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Calabi-Yau pointed Hopf algebras of finite Cartan type

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Samenvatting

Het hoofddoel van deze thesis is de studie van de Calabi-Yau (CY) eigenschap van pointed Hopf algebra's $U(\mathcal{D}, \lambda)$ van eindig Cartan type en hun overeenkomstige Nichols algebra's $\mathcal{B}(V)$.

We gebruiken de homological integral om de rigid dualizing complex van een pointed Hopf algebra $U(\mathcal{D}, \lambda)$ te berekenen. We geven een nodige en voldoende voorwaarde voor $U(\mathcal{D}, \lambda)$ om een CY algebra te zijn. CY pointed Hopf algebra's $U(\mathcal{D}, \lambda)$ met dimensie lager dan 5 worden geclassificeerd.

Een Nichols algebra $\mathcal{B}(V)$ van eindig Cartan type is een $\mathbb{N}^p$-gegradeerde algebra ($p \in \mathbb{N}$), zodat de geassocieerde gegradeerde algebra $\mathbb{G}\mathrm{r}\mathcal{B}(V)$ een quantum veelterm algebra is. We verkrijgen de rigid dualizing complex van $\mathcal{B}(V)$ door middel van een analyse van de relaties met de rigid dualizing complexen van $\mathbb{G}\mathrm{r}\mathcal{B}(V)$ en $U(\mathcal{D}, \lambda)$. We geven een nodige en voldoende voorwaarde voor $\mathcal{B}(V)$ om een CY algebra te zijn.

Het blijkt dat de algebra's $U(\mathcal{D}, \lambda)$ en $\mathcal{B}(V)$ niet gelijktijdig CY algebra's kunnen zijn. Zij H een eindig dimensionale Hopf algebra en zij R een braided Hopf algebra in de category ${}^H_H\mathcal{YD}$ van Yetter-Drinfeld modulen. Het verband tussen de CY eigenschap van R en van $R\#H$ wordt besproken. In het geval dat R CY is en H semi-simpel is, berekenen we de homological integral van $R\#H$ en geven we een nodige en voldoende voorwaarde opdat $R\#H$ een CY algebra is. Indien H de groep algebra $\mathbb{k}\Gamma$ van een eindige groep Γ is en $R\#\mathbb{k}\Gamma$ een CY algebra is, geven we een nodige en voldoende voorwaarde voor R om een CY algebra te zijn, door middel van de rigid dualizing complex van R te berekenen.

Ten slotte bestuderen we de eigenschappen van een eindig dimensionale pointed

Hopf algebra $u(\mathcal{D}, \lambda, \mu)$. Door gebruik te maken van de Hochschild-Serre spectraal reeks, beschrijven we de volledige structuur van de Ext algebra van een Nichols algebra van type A_2 in termen van voortbrengers en relaties. Als toepassing bewijzen we dat pointed Hopf algebra's $u(\mathcal{D}, \lambda, \mu)$ met Dynkin diagrammen van type A, D of E, met uitzondering van type A_1 en $A_1 \times A_1$ met orde $N_J > 2$ voor minstens één component J, wild zijn. Ten tweede bestuderen we de CY eigenschap van een eindig dimensionale Hopf algebra $u(\mathcal{D}, \lambda, \mu)$. Dit is onmogelijk een CY algebra. Door een analyse van de structuur van de Ext algebra, bekomen we dat de bijhorende stable categorie geen CY categorie is.

Introduction

The notion of a Calabi-Yau (CY) category has its origin in algebraic geometry. The bounded derived category of coherent sheaves on a CY manifold has a Serre functor which is isomorphic to a power of the shift functor. A triangulated category satisfying this condition was defined to be a CY category by Kontsevich [47]. He used CY categories to study the homological mirror symmetry.

In this thesis, we follow Ginzburg's definition of a CY algebra [32] (Definition 1.5.6). This definition is a non-categorical definition, and was inspired by Van den Bergh's duality theorem [67]. If A is a CY algebra of dimension d, then the category $D^b_{fd}(A)$ is a CY category by [44, Lemma 4.1], where $D^b_{fd}(A)$ is the full triangulated subcategory of the derived category of A consisting of complexes whose homology is of finite total dimension.

In recent years, CY algebras (categories) have attracted lots of attention. In the representation theory of finite dimensional algebras, cluster categories are 2-CY categories. Thus CY categories (algebras) have found their applications in Fomin-Zelevinsky's cluster-tilting theory [1], [36], [42], [43] etc.. Besides, CY categories have been also applied to mathematical physics, notably to String Theory and to the coformal field theory [24], [48] etc..

In [35], He, Van Oystaeyen and Zhang discussed the CY property of cocommutative Hopf algebras by using the homological integral. A necessary and sufficient condition for a Noetherian Hopf algebra to be a CY algebra was given there. In addition, Noetherian cocommutative CY Hopf algebras of dimension not greater than 3 were classified. The notion of a homological integral was introduced by Lu, Wu and Zhang in order to study infinite dimensional AS-Gorenstein (Definition 1.5.17) Hopf algebras [53]. It generalizes the notion of an integral of a finite dimensional Hopf algebra.

CY algebras are closely related to algebras having a rigid dualizing complex (Definition 1.5.10). An algebra A is a CY algebra of dimension d if and only if A is homologically smooth and has a rigid dualizing complex $A[d]$ (cf. Corollary 1.5.12). In fact, dualizing complexes are of great interest of their own. The non-commutative version of a dualizing complex (Definition 1.5.9) was introduced by Yekutieli in 1990 [74]. Since then it became a useful tool to study homological properties of non-commutative algebras [37], [38], [75], [77] etc.. Roughly speaking, a dualizing complex is a complex which induces a duality between certain subcategories of derived categories of modules (cf. (1.18)). However, dualizing complexes are not unique. To overcome this weakness, Van den Bergh introduced the notion of a rigid dualzing complex in [68], which is unique up to isomorphism. Brown and Zhang gave the rigid dualizing complex of an AS-Gorenstein Hopf algebra [20]. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. The rigid dualizing complex of the quantized enveloping algebra $U_q(\mathfrak{g})$ was given by Chemla [23]. It turns out that the algebra $U_q(\mathfrak{g})$ is a CY algebra. As far as we know, there are no literature concerning the CY property or dualizing complexes of braided Hopf algebras.

A Hopf algebra A is called pointed, if all its simple left or right comodules are 1-dimensional. This is equivalent to saying that the coradical of A is a group algebra. For example, group algebras, universal enveloping algebras of Lie algebras, and quantized enveloping algebras of finite dimensional semisimple Lie algebras are all pointed Hopf algebras. For a pointed Hopf algebra A, its coradical filtration is a Hopf algebra filtration. Let $\operatorname{Gr} A$ be its associated graded Hopf algebra. Its degree zero part is $\Bbbk\Gamma$, where Γ is the group of all group-like elements of A. Then there is a Hopf algebra projection from $\operatorname{Gr} A$ onto $\Bbbk\Gamma$. By a theorem of Radford [62], $\operatorname{Gr} A$ is the biproduct or the bosonization

$$\operatorname{Gr} A \cong R\#\Bbbk\Gamma,$$

where R is a braided Hopf algebra in the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$.

The vector space V consisting of primitive elements of R is a Yetter-Drinfeld module over $\Bbbk\Gamma$. The algebra $\mathcal{B}(V)$ generated by V is a braided Hopf subalgebra of R. Its algebra structure and coalgebra structure depend only on the braiding of V. Now the algebra $\mathcal{B}(V)$ is called the Nichols algebra of V. The structure of a Nichols algebra first appeared in [60] and was rediscovered independently by several authors later [71], [54].

Andruskiewitsch and Schneider made a lot of contribution to the classification of pointed Hopf algebras. Their method consists of three parts. First, determine all possible Nichols algebras $\mathcal{B}(V)$, and then determine the structure of all pointed Hopf algebras A with Γ being the group of group-like elements of A such that $\mathrm{Gr}\, A \cong \mathcal{B}(V)\#\Bbbk\Gamma$. Finally, decide which Hopf algebras A are generated by group-like elements and skew-primitive elements.

The Hopf algebras $U(\mathcal{D}, \lambda)$ constructed in [7] constitute a large class of pointed Hopf algebras with finite Gelfand-Kirillov dimension, whose group-like elements form an abelian group. Such a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ is viewed as a generalization of the quantized enveloping algebra $U_q(\mathfrak{g})$, $\mathfrak{g}$ a finite dimensional semisimple Lie algebra. The main purpose of this thesis is to study the CY property of the algebras $U(\mathcal{D}, \lambda)$ and their corresponding Nichols algebras.

Now we formulate our main results. We work over a fixed algebraically closed field $\Bbbk$ with characteristic 0. Let Γ be a free abelian group of finite rank,

$$\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant \theta}, (\chi_i)_{1\leqslant i\leqslant \theta}, (a_{ij})_{1\leqslant i,j\leqslant \theta})$$

a generic datum of finite Cartan type for Γ (Definition 1.4.1), and λ a family of linking parameters for $\mathcal{D}$ (Definition 1.4.3). For simplicity, we define $q_{ij} = \chi_j(g_i)$, $1 \leqslant i, j \leqslant \theta$.

Let Φ be the root system of the Cartan matrix (a_{ij}), $\{\alpha_1, \cdots, \alpha_\theta\}$ a set of simple roots, and $\mathcal{X}$ the set of connected components of the Dynkin diagram. Let Φ_J, $J \in \mathcal{X}$, be the root system of the component J. For $1 \leqslant i, j \leqslant \theta$, we write $i \sim j$ if they belong to the same connected component. Assume that $w_0 = s_{i_1} \cdots s_{i_p}$ is a reduced decomposition of the longest element in the Weyl group $\mathcal{W}$ as a product of simple reflections. Then

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$$

are the positive roots. If $\beta_i = \sum_{i=1}^{\theta} m_i\alpha_i$, then we define

$$g_{\beta_i} = g_1^{m_1} \cdots g_\theta^{m_\theta} \text{ and } \chi_{\beta_i} = \chi_1^{m_1} \cdots \chi_\theta^{m_\theta}.$$

The CY property of $U(\mathcal{D}, \lambda)$ is discussed in Chapter 2. First, we obtain the following theorem.

Theorem 1. (Theorem 2.1.5) *Let $\mathcal{D}$ be a generic datum of finite Cartan type for a free abelian group Γ of rank s, and λ a family of linking parameters for $\mathcal{D}$.*

(1) The rigid dualizing complex of the Hopf algebra $A = U(\mathcal{D}, \lambda)$ is ${}_{\psi}A[p+s]$, where p is the number of the positive roots and s is the rank of Γ. The algebra automorphism ψ is defined by $\psi(x_k) = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$, and $\psi(g) = (\prod_{i=1}^{p} \chi_{\beta_i})(g)$ for any $g \in \Gamma$, where each j_k is the integer such that $\beta_{j_k} = \alpha_k$.

(2) The algebra A is CY if and only if $\prod_{i=1}^{p} \chi_{\beta_i} = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism.

The proof is based on the homological integral of $U(\mathcal{D}, \lambda)$. Note that the algebra $U(\mathcal{D}, \lambda)$ is a CY algebra if and only if its associated graded algebra $U(\mathcal{D}, 0)$ (with respect to the coradical filtration) is a CY algebra.

In Section 2.2, we classify CY pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimensions less than 5. It turns out that $U_q(\mathfrak{sl}_2)$ is the only quantized enveloping algebra appearing in the classification.

Let V be the braided vector space with basis $\{x_1, \cdots, x_\theta\}$ whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$

for any $1 \leqslant i, j \leqslant \theta$. Then the associated graded algebra of $U(\mathcal{D}, \lambda)$ with respect to the coradical filtration is $U(\mathcal{D}, 0) \cong \mathcal{B}(V)\#\Bbbk\Gamma$ (cf. Theorem 1.4.7).

In Chapter 3 we discuss the Nichols algebra $\mathcal{B}(V)$. It is a braided Hopf algebra. The general method of computing the rigid dualizing complex of an AS-Gorenstein Hopf algebra can not be easily modified to suit the case of a braided Hopf algebra. For the special case of $\mathcal{B}(V)$, we prove that the algebra $\mathcal{B}(V)$ is a connected AS-regular algebra (Proposition 3.1.5). In addition, it is an $\mathbb{N}^{p+1}$-filtered algebra whose associated graded algebra $\mathbb{G}\mathrm{r}R$ is a quantum polynomial algebra (Corollary 3.1.2). Since a quantum polynomial algebra is a Koszul AS-regular algebra, it is not difficult to obtain its rigid dualizing complex. When A is an $\mathbb{N}$-filtered algebra whose associated graded algebra $\operatorname{Gr} A$ is a connected AS-Gorenstein algebra (Definition 1.5.16), the relation between the rigid dualizing complex of A and the one of $\operatorname{Gr} A$ was discussed in [68, Prop. 8.2], [76, Prop. 1.1] and [77, Prop. 6.18]. Since these results only work for an $\mathbb{N}$-filtered algebra, we construct a sequence of algebras

$$R = R^{(0)},\ \ R^{(1)}, \cdots, \mathbb{G}\mathrm{r} R = R^{(p)},$$

such that each of which is the associated graded algebra of the previous one with respect to an $\mathbb{N}$-filtration. Furthermore, with the relation between the rigid dualizing complex of R and the one of $U(\mathcal{D}, 0) = R\#\Bbbk\Gamma$, we obtain the rigid dualizing complex of R.

Theorem 2. (Theorem 3.1.9 and Lemma 3.2.1) *Let V be a generic braided vector space of finite Cartan type, and $R = \mathcal{B}(V)$ the Nichols algebra of V. For each $1 \leqslant k \leqslant \theta$, let j_k be the integer such that $\beta_{j_k} = \alpha_k$.*

(1) The rigid dualizing complex is isomorphic to ${}_\varphi R[p]$, where φ is the algebra automorphism defined by

$$\varphi(x_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k)x_k,$$

for any $1 \leqslant k \leqslant \theta$.

(2) The algebra R is a CY algebra if and only if

$$\prod_{i=1}^{j_k-1} \chi_k(g_{\beta_i}) = \prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k),$$

for any $1 \leqslant k \leqslant \theta$.

After obtaining the aforementioned theorems, we compare the CY property of a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ and the corresponding Nichols algebra $\mathcal{B}(V)$ in Section 3.2. It turns out that if one of them is CY, then the other one is not. This leads to the following question:

Let H be a Hopf algebra, and R a braided Hopf algebra in the category of Yetter-Drinfeld modules over H. What is the relation between the CY property of R and that of $R\#H$?

Let R be a Koszul CY algebra (not necessarily a braided Hopf algebra) and H the group algebra $\Bbbk\Gamma$, where Γ is a finite group of automorphisms of R. In [72], Wu and Zhu showed that the smash product $R\#H$ is CY if and only if the homological determinant (Definition 4.1.6) of the H-action is trivial. Later, this result was generalized to the case where R is a p-Koszul CY algebra and H is an involutory CY Hopf algebra [52]. The authors defined an H-module structure on the Koszul bimodule complex of R, and they computed the H-module structures on the Hochschild cohomologies. The homological determinant for graded automorphisms of an AS-Gorenstein algebra was first defined by Jørgensen and Zhang in order to study the AS-Gorenstein property of invariant subrings [39]. A Hopf algebra version was introduced later in [46].

Let H be a finite dimensional Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules. Inspired by Wu and Zhu's work, in Section 4.1, we use the homological determinant of the H-action to describe the homological integral of $R\#H$. We then give a necessary and sufficient condition for $R\#H$ to be a CY algebra, when R is CY and H is semisimple.

Theorem 3. (Theorem 4.1.11) *Let H be a semisimple Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules. Suppose that the algebra R is CY of dimension d_R. Then $R\#H$ is CY if and only if the homological determinant of R is trivial and the algebra automorphism ϕ defined by*

$$\phi(r\#h) = \mathcal{S}_H(r_{(-1)})(\mathcal{S}_R^2(r_{(0)}))\mathcal{S}_H^2(h)$$

for any $r\#h \in R\#H$ is an inner automorphism.

We can also ask ourselves when R is CY, if $R\#H$ is CY? In Section 4.2, we answer this question when $H = \Bbbk\Gamma$ is the group algebra of a finite group. We first construct a bimodule resolution of R from a projective resolution of $\Bbbk$ over the algebra $R\#\Bbbk\Gamma$. Based on this, we obtain the rigid dualizing complex of R when R is AS-Gorenstein.

We explain some notations first. We use $\Delta(r) = r^1 \otimes r^2$ to denote the comultiplication for a braided Hopf algebra (cf. (1.1)). The algebra R is a Γ-comodule, so R is a Γ-graded module. Let δ denote the Γ-comodule structure. Then $R = \oplus_{g\in\Gamma} R_g$, where $R_g = \{r \in R \mid \delta(r) = g \otimes r\}$. If $r = \sum_{g\in\Gamma} r_g$ with $r_g \in R_g$, then $\delta(r) = \sum_{g\in\Gamma} g \otimes r_g$. See Remark 4.2.1 for detail.

Theorem 4. (Theorem 4.2.9) *Let Γ be a finite group and R a braided Hopf algebra* *in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ of Yetter-Drinfeld modules. Assume that R is an AS-Gorenstein algebra with injective dimension d. If $\int_R^l \cong \Bbbk_{\xi_R}$, for some algebra homomorphism $\xi_R : R \to \Bbbk$, then R has a rigid dualizing complex ${}_{\varphi}R[d]$, where φ is the algebra automorphism defined by $\varphi(r) = \sum_{g\in\Gamma} \xi_R(r^1)\,\mathrm{hdet}(g)g^{-1}(\mathcal{S}_R^2((r^2)_g))$ for all $r \in R$. Here* hdet *denotes the homological determinant of the group action.*

Following the foregoing theorem, we obtain the following result, characterizing the CY property of R when $R\#\Bbbk\Gamma$ is CY.

Theorem 5. (Theorem 4.2.11) *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ of Yetter-Drinfeld modules. Define an algebra automorphism φ of R by*

$$\varphi(r) = \sum_{g\in\Gamma} g^{-1}(\mathcal{S}_R^2(r_g)),$$

for any $r \in R$. If $R\#\Bbbk\Gamma$ is a CY algebra, then R is CY if and only if the algebra automorphism φ is an inner automorphism.

The groups of group-like elements of pointed Hopf algebras discussed in Chapter 2 are all infinite. There are CY pointed Hopf algebras with a finite abelian group of group-like elements. We provide some examples at the end of Chapter 4.

Now we turn to finite dimensional pointed Hopf algebras. Let

$$\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$$

be a datum of finite Cartan type such that Γ is a finite abelian group. We assume that the order $\chi_i(q_i) = q_{ii}$ is odd for any $1 \leqslant i \leqslant \theta$, and that the order of q_{ii} is prime to 3 for all i in a connected component of type G_2. Then by equation (1.7), the order N_i of q_{ii} is constant in each component $J \in \mathcal{X}$. Denote this common order by N_J.

Let λ be a family of linking parameters for $\mathcal{D}$ and μ a family of root vector parameters (Definition 1.4.9). The finite dimension Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is a deformation of the bosonization of a finite dimensional Nichols algebra by $\Bbbk\Gamma$. Andruskiewitsch and Schneider [8] proved that $u(\mathcal{D}, \lambda, \mu)$ is finite dimensional and pointed. Conversely, if H is a finite dimensional pointed Hopf algebra with an abelian group of grouplike elements with order not divisible by primes less than 11, then $H \cong u(\mathcal{D}, \lambda, \mu)$ for some $\mathcal{D}$, λ, μ.

The homological properties of an algebra R over a field $\mathbb{k}$ rely exclusively on the structure of its Ext algebra $\mathrm{Ext}_R^*(\mathbb{k}, \mathbb{k})$. In two recent papers [29, 61] support varieties of modules over finite dimensional Hopf algebras were introduced. It turns out that support varieties are useful tools to study homological properties and representations of finite dimensional (braided) Hopf algebras. To define and to compute support varieties over a (braided) Hopf algebra we need first to understand the Ext algebra of the (braided) Hopf algebra. These motivate us to study the structure of the Ext algebra of a finite dimensional Nichols algebra. In Chapter 5, we give the full structure of the Ext algebra of a Nichols algebra of type A_2 in terms of generators and relations (Proposition 5.1.1, Theorem 5.1.11 and Theorem 5.1.12).

Using these structures, we can show that for a finite dimensional pointed Hopf algebra A of type A_2, the support variety of $\mathbb{k}$ over A is isomorphic to the variety of $\mathbb{k}$ over the associated graded algebra with respect to the coradical filtration of A. We then apply the main theorems in Chapter 5 to show that if the components of the Dynkin diagram of a pointed Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ are of type A, D, or E, except for A_1 and $A_1 \times A_1$, and the order $N_J > 2$ for at least one component, then $u(\mathcal{D}, \lambda, \mu)$ is wild (Proposition 5.2.2).

This thesis mainly discuss the CY property. A finite dimensional CY algebra must be semisimple. So a finite dimensional algebra $u(\mathcal{D}, \lambda, \mu)$ is not a CY algebra. But a finite dimensional Hopf algebra is Frobenius, so its stable category is a triangulated category. A natural question now arises: is the stable category of a pointed Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ a CY category? By analyzing the structure of the Ext algebra of $u(\mathcal{D}, \lambda, \mu)$, it turns out that in most cases, the answer to this question is negative. The details can be found at the end of Section 5.2.

Notations and conventions

Throughout, $\Bbbk$ is a fixed algebraically closed field with characteristic 0. All vector spaces and algebras are assumed to be over $\Bbbk$. $(-)^*$ denotes the functor $\mathrm{Hom}_{\Bbbk}(-, \Bbbk)$. The unfinished tensor $\otimes$ means $\otimes_{\Bbbk}$.

Without otherwise stated, a Hopf algebra means a Hopf algebra with a bijective antipode. We use the (sumless) Sweedler's notation for the comultiplication and coaction. The comultiplication for a braided Hopf algebra R is denoted by

$$\Delta(r) = r^1 \otimes r^2.$$

Let A be a Hopf algebra, and $\xi : A \to \Bbbk$ an algebra homomorphism. We write $[\xi]$ to be the winding homomorphism of ξ defined by

$$[\xi](a) = \xi(a_1)a_2,$$

for any $a \in A$.

Given an algebra A, we write A^{op} for the opposite algebra of A and A^e for the enveloping algebra $A \otimes A^{op}$ of A. For any bimodule mentioned in this thesis, we assume that $\Bbbk$ acts centrally on it. Then the category of A-A-bimodules is equivalent to the category of left (or right) A^e-modules. $\mathrm{Mod}A$ denotes the category of left A-modules. We use $\mathrm{Mod}A^{op}$ to denote the category of right A-modules.

For a left A-module M and an algebra automorphism $\phi : A \to A$, ${}_{\phi}M$ stands for the left A-module twisted by the automorphism ϕ. Similarly, for a right A-module N, we have N_{ϕ}. Observe that $A_{\phi} \cong {}_{\phi^{-1}}A$ as A-A-bimodules. $A_{\phi} \cong A$ as A-A-bimodules if and only if ϕ is an inner automorphism.

A Noetherian algebra in this thesis means a left and right Noetherian algebra. If the injective dimension of ${}_AA$ and A_A are both finite, then these two integers are equal by [78, Lemma A]. We call this common value the injective dimension of A. The left global dimension and the right global dimension of a Noetherian algebra are equal [70, Exe. 4.1.1]. When the global dimension is finite, then it is equal to the injective dimension.

Chapter 1

Preliminaries

In this chapter, we first recall the definitions of a braided Hopf algebra and a Nichols algebra which are important in the classification theory of pointed Hopf algebras. In this thesis, we will concentrate ourselves on pointed Hopf algebras of finite Cartan type. We explain the definition in Section 1.4. At last, we recall the definition and basic properties of Calabi-Yau algebras.

1.1 Graded and filtered algebras

In this section, we fix some notations of graded algebras and filtered algebras. Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a graded algebra. We denote by $\mathrm{GrMod}(A)$ the category of graded left A-modules with graded homomorphisms of degree 0. Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded module, we denote by $M(l)$ the l-th degree shift of M. That is, $M(l) = \oplus_{i\in\mathbb{Z}} M(l)_i$ and $M(l)_i = M_{i+l}$ for each $i \in \mathbb{Z}$. A module F in $\mathrm{GrMod}(A)$ is called *graded free*, if there is an index set Λ such that $F = \oplus_{i\in\Lambda} F_i$ and each F_i is a shift of A. For graded modules M and N, we write $\mathrm{Hom}_A(M, N)$ for the space of all A-module homomorphisms. Set

$$\mathrm{HOM}_A(M, N)_k := \{f \in \mathrm{Hom}_A(M, N) | f(M_i) \subseteq N_{i+k}\},$$

and

$$\mathrm{HOM}_A(M, N) := \oplus_{k\in\mathbb{Z}} \mathrm{HOM}_A(M, N)_k.$$

Let $\mathrm{EXT}_A^i(-,-)$ be the derived functor of $\mathrm{HOM}_A(-,-)$. If M is finitely generated, then $\mathrm{Hom}_A(M,N) = \mathrm{HOM}_A(M,N)$. If A is in addition Noetherian, then $\mathrm{Ext}_A^i(M,N) = \mathrm{EXT}_A^i(M,N)$ for $i \geqslant 0$.

A $\mathbb{Z}$-filtration on an algebra A is given by an ascending chain of vector subspaces of A, $FA = \{F_nA \mid n \in \mathbb{Z}\}$ such that $1 \in F_0A$ and $F_nAF_mA \subseteq F_{n+m}A$, for all $n, m \in \mathbb{Z}$. If there is a filtration $FM = \{F_nM \mid n \in \mathbb{Z}\}$ on an A-module M, such that $F_nAF_mM \subseteq F_{n+m}M$, for all $n, m \in \mathbb{Z}$, then M is called a *filtered* module. A filtration FM is *exhaustive* if $M = \cup_{i\in\mathbb{Z}} F_nM$. If $\cap_{i\in\mathbb{Z}} F_nM = 0$, then FM is called *separated*. All filtration considered in this thesis are exhaustive and separated. We write $\mathrm{FiltMod}(A)$ for the category of filtered A-modules and filtered homomorphisms of degree 0. Shift of filtered modules and filtered free modules can be defined similarly to the case of graded modules. Let M and N be two filtered A-modules with filtration $FM = \{F_iM \mid i \in \mathbb{Z}\}$ and $FN = \{F_iN \mid i \in \mathbb{Z}\}$ respectively. We denote by $\mathrm{Hom}_A(M,N)$ for the vector space of A-module homomorphisms from M to N. We define

$$\mathrm{HOM}_A(M,N)_k := \{f \in \mathrm{Hom}_A(M,N) \mid f(F_iM) \subseteq F_{i+k}N\}$$

and

$$\mathrm{HOM}_A(M,N) := \cup_{k\in\mathbb{Z}} \mathrm{HOM}_A(M,N)_k.$$

Now the vector space $\mathrm{HOM}_A(M,N)$ is filtered by $F_k \mathrm{HOM}_A(M,N) = \mathrm{HOM}_A(M,N)_k$. We denote by $\mathrm{EXT}_A^i(-,-)$ the derived functor of $\mathrm{HOM}_A(-,-)$. Similarly, if M is finitely generated, then $\mathrm{Hom}_A(M,N) = \mathrm{HOM}_A(M,N)$. If A is in addition Noetherian, then $\mathrm{Ext}_A^i(M,N) = \mathrm{EXT}_A^i(M,N)$ for $i \geqslant 0$.

Let M be a filtered module with filtration FM. If there exist $m_1, \cdots, m_s \in M$ and $k_1, \cdots, k_s \in \mathbb{Z}$, such that for all $i \in \mathbb{Z}$

$$F_iM = \sum_{j=1}^{s} (F_{i-k_j}A)m_j,$$

then FM is called a *good filtration* on M.

If M is a filtered module with a good filtration FM, then M is finitely generated. However, the converse is not necessarily true (see [51, Rem. I.5.2] for an example).

1.2 Braided Hopf algebras

1.2.1 Braided tensor categories

In this subsection, we briefly recall the definition of a braided tensor category which is the appropriate setting for a braided Hopf algebra. For more detail about braided tensor categories, one refers to [41].

Definition 1.2.1. A *tensor category* $(\mathscr{C}, \otimes, I, a, l, r)$ is a category $\mathscr{C}$ equipped with

- a tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$;
- an object I, called the *unit* of the tensor category;
- a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, called the *associativity constraint*;
- a natural isomorphism $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$, called the *left unit constraint* with respect to I;
- a natural isomorphism $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$, called the *right unit constraint* with respect to I;

such that the Pentagon Axiom and the Triangle Axiom are satisfied. That is, the following two diagrams

$$\begin{tikzcd}
(U \otimes (V \otimes W)) \otimes X \arrow[dd, "a_{U,V \otimes W,X}"'] & ((U \otimes V) \otimes W) \otimes X \arrow[l, "a_{U,V,W} \otimes \mathrm{id}_X"'] \arrow[d, "a_{U \otimes V,W,X}"] \\
 & (U \otimes V) \otimes (W \otimes X) \arrow[d, "a_{U,V,W \otimes X}"] \\
U \otimes ((V \otimes W) \otimes X) \arrow[r, "\mathrm{id} \otimes a_{V,W,X}"] & U \otimes (V \otimes (W \otimes X))
\end{tikzcd}$$

$$\begin{tikzcd}
(V \otimes I) \otimes W \arrow[rr, "a_{V,I,W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (I \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
 & V \otimes W &
\end{tikzcd}$$

are commutative for all objects U, V, W and X in $\mathscr{C}$.

The most fundamental example of a tensor category is given by the category of vector spaces over a field $\Bbbk$. It is equipped with the usual tensor product, the unit object I is the field $\Bbbk$ itself.

Let $\mathscr{C}$ be a tensor category with a tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. Denote by $\tau : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ the flip functor. That is, $\tau(V, W) = (W, V)$ for any V, W in $\mathscr{C}$. A *commutativity constraint* c is a natural isomorphism $c : \otimes \to \otimes\tau$.

Definition 1.2.2. Let $(\mathscr{C}, \otimes, I, a, l, r)$ be a tensor category.

(1) A *braiding* is a commutativity constraint c satisfying the Hexagon Axiom. That is, the following two diagrams

$$\begin{tikzcd}
 & U\otimes(V\otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V\otimes W)\otimes U \arrow[dr, "a_{V,W,U}"] & \\
(U\otimes V)\otimes W \arrow[ur, "a_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes \mathrm{id}_W"'] & & & V\otimes(W\otimes U) \\
 & (V\otimes U)\otimes W \arrow[r, "a_{V,U,W}"] & V\otimes(U\otimes W) \arrow[ur, "\mathrm{id}_V\otimes c_{U,W}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (U\otimes V)\otimes W \arrow[r, "c_{U\otimes V,W}"] & W\otimes(U\otimes V) \arrow[dr, "a^{-1}_{W,U,V}"] & \\
U\otimes(V\otimes W) \arrow[ur, "a^{-1}_{U,V,W}"] \arrow[dr, "\mathrm{id}_U\otimes c_{V,W}"'] & & & (W\otimes U)\otimes V \\
 & U\otimes(W\otimes V) \arrow[r, "a^{-1}_{U,W,V}"] & (U\otimes W)\otimes V \arrow[ur, "c_{U,W}\otimes \mathrm{id}_V"'] &
\end{tikzcd}$$

commute for any objects U, V and W in $\mathscr{C}$.

(2) A *braided tensor category* $(\mathscr{C}, \otimes, I, a, l, r, c)$ is a tensor category with a braiding.

As a consequence of the Hexagon Axiom, the following equation holds for any objects U, V and W in $\mathscr{C}$ (we have omitted the associativity morphisms)

$$(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}).$$

1.2.2 Braided vector spaces and Yetter-Drinfeld modules

Definition 1.2.3. Let V be a vector space and $c : V \otimes V \to V \otimes V$ a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the following braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

An easy and important example is a braided vector space of diagonal type. A braided vector space (V, c) is said to be of *diagonal type* if there is a basis $\{x_i \mid i \in I\}$ of V and a family of non-zero scalars $q_{ij} \in \Bbbk$, $i, j \in I$, such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i,$$

for all $i, j \in I$.

We mainly discuss examples of braided vector spaces related to the notion of a Yetter-Drinfeld module.

Definition 1.2.4. Let H be a Hopf algebra. A (left-left) *Yetter-Drinfeld module* V over H is simultaneously a left H-module and a left H-comodule satisfying the compatibility condition

$$\delta(h \cdot v) = h_1 v_{(-1)} \mathcal{S} h_3 \otimes h_2 \cdot v_{(0)},$$

for any $v \in V$, $h \in H$.

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter-Drinfeld modules over H with morphisms given by H-linear and H-colinear maps.

The tensor product of two Yetter-Drinfeld modules M and N is again a Yetter-Drinfeld module with the module and comodule structures given as follows

$$h(m \otimes n) = h_1 m \otimes h_2 n \text{ and } \delta(m \otimes n) = m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)},$$

for any $h \in H$, $m \in M$ and $n \in N$. This turns the category of Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$ into a braided tensor category.

For any two Yetter-Drinfeld modules M and N, the braiding $c_{M,N} : M \otimes N \to N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)},$$

for any $m \in M$ and $n \in N$.

Yetter-Drinfeld modules over a group algebra are important in this thesis. Let Γ

be a group. We abbreviate ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ to ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

A $\Bbbk\Gamma$-comodule V is just a Γ-graded vector space: $V = \oplus_{g\in\Gamma} V_g$, where $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$. From the definition of a Yetter-Drinfeld module, we obtain that $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ if and only if $gV_h \subseteq V_{ghg^{-1}}$, for all $g, h \in \Gamma$. In particular, if Γ is abelian, then a Yetter-Drinfeld module over $\Bbbk\Gamma$ is nothing but a Γ-graded Γ-module.

1.2.3 Braided Hopf algebras

We deal in this thesis only with braided Hopf algebras in categories of Yetter-Drinfeld modules.

Definition 1.2.5. Let H be a Hopf algebra.

(1) An *algebra* (R, m, u) in ${}^{H}_{H}\mathcal{YD}$ is an algebra (R, m, u), where $m : R \otimes R \to R$ is the multiplication, and $u : \Bbbk \to R$ is the unit, such that $R \in {}^{H}_{H}\mathcal{YD}$ and both m and u are morphisms in ${}^{H}_{H}\mathcal{YD}$.

(2) A *coalgebra* (R, Δ, ε) in ${}^{H}_{H}\mathcal{YD}$ is a coalgebra (R, Δ, ε), where $\Delta : R \to R \otimes R$ is the comultiplication, and $\varepsilon : R \to \Bbbk$ is the counit, such that $R \in {}^{H}_{H}\mathcal{YD}$ and both Δ and ε are morphisms in ${}^{H}_{H}\mathcal{YD}$.

Let R and S be two algebras in ${}^{H}_{H}\mathcal{YD}$. Then $R \otimes S$ is a Yetter-Drinfeld module in ${}^{H}_{H}\mathcal{YD}$, and becomes an algebra in the category ${}^{H}_{H}\mathcal{YD}$ with the multiplication $m_{R\underline{\otimes}S}$ defined by

$$m_{R\underline{\otimes}S} := (m_R \otimes m_S)(\mathrm{id} \otimes c \otimes \mathrm{id}).$$

Denote this algebra by $R\underline{\otimes}S$.

Definition 1.2.6. Let H be a Hopf algebra. A *braided bialgebra* in ${}^{H}_{H}\mathcal{YD}$ is a collection $(R, m, u, \Delta, \varepsilon)$, where

- (R, m, u) is an algebra in ${}^{H}_{H}\mathcal{YD}$.
- (R, Δ, ε) is a coalgebra in ${}^{H}_{H}\mathcal{YD}$.
- $\Delta : R \to R\underline{\otimes}R$ and $\varepsilon : R \to \Bbbk$ are morphisms of algebras.

If in addition, the identity is convolution invertible in $\mathrm{End}(R)$, then R is called a *braided Hopf algebra* in ${}^{H}_{H}\mathcal{YD}$. The inverse of the identity is called the *antipode* of R.

A *graded* braided Hopf algebra in ${}^H_H\mathcal{YD}$ is a braided Hopf algebra $R = \oplus_{i \geqslant 0} R_i$ in ${}^H_H\mathcal{YD}$, such that R is a (positively) graded algebra and a graded coalgebra, with each R_i a Yetter-Drinfeld module.

In order to distinguish comultiplications of braided Hopf algebras from those of usual Hopf algebras, we use Sweedler notation with upper indices for braided Hopf algebras

$$\Delta(r) = r^1 \otimes r^2. \tag{1.1}$$

If A is a Hopf algebra, the adjoint representation "ad" is defined by

$$\mathrm{ad}(x)(y) = x_1 y \mathcal{S}_A(x_2),$$

for all $x, y \in A$. Similarly, the *braided adjoint representation* "ad_c" of a braided Hopf algebra R in ${}^H_H\mathcal{YD}$ is given by

$$\mathrm{ad}_c(x)(y) = m(m \otimes \mathcal{S}_R)(\mathrm{id} \otimes c)(\Delta_R \otimes \mathrm{id})(x \otimes y), \tag{1.2}$$

for all $x, y \in R$.

If x is a primitive element, then the braided adjoint representation of x is just

$$\mathrm{ad}_c(x)(y) = m(\mathrm{id} - c)(x \otimes y) := [x, y]_c.$$

$[x, y]_c$ is called a *braided commutator.*

Let H be a Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then $R\#H$ is a usual Hopf algebra with the following structure [62]:

- The multiplication is given by

$$(r\#g)(s\#h) := r g_1(s)\#g_2 h \tag{1.3}$$

  with unit $u_R \otimes u_H$.

- The comultiplication is given by

$$\Delta(r\#h) := r^1\#(r^2)_{(-1)} h_1 \otimes (r^2)_{(0)}\#h_2 \tag{1.4}$$

  with counit $\varepsilon_R \otimes \varepsilon_H$.

- The antipode is given by

$$\mathcal{S}_{R\#H}(r\#h) = (1\#\mathcal{S}_H(r_{(-1)}h))(\mathcal{S}_R(r_{(0)})\#1). \tag{1.5}$$

The algebra $R\#H$ is called the *Radford biproduct* or *bosonization* of R by H. R is a subalgebra of $R\#H$ and H is Hopf subalgebra of $R\#H$.

Conversely, let A and H be two Hopf algebras and $\pi : A \to H$, $\iota : H \to A$ Hopf algebra homomorphisms such that $\pi\iota = \mathrm{id}_H$. In this case the algebra of right coinvariants with respect to π

$$R = A^{co\pi} := \{a \in A \mid (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\},$$

is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, with the following structure [62]:

- The action of H on R is the restriction of the adjoint action (composed with ι).
- The coaction is $(\pi \otimes \mathrm{id})\Delta$.
- R is a subalgebra of A.
- The comultiplication is given by

$$\Delta_R(r) = r_1\iota\mathcal{S}_H\pi(r_2) \otimes r_3.$$

- The antipode is given by

$$\mathcal{S}_R(r) = \pi(r_1)\mathcal{S}_A(r_2).$$

Define a linear map $\rho : A \to R$ by

$$\rho(a) = a_1\iota\mathcal{S}_H\pi(a_2),$$

for all $a \in R$.

Theorem 1.2.7. *[62] The morphisms $\Psi : A \to R\#H$ and $\Phi : R\#H \to A$ defined by*

$$\Psi(a) = \rho(a_1)\#\pi(a_2) \text{ and } \Phi(r\#h) = r\iota(h)$$

are mutually inverse isomorphisms of Hopf algebras.

1.3 Nichols algebras

An important class of braided Hopf algebras generated by primitive elements is formed by Nichols algebras $\mathcal{B}(V)$ of braided vector spaces V. They appeared first in the paper [60] of Nichols.

Definition 1.3.1. Let V be a Yetter-Drinfeld module over a Hopf algebra H. A graded braided Hopf algebra $R = \oplus_{i \geqslant 0} R_i$ in the category ${}^H_H\mathcal{YD}$ is called a *Nichols algebra* of V if the following conditions hold:

- $R_0 \cong \Bbbk$ and $R_1 \cong V$.
- $R_1 = P(R)$, the primitive elements in R.
- R is generated as an algebra by R_1.

We denote the algebra R by $\mathcal{B}(V)$.

Proposition 1.3.2. [5] *Given a Yetter-Drinfeld module V, a Nichols algebra of V exists and is unique up to isomorphism.*

Example 1.3.3. Let V be a finite dimensional vector space and $\tau : V \otimes V \to V \otimes V$ the flip map. The braided vector space (V, τ) can be viewed as a Yetter-Drinfeld module over any Hopf algebra H with trivial action and trivial coaction. The Nichols algebra $\mathcal{B}(V)$ is isomorphic to $S(V)$, the symmetric algebra of V.

In some sense, a Nichols algebra is a generalization of the symmetric algebra of a vector space where the flip map is replaced by a general braiding.

Example 1.3.4. Let (V, c) be a braided vector space of Hecke type, that is, there is a scalar $q \in \Bbbk$ such that

$$(c - q)(c + 1) = 0.$$

By [5, Prop. 3.4], if the scalar q is either 1 or not a root of unity, then $\mathcal{B}(V) \cong T(V)/I$, where $T(V)$ is the tensor algebra of the vector space V, and I is the ideal generated by $\mathrm{Im}(c - q)$.

1.4 Pointed Hopf algebras of finite Cartan type

A Hopf algebra A is called *pointed*, if all its simple left or right comodules are 1-dimensional. This is equivalent to saying that the coradical of A is a group algebra.

The *coradical filtration* $\{A_i \mid i \geqslant 0\}$ of a Hopf algebra is defined inductively as follows. A_0 is the coradical of A. For each $i \geqslant 1$, define

$$A_i = \Delta^{-1}(A \otimes A_{i-1} + A_0 \otimes A).$$

If A is a pointed Hopf algebra, then its coradical filtration is a Hopf algebra filtration (cf. [58, Lemma 5.2.8]). Coradical filtration is important in the classification of pointed Hopf algebras, more detail can be found in [5], [8] etc.

A large classes of pointed Hopf algebras with an abelian group of group-like elements consists of the pointed Hopf algebras of finite Cartan type. The corresponding Nichols algebras provide examples of Nichols algebras of diagonal type.

For a datum of finite Cartan type, we follow the notations in [63] and [8], which are slightly different from the ones in [7]. We need the following terminology:

- an abelian group Γ;
- a Cartan matrix $(a_{ij}) \in \mathbb{Z}^{\theta\times\theta}$ of finite type, where $\theta \in \mathbb{N}$;
- a set $\mathcal{X}$ of connected components of the Dynkin diagram corresponding to the Cartan matrix (a_{ij}). If $1 \leqslant i, j \leqslant \theta$, then $i \sim j$ means that they belong to the same connected component;
- elements $g_1, \cdots, g_\theta \in \Gamma$ and characters $\chi_1, \cdots, \chi_\theta \in \widehat{\Gamma}$ such that

$$\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}},\ \chi_i(g_i) \neq 1,\ \text{for all } 1 \leqslant i, j \leqslant \theta. \tag{1.6}$$

Definition 1.4.1. The collection $\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$ is called a *datum of finite Cartan type* for Γ.

For simplicity, we define $q_{ij} = \chi_j(g_i)$. Then equation (1.6) reads as

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}},\ \ q_{ii} \neq 1,\ \text{for all } 1 \leqslant i, j \leqslant \theta. \tag{1.7}$$

A datum $\mathcal{D}$ is called *generic* if q_{ii} is not a root of unity for all $1 \leqslant i \leqslant \theta$ and Γ is a free abelian group of finite rank (cf. [7]).

Remark 1.4.2. In [63], a generic datum only requires that q_{ii} is not root of unity for each $1 \leqslant i \leqslant \theta$. In this thesis, we mainly discuss the algebras constructed in [7]. For convenience, we further assume that Γ is a free abelian group of finite rank.

Given a datum $\mathcal{D}$, we fix a braided vector space defined as follows. Let V be a Yetter-Drinfeld module over the group algebra $\Bbbk\Gamma$ with basis $x_i \in V_{g_i}^{\chi_i}$, $1 \leqslant i \leqslant \theta$. Then V is a braided vector space of diagonal type whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad 1 \leqslant i, j \leqslant \theta. \tag{1.8}$$

The braiding is called *generic* if q_{ii} is not a root of unity for all $1 \leqslant i \leqslant \theta$.

Since the Cartan matrix is of finite type and

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}}, \quad 1 \leqslant i, j \leqslant \theta, \tag{1.9}$$

there are $d_i \in \{1, 2, 3\}$, $1 \leqslant i \leqslant \theta$, and $q_J \in \Bbbk$, $J \in \mathcal{X}$, such that

$$q_{ii} = q_J^{2d_i}, \quad d_i a_{ij} = d_j a_{ji}, \tag{1.10}$$

for all $J \in \mathcal{X}$ and $i, j \in J$ (cf. [8, Lemma 2.3]). Set

$$\hat{q}_{ij} = \begin{cases} q_J^{d_i a_{ij}} & i, j \in J; \\ 1 & i \nsim j. \end{cases}$$

Then

$$q_{ij} q_{ji} = \hat{q}_{ij} \hat{q}_{ji}, \quad q_{ii} = \hat{q}_{ii},$$

for all $1 \leqslant i, j \leqslant \theta$.

Therefore, when the braiding is generic, it is twist equivalent to a braiding of *DJ-type* (Drinfeld-Jimbo type) [7, Sec. 1]. When the group Γ is a finite abelian group, the braiding is twist equivalent to a braiding of *FL-type* (Frobenius-Lusztig type) [5, Defn. 4.5].

Definition 1.4.3. Vertices $1 \leqslant i, j \leqslant \theta$ are called *linkable* if $i \nsim j$, $g_i g_j \neq 1$ and $\chi_i \chi_j = \varepsilon$.

A *family of linking parameters for* $\mathcal{D}$ is a family $\lambda = (\lambda_{ij})_{1\leqslant i<j\leqslant\theta}$ of elements in $\Bbbk$ such that the following conditions are satisfied for all $1 \leqslant i < j \leqslant \theta$,

$$\text{if } i \text{ and } j \text{ are not linkable then } \lambda_{ij} = 0.$$

Lemma 1.4.4. [6, lemma 5.6] *Any vertex* $1 \leqslant i \leqslant \theta$ *is linkable to at most one vertex.*

Let Φ be the root system corresponding to the Cartan matrix (a_{ij}) with $\Pi = \{\alpha_1, \cdots, \alpha_\theta\}$ a set of fixed simple roots. Let $\mathcal{W}$ be the Weyl group of the root system Φ. We fix a reduced decomposition of the longest element

$$w_0 = s_{i_1} \cdots s_{i_p}$$

of $\mathcal{W}$ as a product of simple reflections. Then the positive roots Φ^+ are precisely the followings

$$\beta_1 = \alpha_{i_1}, \ \ \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}).$$

If $\beta_i = \sum_{i=1}^{\theta} m_i\alpha_i$, then we define

$$g_{\beta_i} = g_1^{m_1} \cdots g_\theta^{m_\theta} \text{ and } \chi_{\beta_i} = \chi_1^{m_1} \cdots \chi_\theta^{m_\theta}.$$

Similarly, we write $q_{\beta_j\beta_i} = \chi_{\beta_i}(g_{\beta_j})$.

1.4.1 Hopf algebras $U(\mathcal{D}, \lambda)$

Definition 1.4.5. Let $\mathcal{D}$ be a datum of finite Cartan type for a group Γ, and λ a family of linking parameters for $\mathcal{D}$. The algebra $U(\mathcal{D}, \lambda)$ is defined to be the quotient Hopf algebra of the smash product $\Bbbk\langle x_1, \cdots, x_\theta\rangle \# \Bbbk\Gamma$ modulo the ideal generated by the following relations

$$\begin{array}{lll} \text{(Serre relations)} & (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, & 1 \leqslant i, j \leqslant \theta, \ \ i \neq j, \ \ i \sim j, \\ \text{(linking relations)} & x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j), & 1 \leqslant i < j \leqslant \theta, \ \ i \nsim j, \end{array}$$

where ad_c is the braided adjoint representation (cf. (1.2)).

The comultiplication structure of $U(\mathcal{D}, \lambda)$ is given by

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \ \ \Delta(g) = g \otimes g,$$

for all $1 \leqslant i \leqslant \theta$ and $g \in \Gamma$.

In the rest of this subsection, we assume that the datum $\mathcal{D}$ is generic. In this case, $U(\mathcal{D},\lambda)$ is the algebra generated by $x_1, \cdots, x_\theta$ and $y_1^{\pm 1}, \cdots, y_s^{\pm 1}$, subject to the following relations (cf. [7, Sec. 4])

$$\begin{array}{lll} & y_m^{\pm 1} y_h^{\pm 1} = y_h^{\pm 1} y_m^{\pm 1}, \quad y_m^{\pm 1} y_m^{\mp 1} = 1, & 1 \leqslant m, h \leqslant s, \\ \text{(group action)} & y_h x_j = \chi_j(y_h) x_j y_h, & 1 \leqslant j \leqslant \theta, \ \ 1 \leqslant h \leqslant s, \\ \text{(Serre relations)} & (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, & 1 \leqslant i, j \leqslant \theta, \ \ i \neq j, \ \ i \sim j, \\ \text{(linking relations)} & x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j), & 1 \leqslant i < j \leqslant \theta, \ \ i \nsim j. \end{array} \tag{1.11}$$

Let V be the braided vector space as defined in (1.8). It can be easily derived from the proof of [7, Thm. 4.3] that the Nichols algebra $\mathcal{B}(V)$ is the algebra generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

$$\mathrm{ad}_c(x_i)^{1-a_{ij}} x_j = 0, \ \ 1 \leqslant i, j \leqslant \theta, \ \ i \neq j.$$

Root vectors for a quantum group $U_q(\mathfrak{g})$ were defined by Lusztig [54]. Up to a non-zero scalar, each root vector can be expressed as an iterated braided commutator. As in [6, Sec. 4.1], this definition can be generalized to a pointed Hopf algebra $U(\mathcal{D},\lambda)$. For each positive root β_i, $1 \leqslant i \leqslant p$, the root vector x_{β_i} is defined by the same iterated braided commutator of the elements $x_1, \cdots, x_\theta$, but with respect to the general braiding.

Remark 1.4.6. If $\beta_j = \alpha_l$, then $x_{\beta_j} = x_l$. That is, $x_1, \cdots, x_\theta$ are the simple root vectors.

Theorem 1.4.7. [7, Thm. 4.3] *Let $\mathcal{D} = (\Gamma, (g_i), (\chi_i), (a_{ij}))$ be a generic datum of finite Cartan type and λ a family of linking parameters for $\mathcal{D}$. The algebra $U(\mathcal{D},\lambda)$ is a pointed Hopf algebra with comultiplication determined by*

$$\Delta(y_h) = y_h \otimes y_h, \ \ \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \ \ 1 \leqslant h \leqslant s, 1 \leqslant i \leqslant \theta.$$

Furthermore, $U(\mathcal{D},\lambda)$ has a PBW-basis given by monomials in the root vectors

$$\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} g \mid a_i \geqslant 0, \ 1 \leqslant i \leqslant p \text{ and } g \in \Gamma\}. \tag{1.12}$$

There is an isomorphism of graded Hopf algebras $\mathrm{Gr}\, U(\mathcal{D},\lambda) \cong \mathcal{B}(V) \# \Bbbk\Gamma \cong U(\mathcal{D}, 0)$, *where* $\operatorname{Gr} U(\mathcal{D},\lambda)$ *is the associated graded algebra of* $U(\mathcal{D},\lambda)$ *with respect to the coradical filtration. The algebra* $U(\mathcal{D},\lambda)$ *has finite Gelfand-Kirillov dimension and is a domain.*

Example 1.4.8. Let $\mathcal{D}$ be a datum of finite Cartan type given by

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$, a free abelian group of rank 2;
- the Cartan matrix is of type A_2, that is,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix};$$

- $g_i = y_i$, $\chi_i(g_j) = q^{a_{ij}}$, $1 \leqslant i,j \leqslant 2$, where $q \in \Bbbk$ is not a root of unity.

Then the algebra $U(\mathcal{D},0)$ is generated by x_i, $y_i^{\pm 1}$, $1 \leqslant i \leqslant 2$, subject to the relations

$$y_m^{\pm 1} y_h^{\pm 1} = y_h^{\pm 1} y_m^{\pm 1}, \quad y_m^{\pm 1} y_m^{\mp 1} = 1, \quad 1 \leqslant m,h \leqslant 2,$$

$$y_i x_j = q^{a_{ji}} x_j y_i, \quad 1 \leqslant i,j \leqslant 2,$$
$$x_1^2 x_2 - q^{-1} x_1 x_2 x_1 - q x_1 x_2 x_1 + x_2 x_1^2,$$
$$x_2^2 x_1 - q^{-1} x_2 x_1 x_2 - q x_2 x_1 x_2 + x_1 x_2^2.$$

The element $s_1 s_2 s_1$ is the longest element in the Weyl group $\mathcal{W}$ and

$$\alpha_1, \quad \alpha_1 + \alpha_2, \quad \alpha_2$$

are the positive roots. The corresponding root vectors are

$$x_1, \quad x_{12} = [x_1, x_2]_c = x_1 x_2 - q^{-1} x_2 x_1, \quad x_2.$$

1.4.2 Hopf algebras $u(\mathcal{D},\lambda,\mu)$

In this subsection, we assume that $\mathcal{D}$ is a datum of finite Cartan type for a finite abelian group Γ such that for all $1 \leqslant i \leqslant \theta$,

$$\begin{aligned} &q_{ii} \text{ has odd order, and} \\ &\text{the order of } q_{ii} \text{ is prime to } 3, \text{ if } i \text{ lies in a component } G_2. \end{aligned} \tag{1.13}$$

Since $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$, $1 \leqslant i, j \leqslant \theta$, the order of q_{ii} is constant in each component J of the Dynkin diagram. Let N_J denote this common order. Let λ be a family of linking parameters for $\mathcal{D}$. The algebra $U(\mathcal{D}, \lambda)$ is defined in Definition 1.4.5. The root vector $x_{\beta_i} \in U(\mathcal{D}, \lambda)$ corresponding to the positive root β_i, $1 \leqslant i \leqslant p$, can be defined in the same way as in the generic case.

Definition 1.4.9. Let $(\mu_\alpha)_{\alpha\in\Phi^+}$ be a set of scalars, such that for all $\alpha \in \Phi_J$, $J \in \mathcal{X}$,

$$\mu_\alpha = 0 \text{ if } g_\alpha^{N_J} = 1 \text{ or } \chi_\alpha^{N_J} \neq \varepsilon.$$

This set of scalars are called *root vector parameters*.

Definition 1.4.10. The finite dimensional Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is the quotient of $U(\mathcal{D}, \lambda)$ modulo the ideal generated by

$$\text{(root vector relations)} \qquad x_\alpha^{N_J} - u_\alpha(\mu), \qquad \alpha \in \Phi_J^+, \ \ J \in \mathcal{X},$$

where $u_\alpha(\mu) \in \Bbbk G$ is defined inductively on Φ^+ as in [8, Sec. 4.2].

Let V be the braided vector space as defined in (1.8). It follows from [8, Thm. 5.1] that the Nichols algebra $\mathcal{B}(V)$ is generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

$$(\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, \ \ 1 \leqslant i, j \leqslant \theta, \ \ i \neq j,$$

$$x_\alpha^{N_J} = 0, \ \ \alpha \in \Phi_J^+, \ \ J \in \mathcal{X}.$$

The following theorem describes the structure of the algebra $u(\mathcal{D}, \lambda, \mu)$.

Theorem 1.4.11. [*8*, Thm. 4.5 and Cor. 5.2] *The algebra $u(\mathcal{D}, \lambda, \mu)$ is a quotient Hopf algebra of $U(\mathcal{D}, \lambda)$ with Γ the group of group-like elements. The following elements form a PBW basis of $u(\mathcal{D}, \lambda, \mu)$,*

$$\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} g \mid 0 \leqslant a_i < N_J\ ,\ \ \beta_i \in \Phi_J^+, \ \ 1 \leqslant i \leqslant p \text{ and } g \in \Gamma\}. \tag{1.14}$$

In particular,

$$\dim u(\mathcal{D}, \lambda, \mu) = (\prod_{J\in\mathcal{X}} N_J^{n_J})|\Gamma|,$$

where n_J is the number of positive roots in component J. There is an isomorphism of graded Hopf algebras $\mathrm{Gr}\, u(\mathcal{D}, \lambda, \mu) \cong \mathcal{B}(V)\#\Bbbk\Gamma \cong u(\mathcal{D}, 0, 0)$, *where* $\mathrm{Gr}\, u(\mathcal{D}, \lambda, \mu)$ *is the associated graded algebra of $u(\mathcal{D}, \lambda, \mu)$ with respect to the coradical filtration.*

1.5 Calabi-Yau categories and Calabi-Yau algebras

1.5.1 Triangulated categories

Before we explain the concept of a Calabi-Yau category, we briefly recall the definition a triangulated category. More detailed discussion about triangulated categories can be found in [31], [34] or [59].

Let $\mathscr{C}$ be an additive category with an automorphism T. The functor T is usually called the *shift functor*.

A *triangle* in $\mathscr{C}$ is a sixtuple (X, Y, Z, u, v, w), where X, Y and Z are objects in $\mathscr{C}$ and $u : X \to Y$, $v : Y \to Z$ and $w : Z \to T(X)$ are morphisms in $\mathscr{C}$. A triangle is usually denoted by the diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

A *morphism* between two triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

and

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} T(X')$$

is a triple (f, g, h) with f, g, h morphisms in $\mathscr{C}$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X').
\end{tikzcd}$$

If f, g and h are all isomorphisms, then the morphism (f, g, h) is called an *isomorphism*.

Definition 1.5.1. A *triangulated category* $(\mathscr{C}, T, \mathcal{E})$ (or simply $\mathscr{C}$) is an additive category equipped with the shift functor T and a family of triangles $\mathcal{E}$, called *distinguished triangles*, satisfying the following axioms.

(TR1a) Any triangle isomorphic to a distinguished triangle is a distinguished triangle;

(TR1b) Every morphism $u : X \to Y$ in $\mathscr{C}$ can be embedded into a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$;

(TR1c) For any object X in $\mathscr{C}$, the triangle $X \xrightarrow{id_X} X \to 0 \to T(X)$ is a distinguished triangle.

(TR2) (*Turning of triangles axiom*) If

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a distinguished triangle, then

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

is a distinguished triangle.

(TR3) Let

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X')
\end{tikzcd}$$

be a diagram where the rows are distinguished triangles and the first square is commutative. Then there exists a morphism $h : Z \to Z'$, such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X').
\end{tikzcd}$$

(TR4) (*Octahedral axiom*) Let

$$X \xrightarrow{u} Y \xrightarrow{u'} Z' \xrightarrow{u''} T(X),$$

$$X \xrightarrow{w} Z \xrightarrow{w'} Y' \xrightarrow{w''} T(X)$$

and

$$Y \xrightarrow{v} Z \xrightarrow{v'} X' \xrightarrow{v''} T(Y)$$

be three distinguished triangles, such that $w = vu$. Then there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "u'"] \arrow[d, "v"] & Z' \arrow[r, "u''"] \arrow[d, dashed, "a"] & T(X) \arrow[d, equal] \\
X \arrow[r, "w"] \arrow[d, "u"] & Z \arrow[r, "w'"] \arrow[d, equal] & Y' \arrow[r, "w''"] \arrow[d, dashed, "b"] & T(X) \arrow[d, "T(u)"] \\
Y \arrow[r, "v"] \arrow[d, "u'"] & Z \arrow[r, "v'"] \arrow[d, "w'"] & X' \arrow[r, "v''"] \arrow[d, equal] & TY \arrow[d, "T(u')"] \\
Z' \arrow[r, dashed, "a"] & Y' \arrow[r, dashed, "b"] & X' \arrow[r] & TZ'
\end{tikzcd}$$

where the rows are distinguished triangles.

Let A be an algebra. The derived category $D(A)$ of the abelian category Mod$A$ is a triangulated category. Detailed definition of a derived category can be found, for example, in [31]. Roughly speaking, $D(A)$ has all complexes of A-modules as its objects. The morphisms are obtained by formally inverting quasi-isomorphisms. Let

$$\mathcal{M} : \cdots \to M^i \xrightarrow{d^i_{\mathcal{M}}} M^{i+1} \to \cdots$$

be a complex. The complex $T\mathcal{M}$ is a complex such that $(T\mathcal{M})^i = \mathcal{M}^{i+1}$ and $d^i_{T\mathcal{M}} = -d^{i+1}_{\mathcal{M}}$. Each short exact sequence of complexes

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{L} \to 0$$

canonically determines a standard triangle

$$\mathcal{M} \to \mathcal{N} \to \mathcal{L} \to T(\mathcal{M}).$$

The distinguished triangles are the ones isomorphic to standard triangles.

We denote by $D^b(A)$ the full triangulated subcategory of $D(A)$ consisting of bounded complexes. In the following, the shift functor T^d is denoted by $[d]$.

Definition 1.5.2. Let $(\mathscr{C}, T, \mathcal{E})$ and $(\mathscr{D}, T', \mathcal{F})$ be two triangulated categories. A *triangle functor* from $\mathscr{C}$ to $\mathscr{D}$ is a pair (F, α), where $F : \mathscr{C} \to \mathscr{D}$ is an additive functor and $\alpha : FT \to T'F$ is a natural isomorphism, such that F maps a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

in $\mathscr{C}$ to a distinguished triangle

$$FX \xrightarrow{F(u)} FY \xrightarrow{F(v)} FZ \xrightarrow{\alpha_X F(w)} T'F(X)$$

in $\mathscr{D}$.

Using the turning of triangles axiom, one can obtain that if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a distinguished triangle, then

$$T(X) \xrightarrow{Tu} T(Y) \xrightarrow{Tv} T(Z) \xrightarrow{-Tw} T^2(X)$$

is a distinguished triangle. Therefore, $(T, -1_{T^2})$ is a triangle functor.

Two triangle functors (F, α) and (F', α') are natural isomorphic if there is a natural isomorphism $\theta : F \to F'$ such that the following diagram commutes for any $X \in \mathscr{C}$:

$$\begin{tikzcd}
FT(X) \arrow[r, "\theta_{T(X)}"] \arrow[d, "\alpha_X"'] & F'T(X) \arrow[d, "\alpha'_X"] \\
T'F(X) \arrow[r, "T'(\theta_X)"] & T'F'(X).
\end{tikzcd}$$

1.5.2 Calabi-Yau categories

Let $\mathscr{C}$ be a $\Bbbk$-linear category. It is called Hom-finite if $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is finite dimensional for any X and Y in $\mathscr{C}$.

Definition 1.5.3. (cf. [44] and [45]) A *right Serre functor* for a Hom-finite $\Bbbk$-linear triangulated category $(\mathscr{C}, T)$ is a triangle functor $(S, \alpha) : \mathscr{C} \to \mathscr{C}$ together with isomorphisms

$$\zeta_X : \mathrm{Hom}_{\mathscr{C}}(-, SX) \to \mathrm{Hom}_{\mathscr{C}}(X, -)^* \tag{1.15}$$

which are natural in X and satisfying the following equations:

$$\zeta_X \circ T^{-1} \circ (\alpha_X)_* = -(T)^* \circ (\zeta_{TX}) \tag{1.16}$$

for any $X \in \mathscr{C}$.

A *Serre functor* is a right Serre functor which is in addition an equivalence.

Remark 1.5.4. The definition of a Serre functor of an additive category can be found in [19] or [64].

Definition 1.5.3 is equivalent to saying that a Serre functor $(S, \alpha) : (\mathscr{C}, T) \to (\mathscr{C}, T)$ is a triangle functor which in addition is an equivalence such that there are natural isomorphisms

$$\zeta_{X,Y} : \ \mathrm{Hom}_{\mathscr{C}}(X,\ Y) \to \mathrm{Hom}_{\mathscr{C}}(Y,\ SX)^*,$$

for any X, $Y \in \mathscr{C}$, and the following diagram anti-commutes

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathscr{C}}(TY, (ST)(X)) \arrow[r, "\zeta^*_{TX,TY}"] \arrow[dl, "(\alpha_X)_*"'] & \mathrm{Hom}_{\mathscr{C}}(TX, TY)^* \arrow[dd, "(T)^*"] \\
\mathrm{Hom}_{\mathscr{C}}(TY, (TS)(X)) \arrow[dr, "T^{-1}"] & & \\
 & \mathrm{Hom}_{\mathscr{C}}(Y, SX) \arrow[r, "\zeta^*_{X,Y}"] & \mathrm{Hom}_{\mathscr{C}}(X, Y)^*
\end{tikzcd} \tag{1.17}$$

Definition 1.5.5. A *d-Calabi-Yau category* is a Hom-finite $\Bbbk$-linear triangulated category $(\mathscr{C}, T)$, such that it admits a Serre functor (S, α) and there is a natural isomorphism of triangle functors

$$(S, \alpha) \cong (T^d, (-1)^d),$$

for some $d \in \mathbb{Z}$. The Calabi-Yau dimension of $\mathscr{C}$ is the smallest non-negative integer d satisfying the above condition.

1.5.3 Calabi-Yau algebras

We follow Ginzburg's definition of a Calabi-Yau algebra [32].

Definition 1.5.6. An algebra A is called a *Calabi-Yau algebra of dimension d* if

(i) A is *homologically smooth*. That is, A has a bounded resolution of finitely generated projective A-A-bimodules.

(ii) There are A-A-bimodule isomorphisms

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A, & i = d. \end{cases}$$

In the sequel, Calabi-Yau will be abbreviated as CY for short.

Remark 1.5.7. Let A be an algebra. Denote by $D^b_{fd}(A)$ the full triangulated subcategory of the derived category of A consisting of complexes whose homology is of finite total dimension. By [44, lemma 4.1], if A is a CY algebra of dimension d, then the category $D^b_{fd}(A)$ is a d-CY category. Sometimes an algebra A is called a CY algebra of dimension d if the category $D^b_{fd}(A)$ is a CY category of dimension d (see e.g. [17]).

Example 1.5.8. We list some examples of CY algebras.

(1) The polynomial algebra $\Bbbk[x_1, \cdots, x_n]$ with n variables is a CY algebra of dimension n.

(2) Any Sridharan enveloping algebra of an n-dimensional abelian Lie algebra is a CY algebra of dimension n [12, Thm. 6.5].

(3) Let A be the algebra $\Bbbk\langle x_0, x_1, x_2, x_3\rangle/I$, where the ideal I is generated by the following relations

$$\begin{array}{ll} x_0x_1 - x_1x_0 - \alpha(x_2x_3 + x_3x_2), & x_0x_1 + x_1x_0 - (x_2x_3 - x_3x_2), \\ x_0x_2 - x_2x_0 - \beta(x_3x_1 + x_1x_3), & x_0x_2 + x_2x_0 - (x_3x_1 - x_1x_3), \\ x_0x_3 - x_3x_0 - \gamma(x_1x_2 + x_2x_1), & x_0x_3 + x_3x_0 - (x_1x_2 - x_2x_1), \end{array}$$

$\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ and $(\alpha, \beta, \gamma) \notin \{(\alpha, -1, 1), (1, \beta, -1), (-1, 1, \gamma)\}$. The algebra A is a 4-dimensional Sklyanian algebra and a CY algebra of dimension 4 [18, Prop. 7.1].

CY algebras are closely related to rigid dualizing complexes. The non-commutative version of a dualizing complex was first introduced by Yekutieli.

Definition 1.5.9. [74] (cf. [68, Defn. 6.1]) Assume that A is a (graded) Noetherian algebra. Then an object $\mathscr{R}$ of $D^b(A^e)$ $(D^b(\mathrm{GrMod}(A^e)))$is called a *dualizing complex* (in the graded sense) if it satisfies the following conditions:

(i) $\mathscr{R}$ is of finite injective dimension over A and A^{op}.

(ii) The cohomology of $\mathscr{R}$ is given by bimodules which are finitely generated on both sides.

(iii) The natural morphisms $A \to \mathrm{RHom}_A(\mathscr{R}, \mathscr{R})$ and $A \to \mathrm{RHom}_{A^{op}}(\mathscr{R}, \mathscr{R})$ are isomorphisms in $D(A^e)$ $(D(\mathrm{GrMod}(A^e)))$.

Roughly speaking, a dualizing complex is a complex $\mathscr{R} \in D^b(A^e)$ such that the functor

$$\mathrm{RHom}_A(-, \mathscr{R}) : D^b_{fg}(A) \to D^b_{fg}(A^{op}) \tag{1.18}$$

is a duality, with adjoint $\mathrm{RHom}_{A^{op}}(-, \mathscr{R})$ (cf. [74, Prop. 3.4 and Prop. 3.5]). Here $D^b_{fg}(A)$ is the full triangulated subcategory of $D(A)$ consisting of bounded complexes with finitely generated cohomology modules.

In the above definition, the algebra A is a Noetherian algebra. In this case, a dualizing complex in the graded sense is also a dualizing complex in the usual sense.

Dualizing complexes are not unique up to isomorphism. To overcome this weakness, Van den Bergh introduced the concept of a rigid dualizing complex in [68, Defn. 8.1].

Definition 1.5.10. Let A be a (graded) Noetherian algebra. A dualizing complex $\mathscr{R}$ over A is called *rigid* (in the graded sense) if

$$\mathrm{RHom}_{A^e}(A, {}_A\mathscr{R} \otimes \mathscr{R}_A) \cong \mathscr{R}$$

in $D(A^e)$ $(D(\mathrm{GrMod}(A^e)))$.

Note again that if A^e is Noetherian then the graded version of this definition implies the ungraded version.

Lemma 1.5.11. (cf.[*20*, Prop. 4.3] *and* [*68*, Prop. 8.4]) *Let A be a Noetherian algebra. Then the following two conditions are equivalent:*

(1) A has a rigid dualizing complex $\mathscr{R} = A_\psi[s]$, where ψ is an algebra automorphism and $s \in \mathbb{Z}$.

(2) A has finite injective dimension d and there is an algebra automorphism ϕ such

that

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A_\phi & i = d \end{cases}$$

as A-A-bimodules.

If one of the two conditions holds, then $\phi\psi$ is an inner automorphism and $s = d$.

This Lemma 1.5.11 will be frequently used in this thesis. The following corollary follows directly from Lemma 1.5.11 and the definition of a CY algebra. It gives the relation between CY algebras and rigid dualizing complexes.

Corollary 1.5.12. *Let A be a Noetherian algebra which is homologically smooth. Then A is a CY algebra of dimension d if and only if A has a rigid dualizing complex $A[d]$.*

Now we take the Koszul complex and the rigid dualizing complex of a quantum polynomial algebra as an example. They are also preparations for Chapter 2 and Chapter 3.

Let S be the algebra

$$\Bbbk\langle x_1, x_2, \cdots, x_n \mid x_i x_j - q_{ij} x_j x_i, 1 \leqslant i < j \leqslant n\rangle,$$

where $q_{ij} \in \Bbbk$.

The algebra S is a quadratic algebra, its quadratic dual $S^!$ is isomorphic to

$$\Bbbk\langle x_1^*, x_2^*, \cdots, x_n^* \mid q_{ij} x_i^* x_j^* + x_j^* x_i^*, 1 \leqslant i < j \leqslant n\rangle.$$

The algebra $S^!$ is a Frobenius algebra with the Nakayama automorphism η defined by

$$\eta(x_i^*) = (-1)^{n-1} q_{1i} \cdots q_{(i-1)i} q_{i(i+1)}^{-1} \cdots q_{in}^{-1} x_i^*,$$

for all $1 \leqslant i \leqslant n$.

It is easy to see that S is a Koszul algebra. The following complex is the minimal projective resolution of $\Bbbk$ over S,

$$0 \to S \otimes S_n^{!*} \to \cdots \to S \otimes S_j^{!*} \xrightarrow{d_j} S \otimes S_{j-1}^{!*} \to \cdots \to S \otimes S_1^{!*} \to S \to \Bbbk \to 0. \quad (1.19)$$

The differentials d_j, $1 \leqslant j \leqslant n$, are defined by $d_j(1 \otimes a) = \sum_{i=1}^{n} x_i \otimes a \cdot x_i^*$, for $a \in S_j^{!*}$, where "$\cdot$" denotes the right $S^!$-action on $S^{!*}$. The complex (1.19) is a quantum version of the classical Koszul complex of a polynomial algebra.

Lemma 1.5.13. *(1) The algebra S is homologically smooth.*

(2) The rigid dualizing complex of S is isomorphic to $S_\zeta(-n)[n]$, where ζ is the algebra automorphism defined by

$$\zeta(x_i) = q_{1i} \cdots q_{(i-1)i} q_{i(i+1)}^{-1} \cdots q_{in}^{-1} x_i,$$

for all $1 \leqslant i \leqslant n$.

Proof. The Koszul bimodule complex (cf. [68, Thm. 9.1]) of S is as follows

$$\mathcal{K} : 0 \to S \otimes S_n^{!*} \otimes S \to \cdots \to S \otimes S_j^{!*} \otimes S \xrightarrow{D_j} S \otimes S_{j-1}^{!*} \otimes S \to \cdots \to S \otimes S \to 0. \quad (1.20)$$

The differentials $D_j : S \otimes S_j^{!*} \otimes S \to S \otimes S_{j-1}^{!*} \otimes S$, $1 \leqslant j \leqslant n$, are defined by $D_j = d_j^l + (-1)^j d_j^r$, where $d_j^l(1 \otimes a \otimes 1) = \sum_{i=1}^{n} x_i \otimes a \cdot x_i^* \otimes 1$ and $d_j^r(1 \otimes a \otimes 1) = \sum_{i=1}^{n} 1 \otimes x_i^* \cdot a \otimes x_i$, for any $1 \otimes a \otimes 1 \in S \otimes S_j^{!*} \otimes S$. We have that $\mathcal{K} \to S \to 0$ is exact. This shows that the algebra S is homologically smooth.

The algebra $S^!$ is Frobenius, so S is AS-regular. By [68, Thm. 9.2 and Prop. 8.2], the rigid dualizing complex of S is isomorphic to $S_{\epsilon^{n+1}\phi}(-n)[n]$. The automorphism ϵ is the multiplication by $(-1)^m$ on S_m. The automorphism ϕ satisfies that $\phi|_{S_1}$ is dual to $\eta|_{S_1^!}$. So

$$\epsilon^{n+1}\phi(x_i) = q_{1i} \cdots q_{(i-1)i} q_{i(i+1)}^{-1} \cdots q_{in}^{-1} x_i,$$

for all $1 \leqslant i \leqslant n$. □

Remark 1.5.14. The algebra S is a connected graded algebra. So $S_\zeta \cong S$ as bimodules if and only if $\zeta = \mathrm{id}$. Therefore, the algebra S is CY of dimension n if and only if

$$q_{1i} \cdots q_{(i-1)i} q_{i(i+1)}^{-1} \cdots q_{in}^{-1} = 1$$

for all $1 \leqslant i \leqslant n$.

The algebra S^e is Noetherian. Therefore, the rigid dualizing complex of S in the ungraded sense is just $S_\zeta[n]$. We also have

$$\operatorname{Ext}^i_{S^e}(S, S^e) \cong \begin{cases} 0, & i \neq n; \\ {}_\zeta S, & i = n. \end{cases}$$

For an A-A-bimodule M, the Hochschild homology and cohomology of A with coefficients in M are defined to be $\operatorname{Tor}^{A^e}_*(A, M)$ and $\operatorname{Ext}^*_{A^e}(A, M)$ respectively. Denote them by $\mathrm{HH}_*(A, M)$ and $\mathrm{HH}^*(A, M)$. An A-A-bimodule U is said to be *invertible* if there exists another A-A-bimodule V, such that

$$U \otimes_A V \cong V \otimes_A U \cong A$$

as A-A-bimodule.

In [67], Van den Bergh proved the following duality between Hochschild homology and cohomology.

Theorem 1.5.15. *Assume A is a homologically smooth algebra. If there is an integer d and an invertible bimodule U such that*

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ U, & i = d. \end{cases}$$

as bimodules, then for any A-A-bimodule M,

$$\mathrm{HH}^i(A, M) = \mathrm{HH}_{d-i}(A, U \otimes_A M), \ \ 0 \leqslant i \leqslant d.$$

This theorem is usually called Van den Bergh's duality theorem.

1.5.4 Calabi-Yau property of Hopf algebras

It turns out that CY algebras are Artin-Schelter algebras. For Hopf algebras, the CY property can be characterized via homological integrals of Artin-Gorenstein algebras.

Let us recall the definition of an Artin-Schelter Gorenstein (regular) algebra first.

An $\mathbb{N}$-graded algebra $A = \oplus_{i \geqslant 0} A_i$ is called *connected* if $A_0 = \Bbbk$.

Definition 1.5.16. A connected $\mathbb{N}$-graded algebra A is called *Artin-Schelter regular* (AS-regular for short) if the following three conditions hold:

(i) A has finite global dimension d.

(ii) There is some integer l, such that

$$\mathrm{EXT}^i_A(\Bbbk, A) \cong \begin{cases} 0, & \text{if } i \neq d; \\ \Bbbk(l) & \text{if } i = d. \end{cases}$$

(iii) A has finite Gelfand-Kirillov dimension, that is, there is a positive number c such that $\dim A_i < cn^c$ for all $i \in \mathbb{N}$.

Recall that (-) denotes the degree shift. If a connected graded algebra A satisfies condition (ii), then A is called *Artin-Schelter Gorenstein* (AS-Gorenstein for short).

In [20], the notion of an AS-Gorenstein (regular) algebra was defined for a general augmented algebra.

Definition 1.5.17. (1) Let A be a left Noetherian augmented algebra with a fixed augmentation map $\varepsilon : A \to \Bbbk$. A is said to be *left AS-Gorenstein*, if

(i) $\operatorname{injdim}_A A = d < \infty$,

(ii) $\dim \mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) = \begin{cases} 0, & i \neq d; \\ 1, & i = d, \end{cases}$

where injdim stands for injective dimension.

A *Right AS-Gorenstein algebras* can be defined similarly.

(2) An algebra A is said to be *AS-Gorenstein* if it is both left and right AS-Gorenstein (relative to the same augmentation map ε).

(3) An AS-Gorenstein algebra A is said to be *regular* if in addition, the global dimension of A is finite.

The concept of a homological integral for an AS-Gorenstein Hopf algebra was introduced by Lu, Wu and Zhang in [53]. It is a generalization of an integral of a finite dimensional Hopf algebra. In [20], homological integrals were defined for general AS-Gorenstein algebras.

Definition 1.5.18. [20] Let A be a left AS-Gorenstein algebra with $\operatorname{injdim}{}_A A = d$. Then $\operatorname{Ext}_A^d({}_A\Bbbk, {}_A A)$ is a 1-dimensional right A-module. Any non-zero element in $\operatorname{Ext}_A^d({}_A\Bbbk, {}_A A)$ is called a *left homological integral* of A. We write $\int_A^l$ for $\operatorname{Ext}_A^d({}_A\Bbbk, {}_A A)$.

Similarly, if A is right AS-Gorenstein, any non-zero element in $\operatorname{Ext}_A^d(\Bbbk_A, A_A)$ is called a *right homological integral* of A. Write $\int_A^r$ for $\operatorname{Ext}_A^d(\Bbbk_A, A_A)$.

$\int_A^l$ and $\int_A^r$ are called left and right homological integral modules of A respectively.

The left integral module $\int_A^l$ is a 1-dimensional right A-module. Thus $\int_A^l \cong \Bbbk_\xi$ for some algebra homomorphism $\xi : A \to \Bbbk$.

Proposition 1.5.19. *Let A be a Noetherian augmented algebra such that A is CY of dimension d. Then A is AS-regular of global dimension d. In addition, $\int_A^l \cong \Bbbk$ as right A-modules.*

Proof. If A is an augmented algebra, then ${}_A\Bbbk$ is a finite dimensional module. By [13, Remark 2.8], A has global dimension d.

It follows from [13, Prop. 2.2] that A admits a projective bimodule resolution

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to A \to 0,$$

where each P_i is finitely generated as an A-A-bimodule. Tensoring with functor $\otimes_A \Bbbk$, we obtain a projective resolution of ${}_A\Bbbk$:

$$0 \to P_d \otimes_A \Bbbk \to \cdots \to P_1 \otimes_A \Bbbk \to P_0 \otimes_A \Bbbk \to {}_A\Bbbk \to 0.$$

Since each P_i is finitely generated, the isomorphism

$$\Bbbk \otimes_A \operatorname{Hom}_{A^e}(P_i, A^e) \cong \operatorname{Hom}_A(P_i \otimes_A \Bbbk, A)$$

holds in $\operatorname{Mod} A^{op}$. Therefore, the complex $\operatorname{Hom}_A(P_\bullet \otimes_A \Bbbk, A)$ is isomorphic to the complex $\Bbbk \otimes_A \operatorname{Hom}_{A^e}(P_\bullet, A^e)$. The algebra A is CY of dimension d. So the following A-A-bimodule complex is exact,

$$0 \to \operatorname{Hom}_{A^e}(P_0, A^e) \to \cdots \to \operatorname{Hom}_{A^e}(P_{d-1}, A^e) \to \operatorname{Hom}_{A^e}(P_d, A^e) \to A \to 0.$$

Thus the complex $\Bbbk \otimes_A \operatorname{Hom}_{A^e}(P_\bullet, A^e)$ is exact except at $\Bbbk \otimes_A \operatorname{Hom}_{A^e}(P_d, A^e)$, whose homology is $\Bbbk$. It follows that the isomorphisms

$$\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \cong \begin{cases} 0, & i \neq d; \\ \Bbbk, & i = d \end{cases}$$

hold in $\mathrm{Mod}A^{op}$. Similarly, we have isomorphisms

$$\mathrm{Ext}^i_A(\Bbbk_A, A_A) \cong \begin{cases} 0, & i \neq d; \\ \Bbbk, & i = d \end{cases}$$

in $\mathrm{Mod}A$. We conclude that A is AS-regular and $\int_A^l \cong \Bbbk$. □

Remark 1.5.20. From the proof of Proposition 1.5.19 we can see that if A is a Noetherian augmented algebra such that

(i) A is homologically smooth, and

(ii) there is an integer d and an algebra automorphism ψ, such that

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A_\psi, & i = d \end{cases}$$

as A-A-bimodules,

then A is AS-regular of global dimension d. In this case, $\int_A^l \cong \Bbbk_\xi$. The algebra homomorphism ξ is defined by $\xi(a) = \varepsilon(\psi(a))$ for all $a \in A$, where ε is the augmentation map of A.

Let A be a Hopf algebra, and $\xi : A \to \Bbbk$ an algebra homomorphism. We let $[\xi]$ be the winding homomorphism of ξ defined by

$$[\xi](a) = \xi(a_1)a_2,$$

for all $a \in A$. Then we have the following.

Proposition 1.5.21. [*20*, Prop. 4.5] *Let A be a Noetherian AS-Gorenstein Hopf algebra with injective dimension d. Let $\int_A^l = \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is an algebra homomorphism. Then the rigid dualizing complex of A is ${}_{[\xi]S_A^2}A[d]$.*

The following theorem characterizes the CY property of Noetherian Hopf algebras.

Theorem 1.5.22. [*35*, Thm. 2.3] *Let A be a Noetherian AS-Gorenstein Hopf algebra. Then A is CY algebra of dimension d if and only if*

(ii) A is AS-regular with global dimension d and $\int_A^l \cong \Bbbk$ as right A-modules.

(ii) $\mathcal{S}_A^2$ is an inner automorphism of A.

Chapter 2

Calabi-Yau pointed Hopf algebras $U(\mathcal{D}, \lambda)$

Chemla calculated the rigid dualizing complex of the quantized enveloping algebra $U_q(\mathfrak{g})$ of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ [23]. As a result, $U_q(\mathfrak{g})$ is a CY algebra. A pointed Hopf algebra $U(\mathcal{D}, \lambda)$ is not necessarily a CY algebra. In Section 2.1, we calculate the rigid dualizing complex of a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ and give a necessarily and sufficient condition for $U(\mathcal{D}, \lambda)$ to be a CY algebra. This result is also a preparation for computing the rigid dualizing complex of the corresponding Nichols algebra in Chapter 3. The CY pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimensions less than 5 are classified in Section 2.2.

2.1 Rigid dualizing complexes of pointed Hopf algebras $U(\mathcal{D}, \lambda)$

In this section we fix a generic datum of finite Cartan type

$$\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$$

for Γ and a family of linking parameters $\lambda = (\lambda_{ij})_{1\leqslant i<j\leqslant\theta, i\nsim j}$ for $\mathcal{D}$, where Γ is a free abelian group of rank s.

Let $\{\alpha_1,\cdots,\alpha_\theta\}$ be a fixed set of simple roots of the root system corresponding to the Cartan matrix (a_{ij}). We also fix a reduced decomposition $w_0 = s_{i_1}\cdots s_{i_p}$ of the longest element w_0 in the Weyl group $\mathcal{W}$ as a product of simple reflections. Then

$$\beta_1 = \alpha_{i_1},\ \ \beta_2 = s_{i_2}(\alpha_{i_1}),\cdots,\beta_p = s_{i_1}\cdots s_{i_{p-1}}(\alpha_{i_p})$$

are the positive roots. Let x_{β_i}, $1\leqslant i\leqslant p$, be the corresponding root vectors. There are $1\leqslant j_k\leqslant p$, $1\leqslant k\leqslant\theta$, such that $\beta_{j_k}=\alpha_k$. Then $x_{\beta_{j_k}} = x_k$, $1\leqslant k\leqslant\theta$.

The algebra $U(\mathcal{D},\lambda)$ is defined in Section 1.4. Following from Theorem 1.4.7, the set

$$\{x_{\beta_1}^{a_1}\cdots x_{\beta_p}^{a_p}g \mid a_i\geqslant 0, 1\leqslant i\leqslant p, g\in\Gamma\}$$

forms a PBW basis of the algebra $U(\mathcal{D},\lambda)$. As in [7], the degrees are defined as follows

$$\deg(x_{\beta_1}^{a_1}\cdots x_{\beta_p}^{a_p}g) = (a_1,\cdots,a_p,\sum_{i=1}^{p} a_i ht(\beta_i)) \in (\mathbb{Z}^{\geqslant 0})^{p+1}, \tag{2.1}$$

where $ht(\beta_i)$ is the height of the root β_i. That is, if $\beta_i=\sum_{i=1}^{\theta} m_i\alpha_i$, then $ht(\beta_i) = \sum_{i=1}^{\theta} m_i$. In this thesis, we always order the elements in $\mathbb{N}^{p+1}$ as follows

$$\begin{aligned}&(a_1,\cdots,a_p,a_{p+1}) < (b_1,\cdots,b_p,b_{p+1}) \text{ if and only if there is some}\\ &1\leqslant k\leqslant p+1, \text{ such that } a_i=b_i \text{ for } i\geqslant k \text{ and } a_{k-1}<b_{k-1}.\end{aligned} \tag{2.2}$$

Given $\mathbb{m}\in\mathbb{N}^{p+1}$, let $F_{\mathbb{m}}U(\mathcal{D},\lambda)$ be the space spanned by the monomials $x_{\beta_1}^{a_1}\cdots x_{\beta_p}^{a_p}g$ such that $\deg(x_{\beta_1}^{a_1}\cdots x_{\beta_p}^{a_p}g)\leqslant\mathbb{m}$. We claim that this gives an algebra filtration on $U(\mathcal{D},\lambda)$.

Lemma 2.1.1. *If the root vectors x_{β_i}, x_{β_j} belong to the same connected component and $j>i$, then*

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{\boldsymbol{a}\in\mathbb{N}^p}\rho_{\boldsymbol{a}} x_{\beta_1}^{a_1}\cdots x_{\beta_p}^{a_p}, \tag{2.3}$$

where $\rho_{\boldsymbol{a}}\in\Bbbk$ and $\rho_{\boldsymbol{a}}\neq 0$ only when $\boldsymbol{a}=(a_1,\cdots,a_p)$ satisfies that $a_k=0$ for $k\leqslant i$ and $k\geqslant j$. In particular, the equation (2.3) holds for all root vectors x_{β_i}, x_{β_j} with $i<j$ in $U(\mathcal{D},0)$.

Proof. This follows from [7, Prop. 2.2] and the classical relations that hold for a quantum group $U_q(\mathfrak{g})$ (see [25, Thm. 9.3] for example). It was actually proved in Step VI of the proof of Theorem 4.3 in [7]. $\square$

Lemma 2.1.2. *(1) The filtration defined on the PBW basis is an algebra filtration.*

(2) The associated graded algebra $\mathbb{G}\mathrm{r}U(\mathcal{D},\lambda)$ is generated by x_{β_i}, $1 \leqslant i \leqslant p$, and y_h, $1 \leqslant h \leqslant s$, subject to the relations

$$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, \quad y_h^{\pm 1} y_h^{\mp 1} = 1, \quad 1 \leqslant h, m \leqslant s,$$

$$y_h x_{\beta_i} = \chi_{\beta_i}(y_h) x_{\beta_i} y_h, \quad 1 \leqslant i \leqslant p, \ 1 \leqslant h \leqslant s,$$

$$x_{\beta_i} x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i}, \ 1 \leqslant i < j \leqslant p.$$

Proof. This follows from Lemma 2.1.1 and the linking relations. □

Note that the associated graded algebra $\mathbb{G}\mathrm{r}U(\mathcal{D},\lambda)$ is an $\mathbb{N}^{p+1}$-graded algebra.

Lemma 2.1.3. *The Hopf algebra $A = U(\mathcal{D},\lambda)$ is Noetherian with finite global dimension bounded by $p+s$.*

Proof. The group algebra $\Bbbk\Gamma$ is isomorphic to a Laurent polynomial algebra with s variables. So $\Bbbk\Gamma$ is Noetherian of global dimension s. By Lemma 2.1.1, the algebra $\mathrm{Gr}\, A \cong U(\mathcal{D}, 0)$ is an iterated Ore extension of $\Bbbk\Gamma$, where $\mathrm{Gr}\, A$ is the associated graded algebra of A with respect to the coradical filtration (cf. Theorem 1.4.7). Indeed, if $x_{\beta_1}, \cdots, x_{\beta_p}$ are the root vectors of A, then

$$\mathrm{Gr}\, A \cong \Bbbk\Gamma[x_{\beta_1}; \tau_1, \delta_1][x_{\beta_2}; \tau_2, \delta_2] \cdots [x_{\beta_p}; \tau_p, \delta_p].$$

For $1 \leqslant j \leqslant p$, τ_j is an algebra automorphism such that its action on each x_{β_i}, $i < j$, and $g \in \Gamma$ is a scalar multiplication. δ_j is a τ_j-derivation such that $\delta_j(g) = 0$, $g \in \Bbbk\Gamma$ and $\delta_j(x_{\beta_i})$, $i < j$, is a linear combination of monomials in $x_{\beta_{i+1}}, \cdots, x_{\beta_{j-1}}$. By [57, Thm. 1.2.9 and Thm. 7.5.3], we have that $\mathrm{Gr}\, A$ is Noetherian of global dimension less than $p+s$. Following from [57, Thm. 1.6.9 and Cor. 7.6.18], the algebra A is Noetherian of global dimension less than $p+s$. □

Theorem 2.1.4. *Let $\mathcal{D}$ be a generic datum of finite Cartan type for a group Γ, λ a family of linking parameters for $\mathcal{D}$, and A the Hopf algebra $U(\mathcal{D},\lambda)$. Then A is Noetherian AS-regular of global dimension $p+s$, where s is the rank of Γ and p is the number of the positive roots of the Cartan matrix. The left homological integral module $\int_A^l$ of A is isomorphic to $\Bbbk_\xi$, where $\xi : A \to \Bbbk$ is an algebra homomorphism defined by $\xi(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$ and $\xi(x_i) = 0$ for all $1 \leqslant i \leqslant \theta$.*

Proof. We first show that

$$\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \cong \begin{cases} 0, & i \neq p+s; \\ \Bbbk_\xi, & i = p+s. \end{cases}$$

With Lemma 2.1.2 and Lemma 2.1.3, the method in [23, Prop. 3.2.1] for computing the group $\mathrm{Ext}^*_{U_q(\mathfrak{g})}({}_{U_q(\mathfrak{g})}\Bbbk, {}_{U_q(\mathfrak{g})}U_q(\mathfrak{g}))$ also works in the case of $U(\mathcal{D},\lambda)$. The difference is that the right A-module structure on $\mathrm{Ext}^{p+s}_A({}_A\Bbbk, {}_AA)$ is not trivial in the case of $U(\mathcal{D},\lambda)$. Put $C = \mathbb{G}\mathrm{r}U(\mathcal{D},\lambda)$. We also have that $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) = 0$ for $i \neq p+s$ and $\mathrm{Ext}^{p+s}_C({}_C\Bbbk, {}_CC) \cong \mathrm{Ext}^{p+s}_A({}_A\Bbbk, {}_AA)$ as right Γ-modules.

We now give the structure of $\mathrm{Ext}^*_C({}_C\Bbbk, {}_CC)$. Let B be the following algebra,

$$\Bbbk\langle x_{\beta_1}, \cdots, x_{\beta_p} \mid x_{\beta_i}x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i})x_{\beta_j}x_{\beta_i}, 1 \leqslant i < j \leqslant p\rangle.$$

Then $C = B\#\Bbbk\Gamma$. We have the following isomorphisms

$$\begin{aligned} \mathrm{RHom}_C(\Bbbk, C) &\cong \mathrm{RHom}_C(\Bbbk\Gamma \otimes_{\Bbbk\Gamma} \Bbbk, C) \\ &\cong \mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \mathrm{RHom}_C(\Bbbk\Gamma, C)) \\ &\cong \mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \Bbbk\Gamma) \otimes^L_{\Bbbk\Gamma} \mathrm{RHom}_C(\Bbbk\Gamma, C). \end{aligned}$$

Let

$$0 \to B \otimes B^{!*}_p \to \cdots B \otimes B^{!*}_j \to \cdots \to B \otimes B^{!*}_1 \to B \ \to \Bbbk \to 0 \tag{2.4}$$

be the Koszul complex of B (cf. complex (1.19)). It is a projective resolution of $\Bbbk$. Each $B^{!*}_j$ is a left $\Bbbk\Gamma$-module defined by

$$\begin{aligned} [g(\beta)](x^*_{\beta_{i_1}} \wedge \cdots \wedge x^*_{\beta_{i_j}}) &= \beta(g^{-1}(x^*_{\beta_{i_1}} \wedge \cdots \wedge x^*_{\beta_{i_j}})) \\ &= \beta(g^{-1}(x^*_{\beta_{i_1}}) \wedge \cdots \wedge g^{-1}(x^*_{\beta_{i_j}})) \\ &= \textstyle\prod_{t=1}^j \chi_{\beta_{i_t}}(g)\beta(x^*_{\beta_{i_1}} \wedge \cdots \wedge x^*_{\beta_{i_j}}). \end{aligned}$$

Thus, each $B \otimes B^{!*}_j$ is a $B\#\Bbbk\Gamma$-module defined by

$$(c\#g) \cdot (b \otimes \beta) = (c\#g)(b) \otimes g(\beta),$$

for any $b \otimes \beta \in B \otimes B^{!*}_j$ and $c\#g \in B\#\Bbbk\Gamma$. It is not difficult to see that the complex (2.4) is an exact sequence of $B\#\Bbbk\Gamma$ modules. Tensoring it with $\Bbbk\Gamma$, we obtain the following exact sequence of $B\#\Bbbk\Gamma$-modules

$$0 \to B \otimes B_p^{!*} \otimes \Bbbk\Gamma \to \cdots B \otimes B_j^{!*} \otimes \Bbbk\Gamma \to \cdots \to B \otimes B_1^{!*} \otimes \Bbbk\Gamma \to B \otimes \Bbbk\Gamma \to \Bbbk\Gamma \to 0,$$

where the Γ-action is diagonal. Each $B \otimes B_t^{!*} \otimes \Bbbk\Gamma$ is a free $B\#\Bbbk\Gamma$-module. Therefore, we obtain a projective resolution of $\Bbbk\Gamma$ over $B\#\Bbbk\Gamma$.

The complex

$$0 \to \mathrm{Hom}_C(B\otimes\Bbbk\Gamma, C) \to \mathrm{Hom}_C(B\otimes B_1^{!*}\otimes\Bbbk\Gamma, C) \to \cdots \to \mathrm{Hom}_C(B\otimes B_p^{!*}\otimes\Bbbk\Gamma, C) \to 0$$

is isomorphic to the following complex

$$0 \to C \to B_1^! \otimes C \to \cdots \to B_{p-1}^! \otimes C \xrightarrow{\delta_p} B_p^! \otimes C \to 0.$$

This complex is exact except at $B_p^! \otimes C$, whose cohomology is isomorphic to $B_p^! \otimes \Bbbk\Gamma$. So $\mathrm{RHom}_C(\Bbbk\Gamma, C) \cong B_p^! \otimes \Bbbk\Gamma[p]$. We have

$$(x^*_{\beta_1} \wedge \cdots \wedge x^*_{\beta_p}) \otimes g = (\prod_{i=1}^{p} \chi_{\beta_i})(g) g((x^*_{\beta_1} \wedge \cdots \wedge x^*_{\beta_p}) \otimes 1),$$

for all $g \in \Gamma$. The group Γ is a free abelian group of rank s, so $\mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \Bbbk\Gamma) \cong \Bbbk[s]$. Therefore, we obtain that

$$\mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \Bbbk\Gamma) \otimes^L_{\Bbbk\Gamma} \mathrm{RHom}_C(\Bbbk\Gamma, C) \cong \Bbbk_{\xi'}[p+s],$$

where ξ' is defined by $\xi'(g) = (\prod_{i=1}^{p} \chi_{\beta_i})(g)$ for all $g \in \Gamma$ and $\xi'(x_{\beta_j}) = 0$ for all $1 \leqslant j \leqslant p$. That is,

$$\mathrm{Ext}^i_C({}_C\Bbbk, {}_CC) \cong \begin{cases} 0, & i \neq p+s; \\ \Bbbk_{\xi'}, & i = p+s. \end{cases}$$

$\mathrm{Ext}^{p+s}_A({}_A\Bbbk, {}_AA)$ is a 1-dimensional right A-module. Let m be a basis of the module $\mathrm{Ext}^{p+s}_A({}_A\Bbbk, {}_AA)$. It follows from the right version of [63, lemma 2.13 (1)] that $m \cdot x_i = 0$ for all $1 \leqslant i \leqslant \theta$. Since $\mathrm{Ext}^{p+s}_C({}_C\Bbbk, {}_CC) \cong \mathrm{Ext}^{p+s}_A({}_A\Bbbk, {}_AA)$ as right Γ-modules, we have showed that

$$\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \cong \begin{cases} 0, & i \neq p+s; \\ \Bbbk_{\xi}, & i = p+s. \end{cases}$$

Similarly, we have

$$\dim \operatorname{Ext}^i_A(\Bbbk_A, A_A) = \begin{cases} 0, & i \neq p+s; \\ 1, & i = p+s. \end{cases}$$

By Lemma 2.1.3, the algebra A is AS-regular of global dimension $p+s$. □

Now we can give a necessary and sufficient condition for a pointed Hopf algebra $U(\mathcal{D},\lambda)$ to be CY.

Theorem 2.1.5. *Let $\mathcal{D}$ be a generic datum of finite Cartan type for a group Γ, λ a family of linking parameters for $\mathcal{D}$ and A the Hopf algebra $U(\mathcal{D},\lambda)$.*

(1) The rigid dualizing complex of the Hopf algebra $A = U(\mathcal{D},\lambda)$ is ${}_\psi A[p+s]$, where ψ is defined by $\psi(x_k) = \prod_{i=1, i\neq j_k}^p \chi_{\beta_i}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$, and $\psi(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$ where each j_k, $1 \leqslant k \leqslant \theta$, is the integer such that $\beta_{j_k} = \alpha_k$.

(2) The algebra A is CY if and only if $\prod_{i=1}^p \chi_{\beta_i} = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism.

Proof. (1) By Proposition 1.5.21 and Theorem 2.1.4, the rigid dualizing complex of A is isomorphic to ${}_{[\xi]\mathcal{S}_A^2}A[p+s]$, where ξ is the algebra homomorphism defined in Theorem 2.1.4. It is not difficult to see that

$$([\xi]\mathcal{S}_A^2)(g) = (\prod_{i=1}^p \chi_{\beta_i})(g),$$

for all $g \in \Gamma$. For $1 \leqslant k \leqslant \theta$, we have $\Delta(x_k) = x_k \otimes 1 + g_k \otimes x_k$ and $\mathcal{S}_A^2(x_k) = \chi_k(g_k^{-1})x_k$. If j_k is the integer such that $\beta_{j_k} = \alpha_k$, then $\chi_{\beta_{j_k}}(g_k) = \chi_k(g_k)$. So

$$\begin{aligned} ([\xi]\mathcal{S}_A^2)(x_k) &= \chi_k(g_k^{-1})[\xi](x_k) \\ &= \chi_k(g_k^{-1}) \textstyle\prod_{i=1}^p \chi_{\beta_i}(g_k)(x_k) \\ &= \textstyle\prod_{i=1, i\neq j_k}^p \chi_{\beta_i}(g_k)(x_k). \end{aligned}$$

(2) follows from Theorem 2.1.4 and Theorem 1.5.22. □

Remark 2.1.6. From Theorem 2.1.5, we can see that for a pointed Hopf algebra $U(\mathcal{D},\lambda)$, it is CY if and only if its associated graded algebra $U(\mathcal{D},0)$ is CY.

Corollary 2.1.7. *Assume that $A = U(\mathcal{D}, \lambda)$. For every A-A-bimodule M, there are isomorphisms:*

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_{p+s-i}(A, {}_{\psi^{-1}}M), \quad 0 \leqslant i \leqslant p+s, \tag{2.5}$$

where ψ is the algebra automorphism defined in Theorem 2.1.5.

Proof. This follows from [20, Cor. 5.2] and Theorem 2.1.4. $\square$

2.2 Calabi-Yau pointed Hopf algebras $U(\mathcal{D},\lambda)$ of low dimensions

In this section, we assume that $\Bbbk = \mathbb{C}$. We shall classify CY pointed Hopf algebras $U(\mathcal{D},\lambda)$ of dimension less than 5, where $\mathcal{D}$ is a generic datum of finite Cartan type for a group Γ and λ is a family of liking parameters for $\mathcal{D}$.

Remark 2.2.1. Let $\mathcal{D}(\Gamma,(g_i),(\chi_i),(a_{ij}))$ be a generic datum of finite Cartan type. Then $\chi_i(g_i)$ are not to be roots of unity for $1 \leqslant i \leqslant \theta$. Hence, in the classification, we exclude the case where the group is trivial. If the group Γ in a datum $\mathcal{D}(\Gamma,(g_i),(\chi_i),(a_{ij}))$ is trivial, then the algebra $U(\mathcal{D},0)$ (in this case, $U(\mathcal{D},0)$ has no non-trivial lifting) is the universal enveloping algebra $U(\mathfrak{g})$, where the Lie algebra $\mathfrak{g}$ is generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

$$(\operatorname{ad} x_i)^{1-a_{ij}} x_j = 0,\ \ 1 \leqslant i,j \leqslant \theta,\ \ i \neq j.$$

We have $\operatorname{tr}(\operatorname{ad} x) = 0$ for all $x \in \mathfrak{g}$. Therefore, $U(\mathfrak{g})$ is CY by [35, Lemma 4.1]. We list those of dimension less than 5 in the following table.

Case	CY dimension	Cartan matrix	Lie algebra	
			bases	relations
1	1	A_1	x	
2	2	$A_1 \times A_1$	x, y	abelian Lie algebra
3	3	$A_1 \times A_1 \times A_1$	x, y, z	abelian Lie algebra
4	3	A_2	x, y, z	$[x,y]=z, [x,z]=[y,z]=0$
5	4	$A_1 \times \cdots \times A_1$	x, y, z, w	abelian Lie algebra
6	4	$A_1 \times A_2$	x, y, z, w	$[x,y]=z, [x,z]=[y,z]=0$ $[x,w]=[y,w]=[z,w]=0$
7	4	B_2	x, y, z, w	$[x,y]=z, [x,z]=w,$ $[x,w]=[y,z]$ $=[y,w]=[z,w]=0$

Remark 2.2.2. The Lie algebra in Case 4 is the Heisenberg algebra. In [35], the authors classified those 3-dimensional Lie algebras whose universal enveloping algebras

are CY algebras. Beside the algebras in Case 3 and Case 4, the other two Lie algebras are

- The 3-dimensional simple Lie algebra $\mathfrak{sl}_2$;
- The Lie algebra $\mathfrak{g}$, where $\mathfrak{g}$ has a basis {x,y,z} such that $[x,y]=y$, $[x,z]=-z$ and $[y,z]=0$.

Definition 2.2.3. Let

$$\mathcal{D}(\Gamma,(g_i)_{1\leqslant i\leqslant\theta},(\chi_i)_{1\leqslant i\leqslant\theta},(a_{ij})_{1\leqslant i,j\leqslant\theta})$$

and

$$\mathcal{D}'(\Gamma',(g'_i)_{1\leqslant i\leqslant\theta',},(\chi'_i)_{1\leqslant i\leqslant\theta'},(a'_{ij})_{1\leqslant i,j\leqslant\theta'})$$

be two generic data of finite Cartan type for groups Γ and Γ', where Γ and Γ' are both free abelian groups of finite rank. Let λ and λ' be two families of linking parameters for $\mathcal{D}$ and $\mathcal{D}'$ respectively.

The data $(\mathcal{D},\lambda)$ and $(\mathcal{D}',\lambda')$ are said to be *isomorphic* if $\theta=\theta'$, and if there exist a group isomorphism $\varphi:\Gamma\to\Gamma'$, a permutation $\sigma\in\mathbb{S}_\theta$, and elements $0\neq\alpha_i\in\Bbbk$, for all $1\leqslant i\leqslant\theta$ subject to the following relations:

(1) $\varphi(g_i)=g'_{\sigma(i)}$, for all $1\leqslant i\leqslant\theta$.

(2) $\chi_i=\chi'_{\sigma(i)}\varphi$, for all $1\leqslant i\leqslant\theta$.

(3) $\lambda_{ij}=\begin{cases}\alpha_i\alpha_j\lambda'_{\sigma(i)\sigma(j)}, & \text{if } \sigma(i)<\sigma(j)\\ -\alpha_i\alpha_j\chi_j(g_i)\lambda'_{\sigma(j)\sigma(i)}, & \text{if } \sigma(i)>\sigma(j)\end{cases},$

for all $1\leqslant i<j\leqslant\theta$ and $i\nsim j$.

In this case the triple $(\varphi,\sigma,(\alpha_i))$ is called an isomorphism from $(\mathcal{D},\lambda)$ to $(\mathcal{D}',\lambda')$.

If $(\mathcal{D},\lambda)$ and $(\mathcal{D}',\lambda)$ are isomorphic, then we can deduce that $a_{ij}=a'_{\sigma(i)\sigma(j)}$ for all $1\leqslant i,j\leqslant\theta$ [7].

The following corollary can be immediately obtained from the definition of isomorphic data.

Corollary 2.2.4. *Let $\mathcal{D}$ be a generic datum of finite Cartan type formed by* $(\Gamma, (g_i), (\chi_i), (a_{ij}))$. *Assume that $\varphi : \Gamma \to \Gamma'$ is a group isomorphism and σ is a permutation in $\mathbb{S}_\theta$. Then $(\mathcal{D}, 0)$ is isomorphic to $(\mathcal{D}', 0)$, where $\mathcal{D}'$ is formed by* $(\Gamma', (\varphi(g_{\sigma^{-1}(i)})), (\chi_{\sigma^{-1}(i)}\varphi^{-1}), (a_{\sigma^{-1}(i)\sigma^{-1}(j)}))$.

Let $\mathcal{D}$ be a generic datum of finite Cartan type and λ a family of linking parameters for $\mathcal{D}$. In the rest of this chapter, we simply call $(\mathcal{D},\lambda)$ a generic datum of finite Cartan type. Following from [7], the pointed Hopf algebra $U(\mathcal{D},\lambda)$ is uniquely determined by datum $(\mathcal{D},\lambda)$. Let $Isom((\mathcal{D},\lambda),(\mathcal{D}',\lambda'))$ be the set of all isomorphisms from $(\mathcal{D},\lambda)$ to $(\mathcal{D}',\lambda')$. Let A, B be two Hopf algebras, we denote by $Isom(A,B)$ the set of all Hopf algebra isomorphisms from A to B.

Lemma 2.2.5. [7, Thm. 4.5] *Let $(\mathcal{D},\lambda)$ and $(\mathcal{D}',\lambda')$ be two generic data of finite Cartan type. Then the Hopf algebras $U(\mathcal{D},\lambda)$ and $U(\mathcal{D}',\lambda')$ are isomorphic if and only if $(\mathcal{D},\lambda)$ is isomorphic to $(\mathcal{D}',\lambda')$. More precisely, let $x_1,\cdots,x_\theta$ (resp. $x'_1,\cdots,x'_\theta$) be the simple root vectors in $U(\mathcal{D},\lambda)$ (resp. $U(\mathcal{D}',\lambda')$), and let $g_1,\cdots,g_\theta$ (resp. $g'_1,\cdots,g'_\theta$) be the group-like elements in $\mathcal{D}$ (resp. $\mathcal{D}'$). Then the map*

$$Isom(U(\mathcal{D},\lambda),U(\mathcal{D}',\lambda')) \to Isom((\mathcal{D},\lambda),(\mathcal{D}',\lambda')),$$

given by $\phi \mapsto (\varphi,\sigma,(\alpha_i))$, where $\varphi(g)=\phi(g)$, $\varphi(g_i)=g'_{\sigma(i)}$, $\phi(x_i)=\alpha_i x'_{\sigma(i)}$, for all $g\in\Gamma$, $1\leqslant i\leqslant\theta$, is bijective.

Let $(\mathcal{D},\lambda)$ be a generic datum of finite Cartan type. By Lemma 1.4.4, any vertex can be linkable to at most one vertex. That is, for any $1\leqslant i\leqslant\theta$, there is at most one $1\leqslant j\leqslant\theta$, such that $i<j$ and $\lambda_{ij}\neq 0$. Thus it is reasonable to set elements $\alpha_i\in\Bbbk$ as follows:

If there is an integer j, such that $i<j$ and $\lambda_{ij}\neq 0$, then set $\alpha_i=\lambda_{ij}$;
Otherwise set $\alpha_i=1$.

Define

$$\lambda'_{ij}=\begin{cases}1, & \lambda_{ij}\neq 0;\\ 0, & \lambda_{ij}=0.\end{cases}$$

Then $(\mathrm{id},\mathrm{id},(\alpha_i))$ is an isomorphism from $(\mathcal{D},\lambda)$ to $(\mathcal{D},\lambda')$. Therefore, we can assume that the family of linking parameters $(\lambda_{ij})_{1\leqslant i<j\leqslant\theta}$ are chosen from $\{0,1\}$.

The following lemma is well-known.

Lemma 2.2.6. *If Γ is a free abelian group of rank s, then the algebra $\Bbbk\Gamma$ is a CY algebra of dimension s.*

If Γ is a free abelian group of finite rank, we denote by $|\Gamma|$ the rank of Γ.

Proposition 2.2.7. *Let A be the algebra $U(\mathcal{D}, \lambda)$, where $(\mathcal{D}, \lambda)$ is a generic datum of finite Cartan type for a group Γ. Then*

(1) A is CY of dimension 1 if and only if $A = \Bbbk\mathbb{Z}$.

(2) A is CY of dimension 2 if and only if $A = \Bbbk\Gamma$, where Γ is a free abelian group of rank 2.

Proof. (1) is clear.

(2) It is sufficient to show that if A is CY of dimension 2, then A is the group algebra of a free abelian group of rank 2. By Theorem 2.1.4, if the global dimension of A is 2. Then the following possibilities arise:

(a) $|\Gamma| = 2$, $A = \Bbbk\Gamma$ is the group algebra of a free abelian group of rank 2;

(b) $|\Gamma| = 1$ and the Cartan matrix of A is of type A_1.

Let A be a pointed Hopf algebra of type (b) and let the datum

$$(\mathcal{D}, \lambda) = (\Gamma, (g_i), (\chi_i), (a_{ij}), (\lambda_{ij}))$$

be as follows

- $\Gamma = \langle y_1 \rangle \cong \mathbb{Z}$;
- $g_1 = y_1^k$, for some $k \in \mathbb{Z}$;
- $\chi_1 \in \widehat{\Gamma}$ is defined by $\chi_1(y_1) = q$, where q is not a root of unity;
- The Cartan matrix is of type A_1;
- $\lambda = 0$.

Observe that in this case, the linking parameter must be 0. In addition, there is only one root vector, that is, the simple root vector x_1. Since $q \neq 1$, we have $\chi_1 \neq \varepsilon$. So the algebra A is not CY by Theorem 2.1.5.

Therefore, if A is CY, then A is of type (a). Hence, the classification is complete. $\square$

Proposition 2.2.8. *Let A be the algebra $U(\mathcal{D},\lambda)$, where $(\mathcal{D},\lambda)$ is a generic datum of finite Cartan type for a group Γ. If A is CY of dimension 3, then the group Γ and the Cartan matrix (a_{ij}) are given by one of the following 2 cases.*

Case	$\lvert\Gamma\rvert$	*Cartan matrix*
1	*3*	*Trivial*
2	*1*	$A_1 \times A_1$

The non-isomorphic classes of CY algebras in each case are given as follows.

Case 1: The group algebra of a free abelian group of rank 3.

Case 2:

(I) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ is given as follows:

- $\Gamma = \langle y_1\rangle \cong \mathbb{Z}$;
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q$, *where $q \in \Bbbk$ is not a root of unity and $0 < |q| < 1$, and* $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 0$.

(II) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ is given as follows:

- $\Gamma = \langle y_1\rangle \cong \mathbb{Z}$;
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q$, *where $q \in \Bbbk$ is not a root of unity and $0 < |q| < 1$, and* $\chi_2 = \chi_1^{-1}$;

- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$*;*
- $\lambda_{12} = 1$.

Proof. By Remark 2.1.6, it is sufficient to discuss the graded case and consider the non-trivial liftings. We first show that the algebras listed in the proposition are all CY. Case 1 follows from Lemma 2.2.6. Now we discuss Case 2. The root system of the Cartan matrix of type $A_1 \times A_1$ has two simple roots, say α_1 and α_2. They are also the positive roots. First we have $\chi_1\chi_2 = \varepsilon$. Since $\mathcal{S}_A^2(x_i) = \chi_i(g_i^{-1})x_i$, $i = 1, 2$, $g_1 = g_2 = y_1^k$, we have $\mathcal{S}_A^2(x_i) = {y_1}^{-k}x_i{y_1}^k$ for $i = 1, 2$. It is easy to see that $\mathcal{S}_A^2(y_1) = y_1$. It follows that $\mathcal{S}_A^2$ is an inner automorphism. Thus the algebras in Case 2 are CY by Theorem 2.1.5.

Now we show that the classification is complete.

If A is of global dimension 3, then the following possibilities for the group Γ and the Cartan matrix (a_{ij}) arise:

(1) $|\Gamma| = 3$, A is the group algebra of a free abelian group of rank 3.

(2) $|\Gamma| = 2$ and the Cartan matrix of A is of type A_1.

(3) $|\Gamma| = 1$ and the Cartan matrix of A is of type $A_1 \times A_1$.

Similar to the case of global dimension 2, A can not be CY if A is of type (2).

Now, let A be a CY graded algebra of type (3). In this case, we have $\chi_2(g_1)\chi_1(g_2) = 1$ (cf. equation (1.6)). In addition, we have $\chi_1\chi_2 = \varepsilon$ by Theorem 2.1.5. It follows that $1 = \chi_2(g_1)\chi_1(g_2) = \chi_1^{-1}(g_1)\chi_1(g_2)$. Let $\Gamma = \langle y_1 \rangle$ and $g_1 = y_1^k$, $g_2 = y_1^l$ for some $k, l \in \mathbb{Z}$. Then $\chi_1(y_1^{l-k}) = 1$. Since $\chi_1(y_1)$ is not a root of unity, we have $k = l$, that is, $g_1 = g_2 = y_1^k$. Therefore, $A \cong U(\mathcal{D}, 0)$, where the datum $\mathcal{D}$ is given by

- $\Gamma = \langle y_1 \rangle \cong \mathbb{Z}$;
- $g_1 = g_2 = y_1^k$, for some $k \in \mathbb{Z}$;
- $\chi_1(y_1) = q$, where $q \in \Bbbk$ is not a root of unity, and $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ is the Cartan matrix of type $A_1 \times A_1$.

Let $\mathcal{D}'$ be another datum given by

- $\Gamma' = \langle y_1' \rangle \cong \mathbb{Z}$;
- $g_1' = g_2' = y_1'^{k'}$, for some $k' \in \mathbb{Z}$;
- $\chi_1'(y_1') = q'$, where $q' \in \Bbbk$ is not a root of unity, and $\chi_2' = \chi_1'^{-1}$;
- $(a'_{ij})_{1\leqslant i,j\leqslant 2}$ is the Cartan matrix of type $A_1 \times A_1$.

Assume that $(\mathcal{D}', 0)$ is isomorphic to $(\mathcal{D}, 0)$ via an isomorphism $(\varphi, \sigma, (\alpha_i))$. Then φ is a group automorphism such that $\varphi(y_1) = y_1'$ or $\varphi(y_1) = y_1'^{-1}$. Since $\sigma \in \mathbb{S}_2$, we have $\sigma = \mathrm{id}$ or $\sigma = (12)$. From an easy computation, there are four possibilities for k' and q',

- $k' = k$ and $q' = q$;
- $k' = -k$ and $q' = q$;
- $k' = k$ and $q' = q^{-1}$;
- $k' = -k$ and $q' = q^{-1}$.

This shows that $A = U(\mathcal{D}, 0)$ is isomorphic to an algebra in (I) of Case 2. In addition, every pair $(k, q) \in \mathbb{Z}^+ \times \Bbbk$, such that $0 < |q| < 1$ determines a non-isomorphic algebra in (I) of Case 2. Each algebra in (I) of Case 2 has only one non-trivial lifting, which is isomorphic to an algebra in (II).

Thus we have completed the classification. □

We list all CY Hopf algebras $U(\mathcal{D},\lambda)$ of dimension 3 in terms of generators and relations in the following table. Note that in each case q is not a root of unity.

Table 4.1: CY algebras of dimension 3

Case	Generators	Relations
Case 1	y_h, y_h^{-1}	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$
	$1 \leqslant h \leqslant 3$	$y_h^{\pm 1} y_h^{\mp 1} = 1$
		$1 \leqslant h, m \leqslant 3$
Case 2 (I)	$y_1^{\pm 1}, x_1, x_2$	$y_1 y_1^{-1} = y_1^{-1} y_1 = 1$
		$y_1 x_1 = q x_1 y_1$
		$y_1 x_2 = q^{-1} x_2 y_1$, $0 < \lvert q \rvert < 1$
		$x_1 x_2 - q^{-k} x_2 x_1 = 0$, $k \in \mathbb{Z}^+$

Case 2 (II)	$y_1^{\pm 1}, x_1, x_2$	$y_1y_1^{-1} = y_1^{-1}y_1 = 1$ $y_1x_1 = qx_1y_1$ $y_1x_2 = q^{-1}x_2y_1$, $0 < \|q\| < 1$ $x_1x_2 - q^{-k}x_2x_1 = (1 - y_1^{2k})$, $k \in \mathbb{Z}^+$

Proposition 2.2.9. *Let A be the algebra $U(\mathcal{D},\lambda)$, where $(\mathcal{D},\lambda)$ is a generic datum of finite Cartan type for a group Γ. If A is CY of dimension 4, then the group Γ and the Cartan matrix (a_{ij}) are given by one of the following 2 cases.*

Case	$\|\Gamma\|$	*Cartan matrix*
1	*4*	*Trivial*
2	*2*	$A_1 \times A_1$

In each case, the non-isomorphic classes of CY algebras are given as follows.

Case 1: The group algebra of a free abelian group of rank 4.

Case 2:

(I) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$*;*
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$*;*
- – $\chi_1(y_1) = q_1$, $\chi_1(y_2) = q_2$, *where* $q_1, q_2 \in \Bbbk$ *satisfies that* $0 < |q_1| < 1$ *and* q_1 *is not a root of unity,*
  – $\chi_2 = \chi_1^{-1}$*;*
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$*;*
- $\lambda_{12} = 0$.

(II) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$*;*
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$*;*
- – $\chi_1(y_1) = q_1$, $\chi_1(y_2) = q_2$, *where* $q_1, q_2 \in \Bbbk$ *satisfies that* $0 < |q_1| < 1$ *and* q_1 *is not a root of unity,*

  - $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 1$.

Let A *and* B *be two algebras in Case (I) (or (II)) defined by triples* (k, q_1, q_2) *and* (k', q'_1, q'_2) *respectively. They are isomorphic if and only if* $k = k'$, $q_1 = q'_1$ *and there is some integer* b, *such that* $q'_2 = q_1^b q_2$ *or* $q'_2 = q_1^b q_2^{-1}$.

(III) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_2^l$ *for some* $k, l \in \mathbb{Z}^+$;
-   - $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k}{l}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$,
    - $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 0$.

(IV) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_2^l$ *for some* $k, l \in \mathbb{Z}^+$;
-   - $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k}{l}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$,
    - $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 1$.

(V) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_1^{l_1} y_2^{l_2}$ *for some* $k, , l_1, l_2 \in \mathbb{Z}^+$, $k \neq l_1$, $0 < l_1 < l_2$;
-   - $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k-l_1}{l_2}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$,
    - $\chi_2 = \chi_1^{-1}$;

- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$*;*
- $\lambda_{12} = 0$.

(VI) The datum $(\mathcal{D},\lambda) = (\Gamma,(g_1,g_2),(\chi_1,\chi_2),(a_{ij})_{1\leqslant i,j\leqslant 2},\lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$*;*
- $g_1 = y_1^k$, $g_2 = y_1^{l_1}y_2^{l_2}$ *for some* $k, l_1, l_2 \in \mathbb{Z}^+$, $k \neq l_1$ *and* $0 < l_1 < l_2$*;*
- – $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k-l_1}{l_2}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$,
  – $\chi_2 = \chi_1^{-1}$*;*
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$*;*
- $\lambda_{12} = 1$.

Proof. We first show that the algebras listed in the proposition are all CY. That the algebra in Case 1 is a CY algebra follows from Lemma 2.2.6. In Case 2, we have $\chi_1\chi_2 = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism in each subcase. Indeed, $\mathcal{S}_A^2(x_i) = g_1^{-1}x_ig_1$ and $\mathcal{S}_A^2(y_i) = g_1^{-1}y_ig_1 = y_i$, $i = 1, 2$. Thus the algebras in Case 2 are CY by Theorem 2.1.5.

Now we show that the classification is complete and the algebras on the list are non-isomorphic to each other.

If A is of global dimension 4, then the group Γ and the Cartan matrix (a_{ij}) must be one of the following types:

(1) $|\Gamma| = 4$ and A is the group algebra of a free abelian group of rank 4.

(2) $|\Gamma| = 3$ and the Cartan matrix of A is of type A_1.

(3) $|\Gamma| = 2$ and the Cartan matrix of A is of type $A_1 \times A_1$.

(4) $|\Gamma| = 1$ and the Cartan matrix of A is of type $A_1 \times A_1 \times A_1$.

(5) $|\Gamma| = 1$ and the Cartan matrix of A is of type A_2.

Let A be a CY algebra of dimension 4. Similar to the case of global dimension 2, A cannot be of type (2). We claim that A cannot be of type (4) and (5) either.

Assume that A is of type (4), put $\Gamma = \langle y_1\rangle$, $g_i = y_1^{m_i}$ for some $0 \neq m_i \in \mathbb{Z}$ and $\chi_i(y_1) = q_i$ for some $q_i \in \Bbbk$, $1 \leqslant i \leqslant 3$. Then $q_{ij} = q_j^{m_i}$, for $1 \leqslant i,j \leqslant 3$. Because each q_{ii} is not a root of unity, each q_i is not a root of unity either. Since $q_{ij}q_{ji}=1$, we have

$$q_1^{m_2}q_2^{m_1}=1,\ \ q_1^{m_3}q_3^{m_1}=1,\ \ q_2^{m_3}q_3^{m_2}=1.$$

Then $q_1^{2m_2m_3}=1$. But q_1 is not a root of unity. So A can not be of type (4).

In case (5), there are 3 positive roots in the root system. They are α_1, α_2 and $\alpha_1+\alpha_2$, where α_1 and α_2 are the simple roots. If A is CY, then $\chi_1^2\chi_2^2=\varepsilon$ by Theorem 2.1.5. So we have $q_{11}^2q_{21}^2=1$ and $q_{12}^2q_{22}^2=1$. However, $q_{21}q_{12}=q_{11}^{-1}$ (equation (1.7)). Thus $q_{22}^2=1$. But q_{22} is not a root of unity. So A cannot be of type (5) either.

Now to show that the classification is complete, we only need to show that if A is a CY pointed Hopf algebra of type (3), then A is isomorphic to an algebra in Case 2. Each algebra in (I), (III) and (V) of Case 2 has only one non-trivial lifting, which is isomorphic to an algebra in (II), (IV) and (VI) respectively. By Remark 2.1.6, it suffices to show that if A is a graded CY pointed Hopf algebra of type (3), then A is isomorphic to an algebra in (I), (III) and (V) of Case 2.

Let $\Gamma=\langle y_1,y_2\rangle$ be a free abelian group of rank 2. We write $\chi_1(y_1)=q_1$, $\chi_1(y_2)=q_2$ and $g_1=y_1^{k_1}y_2^{k_2}$, $g_2=y_1^{l_1}y_2^{l_2}$, where $\chi_1(g_1)=q_1^{k_1}q_2^{k_2}$ is not a root of unity, and $k_1,k_2,l_1,l_2\in\mathbb{Z}$. Following Theorem 2.1.5, we have $\chi_1\chi_2=\varepsilon$. So $q_{21}=q_1^{l_1}q_2^{l_2}$ and $q_{12}=q_1^{-k_1}q_2^{-k_2}$. We also have $q_{12}q_{21}=1$ (equation (1.7)). Thus $q_1^{l_1-k_1}q_2^{l_2-k_2}=1$. Therefore, $A\cong U(\mathcal{D},0)$, where the datum $\mathcal{D}$ is formed by

- $\Gamma=\langle y_1,y_2\rangle\cong\mathbb{Z}^2$;
- (a_{ij}) is the Cartan matrix of type $A_1\times A_1$;
- $g_1=y_1^{k_1}y_2^{k_2}$, $g_2=y_1^{l_1}y_2^{l_2}$, $k_1,k_2,l_1,l_2\in\mathbb{Z}$;
- $\chi_1(y_1)=q_1$, $\chi_1(y_2)=q_2$, where $\chi_1(g_1)=q_1^{k_1}q_2^{k_2}$ is not a root of unity and $q_1^{l_1-k_1}q_2^{l_2-k_2}=1$, and $\chi_2=\chi_1^{-1}$.

In the above datum $\mathcal{D}$, we may assume that $k_1>0$ and $k_2=0$. Then q_1 is not a root of unity. We show that there is a group isomorphism $\varphi:\Gamma\to\Gamma'$, where $\Gamma'=\langle y_1',y_2'\rangle$ is also a free abelian group of rank 2, such that $\varphi(y_1^{k_1}y_2^{k_2})=y_1'^k$ and $k>0$.

The integers k_1 and k_2 can not be both equal to 0. If $k_2=0$ and $k_1>0$, then it is done. If $k_2=0$ and $k_1<0$, then $\varphi(y_1)=y_1'^{-1}$ and $\varphi(y_2)=y_2'^{-1}$ defines a desired isomorphism.

Similarly, we can obtain a desired isomorphism when $k_1 = 0$ and $k_2 \neq 0$.

If $k_1, k_2 \neq 0$, then there are some $k, \bar{k}_1, \bar{k}_2 \in \mathbb{Z}$, such that $k_1 = \bar{k}_1 k$, $k_2 = \bar{k}_2 k$, $k > 0$ and $(\bar{k}_1, \bar{k}_2) = 1$, that is, $\bar{k}_1$ and $\bar{k}_2$ have no common divisors. We can find integers a, b such that $a\bar{k}_1 + b\bar{k}_2 = 1$. Let $\varphi : \Gamma \to \Gamma'$ be the group isomorphism defined by $\varphi(y_1) = {y'_1}^a {y'_2}^{-\bar{k}_2}$ and $\varphi(y_2) = {y'_1}^b {y'_2}^{\bar{k}_1}$. Then $\varphi(y_1^{k_1} y_2^{k_2}) = {y'_1}^k$ and $k > 0$. In conclusion, we have proved the claim.

If $l_2 = 0$, then we have $q_1^{l_1 - k_1} = 1$. Since q_1 is not a root of unity, we have $l_1 = k_1$. Applying a similar argument to the one in Case 2 of Proposition 2.2.8, we find that A is isomorphic to an algebra in (I) of Case 2.

Next, we consider the case when $l_2 \neq 0$. In case $l_1 = 0$, like what we did for k_1 and k_2, we may assume that $l_2 > 0$. If $0 < |q_1| < 1$, then A is isomorphic to an algebra in (III) of Case 2. Otherwise, the datum $(\mathcal{D}, 0)$ is isomorphic to the datum given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g'_1 = y_1^{l_2}$, $g'_2 = y_2^{k_1}$, $k_1, l_2 \in \mathbb{Z}^+$;
- $\chi'_1(y_1) = q_1^{-\frac{k_1}{l_2}}$, $\chi'_1(y_2) = q_1^{-1}$, $\chi'_2 = {\chi'_1}^{-1}$.
- (a_{ij}) is the Cartan matrix of type $A_1 \times A_1$;
- $\lambda_{12} = 0$

via the isomorphism $(\varphi, (12), \alpha_1 = \alpha_2 = 1)$, where φ is the algebra automorphism defined by $\varphi(y_1) = y_2$ and $\varphi(y_2) = y_1$. So A is isomorphic to an algebra in (III) of Case 2 as well.

If $l_1 \neq 0$ and $l_2 > 0$, then there is an integer c, such that $0 \leqslant l_1 + cl_2 < l_2$. Let $\Gamma' = \langle y'_1, y'_2 \rangle$ be a free abelian group of rank 2, and $\varphi : \Gamma \to \Gamma'$ the group isomorphism defined by $\varphi(y_1) = y'_1$ and $\varphi(y_2) = {y'_1}^c y'_2$. Then $\varphi(y_1^{k_1}) = {y'_1}^{k_1}$ and $\varphi(y_1^{l_1} y_2^{l_2}) = {y'_1}^{l_1 + cl_2} {y'_2}^{l_2}$.

If $l_1 \neq 0$ and $l_2 < 0$, then there are integers $\bar{l}_1$, $\bar{l}_2$, such that $l_1 = \bar{l}_1 l$, $l_2 = \bar{l}_2 l$, $l > 0$ and $(\bar{l}_1, \bar{l}_2) = 1$. So $\bar{l}_2 < 0$. We can find integers a, b such that $a\bar{l}_1 + b\bar{l}_2 = 1$. Since for any integer d, $(a + d\bar{l}_2)\bar{l}_1 + (b - d\bar{l}_1)\bar{l}_2 = a\bar{l}_1 + b\bar{l}_2 = 1$, we may assume that $0 \leqslant a < -\bar{l}_2$. Let $\Gamma' = \langle y'_1, y'_2 \rangle$ be a free abelian group of rank 2, and $\varphi : \Gamma \to \Gamma'$ be the group isomorphism defined by $\varphi(y_1) = {y'_1}^a {y'_2}^{-\bar{l}_2}$ and $\varphi(y_2) = {y'_1}^b {y'_2}^{\bar{l}_1}$. Then $\varphi(y_1^{k_1}) = {y'_1}^{ak_1} {y'_2}^{-\bar{l}_2 k_1}$ and $\varphi(y_1^{l_1} y_2^{l_2}) = {y'_1}^l$.

In summary, by Corollary 2.2.4, we may assume that $l_2>0$ and $0\leqslant l_1<l_2$. If $l_1=0$, then we go back to the case we just discussed. If $l_1\neq 0$ and $0<|q_1|<1$, then A is isomorphic to an algebra in (V). Otherwise, $(\mathcal{D},0)$ is isomorphic to the datum given by given by

- $\Gamma=\langle y_1,y_2\rangle\cong\mathbb{Z}^2$;
- $g_1'=y_1^l, g_2'=y_1^{ak_1}y_2^{k_1\bar{l}_2}$. $\bar{l}_1,\bar{l}_2\in\mathbb{Z}^+$ are the integers such that $l\bar{l}_1=l_1$, $l\bar{l}_2=l_2$, and $(\bar{l}_1,\bar{l}_2)=1$. $a,b\in\mathbb{Z}$ are the integers such that $a\bar{l}_1+b\bar{l}_2=1$ and $0<a<\bar{l}_2$.
- $\chi_1'(y_1)=q_1^{-\frac{k_1\bar{l}_2}{l_2}}$, $\chi_1'(y_2)=q_1^{\frac{ak_1-l}{l_2}}$, $\chi_2'=\chi_1'^{-1}$.
- (a_{ij}) is the Cartan matrix of type $A_1\times A_1$;
- $\lambda_{12}=0$

via the isomorphism $(\varphi,(12),\alpha_1=\alpha_2=1)$, where φ is the isomorphism defined by $\varphi(y_1)=y_1^a y_2^{\bar{l}_2}$ and $\varphi(y_2)=y_1^b y_2^{-\bar{l}_1}$. It follows that A is isomorphic to an algebra in (V) as well.

It is clear that the algebras from different cases and subcases are non-isomorphic to each other. It is sufficient to show that the algebras in the same subcases in Case 2 are non-isomorphic. Each algebra in (II), (IV) and (VI) is a lifting of an algebra in (I), (III) and (V) respectively. So it is sufficient to discuss the cases (I), (III) and (V).

First we discuss the case (I). Let $\mathcal{D}$ and $\mathcal{D}'$ be two data given by

- $\Gamma=\langle y_1,y_2\rangle\cong\mathbb{Z}^2$;
- $g_1=g_2=y_1^k$ for some $k\in\mathbb{Z}^+$;
- $\chi_1(y_1)=q_1$, $\chi_1(y_2)=q_2$, where $q_1,q_2\in\Bbbk$ satisfies that $0<|q_1|<1$ and q_1 is not a root of unity, and $\chi_2=\chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ is the Cartan matrix of type $A_1\times A_1$

and

- $\Gamma=\langle y_1',y_2'\rangle\cong\mathbb{Z}^2$;

- $g_1 = g_2 = y_1^{k'}$ for some $k' \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q'_1$, $\chi_1(y_2) = q'_2$, where $q'_1, q'_2 \in \Bbbk$ satisfies that $0 < |q'_1| < 1$ and q'_1 is not a root of unity, and $\chi_2 = \chi_1^{-1}$;
- $(a'_{ij})_{1\leqslant i,j\leqslant 2}$ is the Cartan matrix of type $A_1 \times A_1$

respectively. Assume that $(\varphi, \sigma, \alpha)$ is an isomorphism from $(\mathcal{D}, 0)$ to $(\mathcal{D}', 0)$. Say $\varphi(y_1) = {y'_1}^a {y'_2}^c$ and $\varphi(y_2) = {y'_1}^b {y'_2}^d$. Since $g_1 = g_2$ and $g'_1 = g'_2$, we have $\varphi(y_1^k) = {y'_1}^{k'}$. Moreover, $k, k' > 0$. So $a = 1$, $c = 0$ and $d = \pm 1$. Consequently, we have $k = k'$, $q_1 = q'_1$. If $\sigma = \mathrm{id}$, then $q'_2 = q_1^{-b} q_2$. Otherwise, $q'_2 = q_1^b q_2^{-1}$. We have identified the isomorphic algebras in (I).

Similarly, it is direct to show that each triple $(k, l, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \Bbbk$ such that $0 < |q| < 1$ determines a non-isomorphic algebra in (III).

Now we show that the algebras in (V) are non-isomorphic. Let $\mathcal{D}$ and $\mathcal{D}'$ be the data given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_1^{l_1} y_2^{l_2}$ such that $k, l_1, l_2 \in \mathbb{Z}^+$ and $0 < l_1 < l_2$;
- $\chi_1(y_1) = q$, where $q \in \Bbbk$ is not a root of unity, $0 < |q| < 1$, and $\chi_1(y_2) = q^{\frac{k-l_1}{l_2}}$ and $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$, the Cartan matrix of type $A_1 \times A_1$

and

- $\Gamma' = \langle y'_1, y'_2 \rangle$ is also a free abelian group of rank 2;
- $g'_1 = {y'_1}^{k'}$, $g'_2 = {y'_1}^{l'_1} {y'_2}^{l'_2}$ such that $k', l'_1, l'_2 \in \mathbb{Z}^+$ and $0 < l'_1 < l'_2$;
- $\chi'_1(y'_1) = q'$, where $q' \in \Bbbk$ is not a root of unity, $0 < |q'| < 1$, and $\chi'_1(y'_2) = {q'}^{\frac{k'-l'_1}{l'_2}}$ and $\chi'_2 = {\chi'_1}^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$, the Cartan matrix of type $A_1 \times A_1$

respectively. We claim that $(\mathcal{D}, 0)$ and $(\mathcal{D}', 0)$ are isomorphic if and only if $q = q'$, $k = k'$, $l_1 = l'_1$ and $l_2 = l'_2$.

Assume that $(\mathcal{D},0)$ is isomorphic to $(\mathcal{D}',0)$ via an isomorphism $(\varphi,\sigma,\alpha_1=\alpha_2=1)$. Suppose that $\varphi(y_1)={y_1'}^a{y_2'}^c$ and $\varphi(y_2)={y_1'}^b{y_2'}^d$, with $a,b,c,d\in\mathbb{Z}$.

Either $\sigma=\mathrm{id}$ or $\sigma=(12)$. If $\sigma=\mathrm{id}$, then $\varphi(g_i)=g_i'$, $i=1,2$. So

$$ {y_1'}^{ak}{y_2'}^{ck}={y_1'}^{k'} \text{ and } {y_1'}^{al_1+bl_2}{y_2'}^{cl_1+dl_2}={y_1'}^{l_1'}{y_2'}^{l_2'}. $$

Since φ is an isomorphism, we have $ad-bc=\pm1$. Because, $k,k',l_2,l_2'>0$, $0<l_1<l_2$ and $0<l_1'<l_2'$, it follows that $b=c=0$ and $a=d=1$. Therefore, $k=k'$, $l_1=l_1'$, $l_2=l_2'$, and $q=q'$. Namely, $(\mathcal{D},0)=(\mathcal{D}',0)$

If $\sigma=(12)$, then $\varphi(g_i)=g_{3-i}'$, $i=1,2$. This implies that

$$ {y_1'}^{ak}{y_2'}^{ck}={y_1'}^{l_1'}{y_2'}^{l_2'} \text{ and } {y_1'}^{al_1+bl_2}{y_2'}^{cl_1+dl_2}={y_1'}^{k'}. $$

We can find integers $\bar{l}_1$ and $\bar{l}_2$, such that $l_1=\bar{l}_1l$, $l_2=\bar{l}_2l$, $l>0$ and $(\bar{l}_1,\bar{l}_2)=1$.

Since $ad-bc=\pm1$, we have $(c,d)=1$. From $ck=l_2'>0$ and $cl_1+dl_2=0$, it follows that $c=\bar{l}_2$ and $d=-\bar{l}_1$. If $ad-bc=1$, we have

$$ k'=al_1+bl_2=l(a\bar{l}_1+b\bar{l}_2)=-l(ad-bc)=-l<0, $$

a contradiction!

If $ad-bc=-1$, we have

$$ q'=\chi_1'(y_1')=\chi_2\varphi^{-1}(y_1)=\chi_2(y_1^{\bar{l}_1}y_2^{\bar{l}_2})=q^{-\bar{l}_2\frac{k}{l_2}}. $$

But $\bar{l}_2,k,l_2>0$ and $0<|q|,|q'|<1$. We get a contraction as well. In summary, we have proved the claim. □

Now we list all pointed CY Hopf algebras $U(\mathcal{D},\lambda)$ of dimension 4 in terms of generators and relations in the following table. Note that q_1 and q are not roots of unity.

Table 4.2: CY algebras of dimension 4

Case	Generators	Relations
Case 1	y_h, y_h^{-1}	$y_h^{\pm1}y_m^{\pm1}=y_m^{\pm1}y_h^{\pm1}$
	$1\leqslant h\leqslant 4$	$y_h^{\pm1}y_h^{\mp1}=1$
		$1\leqslant h,m\leqslant 4$

Case 2 (I)	$y_1^{\pm1}, y_2^{\pm1}, x_1, x_2$	$y_h^{\pm1}y_m^{\pm1} = y_m^{\pm1}y_h^{\pm1}$ $y_h^{\pm1}y_h^{\mp1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1x_1 = q_1x_1y_1,\ y_1x_2 = q_1^{-1}x_2y_1$ $y_2x_1 = q_2x_1y_2,\ y_2x_2 = q_2^{-1}x_2y_2$ $0 < \lvert q_1\rvert < 1$ $x_1x_2 - q_1^{-k}x_2x_1 = 0,\ k \in \mathbb{Z}^+$
Case 2 (II)	$y_1^{\pm1}, y_2^{\pm1}, x_1, x_2$	$y_h^{\pm1}y_m^{\pm1} = y_m^{\pm1}y_h^{\pm1}$ $y_h^{\pm1}y_h^{\mp1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1x_1 = q_1x_1y_1,\ y_1x_2 = q_1^{-1}x_2y_1$ $y_2x_1 = q_2x_1y_2,\ y_2x_2 = q_2^{-1}x_2y_2$ $0 < \lvert q_1\rvert < 1$ $x_1x_2 - q_1^{-k}x_2x_1 = 1 - y_1^{2k},\ k \in \mathbb{Z}^+$
Case 2 (III)	$y_1^{\pm1}, y_2^{\pm1}, x_1, x_2$	$y_h^{\pm1}y_m^{\pm1} = y_m^{\pm1}y_h^{\pm1}$ $y_h^{\pm1}y_h^{\mp1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1x_1 = qx_1y_1,\ y_1x_2 = q^{-1}x_2y_1$ $y_2x_1 = q^{\frac{k}{l}}x_1y_2,\ y_2x_2 = q^{-\frac{k}{l}}x_2y_2$ $x_1x_2 - q^{-k}x_2x_1 = 0$ $k, l \in \mathbb{Z}^+,\ 0 < \lvert q\rvert < 1$
Case 2 (IV)	$y_1^{\pm1}, y_2^{\pm1}, x_1, x_2$	$y_h^{\pm1}y_m^{\pm1} = y_m^{\pm1}y_h^{\pm1}$ $y_h^{\pm1}y_h^{\mp1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1x_1 = qx_1y_1,\ y_1x_2 = q^{-1}x_2y_1$ $y_2x_1 = q^{\frac{k}{l}}x_1y_2,\ y_2x_2 = q^{-\frac{k}{l}}x_2y_2$ $x_1x_2 - q^{-k}x_2x_1 = 1 - y_1^ky_2^l$ $k, l \in \mathbb{Z}^+,\ 0 < \lvert q\rvert < 1$
Case 2 (V)	$y_1^{\pm1}, y_2^{\pm1}, x_1, x_2$	$y_h^{\pm1}y_m^{\pm1} = y_m^{\pm1}y_h^{\pm1}$ $y_h^{\pm1}y_h^{\mp1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1x_1 = qx_1y_1,\ y_1x_2 = q^{-1}x_2y_1$ $y_2x_1 = q^{\frac{k-l_1}{l_2}}x_1y_2,\ y_2x_2 = q^{-\frac{k-l_1}{l_2}}x_2y_2$ $x_1x_2 - q^{-k}x_2x_1 = 0$ $k, l_1, l_2 \in \mathbb{Z}^+,\ 0 < l_1 < l_2,\ 0 < \lvert q\rvert < 1$

Case 2 (VI)	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1 x_1 = q x_1 y_1,\ y_1 x_2 = q^{-1} x_2 y_1$ $y_2 x_1 = q^{\frac{k-l_1}{l_2}} x_1 y_2,\ y_2 x_2 = q^{-\frac{k-l_1}{l_2}} x_2 y_2$ $x_1 x_2 - q^{-k} x_2 x_1 = 1 - y_1^{k+l_1} y_2^{l_2}$ $k, l_1, l_2 \in \mathbb{Z}^+,\ 0 < l_1 < l_2,\ 0 < \lvert q \rvert < 1$

Let $\mathfrak{g}$ be a semisimple Lie algebra and $U_q(\mathfrak{g})$ its quantized enveloping algebra. By [20, Prop. 6.4], the global dimension of the algebra $U_q(\mathfrak{g})$ is the dimension of $\mathfrak{g}$. Thus, if $U_q(\mathfrak{g})$ is of global dimension less than 5, then $U_q(\mathfrak{g})$ is isomorphic to $U_q(\mathfrak{sl}_2)$, which is of global dimension 3. That is, among the algebras of the form $U_q(\mathfrak{g})$, only $U_q(\mathfrak{sl}_2)$ appears in the lists of Propositions 2.2.7, 2.2.8 and 2.2.9. The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to $U(\mathcal{D}, \lambda)$ with the datum given by

- $\Gamma = \langle g \rangle$, a free abelian group of rank 1;
- The Cartan matrix is of type $A_1 \times A_1$;
- $g_1 = g_2 = g$;
- $\chi_1(g) = q^{-2}$, $\chi_2(g) = q^2$, where q is not a root of unity;
- $\lambda_{12} = 1$.

It belongs to (II) of Case 2 of Proposition 2.2.8.

The family of pointed Hopf algebras $U(\mathcal{D}, \lambda)$ provide more examples of CY algebras of higher dimensions. From the classification of CY pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimensions less than 5, we see that the Cartan matrices are either trivial or of type $A_1 \times \cdots \times A_1$. The following example provides a CY pointed Hopf algebra of type $A_2 \times A_1$ of dimension 7.

Example 2.2.10. Let A be $U(\mathcal{D}, \lambda)$ with the datum $(\mathcal{D}, \lambda)$ given by

- $\Gamma = \langle y_1, y_2, y_3 \rangle$, a free abelian group of rank 3;

- The Cartan matrix is
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

- $g_i = y_i$, $1 \leqslant i \leqslant 3$;

- χ_i, $1 \leqslant i \leqslant 3$, are given by the following table, where q is not a root of unity.

	y_1	y_2	y_3
χ_1	q	q^{-2}	q^4
χ_2	q	q	q^{-2}
χ_3	q^{-4}	q^2	q^{-4}

- $\lambda = 0$

In other words, A is the algebra with generators x_i, $y_j^{\pm 1}$, $1 \leqslant i, j \leqslant 3$, subject to the relations
$$y_i^{\pm 1} y_j^{\pm 1} = y_j^{\pm 1} y_i^{\pm 1}, \quad y_j^{\pm 1} y_j^{\mp 1} = 1, \quad 1 \leqslant i, j \leqslant 3,$$
$$y_j(x_i) = \chi_i(y_j) x_i y_j, \quad 1 \leqslant i, j \leqslant 3,$$
$$x_1^2 x_2 - q x_1 x_2 x_1 - q^2 x_1 x_2 x_1 + q^3 x_2 x_1^2 = 0,$$
$$x_2^2 x_1 - q^{-2} x_2 x_1 x_2 - q^{-1} x_2 x_1 x_2 + q^{-3} x_1 x_2^2 = 0,$$
$$x_1 x_3 = x_3 x_1.$$

The non-trivial liftings of A are also CY.

Chapter 3

Calabi-Yau Nichols algebras of finite Cartan type

Let $\mathcal{D}$ be a generic datum of finite Cartan type, and λ a family of linking parameters. In Chapter 2, we calculate the rigid dualizing complex of the algebra $U(\mathcal{D}, \lambda)$. Based on this result, in Section 3.1, we give the rigid dualizing complex of the corresponding Nichols algebra $\mathcal{B}(V)$ and characterize its CY property. In Section 3.2, we give the relation between the CY property of the pointed Hopf algebra $U(\mathcal{D}, \lambda)$ and that of the Nichols algebra $\mathcal{B}(V)$.

3.1 Rigid dualizing complexes of Nichols algebras of finite Cartan type

In this section, we fix a generic datum of finite Cartan type

$$\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$$

for a group Γ, where Γ is a free abelian group of rank s. Let V be the generic braided vector space with basis $\{x_1, \cdots, x_\theta\}$ whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$

for all $1 \leqslant i, j \leqslant \theta$, where $q_{ij} = \chi_j(g_i)$. Recall that the Nichols algebra $\mathcal{B}(V)$ is generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

$$\mathrm{ad}_c(x_i)^{1-a_{ij}} x_j = 0, \ \ 1 \leqslant i, j \leqslant \theta, \ \ i \neq j,$$

where ad_c is the braided adjoint representation (1.2). Let $\{\alpha_1, \cdots, \alpha_\theta\}$ be a fixed set of simple roots of the root system corresponding to the Cartan matrix (a_{ij}), and $w_0 = s_{i_1} \cdots s_{i_p}$ a reduced decomposition of the longest element in the Weyl group $\mathcal{W}$ as a product of simple reflections. Then

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_p)$$

are the positive roots. Assume that $x_{\beta_1}, \cdots, x_{\beta_p}$ are the corresponding root vectors. For each $1 \leqslant k \leqslant \theta$, let $1 \leqslant j_k \leqslant p$ be the integer such that $\beta_{j_k} = \alpha_k$. Then we have $x_{\beta_{j_k}} = x_k$.

By Theorem 1.4.7, the Nichols algebra $\mathcal{B}(V)$ is a subalgebra of $U(\mathcal{D}, 0)$, and the following monomials in root vectors

$$\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} \mid a_i \geqslant 0, 1 \leqslant i \leqslant p\}$$

form a PBW basis of the Nichols algebra $\mathcal{B}(V)$. Recall that the degree (cf. (2.1)) of each PBW basis element is defined by

$$\deg(x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}) = (a_1, \cdots, a_p, \sum a_i ht(\beta_i)) \in (\mathbb{Z}^{\geqslant 0})^{p+1},$$

where $ht(\beta_i)$ is the height of β_i.

The following result is a direct consequence of Lemma 2.1.1.

Lemma 3.1.1. *In the Nichols algebra $\mathcal{B}(V)$, for $j > i$, we have*

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{\boldsymbol{a} \in \mathbb{N}^p} \rho_{\boldsymbol{a}} x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}, \tag{3.1}$$

where $\rho_{\boldsymbol{a}} \in \Bbbk$ and $\rho_{\boldsymbol{a}} \neq 0$ only when $\boldsymbol{a} = (a_1, \cdots, a_p)$ satisfies that $a_k = 0$ for $k \leqslant i$ and $k \geqslant j$.

Order the PBW basis elements by degree as in (2.2). By Lemma 3.1.1, we obtain the following corollary.

Corollary 3.1.2. *The Nichols algebra $\mathcal{B}(V)$ is an $\mathbb{N}^{p+1}$-filtered algebra, whose associated graded algebra $\mathbb{G}\mathrm{r}\mathcal{B}(V)$ is isomorphic to the following algebra:*

$$\Bbbk\langle x_{\beta_1}, \cdots, x_{\beta_p} \mid x_{\beta_i} x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i},\ \ 1 \leqslant i < j \leqslant p\rangle,$$

where $x_{\beta_1}, \cdots, x_{\beta_p}$ are the root vectors of $\mathcal{B}(V)$.

For elements $\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}\}$, where $a_1, \cdots, a_p \geqslant 0$, define

$$d_0(x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}) = \textstyle\sum_{i=1}^p a_i ht(\beta_i).$$

Then $R = \mathcal{B}(V)$ is a graded algebra with grading given by d_0. Let $R^{(0)} = R$. Define $d_1(x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}) = a_p$. We obtain an $\mathbb{N}$-filtration on $R^{(0)}$. Let $R^{(1)} = \mathrm{Gr}\, R^{(0)}$ be the associated graded algebra. In a similar way, we define $d_2(x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}) = a_{p-1}$ and let $R^{(2)} = \mathrm{Gr}\, R^{(1)}$ be the associated graded algebra. Inductively, we obtain a sequence of $\mathbb{N}$-filtered algebras $R^{(0)}, \cdots, R^{(p)}$, such that $R^{(i)} = \mathrm{Gr}\, R^{(i-1)}$, for $1 \leqslant i \leqslant p$, and $R^{(p)} = \mathbb{G}\mathrm{r}R$.

The algebra R^e has a PBW basis as follows

$$\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} \otimes x_{\beta_p}^{b_p} \star \cdots \star x_{\beta_1}^{b_1} \mid a_1, \cdots, a_p, b_1, \cdots, b_p \geqslant 0\}, \tag{3.2}$$

where "$\star$" denotes the multiplication in R^{op}. Similarly, define a degree on each element as

$$\begin{aligned} & \deg(x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} \otimes x_{\beta_p}^{b_p} \star \cdots \star x_{\beta_1}^{b_1}) \\ = \ & (a_1 + b_1, \cdots, a_p + b_p, \textstyle\sum (a_i + b_i) ht\beta_i) \in (\mathbb{Z}^{\geqslant 0})^{(p+1)}. \end{aligned}$$

Then R^e is an $\mathbb{N}^{p+1}$-filtered algebra whose associated graded algebra $\mathbb{G}\mathrm{r}(R^e)$ is isomorphic to $(\mathbb{G}\mathrm{r}R)^e$.

In a similar way, we can obtain a sequence of $\mathbb{N}$-filtered algebras $(R^e)^{(0)}, \cdots, (R^e)^{(p)}$, such that $(R^e)^{(i)} = \mathrm{Gr}((R^e)^{(i-1)})$, for $1 \leqslant i \leqslant p$, and $(R^e)^{(p)} = \mathbb{G}\mathrm{r}R^e$. In fact, $(R^e)^{(i)} = (R^{(i)})^e$, for $0 \leqslant i \leqslant p$.

Lemma 3.1.3. *Let $R = \mathcal{B}(V)$ be the Nichols algebra of V. Then the algebra R^e is Noetherian.*

Proof. The sequence $(R^e)^{(0)}, \cdots, (R^e)^{(p)}$ is a sequence of algebras, each of which is the associated graded algebra of the previous one with respect to an $\mathbb{N}$-filtration. The algebra $(R^e)^{(p)}$ is isomorphic to $(\mathbb{G}\mathrm{r}R)^e$, which is Noetherian. By [57, Thm. 1.6.9], the algebra R^e is Noetherian. $\square$

Lemma 3.1.4. *The algebra $R = \mathcal{B}(V)$ is homologically smooth.*

Proof. Since R^e is Noetherian by Lemma 3.1.3 and R is a finitely generated R^e-module, it is sufficient to prove that the projective dimension $\operatorname{projdim}_{R^e} R$ is finite. The filtration on each $(R^{(i)})^e$, $0 \leqslant i \leqslant p-1$, is bounded below. In addition, from the proof of the foregoing Lemma 3.1.3, each $(R^{(i)})^e$ is Noetherian for $0 \leqslant i \leqslant p$. Therefore, $(R^{(i)})^e$ is a Zariskian algebra for each $0 \leqslant i \leqslant p-1$. It is clear that each $R^{(i)}$, $1 \leqslant i \leqslant p-1$, viewed as an $(R^{(i)})^e$-module has a good filtration. By [50, Cor. 5.8], we have

$$\begin{aligned} \operatorname{projdim}_{R^e} R &= \operatorname{projdim}_{(R^{(0)})^e} R^{(0)} \leqslant \operatorname{projdim}_{(R^{(1)})^e} R^{(1)} \leqslant \cdots \\ &\leqslant \operatorname{projdim}_{(R^{(p)})^e} R^{(p)} = \operatorname{projdim}_{(\mathbb{Gr}R)^e} \mathbb{Gr}R. \end{aligned}$$

The algebra $\mathbb{Gr}R$ is a quantum polynomial algebra of q variables. From the Koszul bimodule complex of $\mathbb{Gr}R$ (cf. (1.20)), we obtain that $\operatorname{projdim}_{(\mathbb{Gr}R)^e} \mathbb{Gr}R = p$. Therefore, $\operatorname{projdim}_{R^e} R \leqslant p$ and R is homologically smooth. $\square$

Proposition 3.1.5. *Let $R = \mathcal{B}(V)$ be the Nichols algebra of V.*

(1) R is AS-regular of global dimension p.

(2) The rigid dualizing complex of R in the graded sense is isomorphic to ${}_{\varphi}R(l)[p]$ for some integer l and $(\mathbb{N})$-graded algebra automorphism φ of degree 0.

(3) The rigid dualizing complex in the ungraded sense is just ${}_{\varphi}R[p]$.

Proof. Let $x_{\beta_1}, \cdots, x_{\beta_p}$ be the root vectors. By Lemma 3.1.1, the algebra R is an iterated graded Ore extension of $\mathbb{k}[x_{\beta_1}]$. Indeed,

$$R \cong \mathbb{k}[x_{\beta_1}][x_{\beta_2}; \tau_2, \delta_2] \cdots [x_{\beta_p}; \tau_p, \delta_p],$$

where for $2 \leqslant j \leqslant p$, τ_j is an algebra automorphism such that $\tau_j(x_{\beta_i})$ is just a scalar multiple of x_{β_i} for $i < j$, and δ_j is a τ_j-derivation such that $\delta_j(x_{\beta_i})$, $i < j$, is a linear combination of monomials in $x_{\beta_{i+1}}, \cdots, x_{\beta_{j-1}}$. It is well-known that $\mathbb{k}[x_{\beta_1}]$ is an AS-regular algebra of dimension 1, and the AS-regularity is preserved under graded Ore extension. So R is an AS-regular algebra of dimension p. Therefore, the rigid dualizing complex of R in the graded case is isomorphic to ${}_{\varphi}R(l)[p]$ for some graded algebra automorphism φ and some $l \in \mathbb{Z}$. By Lemma 3.1.3, R^e is Noetherian. Thus the rigid dualizing complex ${}_\varphi R(l)[p]$ in the graded case implies the dualizing complex ${}_\varphi R[p]$ in the ungraded case. $\square$

We claim that the automorphism φ in Proposition 3.1.5 is just a scalar multiplication. We need some preparations to prove this.

If R is a Γ-module algebra, then the algebra R^e is also a Γ-module algebra with the natural action $g(r \otimes s) := g(r) \otimes g(s)$, for all $g \in \Gamma$ and $r, s \in R$.

Lemma 3.1.6. *Let R be a Γ-module algebra, such that $\Bbbk^\times$ is the group of units of R. Assume that U is an $R^e\#\Bbbk\Gamma$-module, and $U \cong R_\phi$ as $R^e\#\Bbbk\Gamma$-modules, where ϕ is an algebra automorphism.*

(1) The algebra automorphism ϕ preserves the Γ-action.

(2) The $R^e\#\Bbbk\Gamma$-module structure on U (up to isomorphism) is parameterized by $\mathrm{Hom}(\Gamma, \Bbbk)$*, the set of group homomorphisms from Γ to $\Bbbk^\times$.*

Proof. Fix an isomorphism $U \cong R_\phi$. Let $u \in U$ be the element mapped to $1 \in R$. Then $U = Ru$ and we have $g(ru) = g(r)g(u)$ for all $r \in R$ and $g \in \Gamma$. So to determine the Γ-action on U, we only need to determine $g(u)$ for $g \in \Gamma$. Since $g(u) \in U$, there is some $r_g \in R$, such that $g(u) = r_g u$. On the other hand, we have

$$U = g(U) = g(Ru).$$

So there is some $s \in R$, such that $u = g(s)r_g u$. Since the element u forms an R-basis of U, the element r_g has a left inverse. Similarly, there is some $s' \in R$, such that $u = r_g u g(s')$. Since $U \cong R_\phi$ as R-R-bimodules, we have

$$\phi(r)u = ur, \tag{3.3}$$

for any $r \in R$. So $u = r_g u g(s') = r_g \phi(g(s'))u$. Thus r_g has a right inverse as well. Consequently, r_g is a unit in R, and $r_g \in \Bbbk^\times$. We also have $g(h(u)) = (gh)(u)$ for $g, h \in \Gamma$. That is, $r_{gh} = r_g r_h$. Therefore, the Γ-action on U defines a group homomorphism from Γ to $\Bbbk^\times$.

Suppose that the Γ-action on U is given by a group homomorphism $\chi : \Gamma \to \Bbbk^\times$. U is an $R^e\#\Bbbk\Gamma$-module, this leads to $g(rus) = g(r)g(u)g(s)$, for any $r, s \in R$ and

$g \in \Gamma$. On one hand, we have

$$\begin{aligned} g(\phi(r)u) &= g(ur) \\ &= g(u)g(r) \\ &= \chi(g)ug(r) \\ &\overset{(3.3)}{=} \chi(g)\phi(g(r))u. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g(\phi(r)u) &= g(\phi(r))g(u) \\ &= \chi(g)g(\phi(r))u. \end{aligned}$$

So $g(\phi(r)) = \phi(g(r))$. That is, the automorphism ϕ preserves the Γ-action. Consequently, (1) is proved.

Let $\chi : \Gamma \to \Bbbk^\times$ be a group homomorphism. Then it is clear that $g(ru) := \chi(g)g(r)u$ defines a Γ-action on U such that U is an $R^e \# \Bbbk\Gamma$-module.

Suppose there are two Γ-actions on U such that they are isomorphic. We write these two actions as $g^{\cdot 1}(u) = r_g u$ and $g^{\cdot 2}(u) = s_g u$. Denote by U_1 and U_2 the Γ-modules with these two actions respectively. Let $f : U_1 \to U_2$ be an $R^e \# \Bbbk\Gamma$-module isomorphism. Then $f(u) = ru$ for some unit $r \in R$. Since the set of units of R is $\Bbbk^\times$, we have $r \in \Bbbk^\times$. On one hand, we have

$$\begin{aligned} f(g^{\cdot 1}(u)) &= f(r_g u) \\ &= r_g r u. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} f(g^{\cdot 1}(u)) &= g^{\cdot 2}(f(u)) \\ &= g^{\cdot 2}(ru) \\ &= r g^{\cdot 2}(u) \\ &= s_g r u. \end{aligned}$$

Therefore, $r_g = s_g$, and (2) follows. □

If U is an $R^e \# \Bbbk\Gamma$-module, then we can define an $(R\#\Bbbk\Gamma)^e$-module $U \# \Bbbk\Gamma$. It is isomorphic to $U \otimes \Gamma$ as vector space with bimodule structure given by

$$(r\#h)(u \otimes g) := rh(u) \otimes hg,$$

$$(u \otimes g)(r\#h) := ug(r) \otimes gh,$$

for any $r\#h \in R\#H$ and $u \otimes g \in U \otimes \Gamma$.

Lemma 3.1.7. *Let R be a Γ-module algebra with $\Bbbk^\times$ being the group of units and U an $R^e\#\Bbbk\Gamma$-module. Assume that $U \cong R_\phi$ as $R^e\#\Bbbk\Gamma$-modules, where ϕ is an algebra automorphism. If the Γ-action on U is defined by a group homomorphism $\chi : \Gamma \to \Bbbk^\times$. Then $U\#\Bbbk\Gamma \cong (R\#\Bbbk\Gamma)_\psi$ as $(R\#\Bbbk\Gamma)^e$-modules, where ψ is the algebra automorphism defined by $\psi(r\#g) = \chi(g^{-1})\phi(r)\#g$ for any $r\#g \in R\#\Bbbk\Gamma$.*

Proof. The homomorphism ψ defined in the lemma is clearly bijective. First we show that it is an algebra homomorphism. For any $r\#g, s\#h \in R\#\Bbbk\Gamma$, we have

$$\begin{array}{rcl} \psi((r\#g)(s\#h)) & = & \psi(rg(s)\#gh) \\ & = & \chi(h^{-1}g^{-1})\phi(rg(s))\#gh \\ & = & \chi(h^{-1}g^{-1})\phi(r)\phi(g(s))\#gh \\ & = & \chi(h^{-1}g^{-1})\phi(r)g(\phi(s))\#gh \\ & = & (\phi(r)\chi(g^{-1})\#g)(\phi(s)\chi(h^{-1})\#h) \\ & = & \psi(r\#g)\psi(s\#h). \end{array}$$

The forth equation holds since ϕ preserves the Γ-action by Lemma 3.1.6.

Next we prove that $U\#\Bbbk\Gamma \cong (R\#\Bbbk\Gamma)_\psi$ as $(R\#\Bbbk\Gamma)^e$-modules. Fix an isomorphism $U \cong R_\phi$ and let $u \in U$ be the element mapped to $1 \in R$. We define a homomorphism $\Phi : U\#\Bbbk\Gamma \to (R\#\Bbbk\Gamma)_\psi$ by $\Phi(ru \otimes g) = \chi(g^{-1})r\#g$. It is easy to see that Φ is an isomorphism of left $R\#\Bbbk\Gamma$-modules. Now we show that it is a right $R\#\Bbbk\Gamma$-module homomorphism. We have

$$\begin{array}{rcl} \Phi(u(r\#g)) & = & \Phi(ur \otimes g) \\ & = & \Phi(\phi(r)u \otimes g) \\ & = & \chi(g^{-1})\phi(r)\#g \\ & = & \Phi(u)\psi(r\#g) \\ & = & \Phi(u) \cdot (r\#g). \end{array}$$

□

Now we can prove the following lemma.

Lemma 3.1.8. *Keep the notations as in Proposition 3.1.5. The actions of φ on generators $x_1, \cdots, x_\theta$ are just scalar multiplications.*

Proof. By Proposition 3.1.5 and Lemma 1.5.11, we have R-R-bimodule isomorphisms

$$\mathrm{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0, & i \neq p; \\ R_\varphi, & i = p. \end{cases}$$

The group Γ is a free abelian group of rank s, so the algebra $\Bbbk\Gamma$ is a CY algebra of dimension s. Following from [28, Sec. 2], R_φ is an $R^e\#\Bbbk\Gamma$-module and there are $(R\#\Bbbk\Gamma)^e$-bimodule isomorphisms

$$\mathrm{Ext}^i_{(R\#\Bbbk\Gamma)^e}(R\#\Bbbk\Gamma, (R\#\Bbbk\Gamma)^e) \cong \begin{cases} 0, & i \neq p+s; \\ (R_\varphi)\#\Bbbk\Gamma, & i = p+s. \end{cases}$$

For the sake of completeness, we sketch the proof here. By Lemma 3.1.4, R is homologically smooth. That is, R has a bimodule projective resolution

$$0 \to P_q \to \cdots \to P_1 \to P_0 \to R \to 0, \tag{3.4}$$

with each P_i finitely generated as an R-R-bimodule.

$\mathrm{Ext}^*_{R^e}(R, R^e)$ are the cohomologies of the complex $\mathrm{Hom}_{R^e}(P_\bullet, R^e)$. The algebra R^e is a $R^e\#\Bbbk\Gamma$-module defined by

$$((c \otimes d)\#g) \cdot (a \otimes b) = g(a)d \otimes cg(b)$$

for any $a \otimes b \in R^e$ and $(c \otimes d)\#g \in R^e\#\Bbbk\Gamma$. Then each $\mathrm{Hom}_{R^e}(P_i, R^e)$ is a $R^e\#\Bbbk\Gamma$ as well:

$$[((c \otimes d)\#g) \cdot f](x) = ((c \otimes d)\#g) \cdot f(x), \tag{3.5}$$

where $(c \otimes d)\#g \in R^e\#\Bbbk\Gamma$, $f \in \mathrm{Hom}_{R^e}(P_i, R^e)$ and $x \in P_i$. Now $\mathrm{Hom}_{R^e}(P_\bullet, R^e)$ is a complex of left $R^e\#\Bbbk\Gamma$-modules. Thus we obtain that $\mathrm{Ext}^p_{R^e}(R, R^e) \cong R_\varphi$ is an $R^e\#\Bbbk\Gamma$-module.

Put $A = R\#\Bbbk\Gamma$. Observe that A^e is an $R^e\#\Bbbk\Gamma$-A^e-bimodule. The left $\Bbbk\Gamma$-module action is defined by

$$g \cdot (a\#h \otimes b\#k) = g(a)gh \otimes b\#kg^{-1}, \tag{3.6}$$

for any $a\#h \otimes b\#k \in A^e$ and $g \in \Gamma$. The left R^e-action and right A^e-action are given by multiplication. Let W be the vector space $\Bbbk\Gamma \otimes \Bbbk\Gamma$. $R^e \otimes W$ is also an $R^e \# \Bbbk\Gamma$-A^e-bimodule defined by

$$((c \otimes d)\#g) \cdot (a \otimes b \otimes h \otimes k) = cg(a) \otimes g(b)d \otimes gh \otimes kg^{-1}$$

and

$$(a \otimes b \otimes h \otimes k) \cdot (c\#h' \otimes d\#k') = ah(c) \otimes ((k^{-1}k'^{-1})d)b \otimes hh' \otimes k'k.$$

It is not difficult to see that the morphism $f : A^e \to R^e \otimes W$ defined by

$$f(a\#h \otimes b\#k) = a \otimes k^{-1}(b) \otimes h \otimes k$$

is an isomorphism of $R^e \# \Bbbk\Gamma$-A^e-bimodules.

Let P be a finitely generated projective R^e-module. The $\Bbbk\Gamma$-A^e-bimodule structure of $R^e \otimes W$ induces a $\Bbbk\Gamma$-A^e-bimodule structure on $\mathrm{Hom}_{R^e}(P, R^e \otimes W)$. We define a $\Bbbk\Gamma$-A^e-bimodule structure on $\mathrm{Hom}_{R^e}(P, R^e) \otimes W$ as follows

$$g \cdot (f \otimes h \otimes k) = g \cdot f \otimes gh \otimes kg^{-1}$$

and

$$(f \otimes h \otimes k) \cdot (c\#h' \otimes d\#k') = (h(c) \otimes (k^{-1}k'^{-1})d) \cdot f \otimes hh' \otimes k'k,$$

where the $R^e \# \Bbbk\Gamma$-module structure on $\mathrm{Hom}_{R^e}(P, R^e)$ is defined in (3.5). Now the canonical isomorphism from $\mathrm{Hom}_{R^e}(P, R^e) \otimes W$ to $\mathrm{Hom}_{R^e}(P, R^e \otimes W)$ is a $\Bbbk\Gamma$-A^e-bimodule isomorphism.

R admits a resolution like (3.4) with each P_i finitely generated. So

$$\mathrm{Ext}^i_{R^e}(R, R^e \otimes W) \cong \mathrm{Ext}^i_{R^e}(R, R^e) \otimes W$$

as $\Bbbk\Gamma$-A^e-bimodules for all $i \geqslant 0$.

We have Stefan's spectral sequence [66]

$$\mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^e}(R, A^e)) \Rightarrow \mathrm{Ext}^{m+n}_{A^e}(A, A^e).$$

For $m, n \geqslant 0$, we have

$$\begin{aligned}\mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^e}(R, A^e)) &\cong \mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^e}(R, R^e \otimes W)) \\ &\cong \mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^e}(R, R^e) \otimes W).\end{aligned}$$

So $\mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^e}(R, A^e)) = 0$ except that $m = s$ and $n = p$. Therefore,

$$\mathrm{Ext}^i_{(R\#\Bbbk\Gamma)^e}(R\#\Bbbk\Gamma, (R\#\Bbbk\Gamma)^e) = 0$$

for $i \neq p + s$ and

$$\mathrm{Ext}^{p+s}_{A^e}(A, A^e) \cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^p_{R^e}(R, A^e)).$$

Let M be a left $\Bbbk\Gamma$-module. One can consider it as a $\Bbbk\Gamma$-$\Bbbk\Gamma$-bimodule M_ε with the trivial right $\Bbbk\Gamma$-module action. The algebra $\Bbbk\Gamma$ is a CY algebra of dimension s, from Van den bergh's duality theorem (1.5.15) we have

$$\begin{aligned}\mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, M) &\cong \mathrm{HH}^s(\Bbbk\Gamma, M_\varepsilon) \\ &\cong \mathrm{HH}_0(\Bbbk\Gamma, M_\varepsilon) \\ &\cong \mathrm{Tor}^{\Bbbk\Gamma}_0(\Bbbk, M).\end{aligned} \tag{3.7}$$

Now we have the following isomorphisms of right A^e-modules

$$\begin{aligned}\mathrm{Ext}^{p+s}_{A^e}(A, A^e) &\cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^p_{R^e}(R, A^e)) \\ &\cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^p_{R^e}(R, R^e) \otimes W) \\ &\cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, R_\varphi \otimes W) \\ &\cong \mathrm{Tor}^{\Bbbk\Gamma}_0(\Bbbk, R_\varphi \otimes W) \\ &\cong \Bbbk \otimes_{\Bbbk\Gamma} R_\varphi \otimes W.\end{aligned}$$

If we look at the $\Bbbk\Gamma$-A^e-bimodule structure on $R_\varphi \otimes W$ carefully, we obtain that

$$\Bbbk \otimes_{\Bbbk\Gamma} R_\varphi \otimes W \cong R_\varphi\#\Bbbk\Gamma$$

as right A^e-modules.

Since the connected graded algebra R is a domain by Theorem 1.4.7, the group of units of R is $\Bbbk^\times$. Following Lemma 3.1.6 and 3.1.7, we have $(R_\varphi)\#\Bbbk\Gamma \cong (R\#\Bbbk\Gamma)_{\bar{\psi}}$, where $\bar{\psi}$ is the algebra automorphism defined by $\bar{\psi}(r\#g) = \varphi(r)\chi(g^{-1})$ for some algebra homomorphism $\chi : \Gamma \to \Bbbk$.

On the other hand, we have $A = R\#\Bbbk\Gamma \cong U(\mathcal{D}, 0)$, and A-A-bimodule isomorphisms

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq p+s; \\ A_\psi, & i = p+s, \end{cases}$$

where ψ is the algebra automorphism defined in Theorem 2.1.5.

Therefore, we have A-A-bimodule isomorphisms $A_{\bar{\psi}} \cong A_\psi$. That is, $\bar{\psi}$ and ψ differ only by an inner automorphism. By Theorem 1.4.7, the graded algebra A is a domain, the invertible elements of A are in $\Bbbk\Gamma$. The actions of ψ and the group actions on generators $x_1, \cdots, x_\theta$ are just scalar multiplications. Thus the actions of $\bar{\psi}$ on $x_1, \cdots, x_\theta$ are also scalar multiplications. Since $\bar{\psi}(x_i) = \varphi(x_i)$ for all $1 \leqslant i \leqslant \theta$, we get our desired result. □

We are ready to prove the main theorem of this section.

Theorem 3.1.9. *Let V be a generic braided vector space of finite Cartan type, and $R = \mathcal{B}(V)$ the Nichols algebra of V. For each $1 \leqslant k \leqslant \theta$, let j_k be the integer such that $\beta_{j_k} = \alpha_k$.*

(1) The rigid dualizing complex is isomorphic to ${}_\varphi R[p]$, where φ is the algebra automorphism defined by

$$\varphi(x_k) = ((\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k,$$

for all $1 \leqslant k \leqslant \theta$.

(2) The algebra R is a CY algebra if and only if

$$\prod_{i=1}^{j_k-1} \chi_k(g_{\beta_i}) = \prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k),$$

for all $1 \leqslant k \leqslant \theta$.

Proof. (1) Note that $\mathbb{Gr}R$ is isomorphic to the following quantum polynomial algebra:

$$\Bbbk\langle x_{\beta_1}, \cdots, x_{\beta_p} \mid x_{\beta_i} x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i},\ \ 1 \leqslant i < j \leqslant p \rangle.$$

By Lemma 1.5.13, $\mathbb{Gr}R$ has a rigid dualizing complex ${}_{\bar{\zeta}}\mathbb{Gr}R[p](\cong \mathbb{Gr}R_{\bar{\zeta}^{-1}}[p])$, where

$\bar{\zeta}$ is defined by

$$\bar{\zeta}(x_{\beta_k}) = \chi_{\beta_k}^{-1}(g_{\beta_1}) \cdots \chi_k^{-1}(g_{\beta_{k-1}})\chi_{\beta_{k+1}}(g_{\beta_k}) \cdots \chi_{\beta_p}(g_{\beta_k})x_{\beta_k},$$

for all $1 \leqslant k \leqslant p$.

On the other hand, it follows from Proposition 3.1.5 and Lemma 3.1.8 that R has a rigid dualizing complex ${}_{\varphi}R$, where φ is an algebra automorphism such that for each $1 \leqslant k \leqslant \theta$, $\varphi(x_k)$ is a scalar multiple of x_k. Assume that $\varphi(x_k) = l_k x_k$, with $l_k \in \mathbb{k}$.

Let $R^{(0)}, \cdots, R^{(p)}$ be the sequence of algebras defined after Corollary 3.1.2. By Lemma 3.1.1, applying a similar argument as in the proof of Proposition 3.1.5, we obtain that each $R^{(i)}$, $0 \leqslant i \leqslant p$, is an iterated Ore extension of the polynomial algebra $\mathbb{k}[x]$. Thus each of them is AS-regular. It follows from [76, Prop. 1.1] that each $R^{(i)}$, $1 \leqslant i \leqslant p$, has a rigid dualizing complex ${}_{\varphi^{(i)}}(R^{(i)})[p]$, where $\varphi^{(i)} = \operatorname{Gr}\varphi^{(i-1)}$ and $\varphi^{(0)} = \varphi$. Since for each $1 \leqslant k \leqslant \theta$, $\varphi(x_k) = l_k x_k$, we have $\varphi^{(p)}(x_k) = l_k x_k$. Because $R^{(p)} = \mathbb{G}\mathrm{r}R$, there is a bimodule isomorphism ${}_{\varphi^{(p)}}(R^{(p)}) \cong {}_{\bar{\zeta}}(\mathbb{G}\mathrm{r}R)$. We obtain that $\varphi^{(p)} = \bar{\zeta}$, as R is connected. Therefore, for each $1 \leqslant k \leqslant \theta$,

$$l_k x_k = \bar{\zeta}(x_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k,$$

where j_k is the integer such that $\beta_{j_k} = \alpha_k$.

Now we conclude that $\varphi(x_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k$, for each $1 \leqslant k \leqslant \theta$.

(2) The algebra R is homologically smooth by Lemma 3.1.4. It follows from Corollary 1.5.12 that R is CY if and only if $R \cong {}_{\varphi}R$ as bimodules. That is, R is CY if and only if $\varphi = \mathrm{id}$. Hence (2) follows from (1). □

Example 3.1.10. Let $\mathcal{D}(\Gamma, (g_i), (\chi_i), (a_{ij}))$ be a generic datum such that the Cartan matrix is of type A_2. This defines a braided vecter space V. Let $\{x_1, x_2\}$ be a basis of V. The braiding of V is given by

$$c(x_i \otimes x_j) = \chi_j(g_i)x_j \otimes x_i, \quad i, j = 1, 2.$$

The Nichols algebra $R = \mathcal{B}(V)$ of V is generated by x_1 and x_2 subject to the relations

$$x_1^2 x_2 - q_{12} x_1 x_2 x_1 - q_{11} q_{12} x_1 x_2 x_1 + q_{11} q_{12}^2 x_2 x_1^2 = 0,$$

$$x_2^2x_1 - q_{21}x_2x_1x_2 - q_{22}q_{21}x_2x_1x_2 + q_{22}q_{21}^2x_1x_2^2 = 0,$$

where $q_{ij} = \chi_j(g_i)$. The element $s_1s_2s_1$ is the longest element in the Weyl group $\mathcal{W}$. Let α_1 and α_2 be the two simple roots. Then the positive roots are as follows

$$\beta_1 = \alpha_1, \;\; \beta_2 = \alpha_1 + \alpha_2, \;\; \beta_3 = \alpha_2.$$

By Theorem 3.1.9, the algebra R is CY if and only if

$$\chi_{\beta_2}(g_1)\chi_{\beta_3}(g_1) = (\chi_1\chi_2^2)(g_1) = 1$$

and

$$\chi_2(g_{\beta_1})\chi_2(g_{\beta_2}) = \chi_2(g_1^2g_2) = 1.$$

That is, $q_{11}q_{12}^2 = q_{22}q_{12}^2 = 1$. By equation (1.6), we have $q_{11}^{-1} = q_{22}^{-1} = q_{12}q_{21}$.

Now we conclude that the algebra R is CY if and only if there is some $q \in \Bbbk^\times$, which is not root of unity such that

$$q_{11} = q_{22} = q^2 \;\text{ and }\; q_{12} = q_{21} = q^{-1}.$$

In other words, the braiding is of DJ-type. Then the algebra R is an AS-regular algebra of type A (see [9] for terminology). This coincides with Proposition 5.4 in [13].

Example 3.1.11. Let R be a Nichols algebra of type B_2. That is, R is generated by x_1 and x_2 subject to the relations

$$\begin{array}{l} x_1^3x_2 - q_{12}x_1^2x_2x_1 - q_{11}q_{12}x_1^2x_2x_1 + q_{11}q_{12}^2x_1x_2x_1^2 \\ -q_{11}^2q_{12}(x_1^2x_2x_1 - q_{12}x_1x_2x_1^2 - q_{11}q_{12}x_1x_2x_1^2 + q_{11}q_{12}^2x_2x_1^3) = 0 \end{array},$$

$$x_2^2x_1 - q_{21}x_2x_1x_2 - q_{22}q_{21}x_2x_1x_2 + q_{22}q_{21}^2x_1x_2^2 = 0,$$

where $q_{ij} \in \Bbbk$ for $1 \leqslant i, j \leqslant 2$ and $q_{12}q_{21} = q_{11}^{-2} = q_{22}^{-1}$. Applying a similar argument, we obtain that R is CY if and only if there is some $q \in \Bbbk^\times$, which is not a root of unity, such that

$$q_{11} = q, \;\; q_{12} = q^{-1}, \;\; q_{21} = q^{-1} \text{ and } q_{22} = q^2.$$

3.2 Relation with pointed Hopf algebras

We keep the notations as in Section 3.1. Let λ be a family of linking parameters for $\mathcal{D}$ and A the algebra $U(\mathcal{D},\lambda)$. In this subsection, we discuss the relation between the CY property of the algebra $U(\mathcal{D},\lambda)$ and that of the corresponding Nichols algebra $\mathcal{B}(V)$. It turns out that if one of them is CY, then the other one is not.

Lemma 3.2.1. *For each $1 \leqslant k \leqslant \theta$, we have*

$$\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)).$$

Proof. Let $\omega_0 = s_{i_1}\cdots s_{i_p}$ be the fixed reduced decomposition of the longest element ω_0 in the Weyl group. It is clear that ω_0^{-1} is also of maximal length. By Lemma 3.11 in [40], for each $1 \leqslant k \leqslant \theta$, there exists $1 \leqslant t \leqslant p$, such that

$$s_k s_{i_1}\cdots s_{i_{t-1}} = s_{i_1}\cdots s_{i_t}.$$

That is, $\omega_0 = s_k s_{i_1}\cdots s_{i_{t-1}} s_{i_{t+1}}\cdots s_{i_p}$. Set

$$\beta'_1 = \alpha_k,\ \ \beta'_2 = s_k(\alpha_{i_1}), \cdots, \beta'_p = s_k s_{i_1}\cdots s_{i_{t-1}} s_{i_{t+1}}\cdots s_{i_{p-1}}(\alpha_{i_p}).$$

Applying a similar argument as in the proof of Theorem 3.1.9, we conclude that the rigid dualizing complex of the algebra $R = \mathcal{B}(V)$ is isomorphic to ${}_{\varphi'}R[p]$. The algebra automorphism φ' is defined by

$$\varphi'(x_l) = (\prod_{i=1}^{j'_l-1} \chi_l^{-1}(g_{\beta'_i}))(\prod_{i=j'_l+1}^{p} \chi_{\beta'_i}(g_l))x_l,$$

for each $1 \leqslant l \leqslant \theta$, where j'_l, $1 \leqslant l \leqslant \theta$, are the integers such that $\beta'_{j'_l} = \alpha_l$. In particular, we have

$$\varphi'(x_k) = (\prod_{i=2}^{p} \chi_{\beta'_i}(g_k))x_k.$$

The rigid duallizing complex is unique up to isomorphism, so ${}_{\varphi'}R \cong {}_{\varphi}R$ as R-R-bimodules, where φ is the algebra automorphism defined in Theorem 3.1.9. Since the graded algebra R is connected, we have $\varphi' = \varphi$. In particular, $\varphi'(x_k) = \varphi(x_k)$, that is,

$$\prod_{i=2}^{p}\chi_{\beta'_i}(g_k) = (\prod_{i=1}^{j_k-1}\chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p}\chi_{\beta_i}(g_k)).$$

Both $\beta_1, \cdots, \beta_p$ and $\beta'_1, \cdots, \beta'_p$ are enumerations of positive roots. We have $\alpha_k = \beta'_1 = \beta_{j_k}$. Therefore,

$$\prod_{i=2}^{p}\chi_{\beta'_i}(g_k) = \prod_{i=1, i\neq j_k}^{p}\chi_{\beta_i}(g_k).$$

It follows that

$$(\prod_{i=1}^{j_k-1}\chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p}\chi_{\beta_i}(g_k)) = \prod_{i=1, i\neq j_k}^{p}\chi_{\beta_i}(g_k).$$

□

Proposition 3.2.2. *If $A = U(\mathcal{D}, \lambda)$ is a CY algebra, then the rigid dualizing complex of the Nichols algebra $R = \mathcal{B}(V)$ is isomorphic to ${}_\varphi R[p]$, where φ is defined by $\varphi(x_k) = \chi_k^{-1}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$.*

Proof. By Theorem 3.1.9, the rigid dualizing complex of R is isomorphic to ${}_\varphi R[p]$, where φ is defined by

$$\varphi(x_k) = (\prod_{i=1}^{j_k-1}\chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p}\chi_{\beta_i}(g_k))x_k,$$

for all $1 \leqslant k \leqslant \theta$. If A is a CY algebra, then $\prod_{i=1}^{p}\chi_{\beta_i} = \varepsilon$ by Theorem 2.1.5. Therefore, for $1 \leqslant k \leqslant \theta$,

$$\begin{aligned}(\textstyle\prod_{i=1}^{j_k-1}\chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p}\chi_{\beta_i}(g_k)) &= \textstyle\prod_{i=1, i\neq j_k}^{p}\chi_{\beta_i}(g_k)\\ &= \chi_k^{-1}(g_k),\end{aligned}$$

where the first equation follows from Lemma 3.2.1. Now $\varphi(x_k) = \chi_k^{-1}(g_k)x_k$ for all $1 \leqslant k \leqslant \theta$. Thus we have completed the proof. □

Since $\chi_k(g_k) \neq 1$ for all $1 \leqslant k \leqslant \theta$, the algebra $R = \mathcal{B}(V)$ is not CY, if $A = U(\mathcal{D}, \lambda)$ is a CY algebra.

Proposition 3.2.3. *If the Nichols algebra $R = \mathcal{B}(V)$ is a CY algebra, then the rigid dualizing complex of $A = U(\mathcal{D}, \lambda)$ is isomorphic to ${}_\psi A[p + s]$, where ψ is defined by*

$\psi(x_k) = x_k$ for all $1 \leqslant k \leqslant \theta$ and $\psi(g) = \prod_{i=1}^{p} \chi_{\beta_i}(g)$ for all $g \in \Gamma$.

Proof. If the algebra R is CY, by Theorem 3.1.9 and Lemma 3.2.1, for each $1 \leqslant k \leqslant \theta$, we have

$$\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)) = 1.$$

Now the statement follows from Theorem 2.1.5. □

With the assumption of Proposition 3.2.3, for all $1 \leqslant k \leqslant \theta$, we have

$$\psi(g_k) = \prod_{i=1}^{p} \chi_{\beta_i}(g_k) = \chi_k(g_k) g_k \neq g_k.$$

Since the invertible elements of A are in $\Bbbk\Gamma$ and Γ is an abelian group, ψ can not be an inner automorphism. So the algebra A is not CY.

Example 3.2.4. Let R be the algebra in Example 3.1.10. Assume that $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$, and $g_i = y_i$, $i = 1, 2$. The characters χ_1 and χ_2 are given by the following table,

	y_1	y_2
χ_1	q^2	q^{-1}
χ_2	q^{-1}	q^2

where q is not a root of unity.

The algebra R is a CY algebra. But the algebra $A = R\#\Bbbk\Gamma$ is not. The rigid dualizing complex of A is isomorphic to ${}_{\psi}A[5]$, where ψ is defined by $\psi(x_i) = x_i$ and $\psi(y_i) = q^2 y_i$ for $i = 1, 2$.

Example 3.2.5. Let A be the algebra in Example 2.2.10. It is a CY algebra. However, its corresponding Nichols algebra R is not CY. Its rigid dualizing complex is isomorphic to ${}_{\varphi}R[7]$, where φ is defined by $\varphi(x_1) = q^{-1}x_1$, $\varphi(x_2) = q^{-1}x_2$ and $\varphi(x_3) = q^4 x_3$.

Example 3.2.6. Let A be an algebra with generators $y_1^{\pm 1}$, $y_2^{\pm 1}$, x_1 and x_2 subject to the relations

$$y_h^{\pm 1} y_h^{\mp 1} = 1, \quad 1 \leqslant h, m \leqslant 2,$$

$$y_1 x_1 = q x_1 y_1, \quad y_1 x_2 = q^{-1} x_2 y_1,$$

$$y_2 x_1 = q^{\frac{k}{l}} x_1 y_2, \quad y_2 x_2 = q^{-\frac{k}{l}} x_2 y_2,$$

$$x_1x_2 - q^{-k}x_2x_1 = 1 - y_1^k y_2^l,$$

where $k, l \in \mathbb{Z}^+$ and $0 < |q| < 1$ is not a root of unity.

By Proposition 2.2.9 (cf. Table 4.2 in Section 2.2), the algebra A is a CY algebra of dimension 4. Let R be the corresponding Nichols algebra of A. The rigid dualizing complex of R is isomorphic to ${}_{\varphi}R[2]$, where φ is defined by $\varphi(x_1) = q^{-k}x_1$ and $\varphi(x_2) = q^k x_2$.

Chapter 4

Rigid dualizing complexes of braided Hopf algebras over finite group algebras

Let V be a vector space of dimension d and Γ a finite subgroup of $\mathrm{GL}_d(\Bbbk)$. The skew group algebra $S(V)\#\Bbbk\Gamma$ is a CY algebra if and only if $\Gamma \subseteq \mathrm{SL}_d(\Bbbk)$, where $S(V)$ is the symmetric algebra of V ([28, Page 427] or [36, Thm. 3.14]). Let R be a Koszul CY algebra and H the group algebra $\Bbbk\Gamma$, where Γ is a finite group of automorphisms of R. In [72], Wu and Zhu showed that the smash product $R\#H$ is CY if and only if the homological determinant (Definition 4.1.6) of the H-action is trivial. Later, this result was generalized to the case where R is a p-Koszul CY algebra and H is an involutory CY Hopf algebra [52].

We mentioned in Example 1.3.3 that the algebra $S(V)$ can be viewed as a braided Hopf algebra. Let H be a finite dimensional Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Those aforementioned examples motivated us to discuss the relation between the CY property of R and that of $R\#H$. Inspired by Wu and Zhu's work, in Section 4.1, we use the homological determinant of the H-action to describe the homological integral of $R\#H$. We then give a necessary and sufficient condition for $R\#H$ to be a CY algebra, in case R is CY and H is semisimple. Conversely, if $R\#H$ is a CY algebra, when R is a CY algebra? In Section 4.2, we will answer this question in case H is the group algebra $\Bbbk\Gamma$, where Γ is a finite group. In fact, we show that an AS-Gorenstein braided Hopf algebra in the category of Yetter-Drinfeld modules over a finite group algebra has a rigid dualizing complex.

The groups of group-like elements of pointed Hopf algebras discussed in Chapter 2 are all infinite. At the end of this chapter, we show that there are CY pointed Hopf algebras with a finite abelian group of group-like elements.

In this chapter, unless otherwise stated, $\Bbbk$ is a just fixed field.

4.1 Calabi-Yau property under Hopf actions

Let H be a Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. For $h \in H$ and $r \in R$, We write $h(r)$ for h acting on r. It is an element in R. On the other hand, we write hr for h multiplying with r. It is an element in $R\#H$. For a left $R\#H$-module M, the vector space $M \otimes H$ is a left $R\#H$-module defined by

$$(r\#h) \cdot (m \otimes g) := (r\#h_1)m \otimes h_2 g,$$

for all $r\#h \in R\#H$ and $m \otimes g \in M \otimes H$. Denote this $R\#H$-module by $M\#H$.

Let M and N be two $R\#H$-modules. Then there is a natural left H-module structure on $\mathrm{Hom}_R(M, N)$ given by the adjoint action

$$(h \rightharpoonup f)(m) := h_2 f(\mathcal{S}_H^{-1}(h_1)m),$$

for all $h \in H$, $f \in \mathrm{Hom}_R(M, N)$ and $m \in M$.

Lemma 4.1.1. *Let M be a left $R\#H$-module. Then $\mathrm{Hom}_R(M, R) \otimes H$ is an H-$R\#H$-bimodule, where the left H-module structure is defined by*

$$h \cdot (f \otimes g) := h_1 \rightharpoonup f \otimes h_2 g$$

and the right $R\#H$-module structure is defined by

$$(f \otimes g) \cdot (r\#h) := f g_1(r) \otimes g_2 h,$$

for all $f \in \mathrm{Hom}_R(M, R)$, $g, h \in H$ and $r \in R$.

Proof. First we show that for all $h \in H$, $f \in \mathrm{Hom}_R(M, R)$ and $r \in R$

$$(h_1 \rightharpoonup f)h_2(r) = h \rightharpoonup (fr). \tag{4.1}$$

For $m \in M$, we have

$$\begin{aligned}
[(h_1 \rightharpoonup f)h_2(r)](m) &= (h_1 \rightharpoonup f)(m)h_2(r)\\
&= h_2(f(\mathcal{S}_H^{-1}(h_1)m))h_3(r)\\
&= h_2(f(\mathcal{S}_H^{-1}(h_1)m)r)\\
&= h_2((fr)(\mathcal{S}_H^{-1}(h_1)m))\\
&= [h \rightharpoonup (fr)](m).
\end{aligned}$$

Now we check that for all $f \otimes g \in \mathrm{Hom}_R(M, R) \otimes H$, $h \in H$ and $r\#k \in R\#H$, $(h \cdot (f \otimes g)) \cdot (r\#k) = h \cdot ((f \otimes g) \cdot (r\#k))$. We have

$$\begin{aligned}
(h \cdot (f \otimes g)) \cdot (r\#k) &= (h_1 \rightharpoonup f \otimes h_2 g) \cdot (r\#k)\\
&= (h_1 \rightharpoonup f)(h_2 g_1)(r) \otimes h_3 g_2 k.
\end{aligned}$$

and

$$\begin{aligned}
h \cdot ((f \otimes g) \cdot (r\#k)) &= h \rightharpoonup (f g_1(r) \otimes g_2 k)\\
&= h_1 \rightharpoonup (f g_1(r)) \otimes h_2 g_2 k\\
&\overset{(4.1)}{=} (h_1 \rightharpoonup f)(h_2 g_1)(r) \otimes h_3 g_2 k.
\end{aligned}$$

□

Let M be an $R\#H$-module. There is a natural right $R\#H$-module structure on $\mathrm{Hom}_{R\#H}(M\#H, R\#H)$. It is also a left H-module defined by

$$(h \cdot f)(m \otimes g) := f(m \otimes gh), \tag{4.2}$$

for all $h \in H$, $f \in \mathrm{Hom}_{R\#H}(M\#H, R\#H)$ and $m \otimes g \in M \otimes H$. Then $\mathrm{Hom}_{R\#H}(M\#H, R\#H)$ is an H-$R\#H$-bimodule.

Proposition 4.1.2. *Let P be an $R\#H$-module, which is finitely generated projective as an R-module. Then*

$$\mathrm{Hom}_R(P, R) \otimes H \cong \mathrm{Hom}_{R\#H}(P\#H, R\#H)$$

as H-$R\#H$-bimodules.

Proof. Let

$$\psi : \mathrm{Hom}_R(P, R) \otimes H \to \mathrm{Hom}_{R\#H}(P\#H, R\#H)$$

be the homomorphism defined by

$$\begin{aligned}[\psi(f \otimes h)](p \otimes g) &= (g_1 \rightharpoonup f)(p)\#g_2h \\ &= g_2(f(\mathcal{S}_H^{-1}(g_1)p))\#g_3h,\end{aligned}$$

for all $f \otimes h \in \mathrm{Hom}_R(P, R) \otimes H$ and $p \otimes g \in P\#H$.

We claim that the image of ψ is contained in $\mathrm{Hom}_{R\#H}(P\#H, R\#H)$. For any $f \otimes h \in \mathrm{Hom}_R(P, R) \otimes H$, $r\#k \in R\#H$ and $p \otimes g \in P\#H$, on one hand, we have

$$\begin{aligned}[\psi(f \otimes h)]((r\#k)(p \otimes g)) &= [\psi(f \otimes h)]((r\#k_1)p \otimes k_2g)) \\ &= (k_3g_2)(f(\mathcal{S}_H^{-1}(k_2g_1)((r\#k_1)p)))\#k_4g_3h \\ &= (k_2g_3)(f(((\mathcal{S}_H^{-1}(k_1g_2))(r))\mathcal{S}_H^{-1}(g_1)p))\#k_3g_4h.\end{aligned}$$

On the other hand,

$$\begin{aligned}(r\#k)[\psi(f \otimes h)](p \otimes g) &= (r\#k)(g_2(f(\mathcal{S}_H^{-1}(g_1)p))\#g_3h) \\ &= r(k_1g_2)(f(\mathcal{S}_H^{-1}(g_1)p))\#k_2g_3h \\ &= (k_2g_3)(\mathcal{S}_H^{-1}(k_1g_2)(r)f(\mathcal{S}_H^{-1}(g_1)p))\#k_2g_3h \\ &= (k_2g_3)(f(((\mathcal{S}_H^{-1}(k_1g_2))(r))\mathcal{S}_H^{-1}(g_1)p))\#k_3g_4h.\end{aligned}$$

Now we show that ψ is an H-$R\#H$-bimodule homomorphism. We have

$$\begin{aligned}[\psi((f \otimes h)(r\#k))](p \otimes g) &= [\psi(fh_1(r) \otimes h_2k))](p \otimes g) \\ &= g_2([fh_1(r)](\mathcal{S}_H^{-1}(g_1)p)) \otimes g_3h_2k \\ &= g_2(f(\mathcal{S}_H^{-1}(g_1)p))(g_3h_1)(r) \otimes g_4h_2k \\ &= (g_2(f(\mathcal{S}_H^{-1}(g_1)p)) \otimes g_3h)(r\#k) \\ &= [\psi(f \otimes h)(r\#k)](p \otimes g)\end{aligned}$$

and

$$\begin{aligned}[\psi(k(f \otimes h))](p \otimes g) &= [\psi(k_1 \rightharpoonup f \otimes k_2h)](p \otimes g) \\ &= g_2((k_1 \rightharpoonup f)(\mathcal{S}_H^{-1}(g_1)p))\#g_3k_2h \\ &= (g_2k_2)(f(\mathcal{S}_H^{-1}(k_1)\mathcal{S}_H^{-1}(g_1)p))\#g_3k_3h \\ &= ((g_1k_1) \rightharpoonup f)(p) \otimes g_2k_2h \\ &= [\psi(f \otimes h)](p \otimes gk) \\ &= [k \cdot \psi(f \otimes h)](p \otimes g).\end{aligned}$$

So $\mathrm{Hom}_R(P,R)\otimes H \cong \mathrm{Hom}_{R\#H}(P\#H,R\#H)$ as H-$R\#H$-bimodules when P is finitely generated projective as an R-module. $\square$

Proposition 4.1.3. *Let H be a finite dimensional Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then*

$$\mathrm{Ext}^i_{R\#H}(H,R\#H)\cong \mathrm{Ext}^i_R(\Bbbk,R)\otimes H$$

as H-$R\#H$-bimodules for all $i\geqslant 0$.

Proof. Since R is Noetherian and H is finite dimensional, $R\#H$ is also Noetherian. Then ${}_{R\#H}\Bbbk$ admits a projective resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to \Bbbk \to 0$$

such that each P_n is a finitely generated $R\#H$-module. Because H is finite dimensional, each P_n is also finitely generated as an R-module. Tensoring with H, we obtain a projective resolution of H over $R\#H$

$$\cdots \to P_n\#H \to \cdots \to P_1\#H \to P_0\#H \to H \to 0.$$

Applying the functor $\mathrm{Hom}_{R\#H}(-,R\#H)$ to this complex, we obtain the following complex

$$\begin{aligned}0 \to \mathrm{Hom}_{R\#H}(P_0\#H,R\#H) \to \mathrm{Hom}_{R\#H}(P_1\#H,R\#H) \to \cdots \\ \to \mathrm{Hom}_{R\#H}(P_n\#H,R\#H) \to \cdots .\end{aligned} \tag{4.3}$$

This is a complex of H-$R\#H$-bimodules, where the left H-module structure is defined as in (4.2). By Lemma 4.1.1 and Proposition 4.1.2, one can check that it is isomorphic to the following complex of H-$R\#H$-bimodules,

$$\begin{aligned}0 \to \mathrm{Hom}_R(P_0,R)\otimes H \to \mathrm{Hom}_R(P_1,R)\otimes H \cdots \\ \to \mathrm{Hom}_R(P_n,R)\otimes H \to \cdots .\end{aligned} \tag{4.4}$$

After taking cohomologies of complex (4.3) and complex (4.4), we arrive at isomorphisms of H-$R\#H$-bimodules

$$\mathrm{Ext}^i_{R\#H}(H, R\#H) \cong \mathrm{Ext}^i_R(\Bbbk, R) \otimes H$$

for all $i \geqslant 0$. $\square$

The algebra R can be viewed as an augmented right H-module algebra through the right H-action: $r \cdot h := \mathcal{S}_H^{-1}(h) \cdot r$, for all $r \in R$ and $h \in H$. The algebra $H\#R$ can be defined in a similar way. The multiplication is given by

$$(h\#s)(k\#r) := hk_2\#(s \cdot k_1)r = hk_2\#(\mathcal{S}_H^{-1}(k_1)(s))r,$$

for all $h\#s$ and $k\#r \in H\#R$. The homomorphism $\varphi : R\#H \to H\#R$ defined by

$$\varphi(r\#k) = k_2\#\mathcal{S}_H^{-1}(k_1)(r)$$

is an algebra isomorphism with its inverse $\psi : H\#R \to R\#H$ defined by

$$\psi(k\#r) = k_1(r)\#k_2.$$

In addition, φ is compatible with the augmentation maps of $R\#H$ and $H\#R$ respectively. Now right $R\#H$-modules can be treated as $H\#R$-modules. Let M and N be two $H\#R$-modules, then $\mathrm{Hom}_R(M, N)$ is a right H-module defined by

$$(f \leftharpoonup h)(m) := f(m\mathcal{S}_H(h_1))h_2,$$

for all $h \in H$, $f \in \mathrm{Hom}_R(M, N)$ and $m \in M$.

Similar to the left case, we have the following proposition.

Proposition 4.1.4. *Let H be a finite dimensional Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then*

$$\mathrm{Ext}^i_{R\#H}(H_{R\#H}, R\#H_{R\#H}) \cong H \otimes \mathrm{Ext}^i_R(\Bbbk_R, R_R)$$

as $R\#H$-H-bimodules for all $i \geqslant 0$.

Lemma 4.1.5. *Let H be a Hopf algebra and R an H-module algebra. If the left global dimensions of R and H are d_R and d_H respectively, then the left global dimension of $A = R\#H$ is not greater than $d_R + d_H$.*

Proof. Let M and N be two A-modules. We have

$$\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_R(M,N)),$$

that is, the functor $\mathrm{Hom}_A(M,-)$ factors through as follows

$$\begin{tikzcd} \mathrm{Mod}A \arrow[rr, "\mathrm{Hom}_R(M,-)"] \arrow[dr, "\mathrm{Hom}_A(M,-)"'] & & \mathrm{Mod}H \arrow[dl, "\mathrm{Hom}_H(\Bbbk,-)"] \\ & \mathrm{Mod}\Bbbk & \end{tikzcd} .$$

To apply the Grothendieck spectral sequence (see e.g. [70, Sec. 5.8]), we need to show that if N is an injective A-module, then $\mathrm{Ext}^q_H(\Bbbk, \mathrm{Hom}_R(M,N)) = 0$ for all $q \geqslant 1$.

Let

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to \Bbbk \to 0$$

be a projective resolution of $\Bbbk$ over H. $\mathrm{Ext}^*_H(\Bbbk, \mathrm{Hom}_R(M,N))$ are the cohomologies of the complex $\mathrm{Hom}_H(P_\bullet, \mathrm{Hom}_R(M,N))$. There are the following isomorphisms

$$\begin{aligned} \mathrm{Hom}_H(P_\bullet, \mathrm{Hom}_R(M,N)) &\cong \mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_\Bbbk(P_\bullet, \mathrm{Hom}_R(M,N))) \\ &\cong \mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_R(P_\bullet \otimes M, N)) \\ &\cong \mathrm{Hom}_{R\#H}(P_\bullet \otimes M, N). \end{aligned}$$

Let P_i be a projective module in the complex $P_\bullet$. Note that the $R\#H$-module structure on $P_i \otimes M$ is given by

$$(r\#h) \cdot (p \otimes h) = h_2 \otimes rh_1 m,$$

for all $r\#h \in R\#H$ and $p \otimes m \in P_i \otimes M$. The complex $P_\bullet$ is exact except at P_0. Since the functors $\mathrm{Hom}_{R\#H}(-,N)$ and $-\otimes M$ are exact, the complex $\mathrm{Hom}_H(P_\bullet, \mathrm{Hom}_R(M,N))$ is also exact except at $\mathrm{Hom}_H(P_0, \mathrm{Hom}_R(M,N))$. It follows that

$$\mathrm{Ext}^q_H(\Bbbk, \mathrm{Hom}_R(M,N)) = 0$$

for all $q \geqslant 1$.

Now we have

$$\mathrm{Ext}^q_H(\Bbbk, \mathrm{Ext}^p_R(M,N)) \Rightarrow \mathrm{Ext}^{p+q}_{R\#H}(M,N).$$

Because the left global dimensions of R and H are d_R and d_H, $\operatorname{Ext}^i_{R\#H}(M,N)=0$ for all $i \geqslant d_R+d_H$. Therefore, the left global dimension of $R\#H$ is not greater than d_R+d_H. □

The homological determinant for graded automorphisms of an AS-Gorenstein algebra was defined by Jørgensen and Zhang [39]. A Hopf algebra version was introduced later in [46]. The homological determinant was used to study the AS-Gorenstein property of invariant subrings.

Definition 4.1.6. (cf. [52], [46]) Let R be an AS-Gorenstein algebra of injective dimension d. There is a left H-action on $\operatorname{Ext}^d_R(\Bbbk, R)$ induced by the left H-action on R. Let $\mathbf{e}$ be a non-zero element in $\operatorname{Ext}^d_R(\Bbbk, R)$. Then there is an algebra homomorphism $\eta : H \to \Bbbk$ satisfying $h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in H$.

(i) The composite map $\eta\mathcal{S}_H : H \to \Bbbk$ is called the *homological determinant* of the H-action on R, and it is denoted by hdet (or more precisely hdet_R).

(ii) The homological determinant hdet_R is said to be *trivial* if $\mathrm{hdet}_R = \varepsilon_H$, where ε_H is the counit of the Hopf algebra H.

Let H be an involutory CY Hopf algebra and R a p-Koszul CY algebra which is a left H-module algebra. As we mentioned in the introduction of this chapter, in [52], Wu and Zhu used the homological determinant of the H-action to characterize the CY property of $R\#H$. They defined an H-module structure on the Koszul bimodule complex of R and computed the H-module structures on the Hochschild cohomologies. Then they proved that $R\#H$ is CY if and only if the homological determinant is trivial. If H is not involutory or R is not a p-Koszul algebra, then is $R\#H$ still a CY algebra when the homological determinant is trivial?

We discuss the question when R is a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$, where H is a finite dimensional Hopf algebra. We use the homological determinant to discuss the homological integral and the rigid dualizing complex of the algebra $A = R\#H$. We then give a necessary and sufficient condition for A to be a CY algebra. The result we obtained is slightly different from what was obtained by Wu and Zhu. We first need the following lemma.

Lemma 4.1.7. *Let H be a Hopf algebra, and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then*
$$\mathcal{S}^2_{R\#H}(r) = \mathcal{S}_H(r_{(-1)})(\mathcal{S}^2_R(r_{(0)})),$$

for any $r \in R$.

Proof. Set $A = R\#H$. By equation (1.5), for any $r \in R$,

$$\mathcal{S}_A(r) = (1\#\mathcal{S}_H(r_{(-1)}))(\mathcal{S}_R(r_{(0)})\#1).$$

Therefore,

$$\begin{aligned}
\mathcal{S}_A^2(r) &= \mathcal{S}_A((1\#\mathcal{S}_H(r_{(-1)}))(\mathcal{S}_R(r_{(0)})\#1)) \\
&= \mathcal{S}_A(\mathcal{S}_R(r_{(0)})\#1)\mathcal{S}_A(1\#\mathcal{S}_H(r_{(-1)})) \\
&= (1\#\mathcal{S}_H(\mathcal{S}_R(r_{(0)})_{(-1)}))(\mathcal{S}_R(\mathcal{S}_R(r_{(0)})_{(0)})\#1)(1\#\mathcal{S}_H^2(r_{(-1)})) \\
&= (1\#\mathcal{S}_H(r_{(0)(-1)}))(\mathcal{S}_R^2(r_{(0)(0)})\#1)(1\#\mathcal{S}_H^2(r_{(-1)})) \\
&= (1\#\mathcal{S}_H(r_{(-1)2}))(\mathcal{S}_R^2(r_{(0)})\#1)(1\#\mathcal{S}_H^2(r_{(-1)1})) \\
&= \mathcal{S}_H(r_{(-1)3})(\mathcal{S}_R^2(r_{(0)}))\#\mathcal{S}_H(r_{(-1)2})\mathcal{S}_H^2(r_{(-1)1}) \\
&= \mathcal{S}_H(r_{(-1)2})(\mathcal{S}_R^2(r_{(0)}))\#\mathcal{S}_H(\varepsilon(r_{(-1)1})) \\
&= \mathcal{S}_H(r_{(-1)})(\mathcal{S}_R^2(r_{(0)})).
\end{aligned}$$

□

Proposition 4.1.8. *Let H be a semisimple Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. If R is an AS-regular algebra of global dimension d_R, then $A = R\#H$ is also AS-regular of global dimension d_R.*

In this case, if $\int_R^l = \Bbbk_{\xi_R}$ and $\int_H^l = \Bbbk_{\xi_H}$, where $\xi_R : R \to \Bbbk$ and $\xi_H : H \to \Bbbk$ are algebra homomorphisms, then $\int_A^l = \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is defined by

$$\xi(r\#h) = \xi_R(r)\,\mathrm{hdet}(h_1)\xi_H(h_2),$$

for all $r\#h \in R\#H$. The rigid dualizing complex of A is isomorphic to ${}_\psi A[d_R]$, where ψ is the algebra automorphism $[\xi]\mathcal{S}_A^2$. To be more precise, ψ is defined by

$$\psi(r\#h) = \xi_R(r^1)\,\mathrm{hdet}((r^2)_{(-1)1}h_1)\mathcal{S}_H((r^2)_{(-1)2})(\mathcal{S}_R^2((r^2)_{(0)}))\#\mathcal{S}_H^2(h_2),$$

for all $r\#h \in R\#H$.

Proof. Let $P_\bullet \to H \to 0$ be a projective A-module resolution of H with each P_i finitely generated. Since H is semisimple, $\Bbbk$ is projective as an H-module. It follows that $\Bbbk \otimes_H P_\bullet \to \Bbbk \to 0$ is a projective A-module resolution of $\Bbbk$. Now the following isomorphism of complexes holds:

$$\mathrm{Hom}_A(\Bbbk \otimes_H P_\bullet, A) \cong \mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_A(P_\bullet, A)).$$

The fact that the trivial module $\Bbbk$ is a finitely generated projective H-module implies that

$$\begin{aligned} \mathrm{Ext}^i_A(\Bbbk, A) &\cong \mathrm{Hom}_H(\Bbbk, \mathrm{Ext}^i_A(H, A)) \\ &\cong \mathrm{Hom}_H(\Bbbk, H) \otimes_H \mathrm{Ext}^i_A(H, A) \end{aligned} \tag{4.5}$$

for all $i > 0$. Following Proposition 4.1.3, we have $\int^l_A \cong \int^l_H \otimes_H \int^l_R \otimes H$ and

$$\dim \mathrm{Ext}^i_A(\Bbbk, {}_AA) = \begin{cases} 0, & i \neq d_R; \\ 1, & i = d_R. \end{cases}$$

Let $\mathbf{e}$ be a non-zero element in $\int^l_R$ and $\mathbf{h}$ a non-zero element in $\int^l_H$. Let $\eta : H \to \Bbbk$ be an algebra homomorphism such that $h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in H$. Then the following equations hold

$$\begin{aligned} (\mathbf{h} \otimes \mathbf{e} \otimes 1) \cdot (r\#h) &= \xi_R(r)\mathbf{h} \otimes \mathbf{e} \otimes h \\ &= \xi_R(r)\mathbf{h} \otimes \varepsilon(h_1)\mathbf{e} \otimes h_2 \\ &= \xi_R(r)\mathbf{h} \otimes \eta(\mathcal{S}_H(h_1))\eta(h_2)\mathbf{e} \otimes h_3 \\ &= \xi_R(r)\eta(\mathcal{S}_H(h_1))\mathbf{h} \otimes h_2 \cdot (\mathbf{e} \otimes 1) \\ &= \xi_R(r)\eta(\mathcal{S}_H(h_1))\varepsilon(h_2)\mathbf{h} \otimes \mathbf{e} \otimes 1 \\ &= \xi_R(r)\,\mathrm{hdet}(h)\mathbf{h} \otimes \mathbf{e} \otimes 1. \end{aligned}$$

This implies that $\int^l_A \cong \Bbbk_\xi$, where ξ is the algebra homomorphism defined in the proposition. Similarly, by Proposition 4.1.4, we have

$$\dim \mathrm{Ext}^i_A(\Bbbk, A_A) = \begin{cases} 0, & i \neq d_R, \\ 1, & i = d_R. \end{cases}$$

Because H is finite dimensional and R is Noetherian, the algebra A is Noetherian as well. Therefore, the left and right global dimensions of A are equal. Since H is semisimple, the global dimension of H is 0. Now it follows from Lemma 4.1.5 that the global dimension of A is d_R. In conclusion, we have proved that A is an AS-regular algebra.

By Proposition 1.5.21, the rigid dualizing complex of A is isomorphic to ${}_{[\xi]\mathcal{S}_A^2}A[d_R]$. For any $r\#h \in R\#H$, we have

$$
\begin{aligned}
& [\xi]\mathcal{S}_A^2(r\#h) \\
\overset{(a)}{=} \;& \mathcal{S}_A^2[\xi](r\#h) \\
\overset{(b)}{=} \;& \xi(r^1\#(r^2)_{(-1)}h_1)\mathcal{S}_A^2((r^2)_{(0)}\#h_2) \\
= \;& \xi_R(r^1)\,\mathrm{hdet}((r^2)_{(-1)}h_1)\mathcal{S}_A^2((r^2)_{(0)})\#\mathcal{S}_H^2(h_2) \\
\overset{(c)}{=} \;& \xi_R(r^1)\,\mathrm{hdet}((r^2)_{(-1)}h_1)\mathcal{S}_H((r^2)_{(0)(-1)})(\mathcal{S}_R^2((r^2)_{(0)(0)}))\#\mathcal{S}_H^2(h_2) \\
= \;& \xi_R(r^1)\,\mathrm{hdet}((r^2)_{(-1)1}h_1)\mathcal{S}_H((r^2)_{(-1)2})(\mathcal{S}_R^2((r^2)_{(0)}))\#\mathcal{S}_H^2(h_2).
\end{aligned}
$$

Equations (a), (b) and (c) follow from [20, Lemma 2.5], Equation (1.4) and Lemma 4.1.7 respectively. Thus the proof is completed. □

Remark 4.1.9. Since ξ is an algebra homomorphism, the following equation holds

$$\xi_R(r)\,\mathrm{hdet}(h) = \xi_R(h_1(r))\,\mathrm{hdet}(h_2).$$

Remark 4.1.10. We show how $\int_{R\#H}^r$ looks like. Let $\mathbf{e}'$ be a non-zero element in $\mathrm{Ext}_R^d(\Bbbk, R)$. There is an algebra homomorphism $\eta' : H \to \Bbbk$ satisfying $\mathbf{e}' \cdot h = \eta'(h)\mathbf{e}'$ for all $h \in H$. Applying a similar argument as in the proof of Proposition 4.1.8, we have that if $\int_R^r = {}_{\xi'_R}\Bbbk$, then $\int_A^r = {}_{\xi'}\Bbbk$, where ξ' is defined by $\xi'(r\#h) = \xi'_R(\mathcal{S}_H^{-1}(h_1)(r))\eta'(\mathcal{S}_H(h_2))$ for all $r\#h \in R\#H$.

Now we give the main theorem of this section.

Theorem 4.1.11. *Let H be a semisimple Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}_H^H\mathcal{YD}$. Suppose that the algebra R is CY of dimension d_R. Then $R\#H$ is CY if and only if the homological determinant of R is trivial and the algebra automorphism ϕ defined by*

$$\phi(r\#h) = \mathcal{S}_H(r_{(-1)})(\mathcal{S}_R^2(r_{(0)}))\mathcal{S}_H^2(h)$$

for all $r\#h \in R\#H$ is an inner automorphism.

Proof. From Proposition 1.5.19, we have that R is AS-regular with $\int_R^l \cong \Bbbk$. In addition, since H is semisimple, the algebra H is unimodular. Thus $\int_H^l = \Bbbk$. Set $A = R\#H$. By Proposition 4.1.8, we obtain that A is AS-regular with $\int_A^l \cong \Bbbk_\xi$, where ξ is the algebra homomorphism defined by $\xi(r\#h) = \varepsilon(r)\,\mathrm{hdet}(h)$ for all $r\#h \in R\#H$. Then following from Proposition 1.5.22, the algebra A is CY if and only if $\xi = \varepsilon$ and

$\mathcal{S}_A^2$ is an inner automorphism. On one hand, $\xi = \varepsilon_H$ if and only if $\mathrm{hdet} = \varepsilon_H$. On the other hand, by Lemma 4.1.7, we have $\mathcal{S}_A^2(r\#h) = \mathcal{S}_H(r_{(-1)})(\mathcal{S}_R^2(r_{(0)}))\mathcal{S}_H^2(h)$, for any $r\#h \in R\#H$. □

Remark 4.1.12. In [52] it is proved that if R is p-Koszul CY and H is involutory, then $R\#H$ is CY if and only if the homological determinant is trivial. Thus in Theorem 4.1.11, if the braided Hopf algebra R is p-Koszul, then we have that the homological determinant is trivial implies that the automorphism ϕ is inner. In the following Example 4.1.13, we see that the automorphism ϕ can be expressed via the homological determinant of the H-action.

Example 4.1.13. Let

$$\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$$

be a datum of finite Cartan type, where Γ is a finite abelian group and (a_{ij}) is of type $A_1 \times \cdots \times A_1$. Assume that V is a braided vector space with a basis $\{x_1, \cdots, x_\theta\}$ whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \ \ 1 \leqslant i, j \leqslant \theta,$$

where $q_{ij} = \chi_j(g_i)$.

Let R be the following algebra:

$$\Bbbk\langle x_1, \cdots, x_\theta \mid x_i x_j = q_{ij} x_j x_i, \ \ 1 \leqslant i < j \leqslant \theta\rangle.$$

It is easy to see that R is a Koszul braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Assume that $\mathcal{K}$ is the Koszul complex (cf. complex (1.19))

$$0 \to R \otimes R_\theta^{!*} \to \cdots R \otimes R_j^{!*} \xrightarrow{d_j} R \otimes R_{j-1}^{!*} \cdots \to R \otimes R_1^{!*} \to R.$$

Then we have that $\mathcal{K} \to {}_R\Bbbk \to 0$ is a projective resolution of $\Bbbk$. Each $R_j^{!*}$ is a left $\Bbbk\Gamma$-module with module structure defined by

$$\begin{aligned}
[g(\beta)](x_{i_1}^* \wedge \cdots \wedge x_{i_j}^*) &= \beta(g^{-1}(x_{i_1}^* \wedge \cdots \wedge x_{i_j}^*)) \\
&= \beta(g^{-1}(x_{i_1}^*) \wedge \cdots \wedge g^{-1}(x_{i_j}^*)) \\
&= (\textstyle\prod_{t=1}^{j} \chi_{i_t}(g))\beta(x_{i_1}^* \wedge \cdots \wedge x_{i_j}^*),
\end{aligned}$$

where $\beta \in S_j^{!*}$. Thus each $R \otimes R_j^{!*}$ is a left $\Bbbk\Gamma$-module. It is not difficult to see that the differentials in the Koszul complex are also left Γ-module homomorphisms. By [23, Prop. 5.0.7], we have that $\int_R^l \cong R_\theta^{!*}$. Therefore, $\mathrm{hdet}(g) = \prod_{i=1}^{\theta} \chi_i(g^{-1})$ for all $g \in \Gamma$.

If for each $1 \leqslant i \leqslant \theta$, $q_{1i} \cdots q_{(i-1)i} = q_{i(i+1)} \cdots q_{i\theta}$, then the algebra R is a CY algebra by Remark 1.5.14. In this case,

$$\begin{aligned} \mathrm{hdet}(g_j) &= \prod_{i=1}^{\theta} \chi_i(g_j^{-1}) \\ &= (\prod_{i=1}^{j-1} \chi_i(g_j^{-1}))\chi_j(g_j^{-1})(\prod_{k=j+1}^{\theta} \chi_k(g_j^{-1})) \\ &= (\prod_{i=1}^{j-1} q_{ij})\chi_j(g_j^{-1})(\prod_{k=j+1}^{\theta} q_{jk}^{-1}) \\ &= \chi_j(g_j^{-1}). \end{aligned}$$

The algebra automorphism ϕ given in Theorem 4.1.11 is defined by

$$\phi(x_j) = \chi_j(g_j^{-1})x_j = \mathrm{hdet}(g_j)x_j$$

for all $1 \leqslant j \leqslant \theta$ and $\phi(g) = g$ for all $g \in \Gamma$. However, $\chi_j(g_j) \neq 1$ for all $1 \leqslant j \leqslant \theta$. The algebra $R\#\Bbbk\Gamma$ is not a CY algebra.

Example 4.1.14. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Assume that there is a group homomorphism $\nu : \Gamma \to Aut_{Lie}(\mathfrak{g})$, where $Aut_{Lie}(\mathfrak{g})$ is the group of Lie algebra automorphisms of $\mathfrak{g}$. Then it is known that $U(\mathfrak{g})\#\Bbbk\Gamma$ is a cocommutative Hopf algebra.

It is proved in [35, Cor. 3.6] that the smash product $U(\mathfrak{g})\#\Bbbk\Gamma$ is CY if and only if $U(\mathfrak{g})$ is CY and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$.

Let d be the dimension of $\mathfrak{g}$. By [35, Lemma 3.1], we have $\int_{U(\mathfrak{g})}^l \cong \wedge^d \mathfrak{g}^*$ as left Γ-modules, where the left Γ-action on $\mathfrak{g}^*$ is defined by $(g \cdot \alpha)(x) = \alpha(g^{-1}x)$ for all $g \in \Gamma$, $\alpha \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$, and Γ acts on $\wedge^d \mathfrak{g}^*$ diagonally. Let $\{x_1, \cdots, x_d\}$ be a basis of $\mathfrak{g}$. Then

$$g(x_1^* \wedge \cdots \wedge x_d^*) = \det(\nu(g^{-1}))(x_1^* \wedge \cdots \wedge x_d^*),$$

for all $g \in \Gamma$. So $\mathrm{hdet}(g) = \det(\nu(g))$. That is, if $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$, then the homological determinant is trivial. The algebra $U(\mathfrak{g})$ is a braided Hopf algebra in the category ${}_\Gamma^\Gamma\mathcal{YD}$ with trivial coaction. So the automorphism ϕ defined in Theorem 4.1.11 is the identity. Therefore, if $U(\mathfrak{g})$ is a CY algebra and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$, by Theorem 4.1.11, the algebra $U(\mathfrak{g})\#\Bbbk\Gamma$ is a CY algebra. This coincides with the result mentioned before.

4.2 Rigid dualizing complexes of braided Hopf algebras over finite group algebras

Before giving the main results of this section, we need some preparations first.

Let A be a Hopf algebra. By [61, Appendix, Lemma 11], A can be viewed as a subalgebra of A^e via the algebra homomorphism $\rho : A \to A^e$ defined by

$$\rho(a) = a_1 \otimes \mathcal{S}(a_2). \tag{4.6}$$

Then A^e is a right A-module via this embedding. We denote this right A-module by $\mathcal{R}(A^e)$. Actually, $\mathcal{R}(A^e)$ is an A^e-A-bimodule. Similarly, A^e is also an A-A^e-bimodule, where the left A-module is induced from the homomorphism ρ. Denote this bimodule by $\mathcal{L}(A^e)$.

In this section, we further assume the characteristic of the base field $\Bbbk$ is 0. From now on, let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with Γ-coaction δ. The bosonization $A = R\#\Bbbk\Gamma$ is a usual Hopf algebra [62]. Let $\mathscr{D}$ be the subalgebra of A^e generated by the elements of the form $(r\#g)\otimes(s\#g^{-1})$ with $r, s \in R$ and $g \in \Gamma$.

Remark 4.2.1. Since R is a Γ-comodule, it is a Γ-graded module: $R = \oplus_{g\in\Gamma} R_g$, where $R_g = \{r \in R \mid \delta(r) = g \otimes r\}$. Therefore, for any $r \in R$, it can be written as $r = \sum_{g\in\Gamma} r_g$ with $r_g \in R_g$. Then $\delta(r) = \sum_{g\in\Gamma} g \otimes r_g$.

Lemma 4.2.2. *The subalgebra $\mathscr{D}$ is a left (resp. right) A-submodule of $\mathcal{L}(A^e)$ (resp. $\mathcal{R}(A^e)$).*

Proof. For any $r\#h \in A$, by equations (1.4) and (1.5), we have

$$\Delta(r\#h) = \sum_{g\in\Gamma} r^1\#gh \otimes (r^2)_g\#h$$

and

$$\mathcal{S}_A(r\#h) = \sum_{g\in\Gamma} h^{-1}g^{-1}\mathcal{S}_R(r_g).$$

Any element in $\mathscr{D}$ can be written as a linear combination of elements of the form

$s\#k \otimes t\#k^{-1} \in \mathscr{D}$ with $s, t \in R$ and $k \in \Gamma$.

$$\begin{array}{rl} & (r\#h) \cdot (s\#k \otimes t\#k^{-1}) \\ = & \sum_{g\in\Gamma}(r^1\#gh)(s\#k) \otimes (t\#k^{-1})\mathcal{S}_A((r^2)_g\#h) \\ = & \sum_{g\in\Gamma}(r^1\#gh)(s\#k) \otimes (t\#k^{-1})h^{-1}g^{-1}\mathcal{S}_R((r^2)_g) \\ = & \sum_{g\in\Gamma}(r^1(gh)(s)\#ghk) \otimes (t(k^{-1}h^{-1}g^{-1})(\mathcal{S}_R((r^2)_g))\#k^{-1}h^{-1}g^{-1}) \\ \in & \mathscr{D}. \end{array}$$

This shows that $\mathscr{D}$ is a left A-submodule of $\mathcal{L}(A^e)$. Similarly, $\mathscr{D}$ is also a right A-submodule of $\mathcal{R}(A^e)$. □

The following lemma is known, we include it for completeness.

Lemma 4.2.3. *(1) Both $\mathcal{L}(A^e)$ and $\mathcal{R}(A^e)$ are free as A-modules.*

(2) $\mathcal{R}(A^e) \otimes_A \Bbbk \cong A$ as left A^e-modules and this isomorphism restricts to a left R^e-isomorphism $\mathscr{D} \otimes_A \Bbbk \cong R$.

(3) If $\xi : A \to \Bbbk$ is an algebra homomorphism, then there is an isomorphism $\Bbbk_\xi \otimes_A \mathcal{L}(A^e) \cong A_{[\xi]\mathcal{S}_A^2}$ of right A^e-modules and the isomorphism restricts to a right R^e-isomorphism $\Bbbk_\xi \otimes_A \mathscr{D} \cong R_{([\xi]\mathcal{S}_A^2)|_R}$.

Proof. (1) was proved in [20, Lemma 2.2]. The module $L(A^e)$ defined in that paper is isomorphic to $\mathcal{R}(A^e)$ as right A-modules. It was proved that $\varphi : A_A \otimes A^{op} \to \mathcal{R}(A^e)$ defined by $\varphi(a \otimes b) = a_1 \otimes b \star \mathcal{S}_A(a_2)$ is an isomorphism, where $\star$ denotes the multiplication in A^{op}. The right A-module structure on $A_A \otimes A^{op}$ is defined by $(a \otimes b) \cdot c = ac \otimes b$ for all a, b and $c \in A$. Similarly, $\mathcal{L}(A^e) \cong {}_AA \otimes A^{op}$ as free left A-module.

(2) $\mathcal{R}(A^e) \otimes_A \Bbbk \cong A$ as left A^e-modules is [61, Appendix, Lemma 11]. The homomorphism $\psi : \mathcal{R}(A^e) \otimes_A \Bbbk \to A$ given by $\psi(a \otimes b \otimes 1) = ab$ is an A^e-isomorphism. It is clear that ψ restricts to an isomorphism from $\mathscr{D} \otimes_A \Bbbk$ to R.

(3) It was proved in [20, Lemma 4.5] that $\Bbbk_\xi \otimes_A \mathcal{L}(A^e) \cong A_{[\xi]\mathcal{S}_A^2}$ as right A^e-modules. Here we give another proof. We construct the the isomorphism explicitly. Define a homomorphism $\Phi : \Bbbk_\xi \otimes_A \mathcal{L}(A^e) \to A_{[\xi]\mathcal{S}_A^2}$ by $\Phi(1 \otimes a \otimes b) = \xi(a_1)b\mathcal{S}_A^2(a_2)$ and a homomorphism $\Psi : A_{[\xi]\mathcal{S}_A^2} \to \Bbbk_\xi \otimes_A A^e$ by $\Psi(a) = 1 \otimes 1 \otimes a$. Note that $[\xi]\mathcal{S}^2 = \mathcal{S}^2[\xi]$

holds by Lemma 2.5 in [20]. For any $x, a, b \in A$, we have

$$\begin{aligned}
\Phi(1 \otimes x_1 a \otimes b\mathcal{S}(x_2)) &= \xi(x_1)\xi(a_1)b\mathcal{S}(x_3)\mathcal{S}^2(x_2)\mathcal{S}^2(a_2) \\
&= \xi(x_1)\xi(a_1)b\mathcal{S}(\varepsilon(x_2))\mathcal{S}^2(a_2) \\
&= \xi(x)\xi(a_1)b\mathcal{S}^2(a_2) \\
&= \xi(x)\Phi(1 \otimes a \otimes b).
\end{aligned}$$

This shows that Φ is well defined. Similar calculations show that Φ and Ψ are right A^e-module homomorphisms and they are inverse to each other.

It is straightforward to check that the isomorphism $\Bbbk_\xi \otimes_A \mathcal{L}(A^e) \cong A_{[\xi]\mathcal{S}_A^2}$ restricts to the isomorphism $\Bbbk_\xi \otimes_A \mathscr{D} \cong R_{([\xi]\mathcal{S}_A^2)|_R}$. $\square$

Lemma 4.2.4. $\mathrm{Hom}_{R^e}(\mathscr{D}, R^e) \cong \mathscr{D}$ *as* A-R^e*-bimodules.*

Proof. The algebra $\mathscr{D}$ is an A-R^e-bimodule. Note that the A-module structure is induced from the homomorphism ρ defined in (4.6). On the other hand, the A-R^e-bimodule structure on $\mathrm{Hom}_{R^e}(\mathscr{D}, R^e)$ is induced from the right A-module structure on $\mathscr{D}$ and the right R^e-module structure on R^e. We have $r\#g = (1\#g)(g^{-1}(r)\#1)$ for any $r\#g \in R\#\Bbbk\Gamma$. Therefore, an element in $\mathscr{D}$ can be expressed of the form $\sum_{g\in\Gamma}(1\#g^{-1})(r^g\#1) \otimes s^g\#g$ with $r^g, s^g \in R$. For simplicity, we write an element $(1\#g)(r\#1)$ with $r \in R$ and $g \in \Gamma$ as gr. Let $\Psi : \mathscr{D} \to \mathrm{Hom}_{R^e}(\mathscr{D}, R^e)$ be a homomorphism defined by

$$[\Psi(\sum_{g\in\Gamma} g^{-1}r^g \otimes (s^g\#g))](h \otimes h^{-1}) = r^h \otimes s^h,$$

for $\sum_{g\in\Gamma} g^{-1}r^g \otimes s^g\#g \in \mathscr{D}, h \in \Gamma$. Next define a homomorphism $\Phi : \mathrm{Hom}_{R^e}(\mathscr{D}, R^e) \to \mathscr{D}$ by

$$\Phi(f) = \sum_{g\in\Gamma}(g^{-1} \otimes g)f(g \otimes g^{-1})$$

for $f \in \mathrm{Hom}_{R^e}(\mathscr{D}, R^e)$. It is clear that Φ is a right R^e-homomorphism. On the other hand, we have

$$\begin{aligned}
\Phi((r\#h)f) &= \textstyle\sum_{g\in\Gamma}(g^{-1} \otimes g)((r\#h)f)(g \otimes g^{-1}) \\
&= \textstyle\sum_{g\in\Gamma}\sum_{k\in\Gamma}(g^{-1} \otimes g)f(g(r^1\#k)h \otimes \mathcal{S}_A((r^2)_k\#h)g^{-1}) \\
&= \textstyle\sum_{g\in\Gamma}\sum_{k\in\Gamma}(g^{-1} \otimes g)f(g(r^1\#k)h \otimes h^{-1}k^{-1}\mathcal{S}_R((r^2)_k)g^{-1}) \\
&= \textstyle\sum_{g\in\Gamma}\sum_{k\in\Gamma}(g^{-1} \otimes g)f(g(r^1)\#gkh \otimes h^{-1}k^{-1}g^{-1}g(\mathcal{S}_R((r^2)_k)))
\end{aligned}$$

and

$$
\begin{array}{rl}
& (r\#h)\Phi(f) \\
= & (\sum_{k\in\Gamma} r_1\#kh\otimes h^{-1}k^{-1}\mathcal{S}_R(r_{2k}))\sum_{g\in\Gamma}(g^{-1}\otimes g)f(g\otimes g^{-1}) \\
= & \sum_{k\in\Gamma}\sum_{g\in\Gamma}(r^1\#khg^{-1}\otimes gh^{-1}k^{-1}\mathcal{S}_R((r^2)_k))f(g\otimes g^{-1}) \\
= & \sum_{k\in\Gamma}\sum_{g\in\Gamma}(khg^{-1}(gh^{-1}k^{-1})(r^1)\otimes(gh^{-1}k^{-1})\mathcal{S}_R((r^2)_k)gh^{-1}k^{-1})f(g\otimes g^{-1}) \\
= & \sum_{k\in\Gamma}\sum_{g\in\Gamma}(khg^{-1}\otimes gh^{-1}k^{-1})f((gh^{-1}k^{-1})(r^1)\#g\otimes g^{-1}(gh^{-1}k^{-1})(\mathcal{S}_R((r^2)_k))) \\
= & \sum_{g\in\Gamma}\sum_{k\in\Gamma}(g^{-1}\otimes g)f(g(r^1)\#gkh\otimes h^{-1}k^{-1}g^{-1}g(\mathcal{S}_R((r^2)_k))).
\end{array}
$$

So Φ is an A-R^e-bimodule homomorphism. It is clear that Φ and Ψ are inverse to each other. Thus Φ is an isomorphism. □

Lemma 4.2.5. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. If $A = R\#\Bbbk\Gamma$ is AS-Gorenstein with $\int_A^l \cong \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is an algebra homomorphism, then we have R-R-bimodule isomorphisms*

$$
\operatorname{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0, & i \neq d; \\ R_{([\xi]\mathcal{S}_A^2)|_R}, & i = d. \end{cases}
$$

Proof. We have the following isomorphisms,

$$
\begin{array}{rcl}
\operatorname{Ext}^i_{R^e}(R, R^e) & \cong & \operatorname{Ext}^i_{R^e}(\mathscr{D}\otimes_A \Bbbk, R^e) \\
& \cong & \operatorname{Ext}^i_A({}_A\Bbbk, \operatorname{Hom}_{R^e}(\mathscr{D}, R^e)) \\
& \cong & \operatorname{Ext}^i_A({}_A\Bbbk, \mathscr{D}) \\
& \cong & \operatorname{Ext}^i_A({}_A\Bbbk, A)\otimes_A \mathscr{D} \\
& \cong & \begin{cases} 0, & i\neq d; \\ \Bbbk_\xi \otimes_A \mathscr{D} \cong R_{([\xi]\mathcal{S}_A^2)|_R}, & i = d. \end{cases}
\end{array}
$$

The first, third and last isomorphism follow from Lemma 4.2.3, Lemma 4.2.4 and Lemma 4.2.3 respectively. The fourth isomorphism follows from the fact that $\mathscr{D}$ is left A-projective. This is because A^e is free as a left A-module by Lemma 4.2.3 and A^e is a direct sum of finite copies of $\mathscr{D}$. Indeed, $A^e \cong \oplus_{h\in\Gamma}\mathscr{D}^h$, where $\mathscr{D}^h$ is the left A-submodule of A^e generated by elements of the form $(r\#gh)\otimes(s\#g^{-1})$ with $r, s \in R$ and $g \in \Gamma$. Moreover, for every $h \in \Gamma$, $\mathscr{D}^h$ is isomorphic to $\mathscr{D}$ as a left A-module. □

Lemma 4.2.6. *If the global dimension of $A = R\#\Bbbk\Gamma$ is finite and R is Noethrian, then R is homologically smooth.*

Proof. By assumption, the algebra A is Noetherian, and ${}_A\Bbbk$ has a finite projective resolution

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \Bbbk \to 0,$$

such that each P_i, $0 \leqslant i \leqslant d$, is a finitely generated projective A-module. By a similar argument to the one in the proof of Lemma 4.2.5, we have that $\mathscr{D}$ is projective as a right A-module. Therefore, the functor $\mathscr{D} \otimes_A -$ is exact. We obtain an exact sequence

$$0 \to \mathscr{D} \otimes_A P_d \to \mathscr{D} \otimes_A P_{d-1} \to \cdots \to \mathscr{D} \otimes_A P_1 \to \mathscr{D} \otimes_A P_0 \to \mathscr{D} \otimes_A \Bbbk \to 0. \quad (4.7)$$

$\mathscr{D}$ is projective as left R^e-module and $\mathscr{D} \otimes_A \Bbbk \cong R$ as left R^e-modules (Lemma 4.2.3). So the complex (4.7) is a projective bimodule resolution of R. Because each P_i is a finitely generated A-module and Γ is a finite group, each $\mathscr{D} \otimes_A P_i$ is a finitely generated left R^e-module. Therefore, we conclude that R is homologically smooth. □

The homological integral of the skew group algebra $R\#\Bbbk\Gamma$ was discussed by He, Van Oystaeyen and Zhang in [35]. Based on their work, here we use the homological determinant of the group action to describe the homological integral of $R\#\Bbbk\Gamma$.

Lemma 4.2.7. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. If R is an AS-Gorenstein algebra with injective dimension d and $\int_R^l \cong \Bbbk_{\xi_R}$, where $\xi_R : R \to \Bbbk$ is an algebra homomorphism, then the algebra $A = R\#\Bbbk\Gamma$ is AS-Gorenstein with injective dimension d as well, and $\int_A^l \cong \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is the algebra homomorphism defined by $\xi(r\#h) = \xi_R(r)\operatorname{hdet}(h)$ for any $r\#h \in R\#\Bbbk\Gamma$.*

Proof. By [35, Prop. 1.1 and 1.3], we have that $A = R\#\Bbbk\Gamma$ is AS-Gorenstein of injective dimension d, $\int_R^l$ is a 1-dimensional left Γ-module, and as right A-modules:

$$\textstyle\int_A^l \cong (\int_R^l \otimes \Bbbk\Gamma)^{\Gamma},$$

where the right A-module structure on $\int_R^l \otimes \Bbbk\Gamma$ is defined by

$$(e \otimes g) \cdot (r\#h) = e(g(r)) \otimes gh,$$

for $g \in \Bbbk\Gamma$, $r\#h \in R\#\Bbbk\Gamma$ and $e \in \int_R^l$, and the left Γ-action on $\int_R^l \otimes \Bbbk\Gamma$ is diagonal. Let $\mathbf{e}$ be a basis of $\int_R^l$. It can be checked directly that the element $\sum_{g\in\Gamma} g(\mathbf{e}) \otimes g$ is a basis of $(\int_R^l \#\Bbbk\Gamma)^{\Gamma}$. Let $\eta : \Bbbk\Gamma \to \Bbbk$ be an algebra homomorphism such that

$h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in \Gamma$. For any $r\#h \in R\#\Bbbk\Gamma$, we have

$$\begin{aligned}
(\textstyle\sum_{g\in\Gamma} g(\mathbf{e})\#g)(r\#h) &= \textstyle\sum_{g\in\Gamma} g(\mathbf{e})g(r)\#gh \\
&= \textstyle\sum_{g\in\Gamma} g(\mathbf{e}r)\#gh \\
&= \xi_R(r)\textstyle\sum_{g\in\Gamma} g(\mathbf{e})\#gh \\
&= \xi_R(r)\eta(h^{-1})\textstyle\sum_{g\in\Gamma} (gh)(\mathbf{e})\#gh \\
&= \xi_R(r)\eta(h^{-1})\textstyle\sum_{g\in\Gamma} g(\mathbf{e})\#g \\
&= \xi_R(r)\,\mathrm{hdet}(h)\textstyle\sum_{g\in\Gamma} g(\mathbf{e})\#g \\
&= \xi(r\#h)\textstyle\sum_{g\in\Gamma} g(\mathbf{e})\#g.
\end{aligned}$$

It implies that $\int_A^l \cong \Bbbk_\xi$. □

There is also a connection between the AS-regularity of R and $R\#\Bbbk\Gamma$.

Proposition 4.2.8. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Then R is AS-regular if and only if $A = R\#\Bbbk\Gamma$ is AS-regular.*

Proof. Assume that R is AS-regular. By Lemma 4.2.7, the algebra A is AS-Gorenstein. To show that A is AS-regular, it suffices to show that the global dimension of A is finite. Since the global dimension of R is finite, there is a finite projective resolution of $\Bbbk$ over R,

$$0 \to P_d \to P_{d-1} \to \cdots P_1 \to P_0 \to \Bbbk \to 0.$$

Note that A is projective as a right R-module. Tensoring this resolution with $A \otimes_R -$, we obtain an exact sequence

$$0 \to A \otimes_R P_d \to A \otimes_R P_{d-1} \to \cdots A \otimes_R P_1 \to A \otimes_R P_0 \to A \otimes_R \Bbbk \to 0.$$

It is clear that each $A \otimes_R P_i$ is projective. This shows that the projective dimension of $A \otimes_R \Bbbk$ is finite. But ${}_A\Bbbk$ is a direct summand of $A \otimes_R \Bbbk$ as an A-module ([11, Lemma III.4.8]). So the projective dimension of ${}_A\Bbbk$ is finite. Since A is a Hopf algebra, the global dimension of A is finite.

Conversely, if A is AS-regular, then R is AS-regular by Lemma 4.2.5, Lemma 4.2.6 and Remark 1.5.20. □

We give the rigid dualzing complex of an AS-Gorenstein braided Hopf algebra in the following theorem.

Theorem 4.2.9. *Let Γ be a finite group and R a braided Hopf algebra in the category*

${}^{\Gamma}_{\Gamma}\mathcal{YD}$. *Assume that R is an AS-Gorenstein algebra with injective dimension d. If $\int_R^l \cong \Bbbk_{\xi_R}$, for some algebra homomorphism $\xi_R : R \to \Bbbk$, then R has a rigid dualizing complex ${}_{\varphi}R[d]$, where φ is the algebra automorphism defined by*

$$\varphi(r) = \sum_{g\in\Gamma} \xi_R(r^1)\,\mathrm{hdet}(g)g^{-1}(\mathcal{S}_R^2((r^2)_g)),$$

for any $r \in R$.

Proof. Put $A = R\#\Bbbk\Gamma$. It follows from Lemma 4.2.7 that A is AS-Gorenstein with $\int_A^l \cong \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is the algebra homomorphism defined by

$$\xi(r\#h) = \xi_R(r)\,\mathrm{hdet}(h)$$

for any $r\#h \in R\#\Bbbk\Gamma$. By Lemma 4.2.5, there are R-R-bimodule isomorphisms

$$\mathrm{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0, & i \neq d; \\ R_{([\xi]\mathcal{S}_A^2)|_R}, & i = d. \end{cases}$$

For any $r \in R$,

$$\begin{array}{rcl} [\xi]\mathcal{S}_A^2(r) & = & \sum_{g\in\Gamma} \xi(r^1\#g)\mathcal{S}_A^2((r^2)_g) \\ & = & \sum_{g\in\Gamma} \xi_R(r^1)\,\mathrm{hdet}(g)\mathcal{S}_A^2((r^2)_g) \\ & = & \sum_{g\in\Gamma} \xi_R(r^1)\,\mathrm{hdet}(g)g^{-1}(\mathcal{S}_R^2((r^2)_g)). \end{array}$$

Now the theorem follows from Lemma 1.5.11. □

Remark 4.2.10. The algebra $A = R\#\Bbbk\Gamma$ has a rigid dualizing complex ${}_{[\xi]\mathcal{S}_A^2}A[d]$ (Proposition 1.5.21). Observe that the algebra automorphism φ given in Theorem 4.2.9 is just the restriction of $[\xi]\mathcal{S}_A^2$ on R.

Now we can characterize the CY property of R in case $R\#\Bbbk\Gamma$ is CY.

Theorem 4.2.11. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Define an algebra automorphism φ of R by*

$$\varphi(r) = \sum_{g\in\Gamma} g^{-1}(\mathcal{S}_R^2(r_g)),$$

for any $r \in R$. If $R\#\Bbbk\Gamma$ is a CY algebra, then R is CY if and only if the algebra automorphism φ is an inner automorphism.

Proof. Assume that $A = R\#\Bbbk\Gamma$ is a CY algebra of dimension d. By Proposition 1.5.19, A is AS-regular of global dimension d and $\int_A^l \cong \Bbbk$. It follows from Lemma 4.2.6 that R is homologically smooth.

Since $\int_A^l \cong \Bbbk$, by Lemma 4.2.5 there are R-R-bimodule isomorphisms

$$\operatorname{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0, & i \neq d; \\ R_{\mathcal{S}_A^2|_R}, & i = d. \end{cases}$$

Following Remark 1.5.20, we obtain that R is AS-regular. Suppose $\int_R^l \cong \Bbbk_{\xi_R}$ for some algebra homomorphism $\xi_R : R \to \Bbbk$. Then by Lemma 4.2.7, $\int_A^l \cong \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is defined by $\xi(r\#h) = \xi_R(r)\,\mathrm{hdet}(h)$ for any $r\#h \in R\#\Bbbk\Gamma$. But $\int_A^l \cong \Bbbk$. Therefore, $\xi_R = \varepsilon_R$ and $\mathrm{hdet} = \varepsilon_H$. It follows from Theorem 4.2.9 that the rigid dualizing complex of R is isomorphic to ${}_\varphi R[d]$, where φ is defined by

$$\begin{aligned} \varphi(r) &= \textstyle\sum_{g\in\Gamma} \xi_R(r^1)\,\mathrm{hdet}(g) g^{-1}(\mathcal{S}_R^2((r^2)_g)) \\ &= \textstyle\sum_{g\in\Gamma} g^{-1}(\mathcal{S}_R^2(r_g)) \end{aligned}$$

for any $r \in R$. Now the theorem follows from Corollary 1.5.12. □

Corollary 4.2.12. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^\Gamma_\Gamma\mathcal{YD}$. Assume that R is an AS-regular algebra. Then the following two conditions are equivalent:*

(1) Both R and $R\#\Bbbk\Gamma$ are CY algebras.

(2) These three conditions are satisfied:

(i) $\int_R^l \cong \Bbbk$;

(ii) The homological determinant of the group action is trivial;

(iii) The algebra automorphism φ defined by

$$\varphi(r) = \sum_{g\in\Gamma} g^{-1}(\mathcal{S}_R^2(r_g))$$

for all $r \in R$ is an inner automorphism.

Proof. (1) $\Rightarrow$ (2) Since R is a CY algebra, by Proposition 1.5.19 we have $\int_R^l \cong \Bbbk$. Because both R and $R\#\Bbbk\Gamma$ are CY, (ii) and (iii) are satisfied by Theorem 4.1.11 and Theorem 4.2.11.

(2) $\Rightarrow$ (1) Since R is AS-regular, $R\#\Bbbk\Gamma$ is AS-regular by Proposition 4.2.8. Then R is homologically smooth (Lemma 4.2.6). By Theorem 4.2.9, if the three conditions in (2) are satisfied, then the rigid dualizing complex of R is isomorphic to $R[d]$, where d is the injective dimension of R. So R is a CY algebra. That the algebra $R\#\Bbbk\Gamma$ is a CY algebra follows from Theorem 4.1.11. $\square$

Example 4.2.13. Let us use the notations in Example 4.1.14. Assume that Γ is a finite group, $\mathfrak{g}$ is a finite dimensional Γ-module Lie algebra, and there is a group homomorphism $\nu : \Gamma \to Aut_{Lie}(\mathfrak{g})$. In Example 4.1.14, we use Theorem 4.1.11 to obtain that if $U(\mathfrak{g})$ is a CY algebra and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$ then $U(\mathfrak{g})\#\Bbbk\Gamma$ is a CY algebra. Now by Theorem 4.2.11, if $U(\mathfrak{g})\#\Bbbk\Gamma$ is a CY algebra, then $U(\mathfrak{g})$ is a CY algebra as well. This is because $U(\mathfrak{g})$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with trivial coaction, the algebra automorphism φ in Theorem 4.2.11 is the identity.

By [20, Prop. 6.3], we have that $\int_{U(\mathfrak{g})}^l = \Bbbk_\xi$, where $\xi(x) = \mathrm{tr}(\mathrm{ad}(x))$ for all $x \in \mathfrak{g}$. We calculate in Example 4.1.14 that $\mathrm{hdet}(g) = \det(\nu(g))$ for $g \in \Gamma$. Therefore, both $U(\mathfrak{g})$ and $U(\mathfrak{g})\#\Bbbk\Gamma$ are CY algebras if and only if $\mathrm{tr}(\mathrm{ad}(x)) = 0$ for all $x \in \mathfrak{g}$ and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$. This coincides with Corollary 3.5 and Lemma 4.1 in [35].

Let

$$\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$$

be a datum of finite Cartan type for a finite abelian group Γ. Let $\{\alpha_1, \cdots, \alpha_\theta\}$ be a set of simple roots of the root system corresponding to the Cartan matrix (a_{ij}). Assume that $w_0 = s_{i_1} \cdots s_{i_p}$ is a reduced decomposition of the longest element in the Weyl group as a product of simple reflections. Then the positive roots are as follows

$$\beta_1 = \alpha_{i_1}, \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}).$$

Let λ be a family of linking parameters for $\mathcal{D}$.

Applying [8, Thm. 3.3] and a similar argument as in the proof of Theorem 2.1.4, we obtain that $A = U(\mathcal{D}, \lambda)$ is AS-regular of global dimension p and $\int_A^l = \Bbbk_\xi$, where ξ is the algebra homomorphism defined by $\xi(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$, for all $g \in \Gamma$ and $\xi(x_i) = 0$ for all $1 \leqslant i \leqslant \theta$. In addition, A has a rigid dualizing complex ${}_{[\xi]\mathcal{S}_A^2}A[p]$.

By Proposition 1.5.22, A is CY if and only if $\prod_{i=1}^{p} \chi_{\beta_i} = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism.

Let R be the algebra generated by $x_1, \cdots, x_\theta$ subject to the relations

$$(\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, 1 \leqslant i, j \leqslant \theta, \ \ i \neq j.$$

Then $U(\mathcal{D}, 0) = R\#\Bbbk\Gamma$. By Lemma 4.2.6 and Lemma 4.2.5, we have that R is homologically smooth, and that it has a rigid dualizing complex ${}_{\varphi}R[p]$, where φ is the restriction of $[\xi]\mathcal{S}_A^2$ on R. That is, φ is defined by $\varphi(x_k) = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k)(x_k)$, $1 \leqslant k \leqslant \theta$, where each $1 \leqslant j_k \leqslant p$ is the integer such that $\beta_{j_k} = \alpha_k$. Therefore, R is CY if and only if $\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = 1$ for each $1 \leqslant k \leqslant \theta$.

One may compare these results with Theorem 2.1.5, Theorem 3.1.9 and Lemma 3.2.1.

Now we give two examples of CY pointed Hopf algebra with a finite group of group-like elements.

Example 4.2.14. Let A be $U(\mathcal{D}, \lambda)$ with the datum $(\mathcal{D}, \lambda)$ given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$;
- The Cartan matrix is of type A_2;
- $g_i = y_i$, $1 \leqslant i \leqslant 2$;
- χ_i, $1 \leqslant i \leqslant 2$, are given by the following table.

	y_1	y_2
χ_1	-1	1
χ_2	-1	-1

- $\lambda = 0$

The algebra A is a CY algebra of dimension 3.

Let R be the algebra generated by x_1 and x_2 subject to relations

$$x_1^2 x_2 - x_2 x_1^2 = 0 \text{ and } x_2^2 x_1 - x_1 x_2^2 = 0.$$

Then $A = R\#\Bbbk\Gamma$. The rigid dualizing complex of R is ${}_{\varphi}R[3]$, where $\varphi = -\,\mathrm{id}$.

Remark 4.2.15. From the proof of Proposition 2.2.9, we can see that if $A = U(\mathcal{D}, \lambda)$ is a CY algebra and $\mathcal{D}$ is a generic datum, then $\mathcal{D}$ cannot be of type A_2.

Example 4.2.16. Let A be $U(\mathcal{D}, \lambda)$ with the datum $(\mathcal{D}, \lambda)$ given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n$;
- The Cartan matrix is of type $A_1 \times A_1$;
- $g_i = y_i$, $i = 1, 2$;
- $\chi_1(y_i) = q$, $\chi_2(y_i) = q^{-1}$, $i = 1, 2$, where $q \in \mathbb{k}$ is an n-th root of unity;
- $\lambda = 1$.

The algebra A is a CY algebra of dimension 2.

Let R be the algebra $\mathbb{k}\langle x_1, x_2 \mid x_1 x_2 = q^{-1} x_2 x_1 \rangle$. Then $A = R \# \mathbb{k}\Gamma$. The rigid dualizing complex of R is ${}_{\varphi}R[3]$, where φ is defined by $\varphi(x_1) = q^{-1} x_1$ and $\varphi(x_2) = q x_2$.

Chapter 5

Ext algebras of Nichols algebras of type A_2

As shown in previous chapters, the homological properties of an algebra R over a field $\Bbbk$ rely exclusively on the structure of its Ext algebra $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$. In two recent papers [29, 61] support varieties of modules over Hopf algebras are introduced. It turns out that support varieties are useful tools to study homological properties and representations of finite dimensional (braided) Hopf algebras. To define and to compute support varieties we need first to understand the Ext algebra of the (braided) Hopf algebra. These motivate us to study the structure of the Ext algebra of a finite dimensional Nichols algebra. As a first attempt to explore the structure of the Ext algebras for further study, we give the full structure of the Ext algebra of a Nichols algebra of type A_2 in terms of generators and relations in this chapter. Using this struture, we can show that for a pointed Hopf algebra A of type A_2, the support variety of $\Bbbk$ over A is isomorphic to the variety of $\Bbbk$ over the associated graded algebra with respect to a certain filtration of A. Then we apply our main results to show that if the components of the Dynkin diagram of a pointed Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ are of type A, D, or E, except for A_1 and $A_1 \times A_1$, and the order $N_J > 2$ for at least one component, then $u(\mathcal{D}, \lambda, \mu)$ is wild.

A finite dimensional CY algebra must be semisimple. So a finite dimensional algebra $u(\mathcal{D}, \lambda, \mu)$ is not a CY algebra. But a finite dimensional Hopf algebra is Frobenius. So its stable category is a triangulated category. A natural question arises: is the stable category of a pointed Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ a CY category? We discuss this question at the end of Section 5.2. It turns out that in most cases, the answer to this question is negative.

5.1 Structures of Ext algebras

Let

$$\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$$

be a datum of finite Cartan type for a finite abelian group Γ. Assume that for $1 \leqslant i \leqslant \theta$, $\chi_i(g_i)$ has odd order and the order of $\chi_i(g_i)$ is prime to 3, if i lies in a component G_2.

Let $\{\alpha_1, \cdots, \alpha_\theta\}$ be a fix set of simple roots of the root system corresponding to the Cartan matrix (a_{ij}). Assume that $w_0 = s_{i_1} \cdots s_{i_p}$ is a reduced decomposition of the longest element w_0 in the Weyl group $\mathcal{W}$ as a product of simple reflections. Then

$$\beta_1 = \alpha_{i_1}, \;\; \beta_2 = s_{i_2}(\alpha_{i_1}), \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$$

are the positive roots. Let x_{β_i}, $1 \leqslant i \leqslant p$, be the corresponding root vectors.

Let V be the braided vector space with basis $\{x_1, \cdots, x_\theta\}$ whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$

for $1 \leqslant i, j \leqslant \theta$, where $q_{ij} = \chi_j(g_i)$.

Recall that the Nichols algebra $\mathcal{B}(V)$ is generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

$$(\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, \;\; 1 \leqslant i, j \leqslant \theta, \;\; i \neq j,$$

$$x_\alpha^{N_J} = 0, \;\; \alpha \in \Phi_J^+, \;\; J \in \mathcal{X},$$

where $\mathcal{X}$ is the set of connected components of the Dynkin diagram, N_J is the common order of q_{ii} with $i \in J$, and Φ_J^+ is set of positive roots of the component J (Section 1.4.2).

The following set

$$\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} \mid 1 \leqslant a_i < N_J, \quad \beta_i \in \Phi_J^+, \quad 1 \leqslant i \leqslant p\}$$

forms a PBW basis of the Nichols algebra $\mathcal{B}(V)$ (Theorem 1.4.11). Define a degree on each element as

$$\deg x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} = (a_1, \cdots, a_p, \sum a_i ht(\beta_i)) \in \mathbb{N}^{p+1},$$

where $ht(\beta_i)$ is the height of the positive root β_i (cf. [55, Sec. 2]).

Lemma 3.1.1 also holds in the case when the group Γ is finite. Therefore, if we order PBW basis elements by degree as in (2.2), we obtain a filtration on the Nichols algebra $\mathcal{B}(V)$. The associated graded algebra $\mathbb{G}\mathrm{r}\mathcal{B}(V)$ is generated by the root vectors x_{β_i}, $1 \leqslant i \leqslant p$, subject to the relations

$$[x_{\beta_i}, x_{\beta_j}]_c = 0, \text{ for all } i < j;$$

$$x_{\beta_i}^{N_J} = 0, \ \ \beta_i \in \Phi_J^+, \ \ 1 \leqslant i \leqslant p.$$

It is clear that each Nichols algebra can be written as a twisted tensor product of a set of Nichols algebras, such that each of them satisfies that the Dynkin diagram associated to the Cartan matrix is connected. In [14], the authors showed that the Ext algebra of a twisted tensor algebra is essentially the twisted tensor algebra of the Ext algebras. Therefore, we only need to discuss the case where the Dynkin diagram is connected. Now we calculate the Ext algebra of a Nichols algebra of type A_2.

Let N be an integer, and let q be a primitive root of 1 of order N. Let q_{ij}, $1 \leqslant i, j \leqslant 2$ be roots of 1, such that

$$q_{11} = q_{22} = q, \quad q_{12}q_{21} = q^{-1}.$$

Let V be a 2-dimensional vector space with basis x_1 and x_2, whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \ \ 1 \leqslant i, j \leqslant 2.$$

Then V is a braided vector space of type A_2.

5.1.1 Case $N = 2$

As discussed in [4], the Nichols algebra $R = \mathcal{B}(V)$ is isomorphic to the algebra generated by x_1 and x_2, with relations

$$x_1x_2x_1x_2 + x_2x_1x_2x_1 = 0, \;\; x_1^2 = x_2^2 = 0.$$

The dimension of R is 8.

Its Ext algebra can be calculated directly via the minimal projective resolution of $\Bbbk$.

Throughout in this chapter, for an algebra R, we write elements in the free module R^n, $n \geqslant 1$, as row vectors. A morphism $f : R^m \to R^n$ is described by an $m \times n$ matrix.

Proposition 5.1.1. *Let $R = \mathcal{B}(V)$ be the algebra mentioned before, then the algebra* $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$ *is generated by $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{b}$ with* $\deg \mathfrak{a}_1 = \deg \mathfrak{a}_2 = 1$ *and* $\deg \mathfrak{b} = 2$*, subject to the relations*

$$\mathfrak{a}_2\mathfrak{a}_1 = \mathfrak{a}_1\mathfrak{a}_2 = 0, \;\; \mathfrak{a}_1\mathfrak{b} = \mathfrak{b}\mathfrak{a}_1, \;\; \mathfrak{a}_2\mathfrak{b} = \mathfrak{b}\mathfrak{a}_2.$$

Proof. We claim that the following complex is the minimal projective resolution of $\Bbbk$.

$$\cdots P_n \xrightarrow{d_n} P_{n-1} \to \cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to \Bbbk, \tag{5.1}$$

where $P_n = R^{n+1}$ and d_n is defined as

$$d_n = \begin{pmatrix} x_1 & & & & & & \\ x_2x_1x_2 & x_1 & & & & & \\ & \cdots & & & & & \\ & & x_2x_1x_2 & x_1 & & & \\ & & & x_2 & x_1x_2x_1 & & \\ & & & & \cdots & & \\ & & & & & x_2 & x_1x_2x_1 \\ & & & & & & x_2 \end{pmatrix},$$

when n is odd and

$$d_n = \begin{pmatrix} x_1 & & & & & & & \\ x_2x_1x_2 & x_1 & & & & & & \\ & \cdots & & & & & & \\ & & x_2x_1x_2 & x_1 & & & & \\ & & & x_2x_1x_2 & x_1x_2x_1 & & & \\ & & & & x_2 & x_1x_2x_1 & & \\ & & & & & \cdots & & \\ & & & & & & x_2 & x_1x_2x_1 \\ & & & & & & & x_2 \end{pmatrix},$$

when n is even. Especially, $d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. It is routine to check that (5.1) is indeed a complex. Now we use induction to prove the exactness. It is clear that the minimal projective resolution starts as

$$R^3 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \to \Bbbk \to 0,$$

where $d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $d_2 = \begin{pmatrix} x_1 & \\ x_2x_1x_2 & x_1x_2x_1 \\ & x_2 \end{pmatrix}$. Assume that the complex (5.1) is exact up to P_n. If n is odd, then

$$\begin{aligned} & \dim(\operatorname{Ker} d_n) \\ = \; & (1 + \dim P_1 + \dim P_3 + \cdots \dim P_n) - (\dim P_0 + \dim P_2 + \cdots \dim P_{n-1}) \\ = \; & 4n + 5. \end{aligned}$$

Since the dimension of R is small, we can calculate the dimension of the submodule $\operatorname{Im} d_{n+1}$ of P_n directly, it is also $4n+5$. Then the complex is exact at P_{n+1}. If n is even, by a similar discussion, we can also conclude that the complex is exact at P_{n+1}, in this case $\dim(\operatorname{Ker} d_n) = 4n + 7$. We have that $\operatorname{Im} d_i \subseteq \operatorname{rad} P_{i-1}$ for each $i \geqslant 0$. Therefore, the complex (5.1) is the minimal projective resolution of $\Bbbk$. Since $\Bbbk$ is a simple module, we have

$$\operatorname{Hom}_R(P_n, \Bbbk) \cong \operatorname{Ext}_R^n(\Bbbk, \Bbbk) \tag{5.2}$$

as vector spaces for each $n \geqslant 0$. Let $\mathfrak{a}_1, \mathfrak{a}_2 \in \operatorname{Hom}_R(P_1, \Bbbk)$ be the functions dual to $(1,0)$ and $(0,1)$ respectively and $\mathfrak{b} \in \operatorname{Hom}_R(P_2, \Bbbk)$ be the function dual to $(0,1,0)$.

Let f_i, g_i and h_i be the morphisms described by the following matrices:

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 0 \\ 0 & x_2x_1 \\ 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$g_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 \\ x_1x_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$h_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have the following commutative diagrams:

$$\begin{tikzcd}
P_3 \arrow[r, "d_3"] \arrow[d, "f_3"'] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"'] \arrow[dr] & P_0 \arrow[r] & \mathbb{k}\ , \\
P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r] & \mathbb{k}
\end{tikzcd}$$

$$\begin{tikzcd}
P_3 \arrow[r, "d_3"] \arrow[d, "g_3"'] & P_2 \arrow[r, "d_2"] \arrow[d, "g_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, "g_1"'] \arrow[dr] & P_0 \arrow[r] & \mathbb{k}\ , \\
P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r] & \mathbb{k}
\end{tikzcd}$$

$$\begin{tikzcd}
P_3 \arrow[r, "d_3"] \arrow[d, "h_3"'] & P_2 \arrow[r, "d_2"] \arrow[d, "h_2"'] \arrow[dr] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r] & \mathbb{k}\ . \\
P_1 \arrow[r, "d_2"] & P_0 \arrow[r, "d_1"] & \mathbb{k}
\end{tikzcd}$$

These commutative diagrams show that the relation listed in the proposition hold.

Let U be the algebra generated by $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{b}$ subject to the relations listed in the proposition. When n is odd, U_n has a basis

$$\{\mathfrak{a}_1^n, \mathfrak{a}_1^{n-2}\mathfrak{b}, \cdots, \mathfrak{a}_1\mathfrak{b}^{\frac{n-1}{2}}, \mathfrak{a}_2\mathfrak{b}^{\frac{n-1}{2}}, \cdots, \mathfrak{a}_2^{n-2}\mathfrak{b}, \mathfrak{a}_2^n\}$$

and when n is even, U_n has a basis

$$\{\mathfrak{a}_1^n, \mathfrak{a}_1^{n-2}\mathfrak{b}, \cdots, \mathfrak{a}_1\mathfrak{b}^{\frac{n}{2}-1}, \mathfrak{b}^{\frac{n}{2}}, \mathfrak{a}_2\mathfrak{b}^{\frac{n}{2}-1}, \cdots, \mathfrak{a}_2^{n-2}\mathfrak{b}, \mathfrak{a}_2^n\}.$$

They are functions dual to $(1, 0 \cdots, 0)$, $\cdots$, $(0, \cdots, 0, 1)$ respectively in the projective resolution (5.1). We have

$$\begin{aligned}
\dim U_n &= n+1 \\
&= \dim \mathrm{Hom}_R(P_n/(\mathrm{rad}\, P_n), \mathbb{k}) \\
&= \dim \mathrm{Hom}_R(P_n, \mathbb{k}) \\
&= \dim \mathrm{Ext}_R^n(\mathbb{k}, \mathbb{k}),
\end{aligned}$$

where the last equation follows from equation (5.2). So we have $\mathrm{Ext}_R^*(\mathbb{k}, \mathbb{k}) = U$, which completes the proof of the proposition. $\square$

5.1.2 Case $N \geqslant 3$

In this case, the Nichols algebra $R = \mathcal{B}(V)$ is the algebra generated by x_1 and x_2 subject to the relations

$$x_1^2 x_2 - (q_{12} + q_{12} q_{11}) x_1 x_2 x_1 + q_{12}^2 q_{22} x_2 x_1^2 = 0,$$

$$x_2^2 x_1 - (q_{21} + q_{21} q_{22}) x_2 x_1 x_2 + q_{21}^2 q_{22} x_2 x_1^2 = 0,$$

$$x_1^N = x_2^N = (x_1 x_2 - q_{12} x_2 x_1)^N = 0.$$

The dimension of R is N^3.

In the rest of this chapter, we set $y = x_1 x_2 - q_{12} x_2 x_1$. From the above relations, we obtain that

$$q_{21} x_1 y - y x_1 = 0, \quad x_2 y - q_{21} y x_2 = 0.$$

Let α_1 and α_2 be the two simple roots. The element $\alpha_1 \alpha_2 \alpha_1$ is a reduced decomposition of the longest element in the Weyl group $\mathcal{W}$ and $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$ are the positive roots. The corresponding root vectors are just x_1, y and x_2. So the set

$$\{x_1^{a_1} y^{a_2} x_2^{a_3},\ 0 \leqslant a_i < N, i = 1, 2, 3\}$$

forms a PBW basis of R. The graded algebra $\mathbb{Gr}R$ corresponding to R is isomorphic to the algebra generated by x_1, y and x_2 subject to the relations

$$x_1 y = q_{21}^{-1} y x_1, \quad x_1 x_2 = q_{12} x_2 x_1, \quad y x_2 = q_{21}^{-1} x_2 y,$$

$$x_1^N = y^N = x_2^N = 0.$$

We first show that the algebra $\mathrm{Ext}_R^*(\Bbbk, \Bbbk)$ is generated in degree 1 and 2.

Let S be the subalgebra of R generated by x_1 and y. To be more precise, it is isomorphic to the algebra generated by x_1 and y subject to the relations

$$y x_1 = q_{21} x_1 y, \qquad x_1^N = y^N = 0.$$

The subalgebra S is a normal subalgebra of R (we refer to [33, Appendix] for the definition of normal subalgebras). Now set $\overline{R} = R/(RS^+)$, where S^+ is the augmentation ideal of S. That is, $\overline{R} = k[x_2]/(x_2^N)$. We use the Hochschild-Serre spectral sequence (cf. [33])

$$E_2^{pq} = \mathrm{Ext}^p_{\overline{R}}(\Bbbk, \mathrm{Ext}^q_S(\Bbbk, \Bbbk)) \Longrightarrow \mathrm{Ext}^{p+q}_R(\Bbbk, \Bbbk) \tag{5.3}$$

to calculate the Ext algebra of R. We show that $E_2 = E_\infty$.

The spectral sequence is constructed as follows. Let

$$\cdots \to Q_1 \to Q_0 \to \Bbbk \to 0$$

and

$$\cdots \to P_1 \to P_0 \to \Bbbk \to 0$$

be free resolutions of ${}_{\overline{R}}\Bbbk$ and ${}_R\Bbbk$ respectively. There is a natural $\overline{R}$-module action on $\mathrm{Hom}_S(P_q, \Bbbk)$ for $q \geqslant 0$. We form a double complex

$$E_0^{pq} = \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Hom}_S(P_q, \Bbbk)).$$

By taking the vertical homology and then the horizontal homology, we have

$$E_1^{pq} = \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Ext}^q_S(\Bbbk, \Bbbk))$$

and

$$E_2^{pq} = \mathrm{Ext}^p_{\overline{R}}(\Bbbk, \mathrm{Ext}^q_S(\Bbbk, \Bbbk)).$$

Now we construct a free resolution of $\Bbbk$ over R, which is a filtered complex. The corresponding graded complex is the minimal projective resolution of $\Bbbk$ over $\mathbb{Gr}R$.

We need some preparation to obtain a projective resolution of ${}_R\Bbbk$. The following lemma is known, see for instance [8] and references therein.

Lemma 5.1.2. *Both the sets*

$$\{x_2^{a_3} y^{a_2} x_1^{a_1}\} \ \text{and} \ \{x_1^{a_1} y^{a_2} x_2^{a_3}\},$$

$0 \leqslant a_i < N, i = 1, 2, 3$ *form bases of the algebra* R.

Let $\sigma, \tau : \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\sigma(a) = \begin{cases} 1, & \text{if } a \text{ is odd;} \\ N-1, & \text{if } a \text{ is even} \end{cases}$$

and

$$\tau(a) = \begin{cases} \frac{a-1}{2}N+1, & \text{if } a \text{ is odd;} \\ \frac{a}{2}N, & \text{if } a \text{ is even.} \end{cases}$$

Lemma 5.1.3. *The element y is a right divisor of $[x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]$.*

(1) If $a_1, a_3 > 0$ are odd, then

$$\tilde{\delta}_2(\Phi(a_1,a_2,a_3)) = -q_{21}^{-\frac{a_2}{2}N}\Phi(a_1-1,a_2+1,a_3-1).$$

(2) If $a_1 > 0$ is odd and $a_3 > 0$ is even, then

$$\begin{aligned} & \tilde{\delta}_2(\Phi(a_1,a_2,a_3)) \\ = \; & q_{12}^{(N-1)\frac{a_1-1}{2}N} q_{21}^{-(N-1)\frac{a_2}{2}N} q q_{21}^{-(N-2)} q_{21}^{\frac{a_1-1}{2}N} x_2^{N-2}\Phi(a_1-1,a_2+1,a_3-1). \end{aligned}$$

(3) If $a_1 > 0$ is even and $a_3 > 0$ is odd, then

$$\tilde{\delta}_2(\Phi(a_1,a_2,a_3)) = q_{21}^{-\frac{a_2}{2}N} x_1^{N-2}\Phi(a_1-1,a_2+1,a_3-1).$$

(4) If $a_1, a_3 > 0$ are even, then

$$\begin{aligned} & \tilde{\delta}_2(\Phi(a_1,a_2,a_3)) \\ = \; & -q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)} q_{21}^{-(N-1)\frac{a_2}{2}N} q_{21}^{\frac{a_1-2}{2}N+1} \\ & \quad (k_1 x_1^{N-2}x_2^{N-2} + \cdots + k_{N-2}y^{N-3}x_1x_2 + k_{N-1}y^{N-2})\Phi(a_1,a_2,a_3) \\ = \; & -q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)} q_{21}^{-(N-1)\frac{a_2}{2}N} q_{21}^{\frac{a_1-2}{2}N+1} \\ & \quad (l_1 x_2^{N-2}x_1^{N-2} + \cdots + l_{N-2}y^{N-3}x_2x_1 + l_{N-1}y^{N-2})\Phi(a_1,a_2,a_3), \end{aligned}$$

where

$$\begin{aligned} [x_1^{N-1}, x_2^{N-1}]_c &= k_1 y x_1^{N-2}x_2^{N-2} + \cdots + k_{N-2}y^{N-2}x_1x_2 + k_{N-1}y^{N-1} \\ &= l_1 y x_2^{N-2}x_1^{N-2} + \cdots + l_{N-2}y^{N-2}x_2x_1 + l_{N-1}y^{N-1}, \end{aligned}$$

with $k_i, l_i \in \Bbbk, 1 \leqslant i \leqslant N-1$.

Proof. (1) is easy to see. (2) and (3) follow from the following two equations,

$$[x_1^{N-1}, x_2]_c = (1 + q^{-1} + \cdots + q^{-N+1})x_1^{N-2}y = -qx_1^{N-2}y$$

and

$$[x_1, x_2^{N-1}]_c = (1 + q^{-1} + \cdots + q^{-N+1})yx_2^{N-2} = -qyx_2^{N-2} = -qq_{21}^{2-N}x_2^{N-2}y.$$

For (4), by Lemma 5.1.2 below, both $\{x_1^{a_1}y^{a_2}x_2^{a_3}\}$ and $\{x_2^{a_3}y^{a_2}x_1^{a_1}\}$, $0 \leqslant a_i < N$, $i = 1, 2, 3$, are bases of R. Using an easy induction, we can see that $[x_1^{N-1}, x_2^{N-1}]_c$ can be expressed as

$$\begin{aligned}
[x_1^{N-1}, x_2^{N-1}]_c &= x_1^{N-1}x_2^{N-1} - q_{12}^{(N-1)^2}x_2^{N-1}x_1^{N-1} \\
&= k_1yx_1^{N-2}x_2^{N-2} + \cdots + k_{N-2}y^{N-2}x_1x_2 + k_{N-1}y^{N-1} \\
&= l_1yx_2^{N-2}x_1^{N-2} + \cdots + l_{N-2}y^{N-2}x_2x_1 + l_{N-1}y^{N-1},
\end{aligned}$$

with $k_i, l_i \in \Bbbk$, $1 \leqslant i \leqslant N - 1$. Observe that y commutes with $x_1^t x_2^t$ and $x_2^t x_1^t$ for $t \geqslant 0$. Then the result follows. $\square$

Let

$$P_\bullet : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \cdots \to P_1 \to P_0 \tag{5.4}$$

be a complex of free R-modules constructed as follows. For each triple (a_1, a_2, a_3), let $\Phi(a_1, a_2, a_3)$ be a free generator for P_n, with $n = a_1 + a_2 + a_3$. Set

$$P_n = \oplus_{a_1+a_2+a_3=n} R\Phi(a_1, a_2, a_3)(-\tau(a_1), -\tau(a_2), -\tau(a_3), -\tau(a_1) - 2\tau(a_2) - \tau(a_3)).$$

Here, (-,-,-,-) denotes the degree shift. The differentials are defined by

$$\partial(\Phi(a_1, a_2, a_3)) = \begin{cases} (\delta_1 + \delta_2 + \delta_3)(\Phi(a_1, a_2, a_3)), & \text{if } a_2 \text{ is odd;} \\ (\delta_1 + \delta_2 + \tilde{\delta}_2 + \delta_3)(\Phi(a_1, a_2, a_3)), & \text{if } a_2 \text{ is even.} \end{cases}$$

The maps δ_i, $1 \leqslant i \leqslant 3$ and $\tilde{\delta}_2$ are defined as follows.

Put

$$\begin{aligned}
\delta_1(\Phi(a_1, a_2, a_3)) &= x_1^{\sigma(a_1)}\Phi(a_1 - 1, a_2, a_3), \quad \text{if } a_1 > 0; \\
\delta_2(\Phi(a_1, a_2, a_3)) &= (-1)^{a_1}q_{21}^{-\sigma(a_2)\tau(a_1)}y^{\sigma(a_2)}\Phi(a_1, a_2 - 1, a_3), \quad \text{if } a_2 > 0; \\
\delta_3(\Phi(a_1, a_2, a_3)) &= (-1)^{a_1+a_2}q_{12}^{\sigma(a_3)\tau(a_1)}q_{21}^{-\sigma(a_3)\tau(a_2)}x_2^{\sigma(a_3)}\Phi(a_1, a_2, a_3 - 1), \text{ if } a_3 > 0; \\
\tilde{\delta}_2(\Phi(a_1, a_2, a_3)) &= D\Phi(a_1 - 1, a_2 + 1, a_3 - 1), \quad \text{if } a_1, a_3 > 0, a_2 \text{ is even,}
\end{aligned}$$

where D is an element in R such that

$$Dy = -q_{21}^{\tau(a_1-1)} q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{21}^{-\sigma(a_3)\tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c.$$

The existence of such element D will be explained in Lemma 5.1.3. For $i = 1, 2, 3$, if $a_i = 0$, set $\delta_i(\Phi(a_1, a_2, a_3)) = 0$. If $a_1 = 0$ or $a_3 = 0$, set $\tilde{\delta}_2(\Phi(a_1, a_2, a_3)) = 0$.

Proposition 5.1.4. *The complex (5.4) is a projective resolution of $\mathbb{k}$ over R, the corresponding graded complex is the minimal projective resolution of $\mathbb{k}$ over $\mathbb{G}\mathrm{r}R$.*

Proof. It is routine to check that (5.4) is indeed a complex. We see it in Appendix 5.3.1. The differentials preserve the filtration and the corresponding graded complex is just the minimal projective resolution of $\mathbb{k}$ over $\mathbb{G}\mathrm{r}R$ as constructed in [55, Sec. 4]. Since the filtration is finite, the complex $P_\bullet$ is exact by [16, Chapter 2, Lemma 3.13]. Therefore, $P_\bullet$ is a free resolution of $\mathbb{k}$ over R. □

In the following, we will forget the shifting on the modules in the complex (5.4). It is clear that it is still a projective resolution of $\mathbb{k}$ over R. The only difference is that the differentials are not of degree 0. We denote this complex by $P_\bullet$ as well.

Proposition 5.1.5. *Let $R = \mathcal{B}(V)$ be the Nichols algebra of V. The Ext algebra $\mathrm{Ext}^*_R(\mathbb{k}, \mathbb{k})$ of R is generated in degree 1 and 2.*

Proof Applying $\mathrm{Hom}_R(-, \mathbb{k})$ to the complex (10), we obtain the complex $\mathrm{Hom}_R(P_\bullet, \mathbb{k})$. The Ext algebra $\mathrm{Ext}^*_R(\mathbb{k}, \mathbb{k})$ is the cohomology of the complex $\mathrm{Hom}_R(P_\bullet, \mathbb{k})$. Let $\xi_i \in \mathrm{Hom}_R(P_2, \mathbb{k})$ be the function dual to $\Phi(0, \cdots, 2, \cdots, 0)$ (the 2 in the i-th place) and $\eta_i \in \mathrm{Hom}_R(P_1, \mathbb{k})$ be the function dual to $\Phi(0, \cdots, 1, \cdots, 0)$ (the 1 in the i-th place). Denote by $\overline{\xi_i}$ and $\overline{\eta_i}$ the corresponding elements in $\mathrm{H}^2(R, \mathbb{k})$ and $\mathrm{H}^1(R, \mathbb{k})$, respectively. In order to show the relations among them, by abuse of notation, we define chain maps $\xi_i : P_n \to P_{n-2}$ and $\eta_i : P_n \to P_{n-1}$ by

$$\xi_1(\Phi(a_1, a_2, a_3)) = q_{21}^{-N\tau(a_2)} q_{12}^{N\tau(a_3)} \Phi(a_1 - 2, a_2, a_3);$$

$$\xi_2(\Phi(a_1, a_2, a_3)) = q_{21}^{-N\tau(a_3)} \Phi(a_1, a_2 - 2, a_3);$$

$$\xi_3(\Phi(a_1, a_2, a_3)) = \Phi(a_1, a_2, a_3 - 2);$$

$$\eta_1(\Phi(a_1,a_2,a_3)) = \begin{cases} (-1)^{a_2+a_3} q_{21}^{-\tau(a_2)} q_{12}^{\tau(a_3)} x_1^{\sigma(a_1)-1}\Phi(a_1-1,a_2,a_3) \\ \quad +Y_1\Phi(a_1-1,a_2+1,a_3-1), \\ \qquad\qquad\qquad\qquad \text{if } a_1,a_2 \text{ is even and } a_3>0; \\ (-1)^{a_2+a_3} q_{21}^{-\tau(a_2)} q_{12}^{\tau(a_3)} x_1^{\sigma(a_1)-1}\Phi(a_1-1,a_2,a_3), \\ \qquad\qquad\qquad\qquad \text{Otherwise;} \end{cases}$$

$$\eta_2(\Phi(a_1,a_2,a_3)) = (-1)^{a_3} q_{21}^{-(\sigma(a_2)-1)\tau(a_1)} q_{21}^{-\tau(a_3)} y^{\sigma(a_2)-1}\Phi(a_1,a_2-1,a_3);$$

$$\eta_3(\Phi(a_1,a_2,a_3)) = \begin{cases} (-1)^{a_3} q_{21}^{-(\sigma(a_2)-1)\tau(a_1)} q_{21}^{-\tau(a_3)} y^{\sigma(a_2)-1}\Phi(a_1,a_2,a_3-1) \\ \quad +Y_2\Phi(a_1-1,a_2+1,a_3-1), \\ \qquad\qquad\qquad\qquad \text{if } a_2,a_3 \text{ is even and } a_1>0; \\ (-1)^{a_3} q_{21}^{-(\sigma(a_2)-1)\tau(a_1)} q_{21}^{-\tau(a_3)} y^{\sigma(a_2)-1}\Phi(a_1,a_2,a_3-1), \\ \qquad\qquad\qquad\qquad \text{Otherwise,} \end{cases}$$

where Y_1 and Y_2 are the elements in R such that

$$Y_1 y = (-1)^{a_3+1} q_{21}^{\sigma(a_2+1)\tau(a_1-1)} q_{21}^{-(\sigma(a_3)+1)\tau(a_2)} q_{12}^{\tau(a_3-1)} q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{12}^{\sigma(a_3)} [x_1^{N-2}, x_2^{\sigma(a_3)}]_c$$

and

$$Y_2 y = (-1)^{a_1+1} q_{21}^{\sigma(a_2+1)\tau(a_1-1)} q_{21}^{-(\sigma(a_3)-1)\tau(a_2)} q_{12}^{(\sigma(a_3)-1)\tau(a_1-1)} [x_1^{\sigma(a_1)}, x_2^{N-2}].$$

Notice that the functions dual to the generators $\Phi(0,\cdots,0,1,0,\cdots,0)$ of P_n form a basis of $\mathrm{Hom}_R(P_n,\Bbbk)$. Thus, with these maps, we obtain that the Ext algebra is generated in degree 1 and 2. $\square$

It is well-known that the following complex is the minimal projective resolution of $\Bbbk$ over $\overline{R} = k[x_2]/(x_2^N)$.

$$Q_\bullet : \cdots \to \overline{R} \xrightarrow{x_2^{N-1}} \overline{R} \xrightarrow{x_2} \overline{R} \xrightarrow{x_2^{N-1}} \overline{R} \xrightarrow{x_2} \overline{R} \to \Bbbk.$$

Therefore, we have

$$\begin{aligned} E_0^{pq} &= \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Hom}_S(P_q,\Bbbk)) \\ &= \mathrm{Hom}_S(\oplus_{a_1+a_2+a_3=q} R\Phi(a_1,a_2,a_3),\Bbbk) \\ &= \oplus_{a_1+a_2+a_3=q}\overline{R}\Phi(a_1,a_2,a_3), \end{aligned}$$

since $\mathrm{Hom}_S(R,\Bbbk) \cong \overline{R}$. The double complex reads as follows

$$\begin{tikzcd}
 & \cdots & & & \cdots & \\
\bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & \cdots \\
\bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & \cdots \\
\bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & \cdots \\
\bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & \bullet \arrow[u] \arrow[r, "x_2^{N-1}"] & \bullet \arrow[u] \arrow[r, "x_2"] & {}
\end{tikzcd}$$

The vertical differentials are induced from the differentials of the complex (5.4).

By taking the vertical homology, we have $E_1^{pq} = \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Ext}_S^q(\Bbbk, \Bbbk))$. Following from [55], the algebra $\mathrm{Ext}_S^*(\Bbbk, \Bbbk)$ is generated by $\mathfrak{u}_1$, $\mathfrak{u}_y$, $\mathfrak{w}_1$ and $\mathfrak{w}_y$, where $\deg \mathfrak{u}_1 = \deg \mathfrak{u}_y = 2$ and $\deg \mathfrak{w}_1 = \deg \mathfrak{w}_y = 1$, subject to the relations

$$\mathfrak{w}_y \mathfrak{w}_1 = -q_{21} \mathfrak{w}_1 \mathfrak{w}_y, \quad \mathfrak{w}_1^2 = \mathfrak{w}_y^2 = 0,$$

$$\mathfrak{w}_y \mathfrak{u}_1 = q_{21}^N \mathfrak{u}_1 \mathfrak{w}_y, \quad \mathfrak{w}_1 \mathfrak{u}_1 = \mathfrak{u}_1 \mathfrak{w}_1, \quad \mathfrak{w}_y \mathfrak{u}_y = \mathfrak{u}_y \mathfrak{w}_y, \quad \mathfrak{w}_1 \mathfrak{u}_y = q_{21}^{-N} \mathfrak{u}_y \mathfrak{w}_1,$$
$$\mathfrak{u}_y \mathfrak{u}_1 = q_{21}^{N^2} \mathfrak{u}_1 \mathfrak{u}_y.$$

We use the notations $\mathfrak{u}_i$ and $\mathfrak{w}_i$ in place of the notations ξ_i and η_i used there. Note that $\mathfrak{w}_1^2 = \mathfrak{w}_y^2 = 0$ holds since we assume that the characteristic of the field $\Bbbk$ is 0. It should also be noticed that the Ext algebra in [55] is the opposite algebra here.

As described in the appendix of [33], there is an action of $\overline{R}$ on $\mathrm{Ext}_S^*(\Bbbk, \Bbbk)$ given by

$$x_2(\mathfrak{u}_y) = x_2(\mathfrak{u}_1) = 0, \quad x_2(\mathfrak{w}_y) = \mathfrak{w}_1, \text{ and } x_2(\mathfrak{w}_1) = 0.$$

This action is a derivation on $\mathrm{Ext}_S^*(\Bbbk, \Bbbk)$. That is, $x_2(\mathfrak{u}\mathfrak{w}) = x_2(\mathfrak{u})\mathfrak{w} + \mathfrak{u} x_2(\mathfrak{w})$ for $\mathfrak{u}, \mathfrak{w} \in \mathrm{Ext}_S^*(\Bbbk, \Bbbk)$.

The following lemma gives a basis of $\mathrm{Ext}_{\overline{R}}^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk, \Bbbk))$.

Lemma 5.1.6. *As a vector space,* $\mathrm{Ext}_{\overline{R}}^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk, \Bbbk))$ *has a basis as follows*

$$\begin{cases} \mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_1, & 2(i+j)+1 = q, \qquad \textit{q is odd and p is even;} \\ \mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_y, & 2(i+j)+1 = q, \qquad \textit{q is odd and p is odd;} \\ \mathfrak{u}_1^i \mathfrak{u}_y^j (\mathfrak{w}_1 \mathfrak{w}_y)^k, & k = 0, 1 \textit{ and } 2(i+j)+2k = q, \quad \textit{q is even.} \end{cases}$$

Proof. Let $E = \mathrm{Ext}_S^*(\Bbbk, \Bbbk)$. The lemma follows directly from the following facts:

(i) If q is odd, then $\{\mathfrak{u}_1^i\mathfrak{u}_y^j\mathfrak{w}_1 | i, j \geqslant 0, 2(i+j)+1 = q\}$ forms a basis of x_2E^q and $\{e \in E^q | x_2 e = 0\}$.

(ii) If q is even, then $x_2E^q = 0$.

(iii) $x_2^{N-1}E = 0$. □

Proposition 5.1.7. *The spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}_R^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk, \Bbbk)) \Longrightarrow \mathrm{Ext}_R^{p+q}(\Bbbk, \Bbbk)$$

satisfies $E_2 = E_\infty$.

Proof. The elements $\mathfrak{u}_1^i\mathfrak{u}_y^j\mathfrak{w}_y$ and $\mathfrak{u}_1^i\mathfrak{u}_y^j\mathfrak{w}_1$ are represented by

$$x_2^{N-2}\Phi(2i+1, 2j, 0) + q_{12}^{-(j+1)}x_2^{N-1}\Phi(2i, 2j+1, 0)$$

and

$$x_2^{N-1}\Phi(2i+1, 2j, 0),$$

while $\mathfrak{u}_1^i\mathfrak{u}_y^j$ and $\mathfrak{u}_1^i\mathfrak{u}_y^j\mathfrak{w}_1\mathfrak{w}_y$ are represented by

$$x_2^{N-1}\Phi(2i, 2j, 0) \text{ and } x_2^{N-1}\Phi(2i+1, 2j+1, 0)$$

in E_0. In other words, all the elements in E_0 representing the elements in E_2 are mapped to 0 under the horizontal differentials. We conclude that $E_2 = E_\infty$. □

We now can determine the dimension of $\mathrm{Ext}_R^*(\Bbbk, \Bbbk)$. This dimension depends on the parity of n.

Corollary 5.1.8. *We have*

$$\dim \mathrm{Ext}_R^n(\Bbbk, \Bbbk) = \begin{cases} \frac{3n^2+8n+5}{8}, & \text{if } n \text{ is odd;} \\ \frac{3n^2+10n+8}{8}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Set $E^n = \oplus_{p+q=n} E_2^{pq} = \oplus_{p+q=n} \mathrm{Ext}_R^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk, \Bbbk))$. By Lemma 5.1.6, we can illustrate the dimensions of E_2^{pq} with the following table:

$$
\begin{array}{ccccccccccccc}
\vert & & & & \cdots & & \cdots & & & & & & \\
4 & & 4 & & 4 & & 4 & & 4 & & 4 & & \cdots \\
\vert & & & & & & & & & & & & \\
7 & & 7 & & 7 & & 7 & & 7 & & 7 & & \cdots \\
\vert & & & & & & & & & & & & \\
3 & & 3 & & 3 & & 3 & & 3 & & 3 & & \cdots \\
\vert & & & & & & & & & & & & \\
5 & & 5 & & 5 & & 5 & & 5 & & 5 & & \cdots \\
\vert & & & & & & & & & & & & \\
2 & & 2 & & 2 & & 2 & & 2 & & 2 & & \cdots \\
\vert & & & & & & & & & & & & \\
3 & & 3 & & 3 & & 3 & & 3 & & 3 & & \cdots \\
\vert & & & & & & & & & & & & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\vert & & & & & & & & & & & & \\
1 & — & 1 & — & 1 & — & 1 & — & 1 & — & 1 & — & \cdots
\end{array}
\quad .
$$

Therefore, when n is odd,

$$
\begin{aligned}
\dim E^n &= (1+2+\cdots+\tfrac{n+1}{2}+1+3+\cdots+n) \\
&= \tfrac{3n^2+8n+5}{8}.
\end{aligned}
$$

When n is even,

$$
\begin{aligned}
\dim E^n &= (1+2+\cdots+\tfrac{n}{2}+1+3+\cdots+n+1) \\
&= \tfrac{3n^2+10n+8}{8}.
\end{aligned}
$$

By Proposition 5.1.7, we have $E_2 = E_\infty$, so $\dim \operatorname{Ext}_R^n(\Bbbk, \Bbbk) = \dim E^n$. This completes the proof. $\square$

Now we give the first segment of the minimal projective resolution of a Nichols algebra of type A_2.

The algebra R is a local algebra. Thus projective R-modules are free. Let

$$
R^{n_4} \to R^{n_3} \to R^{n_2} \to R^{n_1} \to R^{n_0} \to \Bbbk \to 0
$$

be the first segment of the minimal projective resolution. Since $\Bbbk$ is a simple module, we have

$$
\begin{aligned}
\dim \operatorname{Ext}_R^i(\Bbbk, \Bbbk) &= \dim \operatorname{Hom}_R(R^{n_i}, \Bbbk) \\
&= \dim \operatorname{Hom}_R((R/(\operatorname{rad} R))^{n_i}, \Bbbk) \\
&= n_i.
\end{aligned}
$$

From the computation of the dimensions of $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$ in Corollary 5.1.8, we can see that the minimal projective resolution begins as

$$R^{12} \to R^7 \to R^5 \to R^2 \to R \to \Bbbk \to 0.$$

We give the differentials in the following proposition.

As in the construction of $\tilde{\delta}_2$ in §5.1.2, let $\overline{D}$ be the element in R such that $\overline{D}y = [x_1^{N-1}, x_2^{N-1}]_c$.

Proposition 5.1.9. *Let R be a Nichols algebra of A_2 type. The following sequence provides the first segment of the minimal projective resolution of $\Bbbk$ over R,*

$$R^{12} \xrightarrow{d_4} R^7 \xrightarrow{d_3} R^5 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \to \Bbbk \to 0, \tag{5.5}$$

where the differentials are given by the following matrices:

$$d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} x_1^{N-1} & 0 \\ -(q_{12} + qq_{12})x_1x_2 + qq_{12}^2 x_2x_1 & x_1^2 \\ -q_{12}y^{N-1}x_2 & y^{N-1}x_1 \\ x_2^2 & qq_{21}^2 x_1x_2 - (q_{21} + qq_{21})x_2x_1 \\ 0 & x_2^{N-1} \end{pmatrix},$$

$$d_3 = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ q_{12}^N x_2 & x_1^{N-2} & 0 & 0 & 0 \\ 0 & 0 & x_2 & q_{12}q_{21}^{N-1}y^{N-1} & 0 \\ 0 & x_2 & 0 & x_1 & 0 \\ 0 & -q_{21}^{1-N}y^{N-1} & x_1 & 0 & 0 \\ 0 & 0 & 0 & q_{12}^N x_2^{N-2} & x_1 \\ 0 & 0 & 0 & 0 & x_2 \end{pmatrix},$$

$$d_4 = \left(\begin{array}{c:c} A_1 & A_2 \\ \hdashline A_3 & A_4 \end{array}\right),$$

where

$$A_1 = \begin{pmatrix} x_1^{N-1} & 0 & 0 & 0 \\ -(q_{12}^{1+N} + qq_{12}^{1+N})x_1x_2 + qq_{12}^{2+N}x_2x_1 & x_1^2 & 0 & 0 \\ -q_{12}^{1+N}y^{N-1}x_2 & y^{N-1}x_1 & 0 & 0 \\ q_{12}^{N}x_2^2 & qq_{21}^2x_1x_2 - (q_{21} + qq_{21})x_2x_1 & 0 & q_{21}^{N}x_1^{N-1} \\ 0 & x_2^{N-1} & 0 & -q_{12}^{-N^2+2N}\overline{D} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_{12}^{-N^2+N}x_1^{N-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q_{12}^{-N^2+N}x_1^{N-1} & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & x_1^2 & -q^{-1}q_{12}^{N}y^{N-1}x_1 \\ 0 & 0 & y^{N-1}x_1 & 0 \\ 0 & 0 & qq_{21}^2x_1x_2 - (q_{21} + qq_{21})x_2x_1 & q_{21}^{N-1}y^{N-1}x_2 \\ 0 & 0 & 0 & q_{12}^{2N}x_2^{N-1} \\ 0 & 0 & q_{12}^{N^2}x_2^{N-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -(q_{12} + qq_{12})x_1x_2 + qq_{12}^2x_2x_1 & 0 & 0 \\ -q_{12}y^{N-1}x_2 & 0 & 0 \\ x_2^2 & 0 & 0 \\ 0 & -(q_{12} + qq_{12})x_1x_2 + qq_{12}^2x_2x_1 & x_1^2 \\ 0 & -q_{12}y^{N-1}x_2 & y^{N-1}x_1 \\ 0 & x_2^2 & qq_{21}^2x_1x_2 - (q_{21} + qq_{21})x_2x_1 \\ 0 & 0 & x_2^{N-1} \end{pmatrix}.$$

Proof. It is routine to check that (5.5) is indeed a complex. But we need to mention that the following two equations hold

$$\overline{D}x_1 - x_1^{N-1}x_2^{N-2} = 0,$$

$$x_2^{N-1}x_1^{N-2} - q_{12}^{-N^2+2N}\overline{D}x_2 = 0.$$

These equations follow from Lemma 5.1.2 and the equations

$$[x_1^{N-1}, x_2^{N-1}]_c x_1 = yx_1^{N-1}x_2^{N-2},$$

$$[x_1^{N-1}, x_2^{N-1}]_c x_2 = q_{12}^{N^2-2N}yx_2^{N-1}x_1^{N-2}.$$

The complex (5.5) is homotopically equivalent to the first segment of the resolution $P_\bullet$ (without shifting) constructed in Section 2. Therefore, it is exact. $\square$

Remark 5.1.10. In [56, Theorem 6.1.3], the authors give a set of linearly independent 2-cocycles on R, indexed by the positive roots. In the resolution (5.5), the functions dual to $(1,0,0,0,0)$, $(0,0,1,0,0)$ and $(0,0,0,0,1)$ are just those 2-cocycles, corresponding to the positive roots α_1, $\alpha_1+\alpha_2$ and α_2 respectively.

Now we give our main theorems about the structure of the Ext algebra of a Nichols algebra of type A_2.

Theorem 5.1.11. *Let R be a Nichols algebra of type A_2 with $N=3$, then $\operatorname{Ext}^*_R(\Bbbk,\Bbbk)$ is generated by $\mathfrak{a}_i$, $\mathfrak{b}_i$, $\mathfrak{c}_i$, $i=1,2$ and $\mathfrak{b}_y$ with*

$$\deg\mathfrak{a}_i=1,\ \ \deg\mathfrak{b}_i=\deg\mathfrak{b}_y=\deg\mathfrak{c}_i=2,$$

subject to the relations

$$\mathfrak{a}_1^2=\mathfrak{a}_2^2=0,\quad \mathfrak{a}_1\mathfrak{a}_2=\mathfrak{a}_2\mathfrak{a}_1=0,$$
$$\mathfrak{a}_1\mathfrak{b}_1=\mathfrak{b}_1\mathfrak{a}_1,\ \ \mathfrak{a}_1\mathfrak{b}_y=q_{12}^3\mathfrak{b}_y\mathfrak{a}_1,\ \ \mathfrak{a}_1\mathfrak{b}_2=q_{12}^3\mathfrak{b}_2\mathfrak{a}_1,$$
$$\mathfrak{a}_1\mathfrak{c}_1=q^2q_{12}\mathfrak{c}_1\mathfrak{a}_1,\ \ \mathfrak{a}_1\mathfrak{c}_2=qq_{12}^2\mathfrak{c}_2\mathfrak{a}_1,$$
$$q_{12}^3\mathfrak{a}_2\mathfrak{b}_1=\mathfrak{b}_1\mathfrak{a}_2,\ \ q_{12}^3\mathfrak{a}_2\mathfrak{b}_y=\mathfrak{b}_y\mathfrak{a}_2,\ \ \mathfrak{a}_2\mathfrak{b}_2=\mathfrak{b}_2\mathfrak{a}_2,$$
$$\mathfrak{a}_2\mathfrak{c}_1=qq_{21}^2\mathfrak{c}_1\mathfrak{a}_2,\ \ q^2q_{12}\mathfrak{a}_2\mathfrak{c}_2=\mathfrak{c}_2\mathfrak{a}_2,$$
$$q^2q_{12}\mathfrak{a}_2\mathfrak{b}_1=\mathfrak{a}_1\mathfrak{c}_1,\ \ \mathfrak{a}_1\mathfrak{b}_2=q^2q_{12}\mathfrak{a}_2\mathfrak{c}_2,\ \ \mathfrak{c}_1\mathfrak{a}_2=\mathfrak{c}_2\mathfrak{a}_1,$$
$$\mathfrak{b}_1\mathfrak{c}_2=q_{12}^6\mathfrak{c}_1^2,\ \ q_{12}^6\mathfrak{b}_2\mathfrak{c}_1=\mathfrak{c}_2^2,\ \ \mathfrak{b}_1\mathfrak{b}_2=q_{12}^3\mathfrak{c}_1\mathfrak{c}_2,\ \ \mathfrak{c}_1\mathfrak{c}_2=q_{12}^3\mathfrak{c}_2\mathfrak{c}_1,$$
$$\mathfrak{b}_1\mathfrak{b}_y=q_{12}^9\mathfrak{b}_y\mathfrak{b}_1,\ \ \mathfrak{b}_1\mathfrak{b}_2=q_{12}^9\mathfrak{b}_2\mathfrak{b}_1,\ \ \mathfrak{b}_y\mathfrak{b}_2=q_{12}^9\mathfrak{b}_2\mathfrak{b}_y,$$
$$q_{12}^3\mathfrak{c}_1\mathfrak{b}_1=\mathfrak{b}_1\mathfrak{c}_1,\ \ \mathfrak{c}_1\mathfrak{b}_y=q_{12}^3\mathfrak{b}_y\mathfrak{c}_1,\ \ \mathfrak{c}_1\mathfrak{b}_2=q_{12}^6\mathfrak{b}_2\mathfrak{c}_1,$$
$$q_{12}^6\mathfrak{c}_2\mathfrak{b}_1=\mathfrak{b}_1\mathfrak{c}_2,\ \ q_{12}^3\mathfrak{c}_2\mathfrak{b}_y=\mathfrak{b}_y\mathfrak{c}_2,\ \ \mathfrak{c}_2\mathfrak{b}_2=q_{12}^3\mathfrak{b}_2\mathfrak{c}_2.$$

Theorem 5.1.12. *Let R be a Nichols algebra of type A_2 with $N>3$, then $\operatorname{Ext}^*_R(\Bbbk,\Bbbk)$ is generated by $\mathfrak{a}_i$, $\mathfrak{b}_i$ and $\mathfrak{c}_i$, $i=1,2$ and $\mathfrak{b}_y$ with*

$$\deg\mathfrak{a}_i=1,\ \ \deg\mathfrak{b}_i=\deg\mathfrak{b}_y=\deg\mathfrak{c}_i=2,$$

subject to the relations

$$\mathfrak{a}_1^2=\mathfrak{a}_2^2=0,\quad \mathfrak{a}_1\mathfrak{a}_2=\mathfrak{a}_2\mathfrak{a}_1=0,$$
$$\mathfrak{a}_1\mathfrak{b}_1=\mathfrak{b}_1\mathfrak{a}_1,\ \ \mathfrak{a}_1\mathfrak{b}_y=q_{12}^N\mathfrak{b}_y\mathfrak{a}_1,\ \ \mathfrak{a}_1\mathfrak{b}_2=q_{12}^N\mathfrak{b}_2\mathfrak{a}_1,$$
$$q_{12}^N\mathfrak{a}_2\mathfrak{b}_1=\mathfrak{b}_1\mathfrak{a}_2,\ \ q_{12}^N\mathfrak{a}_2\mathfrak{b}_y=\mathfrak{b}_y\mathfrak{a}_2,\ \ \mathfrak{a}_2\mathfrak{b}_2=\mathfrak{b}_2\mathfrak{a}_2,$$
$$\mathfrak{a}_1\mathfrak{c}_2=qq_{12}^2\mathfrak{c}_2\mathfrak{a}_1,\ \ \mathfrak{a}_2\mathfrak{c}_1=qq_{21}^2\mathfrak{c}_1\mathfrak{a}_2,$$
$$\mathfrak{a}_1\mathfrak{c}_1=\mathfrak{c}_1\mathfrak{a}_1=\mathfrak{c}_2\mathfrak{a}_2=\mathfrak{a}_2\mathfrak{c}_2=0,\ \ \mathfrak{c}_1\mathfrak{a}_2=\mathfrak{c}_2\mathfrak{a}_1,$$
$$\mathfrak{c}_1^2=\mathfrak{c}_2^2=\mathfrak{c}_1\mathfrak{c}_2=\mathfrak{c}_2\mathfrak{c}_1=0,$$
$$\mathfrak{b}_1\mathfrak{b}_y=q_{12}^{N^2}\mathfrak{b}_y\mathfrak{b}_1,\ \ \mathfrak{b}_1\mathfrak{b}_2=q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_1,\ \ \mathfrak{b}_y\mathfrak{b}_2=q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_y,$$

$$q_{12}^{N}\mathfrak{c}_1\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{c}_1,\ \ \mathfrak{c}_1\mathfrak{b}_y = q_{12}^{N}\mathfrak{b}_y\mathfrak{c}_1,\ \ \mathfrak{c}_1\mathfrak{b}_2 = q_{12}^{2N}\mathfrak{b}_2\mathfrak{c}_1,$$
$$q_{12}^{2N}\mathfrak{c}_2\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{c}_2,\ \ q_{12}^{N}\mathfrak{c}_2\mathfrak{b}_y = \mathfrak{b}_y\mathfrak{c}_2,\ \ \mathfrak{c}_2\mathfrak{b}_2 = q_{12}^{N}\mathfrak{b}_2\mathfrak{c}_2.$$

Proof of Theorems 5.1.11 and 5.1.12 We prove Theorem 5.1.11. Theorem 5.1.12 can be proved similarly. Consider the minimal resolution (5.5) showed in Proposition 5.1.9, we have $\mathrm{Ext}^1_R(\Bbbk,\Bbbk) = \mathrm{Hom}_R(R^2,\Bbbk)$ and $\mathrm{Ext}^2_R(\Bbbk,\Bbbk) = \mathrm{Hom}_R(R^5,\Bbbk)$, since $\Bbbk$ is a simple module. Let $\mathfrak{a}_1,\mathfrak{a}_2 \in \mathrm{Ext}^1_R(\Bbbk,\Bbbk)$ be the functions dual to $(1,0)$ and $(0,1)$ respectively. Let $\mathfrak{b}_1,\mathfrak{c}_1,\mathfrak{b}_y,\mathfrak{c}_2,\mathfrak{b}_2 \in \mathrm{Ext}^2_R(\Bbbk,\Bbbk)$ be the functions dual to $(1,0,0,0,0),\cdots,(0,0,0,0,1)$ respectively. The relations listed in the theorem can be verified by constructing suitable commutative diagrams, we do this in Appendix 5.3.2. Let U be an algebra generated by $\mathfrak{b}_1$, $\mathfrak{b}_y$, $\mathfrak{b}_2$ and $\mathfrak{a}_i$, $\mathfrak{c}_i$, $i=1,2$, subject to the relations listed in the theorem. Then any element in U can be written as a linear combination of elements of the form $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}\mathfrak{a}_i^{a_i}$, $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_i^{c_i}$ and $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_1\mathfrak{a}_2$, with b_1, b_2, $b_3 \geqslant 0$, a_i, $c_i \in \{0,1\}$, $i=1,2$.

By Proposition 5.1.5, the algebra $\mathrm{Ext}^*_R(\Bbbk,\Bbbk)$ is a quotient of U. When n is odd,

$$\begin{aligned}\dim U_n &= (\tfrac{n-1}{2}+2)(\tfrac{n-1}{2}+1) + \tfrac{1}{2}(\tfrac{n-1}{2})(\tfrac{n-1}{2}+1)\\ &= \tfrac{3n^2+8n+5}{8}.\end{aligned}$$

When n is even,

$$\begin{aligned}\dim U_n &= (\tfrac{n}{2})(\tfrac{n}{2}+1) + \tfrac{1}{2}(\tfrac{n}{2}+1)(\tfrac{n}{2}+2)\\ &= \tfrac{3n^2+10n+8}{8}.\end{aligned}$$

It follows from Corollary 5.1.8 that $\dim U_n = \dim \mathrm{Ext}^n_R(\Bbbk,\Bbbk)$, for all $n \geqslant 0$, so $U = \mathrm{Ext}^*_R(\Bbbk,\Bbbk)$, which completes the proof of the theorem. □

Remark 5.1.13. In [55, Thm 5.4], the authors showed that the Ext algebra of a Nichols algebra of finite Cartan type is braided commutative. This coincides with the results we obtain in Theorems 5.1.11 and 5.1.12.

In [2], the author raised a question of when the Ext algebra of a Nichols algebra is still a Nichols algebra. In general, the answer is negative.

Proposition 5.1.14. *The Ext algebra of a Nichols algebra of type A_2, with natural grading, is not a Nichols algebra.*

Proof. We consider the case $N=2$ first. Denote the Ext algebra by E. From Proposition 5.1.1, E is generated by $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{b}$ subject to the relations

$$\mathfrak{a}_2\mathfrak{a}_1 = \mathfrak{a}_1\mathfrak{a}_2 = 0,\ \ \mathfrak{a}_1\mathfrak{b} = \mathfrak{b}\mathfrak{a}_1,\ \ \mathfrak{a}_2\mathfrak{b} = \mathfrak{b}\mathfrak{a}_2.$$

If E is a Nichols algebra with respect to some braided vector space V, then $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{b}$ should form a basis of V. This is because as an algebra, a Nichols algebra $\mathcal{B}(V)$ is generated by elements in V. With relation $\mathfrak{a}_2\mathfrak{a}_1 = \mathfrak{a}_1\mathfrak{a}_2$, $\mathfrak{a}_1\mathfrak{b} = \mathfrak{b}\mathfrak{a}_1$ and $\mathfrak{a}_2\mathfrak{b} = \mathfrak{b}\mathfrak{a}_2$, the vector space V is of diagonal type. This contradicts the relations $\mathfrak{a}_2\mathfrak{a}_1 = \mathfrak{a}_1\mathfrak{a}_2 = 0$. Therefore, E is not a Nichols algebra. By a similar argument, we can conclude that when $N \geqslant 3$, the Ext algebra is not a Nichols algebra either. $\square$

However, we have the following positive result.

Proposition 5.1.15. *Let R be a Nichols algebra of type A_2 with $N > 3$. Then $\operatorname{Ext}^*_R(\mathbb{k}, \mathbb{k})/\mathcal{N}$ is a Nichols algebra of diagonal type, where $\mathcal{N}$ is the ideal generated by nilpotent elements.*

Proof. From the proof of Theorem 5.1.12, the elements $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}\mathfrak{a}_i^{a_i}$, $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_i^{c_i}$ and $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_1\mathfrak{a}_2$, with b_1, b_2, $b_3 \geqslant 0$, a_i, $c_i \in \{0,1\}$, $i = 1,2$ form a basis of $\operatorname{Ext}^*_R(\mathbb{k}, \mathbb{k})$. With the relations listed in that theorem, the elements $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}\mathfrak{a}_i$, $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_i$ and $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_1\mathfrak{a}_2$ are nilpotent. However, linear combination of elements $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}$ are not nilpotent. Then the algebra $\operatorname{Ext}^*_R(\mathbb{k}, \mathbb{k})/\mathcal{N}$ is generated by $\mathfrak{b}_1$, $\mathfrak{b}_y$ and $\mathfrak{b}_2$ subject to the relations

$$\mathfrak{b}_1\mathfrak{b}_y = q_{12}^{N^2}\mathfrak{b}_y\mathfrak{b}_1, \quad \mathfrak{b}_1\mathfrak{b}_2 = q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_1, \quad \mathfrak{b}_y\mathfrak{b}_2 = q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_y.$$

It is obvious that it is a Nichols algebra of diagonal type with Cartan matrix of type $A_1 \times A_1 \times A_1$. $\square$

5.2 Applications

Before we give some applications of Theorem 5.1.11 and Theorem 5.1.12 in Section 5.1, we recall the definitions of complexities and varieties. We follow the definitions and the notations in [29]. Let A be a finite dimensional Hopf algebra and

$$\mathrm{H}^*(A, \Bbbk) := \mathrm{Ext}_A^*(\Bbbk, \Bbbk).$$

The vector space $\mathrm{H}^*(A, \Bbbk)$ is an associative graded algebra under the Yoneda product. The subalgebra $\mathrm{H}^{ev}(A, \Bbbk)$ of $\mathrm{H}^*(A, \Bbbk)$ is defined as

$$\mathrm{H}^{ev}(A, \Bbbk) = \oplus_{n=0}^{\infty} \mathrm{H}^{2n}(A, \Bbbk).$$

The algebra $\mathrm{H}^{ev}(A, \Bbbk)$ is commutative, since $\mathrm{H}^*(A, \Bbbk)$ is graded commutative. In the following, we say that a Hopf algebra A satisfies the *assumption* (**fg**) if the following conditions hold:

(fg1) The algebra $\mathrm{H}^{ev}(A, \Bbbk)$ is finitely generated;

(fg2) The $\mathrm{H}^{ev}(A, \Bbbk)$-module $\mathrm{Ext}_A^*(M, N)$ is finitely generated for any two finite dimensional A-modules M and N.

Under the assumption (**fg**), the *variety* $\mathcal{V}_A(M, N)$ for A-modules M and N is defined as

$$\mathcal{V}_A(M, N) := \mathrm{MaxSpec}(\mathrm{H}^{ev}(A, \Bbbk)/I(M, N)),$$

where $I(M, N)$ is the annihilator of the action of $\mathrm{H}^{ev}(A, \Bbbk)$ on $\mathrm{Ext}_A^*(M, N)$. It is an homogeneous ideal of $\mathrm{H}^{ev}(A, \Bbbk)$. The *support variety* of M is defined as

$$\mathcal{V}_A(M) = \mathcal{V}_A(M, M).$$

For a graded vector space $V^\bullet = \oplus_{n \in \mathbb{Z}, n \geqslant 0} V^n$, the growth rate $\gamma(V^\bullet)$ is defined as

$$\gamma(V^\bullet) = min\{c \in \mathbb{Z}, c \geqslant 0 \mid \exists b \in \mathbb{R}, \text{such that } \dim V^n \leqslant bn^{c-1}, \text{for all } n \geqslant 0\}.$$

Let M be an A-module and

$$P_\bullet : \; \cdots \to P_1 \to P_0 \to M \to 0$$

the minimal projective resolution of M. Then the growth rate $\gamma(P_\bullet)$ is said to be the *complexity* $\mathrm{cx}_A(M)$ of M.

By [55, Thm. 6.3] a finite dimensional pointed Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ satisfies the assumption (**fg**). The following corollary is a direct consequence of Theorems 5.1.11 and

5.1.12.

Corollary 5.2.1. *Let $A = u(\mathcal{D}, 0, \mu)$ be a pointed Hopf algebra of type A_2 with $N \geqslant 3$ and $R = \mathcal{B}(V)$ the corresponding Nichols algebra. Then*

$$\mathrm{cx}_R(\mathbb{k}) = \mathrm{cx}_A(\mathbb{k}) = 3.$$

In addition, $\mathcal{V}_A(\mathbb{k}) \cong \mathcal{V}_{(\mathbb{G}\mathrm{r}R)\#G}(\mathbb{k})$.

Proof. For the Nichols algebra R, the complexity

$$\mathrm{cx}_R(\mathbb{k}) = \gamma(\mathrm{Ext}^*_R(\mathbb{k}, \mathbb{k})) = 3$$

follows directly from Proposition 5.1.8 or Theorems 5.1.11 and 5.1.12. By [55, Lemma 6.1], $\mathrm{H}^*(u(\mathcal{D}, 0, \mu), \mathbb{k}) \cong \mathrm{H}^*(u(\mathcal{D}, 0, 0), \mathbb{k})$. In addition, we have $\mathrm{Ext}^*_{u(\mathcal{D},0,0)}(\mathbb{k}, \mathbb{k}) \cong \mathrm{Ext}^*_R(\mathbb{k}, \mathbb{k})^G$. Observe that for each positive root α, some power of $\mathfrak{b}_\alpha$ is invariant under the group action. Indeed, from the discussion in Section 6 in [56], each $\mathfrak{b}_\alpha$ (denoted by f_α there) can be expressed as a function $R^+ \times R^+ \to \mathbb{k}$. Then we see that $\mathfrak{b}_\alpha^{M_\alpha}$ is Γ-invariant, where M_α is the integer such that $\chi_\alpha^{M_\alpha} = \varepsilon$. Hence, $\gamma(\mathrm{H}^*(u(\mathcal{D}, 0, 0), \mathbb{k}) = 3$, which implies that $\mathrm{cx}_A(\mathbb{k}) = 3$. With the relations in Theorems 5.1.11 and 5.1.12, we see that

$$\mathcal{V}_A(\mathbb{k}) \cong \mathrm{MaxSpec}(\mathbb{k}[\mathfrak{b}_1^{m_1}, \mathfrak{b}_y^{m_y}, \mathfrak{b}_2^{m_2}]),$$

where m_1, m_y and m_2 are the least integers such that $\mathfrak{b}_1^{m_1}$, $\mathfrak{b}_y^{m_y}$, $\mathfrak{b}_2^{m_2} \in \mathrm{H}^*(u(\mathcal{D}, 0, 0), \mathbb{k})$. That is, $\mathcal{V}_A(\mathbb{k})$ is isomorphic to the maximal spectrum of the polynomial algebra $\mathbb{k}[y_1, y_2, y_3]$. By [55, Thm. 4.1] $\mathcal{V}_{\mathbb{G}\mathrm{r}R\#G}(\mathbb{k})$ is also isomorphic to the maximal spectrum of $\mathbb{k}[y_1, y_2, y_3]$. So $\mathcal{V}_A(\mathbb{k}) \cong \mathcal{V}_{\mathbb{G}\mathrm{r}R\#G}(\mathbb{k})$. □

Now we give an easy application of Theorems 5.1.11 and 5.1.12 . We show that a large class of finite dimensional pointed Hopf algebras of finite Cartan type are wild.

Proposition 5.2.2. *Let $A = u(\mathcal{D}, \lambda, \mu)$ be a pointed Hopf algebra with Dynkin diagrams of type A, D, or E, except for A_1 and $A_1 \times A_1$ with the order $N_J > 2$ for at least one component J. Then A is wild.*

Proof. In view of [29, Thm. 3.1], we only need to prove that $\mathrm{cx}_A(\mathbb{k}) \geqslant 3$. Using [55, Lemma 6.1] again, we have $\mathrm{cx}_A(\mathbb{k}) = \mathrm{cx}_{u(\mathcal{D},\lambda,0)}(\mathbb{k})$. However, $u(\mathcal{D}, \lambda, 0)$ contains a Hopf subalgebra B which is of type A_2 with the order $N \geqslant 3$. Thus $\mathrm{cx}_{u(\mathcal{D},\lambda,0)}(\mathbb{k}) \geqslant \mathrm{cx}_B(\mathbb{k}) \geqslant 3$ by [29, Prop 2.1]. □

A finite dimensional CY algebra is semisimple. Indeed, if R is a finite dimensional CY algebra of dimension d, then by [13, Prop. 2.3], $\mathrm{Ext}^d_R(R, R)^* \cong \mathrm{Hom}_R(R, R)$. Thus we have $d = 0$. So the global dimension of R is 0 ([13, Rem. 2.8]). Therefore, the finite dimensional algebras $u(\mathcal{D},\lambda,\mu)$ are not CY algebras. However, a finite dimensional Hopf algebra A is a Frobenius algebra. Let $\mathrm{mod}A$ be the full subcategory of $\mathrm{Mod}A$ consisting of finitely generated A-modules. Then $\mathrm{mod}A$ is a Frobenius category and its stable category $\underline{\mathrm{mod}}A$ is a triangulated category with the shift functor Ω^{-1}, where Ω is the syzygy functor.

The objects of the category $\underline{\mathrm{mod}}A$ are modules in $\mathrm{mod}A$. For X, $Y \in \underline{\mathrm{mod}}A$, the set of morphisms from X to Y is $\mathrm{Hom}_A(X,Y)/\mathcal{I}(X,Y)$, where

$$\mathcal{I}(X,Y) = \{f \in \mathrm{Hom}_A(X,Y) \mid f \text{ factor through an injective module}\}.$$

We refer to [34] for a detailed discussion about stable categories and Frobenius categories.

Now we wonder whether the stable category of a pointed Hopf algebra $u(\mathcal{D},\lambda,\mu)$ is a CY category. Let A be a Frobenius algebra. By Auslander-Reiten formula (cf. [10, Thm. 2.13] or [11, Cor. IV.4.4]), there are natural isomorphisms

$$\underline{\mathrm{Hom}}_A(X,Y) \cong \mathrm{Ext}^1_A(Y,\tau X)^*,$$

for any $X, Y \in \underline{\mathrm{mod}}A$, where τ is the Auslander-Reiten translate. Therefore, the category $\underline{\mathrm{mod}}A$ has a Serre functor $\tau\Omega^{-1}$. By [11, Prop. IV.3.7], this functor is isomorphic to the functor $\nu\Omega$, where ν is the Nakayama functor of the category $\underline{\mathrm{mod}}A$. That is, $\nu = A^* \otimes_A - \cong \mathrm{Hom}_A(-,A)^*$. Let η be a Nakayama automorphism of A. That is, η is an algebra automorphism such that $A^* \cong A_\eta$ as A-A-bimodules. Thus $\nu \cong A_\eta \otimes_A -$. If $\underline{\mathrm{mod}}A$ is a CY category of dimension d, then there is a natural isomorphism $\nu \cong \Omega^{-d-1}$. The order of a Nakayama automorphism of a finite dimensional Hopf algebra is finite [30, Lemma 1.5]. Therefore, there is some integer n such that $\Omega^n \cong \mathrm{id}$. Now we obtain that if $\underline{\mathrm{mod}}A$ is a CY category, then $\mathrm{cx}_A(\Bbbk) = 1$.

Let $A = u(\mathcal{D},\lambda,\mu)$ be a finite dimensional pointed Hopf algebra of finite Cartan type. If the datum $(\mathcal{D},\lambda,\mu)$ satisfies one of the following conditions:

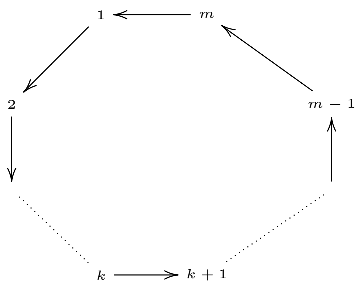
- the Cartan matrix is neither of type A_1 nor of type $A_1 \times A_1$;
- the Cartan matrix is of type $A_1 \times A_1$ and $\lambda = 0$,

then $u(\mathcal{D},\lambda,0)$ contains a subalgebra B, such that $B = u(\mathcal{D}',0,0)$, where the Cartan matrix in $\mathcal{D}'$ is of type $A_1 \times A_1$. By [55, Thm. 4.1], after applying a similar argument as in the proof of Corollary 5.2.1, we obtain that $\mathrm{cx}_B(\Bbbk) = 2$. So

$$\mathrm{cx}_A(\Bbbk) = \mathrm{cx}_{u(\mathcal{D},\lambda,0)} \geqslant \mathrm{cx}_B(\Bbbk) = 2.$$

Therefore, $\underline{\mathrm{mod}}A$ is not a CY category.

If the Cartan matrix is of type A_1, then the non-simple blocks of A are isomorphic to the algebras of the form $\Bbbk Q/I$, where Q is the following quiver,



and I is the ideal generated by all paths of length N such that N divides m [49]. The stable category of $\Bbbk Q/I$ is not a CY category by [22, Thm. 6.1] (cf. [15] and [26]). So $\underline{\mathrm{mod}}A$ is not a CY category.

If the Cartan matrix is of type $A_1 \times A_1$ and the linking parameters are non-zero, we do not know how to calculate the complexity of $\Bbbk$. But we can obtain the complexity in one special case. Assume that A is the small quantum group $u_q(\mathfrak{sl}_2)$, that is, A is generated by E, F and K subject to the relations

$$K^p = 1, \;\; E^p = F^p = 0,$$

$$KE = q^2 EK, \;\; KF = q^{-2} FK, \;\; EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

where p is an odd integer and q is a p-th primitive root of 1. Then by [73, Thm. 3.3.2], the non-simple blocks of A are isomorphic to $\Bbbk Q/I$, where Q is the following quiver,

$$\begin{tikzcd} 1 \arrow[r, bend left=20, "y"] \arrow[r, shift left, "x"] & 2 \arrow[l, shift left, "x"] \arrow[l, bend left=20, "y"] \end{tikzcd}$$

and I is the ideal generated by $x^2 - y^2 = 0$ and $xy = yx = 0$. From a direct computation, we have that $\mathrm{cx}_A(\Bbbk) = 2$. So $\underline{\mathrm{mod}}A$ is not a CY category.

In summary, except the algebra $u(\mathcal{D}, \lambda, \mu)$ of type $A_1 \times A_1$ with nonzero linking parameters, we have proved that the stable category of $u(\mathcal{D}, \lambda, \mu)$ in other cases is not a CY category. This leads us to conjecture that the stable category of $u(\mathcal{D}, \lambda, \mu)$ is not a CY category for any $\mathcal{D}$, λ and μ.

5.3 Appendix

5.3.1

In this subsection, we verify that the complex (5.4) in §5.1.2 is indeed a complex.

The following equations follow directly from Lemma 5.1.3.

$$Dy = \begin{cases} yD, & \text{if } a_1, a_3 \text{ are both even or both odd;} \\ q_{21}^{-N+2} yD, & \text{if } a_1 \text{ even and } a_3 \text{ is odd;} \\ q_{21}^{N-2} yD, & \text{if } a_1 \text{ odd and } a_3 \text{ is even.} \end{cases} \tag{5.6}$$

It is clear that $\delta_i^2 = 0$ for $i = 1, 2, 3$. So if a_2 is odd,

$$\partial^2(\Phi(a_1, a_2, a_3)) = ((\delta_3\delta_1 + \delta_1\delta_3 + \tilde{\delta}_2\delta_2) + (\delta_2\delta_3 + \delta_3\delta_2) + (\delta_1\delta_2 + \delta_2\delta_1))\Phi(a_1, a_2, a_3).$$

Put

$$\begin{aligned} A &= (\delta_3\delta_1 + \delta_1\delta_3 + \tilde{\delta}_2\delta_2)\Phi(a_1, a_2, a_3), \\ B &= (\delta_2\delta_3 + \delta_3\delta_2)\Phi(a_1, a_2, a_3), \\ C &= (\delta_1\delta_2 + \delta_2\delta_1)\Phi(a_1, a_2, a_3). \end{aligned}$$

We show that $A = B = C = 0$.

$$\begin{aligned} A &= (\delta_3\delta_1 + \delta_1\delta_3 + \tilde{\delta}_2\delta_2)\Phi(a_1, a_2, a_3) \\ &= ((-1)^{a_1-1+a_2} q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{21}^{-\sigma(a_3)\tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c \\ &\quad +(-1)^{a_1} q_{21}^{-\tau(a_1)} yD)\Phi(a_1 - 1, a_2, a_3 - 1), \end{aligned}$$

where D satisfies $Dy = -q_{21}^{\tau(a_1-1)} q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{21}^{-\sigma(a_3)\tau(a_2-1)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c$. That is,

$$q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{21}^{-\sigma(a_3)\tau(a_2-1)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + q_{21}^{-\tau(a_1-1)} Dy = 0.$$

Hence,

$$q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{21}^{-\sigma(a_3)\tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + q_{21}^{-\sigma(a_3)} q_{21}^{-\tau(a_1-1)} Dy = 0.$$

By equation (5.6), we have $q_{21}^{-\sigma(a_3)} q_{21}^{-\tau(a_1-1)} Dy = q_{21}^{-\tau(a_1)} yD$. So

$$\begin{aligned} A &= ((-1)^{a_1-1+a_2} q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{21}^{-\sigma(a_3)\tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c \\ &\quad +(-1)^{a_1} q_{21}^{-\tau(a_1)} yD)\Phi(a_1 - 1, a_2, a_3 - 1) \\ &= 0. \end{aligned}$$

The equations $B = 0$ and $C = 0$ can be verified directly. For example,

$$\begin{aligned}
B &= (\delta_2\delta_3 + \delta_3\delta_2)(\Phi(a_1, a_2, a_3)) \\
&= ((-1)^{a_1} q_{21}^{-\sigma(a_2)\tau(a_1)} y^{\sigma(a_2)} (-1)^{a_1+a_2-1} q_{12}^{\sigma(a_3)\tau(a_1)} q_{21}^{-\sigma(a_3)\tau(a_2-1)} x_2^{\sigma(a_3)} \\
&\quad +(-1)^{a_1+a_2} q_{12}^{\sigma(a_3)\tau(a_1)} q_{21}^{-\sigma(a_3)\tau(a_2)} x_2^{\sigma(a_3)} (-1)^{a_1} q_{21}^{-\sigma(a_2)\tau(a_1)} y^{\sigma(a_2)}) \\
&\quad \Phi(a_1, a_2 - 1, a_3 - 1) \\
&= 0,
\end{aligned}$$

since $\tau(a_2 - 1) + \sigma(a_2) = \tau(a_2)$.

If a_2 is even, then

$$\begin{aligned}
\partial^2(\Phi(a_1, a_2, a_3)) &= ((\delta_1\delta_3 + \delta_3\delta_1 + \delta_2\tilde{\delta}_2) + (\delta_1\delta_2 + \delta_2\delta_1) + (\delta_3\delta_2 + \delta_2\delta_3) \\
&\quad +(\tilde{\delta}_2\delta_1 + \delta_1\tilde{\delta}_2) + (\tilde{\delta}_2\delta_3 + \delta_3\tilde{\delta}_2))\Phi(a_1, a_2, a_3).
\end{aligned}$$

The equation $(\delta_1\delta_3 + \delta_3\delta_1 + \delta_2\tilde{\delta}_2)\Phi(a_1, a_2, a_3) = 0$ follows directly from the definition of $\tilde{\delta}_2$. As in the case in which a_2 is odd,

$$(\delta_2\delta_3 + \delta_3\delta_2)\Phi(a_1, a_2, a_3) = 0 \text{ and } (\delta_1\delta_2 + \delta_2\delta_1)\Phi(a_1, a_2, a_3) = 0$$

can be also verified via a straightforward computation. Now, we show that $(\tilde{\delta}_2\delta_1+\delta_1\tilde{\delta}_2)\Phi(a_1, a_2, a_3) = 0$ case by case, using Lemma 5.1.3.

Case (i) a_1 and a_3 are both odd,

$$\begin{aligned}
& (\tilde{\delta}_2\delta_1 + \delta_1\tilde{\delta}_2)\Phi(a_1, a_2, a_3) \\
=\ & (x_1(q_{21}^{-\frac{a_2}{2}} x_1^{N-2}) - q_{21}^{-\frac{a_2}{2}} x_1^{N-1})\Phi(a_1 - 2, a_2 + 1, a_3) \\
=\ & 0.
\end{aligned}$$

Case (ii) a_1 is odd and a_3 is even,

$$\begin{aligned}
& (\tilde{\delta}_2\delta_1 + \delta_1\tilde{\delta}_2)\Phi(a_1, a_2, a_3) \\
=\ & (x_1(-q_{12}^{(N-1)(\frac{a_1-3}{2}N+1)} q_{21}^{-(N-1)\frac{a_2}{2}N} q_{21}^{\frac{a_1-3}{2}N+1})(k_1 x_1^{N-2} x_2^{N-2} + \cdots \\
& \quad +k_{N-2} y^{N-3} x_1 x_2 + k_{N-1} y^{N-2}) \\
& \quad +q_{12}^{(N-1)\frac{a_1-1}{2}N} q_{21}^{-(N-1)\frac{a_2}{2}N} q q_{21}^{-(N-2)} q_{21}^{\frac{a_1-1}{2}N} x_2^{N-2} x_1^{N-1})\Phi(a_1 - 2, a_2 + 1, a_3) \\
=\ & q_{12}^{(N-1)\frac{a_1-1}{2}N} q_{21}^{-(N-1)\frac{a_2}{2}N} q q_{21}^{-(N-2)} q_{21}^{\frac{a_1-1}{2}N} \\
& \quad (-q_{12}^{-N^2+2N} x_1(k_1 x_1^{N-2} x_2^{N-2} + \cdots + k_{N-2} y^{N-3} x_1 x_2 + k_{N-1} y^{N-2}) \\
& \quad +x_2^{N-2} x_1^{N-1})\Phi(a_1 - 2, a_2 + 1, a_3) \\
=\ & 0,
\end{aligned}$$

since $q_{12}^{-N^2+2N} x_1 [x_1^{N-1}, x_2^{N-1}]_c = x_2^{N-2} x_1^{N-1} y$.

Case (iii) a_1 is even and a_3 is odd,

$$\begin{aligned}
& (\tilde{\delta}_2\delta_1+\delta_1\tilde{\delta}_2)\Phi(a_1,a_2,a_3)\\
= \;& -q_{21}^{-\frac{a_2}{2}N}x_1^{N-1}+q_{21}^{-\frac{a_2}{2}N}x_1^{N-1}\Phi(a_1-2,a_2+1,a_3)\\
= \;& 0.
\end{aligned}$$

Case (iv) a_1 and a_3 are both even,

$$\begin{aligned}
& (\tilde{\delta}_2 d_1+d_1\tilde{d}_2)\Phi(a_1,a_2,a_3)\\
= \;& x_1^{N-1}(q_{12}^{(N-1)\frac{a_1-2}{2}N}q_{21}^{-(N-1)\frac{a_2}{2}N}qq_{21}^{-(N-2)}q_{21}^{\frac{a_1-2}{2}N}x_2^{N-2})\\
& +(-q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)}q_{21}^{-(N-1)\frac{a_2}{2}N}q_{21}^{\frac{a_1-2}{2}N+1})(k_1x_1^{N-2}x_2^{N-2}+\cdots\\
& +k_{N-2}y^{N-3}x_1x_2+k_{N-1}y^{N-2})x_1\Phi(a_1-2,a_2+1,a_3)\\
= \;& (q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)}q_{21}^{-(N-1)\frac{a_2}{2}N}q_{21}^{\frac{a_1-2}{2}N+1})(x_1^{N-1}x_2^{N-2}\\
& -(k_1x_1^{N-2}x_2^{N-2}+\cdots+k_{N-2}y^{N-3}x_1x_2+k_{N-1}y^{N-2})x_1)\Phi(a_1-2,a_2+1,a_3)\\
= \;& 0,
\end{aligned}$$

since $[x_1^{N-1},x_2^{N-1}]_c x_1=yx_1^{N-1}x_2^{N-2}$.

Similarly, we can prove that $(\tilde{\delta}_2\delta_3+\delta_3\tilde{\delta}_2)\Phi(a_1,a_2,a_3)=0$.

In conclusion, we have $\partial^2=0$.

5.3.2

In this subsection, we give the necessary commutative diagrams to check the relations in Theorems 5.1.11 and 5.1.12.

Set

$$X_1=q_{12}^{-(N-1)(N-3)}x_1^{N-3}x_2^{N-3}+k_1yx_1^{N-4}x_2^{N-4}+\cdots+k_{N-3}y^{N-3},$$

where $k_i\in\Bbbk$, $1\leqslant i\leqslant N-3$, such that $x_2^{N-1}x_1^{N-3}=X_1x_2^2$, and

$$X_2=q_{12}^{(N-3)(N-1)}x_2^{N-3}x_1^{N-3}+l_1yx_2^{N-4}x_1^{N-4}+l_2y^2x_2^{N-5}x_1^{N-5}+\cdots+l_{N-3}y^{N-3},$$

where $l_i\in\Bbbk$, $1\leqslant i\leqslant N-3$, such that $x_1^{N-1}x_2^{N-3}=X_2x_1^2$.

Let f_1^i, f_2^i , f_3^i and g_1^j, g_2^j , g_3^j, $1\leqslant i\leqslant 5$ and $j=1,2$ be the morphisms described by the following matrices:

f_1^i is the 5×1 matrix with 1 in the i-th position and 0 elsewhere,

$$f_2^1 = \begin{pmatrix} 1 & 0 \\ 0 & q_{12}^N \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, f_2^2 = \begin{pmatrix} 0 & 0 \\ x_1^{N-3} & 0 \\ 0 & 0 \\ 0 & 1 \\ q_{12}\, q_{21}^{1-N} y^{N-2} x_2 & -q_{21}^{1-N} y^{N-2} x_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$f_2^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, f_2^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -q_{12}^2\, q_{21}^{N-1} y^{N-2} x_2 & q_{12}\, q_{21}^{N-1} y^{N-2} x_1 \\ 1 & 0 \\ 0 & 0 \\ 0 & q_{12}^N x_2^{N-3} \\ 0 & 0 \end{pmatrix}, f_2^5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$f_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q_{12}^N & 0 & 0 & 0 \\ 0 & 0 & q_{12}^N & 0 & 0 \\ 0 & 0 & 0 & q_{12}^N & 0 \\ 0 & 0 & 0 & 0 & q_{12}^N \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, f_3^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{12}^{-N^2+N} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^{N^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ q_{21}^{-1} y^{N-2} x_2 & 0 & 0 & 0 & 0 \\ 0 & q_{12}^{-N} x_1^{N-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 & 0 \\ 0 & q_{21}^{1-N} q_{12}^2 y^{N-2} x_2 & q_{12}^N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{12}\, q_{21}^{N-3} y^{N-2} x_2 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^{2N} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_3^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{12}^{-N} & 0 & 0 & 0 & 0 \\ 0 & q_{12}^{-N^2+2N} X_2 & 0 & 0 & 0 \\ 0 & q_{12}\, q_{21}^{-N+3} y^{N-2} x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{21}^N & q_{21}^{N-1} y^{N-2} x_1 & 0 \\ 0 & 0 & 0 & q_{12}^{2N} x_2^{N-3} & 0 \\ 0 & 0 & 0 & 0 & q_{21}^{-N+1} q_{12}^2 y^{N-2} x_1 \\ 0 & 0 & 0 & 0 & q_{12}^N \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_3^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{12}^{-N^2+N} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_1^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, g_1^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$g_2^1 = \begin{pmatrix} x_1^{N-2} & 0 \\ qq_{12}^2 x_2 & (-q_{12} - qq_{12})x_1 \\ 0 & -q_{12} y^{N-1} \\ 0 & x_2 \\ 0 & 0 \end{pmatrix}, g_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (-q_{12} - qq_{12})x_1^{N-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & qq_{12}^2 & 0 \\ 0 & 0 & q_{12}^N & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^N \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_2^2 = \begin{pmatrix} 0 & 0 \\ x_1 & 0 \\ y^{N-1} & 0 \\ (-q_{21} - qq_{21})x_2 & qq_{21}^2 x_1 \\ 0 & x_2^{N-2} \end{pmatrix}, g_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{21}^N & 0 & 0 \\ 0 & qq_{21}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{12}^N(-q_{21} - qq_{21})x_2^{N-3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following commutative diagrams

$$\begin{tikzcd} R^{12} \arrow[r] \arrow[d, "f_3^i"'] & R^7 \arrow[r] \arrow[d, "f_2^i"'] & R^5 \arrow[r] \arrow[d, "f_1^i"'] \arrow[dr] & R^2 \arrow[r] & R \\ R^5 \arrow[r] & R^2 \arrow[r] & R \arrow[r] & \Bbbk \end{tikzcd}, \tag{5.7}$$

$$\begin{tikzcd} R^7 \arrow[r] \arrow[d, "g_3^i"'] & R^5 \arrow[r] \arrow[d, "g_2^i"'] & R^2 \arrow[r] \arrow[d, "g_1^i"'] \arrow[dr] & R \arrow[r] & \Bbbk \\ R^5 \arrow[r] & R^2 \arrow[r] & R \arrow[r] & \Bbbk \end{tikzcd}. \tag{5.8}$$

It is also routine to check the commutativity of the diagrams (5.7) and (5.8). But we need to mention that the following equations hold

$$X_1(qq_{21}^2 x_1 x_2 - (q_{21} + qq_{21})x_2 x_1) = -q_{12}^{-N^2+2N}\overline{D},$$

$$X_2(qq_{12}^2 x_2x_1 - (q_{12} + qq_{12})x_1x_2) = -\overline{D},$$

which follow from Lemma 5.1.2 and the following two equations

$$\begin{aligned} q_{12}^{-N^2+2N}\overline{D}x_2 &= x_2^{N-1}x_1^{N-2} \\ &= X_1x_2^2x_1 \\ &= X_1(-qq_{21}^2x_1x_2 + (q_{21} + qq_{21})x_2x_1)x_2, \end{aligned}$$

$$\begin{aligned} \overline{D}x_1 &= x_1^{N-1}x_2^{N-2} \\ &= X_2x_1^2x_2 \\ &= X_2(-qq_{12}^2x_2x_1 + (q_{12} + qq_{12})x_1x_2)x_1. \end{aligned}$$

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