submodule $N \subset M$, $N \neq 0$, we have* $\mathrm{GK}\dim(M/N) < \mathrm{GK}\dim(M)$.

For further results on critical modules, see [7].

1.4 The non-commutative projective scheme

In this section, we give a definition of $\operatorname{Proj} A$ in the case that $A = k + A_1 + A_2 + \ldots$ is a right noetherian graded k-algebra.

Definition 1.4.1. 1. *A graded right A-module M is called right bounded if $M_n = 0$, for $n >> 0$.*

2. *A graded right A-module M is said to be torsion if it is a direct limit $\varinjlim_{\alpha} M_{(\alpha)}$ in which each $M_{(\alpha)}$ is right bounded.*

Let

$$\begin{aligned} \mathrm{Gr}(A) &:= \text{ the category of graded right } A-\text{modules} \\ \mathrm{Tors}(A) &:= \text{ the full subcategory of } \mathrm{Gr}(A) \text{ of torsion modules} \end{aligned}$$

Morphisms in the category $\mathrm{Gr}(A)$ are the homomorphisms of degree zero. Since $\mathrm{Tors}(A)$ is a dense subcategory of $\mathrm{Gr}(A)$, it makes sense to put

$$\mathrm{QGr}(A) := \text{ the quotient category } \mathrm{Gr}(A)/\mathrm{Tors}(A)$$

So the objects of $\mathrm{QGr}(A)$ are the same as those of $\mathrm{Gr}(A)$ but there are more morphisms.

Let

$$\pi : \mathrm{Gr}(A) \longrightarrow \mathrm{QGr}(A)$$

be the canonical functor and $\mathcal{A}$ the object in $\mathrm{QGr}(A)$ which is the image in $\mathrm{QGr}(A)$ of A_A.

Definition 1.4.2. *The projective scheme of A, denoted by* $\operatorname{Proj} A$, *is the pair* $(\mathrm{QGr}(A), \mathcal{A})$.

This definition of $\operatorname{Proj} A$ is compatible with the classical one for commutative graded rings (see [20]) only under some additional hypotheses, such as that A is generated in degree one. Artin and Zhang, who worked out this way of defining $\operatorname{Proj} A$ in the non-commutative case [9], were inspired by Serre.

Also interested in an extension of a theorem of Serre, Artin and Van den Bergh introduced the twisted homogeneous coordinate rings (see [8]).

VIND JE DIT GEEN SCHITTERENDE OVERGANG ? IK KON NIETS ANDERS BEDENKEN OM OVER TE GAAN NAAR DE TWISTED HOMOGENEOUS COORDINATE RINGS, WEET JIJ MISSCHIEN IETS BETER ?
NOG EEN VRAAGJE, IS HET EEN PROBLEEM DAT PROJ A HIER GEDEFINIEERD IS VOOR RECHTSE MODULEN TERWIJL WE HET IN HOOFDSTUK 3 EIGENLIJK GEBRUIKEN MET LINKSE MODULEN MAAR IN HOOFDSTUK 1 DAN WEER MET RECHTSE ?

Let X be a projective scheme over k, $\mathcal{L}$ the invertible sheaf $\mathcal{O}_X(1)$ and σ an automorphism of X. Denote the pullback $\sigma^*\mathcal{L}$ by $\mathcal{L}^\sigma$ and set, for $n > 0$

$$\mathcal{B}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}}$$

and $\mathcal{B}_0 = \mathcal{O}_X$.

Definition 1.4.3. *The twisted homogeneous coordinate ring associated to the triple* $(X, \sigma, \mathcal{L})$ *is the graded ring*

$$B = \oplus_{n \geq 0} B_n = \oplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{B}_n)$$

with multiplication defined by, if $a \in B_m$, $b \in B_n$, *then* $a \cdot b = a \otimes b^{\sigma^m}$.

In their paper, Artin and Van den Bergh defined the functors

$$\begin{array}{rcrcl} \tilde{(-)} & : & \mathrm{Gr}(B) & \longrightarrow & \mathrm{Qch}(X) \\ \Gamma_* & : & \mathrm{Qch}(X) & \longrightarrow & \mathrm{Gr}(B) \end{array}$$

where

$$\begin{array}{rl} \mathrm{Gr}(B) := & \text{the category of graded right } B-\text{modules} \\ \mathrm{Qch}(X) := & \text{the category of quasi-coherent sheaves} \\ & \text{on the projective scheme } X \end{array}$$

For the definition of Γ_*, let, for all $n > 0$,

$$\mathcal{B}_{-n} = \mathcal{L}^{\sigma^{-1}} \otimes \ldots \otimes \mathcal{L}^{\sigma^{-n}}$$

where $\mathcal{L}^{\sigma^{-1}} = \sigma_*\mathcal{L}$. For a quasi-coherent sheaf $\mathcal{F}$ on X, we define

$$\Gamma_*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{B}_n)$$

Definition 1.4.4. *We say that $\mathcal{L}$ is σ-ample if the functor*

$$s : \mathrm{Qch}(X) \to \mathrm{Qch}(X) : \mathcal{M} \mapsto \mathcal{M}^{\sigma^{-1}} \otimes \mathcal{L}^{\sigma^{-1}}$$

has the property that for every coherent sheaf $\mathcal{M}$, one has $\mathrm{H}^1(X, s^n\mathcal{M}) = 0$ and $s^n\mathcal{M}$ is generated by global sections for large n.

From Theorem 3.12 in [8], it follows that

Theorem 1.4.5. *Let σ be an automorphism of a projective scheme X over k, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let $B = B(X, \sigma, \mathcal{L})$ be the ring defined in Definition 1.4.3, then the functors Γ_* and $\tilde{(-)}$ induce inverse equivalences*

$$\mathrm{Gr}(B)/\operatorname{Tors}(B) \underset{\Gamma_*}{\overset{\tilde{(-)}}{\rightleftarrows}} \mathrm{Qch}(X)$$

where $\operatorname{Tors}(B) :=$ the full subcategory of $\mathrm{Gr}(B)$ of torsion modules.

For a three dimensional regular algebra A, generated in degree one, the twisted homogeneous coordinate ring associated to the triple $\mathcal{T}(A) = (E, \sigma, \mathcal{L})$ is exactly the algebra B described in Proposition 1.2.4. And since in this case $\mathcal{L}$ is σ-ample (Corollary 6.21 in [7]), the functors Γ_* and $\tilde{(-)}$ define an equivalence of the categories $\mathrm{Qch}(E)$ and $\mathrm{QGr}(B)$, the quotient category $\mathrm{Gr}(B)/\operatorname{Tors}(B)$.

Chapter 2

Graded modules of GKdim 1 over a three dimensional Artin-Schelter regular algebra

Let A be a three dimensional Artin-Schelter regular algebra and k an algebraically closed field. We want to give a description of the category of finitely generated A-modules of Gelfand-Kirillov dimension one (modulo those of finite dimension over k).
This is an application of Theorem 2.0.1, stated below.

Theorem 2.0.1. *Let $\mathcal{A}$ be a k-linear locally noetherian Grothendieck category (that is, an abelian category which satisfies AB5 and has a family of noetherian generators). Let $G : \mathcal{A} \to \mathcal{A}$ be an autoequivalence and let $\eta : G \to \mathrm{id}_{\mathcal{A}}$ be a natural transformation such that $\eta(F)$ is surjective for every injective object in $\mathcal{A}$. Let $\mathcal{B}$ be the full subcategory of $\mathcal{A}$ consisting of objects $\mathcal{M}$ with $\eta(\mathcal{M}) = 0$ and let $\mathcal{C}_f$ be the full subcategory of $\mathcal{A}$ consisting of finite length objects whose composition factors lie in $\mathcal{B}$.*

Assume that every simple object in $\mathcal{B}$ has finite injective dimension in $\mathcal{A}$ and furthermore that there is a Cohen-Macauly curve Y over k such that $\mathcal{B}$ is equivalent to $\mathrm{Qch}(Y)$, the category of quasi-coherent $\mathcal{O}_Y$-modules. For $x \in Y$ denote by $\mathcal{P}_x$ the object of $\mathcal{B}$ corresponding to x and define $\tau : Y \to Y$ by $G^{-1}(\mathcal{P}_x) = \mathcal{P}_{\tau x}$. Then we have the following.

1. *$\mathcal{C}_f = \oplus_{z \in Y/\langle\tau\rangle} \mathcal{C}_{f,z}$, where $\mathcal{C}_{f,z}$ is the full subcategory of $\mathcal{C}_f$ consisting of objects whose Jordan-Holder quotients are given by $\mathcal{P}_y$ with $y \in O_\tau(z)$.*

2. *There is a category equivalence F between $\mathcal{C}_{f,z}$ and the category of finite dimensional right modules over a ring C_z. This ring C_z has the following form :*

   (a) *If $|O_\tau(z)| = \infty$ then C_z is given by $\mathbb{Z}\times\mathbb{Z}$ lower triangular matrices with entries in $\hat{\mathcal{O}}_{Y,z}$. In this case z is regular on Y and thus we have $\hat{\mathcal{O}}_{Y,z} \cong k[[x]]$.*

   (b) *If $|O_\tau(z)| = n$ then C_z is given by a ring of $n \times n$ matrices of the form*

$$\begin{pmatrix} R & RU & \dots & RU \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & RU \\ R & \dots & \dots & R \end{pmatrix}$$

   where R is a complete local ring of the form

$$R = k\langle\langle x, y\rangle\rangle/(\psi)$$

   with

$$\psi = yx - qxy + \text{higher order terms} \tag{2.1}$$

   for some $q \in k^*$, or

$$\psi = yx - xy - x^2 + \text{higher order terms} \tag{2.2}$$

   U is a regular normalizing element in $\operatorname{rad}(R)$ such that $R/(U) = \hat{\mathcal{O}}_{Y,z}$.
   If z is not fixed under τ then z is regular on Y and also $U \notin \operatorname{rad}^2(R)$.

3. *Let $I = \mathbb{Z}$ if $|O_\tau(z)| = \infty$ and $I = \mathbb{Z}/n\mathbb{Z}$ if $|O_\tau(z)| = n$. In this way the elements of C_z correspond to $I \times I$-matrices. For $i \in I$, let e_i be the corresponding diagonal idempotent. Then every finite dimensional right C_z-representation W satisfies $W = \oplus_i We_i$.*

4. *Put $S_i = e_iC_z/\operatorname{rad}(e_iC_z)$. Then $F(\mathcal{P}_{\tau^i z}) = S_i$.*

5. *Define the following normal element N of C_z.*

   (a) *If $|O_\tau(z)| = \infty$, then N is given by the matrix whose entries are everywhere zero except on the lower subdiagonal where they are one.*

(b) If $|O_\tau(z)| < \infty$, then

$$N = \begin{pmatrix} 0 & \dots & 0 & U \\ 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

Let $\phi = N \cdot N^{-1}$ then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{f,z} & \xrightarrow{G} & \mathcal{C}_{f,z} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathrm{Mod}_r(C_z) & \xrightarrow{(-)_\phi} & \mathrm{Mod}_r(C_z) \end{array}$$

where $\mathrm{Mod}_r(C_z)$ *denotes the category of right C_z-modules.*

6. *If $\mathcal{M}$ is an object in $\mathcal{C}_{f,z}$ then one has the following commutative diagram.*

$$\begin{array}{ccc} FG(\mathcal{M}) & \xrightarrow{F(\eta(\mathcal{M}))} & F(\mathcal{M}) \\ \| & & \| \\ F(\mathcal{M})_\phi & \xrightarrow{\cdot N} & F(\mathcal{M}) \end{array}$$

7. *Let $\mathcal{C}_{f,z,Y}$ be the pullback of $\mathcal{C}_{f,z}$ in* $\mathrm{Qch}(Y)$*. Thus the objects of $\mathcal{C}_{f,z,Y}$ are the finite length objects in* $\mathrm{Qch}(Y)$ *whose support is contained in the τ-orbit of z. Put $D_z = C_z/(N) = \prod_i \hat{\mathcal{O}}_{Y,\tau^i z}$. Let $(\hat{\ })_z$ be a shorthand for the product of the completion functors $(\hat{\ })_{\tau^i z}$. Then the following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{C}_{f,z,Y} & \xrightarrow{\cong} & \mathcal{C}_{f,z} \\ {\scriptstyle (\hat{\ })_z}\downarrow & & {\scriptstyle F}\downarrow \\ \mathrm{Mod}_r(D_z) & \longrightarrow & \mathrm{Mod}_r(C_z) \end{array}$$

From this theorem we can extract the following corollary.

Corollary 2.0.2. *If z is not a fixed point for τ then z is regular on Y.*

If we think of the curve Y as being embedded in a kind of non-commutative space $\mathcal{A}$ then Theorem 2.0.1 gives us some insight into the structure of $\mathcal{A}$ in a neighbourhood of Y.

The application we have in mind is the following.
Let A be a three dimensional Artin-Schelter regular algebra, generated in degree one. Recall A possesses a regular normalizing element g in degree three or four such that $B = A/(g)$ is a twisted homogeneous coordinate ring associated to a triple $(Y, \sigma, \mathcal{L})$ with Y a plane curve of arithmetic genus one and $\mathcal{L} = \mathcal{O}_Y(1)$. Since $\mathcal{L}$ is σ-ample, the functors

$$\begin{array}{rcl} \mathrm{Gr}(B) & \xrightarrow{\widetilde{(-)}} & \mathrm{Qch}(Y) \\ \mathrm{Qch}(Y) & \xrightarrow{\Gamma_*} & \mathrm{Gr}(B) \end{array} \tag{2.3}$$

factor through the quotient map $\pi : \mathrm{Gr}(A) \to \mathrm{QGr}(A)$ to give inverse equivalences between $\mathrm{QGr}(B)$ and $\mathrm{Qch}(Y)$. The pointmodules of A are all annihilated by g and hence it follows easily from the category equivalence (2.3) that they are of the form $P_x = \Gamma_*(k(x))$, for $x \in Y$. Put $\mathcal{P}_x = \pi P_x$. Let G be the autoequivalence of $\mathrm{Gr}(A)$ given by $- \otimes_A gA$ and denote by the same letter the induced autoequivalence on $\mathrm{QGr}(A)$. The natural transformation $\eta(\mathcal{M})$ is the obvious map $G(\mathcal{M}) \to \mathcal{M}$ obtained from the inclusion $gA \hookrightarrow A$.

It is clear that the hypotheses for Theorem 2.0.1 are satisfied, whence we can apply that theorem in order to give a description of the category of A-modules of Gelfand-Kirillov dimension one modulo those of finite dimension over k.

The proof of Theorem 2.0.1 is based upon a result by Gabriel [18], stating that locally finite categories are dual to pseudocompact rings.

Sections 2.1-2.3, are devoted to some generalities concerning pseudocompact rings. We are especially interested in the relationship between topological and non-topological properties of such rings.
In Sections 2.4, 2.5 we give some classification theorems which are slightly more general than what we need for the proof of Theorem 2.0.1.
Finally in section 2.6, we give the proof of Theorem 2.0.1.

We introduce some extra notations and conventions.
If C is a ring then $\mathrm{Mod}(C)$ refers to the category of left modules over C. The category of right modules is denoted by $\mathrm{Mod}_r(C)$. Note that as before, we denote the category of graded *right*-modules over a graded ring A as $\mathrm{Gr}(A)$. An unspecified module will always be a left module.

If M is a left module over a ring C and ϕ is an automorphism of C then ${}_\phi M$ is the left C-module which is equal to M as a set, but which has its multiplication twisted by ϕ, i.e. $c \cdot m = \phi(c)m$. A similar notation is used for right modules.

2.1 Pseudocompact rings

A ring A provided with a topology, is a *topological ring* if the following are satisfied

(T_1)	The map	$(x, y) \mapsto x + y$	from $A \times A$ to A is continuous.
(T_2)	The map	$x \mapsto -x$	from A to A is continuous.
(T_3)	The map	$(x, y) \mapsto xy$	from $A \times A$ to A is continuous.

(T_1) and (T_2) express that the topology on A is compatible with its structure as a group. We say that a topology on a ring A is compatible with the ringstructure if (T_1), (T_2) and (T_3) are satisfied.

A left *topological module* M over a topological ring A, is a left A-module provided with a topology compatible with its groupstructure and satisfying

(TM) The map $(a, m) \mapsto am$ from $A \times M$ to M is continous.

A left topological module M over a topological ring A is *pseudocompact* if it is Hausdorf, complete and its topology is generated by left submodules of finite colength. A itself is said to be a *pseudocompact ring* is A is pseudocompact as a left A-module.

In the rest of this section A will be a pseudocompact ring.
The category of pseudocompact modules over A is denoted by PC(A). It is an abelian category satisfying AB5* and AB3 [18]. Its dual category is a locally finite category. That is, a Grothendieck category possessing a set of generators of finite length.

Conversely assume that $\mathcal{C}$ is a locally finite category. If $M, N \in \mathcal{C}$ then the *natural* topology on $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is the linear topology generated by the subgroups of the form

$$(S) = \{f : M \to N | f(S) = 0\}$$

where S runs through the objects of finite length in $\mathcal{C}$.
The following result is proved in [18]:

Theorem 2.1.1. *If E is an injective cogenerator for $\mathcal{C}$ then $A = \mathrm{End}_{\mathcal{C}}(E)$, equipped with the natural topology, is a pseudocompact ring, and the functor which sends $M \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(M, E)$ (with the natural topology) is an equivalence of categories between $\mathcal{C}$ and $\mathrm{PC}(A)^0$.*

One easy property of a linear topology will be used repeatedly below.

Lemma 2.1.2. *Assume that M is a topological group with a topology generated by subgroups and $L \subset M$ is an open subgroup. Then L is also closed and the quotient topology on M/L is discrete.*

Proof. L is the complement of the union of cosets of L in M which are not equal to L. Since this union is a union of open sets, it is itself open. Thus L is closed. L is the inverse image of $\bar{e}$ in M/L and hence $\{\bar{e}\}$ is open and closed in the quotient topology. □

The following proposition records for further reference some of the properties of the forgetful functor $\mathrm{PC}(A) \to \mathrm{Mod}(A)$ [18].

Proposition 2.1.3. *The forgetful functor $\mathrm{PC}(A) \to \mathrm{Mod}(A)$ is faithful and commutes with kernels, cokernels and products. In particular, it reflects isomorphism and exactness. If $M \in \mathrm{PC}(A)$ then the subobjects of M in $\mathrm{PC}(A)$ are in one-one correspondence with the subobjects of M in $\mathrm{Mod}(A)$ which are closed.*

Let $\mathrm{Fin}(A)$ be the full subcategory of $\mathrm{Mod}(A)$ consisting of objects which are of finite length and let $\mathrm{PCFin}(A)$ be its pullback in $\mathrm{PC}(A)$.
A module of finite length carrying a linear topology can only be separated if its topology is discrete since the fundamental system of environments of e that generates the linear topology, can't contain infinitely many environments, so $\{e\}$ has to be a member of the fundamental system and therefore the topology must be discrete.
So we conclude immediately that the forgetful functor

$$\mathrm{PCFin}(A) \to \mathrm{Fin}(A)$$

is fully faithfull.

The following lemma gives us more information on $\mathrm{PCFin}(A)$.

Lemma 2.1.4. *1. An object in $\mathrm{PC}(A)$ is simple in $\mathrm{PC}(A)$ if and only if it is simple in $\mathrm{Mod}(A)$.*

2. *The objects in* $\mathrm{PCFin}(A)$ *are precisely the finite length objects in* $\mathrm{PC}(A)$.

Proof. 1. Assume that $0 \neq S \in \mathrm{PC}(A)$ is simple in $\mathrm{PC}(A)$. We want to show that S is simple in $\mathrm{Mod}(A)$. Take $0 \neq x \in S$. Since S is Hausdorf, there exists an open submodule $L \subset S$, not containing x. L is also closed (Lemma 2.1.2) and thus it is pseudocompact if we give it the induced topology. Since S is simple in $\mathrm{PC}(A)$, we obtain $L = 0$. This implies that $S = S/L$ carries the discrete topology. But then every submodule of S is closed and thus it is a subobject of S in $\mathrm{PC}(A)$. Since S is simple in $\mathrm{PC}(A)$, there can be no non-trivial subobjects and thus S is simple in $\mathrm{Mod}(A)$. The other direction is clear since subobjects of an object in $\mathrm{PC}(A)$ come from subobjects of that object in $\mathrm{Mod}(A)$.

2. This follows from 1. □

Let us say that $M \in \mathrm{PC}(A)$ is finitely generated in $\mathrm{PC}(A)$ if there is a surjective map $A^k \to M$ in $\mathrm{PC}(A)$ for some k.

Proposition 2.1.5. *Assume that* $M,\ N \in \mathrm{PC}(A)$, M *finitely generated. Then*

$$\mathrm{Hom}_{\mathrm{PC}(A)}(M,N) \ = \ \mathrm{Hom}_{\mathrm{Mod}(A)}(M,N)$$

Proof. If $n \in N$ then the map $a \mapsto an$ is continuous which yields

$$\mathrm{Hom}_{\mathrm{PC}(A)}(A,N) \ = \ \mathrm{Hom}_{\mathrm{Mod}(A)}(A,N)$$

This proves the proposition for $M = A$ and hence also for $M = A^k$.
Now assume M general. There is an exact sequence in $\mathrm{PC}(A)$

$$0 \to M' \to F \to M \to 0$$

with $F = A^k$. This yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{PC}(A)}(M,N) & \longrightarrow & \mathrm{Hom}_{\mathrm{PC}(A)}(F,N) & \longrightarrow & \mathrm{Hom}_{\mathrm{PC}(A)}(M',N) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{Mod}(A)}(M,N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Mod}(A)}(F,N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Mod}(A)}(M',N)
\end{array}$$

The vertical maps are injective and the middle one is an isomorphism. It follows that the left map must be an isomorphism. □

Corollary 2.1.6. *An object in* $\mathrm{PC}(A)$ *is finitely generated in* $\mathrm{PC}(A)$ *if and only if it is finitely generated in* $\mathrm{Mod}(A)$.

Corollary 2.1.7. *A direct summand in* $\mathrm{Mod}(A)$ *of a finitely generated object in* $\mathrm{PC}(A)$ *is a direct summand in* $\mathrm{PC}(A)$. *In particular, a finitely generated object in* $\mathrm{PC}(A)$ *is projective in* $\mathrm{PC}(A)$ *if and only if it is projective in* $\mathrm{Mod}(A)$.

Proof. If M is a finitely generated object in $\mathrm{PC}(A)$ then a direct summand of M is the image of an idempotent in $\mathrm{End}_{\mathrm{Mod}(A)}(M)$. The result now follows from Proposition 2.1.5 and 2.1.3. □

Corollary 2.1.8. *If* $M \in \mathrm{PC}(A)$ *is finitely generated then a submodule* $L \subset M$ *is open if and only if* $M/L \in \mathrm{PCFin}(A)$.

Proof. If L is open then it is closed by Lemma 2.1.2 and of finite colength. Hence $M/L \in \mathrm{PCFin}(A)$.
Conversely, assume $M/L \in \mathrm{PCFin}(A)$. By Proposition 2.1.5 the quotient map $M \to M/L$ is continous. Since, again by Lemma 2.1.2, M/L carries the discrete topology, $\{\overline{0}\} \subset M/L$ is open and thus so is its inverse image L. □

Since $\mathrm{PC}(A)$ is the dual of a locally finite category, it has projective covers. The projective covers of the pseudocompact simples are the indecomposable projectives. Furthermore every projective in $\mathrm{PC}(A)$ is a product of such indecomposable projectives. By [18] the indecomposable projectives are of the form Ae, where e is a primitive idempotent in A.

Recall also from [18] that if $(e_i)_{i\in I}$ is a summable set of primitive, pairwise orthogonal idempotents with sum 1 then $A = \prod_{i\in I} Ae_i$ and every indecomposable projective in $\mathrm{PC}(A)$ is isomorphic to at least one Ae_i.

Lemma 2.1.9. *The* $(Ae_i)_{i\in I}$ *are the projective covers in* $\mathrm{Mod}(A)$ *of the simple* A*-modules which are pseudocompact.*

Proof. Since the Ae_i are projective covers of simple modules in $\mathrm{PC}(A)$, this follows from the fact that $\mathrm{End}_{\mathrm{Mod}(A)}(Ae_i) = \mathrm{End}_{\mathrm{PC}(A)}(Ae_i)$ is local. □

From Proposition 2.1.3 it follows that M is noetherian in $\mathrm{PC}(A)$ if it satisfies the ascending chain condition on closed subobjects. Thus if M is noetherian in $\mathrm{Mod}(A)$ then it is noetherian in $\mathrm{PC}(A)$.
We now show that the converse holds.

Proposition 2.1.10. *Every subobject in* $\mathrm{Mod}(A)$ *of a noetherian object in* $\mathrm{PC}(A)$ *is closed and hence lies in* $\mathrm{PC}(A)$.

Proof. Let M be a noetherian object in $\mathrm{PC}(A)$. Since the $(Ae_i)_{i\in I}$ form a set of generators for $\mathrm{PC}(A)$, M is a quotient of a direct sum of a finite number of such Ae_i, and in particular is finitely generated.

Assume now that $N \subset M$ in $\mathrm{Mod}(A)$. We have to show that $\overline{N} = N$. Since $\overline{N}$ is a closed submodule of M it is also noetherian in $\mathrm{PC}(A)$. Therefore, without loss of generality, we may assume that $\overline{N} = M$.

We want to show that $N = M$. Assume that this is not so. Consider the partially ordered set

$$\mathcal{P} = \{N \subset N' \subsetneq M \mid N' \in \mathrm{Mod}(A)\}$$

Since M is finitely generated, $\mathcal{P}$ has a maximal element by Zorn's lemma. Again without loss of generality, we may replace N by this maximal element. In that case M/N is simple. However M/N is not pseudocompact since otherwise by Proposition 2.1.5 $N = \ker(M \to M/N)$ would be pseudocompact and hence closed which is impossible because $\overline{N} = M \neq N$.

Let $\phi : Ae_i \to M$ be a non-zero map. Then either $\phi^{-1}(N) = Ae_i$ or $Ae_i/\phi^{-1}(N)$ is simple but not pseudocompact. The last case is impossible since by Lemma 2.1.9 Ae_i has only one simple quotient, and this simple quotient is pseudocompact. Thus N contains the image of every ϕ and therefore $N = M$. □

Corollary 2.1.11. *An object in* $\mathrm{PC}(A)$ *is noetherian in* $\mathrm{PC}(A)$ *if and only if it is noetherian in* $\mathrm{Mod}(A)$.

Corollary 2.1.12. *Assume that* $M \in \mathrm{PC}(A)$ *is noetherian. Then the topology on* M *is the cofinite topology. That is, a submodule* $L \subset M$ *is open if and only if* M/L *has finite length.*

Proof. Since M is noetherian in $\mathrm{PC}(A)$, it is finitely generated. So if a submodule $L \subset M$ is open, $M/L \in \mathrm{PCFin}(A)$ by Corollary 2.1.8 and therefore it has finite length.
For the other direction, let L be a submodule of M of finite colength. By Proposition 2.1.10 L is closed in M. Hence M/L is pseudocompact and since it is of finite length, it carries the discrete topology. Thus $\{\overline{0}\} \subset M/L$ is open, and so is its inverse image L. □

Let R, I, M be respectively a ring, an ideal in R and an R-module. Then the I-*adic* topology on R is the linear topology generated by the submodules of M of the form I^nM. In a pseudocompact ring A the Jacobson radical $\mathrm{rad}(A)$

is the common annihilator of the simple pseudocompact A-modules [[18], dual of Prop IV.12].
The following is a reformulation of the previous corollary.

Corollary 2.1.13. *Assume that $M \in \mathrm{PC}(A)$ is noetherian. Then the topology on M is given by the $\mathrm{rad}(A)$-adic topology.*

Proof. It suffices to show that $M/\operatorname{rad}(A)M$ is a finite sum of simples. Therefore we look at the subcategory of the semisimple objects in $\mathrm{PC}(A)$, which is precisely $\mathrm{PC}(A/\operatorname{rad}(A))$ and this latter is still the dual category of a locally finite category. So we may replace M by $M/\operatorname{rad}(A)M$ and A by $A/\operatorname{rad}(A)$.
Since M is noetherian in $\mathrm{PC}(A)$, it corresponds to a artinian object in the dual category and therefore it must be a finite direct sum of simples. □

Definition 2.1.14. *We say that A is locally noetherian if the $(Ae_i)_{i\in I}$ are noetherian in $\mathrm{PC}(A)$.*
We say that A is noetherian if A is noetherian in $\mathrm{PC}(A)$.

Proposition 2.1.15. *Let A be locally noetherian and $M,\ N \in \mathrm{PC}(A)$. Assume that M is noetherian. Then*

$$\mathrm{Ext}^i_{\mathrm{PC}(A)}(M,N) \;=\; \mathrm{Ext}^i_{\mathrm{Mod}(A)}(M,N)$$

Proof. The case $i=0$ follows from Proposition 2.1.5.
For $i>0$ we use an exact sequence

$$0 \to M' \to P \to M \to 0$$

in $\mathrm{PC}(A)$ with P a finite direct sum of Ae_i. Since A is locally noetherian, M' is also noetherian and so by induction

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{PC}(A)}(M,N) &= \mathrm{Ext}^i_{\mathrm{PC}(A)}(P/M',N) = \mathrm{Ext}^{i-1}_{\mathrm{PC}(A)}(M',N)\\ &= \mathrm{Ext}^{i-1}_{\mathrm{Mod}(A)}(M',N) = \mathrm{Ext}^i_{\mathrm{Mod}(A)}(P/M',N)\\ &= \mathrm{Ext}^i_{\mathrm{Mod}(A)}(M,N)\end{aligned}$$

□

Proposition 2.1.15 yields the following corollary.

Corollary 2.1.16. *Let* $\mathrm{pc}(A)$ *resp.* $\mathrm{mod}(A)$ *be the full subcategory of* $\mathrm{PC}(A)$ *resp.* $\mathrm{Mod}(A)$ *consisting of noetherian objects. Then the functor*

$$\mathrm{pc}(A) \to \mathrm{mod}(A)$$

is fully faithfull and its essential image, i.e. all the objects in $\mathrm{mod}(A)$ *which are isomorphic to an object in the image, is closed under extensions. In particular,* $\mathrm{PCFin}(A)$ *is closed under extensions inside* $\mathrm{Fin}(A)$.

It also follows from Proposition 2.1.15 that :

Proposition 2.1.17. *If A is locally noetherian and* $M \in \mathrm{PC}(A)$ *is noetherian then*

$$\operatorname{proj\,dim}_{\mathrm{Mod}(A)} M \;=\; \operatorname{proj\,dim}_{\mathrm{PC}(A)} M$$

Now we discuss briefly automorphisms of pseudocompact rings.

Lemma 2.1.18. *Assume that A is a pseudocompact ring. If* $\phi \in \mathrm{Aut}(A)$, *then ϕ is continuous if and only if for every pseudocompact A-module S of finite length, we have that ${}_{\phi}S$ is pseudocompact.*

Proof. "$\Rightarrow$" We only have to prove that left multiplication by an element of A on ${}_{\phi}S$ (which has the discrete topology) is continuous. Since ϕ is assumed to be continuous, this is clear.
"$\Leftarrow$" Assume that $L \subset A$ is an open ideal. Then A/L is pseudocompact of finite length (Corollary 2.1.8) and hence ${}_{\phi}(A/L) \cong A/\phi^{-1}(L)$ is pseudocompact of finite length. Again by Corollary 2.1.8 this implies $\phi^{-1}(L)$ is open in A. $\square$

Corollary 2.1.19. *If A is locally noetherian and* $\phi \in \mathrm{Aut}(A)$ *then ϕ is a homeomorphism.*

Proof. It suffices to show that ϕ is continuous. By Lemma 2.1.18 and Corollary 2.1.16, we must show that if S is pseudocompact simple, then so is ${}_{\phi}S$. Since S has a projective cover of the form Ae, for some primitive idempotent e, A is locally noetherian and $A/\operatorname{rad}(A)$ is pseudocompact semisimpel, it follows from Lemma 2.2, page 218 in [17] that $S = Ae/\operatorname{rad}(Ae)$. So ${}_{\phi}S = A\phi^{-1}(e)/\operatorname{rad}(A\phi^{-1}(e))$ and thus by the same lemma in [17], $A\phi^{-1}(e)$ is a projective cover of ${}_{\phi}S$. It now follows from Lemma 2.1.9 that ${}_{\phi}S$ is pseudocompact simple. $\square$

To close this section, we discuss noetherian pseudocompact rings.

Proposition 2.1.20. *Let A be a noetherian pseudocompact ring. Then the forgetful functor*

$$\mathrm{pc}(A) \to \mathrm{mod}(A) \tag{2.4}$$

is an equivalence of categories.

Proof. By Corollary 2.1.16 we only have to show that (2.4) is essentially surjective.
Let $M \in \mathrm{mod}(A)$. Then M has a resolution

$$F_1 \xrightarrow{\phi} F_0 \to M \to 0$$

where the F_i are finitely generated free A-modules. By Proposition 2.1.5, $\phi \in \mathrm{Hom}_{\mathrm{PC}(A)}(F_1, F_0)$. Therefore $M = \operatorname{coker} \phi \in \mathrm{PC}(A)$. □

Proposition 2.1.21. *Assume that A is a noetherian pseudocompact ring. Put $J = \mathrm{rad}(A)$. Then*

1. *A/J is semisimple.*
2. *A is complete for the J-adic topology.*
3. *The topology on A coincides with the J-adic topology.*

Conversely, if A is a left noetherian ring satisfying 1., 2. *then A is pseudocompact when equipped with the J-adic topology.*

Proof. "$\Leftarrow$" Since A/J is semisimple and J^n is finitely generated, all J^n/J^{n+1} are finite direct sums of simples. Hence A/J^n has finite length for all n. Therefore 2. implies that A is pseudocompact.
"$\Rightarrow$" By [18] A/J is a product of endomorphism rings of vectorspaces. Since A/J is also noetherian, it must be semisimple. This proves 1. Property 2. follows from 3. and 3. is precisely Corollary 2.1.13. □

Proposition 2.1.22. *Let A be a pseudocompact ring, $N \in \mathrm{rad}(A)$ a regular normalizing element. Assume that $M \in \mathrm{PC}(A)$ is such tat M/NM is finitely generated. Then M is finitely generated.*

Proof. The key point is that the N-adic topology on M and A is finer than the given topology. Thus if $(f_i)_i$ is a Cauchy sequence for the N-adic topology, then it is convergent in the given topology.

Let $t_1, \dots, t_n \in M$ be such that $\overline{t_1}, \dots, \overline{t_n}$ generate M/NM. We show that $t_1, \dots, t_n$ generate M.

Take $t \in M$. Then there exist $a_1^{(1)}, \dots, a_n^{(1)} \in A$, $t^{(1)} \in M$ such that

$$t - \sum_{i=1}^{n} a_i^{(1)} t_i = N t^{(1)}$$

Continuing this procedure, we find $a_i^j \in A$ such that

$$t - \left(\sum_i a_i^{(1)} t_i + \sum_i N a_i^{(2)} t_i + \ldots + \sum_i N^{p-1} a_i^p t_i \right) \in N^p M$$

and hence

$$t = \sum_i \sum_j (N^{j-1} a_i^j)\, t_i$$

Thus M is generated by $t_1, \dots, t_n$. □

From this we deduce the following.

Proposition 2.1.23. *Let A be a pseudocompact ring, $N \in \operatorname{rad}(A)$ a regular normalizing element. Assume that $M \in \mathrm{PC}(A)$ is such that M/NM is noetherian. Then M is noetherian.*

Proof. Let $T \subset M$ be an arbitrary submodule. We want to show that T is finitely generated. We define first

$$T^{\mathrm{sat}} = \{ t \in M \mid \exists\, k : N^k t \in T \}$$

Obviously, $(T^{\mathrm{sat}})^{\mathrm{sat}} = T^{\mathrm{sat}}$ and $T^{\mathrm{sat}} \cap N^p M = N^p T^{\mathrm{sat}}$. Since $T^{\mathrm{sat}}/\, N T^{\mathrm{sat}} \subset M/NM$, we find that $T^{\mathrm{sat}}/N T^{\mathrm{sat}}$ is finitely generated and hence so is T^{sat} by the previous proposition.

Since T^{sat} is finitely generated, it follows from the definition of T^{sat} that there exists a k such that $N^k T^{\mathrm{sat}} \subset T$. Thus $T^{\mathrm{sat}} \cap N^k M = N^k T^{\mathrm{sat}} \subset T$. Since $N^k T^{\mathrm{sat}}$ is finitely generated, it now suffices to show that $T/(T^{\mathrm{sat}} \cap N^k M)$ is finitely generated. Now clearly, $M/N^k M$ is noetherian and hence so is the subobject $T/(T^{\mathrm{sat}} \cap N^k M)$. □

2.2 A matrix representation for pseudocompact rings

If A is an arbitrary ring, M a left A-module and $(e_i)_{i=1,\ldots,n}$ a finite set of pairwise orthogonal idempotents with sum 1 then it is classical that A is isomorphic to the matrix ring

$$\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix}$$

With $A_{ij} = e_i A e_j$ and M is isomorphic to the set of column vectors $(M_1, \ldots, M_n)^t$ with $M_j = e_j M$.

An element a in A is sent to the matrix $(e_i a e_j)_{ij}$ and an element m of M is sent to $(e_j m)_j$.
It is clear how to extend this result to the pseudocompact situation.

Lemma 2.2.1. *Assume that $(e_i)_{i\in I}$ is a summable set of orthogonal idempotents in a pseudocompact ring A such that $\sum_i e_i = 1$. Let M be a pseudocompact A-module and put $A_{ij} = e_i A e_j$, $M_i = e_i M$.*
Then A is isomorphic to the ring of doubly infinite matrices $(a_{ij})_{ij} \in (A_{ij})_{ij}$ with summable columns and M is isomorphic to the set of summable column vectors $(m_i)_i \in (M_i)_i$.
The isomorphisms are given by the maps

$$\begin{aligned} (a_{ij})_{ij} &\mapsto \textstyle\sum_{i,j} a_{ij} \\ (m_i)_i &\mapsto \textstyle\sum_i m_i \end{aligned}$$

Note that this lemma only says something about the ring structure on A and the module structure on M, but nothing about the topology.

Below we give the A_{ij} the topology induced from A and M_i the topology induced from M. Since $A_{ij} = e_i A e_j$ is closed in A, it is complete. A similar argument is true for M_i. Furthermore the topology is linear (given by abelian subgroups). We also have multiplication mappings

$$\begin{aligned} A_{ij} \times A_{jk} &\to A_{ik} \\ A_{ij} \times M_i &\to M_j \end{aligned}$$

and since these are induced from the multiplication on A and M, they are continuous. This makes A_{ii} into a topological ring, A_{ij} into a topological $A_{ii} - A_{jj}$-bimodule and M_i into a left topological A_{ii}-module.

Lemma 2.2.2. *1. A_{ii} is a pseudocompact ring and A_{ij} is a pseudocompact A_{ii}-module.*

2. M_i is a pseudocompact A_{ii}-module.

Proof. It suffices to prove 2. Indeed if we take $M = Ae_j$ (and afterwards $j = i$), then we obtain part 1.

Let $L \subset M$ be an open submodule. If T is an A_{ii}-submodule of $e_i M$ containing $L \cap e_i M = e_i L$ then $\tilde{T} = AT + L$ is an A-submodule of M containing L, and furthermore $\tilde{T} \cap e_i M = T$. Thus $\mathrm{length}(T/e_i L) \leq \mathrm{length}(\tilde{T}/L) \leq \mathrm{length}(M/L)$ and so it follows that the length of $e_i M/(e_i M \cap L)$ is bounded by that of M/L. Since the topology on $e_i M$ is induced from that on M we deduce that $e_i M$ is pseudocompact. $\square$

Unfortunately it is not in general true that A and M carry the induced topology from the product topologies on $\prod_{ij}(A_{ij})$ and $\prod_i M_i$. A counter example is given by the endomorphism ring of an infinite dimensional vector space.

Under some mild extra hypotheses, this defect can be repaired. Note that the Ae_i are pseudocompact projectives. Hence they are products of indecomposable pseudocompact projectives.

Proposition 2.2.3. *Let $(e_i)_{i \in I}$ be as in Lemma 2.2.1. Assume that every indecomposable pseudocompact projective is a summand of at most a finite number of Ae_i. Then as topological spaces*

$$A = \prod_{i,j} A_{ij} \tag{2.5}$$

$$M = \prod_{i} M_i \tag{2.6}$$

Proof. We certainly have $A = \prod_j Ae_j$. Hence it suffices to prove (2.6).

We have an inclusion

$$M \subset \prod_i M_i \tag{2.7}$$

which is given by the product of the maps $M \to M_i : m \mapsto e_i m$. These maps are continuous and hence the inclusion is also continuous.

We now show that the topology on M is courser than the induced topology for the inclusion (2.7).

Let $L \subset M$ be an open submodule. Since M/L has finite length, the hypotheses imply that $\mathrm{Hom}_A(Ae_i, M/L)$ is non-zero for at most a finite number of i. Since $\mathrm{Hom}(Ae_i, M/L) = e_i(M/L)$, we deduce that for almost all i, $M_i = e_i M \subset L$. Hence $\prod_i (M_i \cap L)$ is open in $\prod_i M_i$. Observing that $M \cap \prod_i (M_i \cap L) \subset L$ yields that L is the union of cosets of $M \cap \prod_i (M_i \cap L)$ in L. Since this union is a union of sets which are open for the topology induced by (2.7), it is itself open for this topology. This finishes the proof.

The proof we have just given also shows that if $(m_i)_i \in \prod_i M_i$ then $(m_i)_i$ is summable in M. Sending $(m_i)_i$ to $\sum_i m_i$ defines an inverse to the inclusion (2.7).
A similar result holds for A. □

2.3 Global dimension

In this section A is a pseudocompact ring. We define

$$\mathrm{Gl}\dim\ A \ = \ \sup_{M \in \mathrm{PC}(A)} \mathrm{proj}\dim_{\mathrm{PC}(A)}(M)$$

Note that we often have $\mathrm{proj}\dim_{\mathrm{PC}(A)} M = \mathrm{proj}\dim_{\mathrm{Mod}(A)} M$ by Proposition 2.1.17. Therefore, if no confusion can arise, we make no distinction between those two types of projective dimension, and we simply write $\mathrm{proj}\dim(M)$.

Lemma 2.3.1. *We also have*

$$\mathrm{Gl}\dim\ A \ = \ \sup_{\substack{S \in \mathrm{PC}(A) \\ S \text{ simple}}} \mathrm{proj}\dim_{\mathrm{PC}(A)}\ S$$

Proof. By Theorem 2.1.1 it suffices to prove the dual statement for locally finite categories. So assume $\mathcal{C}$ is such a category and $\mathrm{inj}\dim(S) \leq n$, for every simple S in $\mathcal{C}$. Hence for every finite length module F one also has $\mathrm{inj}\dim(F) \leq n$. If $M \in \mathcal{C}$ arbitrary, then by definition, M is a direct limit of finite length objects. By the proof [[19], Thm. 1.10.1], monomorphisms into injectives can be constructed in a functorial way and hence so can injective resolutions. Taking the direct limit of the injective resolutions of the subobjects of finite length of M yields an injective resolution of M of length $\leq n$ ($\mathcal{C}$ is locally noetherian and hence a direct limit of injectives is injective). □

The following result is very classical.

Proposition 2.3.2. *Let $N \in \operatorname{rad}(A)$ be a regular normalizing element in A. Assume that A is locally noetherian. Then*

$$\operatorname{Gl}\dim\ A \ \leq\ \operatorname{Gl}\dim\ A/(N)\ + 1$$

Proof. Let $\operatorname{Gl}\dim\ A/(N) = p$. We have to show that $\operatorname{proj}\dim\ S \leq p+1$, for every pseudocompact simple. Since $S = Ae/\operatorname{rad}(Ae)$ for some primitive idempotent e of A, it suffices to prove that

$$\operatorname{proj}\dim\ \operatorname{rad}(P) \leq p$$

where P runs through the indecomposable projectives in $\operatorname{PC}(A)$.
This follows from Lemma 2.3.3 below. □

Lemma 2.3.3. *Assume that A is locally noetherian and let L be a noetherian pseudocompact A-module which is N-torsion free. Then*

$$\operatorname{proj}\dim\ L = \operatorname{proj}\dim_{A/(N)}\ L/NL$$

Proof. By degree shifting one reduces to the case where L/NL is projective over $A/(N)$. In that case the result follows by an appropriate version of Nakayama's Lemma. □

The following type of result seems to be referred to less often.

Proposition 2.3.4. *Assume that A is locally noetherian and $N \in \operatorname{rad}(A)$ is a regular normalizing element such that for every indecomposable pseudocompact projective one has $NP \not\subset \operatorname{rad}^2(P)$. Then*

$$\operatorname{Gl}\dim\ A/(N)\ + 1 \leq \operatorname{Gl}\dim\ A \tag{2.8}$$

Proof. This is an immediate generalization of the proof by Serre that local rings of finite global dimension are regular.

Let $\phi = N \cdot N^{-1}$. By Corollary 2.1.19 ϕ is a homeomorphism.

Let P be an indecomposable pseudocompact projective over A with cosocle S. We have an inclusion

$$_{\phi^{-1}}S \cong NP/N\operatorname{rad}(P) \hookrightarrow \operatorname{rad}(P)/N\operatorname{rad}(P) \tag{2.9}$$

Now we also have $NP \cap \operatorname{rad}^2(P) = N \operatorname{rad}(P)$ and thus there is an inclusion

$$NP/N\operatorname{rad}(P) \hookrightarrow \operatorname{rad}(P)/\operatorname{rad}^2(P) \tag{2.10}$$

$\operatorname{rad}(P)/\operatorname{rad}^2(P)$ is a finite sum of simples and hence this inclusion splits. From the commutative diagram

$$\begin{array}{ccc} NP/N\operatorname{rad}(P) & \longrightarrow & \operatorname{rad}(P)/N\operatorname{rad}(P) \\ \| & & \downarrow \\ NP/N\operatorname{rad}(P) & \longrightarrow & \operatorname{rad}(P)/\operatorname{rad}^2(P) \end{array}$$

we deduce that (2.9) is also split.
Thus

$$\begin{aligned} \operatorname{proj}\dim_A \ S \ -1 &= \operatorname{proj}\dim_A \ \operatorname{rad}(P) \\ &= \operatorname{proj}\dim_{A/(N)} \operatorname{rad}(P)/N\operatorname{rad}(P) \qquad \text{(Lemma 2.3.3)} \\ &\geq \operatorname{proj}\dim_{A/(N)} \ {}_{\phi^{-1}}S \end{aligned}$$

Taking the supremum over all S yields (2.8). □

2.4 A classification problem

Let I be either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. In this section we aim to classify the following data.

(A) A pseudocompact ring A with a summable set of primitive orthogonal idempotents $(e_i)_{i\in I}$ such that $\sum_i e_i = 1$ and $Ae_i \not\cong Ae_j$ for $i \neq j$.

(B) A regular normalizing element $N \in \operatorname{rad}(A)$, inducing a homeomorphism $\phi = N \cdot N^{-1}$ such that $\phi(e_i) = e_{i+1}$ and such that the image of the $(e_i)_{i\in I}$ becomes central in $B = A/(N)$.

The solution to this classification problem is the following.

Proposition 2.4.1. *1. If $|I| = \mathbb{Z}$ then A is isomorphic to the ring $T_I(R)$ of lower triangular $I \times I$-matrices with entries in a local pseudocompact ring R. The topology on $T_I(R)$ is the product topology. Under the isomorphism the e_i correspond to the diagonal idempotents and N corresponds to the matrix in which every entry is zero except those on the lower subdiagonal, which are one.*

2. *If $|I| = \mathbb{Z}/n\mathbb{Z}$ then A is isomorphic to a ring of $n \times n$-matrices of the form*

$$\begin{pmatrix} R & UR & \dots & \dots & UR \\ \vdots & R & UR & \dots & UR \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & UR \\ R & R & \dots & \dots & R \end{pmatrix} \tag{2.11}$$

where R is a local pseudocompact ring and U is a normalizing element in $\mathrm{rad}(R)$ *inducing a homeomorphism $U \cdot U^{-1}$. The topology on (2.11) is the product topology. Under the isomorphism of A with (2.11) the $(e_i)_i$ correspond to the diagonal idempotents and N corresponds to the matrix*

$$\begin{pmatrix} 0 & \dots & \dots & 0 & U \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \tag{2.12}$$

Proof. It is clear that the rings exhibited in 1., 2. and the corresponding N, $(e_i)_{i\in I}$ satisfy (A) and (B), so we only have to be concerned with the converse. To simplify the notations we put $P_i = Ae_i$ and $S_i = P_i/\mathrm{rad}(P_i)$ will be the unique simple quotient of P_i.

Recall that by Proposition 2.2.3 we have a matrix form $A = (A_{ij})_{ij}$ and we also have $N = \sum_i e_{i+1}Ne_i = \sum_i N_{i+1,i}$. Since N is regular we have injections

$$A_{ij} \xrightarrow{\cdot N_{j,j-1}} A_{i,j-1} \tag{2.13}$$

$$A_{ij} \xrightarrow{N_{i+1,i}\cdot} A_{i+1,j} \tag{2.14}$$

and furthermore since the image of the $(e_i)_i$ is central in B, A/NA is diagonal. This implies that (2.13) and (2.14) are isomorphisms for $i \neq j-1$.

Left and right multiplication by N are continuous. Furthermore since A is pseudocompact, it follows that the final topology on AN for right multiplication, i.e. let g be right multiplication by N on A, $V \subset AN$ is open $\Leftrightarrow g^{-1}(V)$ is open in A, coincides with the induced topology. We claim that this is also the case for left multiplication with N on A. Indeed left multiplication by N is the composition $A \xrightarrow{\cdot N} A \xrightarrow{\phi} A$. The fact that ϕ is a homeomorphism shows what we want.

Since (2.13) and (2.14) are restrictions from left and right multiplication by N, they are continuous. Furthermore the topology on the image coincides with the induced topology. This means in particular that if $i \neq j-1$, they are homeomorphisms.
The fact that N is normalizing also implies

$$N_{i+1,i}\, A_{ij} \;=\; A_{i+1,j+1}\, N_{j+1,j} \tag{2.15}$$

We will first consider the case $|I| < \infty$.
We define $N_0 = 1$ and for $i = 1, \ldots, n-1$, $N_i = N_{i,i-1} \ldots N_{1,0}$ and maps

$$\theta_{ij} : A_{ij} \to A_{00} : a \mapsto N_i^{-1} a N_j$$

Note that $A_{ij} N_j \subset A_{i0}$ and left multiplication by N_i defines a homeomorphism from $A_{00} \to A_{i0}$ since it is a composition of maps of the form (2.14) where $i \neq j-1$. Therefore it makes sense to use N_i^{-1}.

Clearly if $a \in A_{ij}$, $b \in A_{jk}$, then $\theta_{ij}(a)\,\theta_{jk}(b) = \theta_{ik}(ab)$ and hence $\theta = (\theta_{ij})_{ij}$ defines an inclusion of A into $M_n(A_{00})$. We want to understand its image.

If $i \geq j$ then θ_{ij} is a homeomorphism since in this case right multiplication by N_j is a homeomorphism as it is a composition of maps of the form (2.13) where $i \neq j-1$. Thus $\theta_{ij}(A_{ij}) = A_{00}$. Hence we look at the case $i < j$. We have

$$\theta_{ij}(A_{ij}) = N_i^{-1} A_{ij} N_j = A_{0j} N_j$$

since $N_i A_{0j} = A_{ij}$ in this case. We also have, using maps of the form (2.13) where $i \neq j-1$

$$A_{0j} N_j = A_{0,j+1} N_{j+1,j} N_j = A_{0,j+1} N_{j+1} = \ldots = A_{0,n-1} N_{n-1} = A_{00} U$$

with $U = N_{0,n-1} N_{n-1,n-2} \ldots N_{1,0}$. Thus

$$\theta_{ij}(A_{ij}) = A_{00}\, U$$

By (2.15), we have $A_{00} U = U A_{00}$ and thus U is a regular normalizing element in A_{00}. Hence (putting $R = A_{00}$)

$$\theta(A) = \begin{pmatrix} R & UR & \dots & \dots & UR \\ \vdots & R & UR & \dots & UR \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & UR \\ R & R & \dots & \dots & R \end{pmatrix}$$

From the definition of U it is clear that $U \cdot U^{-1}$ is a homeomorphism.

One computes

$$\theta(N)_{ij} = N_i^{-1} N_{ij} N_j = \begin{cases} 0 & \text{if } i \neq j+1 \\ 1 & \text{if } i = j+1,\ j \neq n-1 \\ U & \text{if } i = 0,\ j = n-1 \end{cases}$$

If $U \notin \operatorname{rad}(R)$ then it is easily seen that $N \notin \operatorname{rad}(A)$ and thus (B) would be violated.

Let us now consider the case $|I| = \infty$.
The following three lemmas are standard.

Lemma 2.4.2. $\operatorname{Ext}^1_A(S_i, S_j) = 0$ *for* $i \neq j,\ j-1$
(The hypothesis $|I| = \infty$ *is not used here.)*

Proof. Assume $\operatorname{Ext}^1_A(S_i, S_j) \neq 0$. Then there is a non-trivial extension

$$0 \to S_j \to F \to S_i \to 0 \tag{2.16}$$

There are now two possibilities

* $NF = 0$. In this case (2.16) is an extension as B-modules. Since the $(e_i)_i$ are central in B, they define endomorphisms of F. Left multiplication by e_j is the identity on S_j and annihilates S_i if $i \neq j$. This yields the following

$$\begin{array}{ccccccccc} 0 & \to & S_j & \to & F & \to & S_i & \to & 0 \\ & & \| & & \downarrow e_j\cdot & & \downarrow f & & \\ 0 & \to & S_j & \to & F & \to & S_i & & \end{array}$$

  where f is the zero map. By the snake lemma it then follows that $F = S_i \oplus S_j$, which is impossible since F is a non-trivial extension, so i must be equal to j.

* $NF \neq 0$. In this case left multiplication by N defines a non-trivial map $S_i \to_\phi S_j$ and since S_i, ${}_\phi S_j$ are simple, this map must be an isomorphism. Thus $S_i =_\phi S_j$. Now ${}_\phi S_j$ is a simple quotient of ${}_\phi Ae_j = Ae_{j-1}$. Thus ${}_\phi S_j = S_{j-1}$ and we find $i = j-1$. □

IK ZOU NOG EEN UITLEG WILLEN SCHRIJVEN BIJ DIE MAP VAN S_i NAAR ${}_\phi S_j$. DE EERSTE KEER LIFTEN IS VOLGENS MIJ HET VOLGENDE:

$$\begin{array}{ccc} F & \rightarrow & F/S_j \\ & \searrow & \\ & N\cdot & \downarrow \\ & & F \end{array}$$

WAT EEN NIET NUL MAP GEEFT VAN S_i NAAR F. MAAR WAT IS DE TWEEDE LIFT OM IN S_j TERECHT TE KOMEN ?

Lemma 2.4.3. *Assume* $|I| = \infty$. *Let* M *be a finite length module in* $\operatorname{mod}(A)$ *with composition factors among the* $(S_j)_j$. *Assume* $M/\operatorname{rad}(M) = S_i$. *Then the composition factors of* M *are of the form* $S_k,\ k \geq i$.

Proof. We prove this by induction on the length of M. Let $S_t \subset M$ be a simple submodule. By induction, the subquotients of M/S_t are of the form $S_k,\ k \geq i$.

Hence if $t < i$, then it follows from Lemma 2.4.2 that $\operatorname{Ext}^1_A(M/S_t, S_t) = 0$ and thus $M = M/S_t \oplus S_t$. In particular, S_t is a simple quotient of M, different from S_i, contradicting the hypotheses. □

Lemma 2.4.4. *Assume* $|I| = \infty$. *Then* $\operatorname{Hom}(P_i, P_j) = 0$ *for* $i < j$.

Proof. Assume there is a non-trivial map $P_i \xrightarrow{\psi} P_j$. Since P_j is separated there exists an open submodule $L \subsetneq P_j$ such that $P_i/\psi^{-1}(L)$ is non zero. P_j/L has finite length and is modulo its radical, equal to S_j. $P_i/\psi^{-1}(L)$ is a subobject of P_j/L and since S_i is a quotient of $P_i/\psi^{-1}(L)$, it follows that S_i is a subquotient of P_j/L. This implies that $i \geq j$ by Lemma 2.4.3 and we are done. □

We now finish the proof of Proposition 2.4.1.
Since $\operatorname{Hom}_A(P_i, P_j) = e_i A e_j = A_{ij}$ this last lemma implies that in the case $|I| = \infty$ the matrix form for A is lower triangular.

For every $(i, j),\ i \geq j$ there is a homeomorphism

$$\theta_{ij} : A_{ij} \to A_{00}$$

obtained by composing homeomorphisms of the form (2.13) and (2.14). One checks that θ_{ij} is uniquely determined in this way. By a verification as in the case $|I| < \infty$ (but somewhat more complicated) one also shows that θ_{ij} is compatible with multiplication and $\theta_{i+1,i}(N_{i+1,i}) = 1$.

Thus A is isomorphic to the ring of lower triangular matrices with entries in $R = A_{00}$ and N has the required form.
This finishes the proof of Proposition 2.4.1. □

We now exhibit when pseudocompact rings as in Proposition 2.4.1 are locally noetherian and have finite global dimension.

Proposition 2.4.5. *Let A be a pseudocompact ring as in Proposition 2.4.1. Then A is locally noetherian if and only if R is noetherian. Furthermore if R is noetherian then*

$$\operatorname{Gl\,dim} A = \begin{cases} \operatorname{Gl\,dim} R + 1 & \text{if } |I| = \infty \\ \operatorname{Gl\,dim} R/(U) + 1 & \text{if } 2 \leq |I| < \infty \\ \operatorname{Gl\,dim} R & \text{if } |I| = 1 \end{cases} \tag{2.17}$$

Proof. We have

$$A/(N) = \begin{cases} R/(U)^I & \text{if } |I| < \infty \\ R^I & \text{if } |I| = \infty \end{cases} \tag{2.18}$$

The condition for A to be noetherian then follows from Proposition 2.1.23.

The statement about the global dimension is clear in the case $|I| = 1$. For $|I| > 1$ we notice that N satisfies the hypotheses of Proposition 2.3.2 and 2.3.4. Then (2.17) follows from (2.18). □

2.5 More classification

In this section we classify rings R satisfying

(C) R is local, complete and contains an algebraically closed field k, isomorphic to its residue field.

(D) Let m be the maximal ideal of R. We require that m contains a regular normalizing element U such that $R/(U)$ is a commutative noetherian Cohen-Macauly local ring of Krull dimension one.

(E) $\operatorname{proj\,dim}_R\ R/m\ < \infty$.

The solution to this classification problem is as follows.

Proposition 2.5.1. *Assume that R satisfies* (C), (D) *and* (E) *above. Then*

$$R \cong k\langle\langle x, y\rangle\rangle/(\phi) \tag{2.19}$$

where

$$\phi = yx - qxy + \text{higher order terms} \tag{2.20}$$

for some $q \in k^*$ or

$$\phi = yx - xy - x^2 + \text{higher order terms} \tag{2.21}$$

Conversely, every such ring satisfies (C), (D) *and* (E).

Proof. Let us first show that a ring R of the form (2.19) with ϕ of the form (2.20) or (2.21) does indeed satisfy (C), (D) and (E).

It is clear that (C) is satisfied. For (E) observe that we have $\phi = ux + vy$ for some u, $v \in m$ such that $(\overline{u}, \overline{v})$ form a basis of m/m^2. This means that we have a complex

$$0 \longrightarrow R \xrightarrow{(u\,v)} R^2 \xrightarrow{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} R \longrightarrow R/m \longrightarrow 0 \tag{2.22}$$

and we have to show that this complex is exact.
We filter R with the m-adic filtration. For this filtration it is easy to see that

$$\operatorname{gr}\ R = k\langle x, y\rangle/(\theta) \tag{2.23}$$

where θ consists of the quadratic part of ϕ.

The exactness of

$$0 \longrightarrow \operatorname{gr}\ R \xrightarrow{(\overline{u},\overline{v})} (\operatorname{gr}\ R)^2 \xrightarrow{\left(\begin{smallmatrix} \overline{x} \\ \overline{y} \end{smallmatrix}\right)} \operatorname{gr}\ R \longrightarrow R/m \longrightarrow 0 \tag{2.24}$$

which follows from the fact that all the morphisms are graded, implies the exactness of (2.22). This proves (E).

Now let us consider (D). We assume that R is not commutative since otherwise (D) is trivial. From (2.23) it follows that R is a domain, so every element of R is regular. Put $U = [y, x]$. We claim that U is normalizing. This was independently observed by Artin and Stafford. Assume first that we are in the case (2.20). One computes

$$\begin{aligned} Ux &= qxU + [x,\gamma] \\ Uy &= q^{-1}yU + q^{-1}[y,\gamma] \end{aligned} \tag{2.25}$$

where γ represents the non-quadratic terms of ϕ. Now clearly

$$\begin{aligned} [x,\gamma] &= \textstyle\sum_i u_i U v_i \\ [y,\gamma] &= \textstyle\sum_i u_i' U v_i' \end{aligned} \tag{2.26}$$

for appropriate u_i, v_i, u_i', $v_i' \in R$. Substituting (2.26) into (2.25) and then substituting the resulting equations repeatedly into themselves, yields the formulas

$$\begin{aligned} Ux &= (qx + \ldots)U \\ Uy &= (q^{-1}y + \ldots)U \end{aligned}$$

Thus U is a normalizing element.

Case (2.21) is treated similarly starting from

$$\begin{aligned} Ux &= xU + [x,\gamma] \\ Uy &= (y-2x)U + [y,\gamma] - [x,\gamma] \end{aligned}$$

Since $R/[y,x] = k[[x,y]]/(\phi)$ is clearly Cohen-Macauly of Krull dimension one, we have shown that R satisfies (D).

Now we prove the converse. Note that by (D), R is automatically left and right noetherian.

Step 1. $\operatorname{proj}\dim_R\ R/m = 2$

Proof. We have

$$\begin{aligned} \operatorname{proj}\dim_R\ R/m &= 1 + \operatorname{proj}\dim_R\ m \\ &= 1 + \operatorname{proj}\dim_{R/(U)}\ m/Um \quad \text{(Lemma 2.3.3)} \end{aligned}$$

In particular, $\operatorname{proj}\dim_{R/(U)}\ m/Um$ is finite. Since $R/(U)$ is commutative of Krull dimension one, this implies $\operatorname{proj}\dim_{R/(U)}\ m/Um \leq 1$.
Thus $\operatorname{proj}\dim_R\ R/m \leq 2$.

Assume that the projective dimension of R/m is strictly less than 2. It cannot be 0, hence it must be one. This means that there is a resolution

$$0 \to R^n \to R \to R/m \to 0$$

which easily yields that R is the completion of a free k-algebra in n variables. If $n > 1$ then R is not noetherian and if $n = 1$ then R is a discrete valuation ring and hence (D) is not satisfied. $\square$

Step 2. *The minimal resolution of R/m looks like*

$$0 \longrightarrow R \xrightarrow{(u\,v)} R^2 \xrightarrow{\binom{x}{y}} R \longrightarrow R/m \longrightarrow 0$$

where $(\overline{x}, \overline{y})$, $(\overline{u}, \overline{v})$ form bases for m/m^2.

Proof. The minimal resolution of R/m looks like

$$0 \longrightarrow R^b \xrightarrow{g} R^a \xrightarrow{f} R \longrightarrow R/m \longrightarrow 0 \tag{2.27}$$

Tensoring with $R/(U) = \overline{R}$ yields an exact sequence

$$0 \to \overline{R}^b \to \overline{R}^a \to \overline{R} \to 0$$

By taking ranks, it then follows that $a = b + 1$.

Since $R/(U)$ is Cohen-Macauly we have, for $i \leq 1$

$$\operatorname{Ext}^i_R(R/m, R) = \operatorname{Ext}^{i-1}_{R/(U)}(R/m, R/(U)) = 0$$

By dualizing (2.27) and using the previous, Step 1 and the fact that R is local, we find a minimal resolution of $\operatorname{Ext}^2_R(R/m, R)$ as right R-module

$$0 \longrightarrow R \xrightarrow{f^t} R^a \xrightarrow{g^t} R^b \longrightarrow \operatorname{Ext}^2_R(R/m, R) \longrightarrow 0 \tag{2.28}$$

Now $\operatorname{Ext}^2_R(R/m, R)$ is annihilated by m and thus $\dim_k \operatorname{Ext}^2_R(R/m, R) = b$. Hence if $b \neq 1$, we see that (2.28) decomposes as a direct sum of subcomplexes. But then so does the dual complex (2.27), which is impossible since this is a minimal projective resolution of a simple R-module. We conclude that $b = 1$, $a = 2$.

We now find that the minimal resolution of R/m looks like (2.27) with $(\overline{x}, \overline{y})$ a basis for m/m^2. Since the dual complex of (2.27) is a minimal resolution of R/m (as right module), we find that $(\overline{u}, \overline{v})$ is also a basis for m/m^2. $\square$

We can now conclude the proof of Theorem 2.5.1. From Step 2 it follows that R is as in (2.19) with $\phi = ux + vy$. It is now easy to see that ϕ can be put in one of the standard forms (2.20) (2.21). □

Proposition 2.5.2. *Let R, m, U be as in Proposition 2.5.1. Then $R/(U)$ is regular if and only if $U \notin m^2$.*

Proof. If $U \in m/m^2$ then by Proposition 2.3.4 $R/(U)$ is regular. Conversely, assume that $R/(U)$ is regular. Then

$$1 = \dim_k (m/(U))/(m/(U))^2 = \dim_k\ m/((U) + m^2)$$

whence $U \notin m^2$. □

Remark 2.5.3. This result is false in higher dimension. Consider for example

$$R = k\langle\langle x, y\rangle\rangle/([x, [x, y]], [y, [x, y]])$$

Then $R/([x, y]) = k[[x, y]]$ is regular, but $[x, y] \in \mathrm{rad}^2(R)$.

2.6 Proof of Theorem 2.0.1

We start by discussing things a bit more generally.
Let $\mathcal{A}$ be a Grothendieck category, $G : \mathcal{A} \to \mathcal{A}$ an autoequivalence and $\eta : G \to \mathrm{id}_{\mathcal{A}}$ a natural transformation such that

$$\eta(G(A)) = G(\eta(A)) \tag{2.29}$$

for all $A \in \mathcal{A}$. Define

$$\mathcal{B} = \{A \in \mathcal{A} |\ \eta(A) = 0\}$$

Then the following properties are easily verified.

Lemma 2.6.1. *1. $\mathcal{B}$ is closed under subquotients, direct sums and direct products (and hence under limits and colimits).*

2. $\mathcal{B}$ is closed under G, G^{-1} and if $A \in \mathcal{A}$ then $\ker \eta(A)$, $\operatorname{coker} \eta(A) \in \mathcal{B}$.

3. Let $i_* : \mathcal{B} \to \mathcal{A}$ be the inclusion functor. The functors $i^!$, $i^* : \mathcal{A} \to \mathcal{B}$ defined by

$$\begin{aligned} i^!(A) &= \ker(A \xrightarrow{\eta(G^{-1}(A))} G^{-1}(A)) \\ i^*(A) &= \operatorname{coker}(G(A) \xrightarrow{\eta(A)} A) \end{aligned}$$

are respectively the right and the left adjoint of i_*.

Remark 2.6.2. The condition (2.29) is not automatic. A counter example is given by $\mathcal{A} = \mathrm{Mod}(A)$ with $A = k \oplus V$, where V is a k-vectorspace such that $V^2 = 0$ in A. For G we take $M \mapsto_\psi M$ for some $\psi \in \mathrm{GL}(V)$, which we extend in the obvious way to A. To define η, we take $v \in V$, not ψ-invariant and we define $\phi :_\psi A \to A$ as the bimodule map which sends 1 to v. Then we put $\eta(-) = \phi \otimes_A -$. In this case $G(\eta(A)) \neq \eta(G(A))$, and in particular $\mathcal{B}$ is not G-invariant.

Nevertheless (2.29) holds in the case we are interested in as the following lemma shows.

Lemma 2.6.3. *Assume that for all injectives $E \in \mathcal{A}$ we have that $\eta(E)$ is surjective. Then (2.29) holds.*

Proof. We have that $\eta\, G$ is a natural transformation $G^2 \to G$. Applying this to the map $G^{-1}\eta G : GE \to E$, we get a commutative diagram (using $E = G(G^{-1}(E))$).

$$\begin{array}{ccc} G^2(E) & \xrightarrow{\eta(GE)} & GE \\ {\scriptstyle \eta(GE)}\downarrow & & \downarrow{\scriptstyle \eta(E)} \\ GE & \xrightarrow[G^{-1}\eta(GE)]{} & E \end{array}$$

Applying this diagram with E injective and using the surjectivity hypothesis we find that $G^{-1}(\eta(GE)) = \eta(E)$.

Now let $A \in \mathcal{A}$ be arbitrary and let

$$0 \to A \to E \to F$$

be an injective resolution of A. This yields commutative diagrams.

$$\begin{array}{ccccccc} 0 & \longrightarrow & GA & \longrightarrow & GE & \longrightarrow & GF \\ & & {\scriptstyle G(\eta(A))}\uparrow & & {\scriptstyle G(\eta(E))}\uparrow & & {\scriptstyle G(\eta(F))}\uparrow \\ 0 & \longrightarrow & G^2A & \longrightarrow & G^2E & \longrightarrow & G^2F \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & GA & \longrightarrow & GE & \longrightarrow & GF \\ & & {\scriptstyle \eta(G(A))}\uparrow & & {\scriptstyle \eta(G(E))}\uparrow & & {\scriptstyle \eta(G(F))}\uparrow \\ 0 & \longrightarrow & G^2A & \longrightarrow & G^2E & \longrightarrow & G^2F \end{array}$$

The fact that the rightmost squares of these diagrams are commutative, yields the result in general. □

Now let $\mathcal{D} \subset \mathcal{B}$ be a G, G^{-1}-stable localizing subcategory (that is, closed under subquotients, extensions and direct sums) and define $\mathcal{D}^\infty$ as the full subcategory of $\mathcal{A}$ consisting of objects A having an ascending filtration $(F_iA)_{i\in\mathbb{N}}$ such that

$$F_0A = 0, \quad F_{n+1}A/F_nA \in \mathcal{D}, \quad A = \cup_n F_nA \tag{2.30}$$

If there is such a filtration with $F_nA = F_{n+1}A = \dots$, then we say that $A \in \mathcal{D}^n$. Note that $\mathcal{D}^n = \mathcal{B}^n \cap \mathcal{D}^\infty$.

If $A \in \mathcal{A}$ then there is a maximal filtration $(R_nA)_n$ on A satisfying the first two properties in (2.30) with $\mathcal{D} = \mathcal{B}$. This filtration is given by

$$R_nA = \ker(A \xrightarrow{\eta^n} G^{-n}A)$$

An object $A \in \mathcal{A}$ is in $\mathcal{B}^n$ if $R_nA = A$ and it is in $\mathcal{B}^\infty$ if $\cup_n R_nA = A$. A is in $\mathcal{D}^\infty$ if in addition $R_{n+1}A/R_nA \in \mathcal{D}$.

We also consider the descending filtration on A given by

$$L_nA = \mathrm{im}(G^nA \xrightarrow{\eta^n} A)$$

This filtration satisfies $L_nA/L_{n+1}A \in \mathcal{B}$. If $A \in \mathcal{B}_m$, then $L_nA \subset R_{m-n}A$, for all $n \geq m$.

Proposition 2.6.4. *1. $\mathcal{D}^\infty$ is a localizing subcategory in $\mathcal{A}$.*

2. Assume that $\mathcal{A}$ is locally noetherian. If $\mathcal{D}$ is closed under injective hulls in $\mathcal{B}$, then $\mathcal{D}^\infty$ is closed under injective hulls in $\mathcal{A}$.

Proof. 1. Only the closedness under extensions is not immediately clear. Let

$$0 \to D_1 \to A \xrightarrow{\psi} D_2 \to 0$$

be an extension such that $D_1,\ D_2 \in \mathcal{D}^\infty$.

We consider four cases.

(a) If $D_1 \in \mathcal{D}^m$ and $D_2 \in \mathcal{D}^n$, for some m, n then it is easy to see that $A \in \mathcal{D}^{m+n}$.

(b) Assume $D_1 \in \mathcal{D}^n$, for some n. Let F be a filtration on D_2 satisfying (2.30). Then $A = \cup_i \psi^{-1}(F_iD_2)$. Since by (a) all $\psi^{-1}(F_iD_2)$ are in $\mathcal{D}^\infty$, we conclude that this is also true for A.

(c) Assume $D_2 \in \mathcal{D}^n$, for some n. Then $L_n A \subset D_1$ and hence $L_n A \in \mathcal{D}^\infty$. The exact sequence

$$0 \longrightarrow G^n R_n A \longrightarrow G^n(A) \xrightarrow{\eta^n} L_n A \longrightarrow 0$$

combined with (b) ($R_n A \in \mathcal{D}^n$) shows what we want.

(d) Assume now that D_1, D_2 are general. Using (c), we can now use the same reasoning as in (b) to finish the proof.

2. This assertion can be split into two parts.

(a) $\mathcal{B}^\infty$ is closed under injective hulls in $\mathcal{A}$.
To prove this let $B \hookrightarrow A$ be an essential extension with $B \in \mathcal{B}^\infty$ and $A \in \mathcal{A}$. We have to show that $A \in \mathcal{B}^\infty$.
We may clearly assume that A/B contains no subobject in $\mathcal{B}^\infty$.
Assume first that A is noetherian. In that case $B \in \mathcal{B}^n$ for some n. From the exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\eta^n} G^{-n}(L_n A) \longrightarrow 0$$

we deduce that $A/B \cong G^{-n}(L_n A)$. Hence $L_n A$ contains no subobject in $\mathcal{B}^\infty$. Thus $L_n A \cap B = 0$ and hence $L_n A = 0$. This yields $A = B$ and we are through.
Now assume that A is general. By hypothesis $A = \cup_{i \in I} A_i$ where the A_i are noetherian. By looking at the pairs $(B \cap A_i, A_i)$, we find that $A_i \in \mathcal{B}^\infty$. Hence $A \in \mathcal{B}^\infty$.

(b) $\mathcal{D}^\infty$ is closed under injective hulls in $\mathcal{B}^\infty$.
To prove this, assume that $D \hookrightarrow B$ is an essential extension with $D \in \mathcal{D}^\infty$ and $B \in \mathcal{B}^\infty$. Since $B = \cup_{n \in \mathbb{N}} R_n B$, by considering all the pairs $(R_n B \cap D, R_n B)$, we may reduce to the case $B \in \mathcal{B}^n$. We then use induction on n. If $n = 1$ then $B \in \mathcal{B}$, $D \in \mathcal{D}$ and the result follows from the hypotheses on $\mathcal{D}$.
Assume now $n > 1$. We have the standard exact sequence

$$0 \longrightarrow R_1 B \longrightarrow B \xrightarrow{\eta} G^{-1}(L_1 B) \longrightarrow 0 \tag{2.31}$$

Since $B \in \mathcal{B}^n$, it follows that $R_n B = B$. Therefore the exact sequence

$$0 \longrightarrow G^n(R_n B) \longrightarrow G^n B \xrightarrow{\eta^n} L_n B \longrightarrow 0$$

yields that $L_nB = 0$, so $B \in B_n$. This implies that $L_1B \subset R_{n-1}B$, thus $L_1B \in \mathcal{B}^{n-1}$. Clearly $R_1B \in \mathcal{B}$. So looking at the pairs $(R_1B \cap D, R_1B)$ and $(L_1B \cap D, L_1B)$ and induction, reveals that R_1B, $L_1B \in \mathcal{D}^\infty$. Hence from (2.31), we deduce that $B \in \mathcal{D}^\infty$. □

From here on we assume that $\mathcal{A}$ is locally noetherian.
Let $(T_i)_{i\in J}$ be the simple objects in $\mathcal{B}$. It is easy to see that these are also the simple objects $\mathcal{B}^\infty$.
Define $t : J \to J$ by $G^{-1}(T_i) = T_{ti}$. Clearly t is a permutation of J.

We let $\mathcal{D}$, $\mathcal{C}$ be the minimal localizing subcategories of $\mathcal{B}$ and $\mathcal{B}^\infty$ containing $(T_i)_{i\in J}$. Clearly $\mathcal{C} = \mathcal{D}^\infty$.
For i, $j \in J$, we write $i \sim_\mathcal{D} j$, $i \sim_\mathcal{C} j$ if T_i, T_j are respectively in the same connected component of $\mathcal{D}$ and $\mathcal{C}$.
With a reasoning similar to Lemma 2.4.2, one shows that

$$i \sim_\mathcal{C} j \Rightarrow \exists j \in \mathbb{Z} : i \sim_\mathcal{D} t^p j \tag{2.32}$$

Let $K \subset J$ be a union of equivalence classes for $\sim_\mathcal{D}$, stable under t, t^{-1}. By (2.32), K is then also a union of equivalence classes for $\sim_\mathcal{C}$.

We denote by $\mathcal{D}_K$, $\mathcal{C}_K$ the minimal localizing subcategories of $\mathcal{D}$ and $\mathcal{C}$ containing $(T_i)_{i\in K}$. Clearly

$$\begin{aligned} \mathcal{C} &= \oplus_{(K\in J/\sim_\mathcal{C})}\mathcal{C}_K \\ \mathcal{D} &= \oplus_{(K\in J/\sim_\mathcal{D})}\mathcal{D}_K \end{aligned} \tag{2.33}$$

Let E_i be the injective hull of T_i in $\mathcal{C}$. Put $E_K = \oplus_{i\in K} E_i$. Then E_K is an injective cogenerator of $\mathcal{C}_K$. The injective hull of T_i in $\mathcal{D}$ is given by $F_i = R_1E_i$. We also put $F_K = \oplus_{i\in K} F_i$.

Proposition 2.6.5. *Assume that $\eta(E_K)$ is surjective. Let $C_K = \mathrm{End}_\mathcal{C}(E_K)$, $D_K = \mathrm{End}_\mathcal{D}(F_K)$ with the natural topology (as in Theorem 2.1.1). Then there is a regular normalizing element $N \in \mathrm{rad}(C_K)$ with the following properties.*

1. *$D_K = C_K/(N)$ as pseudocompact rings.*
2. *Put $\phi = N \cdot N^{-1}$. Let $e_i \in C_K$ be the idempotent corresponding to the projection $E_K \to E_i$. Then $\phi(e_i) = e_{ti}$.*
3. *Let $U \in \mathcal{C}_K$. There is an isomorphism as C_K-modules*

$$p :_\phi \mathrm{Hom}(U, E_K) \to \mathrm{Hom}(GU, E_K)$$

which is functorial in U.

4. *There is a commutative diagram.*

$$\begin{array}{ccc} \mathrm{Hom}(U, E_K) & \xrightarrow{N\cdot} & {}_\phi\,\mathrm{Hom}(U, E_K) \\ \Big\| & & p\Big\downarrow \\ \mathrm{Hom}(U, E_K) & \xrightarrow{\mathrm{Hom}(\eta(U),E_K)} & \mathrm{Hom}(GU, E_K) \end{array} \tag{2.34}$$

5. *ϕ is a homeomorphism.*

Proof. Since $\ker(\eta(G^{-1}(E_K))) = R_1E_K = F_K$ and $\eta(E_K)$ is surjective which yields that $\eta(G^{-1}(E_K)) = G^{-1}(\eta(E_K))$ is also surjective, we have an exact sequence

$$0 \longrightarrow F_K \longrightarrow E_K \xrightarrow{\eta(G^{-1}(E_K))} G^{-1}(E_K) \longrightarrow 0$$

Applying $\mathrm{Hom}_{\mathcal{C}}(-, E_K)$ and using the fact that $\mathrm{Hom}_{\mathcal{C}}(F_K, E_K) = \mathrm{Hom}_{\mathcal{D}}(F_K, F_K)$ by Lemma 2.6.1 (3.), we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(G^{-1}(E_K), E_K) \xrightarrow{r} C_K \xrightarrow{s} D_K \longrightarrow 0 \tag{2.35}$$

Here $r(f) = f \circ \eta(G^{-1}(E_K))$ and $s(g) = g|F_K$.
If U is a finite length object in $\mathcal{D}$, then one checks that $s^{-1}({}_{D_K}(U)) = {}_{C_K}(U)$ and hence s is continuous.

Now choose isomorphisms $\mu_i : G^{-1}(E_i) \to E_{ti}$ and let $\mu = \oplus_{i\in K}\mu_i$. The map which sends h to $h\circ\mu$ defines an isomorphism $C_K \to \mathrm{Hom}_{\mathcal{C}}(G^{-1}(E_K), E_K)$. Put $N = \mu \circ \eta(G^{-1}(E_K))$, as element of C_K. Then (2.35) yields an exact sequence

$$0 \longrightarrow C_K \xrightarrow{\cdot N} C_K \longrightarrow D_K \longrightarrow 0$$

from which we deduce that N is regular and normalizing.

The simple pseudocompact C_K-modules are of the form $\mathrm{Hom}_{\mathcal{C}}(T_i, E_K)$ and if $f : T_i \to E_K$ is a map in $\mathcal{C}$, then f has its image in F_K (DIT BEGRIJP IK NIET !!) and thus is annihilated by η. Hence $Nf = \mu\eta f = 0$ and thus $N \in \mathrm{rad}(C_K)$.

We now show that N satisfies 2. e_i is the composition of the projection $p_i : E_K \to E_i$ and the injection $q_i : E_i \to E_K$. The fact that $Ne_iN^{-1} = e_{ti}$

now follows from the following commutative diagram.

$$\begin{array}{ccccc}
E_K & \xrightarrow{p_i} & E_i & \xrightarrow{q_i} & E_K \\
\big\downarrow{\scriptstyle \eta(G^{-1}(E_K))} & & \big\downarrow{\scriptstyle \eta(G^{-1}(E_i))} & & \big\downarrow{\scriptstyle \eta(G^{-1}(E_K))} \\
G^{-1}(E_K) & \xrightarrow{G^{-1}(p_i)} & G^{-1}(E_i) & \xrightarrow{G^{-1}(q_i)} & G^{-1}(E_K) \\
\big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu_i} & & \big\downarrow{\scriptstyle \mu} \\
E_K & \xrightarrow{p_{ti}} & E_{ti} & \xrightarrow{q_{ti}} & E_K
\end{array}$$

Now we prove 3. Define the map

$$p : \mathrm{Hom}(U, E_K) \to \mathrm{Hom}(GU, E_K) : f \mapsto G(\mu^{-1} f)$$

We investigate the behaviour of p with respect to left multiplication by an element g of C_K. We find $p(gf) = G(\mu^{-1} gf) = G(\mu^{-1} g\mu\mu^{-1} f) = G(\mu^{-1} g\mu)p(f)$. Now we look at the following commutative diagram.

$$\begin{array}{ccc}
E_K & \xrightarrow{G(\mu^{-1}g\mu)} & E_K \\
\big\downarrow{\scriptstyle \eta(G^{-1}(E_K))} & & \big\downarrow{\scriptstyle \eta(G^{-1}(E_K))} \\
G^{-1}(E_K) & \xrightarrow{\mu^{-1}g\mu} & G^{-1}(E_K) \\
\big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu} \\
E_K & \xrightarrow{g} & E_K
\end{array}$$

From this diagram we deduce that $G(\mu^{-1} g\mu) = N^{-1} gN = \phi^{-1}(g)$. So we conclude that to make p a map of C_K-modules, it suffices to twist $\mathrm{Hom}(U, E_K)$ by ϕ.

Now we prove 4. The commutativity of (2.34) amounts to the identity $G(\mu N f) = f\eta(U)$, for f in $\mathrm{Hom}(U, E_K)$. Since $G(\mu^{-1} N f) = G(\eta(G^{-1}(E_K))f) = \eta(E_K)G(f)$, this follows from the fact that η is a natural transformation.

Finally we note that 5. follows from Lemma 2.1.18 and 3. $\square$

Now we specialize to the situation of Theorem 2.0.1.
Thus $\mathcal{B} = \mathrm{Qch}(Y)$ for a Cohen-Macauly curve Y and $J = Y$, since the simple objects in $\mathrm{Qch}(Y)$ are the pointmodules, which correspond to the points of Y. This also implies that $T_x = \mathcal{P}_x$ and $t = \tau$.
It is also clear that $x \sim_{\mathcal{D}} y \Leftrightarrow x = y$ and thus the equivalence classes for $\sim_{\mathcal{D}}$

are singletons. From (2.32), it follows that the equivalence classes for $\sim_{\mathcal{C}}$ are given by the τ-orbits.
Finally we have for $K \subset Y$

$$D_K = \prod_{x \in K} \hat{\mathcal{O}}_{Y,x} \tag{2.36}$$

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With these data, the proof of Theorem 2.0.1 is now a simple matter of translation using the results of the section 2.4, 2.5,....

Proof. 1. This follows from (2.33).

2.,3. By Theorem 2.1.1, the functor F° given by $\mathcal{M} \mapsto \mathrm{Hom}(\mathcal{M}, E_K)$ defines an equivalence between the dual of $\mathcal{C}_{f,z}$ and the category of left pseudo-compact modules over the ring $C_z = C_K$, where K is the τ-orbit of z.
By Proposition 2.6.5, $D_K = C_K/(N)$ for $N \in \mathrm{rad}(C_K)$ such that $\phi = N \cdot N^{-1}$ is a homeomorphism and such that $\phi(e_y) = e_{\tau y}$. Thus C_K satisfies the hypotheses of Proposition 2.4.1. From that proposition, it follows that we can put C_K and N in the required matrix forms and that we have $R = \hat{\mathcal{O}}_{Y,z}$ if $|K| = \infty$ and $R/(U) = \hat{\mathcal{O}}_{Y,z}$ if $|K| < \infty$.
To find out the exact form of R, we first note that by (2.36), D_K is locally noetherian and hence so is C_K by Proposition 2.1.23. Furthermore, by Proposition 2.6.4, every object in $\mathcal{C}_K$ has finite injective dimension. Thus C_K has finite global dimension. Hence the hypotheses for Proposition 2.5.1 are satisfied and thus R does indeed have the form (2.1) or (2.2).

Now note that if $2 \leq |I| < \infty$, then Proposition 2.4.5 actually tells us that $\mathrm{Gl\,dim}\ \hat{\mathcal{O}}_{Y,z} < \infty$. Thus z is regular on Y. Also by Proposition 2.5.2, this implies that $U \notin \mathrm{rad}^2(R)$.

The essential image of F° is given by the pseudocompact left C_z-modules of finite length. From Proposition 2.2.3, it follows that such modules correspond precisely to the finite dimensional left modules over C_z satisfying $V = \oplus_i e_i V$. KAN JE DIT AUB EVEN VERDUIDELIJKEN ?
Under the duality $V \mapsto V^*$, such modules correspond to the finite dimensional right C_K-modules W satisfying $W = \oplus_i W e_i$.

We now claim that in fact every finite dimensional C_z representation is pseudocompact. This is clear if $|O_\tau(z)| < \infty$, so assume $|O_\tau(z)| = \infty$.

In that case the statement depends upon the fact that $\mathrm{card}(k) = \infty$. Clearly we may reduce to the case that W is simple. Then W is annihilated by the Jacobson radical of C_z, which according to [18] is precisely given by the common annihilator of the pseudocompact simple modules. In other words, $\mathrm{rad}(C_z)$ is given by the lower triangular matrices, having only non-units on the diagonal. Thus W is a $\prod_i k$-module. A finite dimensional simple module over a commutative k-algebra is clearly one dimensional. Hence $\dim(W) = 1$ and we have a corresponding character $\chi : \prod_i k \to k$. Choose $a = (a_i)_i \in \prod_i k$ in such a way that $a_i \neq a_j$ if $i \neq j$. Then there exists $b \in k$ such that $\chi(a - b) = 0$. But $a_i - b = 0$ for at most one i, and hence the ideal generated by $a - b$ is either improper or the kernel of the projection map $\mathrm{pr}_i : \prod_i k \to k$. The first case is clearly impossible and the second case implies that χ is given by projection on the i'th factor. Hence W is pseudocompact.

Putting $F(\mathcal{M}) = \mathrm{Hom}(\mathcal{M}, E_K)^*$ finishes the proof of 2., 3.

4. Since $F^\circ(\mathcal{P}_{\tau^i z})$ is by construction the i'th simple module of C_z, it is given by $C_z e_i / \mathrm{rad}(C_z e_i)$. Hence $F(\mathcal{P}_{\tau^i z}) = (C_z e_i / \mathrm{rad}(C_z e_i))^* = e_i C_z / \mathrm{rad}(e_i C_z)$.

5. This amounts to the construction of a natural isomorphism between $FG(\mathcal{M})$ and $(F\mathcal{M})_\phi$ for $\mathcal{M} \in \mathcal{C}_{f,z}$.
Since $FG(\mathcal{M}) = \mathrm{Hom}(G\mathcal{M}, E_K)^*$ and $(F\mathcal{M})_\phi = (\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, E_K)^*)_\phi = {}_\phi(\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, E_K)^*)$, we can use p^* with p as in Proposition 2.6.5 (3.).

6. This diagram can be obtained by dualizing (2.34).

7. Let $\mathcal{M} \in \mathrm{Qch}(Y)$. Then $F\mathcal{M} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, E_K)^* = \mathrm{Hom}_{\mathrm{Qch}(Y)}(\mathcal{M}, F_K)^* = \prod_i \mathrm{Hom}_{\mathrm{Qch}(Y)}(\mathcal{M}, F_i)^*$ where as before F_i is the injective hull of $k(\tau^i z)$ in $\mathrm{Qch}(Y)$. It follows from Matlis duality that $\mathrm{Hom}_{\mathrm{Qch}(Y)}(\mathcal{M}, F_i)^*$ is the completion of $\mathcal{M}$ at $\tau^i z$. $\square$

Chapter 3

The center of the Proj of a three dimensional Artin-Schelter regular algebra

In this chapter we will prove that in characteristic 0, the center of Proj A, where A is a three dimensional three generator Artin-Schelter regular algebra which is finite over its center, is the projective plane $\mathbb{P}^2$, using the main theorem (Theorem 2.0.1) of the previous chapter in a slightly adjusted version.

Let us indicate the general settings of this chapter.
As was pointed out in the first chapter, one associates to a triple $(Y, \sigma, \mathcal{L})$ where $Y \subset \mathbb{P}^2$ is a cubic divisor, $\sigma \in \mathrm{Aut}(Y)$ and $\mathcal{L} = \mathcal{O}_Y(1)$ which has the additional property $\mathcal{L}^{(1-\sigma)^2} \cong \mathcal{O}_Y$, a three dimensional regular algebra A [6]. We recall that the homological properties of A closely resemble those of a polynomial ring in three variables and in the framework of [9], it is natural to think of Proj A as a non-commutative $\mathbb{P}^2$.

Assume now that σ has finite order. In that case it has been shown in [7] that A is finite over its center R. Let X be the classical Proj of R [20]. To A we may associate a sheaf of $\mathcal{O}_X$-algebras $\mathcal{O}_\Delta$. Let $\mathcal{Z} = Z(\mathcal{O}_\Delta)$. In general $\mathcal{Z}$ will not be equal to $\mathcal{O}_X$, so we define $Z = \underline{\mathrm{Spec}}\mathcal{Z}$. We call Z the center of Proj A.

Motivated by many examples, we dare to make the following conjecture.

Conjecture 3.0.1. $Z \cong \mathbb{P}^2$.

As was said before, we will show that Conjecture 3.0.1 is true in characteristic 0.

In Section 3.1 we develop the rudiments of an intersection theory for $\mathcal{O}_\Delta$. Section 3.2 is dedicated the the Proj of graded rings which are finite over a commutative graded ring whereas Section 3.3 specifies to our situation. In this last section we will use the intersection theory developed in the first section to prove the conjecture in characteristic 0.

3.1 Intersection theory on orders over surfaces

In these notes k will be an algebraically closed basefield.

Below X will be a normal projective surface with function field K. Let D be a central simple algebra over K and let $\mathcal{O}_\Delta$ be a maximal $\mathcal{O}_X$-order.

Our aim is to develop the rudiments of an intersection theory for $\mathcal{O}_\Delta$. More precisely, define the following sets.

$$\begin{aligned}
\operatorname{Div}_r \mathcal{O}_\Delta &= \{\text{locally free fractional right } \mathcal{O}_\Delta\text{-ideals}\}\\
\operatorname{Div}_l \mathcal{O}_\Delta &= \{\text{locally free fractional left } \mathcal{O}_\Delta\text{-ideals}\}\\
\operatorname{Div} \mathcal{O}_\Delta &= \{\text{locally free fractional twosided } \mathcal{O}_\Delta\text{-ideals}\}
\end{aligned}$$

We shall use the notation $\operatorname{Div}_* \mathcal{O}_\Delta$ where $* = r, l$ or $\emptyset$. Recall that T is a fractional right $\mathcal{O}_\Delta$-ideal if for all $x \in X$, T_x is a right $\mathcal{O}_{\Delta,x}$- submodule in D which is also a right $\mathcal{O}_{X,x}$-lattice such that $T_x K = D$. In this case, we say that T is locally free if for all $x \in X$, T_x is a free right $\mathcal{O}_{\Delta,x}$-module. In a similar way, we define locally free fractional left or twosided $\mathcal{O}_\Delta$-ideals.
In the definition of $\operatorname{Div} \mathcal{O}_\Delta$ "free" refers to either left free or right free. It is well-known that these are equivalent. For completeness we include a proof here.

Lemma 3.1.1. *Assume that T is a twosided fractional $\mathcal{O}_\Delta$-ideal. If T is locally left free then it is locally right free and vice versa.*

Proof. It is sufficient to check this locally. So let $x \in X$ be a closed point. Assume that T is locally left free. This means that $T_x = \mathcal{O}_{\Delta,x} a$ for some regular $a \in D$. Since $a \in T_x$ and T_x is also a right ideal, we obtain $a\mathcal{O}_{\Delta,x} \subset \mathcal{O}_{\Delta,x} a$ and thus $\mathcal{O}_{\Delta,x} \subset a^{-1}\mathcal{O}_{\Delta,x} a$. This is an inclusion of two orders and since $\mathcal{O}_{\Delta,x}$ is maximal, this must be an equality. Thus in fact $a\mathcal{O}_{\Delta,x} = \mathcal{O}_{\Delta,x} a$. This proves what we want. □

We also put
$$\operatorname{Div}_{*,+} \mathcal{O}_\Delta = \{I \in \operatorname{Div}_* \mathcal{O}_\Delta \mid I \subset \mathcal{O}_\Delta\}$$

where $* = r, l, \emptyset$.

As usual the product between lattices in D restricts to products

$$\mathrm{Div}_r\,\mathcal{O}_\Delta \times \mathrm{Div}\,\mathcal{O}_\Delta \to \mathrm{Div}_r\,\mathcal{O}_\Delta \tag{3.1}$$

$$\mathrm{Div}\,\mathcal{O}_\Delta \times \mathrm{Div}_l\,\mathcal{O}_\Delta \to \mathrm{Div}_l\,\mathcal{O}_\Delta \tag{3.2}$$

The following lemma is also well-known

Lemma 3.1.2. *If $L \in \mathrm{Div}_l\,\mathcal{O}_\Delta$, and if R is a fractional right $\mathcal{O}_\Delta$-ideal then the canonical map $R \otimes_{\mathcal{O}_\Delta} L \to RL$ is an isomorphism.*

Proof. We may check this locally in a point $x \in X$. Tensoring the inclusion $R_x \hookrightarrow D$ with L_x yields the following commutative diagram.

$$\begin{array}{ccc} R_x \otimes_{\mathcal{O}_{\Delta,x}} L_x & \xrightarrow{\alpha} & D \otimes_{\mathcal{O}_{\Delta,x}} L_x \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ R_x L_x & \xrightarrow{\delta} & DL_x \end{array}$$

Here all maps are the canonical ones. Since L_x is free, it follows that α is an injection. The vertical map γ is an isomorphism, since it is clearly surjective and $\dim D \otimes_{\mathcal{O}_{\Delta,x}} L_x = 1 = \dim DL_x$. Since $\gamma\alpha = \delta\beta$ it follows that β is injective. Since it is also clearly surjective, we are done. $\square$

For $L \in \mathrm{Div}_l\,\mathcal{O}_\Delta$ we define L^* by

$$L^*(U) = \{a \in D \mid L(U)a \subset \mathcal{O}_\Delta(U)\}$$

for affine $U \subset X$.

Take $x \in X$, if $L_x = \mathcal{O}_{\Delta,x}a$, for some regular $a \in D$, then $L_x^* = a^{-1}\mathcal{O}_{\Delta,x}$. From this it follows that $L^* \in \mathrm{Div}_r\,\mathcal{O}_\Delta$.
A similar operation, also denoted by $(-)^*$ is defined on $\mathrm{Div}_r\,\mathcal{O}_\Delta$. Clearly $(-)^*$ defines a bijection between $\mathrm{Div}_l\,\mathcal{O}_\Delta$ and $\mathrm{Div}_r\,\mathcal{O}_\Delta$.

We recall the following.

Lemma 3.1.3.

1. *If $T \in \mathrm{Div}\,\mathcal{O}_\Delta$ then T^*, computed as left or as right fractional ideal is the same and lies again in* $\mathrm{Div}\,\mathcal{O}_\Delta$.
2. $\mathrm{Div}\,\mathcal{O}_\Delta$ *equipped with the lattice product is a commutative group. The inverse is given by $(-)^*$.*

Proof. $\operatorname{Div}\mathcal{O}_\Delta$ is clearly a subsemi-group of the group $\mathcal{D}$ of reflexive divisorial $\mathcal{O}_\Delta$-ideals. It is well known that the analogues of 1.,2. hold for $\mathcal{D}$ [35, Thm 2.3].
Now let $T \in \operatorname{Div}\mathcal{O}_\Delta$ and assume that we compute T^* using the fact that $T \in \operatorname{Div}_l \mathcal{O}_\Delta$. For $x \in X$, let $r \in \mathcal{O}_{\Delta,x}$ and $a \in T_x^*$. Since T_x is a right $\mathcal{O}_{\Delta,x}$-module, it follows that $T_x(ra) \subset \mathcal{O}_{\Delta,x}$, so T_x^* is a left $\mathcal{O}_{\Delta,x}$-module. Similarly one checks that T_x^* is also a left $\mathcal{O}_{X,x}$-lattice such that $KT_x^* = D$. From Lemma 3.1.1 and the fact that T^* is locally right free, it then follows that $T^* \in \operatorname{Div}\mathcal{O}_\Delta$. Thus $\operatorname{Div}\mathcal{O}_\Delta$ is closed under $(-)^*$, which yields that 1.,2. hold also for $\operatorname{Div}\mathcal{O}_\Delta$. $\square$

We want to define a pairing

$$(-,-) : \operatorname{Div}_r \mathcal{O}_\Delta \times \operatorname{Div}_l \mathcal{O}_\Delta \longrightarrow \mathbb{Z}$$

having the following properties for $L \in \operatorname{Div}_l \mathcal{O}_\Delta$, $R \in \operatorname{Div}_r \mathcal{O}_\Delta$ and $T \in \operatorname{Div}\mathcal{O}_\Delta$

(I1) $(RT, L) = (R, L) + (T, L)$

(I2) $(R, TL) = (R, T) + (R, L)$

(I3) (R, L) depends only on the isomorphism class of R and L.

Of course these conditions have to be supplemented with a condition which tells us what happens in the case that R, L are "transversal" in some sense.

If $L \in \operatorname{Div}_{*,+} \mathcal{O}_\Delta$ then we define $\operatorname{Supp} L \subset X$ as the support of the coherent $\mathcal{O}_X$-module $\mathcal{O}_\Delta/L$. Clearly we have

$$x \in \operatorname{Supp} L \iff L_x \neq \mathcal{O}_{\Delta,x}$$

We can now add a further desirable condition for $(-,-)$.

(I4) If $R \in \operatorname{Div}_{r,+} \mathcal{O}_\Delta$ and $L \in \operatorname{Div}_{l,+} \mathcal{O}_\Delta$ and if $\operatorname{Supp} R$ and $\operatorname{Supp} L$ have finite intersection then $(R, L) = \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L$

Note that "dim" makes sense here since we are applying it to a sheaf with finite support ($\operatorname{Supp} \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L = \operatorname{Supp} R \cap \operatorname{Supp} L$).

The classical way of defining intersection numbers is through a moving lemma. So our next aim will be to develop a substitute for this.

Let $\operatorname{Div}_{*,++} \mathcal{O}_\Delta$ for $* = r, l, \emptyset$ be the subset of $\operatorname{Div}_{*,+} \mathcal{O}_\Delta$ consisting of fractional ideals whose dual is generated by global sections as $\mathcal{O}_X$-module.

Lemma 3.1.4. *(Moving Lemma) Assume that* $L \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$*. Let* E *be an effective divisor on* X *and* S *a finite subset of* X*. Then there exists* $L' \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$ *isomorphic to* L *such that* E *and* $\operatorname{Supp} L'$ *have finite intersection and such that* $\operatorname{Supp} L' \cap S = \emptyset$*.*

Proof.

Step 1. Let $x \in X$ we claim that

$$\{a \in \Gamma(X, L^*) \mid x \in \operatorname{Supp}(La)\} \tag{3.3}$$

is a closed subset of $\Gamma(X, L^*)$.

We have $L_x = \mathcal{O}_{\Delta,x} c$. If $a \in \Gamma(X, L^*)$ then $\operatorname{Supp}(La)$ will contain x if and only if $\mathcal{O}_{\Delta,x} ca \neq \mathcal{O}_{\Delta,x}$. This is equivalent with ca not being a unit in $\mathcal{O}_{\Delta,x}$. Finally the latter is equivalent with $\overline{ca}$ not being a unit in $P = \mathcal{O}_{\Delta,x} / \operatorname{rad}(\mathcal{O}_{\Delta,x})$.

So we obtain that the set from (3.3) is the inverse image of the non-units in P under the linear map $a \mapsto \overline{ca}$. Since the non-units form a closed subset of the semi-simple k-algebra P, we are through (note that we didn't use that L^* is generated by global sections).

Step 2. Now we show that if $x \in X$ is a closed point then there exists $a \in \Gamma(X, L^*)$ such that $x \notin \operatorname{Supp}(La)$.

Since L^* is generated by global sections there exist $a \in \Gamma(X, L^*)$ such that $L^*_x = a\mathcal{O}_{\Delta,x}$. Then $L_x = \mathcal{O}_{\Delta,x} a^{-1}$ and hence $(La)_x = \mathcal{O}_{\Delta,x}$. This is precisely what we want.

Step 3. Now we prove the lemma. For the intersection of E and $\operatorname{Supp} L'$ to be finite it is sufficient to prove that $\operatorname{Supp} L'$ has no common component with E, since in that case $\operatorname{Supp} L'$ has at most one point in common with each component of E. So the intersection of E and $\operatorname{Supp} L'$ contains less points then there are components of E and since E is an effective divisor it has finitely many components.

Choose $x_1, \ldots, x_n \in X$, such that every component of E contains at least one of the x_i. Then for each $y \in \{x_1, \ldots, x_n\} \cup S$ the set

$$S_y = \{a \in \Gamma(X, L^*) \mid y \notin \operatorname{Supp}(La)\}$$

is open in $\Gamma(X, L^*)$ and non-empty (by the previous steps). So S_y is a dense open subset in $\Gamma(X, L^*)$. Hence, since $\Gamma(X, L^*)$ is irreducible, there exist an element b in the intersection of the S_y. Put $L' = Lb$. It is clear that $L' \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$. Since for all i, $b \in S_{x_i}$ it follows that E and $\operatorname{Supp} L'$ have no common component and $b \in S_y$, for all $y \in S$ yields that $S \cap \operatorname{Supp} L' = \emptyset$. $\square$

Now we prove our main theorem.

Theorem 3.1.5. *There is a unique pairing*

$$(-,-) : \mathrm{Div}_r\, \mathcal{O}_\Delta \times \mathrm{Div}_l\, \mathcal{O}_\Delta \to \mathbb{Z}$$

satisfying the properties (I1)-(I4) *above.*

Before we start the proof of this theorem we give a lemma that will be used many times.

Lemma 3.1.6. *1. Let $L \in \mathrm{Div}_l\, \mathcal{O}_\Delta$. Then there exist $T \in \mathrm{Div}_{++}\, \mathcal{O}_\Delta$ as well as $L_1 \in \mathrm{Div}_{l,++}\, \mathcal{O}_\Delta$ such that $L \cong T^* L_1$.*

2. Let E be an effective divisor on X and S a finite subset of X. Then there exists L_1, T as in 1. such that $E \cap \mathrm{Supp}\, L_1$, $E \cap \mathrm{Supp}\, T$ are finite and such that $\mathrm{Supp}\, L_1 \cap S = \mathrm{Supp}\, T \cap S = \emptyset$.

3. Let $T \in \mathrm{Div}\, \mathcal{O}_\Delta$. Then there exist $T_1, T_2 \in \mathrm{Div}_{++}\, \mathcal{O}_\Delta$ such that $T \cong T_1^* T_2$.

Proof. 1. Choose an ample M on $\mathcal{O}_X$. For n large enough, M^n is generated by global sections, so in particular there exists a global section f of M^n. Since M is invertible, it is locally free. Hence M^n is also locally free and in particular torsion free, so f is injective. Replace M^n by M.
Let K be the function field of X. Then K is flat over $\mathcal{O}_X$, so f induces a monomorphism $g : K \to M \otimes_{\mathcal{O}_X} K$ which is actually an isomorphism since locally the sheaves are one-dimensional vectorspaces over K.
The canonical map $i : M \to M \otimes_{\mathcal{O}_X} K$ is injective, since M is torsion free. Put $M' = g^{-1}(iM)$. M' is generated by global sections and furthermore M' is embedded in K and contains $\mathcal{O}_X$. Finally replace M' by M.

Since M is ample, for m large enough, $\mathcal{O}_\Delta M^m = \mathcal{O}_\Delta \otimes_{\mathcal{O}_X} M^m$, $L^* M^m = L^* \otimes_{\mathcal{O}_X} M^m$ will be generated by global sections. Hence by a similar argument as above, there will be a fractional right ideal I in D containing $\mathcal{O}_\Delta$, which is isomorphic to $L^* M^m$.
Put $L_1 = I^*$ and $T = \mathcal{O}_\Delta M^{-m}$. Then $L \cong T^* L_1$.

2. By the classical moving lemma [20, Lemma V.1.2] we may choose M in 1. in such a way that T has the correct properties. Then we can do the same with L_1 using Lemma 3.1.4.

3. This is proved in a similar way as 1. except that we may not replace an element in $\operatorname{Div}\mathcal{O}_\Delta$ by an isomorphic one since then it will usually be no longer in $\operatorname{Div}\mathcal{O}_\Delta$.

   As above let $\mathcal{D}$ be the group of divisorial fractional $\mathcal{O}_\Delta$ ideals. As already said above $\operatorname{Div}\mathcal{O}_\Delta \subset \mathcal{D}$. It is known [35, Thm 2.3] that $\mathcal{D}$ is the free group on the height one prime ideals in $\mathcal{O}_\Delta$, and furthermore that for any $I \in \mathcal{D}$ there exist some $n > 0$ and $J \in \operatorname{Div}\mathcal{O}_X$ such that $I^n = (J\mathcal{O}_\Delta)^{**}$. Note that we take the dual here in $\operatorname{Div}\mathcal{O}_X$.

   Let $T \in \operatorname{Div}\mathcal{O}_\Delta$ and let $n > 0$, $J \in \operatorname{Div}\mathcal{O}_X$ be such that $T^n = (J\mathcal{O}_\Delta)^{**}$. Choose an ample line bundle, M on X contained in K. For some m the product $(J \cap \mathcal{O}_X)M^m$ will have a section a. In other words $\mathcal{O}_X \subset (J \cap \mathcal{O}_X) \cdot a^{-1}M^m$. We now replace M by $a^{-1}M^m$. Then $\mathcal{O}_X \subset M$ and $\mathcal{O}_X \subset JM$.

   The fact that $\mathcal{O}_X \subset JM$ and $\mathcal{O}_\Delta$ is a flat $\mathcal{O}_X$-module, implies $\mathcal{O}_\Delta \subset T^nM$. From this, also using $\mathcal{O}_X \subset M$, we obtain $\mathcal{O}_\Delta \subset T^nM^n$. Using the structure of the ordered group $\mathcal{D}$ this implies $\mathcal{O}_\Delta \subset TM$.
   We still have $TM \subset TM^2 \subset \cdots$, whence by replacing M with a sufficiently high power we may assume that both $\mathcal{O}_\Delta M$ and TM are generated by global sections.
   We now take $T_1 = (TM)^*$ and $T_2 = \mathcal{O}_\Delta M^*$. □

Proof of Theorem 3.1.5. We follow the same strategy as the proof of the corresponding commutative result in [20].

First we prove uniqueness.
First note that if $L \in \operatorname{Div}_l \mathcal{O}_\Delta$, $R \in \operatorname{Div}_r \mathcal{O}_\Delta$ then by (I1) we have

$$(R, \mathcal{O}_\Delta) = (R, \mathcal{O}_\Delta) + (R, \mathcal{O}_\Delta)$$

and hence

$$(R, \mathcal{O}_\Delta) = 0 \tag{3.4}$$

Similarly we have $(\mathcal{O}_\Delta, L) = 0$.

Now take in addition $T \in \operatorname{Div}\mathcal{O}_\Delta$. Then by Lemma 3.1.3 we have $TT^* = \mathcal{O}_\Delta$. It follows again from (I1) that

$$0 = (R, TT^*) = (R, T) + (R, T^*)$$

and thus

$$(R, T^*) = -(R, T) \tag{3.5}$$

Similarly we have $(T^*, L) = -(T, L)$.

It follows from Lemma 3.1.6 as well as (3.5) and (I1)(I2) that $(-,-)$ is completely determined by its value on elements of $\operatorname{Div}_{*,++}\mathcal{O}_\Delta$.

However in that case we may apply the moving lemma to reduce ourselves to a computation of (R, L) such that $\operatorname{Supp} R \cap \operatorname{Supp} L$ is finite. In that case (R, L) is determined by (I4).

To prove the existence, we follow the same method and check that everything is well-defined.

Step 1. We start by defining the intersection pairing on $\operatorname{Div}_{*,++}\mathcal{O}_\Delta$.

Let $R \in \operatorname{Div}_{r,++}\mathcal{O}_\Delta$ and $L \in \operatorname{Div}_{l,++}\mathcal{O}_\Delta$. We define

$$(R, L) = \dim_k \mathcal{O}_\Delta/R' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L' \tag{3.6}$$

where $R' \in \operatorname{Div}_{r,++}\mathcal{O}_\Delta$ and $L' \in \operatorname{Div}_{l,++}\mathcal{O}_\Delta$ are chosen in such a way (using the moving lemma) that $R' \cong R$, $L' \cong L$ and $\operatorname{Supp} L' \cap \operatorname{Supp} R'$ is finite.
Of course we have to check that this is independent of the choice of L', R'.

To prove this we claim we verify the following : if $R \in \operatorname{Div}_{r,+}\mathcal{O}_\Delta$ and $L \in \operatorname{Div}_{l,+}\mathcal{O}_\Delta$ are such that $\operatorname{Supp} L \cap \operatorname{Supp} R$ is finite then

$$\dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L = \chi(R \otimes_{\mathcal{O}_\Delta} L) - \chi(R) - \chi(L) + \chi(\mathcal{O}_\Delta) \tag{3.7}$$

where χ is the Euler characteristic.
The right hand side of this equation is clearly independent of the isomorphism classes of L and R.

We start by tensoring the obvious locally free resolution of $\mathcal{O}_\Delta/R$

$$0 \to R \to \mathcal{O}_\Delta \to \mathcal{O}_\Delta/R \to 0$$

by $\mathcal{O}_\Delta/L$. This yields the complex

$$R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L \to \mathcal{O}_\Delta/L$$

By definition the homology of the previous complex is $\mathcal{T}or_i^{\mathcal{O}_\Delta}(\mathcal{O}_\Delta/R, \mathcal{O}_\Delta/L)$ with $i = 0, 1$. Thus we obtain from the additivity of χ

$$\begin{aligned}\chi(\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L) - \chi(\mathcal{T}or_1^{\mathcal{O}_\Delta}(\mathcal{O}_\Delta/R, \mathcal{O}_\Delta/L))\\ = \chi(\mathcal{O}_\Delta/L) - \chi(R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L)\end{aligned}$$

We clearly have

$$\chi(\mathcal{O}_\Delta/L) = \chi(\mathcal{O}_\Delta) - \chi(L)$$

Furthermore by tensoring

$$0 \to L \to \mathcal{O}_\Delta \to \mathcal{O}_\Delta/L \to 0$$

on the left with R we find

$$\chi(R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L) = \chi(R) - \chi(R \otimes_{\mathcal{O}_\Delta} L)$$

Summarizing everything, we obtain, whatever the choice of L, R, that $\chi(\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L) - \chi(\mathcal{T}or_1^{\mathcal{O}_\Delta}(\mathcal{O}_\Delta/R, \mathcal{O}_\Delta/L))$ is equal to the righthand side of (3.7).

In our case $\operatorname{Supp} \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L$ is finite and hence

$$\chi(\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L) = \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L$$

Thus to prove (3.7), we have to show that

$$\mathcal{T}or_1^{\mathcal{O}_\Delta}(\mathcal{O}_\Delta/R, \mathcal{O}_\Delta/L) = 0 \tag{3.8}$$

Pick $x \in X$. We have $R_x = a\mathcal{O}_{\Delta,x}$, $L_x = \mathcal{O}_{\Delta,x}b$, for some regular a, $b \in D$. Then

$$\begin{aligned} \mathcal{T}or_1^{\mathcal{O}_\Delta}(\mathcal{O}_\Delta/R, \mathcal{O}_\Delta/L)_x &= \operatorname{Tor}_1(\mathcal{O}_{\Delta,x}/R_x, \mathcal{O}_{\Delta,x}/L_x) \\ &= \ker(\mathcal{O}_{\Delta,x}/a\mathcal{O}_{\Delta,x} \xrightarrow{\times b} \mathcal{O}_{\Delta,x}/a\mathcal{O}_{\Delta,x}) \end{aligned}$$

Now $\operatorname{Supp} R \cap \operatorname{Supp} L$ is finite so $\operatorname{Tor}_1(\mathcal{O}_{\Delta,x}/R_x, \mathcal{O}_{\Delta,x}/L_x)$ is a finite dimensional $\mathcal{O}_{X,x}$-submodule of $\mathcal{O}_{\Delta,x}/a\mathcal{O}_{\Delta,x}$. This must then be zero by the assumption that $\mathcal{O}_{\Delta,x}$ is reflexive (and hence Cohen-Macauly).

Now assume that $R \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$, $T \in \operatorname{Div}_{++} \mathcal{O}_\Delta$, $L \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$. We verify (I2) in this case.

By the above discussion we already know that (R, TL) depends only on the isomorphism classes of R and TL. Hence by the moving lemma we may assume that $\operatorname{Supp} R \cap \operatorname{Supp}(TL)$ is finite.

Consider the following exact sequence

$$0 \to L/TL \to \mathcal{O}_\Delta/TL \to \mathcal{O}_\Delta/L \to 0 \tag{3.9}$$

and also the identity $L/TL = \mathcal{O}_\Delta/T \otimes_{\mathcal{O}_\Delta} L$ which is obtained from tensoring the obvious resolution of $\mathcal{O}_\Delta/T$ by L and using $T \otimes_{\mathcal{O}_\Delta} L \cong TL$ by Lemma 3.1.2.

By (3.8), the sequence (3.9) remains exact after applying $\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} -$. Hence we obtain the exact sequence

$$0 \to \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T \otimes_{\mathcal{O}_\Delta} L \to \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL \to \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L \to 0$$

From the fact that L is locally free we deduce that for all $x \in X$ we have

$$((\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T) \otimes_{\mathcal{O}_\Delta} L)_x \cong (\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T)_x$$

This implies in particular, that $\operatorname{Supp}(\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T \otimes_{\mathcal{O}_\Delta} L) = \operatorname{Supp} R \cap \operatorname{Supp} T \subset \operatorname{Supp} R \cap \operatorname{Supp}(TL)$ which is finite. Since $\operatorname{Supp} R \cap \operatorname{Supp} L$ is also finite, it follows from the previous and (3.6) that

$$\begin{aligned}
(R, TL) &= \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL \\
&= \dim_k(\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T \otimes_{\mathcal{O}_\Delta} L) + (R, L) \\
&= \dim_k(\mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T) + (R, L) \\
&= (R, T) + (R, L)
\end{aligned}$$

So (I2) holds in this case.

To complete the first step we still have to verify (I1) in this case, this can be obtained using the same method under similar hypotheses.

Step 2. Now let $L \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$ and let R be arbitrary. Using Lemma 3.1.6 (or rather its version for right ideals) we find decompositions $R \cong R_1 T^*$ with $R_1 \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$ and $T \in \operatorname{Div}_{++} \mathcal{O}_\Delta$. Then we define

$$(R, L) = (R_1, L) - (T, L) \tag{3.10}$$

Of course this could depend on the choice of R_1, T. To see that this is not the case write $R \cong R_2 T'^*$ with $R_2 \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$ and $T' \in \operatorname{Div}_{++} \mathcal{O}_\Delta$. We have to show that

$$(R_1, L) - (T, L) = (R_2, L) - (T', L)$$

which by the part of (I1) from the previous step, is equivalent to

$$(R_1 T', L) = (R_2 T, L)$$

The equality now follows from the fact that, using the commutativity of the multiplication in $\operatorname{Div} \mathcal{O}_\Delta$, we have $R_2 T = R_1 T'$.

Now we claim that the definition (3.10) satisfies (I1) for those R, T, L where it makes sense.

So assume that $R \in \operatorname{Div}_r \mathcal{O}_\Delta$, $T \in \operatorname{Div} \mathcal{O}_\Delta$ and $L \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$. According to Lemma 3.1.6 we may write $R = R_1 T_1^*$, $T = T_2 T_3^*$ with $R_1 \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$ and $T_1, T_2, T_3 \in \operatorname{Div}_{++} \mathcal{O}_\Delta$. Then we find using the definition (3.10) and the part of (I1) from Step 1

$$\begin{aligned}
(RT, L) &= (R_1 T_1^* T_2 T_3^*, L) \\
&= (R_1 T_2 (T_1 T_3)^*, L) \\
&= (R_1 T_2, L) - (T_1 T_3, L) \\
&= (R_1, L) + (T_2, L) - (T_1, L) - (T_3, L) \\
&= (R_1 T_1^*, L) + (T_2 T_3^*, L) \\
&= (R, L) + (T, L)
\end{aligned}$$

To verify (I2) in this case, we take $R \in \operatorname{Div}_r \mathcal{O}_\Delta$, $T \in \operatorname{Div}_{++} \mathcal{O}_\Delta$ and $L \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$. Again by Lemma 3.1.6, we may write $R = R_1 T_1^*$ with $R_1 \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$ and $T_1 \in \operatorname{Div}_{++} \mathcal{O}_\Delta$. It now follows from the definition (3.10) and the part of (I2) from Step 1 that

$$\begin{aligned}
(R, TL) &= (R_1 T_1^*, TL) \\
&= (R_1, TL) - (T_1, TL) \\
&= (R_1, T) + (R_1, L) - (T_1, T) - (T_1, L) \\
&= (R_1 T_1^*, T) + (R_1 T_1^*, L) \\
&= (R, T) + (R, L)
\end{aligned}$$

Finally we check (I3) for definition (3.10).
Assume that $R \cong R'$, for R, $R' \in \operatorname{Div}_r \mathcal{O}_\Delta$ and $L \cong L'$, for L, $L' \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$. Using Lemma 3.1.6, we find decompositions $R' \cong R \cong R_1 T^*$ with $R_1 \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$ and $T \in \operatorname{Div}_{++} \mathcal{O}_\Delta$. It follows that

$$\begin{aligned}
(R', L') &= (R_1, L') - (T_1, L') \\
&= (R_1, L) - (T_1, L) \text{ (by (I3) from Step 1)} \\
&= (R, L)
\end{aligned}$$

Step 3. Finally assume now that $L \in \operatorname{Div}_l \mathcal{O}_\Delta$ and $R \in \operatorname{Div}_r \mathcal{O}_\Delta$. We write $L = T^* L_1$ by Lemma 3.1.6 and we define $(R, L) = (R, L_1) - (R, T)$.
One now verifies exactly as in the previous step that this is well defined and furthermore that the properties (I1)(I2) and (I3) are satisfied.

Step 4. We still have to verify that the definition of (R, L) satisfies (I4) under the hypotheses that $R \in \operatorname{Div}_{r,+} \mathcal{O}_\Delta$, $L \in \operatorname{Div}_{l,+} \mathcal{O}_\Delta$ and $\operatorname{Supp} L \cap \operatorname{Supp} R$ is finite.

As usual, we can find $L_1 \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$ and $T \in \operatorname{Div}_{++} \mathcal{O}_\Delta$ such that $L \cong T^* L_1$ and $\operatorname{Supp} R \cap \operatorname{Supp} L_1$ is finite. So $TL \cong L_1 \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$ and $\operatorname{Supp} R \cap \operatorname{Supp}(TL)$ is finite.
Then an exact sequence as in (3.9) shows, using (3.8) which is satisfied since $\operatorname{Supp} R \cap \operatorname{Supp} L$ is finite, that

$$\begin{aligned}
&\dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL \\
&\qquad = \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L + \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T
\end{aligned}$$

Since we can also find $T' \in \operatorname{Div}_{++} \mathcal{O}_\Delta$ such that $RT' \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$ and $\operatorname{Supp}(RT') \cap \operatorname{Supp}(TL)$ is finite, one checks with the same reasoning as at the end of Step 1 that

$$\begin{aligned}
&\dim_k \mathcal{O}_\Delta/RT' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL \\
&\qquad = \dim_k \mathcal{O}_\Delta/T' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL + \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL
\end{aligned}$$

Finally since $\operatorname{Supp}(RT') \cap \operatorname{Supp} T$ is finite, we have

$$\begin{aligned}
&\dim_k \mathcal{O}_\Delta/RT' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T \\
&\qquad = \dim_k \mathcal{O}_\Delta/T' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T + \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T
\end{aligned}$$

Summarizing everything, we obtain

$$\begin{aligned}
&\dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/L \\
&\qquad = \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL - \dim_k \mathcal{O}_\Delta/R \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T \\
&\qquad = \dim_k \mathcal{O}_\Delta/RT' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL - \dim_k \mathcal{O}_\Delta/T' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/TL \\
&\qquad\qquad - \dim_k \mathcal{O}_\Delta/RT' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T + \dim_k \mathcal{O}_\Delta/T' \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/T \\
&\qquad = (RT', TL) - (T', TL) - (RT', T) + (T', T) \text{ (by Step 1)} \\
&\qquad = (R, L) \text{ (using (I1) and (I2))}
\end{aligned}$$

$\square$

The following proposition provides an additional property of the intersection pairing.

Proposition 3.1.7. *Let $L \in \operatorname{Div}_l \mathcal{O}_\Delta$, $R \in \operatorname{Div}_r \mathcal{O}_\Delta$. Then we have*

$$(R, L) = \chi(R \otimes_{\mathcal{O}_\Delta} L) - \chi(R) - \chi(L) + \chi(\mathcal{O}_\Delta)$$

In particular, if $T, T' \in \operatorname{Div} \mathcal{O}_\Delta$ then $(T, T') = (T', T)$.

Proof. Using Lemma 3.1.6, we write $R = R'T_2^*$, $L = T_1^* L'$ with $T_1, T_2 \in \operatorname{Div}_{++} \mathcal{O}_\Delta$, $L' \in \operatorname{Div}_{l,++} \mathcal{O}_\Delta$ and $R' \in \operatorname{Div}_{r,++} \mathcal{O}_\Delta$ in such a way that the supports of R', L', T_1, T_2 have finite pairwise intersection.
This has the effect that we can neglect the higher $\mathcal{T}or$'s in the computation below (by (3.8)). For simplicity we also write "$\otimes$" for "$\otimes_{\mathcal{O}_\Delta}$" and "$\mathcal{O}$" for $\mathcal{O}_\Delta$.

We have

$$\begin{aligned}
\chi(R'T_2^* \otimes T_1^* L') - \chi(R'T_2^*) - \chi(T_1^* L') + \chi(\mathcal{O}) \\
&= (\chi(R'T_2^* \otimes L') + \chi(R'T_2^* \otimes T_1^*/\mathcal{O} \otimes L')) \\
&\qquad - \chi(R'T_2^*) - \chi(T_1^* L') + \chi(\mathcal{O}) \\
&= (\chi(R' \otimes L') + \chi(R' \otimes T_2^*/\mathcal{O} \otimes L')) \\
&\qquad + (\chi(R' \otimes T_1^*/\mathcal{O} \otimes L') + \chi(R' \otimes T_2^*/\mathcal{O} \otimes T_1^*/\mathcal{O} \otimes L')) \\
&\qquad - (\chi(R') + \chi(R' \otimes T_2^*/\mathcal{O})) \\
&\qquad - (\chi(T_1^*/\mathcal{O} \otimes L') + \chi(L')) + \chi(\mathcal{O}) \\
&= \chi(R' \otimes T_2^*/\mathcal{O} \otimes T_1^*/\mathcal{O} \otimes L') \\
&\qquad + (\chi(R' \otimes T_2^*/\mathcal{O} \otimes L') - \chi(R' \otimes T_2^*/\mathcal{O})) \\
&\qquad + (\chi(R' \otimes T_1^*/\mathcal{O} \otimes L') - \chi(T_1^*/\mathcal{O} \otimes L')) \\
&\qquad + \chi(R' \otimes L') - \chi(R') - \chi(L') + \chi(\mathcal{O}) \\
&= \chi(R' \otimes T_2^*/\mathcal{O} \otimes T_1^*/\mathcal{O} \otimes L') \\
&\qquad - \chi(R' \otimes T_2^*/\mathcal{O} \otimes \mathcal{O}/L') - \chi(\mathcal{O}/R' \otimes T_1^*/\mathcal{O} \otimes L') \\
&\qquad + \chi(R' \otimes L') - \chi(R') - \chi(L') + \chi(\mathcal{O})
\end{aligned}$$

Now we use again the hypothesis on the support of R', L', T_1, T_2. This allows us to replace some of the "χ" by "dim" in the above formula.
Furthermore we can compute this dimension by looking at stalks. Using the

fact that R', L', T_1, T_2 are locally free we obtain

$$\begin{aligned}
&\chi(R' \otimes T_2^*/\mathcal{O} \otimes T_1^*/\mathcal{O} \otimes L') \\
&\qquad - \chi(R' \otimes T_2^*/\mathcal{O} \otimes \mathcal{O}/L') - \chi(\mathcal{O}/R' \otimes T_1^*/\mathcal{O} \otimes L') \\
&\qquad + \chi(R' \otimes L') - \chi(R') - \chi(L') + \chi(\mathcal{O}) \\
&= \dim_k \mathcal{O}/T_2 \otimes \mathcal{O}/T_1 - \dim_k \mathcal{O}/T_2 \otimes \mathcal{O}/L' \\
&\qquad - \dim_k \mathcal{O}/R' \otimes \mathcal{O}/T_1 + \dim_k \mathcal{O}/R' \otimes \mathcal{O}/L' \\
&= (T_2, T_1) - (T_2, L') - (R', T_1) + (R', L') \\
&= (R'T_2^*, T_1^*L') \\
&= (R, L)
\end{aligned}$$

In the first equality we have used (3.7). □

We will also need the following.

Lemma 3.1.8. *Assume that $M, M' \in \operatorname{Div} \mathcal{O}_X$ and assume that $\mathcal{O}_\Delta$ has PI-degree s. Then $(\mathcal{O}_\Delta M, \mathcal{O}_\Delta M') = s^2(M, M')$.*

Proof. By additivity and the classical moving lemma we may assume that $M, M' \in \operatorname{Div}_{++} \mathcal{O}_X$ and furthermore that $\operatorname{Supp} M$ and $\operatorname{Supp} M'$ intersect in a finite number of points where $\mathcal{O}_\Delta$ is free of rank s^2 over $\mathcal{O}_X$.

Since $\operatorname{Supp}(\mathcal{O}_\Delta M) \cap \operatorname{Supp}(\mathcal{O}_\Delta M') \subset \operatorname{Supp} M \cap \operatorname{Supp} M'$, it follows from (I4) that

$$\begin{aligned}
(\mathcal{O}_\Delta M, \mathcal{O}_\Delta M') &= \dim_k \mathcal{O}_\Delta/\mathcal{O}_\Delta M \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta/\mathcal{O}_\Delta M' \\
&= \dim_k \mathcal{O}_\Delta \otimes_{\mathcal{O}_X} (\mathcal{O}_X/M \otimes_{\mathcal{O}_X} \mathcal{O}_X/M')
\end{aligned}$$

Now if $x \in \operatorname{Supp} M \cap \operatorname{Supp} M'$ then clearly

$$\dim_k(\mathcal{O}_\Delta \otimes_{\mathcal{O}_X} (\mathcal{O}_X/M \otimes_{\mathcal{O}_X} \mathcal{O}_X/M'))_x = s^2 \dim_k(\mathcal{O}_X/M \otimes_{\mathcal{O}_X} \mathcal{O}_X/M')_x$$

Summing over the points in $\operatorname{Supp} M \cap \operatorname{Supp} M'$ proves what we want. □

3.2 The Proj of graded rings finite over a commutative graded ring

Let $A = k + A_1 + A_2 + \cdots$ be a twosided noetherian graded ring which is finite over a commutative graded ring R. By the Artin-Tate lemma, R is finitely generated. By m we denote the ideal $R_{>0}$.

Throughout assume that A is *generated in degree one*. Let X be the classical Proj of R (see [20]). If M is a graded R-module then we denote the associated quasi-coherent $\mathcal{O}_X$-module by $\tilde{M}$ [20].

Using the definition of $(\tilde{-})$ we see that $\tilde{A}$ defines a sheaf of $\mathcal{O}_X$-algebras. We denote this sheaf by $\mathcal{O}_\Delta$ and by $\mathrm{Qch}(\mathcal{O}_\Delta)$ we denote the category of quasi-coherent $\mathcal{O}_\Delta$-modules, these are quasi-coherent $\mathcal{O}_X$-modules which are also $\mathcal{O}_\Delta$-modules.

If M is a graded A-module then $\tilde{M}$ will be a $\mathcal{O}_\Delta$-module. As usual, for a graded A (or R-module) M, we let $M(n)$ be the graded module whose grading is defined by $M(n)_m = M_{m+n}$. Put $\mathcal{O}_\Delta(n) = \widetilde{A(n)}$. For $\mathcal{M}$ a $\mathcal{O}_\Delta$-module we put $\mathcal{M}(n) = \mathcal{O}_\Delta(n) \otimes_{\mathcal{O}_\Delta} \mathcal{M}$.

Now recall that for any noetherian graded ring $A = k + A_1 + \cdots$ the category $\mathrm{QGr}(A)$ is defined as $\mathrm{Gr}(A)/\operatorname{Tors}(A)$ where $\operatorname{Tors}(A)$ represents the locally right bounded graded A-modules. $\pi : \mathrm{Gr}(A) \to \mathrm{QGr}(A)$ is the quotient functor and $\operatorname{Proj} A$ is the pair $(\mathrm{QGr}(A), \pi({}_A A))$.

Now $M \mapsto \tilde{M}$ defines a functor $\mathrm{Gr}(A) \to \mathrm{Qch}(\mathcal{O}_\Delta)$ such that ${}_A A$ is sent to $\mathcal{O}_\Delta$ and which factors through $\mathrm{QGr}(A)$. This functor defines an equivalence of $\operatorname{Proj} A$ with the pair $(\mathrm{Qch}(\mathcal{O}_\Delta), \mathcal{O}_\Delta)$. The key observation to prove this, is the following:

(*) If $f \in R$ is a homogeneous element of strictly positive degree then A_f is strongly graded (see [27]).

Note that for (*) we need essentially that A is generated in degree one.
For the functor to define an equivalence, one must prove that $\mathcal{O}_\Delta(1) = \widetilde{A(1)}$ is invertible, which is true if for $f \in m$, $(A_f)_1$ is invertible over $(A_f)_0$. Since A_f is strongly graded we are through.

In the sequel we will also use the notation $(\tilde{-})$ when M is a graded R-central A-bimodule. In that case $\tilde{M}$ is canonically an $\mathcal{O}_X$-central $\mathcal{O}_\Delta$-bimodule.

Using the property (*) exhibited above one easily proves for a graded A-module M, that

$$\widetilde{M(n)} = \mathcal{O}_\Delta(n) \otimes_{\mathcal{O}_\Delta} \tilde{M} \tag{3.11}$$

If M is an A-module then we define the local cohomology modules of M by

$$H^i_{A_{>0}}(M) = \operatorname{inj\,lim}_n \operatorname{Ext}^i_A(A/A_{\geq n}, M)$$

It is well-known that $H^i_m(M) = H^i_{A_{>0}}(M)$ (see for example [9, lemma 8.2.(3)]). Hence if M is a R-central graded A-bimodule then so is $H^i_{A_{>0}}(M)$.

We define $R^{(n)}$ by $\oplus_{r\in\mathbb{N}} R_{rn}$ and $M^{(n)}$ by $\oplus_{r\in\mathbb{N}} M_{rn}$. If $n > 0$ then R and $R^{(n)}$ have the same classical Proj. Furthermore it is easily seen that for $M \in \mathrm{Gr}(R)$ we have $\tilde{M} = (M^{(n)})\tilde{}$.

Finally note the following lemma.

Lemma 3.2.1. *Assume that $M \in \mathrm{Gr}(R)$. Then $H^i_m(M)^{(n)} = H^i_{m^{(n)}}(M^{(n)})$.*

Proof. Clearly R is finitely generated as a module over $R^{(n)}$. Put $p = m^{(n)}$. It follows that we have

$$H^i_m(M) = H^i_p(M) = \oplus_j H^i_p(M^{(n,j)})$$

where $M^{(n,j)} = \oplus_{t=j \bmod n} M_t$.
Now the definition of local cohomology easily yields that $H^i_p(M^{(n,j)})$ has its grading concentrated in degree $n\mathbb{Z} + j$. Thus we obtain $(H^i_m(M))^{(n,j)} = H^i_p(M^{(n,j)})$ and the result we were proving is a particular case of this. □

If M is a finitely generated graded A-module, then $H^i_m(M)$ is finite dimensional in every degree (condition "χ", see for example [9, Theorem 8.3]).
In general a graded k-vectorspace V which is finite dimensional in every degree is called *locally finite*. In that case we define V' as the k-dual of V.
It is clear that $(-)'$ is a functor which sends graded left A-modules to graded right A-modules and also graded bimodules to graded bimodules.

Assume that R has dimension n. Since A is a finitely generated R-module and $R \subset A$ it follows that A has dimension n also.
It is well-known that for any graded R-module M one has $H^l_m(M) = 0$ for $l > n$.

We define the dualizing module of $\mathcal{O}_\Delta$ by

$$\omega_\Delta = (H^n_m(A)')\tilde{}$$

It is well-known that the corresponding definition for ω_X yields the classical dualizing module (not the dualizing complex!). This can for example be obtained from [42] where it is shown using local duality, that $(H^n_m(R)')\tilde{}$ represents the functor $H^{n-1}(X,-)^*$. Since representing objects are unique, Serre duality [20, Prop III 7.5] implies what we want.

We have the following lemma.

Lemma 3.2.2. *One has*

$$\omega_\Delta = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_\Delta, \omega_X)$$

Proof. We replace A and R by some Veronese such that R is generated in degree one. By Lemma 3.2.1 this does not affect ω_X and ω_Δ.

Now for finitely generated graded R-modules M, N we have

$$\mathcal{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) = \operatorname{Hom}_R(M, N)\tilde{}$$

and this isomorphism is compatible with possible A-module structure on M and N. To see this we may verify it on affine opens and there we can use (*).

Specializing to $M = A$ and $N = H^n_m(R)'$ yields

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_\Delta, \omega_X) = \operatorname{Hom}_R(A, H^n_m(R)')\tilde{}$$

Now we claim that for any finitely generated graded R-module M we have a natural isomorphism:

$$\operatorname{Hom}_R(M, H^n_m(R)') = H^n_m(M)' \tag{3.12}$$

In fact since we are comparing two left exact contravariant functors it suffices to take $M = R$ and this is clear. The fact that (3.12) is natural in M implies that in case M is a bimodule it is compatible with the bimodule structure. This finishes the proof. $\square$

3.3 Three dimensional Artin-Schelter regular algebras

In this section A will be a three-dimensional three generator Artin-Schelter regular algebra finite over its center R. Let $(Y, \sigma, \mathcal{L})$ be the associated triple. It follows that σ has finite order.

Now we use the notations which where introduced in the beginning of this chapter. So X is the classical Proj of R and we put $\mathcal{O}_\Delta = \tilde{A}$.

Let $\mathcal{Z} = Z(\mathcal{O}_\Delta)$. And denote by Z the projective scheme $\underline{\operatorname{Spec}}_{\mathcal{O}_X} \mathcal{Z}$ which has $\mathcal{Z}$ as its structure sheaf (see [3]). Since $\mathcal{O}_X \subset \mathcal{Z} \subset \mathcal{O}_\Delta$, Z is a covering of X and we consider $\mathcal{O}_\Delta$ as a sheaf on Z.

By D we denote the degree zero part of the graded quotient ring of A. Thus $\mathcal{O}_\Delta$ is an $\mathcal{O}_X$-order in D.

Our aim is now to give a proof of Conjecture 3.0.1 in characteristic zero. *So from now on we assume that k is an algebraically closed field of characteristic zero.*

Assume that A is elliptic. In that case Y is given by a divisor of degree three in $\mathbb{P}^2$.

As before, put $B = A/(g)$ where g is a normalizing element in A of degree three. So B is the twisted homogeneous coordinate ring associated to the triple $(Y, \sigma, \mathcal{L})$. In particular one has that Proj B is equivalent to $(Y, \mathcal{O}_Y)$.

We need the following lemma.

Lemma 3.3.1. *$\tilde{B}$ is a commutative sheaf of $\mathcal{O}_X$-algebras and furthermore $\underline{\mathrm{Spec}}\tilde{B}$ is isomorphic to Y.*

Proof. We recall the definition of B as

$$\bigoplus_n \Gamma(Y, \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}})$$

with multiplication $a \cdot b = ab^{\sigma^m}$ for $a \in B_m$, $b \in B_n$.
Let t be the order of σ and put $\mathcal{M} = \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{t-1}}$. Since $\mathcal{L}$ is σ-ample [6], it follows that $\mathcal{M}$ is ample.
Furthermore it is clear that $B^{(t)} = \oplus_{n \geq 0} H^0(Y, \mathcal{M}^{\otimes n})$, the classical homogeneous coordinate ring of Y associated to $\mathcal{M}$. In particular $B^{(t)}$ is commutative and the classical Proj of $B^{(t)}$ is Y.

Since $\tilde{B} = (B^{(t)})\tilde{}$, we deduce that $\tilde{B}$ is commutative. Furthermore one easily verifies from the definitions that $\underline{\mathrm{Spec}}\tilde{B}$ is nothing but the classical Proj of B (see for example [3]). Hence $\underline{\mathrm{Spec}}\tilde{B}$ is isomorphic to Y. $\square$

Remark 3.3.2. In case A is linear, it is easy to see that with an argument as in the previous lemma, one has that $\tilde{B}$ is commutative. Since in the linear case $A \cong B$, it follows that $\mathcal{Z} = Z(\mathcal{O}_\Delta) = Z(\tilde{A}) = Z(\tilde{B}) = \tilde{B}$. This yields that $Z = Y = \mathbb{P}^2$ and Conjecture 3.0.1 is proved in this case. Thus we may assume in the rest of this chapter that A is elliptic.

In view of Lemma 3.3.1, we will commit a slight abuse of notation by writing $\mathcal{O}_Y$ for $\tilde{B}$.

By the fact that $\mathrm{Qch}(\mathcal{O}_\Delta)$ is equivalent to $\mathrm{QGr}(A)$ it follows that the simple $\mathcal{O}_\Delta$-modules are in one-one correspondence with the simple objects in $\mathrm{QGr}(A)$. Let $M \in \mathrm{Gr}(A)$ represent such a simple object. Clearly we may assume that M is finitely generated and critical. In particular, the multiplication by g is either injective or the zero map, since a critical module is also monoform, which implies that a non-zero map is always injective.

In the first case the simplicity of πM implies that M/gM is finite dimensional. Hence from the exact sequence

$$0 \longrightarrow M(-3)_n \xrightarrow{h} M_n \longrightarrow (M/gM)_n \longrightarrow 0$$

it follows that $\dim M_{n-3} = \dim M_n$, exept for finitely many n. Thus M has almost a periodic Hilbert function.
In the second case M is a B-module. Recall that using the description of B as twisted homogeneous coordinate ring, we find that M corresponds to a simple $\mathcal{O}_Y$-module, that is to a point on Y. Using the explicit description of the equivalence of $\mathrm{Qch}(Y)$ and $\mathrm{QGr}(B)$ it follows that M is equivalent modulo $\mathrm{Tors}(A)$ to a point module.

In any case we can say that M is critical of Gelfand-Kirilov dimension one. Conversely it is clear that such an M will give rise to simple object in $\mathrm{QGr}(A)$ since $\mathrm{GK}\dim(M/M') = 0$ implies that M/M' is torsion.

Critical M of Gelfand-Kirilov dimension were studied in [6]. It was shown that their multiplicity is either equal to 1 or some number $m > 1$ and that their Hilbert function is constant for sufficiently large n. Those that have multiplicity 1 are precisely those that are annihilated by g.

If M is as in the previous paragraph then $\tilde{M}$ is a simple $\mathcal{O}_\Delta$-module and in particular it is supported in a point $x \in X$. Then x lies in some standard open $\mathrm{D}_+(f)$ associated to a homogeneous element $f \in R$ of positive degree. By definition we have $\tilde{M} \mid \mathrm{D}_+(f) = (M_f)_0$.
Now M is critical and multiplication by f on M is not the zero-map, so it must be injective. Since the Hilbert function of M is constant for sufficiently large n, it follows that multiplication by f on M must be an isomorphism in high degree.
It follows that $\dim(M_f)_0 = \dim M_n$ for $n \gg 0$. Thus if M has multiplicity e then $\dim \tilde{M} = e$.

Put $\mathcal{J} = (gA)\tilde{}$. Assume that $\mathcal{O}_\Delta$ has PI-degree s. Since we are assuming that A is not linear, it follows that $s > 1$.
By the above discussion the simple $\mathcal{O}_\Delta$-modules are either 1 or s-dimensional and the one-dimensional ones are anihilated by $\mathcal{J}$. Note that if they are not 1-dimensional, they must be s-dimensional since the multiplicity $m > 1$ from the above discussion is now exactly s. This follows from the fact that over an algebraically closed field k, the PI-degree is the maximum of the dimensions of the simple representations which is m is our case.

By the Artin-Procesi theorem it follows that $\mathcal{J}$ defines the ramification of $\mathcal{O}_\Delta$. More precisely, put $\mathcal{I} = \mathcal{J} \cap \mathcal{O}_Z$. Then $\mathcal{V}(\mathcal{I})$ is the non-Azumaya locus of $\mathcal{O}_\Delta$ in Z, that is the set of $x \in Z$ such that $\mathcal{O}_{\Delta,x}$ is not Azumaya.
Put $S = \mathcal{V}(\mathcal{I})_{\text{red}}$.

Proposition 3.3.3. *1. Z is isomorphic to $\mathbb{P}^2$.*

2. Y and S with their natural embeddings into $\mathbb{P}^2$ have one of the following forms:

- *A smooth elliptic curve.*
- *The union of a line and a conic which intersect in two distinct points.*
- *The union of three lines which intersect in three different points.*
- *An elliptic curve with a node.*

Let $q \in S$. Then $\hat{\mathcal{O}}_{\Delta,q}$ has the following form

- *If q is a node then $\hat{\mathcal{O}}_{\Delta,q} = k\langle\langle x, y\rangle\rangle/(yx - \zeta xy)$, where ζ is an s'th root of unity.*
- *If q is smooth then*

$$\hat{\mathcal{O}}_{\Delta,q} \cong \begin{pmatrix} T & Tx & \cdots & Tx \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & Tx \\ T & \cdots & \cdots & T \end{pmatrix}$$

($s \times s$-matrices) where $T = k[[x, y]]$.

3. The obvious map $Y \to S$ is one-one on singular points and outside the singular points defines an s-sheeted covering.

The proof will consist of a number of lemmas

Lemma 3.3.4. *Assume that Λ is an order of PI-degree $s > 1$ in a central simple algebra E over the quotient field L of the two dimensional complete local ring $P = Z(\Lambda)$ containing a copy of its residue field k. Assume in addition that Λ is basic, and furthermore that there is an invertible ideal J in Λ such that Λ/J is commutative. Finally assume that Λ is reflexive and of finite global dimension.*
Then the following hold.

1. *J is generated by a normalizing element N.*

2. *Λ is isomorphic to*

$$\begin{pmatrix} T & Tx & \cdots & Tx \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & Tx \\ T & \cdots & \cdots & T \end{pmatrix} \tag{3.13}$$

($t \times t$-matrices) where T is a complete local ring of the form

$$T = k\langle\langle x, y\rangle\rangle/(\psi)$$

with

$$\psi = yx - \zeta xy \tag{3.14}$$

for some p'th root of unity ζ, such that pt equals s.

3. *P is regular.*

4. *If Λ has more then one simple module then N and the above isomorphism may be chosen in such a way that N corresponds to*

$$\begin{pmatrix} 0 & \cdots & 0 & x \\ 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \tag{3.15}$$

5. *If Λ has exactly one simple module then N and the above isomorphism may be chosen in such a way that N corresponds to x, or xy.*

6. *If Λ has more then one simple module then $\Lambda/(N)$ is a direct sum of discrete valuation rings. If Λ has exactly one simple module then $\Lambda/(N)$ is either a discrete valuation ring, or isomorphic to $k[[x, y]]/(xy)$.*

Proof. There are many ways of proving this. For example, with a little bit of work, we could deduce it from [30] or [2]. It can also be proved with a computation similar to [38, §3].

We prefer to prove it using a slightly extended version of Theorem 2.0.1.

If $\mathcal{C}$ is a finite length category, then $\widetilde{\mathcal{C}^\circ}$, where $\widetilde{(-)}$ is the closure under direct limits, is a locally finite category. So Gabriel associates to $\widetilde{\mathcal{C}^\circ}$ a pseudocompact ring A such that $\widetilde{\mathcal{C}^\circ}$ is equivalent to the dual of $\mathrm{PC}(A)$, the category of pseudocompact modules over A. From this it follows that $\mathcal{C}$ is equivalent to the category of pseudocompact finite length modules over A. Furthermore if A is basic then an A with this property is unique up to (non-unique) isomorphism.

We apply this with $\mathcal{C}$ being the finite length modules over Λ. Then it is clear that the associated pseudocompact ring is Λ itself.

On the other hand $\mathrm{Mod}(\Lambda)$ almost satisfies the hypotheses of Theorem 2.0.1 with the functor $G = J \otimes_\Lambda -$ except that Λ/J is not quite a "Cohen-Macaulay curve". However from the fact that Λ is reflexive it follows that Λ/J is a one-dimensional Cohen-Macaulay ring, and this is sufficient for the proof.

Using this observation, Theorem 2.0.1 now yields that $\mathcal{C}$ is equivalent to the category of pseudocompact finite length modules over a finite direct sum of rings

$$\begin{pmatrix} T & TU & \cdots & TU \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & TU \\ T & \cdots & \cdots & T \end{pmatrix} \tag{3.16}$$

where T is a (in general non-commutative) complete local ring of global dimension 2.

Hence by the uniqueness alluded to above, Λ is isomorphic to a direct sum of such rings. However Λ is prime, so it must be isomorphic to exactly one ring of the form (3.16).
Note that it is exactly the form (3.16) we get from Theorem 2.0.1 since Λ has finite global dimension.

The fact that T has the form (3.14) follows from the fact that T is finite over its center, as well as the fact that we are in characteristic zero (for example using [2]).
From this we also obtain that $P = Z(\Lambda) \cong k[[x^p, y^p]]$. Thus P is regular.

Now we still need to find the explicit form of N.

If Λ has more than one simple then according to Theorem 2.0.1 2(b) ("n">

1 in this case) N is of the form

$$N = \begin{pmatrix} 0 & \cdots & 0 & U \\ 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

with U a normalizing element in T contained in $\mathrm{rad}(T) - \mathrm{rad}^2(T)$.
If $\zeta = 1$ then we can take $x = U$. If $\zeta \neq 1$ then the only normalizing elements in T not contained in $\mathrm{rad}^2(T)$ are x and y (up to a unit). So possibly after interchanging x and y we may assume that $U = x$.

Assume now that Λ has exactly one simple (so "n"$= 1$). In that case $\Lambda = T$. Since we had assumed $s > 1$ it follows that T is not commutative, so $\zeta \neq 1$. Since N has the property that $T/(N)$ is commutative, (N) must contain $[y, x]$. So $(N)/(xy)$ must be an ideal in $k[[x, y]]/(xy)$ such that the quotient has no finite dimensional submodules (this follows from the corresponding property of $\Lambda/(N)$ which in turn follows from the fact that Λ was assumed reflexive). It follows that the only possibilities for (N) are (x), (y) or (xy). Since we are free to change N by a unit and to interchange x and y it follows that we may take $N = x$ or $N = xy$.

To end the proof we need to prove 6. However this is a simple consequence of 4. and 5. □

Lemma 3.3.5. *Assume that Λ is a hereditary order in a central simple algebra E over the quotient field L of a discrete valuation ring P. Assume that Λ contains an invertible ideal J such that Λ/J is a field. Then Λ is a maximal order and (assuming characteristic zero) the ramification index of Λ is equal to the PI-degree of Λ.*

Proof. We want to work with completions, so let $\hat{P}$ be the m-adic completion of P, where m is the unique maximal ideal of P. Let $\hat{L}$ be the quotient field of $\hat{P}$. Then $\hat{E} = \hat{L} \otimes_L E$ is a central simple $\hat{L}$-algebra and $\hat{\Lambda} = \hat{P} \otimes_P \Lambda$ is a hereditary $\hat{P}$-order in $\hat{E}$. Since $\hat{J} = \hat{P} \otimes_P J$ is an invertible ideal of $\hat{\Lambda}$ and $\hat{\Lambda}/\hat{J}$ is a field, we obtain from Theorem (39.14) in [29] that $\hat{\Lambda}$ is a maximal order in $\hat{E}$ and thus Λ is a maximal order in E. □

Lemma 3.3.6. *Z is smooth. Y is as stated in Proposition 3.3.3 and S has normal crossings.*

Proof. First we make a few general observations:

1. $\mathcal{O}_\Delta$ locally has finite global dimension. This follows from the fact that A has finite global dimension and is generated in degree one (using (*)).

2. $\mathcal{O}_\Delta$ is reflexive as $\mathcal{O}_Z$-module. This follows for example from the fact that A has no non-trivial extensions by one-dimensional modules (see [7, Thm 4.1]).

First we prove that Z is smooth.

Let $q \in Z - S$. By the above $\mathcal{O}_{\Delta,q}$ is Azumaya of finite global dimension and hence the center of $\mathcal{O}_{\Delta,q}$, which is equal to $\mathcal{O}_{Z,q}$ also has finite global dimension. Thus Z is smooth at q.

Now we consider the case $q \in S$. Applying $\tilde{(-)}$ to the exact sequence

$$0 \to gA \to A \to B \to 0$$

using Lemma 3.3.1 yields an exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_\Delta \to \mathcal{O}_Y \to 0$$

which yields an exact sequence

$$0 \to \hat{\mathcal{J}}_q \to \hat{\mathcal{O}}_{\Delta,q} \to \hat{\mathcal{O}}_{Y,q} \to 0$$

So $\hat{\mathcal{O}}_{\Delta,q}/\hat{\mathcal{J}}_q$ is commutative by Lemma 3.3.1.

We claim that $\hat{\mathcal{O}}_{\Delta,q}$ is basic. If it weren't then by the fact that the simple representations of $\mathcal{O}_\Delta$ have either dimension 1 or s, it follows that $\hat{\mathcal{O}}_{\Delta,q}$ must have a simple representation of dimension s. However, since $q \in S$, $\hat{\mathcal{O}}_{\Delta,q}$ also has a simple representation of dimension 1. This contradicts the additivity principle for PI-degree [11].

Using the observations 1. and 2. above, it follows that the $\hat{\mathcal{O}}_{Z,q}$-order $\hat{\mathcal{O}}_{\Delta,q}$ satisfies the hypotheses for Lemma 3.3.4. Thus $\hat{\mathcal{O}}_{Z,q}$ is regular and hence Z is smooth in q.

Since $\hat{\mathcal{O}}_{Y,q} \cong \hat{\mathcal{O}}_{\Delta,q}/\hat{\mathcal{J}}_q$, Lemma 3.3.4 (6.) implies that Y has at most a node in q. Furthermore we see immediately that the ramification locus of $\hat{\mathcal{O}}_{\Delta,q}$ has normal crossings. Hence it follows that S has normal crossings in q.

Using the fact that Y is a cubic divisor in $\mathbb{P}^2$ we now deduce from (4.13) in [6], that Y is as stated in the proposition. $\square$

Lemma 3.3.7. *1. The irreducible components of Y define invertible ideals in $\mathcal{O}_\Delta$.*

2. $\mathcal{O}_\Delta$ is a maximal order.

3. $\mathcal{O}_\Delta$ has ramification index s in all components of S.

Proof. For some comments on why these assertions need some kind of proof see Remark 3.3.8 below.

1. Since $\mathcal{J}$ is invertible by construction and Y is reduced by Lemma 3.3.6, this is clear if Y consists of one component. So assume that this is not the case and let Y_1 be an irreducible component of Y with $Y_1 \neq Y$. Let $\mathcal{J}_1$ be the corresponding ideal in $\mathcal{O}_\Delta$.

   We first prove that $\mathcal{J}_1$ is reflexive as $\mathcal{O}_Z$-module. To see this let $\mathcal{J}_1'$ be the bidual of $\mathcal{J}_1$ with respect to $\mathcal{O}_Z$. Since Z is smooth, $\mathcal{J}_1$ is equal to $\mathcal{J}_1'$ in height one primes. In other words $\mathcal{J}_1'/\mathcal{J}_1$ is a zero dimensional submodule of $\mathcal{O}_{Y_1}$. Since $\mathcal{O}_{Y_1}$ is irreducible, it contains no such submodules. It follows that $\mathcal{J}_1' = \mathcal{J}_1$.

   Since the explicit models given in Lemma 3.3.4 yield that $\mathcal{O}_\Delta$ locally has global dimension two (this is easily seen directly), standard arguments now imply that $\mathcal{J}_1$ is at least locally projective (on either side) as $\mathcal{O}_\Delta$-module.

   Let p be a point in Y_1 which is singular in Y (such a point exists since Y is connected by Lemma 3.3.6) and let q be the corresponding point in S. Since p is singular in Y, $\hat{\mathcal{O}}_{\Delta,q}$ has only one simple module by Lemma 3.3.4 (6.), so by Lemma 3.3.4 (2.), $\hat{\mathcal{O}}_{\Delta,q} = k\langle\langle x, y\rangle\rangle/(yx - \zeta xy)$. Thus for example $\hat{\mathcal{J}}_{1,q} = (x)$ (Lemma 3.3.4 (5.)). In particular $\hat{\mathcal{J}}_{1,q}$ is invertible.

   Our aim is now to show that the cokernel $\mathcal{K}$ of

   $$\mathcal{J}_1 \otimes_{\mathcal{O}_\Delta} \mathcal{J}_1^* \to \mathcal{O}_\Delta$$

   is zero.
   By the previous discussion $\hat{\mathcal{K}}_q$ is clearly zero. Since as usual $\hat{\mathcal{O}}_{\Delta,q}$ is faithfully flat as $\mathcal{O}_{\Delta,q}$-module, it follows that $\mathcal{K}_q$ is also zero. Now let S_1 be the image of Y_1 in S, it then follows by semicontinuity, that $\mathcal{K} \mid U = 0$ for an open neighbourhood U of q in S_1. Since S_1 is irreducible and obviously also $\mathcal{K} \mid Z - S_1 = 0$, we conclude that the support of $\mathcal{K}$ is

zero-dimensional.

On the other hand since $\mathcal{J}_1$ is locally projective as $\mathcal{O}_\Delta$-module, it easily follows that $\mathcal{J}_1 \otimes_{\mathcal{O}_\Delta} \mathcal{J}_1^*$ is at least reflexive as $\mathcal{O}_Z$-module. So in particular it has no extensions by sheaves of finite support. We conclude that $\mathcal{K} = 0$ and so $\mathcal{J}_1$ is an invertible ideal in $\mathcal{O}_\Delta$.

2. Since $\mathcal{O}_\Delta$ is reflexive it suffices to show that $\mathcal{O}_\Delta$ is a maximal order in codimension one, i.e. that localisation on height one primes is a maximal order. The explicit local models of $\mathcal{O}_\Delta$ yield that $\mathcal{O}_\Delta$ is homologically homogeneous [16]. From this it follows that localizations of $\mathcal{O}_\Delta$ at generic points of irreducible curves are hereditary. Thus what we want to prove follows from Lemma 3.3.5 and (1.).

3. This follows also from Lemma 3.3.5. $\square$

Remark 3.3.8. First we remark that from the fact that $\mathcal{J}$ is invertible, it does not trivially follow that $\mathcal{J}_1$ is invertible. Indeed consider the following example

$$\mathcal{O}_\Delta = \begin{pmatrix} \mathcal{O}_W & \mathcal{O}_W(-w) \\ \mathcal{O}_W & \mathcal{O}_W \end{pmatrix} \tag{3.17}$$

where $W = \mathbb{P}^1$, $w \in \mathbb{P}^1$ and let

$$\mathcal{J} = \begin{pmatrix} \mathcal{O}_W(-w) & \mathcal{O}_W(-w) \\ \mathcal{O}_W & \mathcal{O}_W(-w) \end{pmatrix}$$

then clearly $\mathcal{J}$ is invertible and $\mathcal{O}_\Delta/\mathcal{J}$ defines two copies of the point w. If we take one copy to define $\mathcal{J}_1$, then we find

$$\mathcal{J}_1 = \begin{pmatrix} \mathcal{O}_W(-w) & \mathcal{O}_W(-w) \\ \mathcal{O}_W & \mathcal{O}_W \end{pmatrix}$$

which is not invertible.

The fact that $\mathcal{O}_\Delta$ is maximal also does not follow for trivial reasons. Note that it was shown in [37] that A is a maximal order. However this does not a priori imply that $\tilde{A}$ is a maximal order. To illustrate this we can use the same counterexample.

Let $A = \oplus_{n\in\mathbb{Z}} A_n$ with $A_n = \Gamma(W, \mathcal{J}^{-n})$. Then

$$A \cong \begin{pmatrix} k[x,y] & k[x,y](-1) \\ k[x,y](1) & k[x,y] \end{pmatrix}$$

where $k[x, y]$ is graded by $\deg x = 1$, $\deg y = 2$. Obviously A is a maximal order and one easily verifies that $\tilde{A} = \mathcal{O}_\Delta$. On the other hand $\mathcal{O}_\Delta$ is clearly non-maximal.

In dimension one, one cannot construct such an example with A a domain by Tsen's theorem. However this is possible in higher dimension. Start for example with a maximal order $\mathcal{O}_\Gamma$ in a division algebra over a surface W and then take a "tame" suborder $\mathcal{O}_\Delta$ [35] of $\mathcal{O}_\Gamma$ ramified in a curve over which $\mathcal{O}_\Gamma$ itself is unramified.

Nevertheless it is conceivable that the methods in [37] may be adapted to prove that $\tilde{A}$ is a maximal order in a reasonable level of generality. However for simplicity we have prefered to give a direct proof in our special case.

Lemma 3.3.9. *1. If $q \in S$, then $\hat{\mathcal{O}}_{\Delta,q}$ has the form indicated in the statement of Proposition 3.3.3.*

2. The obvious map $Y \to S$ is one-one on singular points and outside the singular points defines an s-sheeted covering.

Proof. 1. If q is a node on Y, then $\hat{\mathcal{O}}_{\Delta,q}$ has one simple and so it has the required form by Lemma 3.3.4 (2.). If q is smooth on Y, then $\hat{\mathcal{O}}_{\Delta,q}$ has more than one simple and with the notation of Lemma 3.3.4 (2.) "t" must be exactly s, since all components of S have equal ramification index s by the previous lemma. It then follows from Lemma 3.3.4 (2.) that T is the commutative ring $k[[x, y]]$.
KAN JE AUB EVEN UITLEGGEN WAAROM ER IN HET GEVAL VAN SMOOTH ALTIJD MEER DAN 1 SIMPEL IS, UIT DE STELLING VAN ARTIKEL 1 VOLGT TOCH MAAR ALLEEN (NOT REGULAR $\Rightarrow$ 1 SIMPEL), DUS EEN ENKELE PIJL ?

2. This can now be verified directly using the explicit form of $\hat{\mathcal{O}}_{\Delta,q}$. □

Proof of Proposition 3.3.3. The only parts of Proposition 3.3.3 that still need proof are the fact that $Z = \mathbb{P}^2$ and the explicit form of S. However once we have shown that $Z = \mathbb{P}^2$ then the form of S can be verified case by case using Lemma 3.3.9 (2.) and the fact that S has normal crossings.

So it remains to show that $Z = \mathbb{P}^2$. To do this we compute ω_Δ. Using standard theory of maximal orders and Lemma 3.2.2, we find

$$\omega_\Delta = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_\Delta, \omega_Z) = \mathcal{J}^{1-s} \otimes_{\mathcal{O}_Z} \omega_Z \tag{3.18}$$

On the other hand by definition $\omega_\Delta = \widetilde{H^3_m(A)'}$. Now it has been shown by Yekutieli [41] that $H^3_m(A)'$ is equal to $(gA)_\lambda$ where λ is an automorphism which on A_n, is multiplication by α^n for some scalar α. It is clear that λ disappears when we apply $(\tilde{-})$ and hence $\omega_\Delta = (gA)\tilde{} = \mathcal{J}$.

Comparing with (3.18) yields that

$$\mathcal{O}_\Delta \otimes_{\mathcal{O}_Z} \omega_Z = \mathcal{J}^s \tag{3.19}$$

Using the fact that gA only differs by a graded automorphism from $A(-3)$, it is easy to see that $\mathcal{J}^{-1}$ is ample (in the sense of [9]).
Since we are in characteristic zero, the trace map shows that $\mathcal{O}_Z$ is a direct summand of $\mathcal{O}_\Delta$.
From this we deduces the following:

Sublemma . ω_Z^{-1} *is ample on* Z.

Proof. Let $\mathcal{M}$ be a coherent sheaf on Z. We have to show that for $i > 0$ and for large n $H^i(Z, \omega_Z^{-n} \otimes_{\mathcal{O}_Z} \mathcal{M}) = 0$ [20, Prop III.5.3]. Since $\mathcal{O}_Z$ is a direct summand of $\mathcal{O}_\Delta$, it is sufficient to show that $H^i(Z, \mathcal{O}_\Delta \otimes_{\mathcal{O}_Z} \omega_Z^{-n} \otimes_{\mathcal{O}_Z} \mathcal{M}) = 0$ for large n. Now by (3.19), we have $\mathcal{O}_\Delta \otimes_{\mathcal{O}_Z} \omega_Z^{-n} = \mathcal{J}^{-ns}$. Since $\mathcal{J}^{-1}$ is ample this proves what we want. $\square$

From this sublemma, we deduce, using [10, Ex. V.1], that Z is either $\mathbb{P}^1 \times \mathbb{P}^1$ or else is obtained by blowing up at most eight points in $\mathbb{P}^2$ in general position.

To find out the actual form of Z, we compute the self intersection (ω_Z, ω_Z).

$$\begin{aligned}
(\omega_Z, \omega_Z) &= \tfrac{1}{s^2}(\mathcal{J}^s, \mathcal{J}^s) && \text{by Lemma 3.1.8 and (3.19)}\\
&= \tfrac{1}{s}(\mathcal{J}, \mathcal{J}^s) && \text{using repeatedly (I1)}\\
&= (\mathcal{J}, \mathcal{J}) && \text{using repeatedly (I2)}\\
&= \chi(\mathcal{J}^2) - 2\chi(\mathcal{J}) + \chi(\mathcal{O}_\Delta) && \text{by Proposition 3.1.7}\\
&= \chi(\mathcal{O}_\Delta/\mathcal{J}) - \chi(\mathcal{J}/\mathcal{J}^2) && \text{using the obvious exact sequences}
\end{aligned}$$

Now we have $\mathcal{O}_\Delta/\mathcal{J} = \mathcal{O}_Y$ and thus by the fact that Y has arithmetic genus one, we deduce $\chi(\mathcal{O}_\Delta/\mathcal{J}) = 0$.
Furthermore $\mathcal{J}/\mathcal{J}^2 = (gA/gA^2)\tilde{} \cong (A/gA)(-3)\tilde{}$ (as left modules). To compute the Euler characteristic of A/gA, we use that it is a twisted homogeneous coordinate ring of Y associated to a line bundle of degree three (and an automorphism). Under the equivalence explained in chapter 1, $(A/Ag)(-3)$

corresponds to a line bundle of degree -9 on Y. Thus $\chi(\mathcal{J}/\mathcal{J}^2) = -9$, since Y has arithmetic genus 1.
Combining everything we find that $(\omega_Z, \omega_Z) = 0 - (-9) = 9$.

Now we have $(\omega_{\mathbb{P}^2}, \omega_{\mathbb{P}^2}) = 9$ and $(\omega_{\mathbb{P}^1 \times \mathbb{P}^1}, \omega_{\mathbb{P}^1 \times \mathbb{P}^1}) = 8$. Since blowing up further reduces (ω, ω) [20, Prop V.3.3] the only possibility is $Z \cong \mathbb{P}^2$. $\square$

Chapter 4

The center of a non-commutative regular local ring of dimension two

By the previous chapter, it is clear that we are interested in centers. In this final chapter we will look at the center of a class of rings, which includes the ones mentioned in Proposition 2.5.1. MICHEL, KAN JE DE PARAGRAAF HIERBOVEN MISSCHIEN DOOR IETS BETERS VERVANGEN ? IK KON NIETS ANDERS DAN DIT VERZINNEN.

Let us be more specific. Below k is a field. The rings we will be concerned with, are of the form

$$C = k\langle\langle x, y\rangle\rangle/(\psi)$$

where ψ only has term of total degree ≥ 2 and where the quadratic part of ψ is non-degenerate.

By the proof of Proposition 2.5.1, such rings have global dimension two and it may be argued that they are the non-commutative analogues of two-dimensional regular local rings. This explains at once the title of this chapter.

We propose the following conjecture:

Conjecture 4.0.1. *Let C be as above. Then the center of C is either trivial, or else it is a formal power series ring in two variables. If the quadratic part of ψ is of the form $yx - xy$ and the characteristic p of k is > 0 then $Z(C)$ is generated by elements of the form $x^{p^n} + \varphi(x)$ and $y^{p^n} + \theta(x, y)$ for some $n > 0$ and where φ and θ are trivial or contain only terms in x, y of total degree $> p^n$.*

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In this chapter, we will prove this conjecture in the case that C is given by an Ore extension. That is

$$C = B[[y; \sigma, \delta]]$$

where B is $k[[x]]$, σ is a k-linear automorphism of B and δ is a k-linear σ-derivation of B. Thus δ satisfies

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

and C is obtained from B by adjoining the variable y, subject to the commutation rule

$$yb = \sigma(b)y + \delta(b) \tag{4.1}$$

In other words $C = k\langle\langle x, y\rangle\rangle/(\psi)$ where ψ is given by $yx - \sigma(x)y - \delta(x)$. Thus for ψ to have only terms of degree ≥ 2, it is necessary that $\delta(x)$ contains only terms of degree ≥ 2, if δ is non trivial. We assume this throughout.

We will prove the following theorem:

Theorem 4.0.2. *If C is an Ore extension as above then Conjecture 4.0.1 holds.*

Our treatment of the case where σ is trivial relied originally on the following combinatorial result by G. Baron and A. Schinzel in [12].

Proposition 4.0.3. *For any prime p and any residues x_i mod p, we have:*

$$\sum_{\sigma \in S_{p-1}} x_{\sigma(1)}(x_{\sigma(1)} + x_{\sigma(2)}) \dots (x_{\sigma(1)} + \dots + x_{\sigma(p-1)})$$

$$\equiv (x_1 + \cdots + x_{p-1})^{p-1} \pmod{p}$$

where S_{p-1} is the group of all permutations σ of $\{1, \dots, p-1\}$.

Afterwards we discovered a new approach which is independent of the above result. It turns out that we can now even give a new proof of the result by G. Baron and A. Schinzel, using Lemma 4.1.1. This proof is produced in the final section of this chapter.

Let us give an outline of the strategy used in the following for proving Theorem 4.0.2.

First we dispense with some trivial cases. If σ is trivial and $\delta = 0$ then there is nothing to prove (since we do not consider this case). In addition, it is easy to prove that in the following cases the center of C is trivial.

1. σ is trivial, δ is not trivial and $p = 0$.

2. The order of σ is infinite.

In subsequent sections we deal with the remaining cases. In Section 4.1, we discuss the case where σ is the identity and $p > 0$. In Section 4.2, we focus on the case where δ is trivial and σ is not trivial but has finite order. Finally in Section 4.3 we deal with the case where both σ and δ are non-trivial and σ has finite order.
In this last case our approach is somewhat indirect and we do not obtain nice expressions for the elements generating the center.

4.1 The case where σ is the identity and $p > 0$

It follows from (4.1) that in this case the commutation relation between y and x is given by
$$y\,x \;=\; x\,y \;+\; \delta(x) \tag{4.2}$$

In this case we prove
$$Z(C) = k[[z, w]]$$
where $z = x^p$ and $w = y^p - c_p(x)y$, with
$$c_p(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\dots\left(\frac{\partial\,\delta(x)}{\partial x}\cdot\delta(x)\right)\dots\cdot\delta(x)\right)\cdot\delta(x)\right)$$
in which $\dfrac{\partial}{\partial x}$ and $\delta(x)$ occur $(p-1)$ times.

It is obvious that $[x, z] = 0$. Furthermore from
$$\begin{aligned}[y,z] &= \delta(x^p) = x\delta(x^{p-1}) + \delta(x)x^{p-1} = \dots \\ &= \sum_{\substack{a+b=p-1\\ a,b\geq 0}} x^a\delta(x)x^b = p\delta(x)x^{p-1} = 0\end{aligned}$$
we deduce that z also commutes with y. Hence z is in the center of C.

To prove that w is in the center of C we use the following key-lemma. This lemma will also be used in the new proof of Proposition 4.0.3.

Lemma 4.1.1. *Let $f \in B$ and put*

$$g = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\cdots\left(\frac{\partial f}{\partial x}\cdot f\right)\cdots\cdot f\right)\cdot f\right) \in B$$

where both $\dfrac{\partial}{\partial x}$ *and f occur $(p-1)$ times. Then* $\dfrac{\partial g}{\partial x} = 0$.

Proof. Without loss of generality we may assume that $f \neq 0$. Define the derivation d of B by

$$d(b) := \frac{\partial b}{\partial x}\cdot f$$

and consider the differential operator $e = d^p - g \cdot d$ on B.

We have

$$d^p(x^{i+j}) = \sum_{l=0}^{p}\binom{p}{l} d^l(x^i)d^{p-l}(x^j)$$

This sum reduces to $d^p(x^i)x^j + x^i d^p(x^j)$ in characteristic p since p divides $\binom{p}{l}$, for all $1 \leq l \leq p-1$. It follows that e is also a derivation of B.

If we evaluate e in x, we get

$$\begin{aligned} e(x) &= d^p(x) - g\cdot d(x) \\ &= f\cdot\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\cdots\left(\frac{\partial f}{\partial x}\cdot f\right)\cdots\cdot f\right)\cdot f\right) - g\cdot f \\ &= f\cdot g - g\cdot f \\ &= 0 \end{aligned}$$

and so e is identically zero on B. In particular, e commutes with d.

On the other hand we have, for all $i > 0$

$$\begin{aligned} [e,d](x^i) &= e(d(x^i)) \\ &= ix^{i-1}e(f) \\ &= ix^{i-1}\left(f\frac{\partial}{\partial x}(fg) - gf\frac{\partial f}{\partial x}\right) \\ &= ix^{i-1}f^2\frac{\partial g}{\partial x} \\ &= f\frac{\partial g}{\partial x}d(x^i) \end{aligned}$$

and

$$[e,d](x) = e(d(x)) = e(f) = f^2\frac{\partial\, g}{\partial x} = f\frac{\partial\, g}{\partial x}d(x)$$

It follows that

$$f\frac{\partial\, g}{\partial x}d = 0$$

on B. Evaluating at x and using the fact that $f \neq 0$, this yields $\dfrac{\partial\, g}{\partial x} = 0$. $\square$

Let y_l, respectively y_r be left, respectively right multiplication by y on B. Because y_l and y_r commute, we see that

$$[y,-]^p = \sum_{i=0}^{p} \binom{p}{i} y_l^i(-y_r)^{p-i} = y_l^p - y_r^p = [y^p,-]$$

It follows that we have

$$[y^p, x] = [y, [y, \ldots, [y, \delta(x)] \ldots]] \quad ((p-1) \text{ times } y)$$

By repeatedly using the fact that, for all $f(x) \in B$

$$[y, f(x)] = \frac{\partial\, f(x)}{\partial x}[y,x] = \frac{\partial\, f(x)}{\partial x} \cdot \delta(x)$$

we deduce, for $f(x) = \delta(x)$

$$[y^p, x] = c_p(x)\,[y,x]$$

It follows that w commutes with x.

Furthermore, applying Lemma 4.1.1 with $f = \delta(x) \in B$, we deduce

$$[y,w] = [y, c_p(x)]\,y = \frac{\partial\, c_p(x)}{\partial x}\,[y,x]\,y = 0$$

Thus w commutes also with y and we obtain $k[[z,w]] \subset Z(C)$.

Let $Q(Z(C))$ and $Q(C)$ be respectively the quotientfields of $Z(C)$ and C. Since $\{x^a y^b \mid 0 \le a, b \le p-1\}$ is a basis of C over $k[[z,w]]$, we see that C is free of rank p^2 over $k[[z,w]]$. This implies

$$p^2 = \dim_{k((z,w))} Q(Z(C))\ \dim_{Q(Z(C))} Q(C)$$

whence

$$\dim_{Q(Z(C))} Q(C) \in \{1, p, p^2\}$$

Since C is not commutative and $\dim_{Q(Z(C))} Q(C)$ is a square according to [17], it follows that

$$\dim_{Q(Z(C))} Q(C) = p^2$$

and furthermore that $Z(C)$ and $k[[z,w]]$ have the same quotientfield.

As indicated above C is free of rank p^2 over $k[[z,w]]$. In particular C is finitely generated as a module over $k[[z,w]]$. It follows that $Z(C)$ is also finitely generated as a module over $k[[z,w]]$ and thus $Z(C)$ is integral over $k[[z,w]]$. Since $k[[z,w]]$ is integrally closed, this yields $Z(C) = k[[z,w]]$.

So in order to complete the proof Conjecture 4.0.1 in this special case, we have to show that if $v(\delta(x)) \geq 3$, $v(c_p(x)) > p-1$, where v is the x-adic valuation on B. Therefore, put, for all $r \geq 2$

$$c_r(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\dots\left(\frac{\partial\,\delta(x)}{\partial x}\cdot\delta(x)\right)\dots\cdot\delta(x)\right)\cdot\delta(x)\right)$$

in which $\dfrac{\partial}{\partial x}$ and $\delta(x)$ occur $(r-1)$ times.
We prove by induction that $v(c_r(x)) \geq 2(r-1)$.

Since $v(\delta(x)) \geq 3$, we get

$$v(c_2(x)) = v\left(\frac{\partial\,\delta(x)}{\partial x}\right) \geq 2$$

By induction, we have

$$\begin{aligned} v(c_r(x)) &= v\left(\frac{\partial}{\partial x}\left(c_{r-1}(x)\cdot\delta(x)\right)\right) \\ &= v(c_{r-1}(x)) + v(\delta(x)) - 1 \\ &\geq 2(r-2) + 3 - 1 \\ &= 2(r-1) \end{aligned}$$

So $v(c_p(x)) \geq 2(p-1) > p-1$.

4.2 The case where $\delta = 0$ and σ is not trivial but has finite order

In this case the commutation relation between y and x is given by:

$$y\,x \;=\; \sigma(x)\,y \tag{4.3}$$

We will denote the order of σ by n and put $A = B^{\sigma}$. Let K, L be the quotientfields of A, B respectively. We prove that

$$Z(C) = k[[z, y^n]]$$

where $z = x\,\sigma(x)\ldots\sigma^{n-1}(x)$.

Let us first discuss the structure of A.

Lemma 4.2.1. *$A = k[[z]]$, with z as above.*

Proof. It is obvious that A is a complete discrete valuation ring and k is a copy of its residue field. So A is a formal power series ring $k[[u]]$, where u is a uniformizing element.

Since K is complete under a discrete valuation and L is a finite extension of K, the uniformizing element u must be of the form x^e + higher terms, where e is the ramification index. Furthermore, since the residue class degree equals 1, we conclude that $e = [L : K] = n$.

It is easy to see that since σ is k-linear,

$$\sigma(x) = \zeta\,x + \text{ higher terms}$$

where ζ is an nth root of unity.
So $z = x\,\sigma(x)\ldots\sigma^{n-1}(x)$ is of the form $x^n +$ higher terms. Therefore z is also a uniformizing element and furthermore $A = k[[z]]$. □

It is clear that $A \subset Z(C)$ and that y^n belongs to the center of C. We now look at the other inclusion.

Let f be in $Z(C)$. We can write f, in a unique way, in the form $\sum_{i\geq 0} a_i\,y^i$, where $a_i \in B$. Since $f \in Z(C)$, we have (using (4.3))

$$0 = [x, f] = \sum_{i\geq 0} a_i(x - \sigma^i(x))y^i$$

Hence, for all $i \in \mathbb{N}$, if $a_i \neq 0,\ x = \sigma^i(x)$, so n divides i.
On the other hand we have

$$0 = [y, f] = \sum_{i \geq 0} (\sigma(a_i) - a_i) y^{i+1}$$

so $\sigma(a_i) = a_i$, for all i in $\mathbb{N}$. This yields $a_i \in A$, for all i in $\mathbb{N}$.
It follows that $f \in k[[z, y^n]]$.

We have now proved that $Z(C)$ is a formal power series ring in the two variables z, w. The remaining claim of Conjecture 4.0.1 follows from the fact that if $\sigma(x)$ is of the form $x+$ higher terms, then

- If $p = 0$ and σ is non-trivial then its order is infinite (easily proved).
- If $p > 0$ and if the order of σ is finite then it is a power of p [36].

4.3 The case where σ and δ are non trivial and σ has finite order

Here we have the following commutationrelation between y and x :

$$y\,x \;=\; \sigma(x)\,y \;+\; \delta(x) \tag{4.4}$$

As before we denote the order of σ by n and we assume $n \neq 1$. We put $A = B^\sigma$ and we let K and L be respectively the quotientfields of A and B. We extend the action of σ and δ to L and we denote these extended maps also by σ and δ.

It was shown in Lemma 4.2.1, that A is the ring of power series over k in $z = x\,\sigma(x)\ldots\sigma^{n-1}(x) \,\in\, B$.

For convenience we will first work in the polynomial Ore extension $S = B[y; \sigma, \delta]$. We will prove the following theorem.

Theorem 4.3.1. *The center $Z(S)$ of S is the ring of polynomials $A[w]$, where w is a monic (skew) polynomial of degree n in y with coefficients in B. In particular, we find that S is free of rank n^2 over $Z(S)$.*

The proof of this theorem depends on the following lemma:

Lemma 4.3.2. *Let D, D' be central simple algebras of the same PI-degree with centers Z, Z', respectively. Assume that $D \subseteq D'$. Then $Z \subseteq Z'$ and furthermore the map $\varphi : D \otimes_Z Z' \to D'$, defined by $\varphi(d \otimes z') := dz'$, is an isomorphism.*

Proof. Denote the PI-degree of D and D' by m. Then the PI-degree of DZ' is also equal to m since we have inclusions $D \subseteq DZ' \subseteq D'$.

Furthermore, from $Z' \subseteq Z(DZ') \subseteq DZ' \subseteq D'$ (where $Z(DZ')$ is the center of DZ'), we deduce

$$m^2 = [DZ' : Z(DZ')] \leq [DZ' : Z'] \leq [D' : Z'] = m^2$$

This yields $[DZ' : Z'] = m^2 = [D' : Z']$. And it follows that $DZ' = D'$ and in particular

$$Z \subseteq Z(DZ') = Z(D') = Z'$$

From $DZ' = D'$ we also conclude that the map $\varphi : D \otimes_Z Z' \to D'$ defined above, is an epimorphism. Since D is a central simple algebra, the same holds for $D \otimes_Z Z'$. So $D \otimes_Z Z'$ is simple, which implies that φ has to be an isomorphism. $\square$

Proof of Theorem 4.3.1. For all $f \in B$, we have, working out the identity $\delta(x\, f) = \delta(f\, x)$

$$\delta(f) \;=\; \frac{\sigma(f) \,-\, f}{\sigma(x) \,-\, x} \cdot \delta(x) \tag{4.5}$$

This implies $\delta(f) = 0$, for all $f \in A$. So the polynomial ring $R = A[y]$ is a commutative subring of S.

Now consider S as right R-module. Since B is free of rank n over $A = k[[x^n +$ higher terms$]]$, S is free of rank n over R. Furthermore, left multiplication yields an injective ringhomomorphism

$$S \hookrightarrow \mathrm{End}_R(S_R) \tag{4.6}$$

It follows that S is isomorphic to a subring of the matrix ring $M_n(R)$, which is a PI-ring since R is commutative. So S satisfies a polynomial identity and furthermore, the PI-degree of S is less or equal to the PI-degree of $M_n(R)$ which is n. We claim that it is exactly n.

To see this, filter S by y degree and denote the associated graded ring by $\mathrm{gr}\, S$. Since $\mathrm{gr}\, S = B[\overline{y}; \sigma]$, we see that $\mathrm{gr}\, S$ is a domain and furthermore

$Z(\operatorname{gr} S) = A[\overline{y}^n]$ by Section 4.2. So $\operatorname{gr} S$ is a prime ring of rank n^2 over its center which implies that its PI-degree is equal to n. Since the PI-degree of $S \geq$ PI-degree of $\operatorname{gr} S$, it now follows that the PI-degree of S is exactly n.

Let E be the quotientfield of S. As in (4.6) we have an inclusion

$$i : E \hookrightarrow \operatorname{End}_{K(y)}(E_{K(y)}) \tag{4.7}$$

Since E and $\operatorname{End}_{K(y)}(E_{K(y)})$ are both central simple algebras of PI-degree n, (4.7) induces, by Lemma 4.3.2, an isomorphism

$$\varphi : E \otimes_{Z(E)} K(y) \longrightarrow \operatorname{End}_{K(y)}(E_{K(y)}) \tag{4.8}$$

defined by $\varphi(e \otimes f) = i(e)\, f$.
This means that we can compute the characteristic polynomial of each $e \in E$, in $\operatorname{End}_{K(y)}(E_{K(y)})$.

Since S is an Ore extension, it is also a maximal order by [23] and so it is closed under taking coefficients of reduced characteristic polynomials. Using this observation we can now explicitly construct elements in the center of S since the coefficients of reduced characteristic polynomials are central elements of E. The coefficient we are interested in, is the reduced norm of y.

By definition this reduced norm may be computed by taking the image of y in $\operatorname{End}_{K(y)}(E_{K(y)})$ under (4.8), i.e. $\varphi(y \otimes 1) = i(y)$, where $i(y)$ is left multiplication by y, and then computing the determinant of $i(y)$ in $\operatorname{End}_{K(y)}(E_{K(y)})$.

To perform this computation we need a suitable basis for $E/K(y)$. We pick a normal basis $\{f, \sigma(f), \ldots, \sigma^{n-1}(f)\}$ for L/K, for some $f \in L$ [17]. This is still a basis for $E/K(y)$.

We now compute the matrix of $i(y)$ explicitly.
By (4.4) we get, for all $0 \leq j \leq n-1$,

$$i(y)(\sigma^j(f)) = \sigma^{j+1}(f)y + \delta(\sigma^j(f))$$

Since $\{f, \sigma(f), \ldots, \sigma^{n-1}(f)\}$ is a basis for L/K

$$i(y)(\sigma^j(f)) = \sigma^{j+1}(f)y + \sum_{i=0}^{n-1} \sigma^i(f) a_{ji}$$

for certain $a_{ji} \in K$.

It follows that the matrix of $i(y) = D + C\,y$, where $D = (a_{ji}) \in M_n(K)$ and

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

the matrix of a cyclic permutation.

This yields

$$\mathrm{Nrd}(y) = \det(D + Cy) = (-1)^{n+1} y^n + \text{ lower terms in } y$$

Put $w = (-1)^{n+1}\,\mathrm{Nrd}(y)$. Clearly $A[w] \subset Z(S)$. Since B is free of rank n over A and $w = y^n +$ lower terms in y, S is free of rank n^2 over $A[w]$. In particular, $Z(S)$ is integral over $A[w]$. Since $A[w] \subset Z(S) \subset S$, it follows that

$$K(w) \subset Q(Z(S)) \subset E \tag{4.9}$$

where $Q(Z(S))$ is the quotientfield of $Z(S)$.
Since S is free of rank n^2 over $A[w]$ and E is a central simple algebra of PI-degree n, (4.9) yields

$$\dim_{K(w)} Q(Z(S)) = 1$$

Thus $A[w]$ and $Z(S)$ have the same quotientfield. The fact that $A[w]$ is integrally closed and $Z(S)$ is integral over $A[w]$, now implies $A[w] = Z(S)$. $\square$

In the next proposition we will obtain more information on the element w constructed in the above theorem. Let v be the x-adic valuation on B.

Proposition 4.3.3. *Assume that $v(\delta(x)) = a$. If $w = y^n + \sum_{i=0}^{n-1} f_i(x)\,y^i$, then for $i > 0$ we have*

$$v(f_i) \geq (a-1)(n-i)$$

Furthermore there exists an element $q_0(z) \in k[[z]]$ such that

$$v(f_0 + q_0(z)) \geq (a-1)n$$

In the proof of this proposition we need the result of the following lemma:

Lemma 4.3.4. *If $f \in B$, then*

$$v\left(\frac{\sigma(f)-f}{\sigma(x)-x}\right) \geq v(f)-1$$

Proof. Put $r = v(f)$ and $f = \sum_{i=r}^{+\infty} a_i x^i$, where $a_i \in k$ and $a_r \neq 0$.

Case 1. $r \geq 1$

If we denote f by $f(x)$ and $\sum_{i=r}^{+\infty} a_i \sigma(x)^i$ by $f(\sigma(x))$, we have

$$\frac{\sigma(f)-f}{\sigma(x)-x} = \frac{f(\sigma(x))-f(x)}{\sigma(x)-x} = \frac{f(x+h)-f(x)}{h}$$

where $h = \sigma(x) - x$.

It is easy to see that

$$\frac{f(x+h)-f(x)}{h} = \sum_{i=0}^{+\infty}\left(\sum_{j=r}^{+\infty} a_j \psi_{i,j} h^{j-i-1}\right) x^i$$

where

$$\psi_{ij} = \begin{cases} 0 & \text{if } i \geq j \\ \frac{j!}{i!(j-i)!} & \text{if } i < j \end{cases}$$

Since σ is of the form $\zeta x +$ higher terms, where ζ in an nth root of unity, $v(h) \geq 1$. It follows that

$$v\left(\frac{\sigma(f)-f}{\sigma(x)-x}\right) = v\left(\frac{f(x+h)-f(x)}{h}\right) \geq \min_i((r-i-1)v(h)+i) \geq r-1$$

Case 2. $r = 0$

Since σ is an automorphism which is also k-linear, it follows that

$$v\left(\frac{\sigma(f)-f}{\sigma(x)-x}\right) = v\left(\frac{\sigma(g)-g}{\sigma(x)-x}\right)$$

where $g = f - a_0$.

So applying Case 1 to g, this yields

$$v\left(\frac{\sigma(f)-f}{\sigma(x)-x}\right) \geq v(g)-1 \geq 0 \geq v(f)-1$$

□

We return now to the proof of Proposition 4.3.3.

Proof of Proposition 4.3.3. Put $\overline{y} = x^{-a+1}y$. If we multiply (4.4) on the left with x^{-a+1}, we obtain

$$\overline{y}\, x \;=\; \sigma(x)\, \overline{y} \;+\; x^{-a+1}\, \delta(x) \tag{4.10}$$

Consider the ring $\overline{S} = B[\overline{y}; \sigma, \overline{\delta}]$, where $\overline{\delta}$ is the σ-derivation of B defined by $\overline{\delta}(b) = x^{-a+1}\delta(b)$. We clearly have inclusions $S \subset \overline{S} \subset L[y; \sigma, \delta]$.

Applying Theorem 4.3.1 to $\overline{S}$, we find that $\overline{S}$ has a central element $\overline{w}$ of the form

$$\overline{w} = \overline{y}^n + \sum_{i=0}^{n-1} g_i(x)\overline{y}^i \tag{4.11}$$

with $g_i(x) \in B$.

Verifying the commutationrelation of x^{-a+1} and y, we find

$$y\, x^{-a+1} = \sigma(x^{-a+1})\, y + \delta(x^{-a+1}) \tag{4.12}$$

Using (4.5) and Lemma 4.3.4, we have for all $f \in B$

$$v(\delta(f)) = v\left(\frac{\sigma(f) - f}{\sigma(x) - x} \cdot \delta(x)\right) = v\left(\frac{\sigma(f) - f}{\sigma(x) - x}\right) + v(\delta(x)) \geq v(f) - 1 + a$$

In particular, it follows that $\delta(x^{-a+1}) \in B$.

Using (4.12), we can rewrite $\overline{w}$ in the following form

$$\overline{w} = z^{-a+1}y^n + h_0(x) + \sum_{i=1}^{n-1}(x\,\sigma(x)\ldots\sigma^{i-1}(x))^{-a+1}h_i(x)y^i$$

where $h_i(x) \in B$, for all $0 \leq i \leq n-1$ and with z the element of A defined in Section 4.2.

Multiplying $\overline{w}$ with z^{a-1}, we get the element

$$y^n + z^{a-1}h_0(x) + \sum_{i=1}^{n-1}(\sigma^i(x)\ldots\sigma^{n-1}(x))^{a-1}h_i(x)y^i$$

which we will denote by w'.

Let us write $p_0(x)$ for $z^{a-1}h_0(x)$ and $p_i(x)$ for $(\sigma^i(x)\ldots\sigma^{n-1}(x))^{a-1}h_i(x)$, for all $1 \leq i \leq n-1$.

It follows that

$$v(p_0(x)) = (a-1)v(z) + v(h_0(x)) \geq (a-1)n \geq 0$$

and, for all $1 \leq i \leq n-1$

$$v(p_i(x)) = (a-1)\left(\sum_{j=i}^{n-1} v(\sigma^j(x))\right) + v(h_i(x)) \geq (a-1)(n-i) \geq 0$$

So w' belongs to S and since it is a central element of $\overline{S}$, $w' \in Z(S)$.

By Theorem 4.3.1, this yields w' has to be of the form

$$w' = \sum q_i(z)w^i \tag{4.13}$$

Looking at the degree of y, reduces (4.13) to $w' = q_0(z) + q_1(z)\,w$ and since the coefficient of $y^n = 1$, it follows that $w' = q_0(z) + w$. Hence, for all $1 \leq i \leq n-1$

$$v(f_i(x)) = v(p_i(x)) \geq (a-1)(n-i)$$

and

$$v(q_0(z) + f_0(x)) = v(p_0(x)) \geq (a-1)n$$

□

Corollary 4.3.5. *Let C be the formal power series ring $k[[x]][[y;\sigma,\delta]]$ and n the order of σ. Then the center of C is equal to $k[[z,w]]$, where $z = x^n + \varphi(x)$ and $w = y^n + \theta(x,y)$. If $v(\delta(x)) \geq 3$, then φ and θ contain only terms in x, y of total degree $> n$.*

Proof. Let $M \subset S$ be the twosided ideal generated by x, y. Clearly C is equal to the M-adic completion of S. Let m be the maximal ideal of $Z(S)$ generated by z, w. It is easy to see that

$$M^{2n} \subset mS \subset M$$
$$m^a S \cap Z(S) = m^a$$

Thus the completion of $Z(S)$ at the induced topology, coincides with the completion at the m-adic topology, which is $k[[z,w]]$.

Since $S \subset C$, the PI-degree of C is $\geq n$. On the other hand, using the properties of completion, every identity in S vanishes in C. So the PI-degree of C is exactly n.

Since $Z(C) \supset k[[z,w]]$, $\mathrm{rk}_{Z(C)}\, C = n^2$ and $k[[z,w]]$ is integrally closed, we prove exactly as before that $Z(C) = k[[z,w]]$.

From Proposition 4.3.3, it follows that we may assume

$$w = y^n + \sum_{i=0}^{n-1} f_i(x) y^i$$

such that for all $0 \leq i \leq n-1$, we have

$$v(f_i(x)) \geq (a-1)(n-i)$$

where $a = v(\delta(x))$. So in the case that $v(\delta(x)) \geq 3$, θ has the required form.

The proof of Lemma 4.2.1 yields the statement about φ. □

To complete the proof of Theorem 4.0.2, we use the fact that if $\sigma(x)$ is of the form $x+$ higher terms and the characteristic $p > 0$, the order of σ is a power of p [32].

Remark 4.3.6. Although we cannot give a nice expression for w in general, we can do so in a few special cases.

- If $n = 2$ and $p = 2$, then w can be taken equal to

$$w = y^2 - \frac{\sigma(\delta(x)) - \delta(x)}{\sigma(x) - x}\, y$$

  Note that by Lemma 4.3.4, w is indeed an element of C.

- If $\sigma(x)$ has the form $x+$ higher terms, $p > 0$ and $\delta(x)/(\sigma(x) - x)$ is σ-invariant, then $w = y^n$.

4.4 A new proof of Proposition 4.0.3

Let k be a field of characteristic $p > 0$ and consider the field $k(t_1, \ldots, t_{p-1})$, where $t_1, \ldots, t_{p-1}$ are variables. Let $f = \sum_{i=1}^{p-1} f_i t_i \in k(t_1, \ldots, t_{p-1})[x]$ be arbitrary.

Since $k(t_1, \ldots, t_{p-1})$ is also a field of characteristic p, it follows from Lemma 4.1.1 that f satisfies

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}\left(\cdots\left(\frac{\partial f}{\partial x}\cdot f\right)\cdots f\right)\cdot f\right) = 0 \tag{4.14}$$

where $\dfrac{\partial}{\partial x}$ occurs p times and f $(p-1)$ times.

We first compute $\frac{\partial f}{\partial x}$. Assume $f_i = \sum_{j=0}^{n_i} a_{ji}x^j \in k[x]$. We can rewrite f in the form

$$f = \sum_{j=0}^{m} \left(\sum_{i=1}^{p-1} a_{ji} t_i \right) x^j$$

where $m = \max\{n_1, \dots, n_{p-1}\}$ and for all $n_i < j \leq m$, $a_{ji} = 0$. It follows that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \sum_{j=1}^{m} j \left(\sum_{i=1}^{p-1} a_{ji} t_i \right) x^{j-1} \\
&= \sum_{i=1}^{p-1} \left(\sum_{j=1}^{n_i} j a_{ji} x^{j-1} \right) t_i \\
&= \sum_{i=1}^{p-1} \frac{\partial f_i}{\partial x} t_i
\end{aligned}$$

This yields, taking the coefficient of $t_1 \dots t_{p-1}$ in (4.14)

$$\sum_{\sigma \in S_{p-1}} \frac{\partial^2}{\partial x^2} \left(\frac{\partial}{\partial x} \left(\dots \left(\frac{\partial f_{\sigma(1)}}{\partial x} \cdot f_{\sigma(2)} \right) \dots \cdot f_{\sigma(p-2)} \right) \cdot f_{\sigma(p-1)} \right) = 0$$

for all polynomials f_i over a field k of characteristic $p > 0$.

Consider the following expression in the variables $f_1, \dots, f_{p-1}$

$$\begin{aligned}
\sum_{\sigma \in S_{p-1}} \Bigg[\frac{\partial^2}{\partial x^2} \left(\frac{\partial}{\partial x} \left(\dots \left(\frac{\partial f_{\sigma(1)}}{\partial x} \cdot f_{\sigma(2)} \right) \dots \cdot f_{\sigma(p-2)} \right) \cdot f_{\sigma(p-1)} \right) \\
- \frac{\partial^p f_{\sigma(1)}}{\partial x^p} \cdot f_{\sigma(2)} \cdot \ldots \cdot f_{\sigma(p-1)} \Bigg]
\end{aligned} \quad (4.15)$$

(4.15) has the following properties

(a) (4.15) $= 0$, if $f_1, \dots, f_{p-1}$ are polynomials over a field k of characteristic $p > 0$.

(b) Over any field, we may rewrite (4.15) in the form

$$\sum_{0\le u_1,\ldots,u_{p-1}\le p-1} a_{u_1\ldots u_{p-1}} \frac{\partial^{u_1} f_1}{\partial x^{u_1}} \cdot \ldots \cdot \frac{\partial^{u_{p-1}} f_{p-1}}{\partial x^{u_{p-1}}} \tag{4.16}$$

such that $a_{u_1\ldots u_{p-1}} \in \mathbb{Z}$.

Using these properties we will prove that the coefficients of (4.16) are multiples of p.

Define for q, $n \in \mathbb{N}$ the symbolic nth power $q^{(n)}$ of q as follows:

$$q^{(n)} = \begin{cases} 1 & \text{if } n = 0 \\ q(q-1)\ldots(q-n+1) & \text{if } n \ge 1 \end{cases}$$

Now let $(q_i)_{i=1,\ldots,p-1} \in \mathbb{N}$ be arbitrary and put $f_i = x^{q_i}$. Then it is easy to see that (4.16) equals

$$\sum_{u_1,\ldots,u_{p-1}} a_{u_1\ldots u_{p-1}}\, q_1^{(u_1)} \ldots q_{p-1}^{(u_{p-1})}\, x^{q_1-u_1} \ldots x^{q_{p-1}-u_{p-1}}$$

Since (4.16) is zero in k by property (a) we deduce:

$$\sum_{u_1,\ldots,u_{p-1}} a_{u_1\ldots u_{p-1}}\, q_1^{(u_1)} \ldots q_{p-1}^{(u_{p-1})} = 0 \text{ (in } k) \tag{4.17}$$

Let X be the k-vectorspace of all functions $h : k^{p-1} \to k$. By [13]

$$\left\{ x_1^{u_1} \ldots x_{p-1}^{u_{p-1}} \mid \text{for all } 1 \le i \le p-1,\, u_i \le p-1 \right\}$$

is a basis for X.
We may transform these 'normal' monomials into 'symbolic' monomials by a triangular matrix whose determinant is equal to 1. It follows that

$$\left\{ x_1^{(u_1)} \ldots x_{p-1}^{(u_{p-1})} \mid \text{for all } 1 \le i \le p-1,\, u_i \le p-1 \right\}$$

is also a basis for X.

Since (4.17) holds for all $q_1, \ldots, q_{p-1} \in \mathbb{N}$, this implies

$$\sum_{u_1,\ldots,u_{p-1}} a_{u_1\ldots u_{p-1}}\, x_1^{(u_1)} \ldots x_{p-1}^{(u_{p-1})} = 0 \text{ (in } k)$$

We conclude that the coefficients $a_{u_1\dots u_{p-1}}$ are zero in k and hence they are divisible by p, as elements of $\mathbb{Z}$.

Let us look now at the difference of (4.15) and (4.16), i.e.

$$\sum_{\sigma\in S_{p-1}}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}\left(\dots\left(\frac{\partial f_{\sigma(1)}}{\partial x}\cdot f_{\sigma(2)}\right)\dots\cdot f_{\sigma(p-2)}\right)\cdot f_{\sigma(p-1)}\right)\right. \tag{4.18}$$
$$\left.-\frac{\partial^p f_{\sigma(1)}}{\partial x^p}\cdot f_{\sigma(2)}\cdot\ldots\cdot f_{\sigma(p-1)}\right]-\sum_{u_1,\dots,u_{p-1}} a_{u_1\dots u_{p-1}}\frac{\partial^{u_1} f_1}{\partial x^{u_1}}\cdot\ldots\cdot\frac{\partial^{u_{p-1}} f_{p-1}}{\partial x^{u_{p-1}}}$$

By definition (4.18) is equal to zero over any field with a derivation. We will consider (4.18) over the complex numbers $\mathbb{C}$. Let $(v_i)_{i=1,\dots,p-1}\in\mathbb{C}$ and put $f_i=e^{v_ix}$. We deduce that

$$\sum_{\sigma\in S_{p-1}}\Big[v_{\sigma(1)}(v_{\sigma(1)}+v_{\sigma(2)})\dots(v_{\sigma(1)}+\dots+v_{\sigma(p-1)})^2\,e^{(v_{\sigma(1)}+\dots+v_{\sigma(p-1)})\,x}$$
$$-v_{\sigma(1)}^p e^{(v_{\sigma(1)}+\dots+v_{\sigma(p-1)})\,x}\Big]-\sum_{u_1,\dots,u_{p-1}} a_{u_1\dots u_{p-1}}\,v_1^{u_1}\dots v_{p-1}^{u_{p-1}}\,e^{(v_1+\dots+v_{p-1})\,x}=0$$

If we divide this by $e^{(v_1+\dots+v_{p-1})\,x}$, we have, for all $v_1,\dots,v_{p-1}\in\mathbb{C}$

$$\sum_{\sigma\in S_{p-1}}\Big[v_{\sigma(1)}\;(v_{\sigma(1)}+v_{\sigma(2)})\dots(v_{\sigma(1)}+\dots+v_{\sigma(p-1)})^2-v_{\sigma(1)}^p\Big]$$
$$-\sum_{u_1,\dots,u_{p-1}} a_{u_1\dots u_{p-1}}\,v_1^{u_1}\dots v_{p-1}^{u_{p-1}}=0$$

So the polynomial

$$\sum_{\sigma\in S_{p-1}}\Big[x_{\sigma(1)}\;(x_{\sigma(1)}+x_{\sigma(2)})\dots(x_{\sigma(1)}+\dots+x_{\sigma(p-1)})^2-x_{\sigma(1)}^p\Big]$$
$$-\sum_{u_1,\dots,u_{p-1}} a_{u_1\dots u_{p-1}}\,x_1^{u_1}\dots x_{p-1}^{u_{p-1}}=0$$

is identically zero.

If we reduce this modulo p, this yields

$$\left[\sum_{\sigma\in S_{p-1}} x_{\sigma(1)}(x_{\sigma(1)}+x_{\sigma(2)})\dots(x_{\sigma(1)}+\dots+x_{\sigma(p-1)})\right](x_1+\dots+x_{p-1})$$
$$\equiv x_1^p+\dots+x_{p-1}^p\equiv(x_1+\dots+x_{p-1})^p\pmod p$$

Hence Proposition 4.0.3 is proved.

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