DOCTORAATSPROEFSCHRIFT

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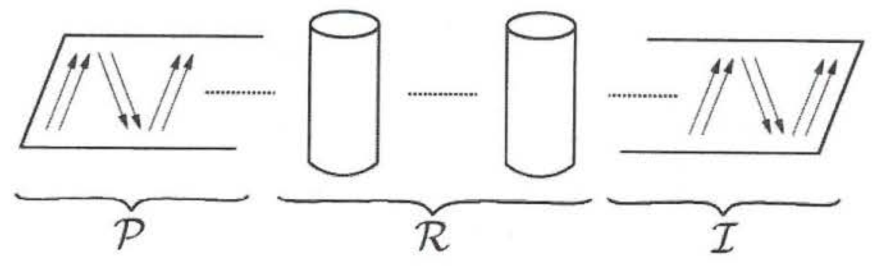

On the classification of hereditary categories

Proefschrift voorgelegd tot het behalen van de graad van
Doctor in de Wetenschappen, richting wiskunde, te verdedigen door:

Adam-Christiaan van Roosmalen

Promotor: prof. dr. M. Van den Bergh

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Contents

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Introduction

A natural approach to noncommutative algebraic geometry is to define a noncommutative scheme not as set with extra structure, but as an abelian category modeled after the category of coherent sheaves on a commutative scheme $\mathbb{X}$. By 'modeled after' we mean here that some of the familiar homological properties of commutative schemes continue to hold.

If $\mathbb{X}$ is noetherian then the category of coherent sheaves on $\mathbb{X}$ has finite homological dimension d if and only if $\mathbb{X}$ is regular and has dimension d. It is therefore natural to study abelian categories of finite homological dimension. If the homological dimension is zero, then the category is semi-simple and there is no distinction between the commutative case and the noncommutative case. So the first interesting case is homological dimension one. Abelian categories with homological dimension at most one are called hereditary. One should think of them as noncommutative (regular) curves and it makes sense to try to classify them.

The first result towards the classification of hereditary categories seems to be [30] where Lenzing showed that a noetherian hereditary Ext-finite category with a tilting complex and without nonzero projectives is equivalent to the category of coherent sheaves on a so-called weighted projective line. A second basic contribution is [23] in which Happel classified all hereditary Ext-finite categories with a tilting complex, up to derived equivalence. The examples occurring here naturally fall into two classes, those derived equivalent with the representations of a quiver, and those derived equivalent to the coherent sheaves on a weighted projective line.

In [40] Reiten and Van den Bergh succeeded in giving a classification of noetherian hereditary Ext-finite categories satisfying a natural homological condition called Serre duality (see §1.5 below). The Serre duality hypothesis is much weaker than the existence of a tilting object. For example in the commutative case all smooth projective curves satisfy Serre duality but the only ones that possess a tilting object are the projectives lines.

Our basic "leitmotiv" in this thesis was to try to obtain a classification of hereditary categories with Serre duality, up to derived equivalence, starting with the results and techniques from [40]. We have not (yet) achieved this goal, but we have performed a number of subclassifications. Among other things this has lead to the discovery of some new interesting examples of hereditary categories.

We first fix some notations which are used throughout this thesis. Let k be an algebraically closed field of arbitrary characteristic. Let $\mathfrak{a}$ be a Hom-finite k-linear category. We define $\operatorname{Mod}\mathfrak{a}$ as the class of contravariant functors $\mathfrak{a} \to \operatorname{Mod} k$. The objects of $\operatorname{Mod}\mathfrak{a}$ are called right $\mathfrak{a}$-modules. With every object $A \in \operatorname{Ob}\mathfrak{a}$, we may associate a "standard" projective (right) module $\mathfrak{a}(-,A)$ and a "standard" injective (right) module $\mathfrak{a}(A,-)^*$. We will denote the category of finitely presented right modules by $\operatorname{mod}\mathfrak{a}$ and the category of finitely presented and cofinitely presented right $\mathfrak{a}$-modules by $\operatorname{mod}^{\mathrm{cfp}}\mathfrak{a}$. If $\operatorname{mod}(\mathfrak{a})$ is an abelian category then we say that $\mathfrak{a}$ is coherent. We will mostly think of k-linear categories as "generalized quivers" with relations and we view $\operatorname{mod}\mathfrak{a}$ and $\operatorname{mod}^{\mathrm{cfp}}\mathfrak{a}$ as certain particularly nice categories of representations of $\mathfrak{a}$.

Directed categories

We classify directed hereditary categories satisfying Serre duality. This project came out of a question in an early version of [40] where the authors asked if their classification would remain valid without the noetherian condition if one works up to derived equivalence.

This question was quickly shown to have a negative answer by Ringel who in [45] gave a class of counter examples. The hereditary categories constructed by Ringel are representations of "infinitely stretched" Dynkin quivers of type A and D such that the resulting categories are locally very nice but are "stretched" too much to be derived equivalent to a noetherian hereditary category. So they cannot be fitted in the classification of [40].

As Ringel's examples are locally representations of Dynkin quivers they are in particular directed. This was for us the inspiration to try to classify directed hereditary categories with Serre duality up to derived equivalence.

This classification is performed in Chapter 3 (see also [53]). Before we can give the main result, we will need to introduce a bit of notation. Below $\mathcal{L}$ will be a linearly ordered set in which every element has an immediate successor and an immediate predecessor. It is easy to see that any such partially ordered set is of the form $\mathcal{T} \overrightarrow{\times} \mathbb{Z}$ where $\mathcal{T}$ is linearly ordered and $\overrightarrow{\times}$ denotes the lexicographically ordered product. Pictorially we may draw $\mathcal{L}$ as

$$\cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots$$

If $\mathcal{L} = \mathbb{Z}$ then $\mathcal{L}$ is sometimes referred to as an A_∞^∞ quiver. Therefore we will usually write $A_\mathcal{L}$ for $\mathcal{L}$. We also define $D_\mathcal{L}$ as the union of $A_\mathcal{L}$ with two distinguished elements which are strictly larger than the elements of $A_\mathcal{L}$ but incomparable with each other.

Schematically:

$$\cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots \begin{array}{l} \nearrow \bullet \\ \\ \searrow \bullet \end{array}$$

Theorem 1. *A connected directed k-linear hereditary category satisfying Serre duality is derived equivalent to* $\operatorname{mod}^{\mathrm{cfp}} k\mathcal{P}$ *where* $\mathcal{P}$ *is a Dynkin quiver,* $A_{\mathcal{L}}$*, or* $D_{\mathcal{L}}$ *and where* $\mathcal{L}$ *is a locally discrete linearly ordered set without maximum or minimum.*

The representations of $A_{\mathcal{L}}$ and $D_{\mathcal{L}}$ where precisely the examples considered by Ringel.

Dualizing k-varieties and hereditary sections

The hereditary categories considered in [40] fall apart in two classes with rather distinct properties. The first class consists of categories without projectives and injectives and is geometric in flavor. The second class consists of categories which are generated by preprojective objects. These categories are more combinatorial in flavor. In particular it is shown that, at least in the noetherian case, these categories are very close to being representations of quivers. Afterwards Idun Reiten suggested ([41]) to try to establish a similar result without the noetherian hypotheses.

Even in the noetherian case the classification of categories generated by preprojectives is somewhat roundabout. However Reiten and Van den Bergh noted that a substantial simplification is possible provided one could show that the categories in question are derived equivalent to the representations of a well-chosen quiver. In joint work with Carl Fredrik Berg ([8]) we have succeeded in establishing this fact.

Theorem 2. *(Corollary 1.56 in the text). Let* $\mathcal{A}$ *be a noetherian k-linear hereditary categories with Serre duality. Assume* $\mathcal{A}$ *has a non-zero projective object, then* $\mathcal{A}$ *is derived equivalent to* $\operatorname{mod} kQ'$ *where* Q' *is strongly locally finite.*

A quiver is said to be strongly locally finite if every indecomposable projective and injective representation has finite length.

The proof of Theorem 2 uses a new combinatorial gadget which we call "round trip distance". It is the pseudo-metric on the vertices of a quiver defined as the minimal number of arrows one has to traverse in the opposite direction on an unoriented path from x to y and back. If the quiver does not have any oriented cycles, then the round trip distance even defines a metric.

The proof of Theorem 2 is based on the following entirely combinatorial result (from Theorem 1.51).

Theorem 3. *Let* Q *be a connected quiver, then the following are equivalent.*

- *The quiver Q has no oriented cycles, and for a certain (equivalently, for all) $x \in Q$ the round trip distance spheres $S_Q(x,n)$ are finite, for all $n \in \mathbb{N}$.*
- *The translation quiver $\mathbb{Z}Q$ has a strongly locally finite section.*

In view of Idun Reiten's question from [41] it is now natural to ask if similar techniques can be used in the nonnoetherian case. More specifically, given the projectives in a hereditary abelian category with Serre duality, can one find a derived equivalent category in which the additive category of the projectives has a better shape? In Chapter 6, we show that this is the case. Before discussing the results from that chapter, we need to introduce some definitions and results from Chapter 2. (Chapter 6 and §2.2 are based on joint work with Carl Fredrik Berg in [7]).

Let $\mathfrak{a}$ be a k-linear additive category with split idempotents. We will say $\mathfrak{a}$ is a finite k-variety if $\dim_k \mathfrak{a}(A,B) < \infty$. We will say that a finite k-variety is a dualizing k-variety if the standard projective modules have a copresentation by standard injectives and the standard injective modules have a presentation by standard projectives (Proposition 2.7). A dualizing k-variety $\mathfrak{a}$ is automatically coherent (Corollary 2.8).

The concepts of dualizing k-varieties and Serre duality are closely related as the following result shows

Theorem 4. *(Theorem 2.9 in the text). Let $\mathfrak{a}$ be a finite k-variety such that every object in* $\operatorname{mod}\mathfrak{a}$ *has finite projective dimension, then* $\operatorname{mod}\mathfrak{a}$ *has Serre duality if and only if $\mathfrak{a}$ is a dualizing k-variety.*

We will say that a k-linear category $\mathfrak{a}$ is semi-hereditary if $\operatorname{mod}\mathfrak{a}$ is abelian and hereditary. Being semi-hereditary is a local property so it is usually easy to verify

Proposition 5. *(Proposition 2.1 in the text). Let $\mathfrak{a}$ be a small preadditive category, then $\mathfrak{a}$ is semi-hereditary if and only if any full subcategory of $\mathfrak{a}$ with a finite number of objects is semi-hereditary.*

We give a combinatorial classification of semi-hereditary dualizing k-varieties by means of so-called thread quivers (see §2.2.2) which are a type of quivers in which some arrows have been marked to represent a locally discrete linearly ordered set (this is a mild generalization of the constructions by Ringel in [45]).

We now return to the results of Chapter 6. Let $\mathcal{A}$ be a hereditary category with Serre duality. We will denote the additive category given by the projective objects by $\mathcal{Q}_{\mathcal{A}}$. In the bounded derived category $D^b\mathcal{A}$, we will denote $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ for the full subcategory given by objects of the form $\tau^n P$ where $P \in \mathcal{Q}_{\mathcal{A}}$ and $n \in \mathbb{Z}$. We will be interested in whether there is a hereditary category $\mathcal{H}$ with Serre duality whose category of projectives is given by $\mathcal{Q}_{\mathcal{H}}$, such that there is an embedding $D^b\mathcal{H} \hookrightarrow D^b\mathcal{A}$ and $\mathbb{Z}\mathcal{Q}_{\mathcal{H}}$ corresponds to $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ under this embedding. We will say $\mathcal{Q}_{\mathcal{H}}$ is a hereditary section in $D^b\mathcal{A}$.

Our main result in this direction assumes a technical condition "(*)" (discussed in §6.4.2). This additional condition is not an artifact of our methods as the next result is false if we do not impose it.

Theorem 6. *(Theorem 6.61 in the text). Let $\mathcal{A}$ be a connected hereditary category satisfying Serre duality with category of projectives $\mathcal{Q}_{\mathcal{A}}$. If $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ is connected and satisfies (*), then $\mathcal{A}$ is derived equivalent to a hereditary category $\mathcal{H}$ satisfying Serre duality and with category of projectives $\mathcal{Q}_{\mathcal{H}}$ such that $\mathcal{Q}_{\mathcal{H}}$ is a dualizing k-variety and the natural embedding $i : D^b \operatorname{mod} \mathcal{Q}_{\mathcal{H}} \to D^b\mathcal{A}$ commutes with Serre duality.*

It seems to be possible to avoid (*) at the (small) cost of adding extra objects of $D^b\mathcal{A}$ to $\mathcal{Q}_{\mathcal{H}}$ (see Theorem 6.64).

When combined with our classification of semi-hereditary dualizing k-varieties in terms of thread quivers the following corollary gives a complete answer to Reiten's question in [41].

Theorem 7. *(Corollary 6.63 in the text). Let $\mathcal{A}$ be a hereditary category with Serre duality. If $\mathcal{A}$ is generated by preprojectives, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} \mathcal{Q}$ *for a semi-hereditary dualizing k-variety $\mathcal{Q}$.*

A natural noncommutative smoothness condition is "saturatedness" [12]. The previous theorem yields the following corollary.

Theorem 8. *(Theorem 6.67 in the text). Let $\mathcal{A}$ be a connected saturated abelian Ext-finite hereditary category with a directing object. Then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} A$ *for a finite dimensional hereditary algebra A.*

Big tubes and uniserial categories

When studying the representation theory of some of the generalized quivers introduced above, a new type of hereditary categories with Serre duality appears naturally as subcategories. It is similar to a tube but it has an infinite number of base objects. Such subcategories are associated to a new kind of hereditary category with Serre duality which we call a "big tube" (see Example 2.18 for such an example where a big tube occurs as a subcategory).

Let $\mathcal{L}$ be a locally discrete linearly ordered set. We will define a "big loop" as the k-linear category $\widehat{k\mathcal{L}^\bullet}$ such that the objects are given by the elements of $\mathcal{L}$ and the morphisms are given by

$$\operatorname{Hom}_{\widehat{k\mathcal{L}^\bullet}}(i,j) = \begin{cases} k[[x]] & \text{if } i \leq j \\ xk[[x]] & \text{if } i > j \end{cases}$$

such that the composition is given by multiplication. It follows easily from Proposition 5 that $\widehat{k\mathcal{L}^\bullet}$ is semi-hereditary.

As $\widehat{k\mathcal{L}^\bullet}$ is not Hom-finite, neither is $\operatorname{mod}(\widehat{k\mathcal{L}^\bullet})$. Nonetheless one easily verifies that $\operatorname{mod}^{\mathrm{cfp}}(\widehat{k\mathcal{L}^\bullet})$ is an Ext-finite hereditary category satisfying Serre duality. It is this category that we call a "big tube."

Big tubes enter in an essential way in the classification of uniserial hereditary categories with Serre duality. Recall that an abelian category is said to be uniserial if the subobjects of any indecomposable object are linearly ordered by inclusion. We have the following result.

Theorem 9. *(Theorem 4.1). Let $\mathcal{A}$ be a connected uniserial hereditary category with Serre duality. Then $\mathcal{A}$ is equivalent to one of the following*

1. $\operatorname{mod} A_n$,
2. $\operatorname{nilp} \tilde{A}_n$,
3. *a big tube,*
4. $\operatorname{mod}^{\mathrm{cfp}} A_{\mathcal{L}}$ *where $\mathcal{L}$ is a linearly ordered locally discrete set .*

Calabi-Yau categories

Recently there has been considerable interest in Calabi-Yau categories. We will say a k-linear abelian category $\mathcal{A}$ is n-Calabi-Yau (or "has Calabi Yau dimension n") if it is Ext-finite and $[n]$ is a Serre functor on $D^b\mathcal{A}$. It is well-known and easy to see that the Calabi-Yau dimension of an abelian Calabi-Yau category equals its global dimension. So in particular a hereditary Calabi-Yau category has Calabi-Yau dimension one and vice versa.

In a 1-Calabi-Yau category, every Auslander-Reiten component is a standard homogeneous tube. Recall that a tube of rank r is an Auslander-Reiten component of the form $\mathbb{Z}A_\infty/\langle\tau^r\rangle$; if $r = 1$ then the tube is a homogeneous tube.

As a prelude to the study of 1-Calabi-Yau categories we make a general study of tubes in hereditary categories with Serre duality (see §4.2). Our main results concerning tubes are

Theorem 10. *(Theorem 4.5). An Auslander-Reiten component in $D^b\mathcal{A}$ is a tube if and only if it contains an indecomposable object X such that $\tau^r X \cong X$, for $r \geq 1$.*

Theorem 11. *(Theorem 4.6). Let $\mathcal{K}$ be a tube in $D^b\mathcal{A}$. Then*

1. *$\mathcal{K}$ is standard,*
2. *$\mathcal{K}$ is directing in the sense that if there is a path $X_0 \to \cdots \to X_n$ in $D^b\mathcal{A}$ with $X_0, X_n \in \mathcal{K}$, then $X_i \in \mathcal{K}$ for all i.*
3. *There exists a τ-invariant t-structure on $D^b\mathcal{A}$ with hereditary heart $\mathcal{H} \supseteq \mathcal{K}$ such that the base objects of $\mathcal{K}$ are simple in $\mathcal{H}$.*

These results are sufficient to classify the abelian 1-Calabi-Yau categories ([52], see Theorem 5.13 in the text).

Theorem 12. *Let $\mathcal{A}$ be a connected abelian 1-Calabi-Yau category, then $\mathcal{A}$ is derived equivalent to one of the following*

1. *the category* $\mathrm{Mod}^{fd} k[[t]]$ *of finite dimensional representations of* $k[[t]]$*, or*

2. *the category* $\operatorname{coh} X$ *of coherent sheaves on an elliptic curve* X.

By extension of the Calabi-Yau property, we will say a k-linear abelian category is *fractionally Calabi-Yau of dimension* $\frac{m}{n}$ if it has a Serre functor F such that there is an $n > 0$ with $F^n \cong [m]$. If $\mathcal{A}$ is fractionally Calabi-Yau of dimension 1, then every Auslander-Reiten component is a tube, but not necessarily a homogeneous one.

Relying on Theorems 10 and 11, we find a classification of hereditary Calabi-Yau categories which are fractionally Calabi-Yau.

Theorem 13. *(Theorem 5.40). Let* $\mathcal{A}$ *be a connected abelian hereditary category which is fractional Calabi-Yau, but not* 1*-Calabi-Yau. Then* $\mathcal{A}$ *is derived equivalent to either*

1. *the category of finite presented modules* $\operatorname{mod} Q$ *over a Dynkin quiver* Q*, or*

2. *the category of nilpotent representations* $\operatorname{nilp} \tilde{A}_n$ *where* $\tilde{A}_n$ *has cyclic orientation and* $n \geq 1$*, or*

3. *the category of coherent sheaves* $\operatorname{coh} \mathbb{X}$ *over a weighted projective line of tubular type.*

The proof is based on the explicit construction of a tilting object (which somewhat strangely only exists in the *non*-1-Calabi-Yau case).

The reduced Grothendieck group

The last chapter (Chapter 7) is about ongoing research which aims to classify the hereditary categories with Serre duality which have a reduced Grothendieck group with small rank.

Recall that the reduced Grothendieck group $K_0^{red}\mathcal{A}$ of an Ext-finite abelian category $\mathcal{A}$ of finite homological dimension is defined as the quotient $K_0\mathcal{A}/\operatorname{rad}\chi$, where $K_0\mathcal{A}$ is the Grothendieck group of $\mathcal{A}$ and $\operatorname{rad}\chi$ is the radical of the Euler form, namely

$$\operatorname{rad}\chi = \{X \in K_0(\mathcal{A}) \mid \chi(X, -) = 0\}.$$

The rank of the free group $K_0^{red}\mathcal{A}$ will be denoted by $\operatorname{Num}\mathcal{A}$.

Theorem 14. *(see Propositions 7.7 and 7.10). Let* $\mathcal{A}$ *be a connected hereditary category with Serre duality. If* $\operatorname{Num}\mathcal{A} = 0$*, then* $\mathcal{A}$ *is equivalent to* $\mathrm{Mod}^{fd} k[[t]]$*. If* $\operatorname{Num}\mathcal{A} = 1$*, then* $\mathcal{A}$ *is equivalent to either*

1. *the category* $\operatorname{mod} k$ *of finite dimensional* k*-vector spaces, or*

2. *the category* $\operatorname{nilp} k\tilde{A}_1$ *where* $\tilde{A}_1$ *has cyclic orientation.*

The interesting cases however start at $\operatorname{Num} \mathcal{A} = 2$. This last condition holds for example for the category of coherent sheaves on a smooth projective curve. In this case we have a result under the additional condition that there is an object E such that $\operatorname{End}(E) = k$ and $\dim \operatorname{Ext}(E, E) \leq 1$. We conjecture that such objects always exist in hereditary categories with Serre duality (we can prove this in some cases).

Theorem 15. *(Theorem 7.11) Let $\mathcal{A}$ be a connected hereditary category with Serre duality. If $\operatorname{Num} \mathcal{A} = 2$ and $\mathcal{A}$ has an object E as described above then $\mathcal{A}$ is derived equivalent to one of the following:*

1. $\operatorname{nilp} k\tilde{A}_2$ *where $\tilde{A}_2$ has cyclic orientation,*
2. $\operatorname{mod} kQ$, *where Q is a generalized Kronecker quiver,*
3. $\operatorname{coh} X$, *for a smooth projective curve X.*

Recall that a generalized Kronecker quiver is a quiver with two vertices, a and b, and a finite number of arrows from a to b.

Samenvatting

Een natuurlijke manier om in niet-commutatieve algebraïsche meetkunde een niet-commutatief schema te definiëren is niet als een verzameling met extra structuur, maar als een abelse categorie gelijkend op de categorie van coherente schoven over een commutatief schema $\mathbb{X}$. Met "gelijkend" bedoelen we hier dat een aantal vertrouwde homologe eigenschappen van commutatieve schema's ook gelden voor deze categorieën.

Als $\mathbb{X}$ noethers is, dan heeft de categorie van coherente schoven over $\mathbb{X}$ eindige homologe dimensie d als en slechts als $\mathbb{X}$ regulier is en dimensie d heeft. Het lijkt dus zinvol om abelse categorieëen met eindige globale dimensie te bestuderen. Als de homologe dimensie nul is, dan is de categorie semi-simpel en is er geen onderscheid tussen het commutatieve en het niet-commutatieve geval. Het eerste interessante geval heeft dus globale dimensie 1. Zo'n categorieën met globale dimensie ten hoogste 1 noemen we hereditair. Ze zijn te beschouwen als niet-commutatieve (reguliere) krommen en het lijkt zowel haalbaar als zinvol om ze te classificeren.

Een eerste stap naar de classificatie van hereditaire categorieën lijkt [30] te zijn, waarin Lenzing aantoont dat een noetherse hereditaire Ext-eindige categorie met een tiltend complex en zonder (niet-nul) projectieve objecten, equivalent is met de categorie van coherente schoven op een zogeheten gewogen projectieve rechte. Een tweede fundamentele contributie staat beschreven in [23] waar Happel alle hereditaire categorieën met een tiltend complex classificieert, op afgeleide equivalentie na. Deze categorieën zijn natuurlijkerwijze in twee klassen onder te verdelen: deze afgeleid equivalent aan de representaties van een gerichte graaf (quiver), en deze afgeleid equivalent aan de coherente schoven op een gewogen projectieve rechte.

In [40] slaagden Reiten en Van den Bergh erin om alle noetherse hereditaire Ext-eindige categorieën te classificeren met een bijkomende natuurlijke homologe eigenschap, genaamd Serre dualiteit (zie §1.5). De hypothese van Serre dualiteit is een veel zwakkere veronderstelling dan het bestaan van een tiltend complex te eisen. In het commutatieve geval, bijvoorbeeld, geldt Serre dualiteit voor all gladde projectieve krommen, maar de enige die een tiltend complex hebben zijn de projectieve rechten.

Het leidmotief in deze thesis is het proberen classificeren van alle hereditary categorieën met Serre dualiteit, op afgeleide equivalentie na, vertrekkende van de technieken en resultaten beschreven in [40]. Dit doel is (nog) niet bereikt, maar wel een aantal deelclassificaties. Deze deelclassificaties hebben onder andere geleid tot nieuwe (interessante) voorbeelden van hereditaire categorieën.

We zullen nu een aantal notaties vastleggen die doorheen de gehele thesis gevolgd worden. Vooreerst zal k altijd een algebraïsch gesloten veld zijn. Zij $\mathfrak{a}$ een Hom-eindige k-lineaire categorie, dan definiëren we $\operatorname{Mod}\mathfrak{a}$ als de klasse van contravariante functoren $\mathfrak{a} \to \operatorname{Mod} k$. De objecten van $\operatorname{Mod}\mathfrak{a}$ noemen we rechtse $\mathfrak{a}$-modulen. Met ieder object $A \in \operatorname{Ob}\mathfrak{a}$ kunnen we een standaard projectief (rechts) moduul $\mathfrak{a}(-, A)$ en een standaard injectief (rechts) moduul $\mathfrak{a}(A, -)^*$ associëren. We zullen de categorie van eindig gepresenteerde rechtse modulen voorstellen door $\operatorname{mod}\mathfrak{a}$ en de categorie van eindig gepresenteerde en coëindig gepresenteerde rechtse modulen door $\operatorname{mod}^{\text{cfp}}\mathfrak{a}$. Als $\operatorname{mod}\mathfrak{a}$ een abelse categorie is, dan zeggen we dat $\mathfrak{a}$ coherent is. Meestal zullen we $\mathfrak{a}$ interpreteren als een "veralgemeende quiver" met relaties, en zullen we $\operatorname{mod}\mathfrak{a}$ en $\operatorname{mod}^{\text{cfp}}\mathfrak{a}$ zien als mooie categorieën van representaties van $\mathfrak{a}$.

Gerichte categorieën

In deze thesis zullen we onder andere de gerichte hereditaire categorieën met Serre dualiteit classificeren. Dit project vond zijn oorsprong in een eerdere versie van [40] waar de auteurs de vraag stellen of hun classificatie eveneens geldig is voor niet-noetherse categorieën indien men bereid is te werken op afgeleide equivalentie na.

Kort daarna toonde Ringel in [45] aan dat het antwoord negatief was door een klasse van tegenvoorbeelden te geven. De hereditaire categorieën die Ringel construeert zijn representaties van "oneindig langgerekte" Dynkin quivers van type A en D zodat de corresponderende categorieëen lokaal een mooie structuur hebben, maar globaal te "uitgerekt" zijn om afgeleid equivalent te zijn met een noetherse hereditaire categorie. In het bijzonder passen ze dus niet in de classificatie van [40].

Ringels voorbeelden zijn lokaal representaties van Dynkin quivers, en hij merkt op dat bijgevolg de corresponderende categorieën gericht zijn. Dit gaf ons de motivatie om zo'n gerichte hereditaire categorieën met Serre dualiteit te classificeren op afgeleide equivalentie na.

Deze classificatie wordt beschreven in §3 (zie eveneens [53]). Voordat we het hoofdresultaat formuleren, introduceren we een aantal notaties. Zij $\mathcal{L}$ een lineair geordende verzameling waar ieder element een directe voorganger en een directe opvolger heeft. Het is eenvoudig in te zien dat zo'n geordende verzameling van de vorm $\mathcal{T} \overrightarrow{\times} \mathbb{Z}$ is, waar $\mathcal{T}$ een lineair geordende verzameling is en $\overrightarrow{\times}$ het lexicografisch geordend product. We kunnen $\mathcal{L}$ grafisch voorstellen door

$$\cdots [\cdots \to \bullet \to \bullet \to \bullet \to \cdots] \cdots [\cdots \to \bullet \to \bullet \to \bullet \to \cdots] \cdots$$

Indien $\mathcal{L} = \mathbb{Z}$, dan noteert men ook wel A_∞^∞ in de plaats van $\mathcal{L}$. We zullen vaak $A_\mathcal{L}$ schrijven voor $\mathcal{L}$. Analoog definiëren we $D_\mathcal{L}$ als the unie van $A_\mathcal{L}$ met twee extra elementen die beide strikt groter zijn dan alle elementen van $A_\mathcal{L}$, maar zelf onderling onvergelijkbaar zijn. We stellen $D_{\mathcal{L}}$ schematisch voor door

$$\begin{tikzcd}[column sep=small] & & & & & & & & & & & & & & & & \bullet \\ \cdots[\cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots]\cdots[\cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots]\cdots \arrow[urrrrrrrr] \arrow[drrrrrrrr] & & & & & & & & \\ & & & & & & & & & & & & & & & & \bullet \end{tikzcd}$$

Stelling 1. *Een samenhangende gerichte k-lineaire hereditaire categorie met Serre dualiteit is afgeleid equivalent met* $\operatorname{mod}^{\mathrm{cfp}} k\mathcal{P}$ *waar $\mathcal{P}$ staat voor een Dynkin quiver, $A_{\mathcal{L}}$, of $D_{\mathcal{L}}$ en waar $\mathcal{L}$ een linear geordende verzameling waar ieder element een directe voorganger en een directe opvolger heeft.*

De representaties van de vorm $A_{\mathcal{L}}$ en $D_{\mathcal{L}}$ zijn juist de tegenvoorbeelden die Ringel geconstrueerd had.

Dualiserende k-variëteiten en hereditare secties

De hereditaire categorieën die in [40] beschouwd werden, vallen uiteen in twee klassen met vrij verschillende eigenschappen. Een eerste klasse bastaat uit categorieën zonder niet-nul projectieven of injectieven en heeft een meetkundig karakter. Een tweede klasse bestaat uit categorieën die voortgebracht worden door preprojectieve objecten; deze categorieën hebben dan een vrij combinatorisch karakter en, in het bijzonder, wordt het aangetoond dat ze –althans in het noetherse geval– heel dicht aanleunen bij de representaties van quivers. Later zal Idun Reiten [41] voorstellen om een soortgelijk resultaat aan te tonen zonder noethers te veronderstellen.

Zelfs in het noetherse geval is de classificatie van categorieën die voortgebracht worden door preprojectieven, vrij indirect. Reiten en Van den Bergh merkten op dat het bewijs substantieel ingekort zou kunnen worden indien je zou kunnen aantonen dat zulke categorieën afgeleid equivalent zijn met de representaties van een goedgekozen quiver. In gezamelijk werk met Carl Fredrik Berg ([8]) zijn we erin geslaagd om dit resultaat aan te tonen.

Stelling 2. *(Gevolg 1.56 in de tekst.) Zij $\mathcal{A}$ een noetherse k-lineaire hereditaire categorie met Serre dualiteit. Indien $\mathcal{A}$ een niet-nul projectief object heeft, dan is $\mathcal{A}$ afgeleid equivalent met* $\operatorname{mod} kQ'$ *voor een sterk lokale quiver Q'.*

We zullen zeggen dat een quiver sterk lokaal is indien alle onontbindbare projectieve en injectieve representaties eindige lengte hebben.

Het bewijs van Stelling 2 gebruikt een nieuwe combinatorische gadget die we de "round trip distance" noemen. Het is een pseudo-metriek op de knopen van de quiver, gedefinieerd als het aantal pijlen die men moet doorlopen op een ongeoriënteerd pad van x naar y en terug. Indien de quiver geen georiënteerde cycles heeft, dan geeft de round trip distance een metriek.

Het bewijs van Stelling 2 is gebaseerd op het volgende zuiver combinatorische resultaat (zie Stelling 1.51).

Stelling 3. *Zij Q een samenhangende quiver, dan zijn de volgende uitspraken equivalent:*

- *de quiver Q heeft geen georiënteerde cycles, en voor een zekere (of equivalent: voor alle) $x \in Q$ zijn de round trip distance bollen $S_Q(x, n)$ eindig, voor alle $n \in \mathbb{N}$.*
- *de stabiele translatiequiver $\mathbb{Z}Q$ heeft een sterk lokaal eindige sectie.*

Met oog op Idun Reitens vraag in [41] is het natuurlijk om af te vragen of soortgelijke technieken ook in het niet-noetherse geval gebruikt kunnen worden. In het bijzonder, gegeven de projectieven in een hereditaire abelse categorie met Serre dualiteit, kan men een afgeleid equivalente categorie vinden waarin de additieve categorie van de projectieven een betere vorm heeft? In Hoofdstuk 6 tonen we aan dat dit inderdaad het geval is. Voordat we onze bevindingen van dat hoofdstuk bespreken, voeren we eerst een aantal begrippen van Hoofdstuk 2 in. (Hoofdstuk 6 en §2.2 zijn gebaseerd op gezamelijk werk met Carl Fredrik Berg in [7]).

Zij $\mathfrak{a}$ een k-lineaire additieve categorie waar idempotente morfismes splitten. We zullen zeggen dat $\mathfrak{a}$ een eindige k-variëteit is indien $\dim_k \mathfrak{a}(A, B) < \infty$. Een eindige k-variëteit is een dualiserende k-variëteit indien de standard projectieven co-eindig gepresentaard zijn en de standaard injectieven eindig gepresenteerd (Eigenschap 2.7). Een dualiserende k-variëteit $\mathfrak{a}$ is altijd coherent (Gevolg 2.8).

De noties van dualiserende k-variëteiten en Serre dualiteit zijn sterk gerelateerd, zoals het volgende resultaat aantoont.

Stelling 4. *(Stelling 2.9 in de tekst). Zij $\mathfrak{a}$ een eindige k-variëteit zodat ieder object in* $\operatorname{mod} \mathfrak{a}$ *eindige projectieve dimensie heeft, dan heeft* $\operatorname{mod} \mathfrak{a}$ *Serre dualiteit als en slechts als $\mathfrak{a}$ een dualiserende k-variëteit is.*

Een k-lineaire categorie is wordt semi-hereditair genoemd indien $\operatorname{mod} \mathfrak{a}$ abels en hereditair is. Omdat deze eigenschap een "lokale eigenschap" is, is het meestal eenvoudig na te gaan.

Eigenschap 5. *(Proposition 2.1 in the tekst) Zij $\mathfrak{a}$ een kleine preadditieve categorie, dan is $\mathfrak{a}$ semi-hereditair indien iedere deelcategorie met een eindig aantal objecten semi-hereditair is.*

We zullen een combinatorische beschrijving geven van semi-hereditaire dualiserende k-variëteiten aan de hand van thread quivers (zie §2.2.2). Dit is een soort quivers waarvan sommige pijlen eigenlijk een lokaal discrete lineair geordende verzameling voorstellen (dit is een lichte veralgemening van de constructies van Ringel in [45]).

We bekijken nu de resultaten uit Hoofdstuk 6. Zij $\mathcal{A}$ een hereditaire category met Serre dualiteit. We zullen de additieve categorie bestaande uit alle projectieve objecten van $\mathcal{A}$ voorstellen door $\mathcal{Q}_\mathcal{A}$. Met $\mathbb{Z}\mathcal{Q}_\mathcal{A}$ bedoelen we de volle additieve deelcategorie van de begrensde afgeleide categorie $D^b\mathcal{A}$ bestaande uit alle elementen van de vorm $\tau^n P$ waar $P \in \mathcal{Q}_\mathcal{A}$ en $n \in \mathbb{Z}$. We kunnen ons nu afvragen of er een hereditaire category $\mathcal{H}$ met Serre dualiteit en additieve categorie van projectieven $\mathcal{Q}_\mathcal{H}$ bestaat samen met een inbedding $D^b\mathcal{H} \hookrightarrow D^b\mathcal{A}$ die $\mathbb{Z}\mathcal{Q}_\mathcal{H}$ en $\mathbb{Z}\mathcal{Q}_\mathcal{A}$ identificeert. In dit geval zullen we zeggen dat $\mathcal{Q}_\mathcal{H}$ een hereditary sectie is van $D^b\mathcal{A}$.

Ons hoofdresultaat is dat we dit altijd kunnen doen, mits we een technische voorwaarde "(*)" opleggen (besproken in §6.4.2). Deze extra voorwaarde is geen nevenverschijnsel van de door ons gebruikte methode vermits het volgende resultaat vals is indien we het niet eisen.

Stelling 6. *(Stelling 6.61 in de tekst) Zij $\mathcal{A}$ een samenhangende hereditaire categorie met Serre dualiteit en met categorie van projectieven $\mathcal{Q}_\mathcal{A}$. Indien $\mathbb{Z}\mathcal{Q}_\mathcal{A}$ samenhangend is en aan voorwaarde (*) voldoet, dan is $\mathcal{A}$ afgeleid equivalent met een hereditaire categorie $\mathcal{H}$ met Serre dualiteit zodat de categorie van projectieve objecten $\mathcal{Q}_\mathcal{H}$ een dualiserende k-variëteit vormen en de natuurlijke inbedding $i : D^b \operatorname{mod} \mathcal{Q}_\mathcal{H} \to D^b\mathcal{A}$ met de Serre functor commuteert.*

Het is mogelijk om de voorwaarde (*) te ontwijken door $\mathcal{Q}_\mathcal{H}$ te vergroten (zie Stelling 6.64). In dit geval zal $\mathcal{Q}_\mathcal{H}$ niet noodzakelijk een hereditaire sectie meer zijn, maar wel steeds een partiële tiltende verzameling (in de zin van §1.9).

Wanneer we de classificatie van semi-hereditaire dualiserende k-variëteiten door thread quivers combineren met onderstaand gevolg van voorgaande stelling, dan geeft dit een volledig antwoord op Reiten haar vraag in [41].

Stelling 7. *(Gevolg 6.63 in de tekst). Zij $\mathcal{A}$ een hereditaire category met Serre dualiteit. Indien $\mathcal{A}$ voortgebracht wordt door preprojectieve objecten, dan is $\mathcal{A}$ afgeleid equivalent aan $\operatorname{mod} \mathcal{Q}$ waar $\mathcal{Q}$ een semi-hereditaire dualiserende k-variëteit is.*

Een natuurlijke niet-commutatieve "gladheidsvoorwaarde" is verzadigdheid [12]. Als gevolg van Stelling 6 vinden deze classificatie.

Stelling 8. *(Stelling 6.67 in de tekst). Zij $\mathcal{A}$ een samenhangende verzadigde abelse Ext-eindige hereditaire category met een gericht object, dan is $\mathcal{A}$ afgeleid equivalent aan $\operatorname{mod} A$ waar A een eindig dimensionale algebra is.*

Grote tubes en uniseriële categorieën

Tijdens het bestuderen van de representaties van veralgemeende quivers zoals hierboven geïntroduceerd, vonden we een nieuw type hereditaire categorieën met Serre dualiteit die voorkomt als deelcategorie. Het lijkt sterk op een tube, maar deze nieuwe categorie heeft oneindig veel "basisobjecten" (zie Voorbeeld 2.18 voor een voorbeeld waar zo een grote tube voorkomt als deelcategorie).

Zij $\mathcal{L}$ een lokaal discrete lineair geordende verzameling. We definiëren een "grote lus" als de k-lineaire categorie $\widehat{k\mathcal{L}^\bullet}$ waar de objecten de elementen van $\mathcal{L}$ zijn en de afbeeldingen gegeven worden door

$$\operatorname{Hom}_{\widehat{k\mathcal{L}^\bullet}}(i,j) = \begin{cases} k[[x]] & \text{indien } i \leq j \\ xk[[x]] & \text{indien } i > j \end{cases}$$

De samenstelling van morfismes komt overeen met het product. Het volgt nu eenvoudig uit Eigenschap 5 dat $\widehat{k\mathcal{L}^\bullet}$ semi-hereditair is.

Vermits $\widehat{k\mathcal{L}^\bullet}$ niet Hom-eindig is, zal $\operatorname{mod}(\widehat{k\mathcal{L}^\bullet})$ het evenmin zijn. Niettegenstaande kan men eenvoudig verifiëren dat $\operatorname{mod}^{\mathrm{cfp}}(\widehat{k\mathcal{L}^\bullet})$ een Ext-eindige hereditaire categorie met Serre dualiteit is. Het is deze categorie die we een "grote tube" noemen.

Grote tubes duiken onvermijdelijk op in de classificatie van uniseriële hereditaire categorieën met Serre dualiteit. We zullen zeggen dat een abelse categorie uniserieel is indien alle deelobjecten van een indecomposable object linear geordend zijn door de inclusie. We hebben het volgende resultaat.

Stelling 9. *(Stelling 4.1) Zij $\mathcal{A}$ een samenhangende uniseriële hereditaire categorie met Serre dualiteit, dan is $\mathcal{A}$ equivalent aan één van de volgende categorieën:*

1. $\operatorname{mod} A_n$,
2. $\operatorname{nilp} \tilde{A}_n$,
3. *een grote tube,*
4. $\operatorname{mod}^{\mathrm{cfp}} A_{\mathcal{L}}$ *waar $\mathcal{L}$ een lineair geordende lokaal discrete werzameling is die ofwel een minimum en een maximum heeft, of geen van de twee.*

Calabi-Yau categorieën

Nieuwe ontwikkelingen zorgen voor een toegenomen interesse in Calabi-Yau categorieën. We zullen zeggen dat een k-lineaire abelse categorie $\mathcal{A}$ een n-Calabi-Yau categorie is (of Calabi-Yau dimensie n heeft) indien ze Ext-eindig is en $[n]$ een Serre functor is op $D^b\mathcal{A}$. Het is geweten dat de Calabi-Yau dimensie van een abelse Calabi-Yau categorie gelijk is aan de globale dimensie. In het bijzonder heeft een (niet semi-simpele) hereditaire Calabi-Yau categorie een Calabi-Yau dimensie van 1 en omgekeerd.

In een abelse 1-Calabi-Yau categorie is iedere Auslander-Reiten component een standaard homogene tube. Een tube van rang r is een Auslander-Reiten component van de vorm $\mathbb{Z}A_\infty/\langle\tau^r\rangle$. Indien $r=1$ spreken we van een homogene tube.

Als inleiding op het bespreken van abelse 1-Calabi-Yau categories, geven we een algemene discussie van tubes in hereditaire categorieën met Serre dualiteit (zie §4.2). Onze hoofdresultaten betreffende tubes zijn de volgende.

Stelling 10. *(Stelling 4.5). Een Auslander-Reiten component in $D^b\mathcal{A}$ is een tube als en slechts als het een indecomposabel object X bevat met $\tau^r X \cong X$ waar $r \geq 1$.*

Stelling 11. *(Stelling 4.6). Zij $\mathcal{K}$ een tube in $D^b\mathcal{A}$, dan*

1. *is $\mathcal{K}$ standaard,*
2. *is $\mathcal{K}$ gericht, waarmee we bedoelen dat als er een path $X_0 \to \cdots \to X_n$ in $D^b\mathcal{A}$ is met $X_0, X_n \in \mathcal{K}$, dat dan $X_i \in \mathcal{K}$ voor alle i.*

3. *bestaat er een τ-invariante t-structuur op $D^b\mathcal{A}$ met hereditair hart $\mathcal{H} \supseteq \mathcal{K}$ waar de randobjecten van $\mathcal{K}$ simpele objecten zijn in $\mathcal{H}$.*

Deze resultaten zijn voldoende om alle abelse 1-Calabi-Yau categorieën te classificeren ([52], zie Stelling 5.13 in de tekst).

Stelling 12. *Zij $\mathcal{A}$ een samenhangende abelse 1-Calabi-Yau categorie, dan is $\mathcal{A}$ afgeleid equivalent aan een van de volgende categorieën:*

1. *de categorie $\operatorname{Mod}^{fd} k[[t]]$ van eindig-dimensionale representaties van $k[[t]]$, of*

2. *de categorie $\operatorname{coh} X$ van coherente schoven over een elliptische kromme X.*

Als uitbreiding van de Calabi-Yau eigenschap, zullen we zeggen dat een k-lineaire abelse categorie fractioneel Calabi-Yau van dimensie $\frac{m}{n}$ is indien $D^b\mathcal{A}$ een Serre functor F heeft en er een $n > 0$ is zodat $F^n \cong [m]$. Indien $\mathcal{A}$ fractioneel Calabi-Yau is van dimensie 1, dan is iedere Auslander-Reiten component een tube, maar niet noodzakelijk homogeen.

We kunnen Stellingen 10 en 11 gebruiken om een classificatie te vinden van alle hereditaire Calabi-Yau categorieën die fractioneel Calabi-Yau zijn.

Stelling 13. *(Stelling 5.40). Zij $\mathcal{A}$ een samenhangende abelse hereditaire categorie die fractioneel Calabi-Yau is, maar niet 1-Calabi-Yau, dan is $\mathcal{A}$ afgeleid equivalent met*

1. *de categorie van eindig gepresenteerde modulen $\operatorname{mod} Q$ van een Dynkin quiver Q, of*

2. *de categorie van nilpotente representaties $\operatorname{nilp} \tilde{A}_n$ vor $n \geq 1$ waar $\tilde{A}_n$ cyclische orientatie heeft, of*

3. *de categorie van coherente schoven $\operatorname{coh} \mathbb{X}$ over een gewogen projectieve rechte van tubulair type.*

Het bewijs van voorgaande stelling steunt op de constructie van een tiltend object dat echter alleen bestaat in het *niet*-1-Calabi-Yau geval.

De gereduceerde Grothendieckgroep

Het laatste hoofdstuk (Hoofdstuk 7) handelt over onafgerond onderzoek waar we proberen de hereditaire categorieën te classificeren die een gereduceerde Grothendieckgroep hebben met kleine rang.

De gereduceerdeGrothendieck groep $K_0^{red}\mathcal{A}$ van een Ext-eindige abelse categorie $\mathcal{A}$ van eindige globale dimensie wordt gedefinieerd als het quotient $K_0\mathcal{A}/\operatorname{rad}\chi$ waar $K_0\mathcal{A}$ de Grothendieckgroep van $\mathcal{A}$ is en $\operatorname{rad}\chi$ het radicaal van de Eulervorm, namelijk

$$\operatorname{rad}\chi = \{X \in K_0(\mathcal{A}) \mid \chi(X,-) = 0\}.$$

We noteren de rang van de vrije groep $K_0^{red}\mathcal{A}$ door $\operatorname{Num}\mathcal{A}$.

Stelling 14. *(zie Eigenschappen 7.7 and 7.10) Zij $\mathcal{A}$ een samenhangende hereditaire categorie met Serre dualiteit. Als* $\operatorname{Num}\mathcal{A} = 0$ *dan $\mathcal{A}$ is equivalent aan* $\operatorname{Mod}^{fd} k[[t]]$. *Indien* $\operatorname{Num}\mathcal{A} = 1$, *dan is $\mathcal{A}$ equivalent met één van volgende categorieën:*

1. *de categorie* $\operatorname{mod} k$ *van eindig dimensionale k-vectorruimten, of*
2. *de categorie* $\operatorname{nilp} k\tilde{A}_1$ *waar $\tilde{A}_1$ cyclische oriëntatie heeft.*

De interessante gevallen zijn echter deze waar $\operatorname{Num}\mathcal{A} = 2$. Deze bevat onder andere de categorieën van coherente schoven over een gladde projectieve kromme. We kunnen ze classificeren indien we een extra voorwaarde opleggen: er is een object X met $\dim \operatorname{Ext}(X, X) \leq 1$. We vermoeden dat zo'n object altijd bestaat in een hereditaire categorie met Serre dualiteit.

Stelling 15. *(Stelling 7.11) Zij $\mathcal{A}$ een samenhangende hereditaire categorie met Serre duality. Als* $\operatorname{Num}\mathcal{A} = 2$ *en $\mathcal{A}$ heeft een object E met* $\dim \operatorname{Ext}(E, E) \leq 1$, *dan is $\mathcal{A}$ afgeleid equivalent aan een van de volgende:*

1. $\operatorname{nilp} k\tilde{A}_2$ *waar $\tilde{A}_2$ cyclische oriëntatie heeft,*
2. $\operatorname{mod} kQ$, *met Q een veralgemeende Kronecker quiver,*
3. $\operatorname{coh} X$, *voor een gladde projectieve kromme X.*

Hier zeggen we dat een quiver een veralgemeende Kronecker quiver is als Q uit twee knopen, a en b, bestaat en er alleen maar pijlen zijn van a naar b.

Chapter 1

Preliminaries

1.1 Additive and k-linear categories

Throughout this thesis k is an algebraically closed field. All categories are assumed to be small and k-linear (see below).

1.1.1 Definitions

Definition 1.1. A *preadditive category* is a category together with an abelian group structure on each of its morphisms set such that the composition of morphisms is bilinear over the integers.

When $\mathfrak{a}$ is a preadditive category, we will usually write $\mathfrak{a}(A, B)$ for $\mathrm{Hom}_{\mathfrak{a}}(-,-)$.

Definition 1.2. A *k-linear category* is a category together with a vector space structure on each of its morphisms set such that the composition of morphisms is bilinear over k.

Definition 1.3. If in a k-linear category $\mathfrak{a}$ each of the morphism sets $\mathfrak{a}(A, B)$ is finite dimensional as k-vector spaces, then we say that $\mathfrak{a}$ is *Hom-finite*.

Example 1.4. With a quiver Q, we may associate a k-linear category as follows: the objects are objects of Q, the morphisms from A to B are k-linear combinations of (oriented) paths from A to B, and the composition of morphisms is induced by the concatenation of paths. This category is called the *path category* of the quiver Q and is denoted by kQ.

Example 1.5. Let $\mathcal{P}$ be a partially ordered set. We associate to $\mathcal{P}$ a preadditive category $k\mathcal{P}$ as follows: the objects of $k\mathcal{P}$ are the elements of $\mathcal{P}$ and

$$(k\mathcal{P})(i,j) = \begin{cases} k & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases}$$

For $i \leq j$ denote the element of $(k\mathcal{P})(i,j)$ corresponding to $1 \in k$ by (i,j). Composition of maps in $k\mathcal{P}$ is defined by $(j,k)(i,j) = (i,k)$.

Example 1.6. A *$\mathbb{Z}$-algebra* is a preadditive category whose object set is given by $\mathbb{Z}$ ([37]).

Definition 1.7. A preadditive category $\mathfrak{a}$ is called *additive* if it has finite direct sums.

It is well-known (see for example [33, Theorem 2] [35, Corollary I.18.2]) that in a preadditive category finite direct sums and finite direct products coincide, thus the natural morphism

$$\oplus_{i \in I} A_i \to \times_{i \in I} A_i$$

is an isomorphism. In this case, $\oplus_i A_i$ is also called a *bi-product*.

Definition 1.8. A preadditive category is *Karoubian* if it has finite direct sums and idempotents split, thus if the endomorphism ring of indecomposable objects are local rings.

1.1.2 Radical and almost split morphisms

We will say a map $A \to B$ is a *radical morphism* if there are no maps $X \to A$ and $B \to X$ such that the composition

$$X \to A \to B \to X$$

is an isomorphism, where X is not zero. The set of all radical maps in $\mathrm{Hom}(A,B)$ is denoted by $\mathrm{rad}(A,B)$; it is a subspace of $\mathrm{Hom}(A,B)$.

The set $\mathrm{rad}^2(A,B) \subseteq \mathrm{rad}(A,B)$ is defined as follows:

$$f \in \mathrm{rad}^2(A,B) \Leftrightarrow \exists g \in \mathrm{rad}(A,X), \exists h \in \mathrm{rad}(X,B) : f = g \circ h$$

where $X \in \mathrm{Ob}\,\mathcal{C}$. If A and B are indecomposable, then $\mathrm{rad}(A,B)$ are just the non-invertible morphisms from A to B. Higher powers of the radical are defined in an obvious way. The infinite radical $\mathrm{rad}^{\infty}(A,B)$ is defined as $\cup_{n\in\mathbb{N}} \mathrm{rad}^n(A,B)$.

A map $f : A \to B$ is said to be *irreducible* if $f \in \mathrm{rad}(A,B) \setminus \mathrm{rad}^2(A,B)$.

We say a non-split map $f : B \to C$ is *right almost split* if any non-split morphism $X \to C$ factors through f. Dually, we define *left almost split* maps.

1.2 Abelian categories and global dimension

1.2.1 Abelian categories

Definition 1.9. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. A morphism $i : K \to X$ is called the *kernel* of f if for every $g : A \to X$ with $f \circ g = 0$, there is a unique morphism $g' : A \to X$ such that $g = i \circ g'$.

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists! g'"'] \arrow[d, "g"] & \\ K \arrow[r, "i"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Remark 1.10. The kernel is defined up to unique isomorphism as subobject of X.

Remark 1.11. Often, we will say $K = \ker f$, with the morphism $K \to X$ being understood.

Cokernels are defined dually.

Definition 1.12. An additive category is called *abelian* if each map has a kernel, a cokernel and for every map $f : X \to Y$ the natural map $\operatorname{coker}\ker f \to \ker\operatorname{coker} f$ is an isomorphism.

Definition 1.13. We define the *image* $\operatorname{im} f$ of f by $\operatorname{im} f = \ker\operatorname{coker} f$.

Remark 1.14. Note that Definition 1.12 is self-dual, thus if a category $\mathcal{A}$ is abelian, so is the opposite category $\mathcal{A}^\circ$.

1.2.2 Representations of preadditive categories

For $\mathfrak{a}$ a small preadditive category we denote by $\operatorname{Mod}(\mathfrak{a})$ the category of right $\mathfrak{a}$-modules. An object M of $\operatorname{Mod}(\mathfrak{a})$ may be represented by a collection of abelian groups $M(A)_{A \in \operatorname{Ob}\mathfrak{a}}$ depending contravariantly in A.

If $f : \mathfrak{a} \to \mathfrak{b}$ is a functor between small preadditive categories then there is an obvious restriction functor

$$(-)_\mathfrak{a} : \operatorname{Mod}(\mathfrak{b}) \to \operatorname{Mod}(\mathfrak{a})$$

which sends $N(B)_{B \in \operatorname{Ob}\mathfrak{b}}$ to $N(f(A))_{A \in \operatorname{Ob}\mathfrak{a}}$. This restriction functor has a left adjoint

$$\mathfrak{b} \otimes_\mathfrak{a} - : \operatorname{Mod}(\mathfrak{a}) \to \operatorname{Mod}(\mathfrak{b})$$

which is the right exact functor which sends the projective generators $\mathfrak{a}(-, A)$ in $\operatorname{Mod}(\mathfrak{a})$ to $\mathfrak{b}(-, f(A))$ in $\operatorname{Mod}(\mathfrak{b})$. As usual if f is fully faithful we have $(\mathfrak{b} \otimes_\mathfrak{a} N)_\mathfrak{a} = N$.

Let M be in $\operatorname{Mod}(\mathfrak{a})$. We will say that M is *finitely generated* if M is a quotient of finitely generated projectives. Similarly we say that M is *finitely presented* if M has a presentation

$$P \to Q \to M \to 0$$

where P, Q are finitely generated projectives. It is easy to see that these notions coincides with the ordinary categorical ones. The full subcategory $\operatorname{Mod}\mathfrak{a}$ spanned by the finitely presented modules will be denoted by $\operatorname{mod}\mathfrak{a}$. If $\operatorname{mod}\mathfrak{a}$ is an abelian category, we will say $\mathfrak{a}$ is *coherent*.

Dually we will say that M is *cofinitely generated* if it is contained in a cofinitely generated injective. *Cofinitely presented* is defined in a similar way.

The categorical interpretation of the latter notions is somewhat less clear. However if $\mathfrak{a}$ is Hom-finite then both finitely and cofinitely presented representations correspond to each other under duality (exchanging $\mathfrak{a}$ and $\mathfrak{a}^\circ$).

With every object A of $\mathfrak{a}$, we may associate a *standard projective* $\mathfrak{a}(-,A)$ and a *standard injective* $\mathfrak{a}(A,-)^*$. It is clear that every finitely generated projective is a direct summand of a standard projective. If $\mathfrak{a}$ has finite direct sums and idempotents split in $\mathfrak{a}$, then every finitely generated projective is isomorphic to a standard projective. Dual notions hold for injective objects.

With an indecomposable object $A \in \operatorname{ind} \mathfrak{a}$, we may associate in a straightforward way a *standard simple* object S_A as the simple top of $\mathfrak{a}(-,A)$ or, equivalently, the simple socle of $\mathfrak{a}(A,-)^*$.

We will say on object $A \in \operatorname{ind} \mathfrak{a}$ is a *source* or a *sinksink* if the corresponding standard projective $\mathfrak{a}(-,A)$ or standard injective $\mathfrak{a}(A,-)^*$, repectively, is simple.

1.2.3 Global dimension

We now come to the definition of global dimension. Since we do not require the category $\mathcal{A}$ to have enough projective or injective objects, we can not use the projective or injective dimension to define the global dimension. We start by the Yoneda definition of the Ext-groups.

Let $\operatorname{Pretext}^n(C,A)$ be the set consisting of isomorphism classes of exact sequences of the form:

$$0 \to A \to B_{n-1} \to B_{n-2} \to \cdots \to B_0 \to C \to 0$$

We may turn $\operatorname{Pretext}^n(C,A)$ into an abelian semigroup by defining $\mathbf{E}+\mathbf{E}'$ as in Figure 1.1 where the lower squares, except the bottom right, are pushouts, and the upper squares, except the upper left, are pullbacks.

$$\begin{tikzcd}
\mathbf{E}+\mathbf{E}': & 0 \arrow[r] & A \arrow[r] & B_{n-1} \arrow[r] & \cdots \arrow[r] & B_0 \arrow[r] & C \arrow[r] & 0 \\
 & 0 \arrow[r] & A \arrow[r] \arrow[u, equal] \arrow[d, "\begin{pmatrix}1\\1\end{pmatrix}"'] & B''_{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & B''_0 \arrow[r] \arrow[u] \arrow[d] & C\oplus C \arrow[r] \arrow[u, "(1\ \ 1)"'] \arrow[d, equal] & 0 \\
\mathbf{E}\oplus\mathbf{E}': & 0 \arrow[r] & A\oplus A \arrow[r] & B_{n-1}\oplus B'_{n-1} \arrow[r] & \cdots \arrow[r] & B_0\oplus B'_0 \arrow[r] & C\oplus C \arrow[r] & 0
\end{tikzcd}$$

Figure 1.1: Diagram defining $\mathbf{E}+\mathbf{E}'$

We will write $\mathbf{E} \sim \mathbf{E}'$ if there is commutative diagram of the form

$$\begin{tikzcd}
\mathbf{E}: & 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & B_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}: & 0 \arrow[r] & A \arrow[r] & B'_{n-1} \arrow[r] & \cdots \arrow[r] & B'_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and define $\operatorname{Ext}^n(C, A)$ as the set of equivalence classes of $\operatorname{Pretext}^n(C, A)$ modulo the equivalent relation $\sim$.

The addition on $\operatorname{Pretext}^n(C, A)$ induces an addition on $\operatorname{Ext}^n(C, A)$ giving it the structure of an abelian group. If $\mathcal{A}$ is k-linear (which we will always assume), then $\operatorname{Ext}^n(C, A)$ is a k-vector space.

We also define Ext^0 as the bi-functor Hom.

Definition 1.15. The largest natural number n such that $\operatorname{Ext}^n(C, A) \neq 0$, for all $C, A \in \operatorname{Ob} \mathcal{A}$ is called the *global dimension* on $\mathcal{A}$. If no such number exists, then we will say the global dimension is infinite. A category with global dimension at most 1 is said to be *hereditary*.

Following proposition is well-known and easy to prove. In here, we define a *bi-cartesian square* as a diagram which is both a pushout and a pullback.

Proposition 1.16. *An abelian category $\mathcal{A}$ is hereditary if and only if for every map $f : X \to Y$, the standard mono-epi factorization $X \to \operatorname{im} f \to Y$ fits in a bi-cartesian square*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & J \arrow[d] \\
\operatorname{im} f \arrow[r] & Y
\end{tikzcd}$$

Definition 1.17. An abelian category is said to be *Ext-finite* if $\dim \operatorname{Ext}^i(X, Y) < \infty$, for all $i \geq 0$ and all $X, Y \in \operatorname{Ob} \mathcal{A}$.

Remark 1.18. Since $\operatorname{Ext}^0(-,-) \cong \operatorname{Hom}(-,-)$, an Ext-finite category is also Hom-finite.

1.2.4 Endo-simple objects

An object E is said to be endo-simple if $\dim_k \operatorname{Hom}(E, E) = 1$. Since we assume our base field k is algebraically closed, in a Hom-finite category, an object E will be endo-simple if and only if every nonzero endomorphism of E is an automorphism.

Proposition 1.19. *Every Hom-finite abelian category has an endo-simple object.*

Proof. Let $X \in \mathrm{Ob}\,\mathcal{A}$ be nonzero such that $\dim \mathrm{End}\, X$ is minimal. If X is not endo-simple, then there is a non-invertible morphism $f : X \to X$, giving the following (non-invertible) maps

$$X \twoheadrightarrow \mathrm{im}\, f \hookrightarrow X.$$

We find an injection $\mathrm{Hom}(\mathrm{im}\, f, \mathrm{im}\, f) \to \mathrm{rad}(X, X)$, and thus $\dim \mathrm{Hom}(\mathrm{im}\, f, \mathrm{im}\, f) < \dim \mathrm{End}(X, X)$. A contradiction. □

1.2.5 An embedding theorem

Let $\mathcal{A}$ be an abelian category. We will denote the closure of $\mathcal{A}$ under (small) direct limits ([17]) by $\mathrm{Ind}\,\mathcal{A}$. It is well-known $\mathrm{Ind}\,\mathcal{A}$ is a *Grothendieck category*, namely a category with a generator and exact filtered direct limits.

The objects in $\mathrm{Ind}\,\mathcal{A}$ are formal direct limits and the Hom-sets are given by

$$\mathrm{Hom}_{\mathrm{Ind}\,\mathcal{A}}(\varinjlim_i A_i, \varinjlim_j B_j) = \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{A}}(A_i, B_j).$$

There is a full and exact embedding $\mathcal{A} \to \mathrm{Ind}\,\mathcal{A}$, which lifts to a full and exact embedding on the level of derived categories (see §1.3.3) $D^b\mathcal{A} \to D^b\,\mathrm{Ind}\,\mathcal{A}$ ([32, Proposition 2.14]).

1.3 Triangulated and derived categories

1.3.1 Triangulated categories

Let $\mathcal{C}$ be an additive category with an auto-equivalence $T : \mathcal{C} \to \mathcal{C}$, called the *translation functor*. We will denote $T^n X$ as $X[n]$ and $T^n f$ as $f[n]$ where X is an object of $\mathcal{C}$ and f a map.

A *triangle* in $\mathcal{C}$ is a sextuple (X, Y, Z, u, v, w) where $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$ and $u : X \to y$, $v : Y \to Z$, and $w : Z \to X[1]$ are morphisms.

Let $\mathcal{T}$ be a class of triangles, called *distinguished triangles*, satisfying following axioms.

1. Let (X, Y, Z, u, v, w) be a distinguished triangle and (X', Y', Z', u', v', w') a triangle. If there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f_X"] & Y \arrow[r, "v"] \arrow[d, "f_Y"] & Z \arrow[r, "w"] \arrow[d, "f_Z"] & X[1] \arrow[d, "f_X[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

where f_X, f_Y, and f_Z are isomorphisms, then (X', Y', Z', u', v', w') is a distinguished triangle. For every map $u : X \to Y$, there is a distinguished triangle (X, Y, Z, u, v, w). The triangle $(X, X, 0, 1, 0, 0)$ is distinguished.

2. The triangle (X, Y, Z, u, v, w) is distinguished, if and only if $(Y, Z, X[1], v, w, -u[1])$ is.

3. If (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') are distinguished triangles, then every commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f_X"] & Y \arrow[r, "v"] \arrow[d, "f_Y"] & Z \arrow[r, "w"] & X[1] \arrow[d, "f_X[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

there is a map $f_Z : Z \to Z'$ such that the extended diagram commutes.

4. Every commutative square

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r] & Y_2 \end{tikzcd}$$

may be embedded into a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & X_1[1] \arrow[d] \\ X_2 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & X_2[1] \arrow[d] \\ X_3 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & X_3[1] \arrow[d] \\ X_1[1] \arrow[r] & Y_1[1] \arrow[r] & Z_1[1] \arrow[r] & X_1[2] \end{tikzcd}$$

where every row and column is a distinguished triangle.

Definition 1.20. An additive category $\mathcal{C}$ together with a translation functor T and a class of distinguished triangles $\mathcal{T}$ satisfying the above axioms is called a *triangulated category*.

Let $u : X \to Y$ be a map in $\mathcal{C}$. The object Z in the distinguished triangle (X, Y, Z, u, v, w) is called the *cone* of u. It is uniquely determined, up to isomorphism, but not functorial.

Remark 1.21. We will often write 'triangle' instead of 'distinguished triangle', if no confusion may arise.

For elementary results about triangulated categories, we refer to [20] and [25].

The following proposition and corollary are slight reformulations of results in [47].

Proposition 1.22. *Let $\mathcal{C}$ be a triangulated category and let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ be a triangle in $\mathcal{C}$. Let $B = \bigoplus_{i=1}^{n} B_i$ where B_i is not necessarily indecomposable for $i = 1, \ldots, n$. Write $g = (g_1, \ldots, g_n)$ and $f = (f_1, \ldots, f_n)$ with maps $f_i : A \to B_i$ and $g_i : B_i \to C$. The following are equivalent:*

1. *g_i is a radical map,*
2. *$f_i : A \to B$ induces nonzero maps from A to every direct summand of B_i.*

Proof. Assume g_i is not a radical map. Let $X \to B_i$ and $C \to X$ be maps such that the composition $X \to B_i \to C \to X$ is an isomorphism. In this case X is a direct summand of B_i and a split epimorphism is given by $B_i \to C \to X$. The composition $A \to B_i \to C \to X$ is zero, and hence $f : A \to B_i$ induces a nonzero map from A to a direct summand of B_i.

For the other direction, let X be a direct summand of B_i such that the composition $A \to B_i \to X$ is zero. We easily find the following morphism of triangles

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "f"] \arrow[d] & \oplus_i B_i \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d] & A[1] \arrow[d] \\
0 \arrow[r] & X \arrow[r, equal] & X \arrow[r] & 0
\end{tikzcd}$$

where the compositions of the vertical morphisms are isomorphisms. We see that the map $X \to B_i \to C \to X$ is an isomorphism, thus $g_i : B_i \to C$ is a radical map. $\square$

Lemma 1.23. *Let $\mathcal{C}$ be a triangulated category and let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a triangle in $\mathcal{C}$ with $h \neq 0$. Let $Y = \bigoplus_{i=1}^{n} Y_i$ where Y_i is not necessarily indecomposable for $i = 1, \ldots, n$. Write $g = (g_1, \ldots, g_n)$ and $f = (f_1, \ldots, f_n)$ with maps $f_i : X \to Y_i$ and $g_i : Y_i \to Z$. Then the following statements are true.*

1. *The morphisms g_i are non-invertible for $i = 1, \ldots, n$.*
2. *If Z is indecomposable, then f_i is nonzero for $i = 1, \ldots, n$.*
3. *The morphisms f_i are non-invertible for $i = 1, \ldots, n$.*
4. *If X is indecomposable, then g_i is nonzero for $i = 1, \ldots, n$.*

Proof. 1. If g_i were invertible, then g would be a split epimorphism and h would be zero.

2. This follows from (1.) and Proposition 1.22.

(3 & 4) Similar.

□

1.3.2 The homotopy category $K^*(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category. A *complex* X is a diagram of the form

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d^i} X^i \xrightarrow{d^{i+1}} X^{i+1} \longrightarrow \cdots$$

such that $d^{i+1} \circ d^i = 0$, for all $i \in \mathbb{Z}$.

A *morphism between complexes* $f : X \to Y$ is a morphism of diagram, thus there are maps $f^i : X^i \to Y^i$ such that the occurring squares commute. The *category of complexes* $C(\mathcal{A})$ given by the complexes and the morphisms between them is denoted by $C(\mathcal{A})$. It is easily checked that it is an abelian category.

Let X be a complex. One defines the category of bounded below complexes $C^+(\mathcal{A})$, the category of bounded above complexes $C^-(\mathcal{A})$, and the category of bounded complexes $C^b(\mathcal{A})$ as the full (abelian) subcategories of $C(\mathcal{A})$ spanned by complexes X with $X^i \cong 0$, for $i << 0$, for $i >> 0$, and for $| i |>> 0$, respectively.

We will use the notation C^*, where $* = \emptyset, +, -, b$.

There is an autoequivalence $T : C^*(\mathcal{A}) \to C^*(\mathcal{A})$ given by shifting 1 degree and changing the sign of the boundary operators, thus the i^{th} degree object $(TX)^i$ of TX is X^{i+1} and $d^i_{TX} = -d^{i+1}_X$. The action on morphisms is clear. We will write $X[n]$ for $T^n X$ and $f[n]$ for $T^n f$, where X is a complex and f is a morphism between complexes.

For every $i \in \mathbb{Z}$, let the i^{th} homology of the complex X be $\mathrm{H}^i X = \ker d^i / \operatorname{im} d^{i-1}$. This defines a functor $\mathrm{H}^i : C^*(\mathcal{A}) \to \mathcal{A}$.

A morphism between complexes $f : X \to Y$ is called *null-homotopic* if, for every $i \in \mathbb{Z}$, there are morphism $k^i : X^i \to Y^{i-1}$ such that $f^i = d^{i-1} \circ k^i + k^{i+1} \circ d^i$. The null-homotopic maps form an ideal in $C^*(\mathcal{A})$; we define the *homotopy category* $K^*(\mathcal{A})$ as the quotient of $C^*(\mathcal{A})$ by this ideal.

It is readily verified that the functor $\mathrm{H}^i : C^*(\mathcal{A}) \to \mathcal{A}$ induces a functor $\mathrm{H}^i : K^*(\mathcal{A}) \to \mathcal{A}$.

Note that the category $K^*(\mathcal{A})$ seldom will be abelian, but it has the structure of a triangulated category.

1.3.3 The derived category

A morphism $f : X \to Y$ in $K(\mathcal{A})$ is a *quasi-isomorphism* if $\mathrm{H}^i f$ is an isomorphism, for all $i \in \mathbb{Z}$. We define the derived category $D\mathcal{A}$ as the localisation of $K\mathcal{A}$ with respect to the quasi-isomorphisms.

Likewise, we define the bounded below derived category $D^+\mathcal{A}$, the bounded above derived category $D^-\mathcal{A}$, and the bounded derived category $D^b\mathcal{A}$ as the localisations

Figure 1.2: Sketch of $D^b\mathcal{A}$ where $\mathcal{A}$ is hereditary

of $K^+\mathcal{A}$, $K^-\mathcal{A}$, and $K^b\mathcal{A}$, respectively. The categories $D^*\mathcal{A}$ inherits the structure of a triangulated category from $K^*\mathcal{A}$.

There is a full embedding $i : \mathcal{A} \to D^*\mathcal{A}$ given by mapping an object A of $\mathcal{A}$ to the complex iA with $(iA)^0 = A$ and 0 in all other degrees. We will often write $\mathcal{A}[0]$ for $i\mathcal{A}$, closed under isomorphisms in $D^*\mathcal{A}$.

There is a nice connection between the Ext-sets of $\mathcal{A}$ and the Hom-sets of $D^b\mathcal{A}$ given by isomorphisms

$$\mathrm{Ext}^n(X, Y) \cong \mathrm{Hom}_{D^b\mathcal{A}}(iX, iY[n])$$

natural in both components. The short exact sequences of $\mathcal{A}$ are turned into distinguished triangles in $D^b\mathcal{A}$: the image of the short exact sequence $0 \to A \to B \to C \to 0$ under i fits into the triangle

$$iA \to iB \to iC \to iA[1]$$

It is well-known that, when $\mathcal{A}$ is a hereditary category, then every object in $D^b\mathcal{A}$ is isomorphic to the direct sum of its homologies (see for example [28, §2.5], [31, Theorem 3.1], or [49, Lemma 8.3.4]). Thus $D^b\mathcal{A} \cong \bigvee_{n\in\mathbb{Z}} \mathcal{A}[n]$, where the right hand side stands for the additive closure of the union of all $\mathcal{A}[n]$. Also, there are no maps from $\mathcal{A}[i]$ to $\mathcal{A}[j]$ if $j \neq i, i+1$ This invokes a drawing as in Figure 1.2, where we have marked $\mathcal{A}[0]$ in grey. Note that the morphisms go in Figure 1.2 go from left to right, which is the standard convention in such pictures.

1.4 Auslander-Reiten sequences and triangles

1.4.1 Auslander-Reiten sequences

Let $\mathcal{A}$ be a Hom-finite abelian category. A short exact sequence $0 \to A \to B \to C \to 0$ is called an *Auslander-Reiten sequence* if one of the following equivalent conditions are satisfied (see [5, Proposition V.1.14])

1. A is indecomposable and $B \to C$ is right almost split,
2. C is indecomposable and $A \to B$ is left almost split.

An Auslander-Reiten sequence $0 \to A \to B \to C \to 0$ is uniquely determined up to isomorphism by C. We write $A = \tau C$ and $\tau^{-1}A = C$.

An Auslander-Reiten sequence is also called an *almost split sequence*.

1.4.2 Auslander-Reiten triangles

Let $\mathcal{C}$ be a triangulated category. A distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is called an *Auslander-Reiten triangle* if the following conditions are satisfied

1. X and Z are indecomposable,
2. w is not zero,
3. any non-split map $W \to Z$ factors through $v : Y \to Z$.

It is shown in [22, Lemma 4.2] that the last condition is equivalent to

3'. any non-split map $X \to W$ factors through $v : X \to Y$.

It is easy to show that an Auslander-Reiten triangle is uniquely determined up to isomorphism by either X or Z. We denote $X = \tau Z$ and $Z = \tau^{-1}X$.

1.5 Serre duality

Let $\mathcal{C}$ be a Hom-finite triangulated k-linear category. A *Serre functor* [12] on $\mathcal{C}$ is an additive auto-equivalence $F : \mathcal{C} \to \mathcal{C}$ such that for every $X, Y \in \operatorname{Ob}\mathcal{C}$ there are isomorphisms

$$\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(Y, FX)^*$$

natural in X and Y, and where $(-)^*$ is the vector-space dual.

We will say $\mathcal{C}$ has *Serre duality* if $\mathcal{C}$ admits a Serre functor. An abelian category $\mathcal{A}$ is said to satisfy Serre duality when the bounded derived category $D^b\mathcal{A}$ has Serre duality.

If a triangulated category $\mathcal{C}$ has a Serre functor F, then F is exact. Two Serre functors of $\mathcal{C}$ are naturally isomorphic.

It has been shown in [40] that $\mathcal{C}$ has Serre duality if and only if $\mathcal{C}$ has Auslander-Reiten triangles. If we denote the Auslander-Reiten shift by τ, then $F \cong \tau[1]$.

For an Ext-finite hereditary abelian category $\mathcal{A}$, the link between Serre duality and Auslander-Reiten sequences is particularly nice (see [40, Theorem I.3.3]): $\mathcal{A}$ has Serre duality if and only if $\mathcal{A}$ has Auslander-Reiten sequences and if there is a 1-1 correspondence between the indecomposable projectives and injectives given by $P \mapsto I$ if the simple top of P and the simple socle of I coincide. This correspondence between the category of projectives $\mathcal{P}$ and the category of injectives $\mathcal{I}$ is known as the *Nakayama functor* $N : \mathcal{P} \to \mathcal{I}$. If $\mathcal{A}$ has Serre duality, then N is an equivalence. Since N is induced by the action of F on $D^b\mathcal{A}$, we will sometimes write F instead of N.

Remark 1.24. In the original definition of a Serre functor [12] there was an extra condition. It may be verified that condition is superfluous.

1.6 Paths and connectedness

Let $\mathcal{A}$ be an additive Krull-Schmidt category, and $X, Y \in \operatorname{ind}\mathcal{A}$. An *(oriented) path* from X to Y is a sequence $X = X_0, X_1, \dots, X_{n-1}, X_n = Y$ where $X_i \in \operatorname{ind}\mathcal{A}$ and $\operatorname{Hom}_{\mathcal{A}}(X_i, X_{i+1}) \neq 0$, for $i \in \{0, 1, \dots n-1\}$.

We define *unoriented paths* in an obvious way.

We will say a (pre)additive/abelian/triangulated category is *connected* or *indecomposable* if it is not the direct product of two nontrivial (pre)additive/abelian/triangulated categories.

While abelian and (pre)additive categories are connected if and only if there is an unoriented path between any two indecomposables, the same is not true for triangulted categories. It follows from [47, Lemma 5] that a triangulated category $\mathcal{C}$ is connected if and only if, for every $X, Y \in \operatorname{ind}\mathcal{A}$, there is an $i \in \mathbb{Z}$ such that there is an unoriented path between X and $Y[i]$.

Example 1.25. The category $D^b \operatorname{mod} k$ is connected as triangulated category, but not as additive category.

For abelian hereditary categories, we may always assume oriented paths have length at most two.

Proposition 1.26. *Let $\mathcal{A}$ be an abelian hereditary Krull-Schmidt category. If there is an oriented path from X to Y, where X and Y are indecomposables, then there is a path $X \to Z \to Y$, where Z is an indecomposable object.*

Proof. It suffices to consider a path $X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 = Y$ of length 3. We may assume that in this path $X_0 \to X_1$ is not an epimorphism, and $X_2 \to X_3$ is not a monomorphism.

By applying Proposition 1.16 sufficiently many times, we obtain following diagram

$$\begin{tikzcd}
 & & & J_0^3 \arrow[dr, two heads] & & & \\
 & & J_0^2 \arrow[ur, hook] \arrow[dr, two heads] & \mathrm{VI} & J_1^2 \arrow[dr, two heads] & & \\
 & J_0^1 \arrow[ur, hook] \arrow[dr, two heads] & \mathrm{IV} & J_1^1 \arrow[ur, hook] \arrow[dr, two heads] & \mathrm{V} & J_2^1 \arrow[dr, two heads] & \\
X_0 \arrow[ur, hook] \arrow[dr, two heads] & \mathrm{I} & X_1 \arrow[ur, hook] \arrow[dr, two heads] & \mathrm{II} & X_2 \arrow[ur, hook] \arrow[dr, two heads] & \mathrm{III} & X_3 \\
 & \operatorname{im} f_0 \arrow[ur, hook] & & \operatorname{im} f_1 \arrow[ur, hook] & & \operatorname{im} f_2 \arrow[ur, hook] &
\end{tikzcd}$$

We wish to show that X_0 maps nonzero to every direct summand of J_0^3 and that every such direct summand maps nonzero to X_3. Therefore, assume $X_0 \to J_0^3$ induces a zero map from X_0 to a direct summand of J_0^3.

Combining the three bi-cartesian squares I, IV, and VI to one bi-cartesian square, it follows from Proposition 1.22 that $J_0^3 \to J_1^2$ is not a radical map. Again it follows from Proposition 1.22 on the square VI that $J_0^2 \to J_0^3$ restricts a zero map to one of the direct summands of J_0^3. Applying Proposition 1.22 once more on the bi-cartesian square VI-V-III yields that the map $J_0^3 \to X_3$ is not radical, and hence is a split epimorphism.

This implies $J_2^1 \to X_3$ in III is a split epimorphism, thus the square III, corresponding to

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_3 \oplus J \arrow[d] \\ \operatorname{im} f_2 \arrow[r] & X_3 \end{tikzcd}$$

where we have written J_2^1 as $X_3 \oplus J$, gives a split exact sequence; we find $J \cong 0$ and $\operatorname{im} f_2 \cong X_2$, hence f_2 is a monomorphism. A contradiction.

Dually, assuming every direct summand of J_0^3 maps nonzero X_3 implies $f_0 : X_0 \to X_1$ is an epimorphism; again a contradiction. $\square$

1.7 Split t-structures

The concept of a t-structure was introduced by Beĭlinson, Bernstein and Deligne in [6] as a means of finding derived equivalent categories. We will be interested in split t-structures of which the heart will be a hereditary category.

These results are taken from joint work with Carl Fredrik Berg [7]. We also refer to [40] and [47] for similar results.

Definition 1.27. A *t-structure* on a triangulated category $\mathcal{C}$ is a pair $(D^{\geq 0}, D^{\leq 0})$ of non-zero full subcategories of $\mathcal{C}$ satisfying the following conditions, where we denote $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$

1. $D^{\leq 0} \subseteq D^{\leq 1}$ and $D^{\geq 1} \subseteq D^{\geq 0}$
2. $\operatorname{Hom}(D^{\leq 0}, D^{\geq 1}) = 0$
3. $\forall Y \in \mathcal{C}$, there exists a triangle $X \to Y \to Z \to X[1]$ with $X \in D^{\leq 0}$ and $Z \in D^{\geq 1}$.

Let $D^{[n,m]} = D^{\geq n} \cap D^{\leq m}$. We will say the t-structure $(D^{\geq 0}, D^{\leq 0})$ is *bounded* if and only if every object of $\mathcal{C}$ is contained in some $D^{[n,m]}$. We call $(D^{\geq 0}, D^{\leq 0})$ *split* if every triangle occurring in (3) is split.

Remark 1.28. If $\mathcal{H}$ is the heart of some t-structure on a triangulated category $\mathcal{C}$, then it is clear from the definition that the t-structure is bounded and split if and only if $\mathcal{C}$ is equivalent to $\bigvee_{n\in\mathbb{Z}}\mathcal{H}[n]$.

It is shown in [6] that the *heart* $\mathcal{H} = D^{\leq 0} \cap D^{\geq 0}$ is an abelian category. Unfortunately, if $\mathcal{A}$ is an abelian category, then not every t-structure on $D^b\mathcal{A}$ defines a heart $\mathcal{H}$ which is derived equivalent to $\mathcal{A}$. Following proposition shows that in our setting we may expect derived equivalence between $\mathcal{A}$ and $\mathcal{H}$.

Proposition 1.29. *Let $\mathcal{A}$ be an abelian category and let $(D^{\geq 0}, D^{\leq 0})$ be a bounded t-structure on $D^b\mathcal{A}$. If all the triangles $X \to Y \to Z \to X[1]$ with $X \in D^{\leq 0}$ and $Z \in D^{\geq 1}$ split, then $D^{\leq 0} \cap D^{\geq 0} = \mathcal{H}$ is hereditary and $D^b\mathcal{A} \cong D^b\mathcal{H}$ as triangulated categories.*

Proof. It is well known that the category $\operatorname{Ind}\mathcal{A}$ of left exact contravariant functors from $\mathcal{A}$ to $\operatorname{Mod} k$ is a k-linear Grothendieck category and that the Yoneda embedding of $\mathcal{A}$ into $\operatorname{Ind}\mathcal{A}$ is a full and exact embedding.

By [32, Proposition 2.14], this embedding extends to a full and exact embedding $D^b\mathcal{A} \to D^b \operatorname{Ind}\mathcal{A}$.

Since all triangles $X \to Y \to Z \to X[1]$ with $X \in D^{\leq 0}$ and $Z \in D^{\geq 1}$ split, we may use [40, Lemma 1.3.5] to see that $\mathcal{H}$ is hereditary.

It is now an easy consequence of [6, Proposition 3.1.16] that $D^b\mathcal{A} \cong D^b\mathcal{H}$ as triangulated categories. □

We will say a subcategory $\mathcal{D}$ of $D^b\mathcal{A}$ is *closed under successors* if it satisfies following property : if $X \in \mathcal{D}$ admits a path to $Y \in \operatorname{ind} D^b\mathcal{A}$, then $Y \in \mathcal{D}$. As following theorem shows, "being closed under successors" is a useful property to find split t-structures.

Theorem 1.30. *Let $\mathcal{A}$ be a connected abelian category which is not semi-simple and let $\mathcal{D}$ be a non-zero full subcategory of $D^b\mathcal{A}$ closed under successors such that $\mathcal{D} \neq D^b\mathcal{A}$, then $(D^{\geq 0}, D^{\leq 0})$ is a bounded and split t-structure on $D^b\mathcal{A}$ where $D^{\geq 0} = \mathcal{D}$ and $D^{\leq 1} = D^b\mathcal{A} \setminus \mathcal{D}$, and the heart $\mathcal{H}$ is a hereditary category derived equivalent to $\mathcal{A}$.*

Proof. It is straightforward to check $(D^{\geq 0}, D^{\leq 0})$ defines a split t-structure. It follows from [47] that the t-structure is bounded. Proposition 1.29 yields the required result. □

Assume that $\mathcal{A}$ is an abelian Ext-finite $\mathcal{A}$ category satisfying Serre duality. We will say a split t-structure is *τ-invariant* if the heart $\mathcal{H}$ is closed under τ-shifts. It is clear the heart $\mathcal{H}$ of a split t-structure has no nonnzero projectives or injectives if and only if the t-structure is τ-invariant.

1.8 Spanning classes and equivalences between triangulated categories

In this section, we give a summary of some results of [13] and [14]. Throughout, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be Hom-finite triangulated categories with Serre duality; the Serre functors of $\mathcal{C}_1$ and $\mathcal{C}_2$ will be denoted by F_1 and F_2, respectively.

Let $G : \mathcal{C}_1 \to \mathcal{C}_2$ be an exact functor. We will say G commutes with the Serre functor is $G \circ F_1 \cong F_2 \circ G$.

Lemma 1.31. *The functor $G : \mathcal{C}_1 \to \mathcal{C}_2$ admits a left adjoint L if and only if it admits a right adjoint R. If G commutes with the Serre functor, then $L \cong R$.*

Proof. If L is a left adjoint of G, then it follows from

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_2}(GX, Y) &\cong \mathrm{Hom}_{\mathcal{C}_2}(F_2^{-1}Y, GX)^* \\ &\cong \mathrm{Hom}_{\mathcal{C}_1}(L \circ F_2^{-1}Y, X)^* \\ &\cong \mathrm{Hom}_{\mathcal{C}_1}(X, F_1 \circ L \circ F_2^{-1}Y) \end{aligned}$$

that $R = F_1 \circ L \circ F_2^{-1}$ is a right adjoint. The other implication is analogous.

If G commutes with the Serre functor, then it is easily verified that $L \cong R$. □

We will use following definition from [13].

Definition 1.32. A subclass Ω of the objects of $\mathcal{C}$ will be called a *spanning class*, if for any object $X \in \mathcal{C}$

$$\begin{aligned} \forall \omega \in \Omega, \forall i \in \mathbb{Z} : \mathrm{Hom}^i(\omega, X) = 0 &\Rightarrow X \cong 0, \\ \forall \omega \in \Omega, \forall i \in \mathbb{Z} : \mathrm{Hom}^i(X, \omega) = 0 &\Rightarrow X \cong 0. \end{aligned}$$

The following result is [14, Theorem 2.3].

Theorem 1.33. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be Hom-finite triangulated categories with Serre duality. Assume $\mathcal{C}_1$ is nontrivial and $\mathcal{C}_2$ is connected. Let $G : \mathcal{C}_1 \to \mathcal{C}_2$ be an exact functor, which has a left adoint. If there is a spanning class Ω of $\mathcal{C}_1$ such that*

$$G : \mathrm{Hom}^i(\omega_1, \omega_2) \xrightarrow{\sim} \mathrm{Hom}^i(G\omega_1, G\omega_2)$$

is an isomorphism for all $i \in \mathbb{Z}$ and all $\omega_1, \omega_2 \in \Omega$, and such that $GF_1(\omega) \cong F_2G(\omega)$, then G is an equivalence of categories.

1.9 Partial Tilting Sets

In this section we shall assume $\mathcal{A}$ is a k-linear abelian Ext-finite category, not necessarily satisfying Serre duality. We will say that the set $\{P_i\}_{i \in I} \subseteq \mathrm{ind}\, D^b\mathcal{A}$ is a *partial tilting set* if $\mathrm{Hom}(P_i, P_j[z]) = 0$, for all $z \in \mathbb{Z}_0$ and all $i, j \in I$.

Recall that a category is called *Karoubian* if the category has finite direct sums and idempotents split. A small preadditive category $\mathfrak{a}$ is *coherent* if the finitely presented objects $\operatorname{mod}(\mathfrak{a})$ in $\operatorname{Mod}(\mathfrak{a})$ form an abelian category.

Theorem 1.34. *[53, Theorem 5.1] Let $\mathcal{A}$ be a k-linear abelian category, $\{P_i\}_{i\in I}$ a partial tilting set of $D^b\mathcal{A}$ and $\mathfrak{a}$ the additive category given by $\{P_i\}_{i\in I}$ as a full subcategory of $D^b\mathcal{A}$. Assume that $\mathfrak{a}$ is Karoubian and coherent, and that every object in $\operatorname{mod}\mathfrak{a}$ has a finite projective resolution, then there is a full exact embedding $D^b(\operatorname{mod}\mathfrak{a}) \to D^b\mathcal{A}$ sending $\operatorname{Hom}(-,P_i)$ to P_i.*

Proof. Due to the conditions on the preadditive category $\mathfrak{a}$, we know that $D^b(\operatorname{mod}\mathfrak{a})$ is equivalent to $K^b\mathfrak{a}$. It thus suffices to construct a full and exact embedding $K^b\mathfrak{a} \to D^b\mathcal{A}$. It is well known that the category $\operatorname{Ind}\mathcal{A}$ of left exact contravariant functors from $\mathcal{A}$ to $\operatorname{Mod} k$ is a k-linear Grothendieck category and that the Yoneda embedding of $\mathcal{A}$ into $\operatorname{Ind}\mathcal{A}$ is a full and exact embedding. By [32, Prop. 2.14], this embedding extends to a full and exact embedding $D^b\mathcal{A} \to D^b\operatorname{Ind}\mathcal{A}$. As a Grothendieck category, $\operatorname{Ind}\mathcal{A}$ has enough injectives and we may, by [42, Prop. 10.1], consider the full and exact embedding $K^b\mathfrak{a} \to D^b\operatorname{Ind}\mathcal{A}$ that extends the embedding of $\mathfrak{a}$ in $D^b\operatorname{Ind}\mathcal{A}$. Induction over triangles shows that the essential image of $K^b\mathfrak{a}$ lies in $D^b\mathcal{A}$. $\square$

1.10 The Auslander-Reiten quiver

1.10.1 Stable translation quivers

Let Q be a quiver with vertex set Q_0 and arrow set Q_1. We will say Q is *locally finite* if for every $x \in Q_0$ if only finitely many arrows start or end at x.

For $x \in Q_0$, we define x^+ as the set of immediate successors, thus

$$x^+ = \{y \in Q_0 \mid \text{there is an arrow } x \to y\}.$$

We define the set of immediate predecessor x^- dually. If Q is locally finite, then x^+ and x^- are finite, for every $x \in Q_0$.

Let (Q,τ) be a pair where Q is a quiver and $\tau : Q_0 \to Q_0$ is a bijection satisfying following property: the number of arrows from x to y is equal to the number of arrows from τy to x. We will say (Q,τ) is a *stable translation quiver*. We will always denote the translation by τ, and will often write Q for the stable translation quiver (Q,τ)

Example 1.35. The following quiver is a stable translation quiver where $\tau x_i = x_{i-1}$.

$$\begin{tikzcd}
\cdots & x_{-1} \arrow[dr] & & x_1 \arrow[ll] \arrow[dr] & & x_3 \arrow[ll] & \cdots \\
x_{-2} \arrow[ur] & & x_0 \arrow[ll] \arrow[ur] & & x_2 \arrow[ll] \arrow[ur] & \cdots &
\end{tikzcd}$$

With every quiver Q, we may associate a stable translation $(\mathbb{Z}Q,\tau)$ quiver as follows:

- $(\mathbb{Z}Q)_0 = \mathbb{Z} \times Q_0$,
- the number of arrows from (x,i) to (y,j) is the number of arrows from x to y is $i=j$, the number of arrows from y to x if $j=i+1$, and zero otherwise.
- $\tau(x,i) = (x,i-2)$

We may turn $\mathbb{Z}Q$ into a stable translation quiver by $\tau(x,i) = (x,i-2)$.

Remark 1.36. One should not confuse the above construction $\mathbb{Z}Q$, which yields a stable translation quiver, and the construction of the path category, kQ, from Example 1.4 which yields a preadditive category.

It has been shown in [43] that every stable translation quiver without loops is of the form $\mathbb{Z}Q/G$, where Q is a tree and G is a well-chosen subgroup of the automorphism group of $\mathbb{Z}Q$.

Non-isomorphic quivers Q and Q' may give rise to isomorphic stable translation quivers $\mathbb{Z}Q$ and $\mathbb{Z}Q'$. We define a section of $\mathbb{Z}Q$ as a full subquiver Q' of $\mathbb{Z}Q$ such that the embedding extends to an isomorphism $\mathbb{Z}Q' \to \mathbb{Z}Q$ of stable translation quivers, hence Q' contains exactly one object from every τ-orbit of $\mathbb{Z}Q$ and if $x \in Q'$ and there is an arrow $x \to y$ in $\mathbb{Z}Q$, then either $y \in Q$ or $\tau y \in Q$ (See [40]).

Our main example of stable translation quivers will be the Auslander-Reiten quiver of $\mathcal{C} = D^b\mathcal{A}$ where $\mathcal{A}$ is an Ext-finite hereditary category with Serre duality.

1.10.2 The Auslander-Reiten quiver

Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt category. We define $\operatorname{ind}\mathcal{C}$ to be a chosen set of representatives of non-isomorphic indecomposable objects of $\mathcal{C}$.

The Auslander-Reiten quiver of $\mathcal{C}$ is defined as follows: the set of vertices is given by $\operatorname{ind}\mathcal{C}$, and for $A, B \in \operatorname{ind}\mathcal{C}$, there are $\dim_k \operatorname{rad}(A,B)/\operatorname{rad}^2(A,B)$ arrows from A to B.

A component of the Auslander-Reiten quiver will be called an *Auslander-Reiten component*. If, for all $A, B \in \operatorname{ind}\mathcal{C}$ lying in the same Auslander-Reiten component, we have $\operatorname{rad}^\infty(A,B) = 0$, then we will say the component is *generalized standard* or just *standard*.

If $\mathcal{C} = D^b\mathcal{A}$ where $\mathcal{A}$ is an Ext-finite abelian category with Serre duality, then $\mathcal{C}$ is a stable translation quiver (see below). The translation τ corresponds with the Auslander-Reiten translate, and the typical 'meshes' of the Auslander-Reiten quiver of $\mathcal{C}$ correspond with the Auslander-Reiten triangles.

In the Auslander-Reiten quivers we will encounter, four kinds of components will often occur: $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$, and $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. When sketching the Auslander-Reiten quiver of a category, we will often represent these component as in Figure 1.3. There is no indication of possible morphisms between these components, other than that morphisms 'generally go from left to right'.

Components of the form $\mathbb{Z}A_\infty$ are also called *wings*, and components of the from $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ with $n \geq 1$ are called *tubes*. If $n=1$, the tube is called *homogeneous*.

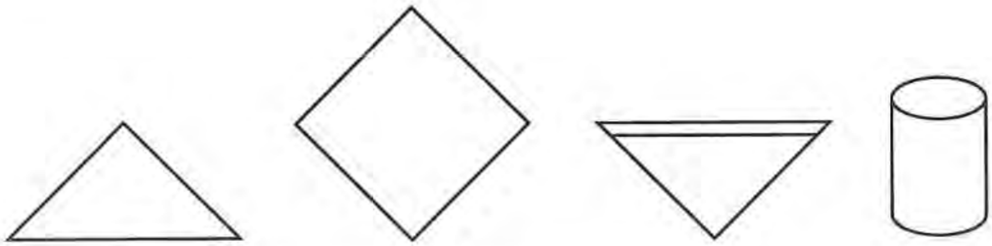


Figure 1.3: Sketches of components of the form $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$ and $\mathbb{Z}A_\infty/\langle\tau^n\rangle$

We will say an indecomposable object $X \in \operatorname{ind}\mathcal{C}$ is *peripheral* if the object M occurring in the Auslander-Reiten triangle $\tau X \to M \to X \to \tau X[1]$ is indecomposable, thus if there is only one arrow ending in X in the Auslander-Reiten quiver of $\mathcal{C}$. Equivalently, there is only one arrow starting in X. A peripheral object lying in a wing are called a *quasi-simple* object.

The full subquiver of the Auslander-Reiten quiver of $\mathcal{A}$ spanned by all projectives or injective objects in $\operatorname{ind}\mathcal{A}$ is called the *quiver of projectives* or *injectives* of $\mathcal{A}$, respectively.

A component of the Auslander-Reiten quiver of $\mathcal{A}$ containing a projective object is called a *preprojective component*. If $\mathcal{A}$ satisfies Serre duality, then the Auslander-Reiten component of $D^b\mathcal{A}$ containing the projective quiver Q is a stable translation quiver of the form $\mathbb{Z}Q$.

1.11 Sectional paths and strongly locally finite quivers

The material of this section has been taken from joint work with Carl Fredrik Berg in [8]. We advise the reader to familiarize himself with the results and techniques in this section before proceeding to Chapter 6 where these results will be generalized. For the benefit of the reader, we start with a rather detailed introduction.

In this section, we will investigate for which quivers Q the stable translation quiver $\mathbb{Z}Q$ admits a *strongly locally finite* section Q', i.e. every vertex of Q' has finitely many neighbors and Q' is without subquivers of the form $\cdot \to \cdot \to \cdots$ or $\cdots \to \cdot \to \cdot$

Before stating our main result, we will need a definition. Let Q be a quiver. For two vertices $x, y \in Q$ we define the *round trip distance* $d(x,y)$ as the least number of arrows that have to be traversed in the opposite direction on a path from x to y and back to x. If Q does not have oriented cycles, then for all $x, y, z \in Q$

1. $d(x,y) \geq 0$ and $d(x,y) = 0 \Longleftrightarrow x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,z) \leq d(x,y) + d(y,z)$

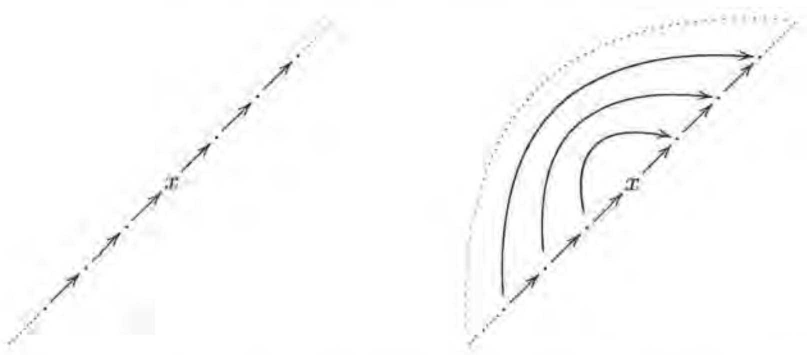


Figure 1.4: A quiver satisfying the equivalent conditions of Theorem 1.37 (left) and one that does not (right)

such that d defines a distance on the vertices of Q (Proposition 1.46 in the text). To the round trip distance, we may associate *round trip distance spheres* as follows

$$S(x,n) = \{y \in Q \mid d(x,y) = n\}.$$

We may now formulate our main theorem (Theorem 1.51 in the text).

Theorem 1.37. *Let Q be a connected quiver, then the following are equivalent.*

- *The quiver Q has no oriented cycles, and for a certain / for all $x \in Q$ the round trip distance spheres $S_Q(x,n)$ are finite, for all $n \in \mathbb{N}$.*
- *The translation quiver $\mathbb{Z}Q$ has a locally finite and path finite section.*

As an example, we see that the left quiver in Figure 1.4 satisfies the first condition of previous theorem, while in the right quiver the round trip distance sphere $S_Q(x,1)$ has infinitely many vertices.

Our main reason to investigate this problem has been a question by Reiten and Van den Bergh in [40]. In said article, Reiten and Van den Bergh classified all k-linear noetherian abelian hereditary Ext-finite categories $\mathcal{A}$ with Serre duality. One type of such categories, characterized by being generated by preprojectives, was constructed by formally inverting a right Serre functor in a the category of finitely presented representations of a certain quiver Q. Reiten and Van den Bergh noted that if a theorem in the sense of Theorem 1.37 were true, then this would give a shorter classification of these kind of categories and another construction (Ringel already gave an alternative way of constructing such categories using ray quivers in [46]). Following these ideas, we show the following result (Corollary 1.56).

Proposition 1.38. *Let $\mathcal{A}$ be a noetherian k-linear abelian Ext-finite hereditary categories with Serre duality. Assume $\mathcal{A}$ is generated by the preprojective objects, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} kQ'$ *where Q' is strongly locally finite.*

The proof of Theorem 1.37 is a constructive one. Let Q be a quiver. In $\mathbb{Z}Q$ we define the *right light cone* centered on a vertex $x \in \mathbb{Z}Q$ as the set of all vertices y such that there is a path from x to y but not to τy. Dually, we define the *left light cone* centered on x as the set of all vertices y such that there is a path from y to x, but not to τx.

Let $y \in \mathbb{Z}Q$ such that $\tau^{-n}y$ lies on the right light cone centered on x, then we will say that the *right light cone distance* $d^{\bullet}(x,y)$ is n. Note that $d^{\bullet}(x,y)$ may be negative, and is not symmetric. Fixing an x, the right light cone distance determines which y we take from a τ-orbit (See for example Figure 1.5).

If $x, y \in Q \in \mathbb{Z}Q$, then the number of τ-shifts one need to go from the right light cone to the left light cone is $d(x,y)$. In Proposition 1.49 we show that in order for a full subquiver Q' of $\mathbb{Z}Q$ to be a section, it suffices that Q' meets every τ-orbit of $\mathbb{Z}Q$ at least once, and that for every vertex x every other vertex y lies between the left and right light cone centered on x. An equivalent way of formulating this is saying that both $d^{\bullet}(x,y)$ and $d^{\bullet}(y,x)$ are positive.

Another useful property of the right light cone distance is that one may see whether a certain section is strongly locally finite or not (Proposition 1.48).

Thus for the quiver Q, we pick any vertex $x \in \mathbb{Z}Q$ and consider the light cone centered on x. In every τ-orbit, we choose a vertex in the middle between the left and right light cone centered on x. Using properties of $d^{\bullet}$ we may then show that the thus defined subquiver of $\mathbb{Z}Q$ is a strongly locally finite quiver, completing the proof of our main result.

1.11.1 Sectional paths

Definition 1.39. Let I be the integers in one of the intervals $]-\infty, n]$, $[n, +\infty[$, $[m, n]$ for $m < n$ or $\{1, \ldots, n\}$ modulo n. Let $\cdots \to A_i \to A_{i+1} \to \cdots$ be a sequence of irreducible morphisms between indecomposables with each index in I. The sequence is said to be *sectional* if $\tau A_{i+2} \not\cong A_i$ whenever both i and $i+2$ are in I. The corresponding path in the Auslander-Reiten quiver is said to be a *sectional path*.

Proposition 1.40. *Let Q be a connected quiver. The following statements are equivalent.*

1. *The quiver Q is locally finite and there are only finitely many sectional paths between any two vertices of $\mathbb{Z}Q$.*
2. *There are only finitely many (possibly non-sectional) paths between any two vertices in $\mathbb{Z}Q$.*
3. *For every vertex $x \in \mathbb{Z}Q$ there are only finitely many paths from x to $\tau^{-n}x$ in $\mathbb{Z}Q$ for all $n \in \mathbb{N}$.*

4. *There is a vertex $x \in \mathbb{Z}Q$ such that there are only finitely many paths from x to $\tau^{-n}x$ in $\mathbb{Z}Q$ for all $n \in \mathbb{N}$.*

Proof.

$(1 \Rightarrow 2)$ Seeking a contradiction to the assumptions in (1), we will assume we may choose x and y such that there are infinitely many paths from x to y. Without loss of generality, we may assume x has coordinates $(0, v_x)$ and y has coordinates (n, v_y), where v_x and v_y are vertices in Q and $n \geq 0$.

Since there are finitely many sectional paths from x to y, an infinite number of the paths between x and y must be non-sectional. If $x \neq \tau y$ then we may turn a non-sectional path into a non-trivial path from x to τy by replacing a part $A_{i-2} \to \tau A_{i+1} \to A_i \to A_{i+1} \to A_{i+2}$ by $A_{i-2} \to \tau A_{i+1} \to \tau A_{i+2}$.

Since the paths from x to y have finite length and Q is locally finite, only finitely many different paths will be turned into the same one by this procedure, thus there are infinitely many paths from x to τy. Repeating this process shows that we either have infinitely many paths from x to $\tau^{n+1}y$ or infinitely many paths from x to $\tau^{-}x$.

The coordinates of $\tau^{n+1}y$ are $(-1, v_y)$, and as such there may be no paths from $x = (0, v_x)$ to $\tau^{n+1}y$.

Therefore assume there are infinitely many paths from x to $\tau^{-}x$. Since Q is locally finite, there may only be a finite number of paths from x to $\tau^{-}x$ of length 2.

All paths from x to $\tau^{-}x$ not of length 2 are sectional, since otherwise we may turn them into paths from x to x using the procedure described before this proposition. Such a path is necessarily sectional. By concatenating such a cycle with itself, we obtain an infinite number of sectional paths from x to x, a contradiction.

Hence we know there are infinitely many sectional paths from x to $\tau^{-}x$, a contradiction to the assumption in (1).

$(2 \Rightarrow 1)$ There is a finite number of paths between x and $\tau^{-1}x$ such that Q is locally finite. The claim about sectional paths is trivial.

$(2 \Rightarrow 3)$ Trivial.

$(3 \Rightarrow 4)$ Trivial.

$(4 \Rightarrow 2)$ Seeking a contradiction, assume there are infinitely many paths from a vertex y to a vertex z of $\mathbb{Z}Q$. Since Q was connected, there is a path from x to $\tau^{n}y$ for an $n \in \mathbb{Z}$. For the same reason there is a path from $\tau^{n}z$ to $\tau^{m}x$ for an $m \in \mathbb{Z}$. Composition gives a path from x to $\tau^{m}x$, hence $m \in -\mathbb{N}$. Since there are infinitely many paths from y to z, composition gives infinitely many paths from x to $\tau^{m}x$, a contradiction to the assumption in (4).

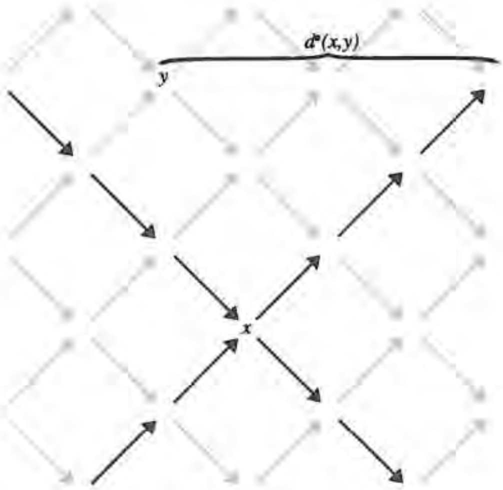


Figure 1.5: Light cones and right light cone distance in $\mathbb{Z}A_\infty^\infty$

□

1.11.2 Light cone distance and round trip distance

Right light cone distances

Let Q be a quiver. In $\mathbb{Z}Q$ we define the *(right) light cone* centered on a vertex $x \in \mathbb{Z}Q$ as the set of all vertices y such that there is a path from x to y but not to τy. It is clear that the right light cone intersect a τ-orbit in at most one vertex. If Q, and hence $\mathbb{Z}Q$, is connected then the right light cone intersect each τ-orbit in exactly one vertex.

Let $y \in \mathbb{Z}Q$. If $\tau^{-n}y$ lies on the right light cone centered on x, then we will say that the *right light cone distance* $d^\bullet(x,y)$ is n. If no such n exists, we define $d^\bullet(x,y) = \infty$. When Q is a connected quiver, then $d^\bullet(x,y)$ is finite for all vertices $x, y \in \mathbb{Z}Q$.

The following lemma is obvious.

Lemma 1.41. *For all $X, Y \in \mathbb{Z}Q$, we have $d^\bullet(X, \tau^n Y) = d^\bullet(X,Y) + n$.*

Note that $d^\bullet(x,y)$ may be negative, and that the function $d^\bullet$ is not symmetric. Next lemma shows the right light cone distance satisfies triangle equality.

Lemma 1.42. *For all vertices $x, y, z \in \mathbb{Z}Q$ we have*

$$d^\bullet(x,z) \leq d^\bullet(x,y) + d^\bullet(y,z)$$

Proof. Assume $d^\bullet(x,y) = n$ and $d^\bullet(y,z) = m$, thus there are paths from x to $\tau^{-n}y$ and from $\tau^{-n}y$ to $\tau^{-n-m}z$. Composition gives a path from x to $\tau^{-n-m}z$, hence

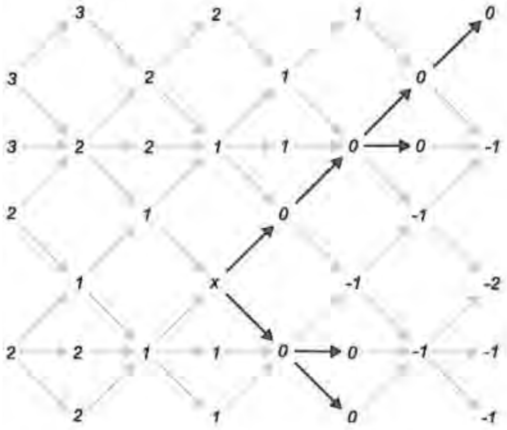


Figure 1.6: A stable translation quiver with the (right) light cone centered on x and the corresponding right light cone distances

$d^\bullet(x,z) \leq n+m$. If either $d^\bullet(x,y)$ or $d^\bullet(y,z)$ is infinite, then the inequality is trivial. □

There is a natural embedding $\epsilon\colon Q \hookrightarrow \mathbb{Z}Q$ induced by the map $\epsilon(x) = (x,0)$. Let x and y be vertices of Q, then we define the *right light cone distance* $d^\bullet_Q(x,y)$ between x and y as the distance $d^\bullet((x,0),(y,0))$.

An equivalent way to describe $d^\bullet_Q(x,y)$ intrinsically on Q is as the minimal number of arrows traversed in the opposite direction over all unoriented paths from x to y.

Next proposition shows $d^\bullet_Q$ defines a hemimetric on a connected quiver Q.

Proposition 1.43. *Let Q be a connected quiver. For all $x,y,z \in Q$ we have*

1. $d^\bullet_Q(x,y) \geq 0$,
2. $d^\bullet_Q(x,x) = 0$,
3. $d^\bullet_Q(x,z) \geq d^\bullet_Q(x,y) + d^\bullet_Q(y,z)$.

If furthermore Q does not have oriented cycles, then we may strengthen (2) *to*

(2') $d^\bullet_Q(x,y) = 0$ and $d^\bullet_Q(y,x) = 0 \iff x = y$.

Proof. This follows directly from the definition of $d^\bullet_Q$ and Lemma 1.42. □

Proposition 1.44. *If $x \to y$ is an arrow in $\mathbb{Z}Q$ for a quiver Q, then $d^\bullet(x,y) = 0$ or $d^\bullet(x,y) = -1$. Furthermore Q has no oriented cycles if and only $d^\bullet(x,y) = 0$ for all arrows $x \to y$.*

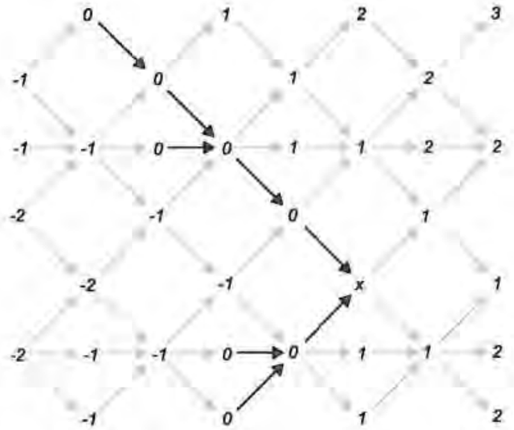


Figure 1.7: A stable translation quiver with the left light cone centered on x and the corresponding left light cone distances

Proof. By the definition of $d^\bullet(x, y)$ and since there is a path from x to y, we have $d^\bullet(x, y) \leq 0$.

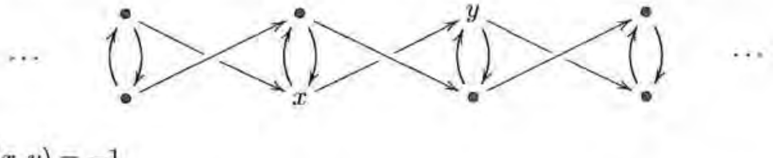
From the arrow $x \to y$ we easily obtain an arrow $\tau^2 y \to \tau x$. A path $x \to \tau^n y$ for $n \geq 2$ would produce a path from x to τx by concatenation with a path from $\tau^n y$ to $\tau^2 y$ and the arrow from $\tau^2 y$ to τx. From the definition of $\mathbb{Z}Q$ we see such a path does not occur, hence $d^\bullet(x, y) \geq -1$.

This shows $d^\bullet(x, y) = 0$ or $d^\bullet(x, y) = -1$.

If we furthermore assume Q, and hence also $\mathbb{Z}Q$, has no oriented cycles, then we may exclude $d^\bullet(x, y) = -1$ since this would give a path from x via τy to x, a contradiction.

If $d^\bullet(x, y) = -1$, then there is a path from x to τy. The arrow $x \to y$ yields an arrow $\tau y \to x$ and we obtain a cycle in $\mathbb{Z}Q$. This implies there is a cycle in Q as well. $\square$

Example 1.45. Let $Q = \tilde{A}_2$ with cyclic orientation, and let $x, y \in \mathbb{Z}Q$ as follows



Then $d^\bullet(x, y) = -1$.

In addition to the right light cone distance one may also define a left light cone and a left light cone distance $d_\bullet \colon \mathbb{Z}Q \times \mathbb{Z}Q \to \mathbb{Z} \cup \{\infty\}$ dually (see Figure 1.7), but since $\leftarrow (x, y) = d^\bullet(y, x)$, the left light cone distance is superfluous.

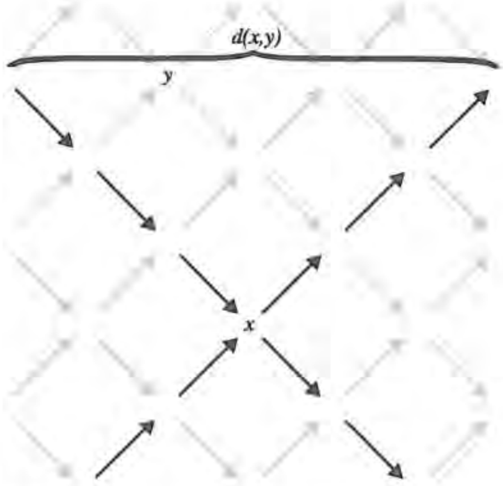


Figure 1.8: Light cones and round trip distance in $\mathbb{Z}A_\infty^\infty$

Round Trip Distances

For two vertices $x, y \in \mathbb{Z}Q$, we define the $d(x,y)$ as

$$r(x,y) = d^\bullet(x,y) + d^\bullet(y,x).$$

It is an immediate consequence of the definition that $d(x,y)$ is the least integer n such that there is a path in $\mathbb{Z}Q$ from x to $\tau^{-n}x$ that contains exactly one vertex from the τ-orbit of y, namely $\tau^{-d^\bullet(x,y)}y$.

Let x and y be vertices of Q, then we define the *round trip distance* $d_Q(x,y)$ between x and y as the distance $d((x,0),(y,0))$ where $(x,0)$ and $(y,0)$ are the vertices in $\mathbb{Z}Q$ corresponding to x and y under the natural embedding $Q \hookrightarrow \mathbb{Z}Q$. Hence

$$\begin{aligned} d_Q(x,y) &= d((x,0),(y,0)) \\ &= d^\bullet((x,0),(y,0)) + d^\bullet((y,0),(x,0)) = d_Q^\bullet(x,y) + d_Q^\bullet(y,x) \end{aligned}$$

As with $d_Q^\bullet$, we may describe $d_Q(x,y)$ intrinsically. If x and y are vertices of Q, then $d_Q(x,y)$ is the least number of arrows traversed in the opposite direction on a path from x to itself passing through y.

Next proposition shows the round trip distance d_Q defines a distance on the vertices of Q when Q is without oriented cycles. If Q has oriented cycles, then d merely defines a pseudodistance.

Proposition 1.46. *Let Q be a connected quiver, then for all $x, y, z \in Q$ we have*

1. $d_Q(x,y) \geq 0$

2. $d_Q(x,x) = 0$

3. $d_Q(x,y) = d_Q(y,x)$

4. $d_Q(x,z) \leq d_Q(x,y) + d_Q(y,z)$

Furthermore, if Q has no oriented cycles then we may strengthen (2) *to*

(2') $d_Q(x,y) = 0 \Leftrightarrow x = y$

Proof. The first three properties follow directly from the definition of d_Q, while triangle inequality follows from Lemma 1.42. If $d_Q(x,y) = 0$, then x and y lie on the same oriented cycle in Q. This proves the last assertion. □

Round Trip Distance Spheres for Quivers

For a vertex x in a quiver Q we define the *round trip distance spheres* $S_Q(x,n)$ where $n \in \mathbb{N}$, as the sets

$$S_Q(x,n) = \{y \in Q \mid d_Q(x,y) = n\}.$$

Similarly we define the *right light cone distance sphere* and the *left light cone distance sphere* as

$$S_Q^{\bullet}(x,n) = \{y \in Q \mid d_Q^{\bullet}(x,y) = n\} \text{ and } S_{\bullet}^{Q}(x,n) = \{y \in Q \mid d_Q^{\bullet}(y,x) = n\}$$

respectively.

We may now extend Proposition 1.40.

Proposition 1.47. *Let Q be a connected quiver. The following statements are equivalent.*

1. *The quiver Q is locally finite and there are only finitely many sectional paths between any two vertices of $\mathbb{Z}Q$.*
2. *There are only finitely many (possibly non-sectional) paths between any two vertices in $\mathbb{Z}Q$.*
3. *For every vertex $x \in \mathbb{Z}Q$ there are only finitely many paths from x to $\tau^{-n}x$ in $\mathbb{Z}Q$ for all $n \in \mathbb{N}$.*
4. *There is a vertex $x \in \mathbb{Z}Q$ such that there are only finitely many paths from x to $\tau^{-n}x$ in $\mathbb{Z}Q$ for all $n \in \mathbb{N}$.*
5. *The quiver Q is without oriented cycles, and for all $x \in Q$ and $n \in \mathbb{N}$ the round trip distance sphere $S_Q(x,n)$ is finite.*
6. *The quiver Q is without oriented cycles, and there is an $x \in Q$ such that the round trip distance sphere $S_Q(x,n)$ is finite, for all $n \in \mathbb{N}$.*

Proof.

($3 \Rightarrow 5$) Since an oriented cycle involving x would give infinitely many paths from x to x, we know Q is without oriented cycles.

Since every vertex $y \in S_Q(x,n)$ has a τ-shift lying on a path from x to $\tau^{-n}x$, and there are only finitely many such paths, it is clear $S_Q(x,n)$ must be finite.

($5 \Rightarrow 2$) For every y on a path from x to $\tau^{-n}x$, we know $d(x,y) \leq n$. Since $S_Q(x,i)$ is finite for all $i \leq n$, there may only be finitely many paths from x to $\tau^{-n}x$.

($5 \Leftrightarrow 6$) This follows directly from triangle inequality.

□

1.11.3 Existence of Strongly Locally Finite Sections

We will now turn our attention to sections in translation quivers of the form $\mathbb{Z}Q$. Our main goal is to find a strongly locally finite section Q' in a stable translation quiver of the form $\mathbb{Z}Q$. To do this we will use the right light cone distance and the round trip distance introduced above.

First, we will give a characterization of strongly locally finite quivers using the light cone distance.

Proposition 1.48. *Let Q be a connected quiver. Then Q is strongly locally finite if and only if Q has no oriented cycles and for any $x \in Q$ all spheres $S^{\bullet}_Q(x,n)$ and $S^Q_{\bullet}(x,n)$ are finite for all $n \in \mathbb{N}$.*

Proof. First, assume Q is strongly locally finite. Since Q is then path finite, it is clear that Q does not have oriented cycles. Seeking a contradiction, we will furthermore assume there to be an $m \in \mathbb{N}$ such that $S^{\bullet}_Q(x,m)$ is infinite for a certain vertex $x \in Q$. Let m be the smallest such integer.

For every $y \in S^{\bullet}_Q(x,m)$ we fix an unoriented path from x to y with exactly m arrows in the opposite direction. In every such path, the right light cone distance will be ascending. In this path, we will consider the first vertex z lying in $S^{\bullet}_Q(x,m) \setminus S^{\bullet}_Q(x,m-1)$.

Such a vertex z admits an oriented path to y and is a neighbor of a vertex in $S^{\bullet}_Q(x,m-1)$. Since this last set is finite and Q is path finite, it is clear that there are only finitely many vertices z. Hence one of these vertices admits oriented paths to an infinite number of vertices in $S^{\bullet}_Q(x,m)$. We conclude that Q has rays.

Dually, one shows $S^Q_{\bullet}(x,n)$ is finite for all $n \in \mathbb{N}$.

For the other implication, assume Q has no oriented cycles and for a certain $x \in Q$ all spheres $S^{\bullet}_Q(x,n)$ and $S^Q_{\bullet}(x,n)$ are finite for all $n \in \mathbb{N}$.

Let $y \in Q$ be any vertex. For all neighbors $z \in Q$ of y, we have either $d^{\bullet}_Q(y,z) = 0$ if there is an arrow $y \to z$ or $d^{\bullet}_Q(y,z) = 1$ if there is an arrow $z \to y$. Using triangle inequality, we find

$$d^\bullet_Q(x,z) \le d^\bullet_Q(x,y) + d^\bullet_Q(y,z) \le d^\bullet_Q(x,y) + 1.$$

Since $S^\bullet_Q(x,n)$ is finite, for all $n \in \mathbb{N}$, we see that y may only have a finite number of neighbors, hence Q is locally finite.

We will now proceed by proving Q is path finite. Assume Q has a ray $a_0 \to a_1 \to \cdots$ as subquiver and denote $d^\bullet_Q(x,a_0) = n$. For $i > j$, triangle inequality gives

$$d^\bullet_Q(x,a_i) \le d^\bullet_Q(x,a_j) + d^\bullet_Q(a_j,a_i) = d^\bullet_Q(x,a_j)$$

since $d^\bullet_Q(a_j,a_i) = 0$, hence the sequence $(d^\bullet_Q(x,a_i))_{i\in\mathbb{N}}$ must stabilize, giving an infinite set $S^\bullet_Q(x,m)$ for an $m \le n$. Thus Q may not have a ray as a subquiver.

Dually, one finds Q may not have a coray as subquiver. □

The next result gives necessary and sufficient conditions for Q' to be a section of $\mathbb{Z}Q$ using the right light cone distance.

Proposition 1.49. *Let Q' be a full subquiver of the stable translation quiver $\mathbb{Z}Q$ that meets every τ-orbit exactly once. Then Q' is a section if and only if $d^\bullet(x,y) \ge 0$ for all vertices $x,y \in Q'$.*

Proof. We will first check that, if $d^\bullet(x,y) \ge 0$ for all vertices $x,y \in Q'$, then Q' is a section. We need to show that for every arrow $x \to z$ in $\mathbb{Z}Q$ with $x \in Q'$ either $z \in Q'$ or $\tau z \in Q'$, and for every arrow $z \to x$ in $\mathbb{Z}Q$ with $x \in Q'$ either $z \in Q'$ or $\tau^{-1}z \in Q'$. We will only show the first part, the second is similar.

So let $x \in Q'$. Since there is an arrow $x \to z$ in $\mathbb{Z}Q$, we know $d^\bullet(x,z) \le 0$, thus the object of the τ-orbit of z belonging to Q' has to be of the form $\tau^n z$ with $n \ge 0$.

An arrow $x \to z$ induces an arrow $\tau z \to x$, hence $d^\bullet(\tau z, x) \le 0$ and thus $n \le 1$. We conclude that either z or τz belongs to Q'.

Conversely, let Q' be a section of $\mathbb{Z}Q$ and let $x,y \in Q'$. Since the injection $Q' \subseteq \mathbb{Z}Q$ lifts to an isomorphism $\mathbb{Z}Q' \to \mathbb{Z}Q$ of translation quivers, Proposition 1.43 yields $d^\bullet(x,y) = d^\bullet_{Q'}(x,y) \ge 0$.

□

Example 1.50. Let x be a vertex of the stable translation quiver $\mathbb{Z}Q$. Using triangle inequality, one easily verifies that the right light cone $S^\bullet(x,0)$ and the left light cone $S_\bullet(x,0)$ are both sections of $\mathbb{Z}Q$.

We now come to the main result of this section.

Theorem 1.51. *Let Q be a connected quiver. The following statements are equivalent.*

1. *The quiver Q is locally finite and there are only finitely many sectional paths between any two vertices of $\mathbb{Z}Q$.*

2. *There are only finitely many (possibly non-sectional) paths between any two vertices in* $\mathbb{Z}Q$.

3. *For every vertex* $x \in \mathbb{Z}Q$ *there are only finitely many paths from* x *to* $\tau^{-n}x$ *in* $\mathbb{Z}Q$ *for all* $n \in \mathbb{N}$.

4. *There is a vertex* $x \in \mathbb{Z}Q$ *such that there are only finitely many paths from* x *to* $\tau^{-n}x$ *in* $\mathbb{Z}Q$ *for all* $n \in \mathbb{N}$.

5. *The quiver* Q *is without oriented cycles, and for all* $x \in Q$ *and* $n \in \mathbb{N}$ *the round trip distance sphere* $S_Q(x,n)$ *is finite.*

6. *The quiver* Q *is without oriented cycles, and there is an* $x \in Q$ *such that the round trip distance sphere* $S_Q(x,n)$ *is finite, for all* $n \in \mathbb{N}$.

7. *The translation quiver* $\mathbb{Z}Q$ *has a strongly locally finite section.*

Proof. The first 6 points are equivalent by Proposition 1.47.

($5 \Rightarrow 7$) We will construct a section Q' in $\mathbb{Z}Q$. Start by fixing a vertex x in $\mathbb{Z}Q$. From every τ-orbit we will choose a vertex y to be in Q' for which $d^\bullet(x,y) = \left\lfloor \frac{d(x,y)}{2} \right\rfloor$, hence also $d^\bullet(y,x) = \left\lceil \frac{d(x,y)}{2} \right\rceil$. We will use Proposition 1.49 to show that the full subquiver Q' picked out this way is a section of $\mathbb{Z}Q$.

Therefore we need to show that for all vertices $y,z \in Q' \subset \mathbb{Z}Q$, we have $d^\bullet_{Q'}(y,z) \geq 0$. We will consider two cases. First, assume $d(x,z) - d(x,y) \geq 0$. Starting with triangle inequality, we have

$$\begin{aligned} d^\bullet(y,z) &\geq d^\bullet(x,z) - d^\bullet(x,y) \\ &= \left\lfloor \frac{d(x,z)}{2} \right\rfloor - \left\lfloor \frac{d(x,y)}{2} \right\rfloor \\ &\geq 0 \end{aligned}$$

Next if $d(x,z) - d(x,y) \leq 0$, we have

$$\begin{aligned} d^\bullet(y,z) &\geq d^\bullet(y,x) - d^\bullet(z,x) \\ &= \left\lceil \frac{d(x,y)}{2} \right\rceil - \left\lceil \frac{d(x,z)}{2} \right\rceil \\ &\geq 0 \end{aligned}$$

Proposition 1.49 then yields that Q' is a section of $\mathbb{Z}Q$.

To show that Q' is path finite, note that $|S^\bullet_{Q'}(x,n)| = |S_Q(x,2n)| + |S_Q(x,2n+1)|$ and $|S^{Q'}_\bullet(x,n)| = |S_Q(x,2n-1)| + |S_Q(x,2n)|$, so by assumption the sets $S^\bullet_{Q'}(x,n)$ and $S^{Q'}_\bullet(x,n)$ are finite. Since Q, and hence also $\mathbb{Z}Q$, is locally finite and has no oriented cycles, we know that the same is true for Q'. Proposition 1.48 now yields Q' is path finite.

($7 \Rightarrow 5$) Let Q' be a strongly locally finite section of $\mathbb{Z}Q$. We may assume there is a vertex $x \in \mathbb{Z}Q$ lying in both Q and Q'. It is then clear that

$$|S_Q(x,n)| = |S_{Q'}(x,n)| = \left| \bigcup_{i+j=n} (S^{\bullet}_{Q'}(x,i) \cap S^{Q'}_{\bullet}(x,j)) \right|.$$

By Proposition 1.48, the right hand side is finite, hence also the left hand side is finite. Since Q' is path finite, it has no oriented cycles, so Q is also without oriented cycles.

$\square$

Example 1.52. Let Q be the quiver

$$\cdots \to x_{-2} \to x_{-1} \to x \to x_1 \to x_2 \to \cdots$$

It is easy to see that Q satisfies statement (5) in Theorem 5.13. The implication (5) $\Rightarrow$ (6) in Theorem 5.13 tells us that $\mathbb{Z}Q$ has a path finite section. Using the construction described in the proof of Theorem 5.13 we find the following vertices in $\mathbb{Z}Q$:

$$\begin{tikzcd}[column sep=small]
\tau^{-3}x_2 \arrow[dr] & & \tau^{-2}x_2 \arrow[dr] & & \tau^{-1}x_2 \arrow[dr] & & x_2 \arrow[dr] & & \tau x_2 \\
& \tau^{-2}x_1 \arrow[ur] \arrow[dr] & & \tau^{-1}x_1 \arrow[ur] \arrow[dr] & & x_1 \arrow[ur] \arrow[dr] & & \tau x_1 \arrow[ur] \arrow[dr] & \\
\tau^{-2}x \arrow[ur] \arrow[dr] & & \tau^{-1}x \arrow[ur] \arrow[dr] & & x \arrow[ur] \arrow[dr] & & \tau x \arrow[ur] \arrow[dr] & & \tau^2 x \\
& \tau^{-1}x_{-1} \arrow[ur] \arrow[dr] & & x_{-1} \arrow[ur] \arrow[dr] & & \tau x_{-1} \arrow[ur] \arrow[dr] & & \tau^2 x_{-1} \arrow[ur] \arrow[dr] & \\
\tau^{-1}x_{-2} \arrow[ur] & & x_{-2} \arrow[ur] & & \tau x_{-2} \arrow[ur] & & \tau^2 x_{-2} \arrow[ur] & & \tau^3 x_{-2}
\end{tikzcd}$$

The full subquiver of $\mathbb{Z}Q$ given by these vertices are the path finite quiver

$$Q': \ \cdots \leftarrow \tau x_{-2} \to \tau x_{-1} \leftarrow x \to x_1 \leftarrow \tau^{-1}x_2 \to \cdots$$

1.11.4 Application for the Auslander-Reiten quiver

In this section, we wish to give an application to Theorem 1.51 to tilting in abelian Ext-finite hereditary categories with Serre duality.

Proposition 1.53. *Let $\mathcal{A}$ be an abelian* Ext*-finite category with Serre duality, then for every X, Y* ind $D^b\mathcal{A}$ *there may only be finitely many sectional paths from X to Y.*

Proof. Assume there are different sectional paths from X to Y. The arrows $A \to B$ in the Auslander-Reiten quiver of $D^b\mathcal{A}$ give a basis of $\operatorname{rad}(A, B)/\operatorname{rad}^2(A, B)$. With such a basis, we may associate linearly independent morphisms of $\operatorname{rad}(X, Y)$. Fix such a morphism for every arrow occurring in an above path from X to Y (if an arrow occurs more than once, we will associate the same morphism with them).

In this way, every sectional path corresponds to a morphism in $\operatorname{Hom}(X, Y)$. We claim different sectional paths give rise to linearly independent morphisms.

Seeking a contradiction, consider the sectional sequences as depicted below

$$\begin{tikzcd}
& A_0^0 \arrow[r, "f_1^0"] & A_1^0 \arrow[r] & \cdots \arrow[r] & A_{n_0}^0 \arrow[ddr, "f_{n_0+1}^0"] & \\
& A_0^1 \arrow[r, "f_1^1"'] & A_1^1 \arrow[r] & \cdots \arrow[r] & A_{n_1}^1 \arrow[dr, "f_{n_1+1}^1"'] & \\
X \arrow[uur, "f_0^0"] \arrow[ur, "f_0^1"'] \arrow[dr, "f_0^m"'] & \vdots & \vdots & & \vdots & Y \\
& A_0^m \arrow[r, "f_1^m"'] & A_1^m \arrow[r] & \cdots \arrow[r] & A_{n_m}^m \arrow[ur, "f_{n_m+1}^m"'] &
\end{tikzcd}$$

such that there is a linear combination of the corresponding maps

$$\sum_{i=0}^{m} \alpha_i \left(\bigcirc_{k=0}^{n_i+1} f_k^i \right) = 0$$

where $\alpha_i \in k \setminus \{0\}$, and where the correct order of composition is understood. Keeping all paths that end with the morphism $f_{n_0+1}^0$ on the left hand side, and moving the others to the right hand side, we find (possibly after renumbering the paths)

$$f_{n_0+1}^0 \circ \left(\sum_{i=0}^{m_1} \alpha_i \left(\bigcirc_{k=0}^{n_i} f_k^i \right) \right) = - \sum_{i=m_1+1}^{m} \alpha_i \left(\bigcirc_{k=0}^{n_i+1} f_k^i \right)$$

Denote $g_1 = \sum_{i=0}^{m_1} \alpha_i \left(\bigcirc_{k=0}^{n_i} f_k^i \right)$. Considering the Auslander-Reiten triangle extending the irreducible map $f_{n_0}^0 : A_{n_0}^0 \to Y$ gives following diagram.

$$\begin{tikzcd}
X \arrow[dr, "g_1"] \arrow[dd, dotted, "\exists h_1"'] \arrow[dddr] & & & \\
& A^0_{n_0} \arrow[dr, "f^0_{n_0}"] & & \\
\tau Y \arrow[ur] \arrow[dr] & & Y \arrow[r] & \tau Y[1] \\
& E_1 \arrow[ur] & &
\end{tikzcd}$$

It follows that $g_1 : X \to A^0_{n_0}$ factors through a map $h_1 : \tau Y \to A^0_{n_0}$. Likewise, we may split the compositions occurring in the definition of g_1 in two groups, putting the ones ending in $f^0_{n_0-1}$ on the left hand side of the equation and moving the others to the right. After possibly renumbering the paths, we get

$$f^0_{n_0} \circ \left(\sum_{i=0}^{m_2} \alpha_i \left(\bigcirc_{k=0}^{n_i-1} f^i_k \right) \right) = - \sum_{i=m_2+1}^{m_1} \alpha_i \left(\bigcirc_{k=0}^{n_i} f^i_k \right).$$

If we write $g_2 = \sum_{i=0}^{m_2} \alpha_i \left(\bigcirc_{k=0}^{n_i-1} f^i_k \right)$, then we see from the Auslander-Reiten triangle built on $A^0_{n_0-1} \oplus \tau Y \to A^0_{n_0}$

$$\begin{tikzcd}
X \arrow[dr, "g_2"] \arrow[dd, dotted, "\exists h_2"'] \arrow[dddr] & & & \\
& A^0_{n_0-1} \arrow[dr, "f^0_{n_0-1}"] & & \\
\tau A^0_{n_0} \arrow[ur] \arrow[dr] & & A^0_{n_0} \arrow[r] & \tau A^0_{n_0}[1] \\
& E_2 \arrow[ur] & &
\end{tikzcd}$$

that g_2 factors through $\tau A^0_{n_0} \to \tau A^0_{n_0-1}$.

Since every considered path is different, iterating this procedure gives a map $g_{n_0-1} = f^0_0 : X \to A^0_0$ which factors through $E \to A^0_0$ as in the Auslander-Reiten triangle

$$\begin{tikzcd}
X \arrow[dr, equal] \arrow[dddr] & & & \\
& X \arrow[dr] & & \\
\tau A_0^0 \arrow[ur] \arrow[dr] & & A_0^0 \arrow[r] & \tau A_1^0[1]^{f_0^0} \\
& E \arrow[ur] & &
\end{tikzcd}$$

which is clearly a contradiction. □

This implies that every stable component of the form $\mathbb{Z}Q$ of the Auslander-Reiten quiver $D^b\mathcal{A}$ satisfies the equivalent conditions of Theorem 5.13. In particular, we have following corollary.

Corollary 1.54. *Let $\mathcal{A}$ be an abelian* Ext*-finite k-linear category with Serre duality. If a component of the Auslander-Reiten quiver of $D^b\mathcal{A}$ is of the form $\mathbb{Z}Q$, then we may choose Q to be strongly locally finite.*

1.11.5 Representation of strongly locally finite quivers

Let Q be a strongly locally finite quiver. It is easy to see this implies there are only finitely many paths between two vertices of Q.

Let $\operatorname{mod} kQ$ be the category of finitely presented representations of Q and denote by $\mathcal{P}$ and $\mathcal{I}$ the full subcategory of projectives and injectives, respectively. With every vertex $x \in Q$ we may associate an indecomposable projective object P_x and an indecomposable injective object I_x. There is also a canonical isomorphism $\nu_{x,y}$: $\operatorname{Hom}(P_x, P_y) \cong \operatorname{Hom}(I_x, I_y)$ since both vector spaces have the paths of y to x as a basis.

We may consider the Nakayama functor $N : \mathcal{P} \to \mathcal{I}$ where $N(P_x) = I_x$ and where the map $\operatorname{Hom}(P_x, P_y) \to \operatorname{Hom}(N(P_x), N(P_y))$ is given by the above isomorphism $\nu_{x,y}$. The Nakayama functor is an equivalence of categories.

It follows from [40, Lemma II.1.2] that the composition

$$F : D^b \operatorname{mod} kQ \cong K^b\mathcal{P} \xrightarrow{N} K^b\mathcal{I} \cong D^b \operatorname{mod} kQ$$

is a right Serre functor. Since F is an equivalence, it is a Serre functor. Hence $\operatorname{mod} kQ$ satisfies Serre duality.

Assume that $D^b\mathcal{A}$ is generated by the connecting component $\mathcal{C}$ and furthermore that the connecting component is standard, thus $\operatorname{rad}^\infty(X, Y) = 0$ for all X and Y in $\operatorname{ind}\mathcal{C}$. If we denote the quiver of projectives in $\mathcal{A}$ by Q, then the Auslander-Reiten quiver of $\mathcal{C}$ will be a stable translation quiver of the form $\mathbb{Z}Q$.

Let X be any vertex in Q and $X \to M_X \to \tau^{-1}X \to X[1]$ be an Auslander-Reiten triangle. Since M_X has as many direct summands as X has direct successors in $\mathbb{Z}Q$, we see that $\mathbb{Z}Q$ and hence also Q must be locally finite. Furthermore, it follows from Proposition 1.53 that there may be only finitely many sectional paths between any two vertices in $\mathbb{Z}Q$, thus by Theorem 5.13 we know $\mathbb{Z}Q$ admits a strongly locally finite section Q'.

Since $\mathcal{C}$ is standard and $\operatorname{Ext}(X,Y) \cong \operatorname{Hom}(Y,\tau X)^*$, Proposition 1.49 yields that the vertices of Q' form a partial tilting set, i.e. $\operatorname{Hom}_{D^b\mathcal{A}}(X, Y[n]) = 0$ for all $n \in \mathbb{Z}\backslash\{0\}$ and all $X, Y \in Q'$. It follows from Theorem 1.34 there is a full and exact embedding $i : D^b \operatorname{mod} kQ' \longrightarrow D^b\mathcal{A}$ mapping P_X to X, where Q' is the dual quiver of Q'.

Considering the exactness of i, and the connection between the Auslander-Reiten translation τ and the Serre functor F, we may check that $i \circ F(P) \cong F \circ i(P)$ for all $P \in Q'$. Since the Serre functor is exact and commutes with i on generators of $D^b \operatorname{mod} k(Q')^{\circ}$, the Serre functor will commute with i. Hence the essential image of i will contain $\mathcal{C}$ and we may conclude i is essentially surjective.

We have proven following theorem.

Theorem 1.55. *Let $\mathcal{A}$ be a k-linear abelian Ext-finite hereditary categories with Serre duality. Assume $D^b\mathcal{A}$ is generated by its connecting component $\mathcal{C}$ and that $\mathcal{C}$ is standard, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} kQ'$ *where Q' is strongly locally finite.*

We now turn our attention to noetherian categories. It has been shown in [40, Theorem II.4.2] that in this case the category $\mathcal{A}$ decomposes as a direct sum $\mathcal{R} \oplus \mathcal{Q}$ where $\mathcal{R}$ has no preprojectives, nor preinjectives, and where $\mathcal{Q}$ is generated by preprojectives. Thus, when $\mathcal{A}$ is a k-linear connected noetherian abelian Ext-finite hereditary categories with Serre duality, saying that $\mathcal{A}$ has at least one non-zero projective object is equivalent to saying that $\mathcal{A}$ is generated by preprojectives.

We have following corollary as answer to a conjecture posed in [40].

Corollary 1.56. *Let $\mathcal{A}$ be a noetherian k-linear abelian Ext-finite hereditary categories with Serre duality. Assume $\mathcal{A}$ has a non-zero projective object, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} kQ'$ *where Q' is strongly locally finite.*

Proof. It has been shown in [40, Proposition II.2.3] that the quiver of projectives Q of $\mathcal{A}$ is locally finite and does not contain any subquivers of the form $\cdot \to \cdot \to \cdot \to \cdots$

Since $\mathcal{A}$ has a non-zero projective object, it is generated by preprojectives and hence $D^b\mathcal{A}$ is generated by the connecting component.

We will show the connecting component $\mathcal{C}$ is standard. Let $X, Y \in \operatorname{ind} \mathcal{C}$ be with coordinates $(0, v_X)$ and (n, v_Y), respectively, in the Auslander-Reiten quiver $\mathbb{Z}Q$ of $\mathcal{C}$. Assume that $\operatorname{rad}^\infty(X,Y) \neq 0$ and that n has been chosen minimal with this property.

Consider the Auslander-Reiten triangle $Y \to M_Y \to \tau^{-1}Y \to Y[1]$. There is at least one indecomposable summand of Y_1 of M_Y such that $\operatorname{rad}^\infty(X, Y_1) \neq 0$. Due to the minimality of n, the coordinates of Y_1 in $\mathbb{Z}Q$ must be (n, v_{Y_1}) where v_{Y_1} is a direct successor of v_Y in Q. Iteration gives an infinite sequence $Y \to Y_1 \to Y_2 \to \cdots$ in Q, a contradiction.

We may now use Theorem 1.55 to see $\mathcal{A}$ is derived equivalent to $\operatorname{mod} kQ'$ where Q' is strongly locally finite. □

1.12 Twist functors

Twist functors have appeared in the literature under different names, for example *shrinking functors* [44], *tubular mutations* [34] and *twist functors* [48]. Similar ideas were the mutations used in [21] in the context of exceptional bundles on projective spaces and, more generally, in [11].

In this section, we will follow the general approach twist functors from [48], but introduce a small generalization. If an object E is endo-simple, then our definition of an associated twist functor coincides with that of [48].

Throughout, let $\mathcal{A}$ be any abelian category.

1.12.1 Definitions

We start by introducing some notations. For $C, D \in C(\mathcal{A})$, let $hom(C,D)$ be the complex given by $hom^i(C,D) = \prod_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(C^j, D^{i+j})$ with $d^i_{hom(C,D)}(\phi) = d_D \circ \phi - (-1)^i \phi \circ d_C$.

Note that the bi-functors Hom and hom are not naturally isomorphic. We do however have $\operatorname{Hom}^i(C,D) = \mathrm{H}^i\, hom(C,D)$, where $\operatorname{Hom}^i(C,D) = \operatorname{Hom}(C, D[i])$.

Let $A = \operatorname{End}(E)$ for an $E \in C(\mathcal{A})$, and let V be a complex of right A-modules. We define the tensor product $V \otimes_A E$ as usual. The right adjoint will be denoted by $lin_A(-,E)$.

If $C \to D$ is a map of complexes, then we denote by $\{C \to D\}$ the associated total complex, obtained by collapsing the bigrading.

Definition 1.57. The category $\mathfrak{K} \subseteq K^+(\operatorname{Ind}\mathcal{A})$ is defined to be the full subcategory whose objects are the bounded below cochain complexes C of $\operatorname{Ind}\mathcal{A}$-injectives which satisfy $H^i(C) \in \mathcal{A}$ for all i, and $H^i(C) = 0$ for $i \gg 0$.

We have the following result ([48, Proposition 2.4])

Proposition 1.58. *There is an exact equivalence (canonical up to natural isomorphism) $D^b\mathcal{A} \cong \mathfrak{K}$.*

1.12.2 Twist functors on $\mathfrak{K}$

Definition 1.59. Let $E \in \operatorname{Ob}\mathfrak{K}$ be an object satisfying following conditions

(S1) E is a bounded complex,

(S2) for any $F \in \mathfrak{K}$, both $\operatorname{Hom}^\bullet_{\mathfrak{K}}(E,F)$ and $\operatorname{Hom}^\bullet_{\mathfrak{K}}(F,E)$ have finite (total) dimension over k.

We will denote $\operatorname{Hom}(E,E)$ by A. With E we associate the *twist functor* $T_E : \mathfrak{K} \to \mathfrak{K}$ defined by

$$T_E = \{hom(E,F) \otimes_A E \xrightarrow{\epsilon} F\}$$

where ϵ is the canonical map, and the *dual twist functor* $T_E^* : \mathfrak{K} \to \mathfrak{K}$ defined by

$$T_E^* = \{F \xrightarrow{\epsilon^*} lin_A(hom(F,E),E)\}$$

where ϵ^* is the canonical map.

Lemma 1.60. *The functor T_E^* is left adjoint to T_E.*

Proof. Analogous to the proof of [48, Lemma 2.8]. □

Definition 1.61. An object $E \in \mathfrak{K}$ satisfying (S1) and (S2) is called *generalized n-spherical* for $n > 0$ if

(S3) $A \cong k^r$ as algebras and

$$\operatorname{Hom}^i_{\mathfrak{K}}(E,E) \cong \begin{cases} A & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

as left A-modules.

(S4) The composition $\operatorname{Hom}^j_{\mathfrak{K}}(F,E) \times \operatorname{Hom}^{n-j}_{\mathfrak{K}}(E,F) \to \operatorname{Hom}^n_{\mathfrak{K}}(E,E)$ is non-degenerate for all $j \in \mathbb{Z}$ and $F \in \mathfrak{K}$, i.e. if $(f,-)$ or $(-,g)$ are zero maps then $f = 0$ or $g = 0$, respectively.

In case $r = 1$, we find the definition of n-spherical object as in [48].

Remark 1.62. We will often call a generalized n-spherical object just an n-spherical object.

Lemma 1.63. *Let E be a generalized n-spherical object, and let $A = \operatorname{End}(E,E)$, then*

$$H^\bullet lin_A(hom(F,E), hom(E,E) \otimes_A E) \cong D \operatorname{Hom}^\bullet(F,E) \otimes_A \operatorname{Hom}^\bullet(E,E) \otimes_A H^\bullet E$$

where D is the dual with respect to A.

Proof. Since A is semi-simple, we may write $hom(F,E) \cong H^\bullet\, hom(F,E) \oplus C$ as left A-modules, where C is contractible. We then find

$$\begin{aligned} H^\bullet lin_A(hom(F,E), hom(E,E) \otimes_A E) &\cong H^\bullet lin_A(\operatorname{Hom}^\bullet(F,E) \oplus C, hom(E,E) \otimes_A E) \\ &\cong D \operatorname{Hom}^\bullet(F,E) \otimes_A H^\bullet(hom(E,E) \otimes_A E) \\ &\cong D \operatorname{Hom}^\bullet(F,E) \otimes_A \operatorname{Hom}^\bullet(E,E) \otimes_A H^\bullet E \end{aligned}$$

□

We have the following generalization of [48, Proposition 2.10]

Proposition 1.64. *Let E be a generalized n-spherical object for some $n > 0$, then the functors $T_E^* T_E$ and $T_E T_E^*$ are both naturally isomorphic to the identity functor on $\mathfrak{K}$.*

Proof. The proof is identical to the proof in [48, Proposition 2.10] using Lemma 1.63 to calculate the homologies. □

1.12.3 Twist functors on derived categories

We will now translate previous result to the setting we will be using in this thesis. Let $\mathcal{A}$ be an abelian category.

Definition 1.65. An object $E \in \operatorname{Ob} D^b\mathcal{A}$ is called *generalized n-spherical* for some $n > 0$ if it satisfies following properties

(S1) E has a finite resolution by $\operatorname{Ind}\mathcal{A}$-injectives.

(S2) $\operatorname{Hom}^\bullet(E, F)$ and $\operatorname{Hom}^\bullet(F, E)$ are finite dimensional for all $F \in D^b\mathcal{A}$

(S3) $A \cong k^r$ as algebras and

$$\operatorname{Hom}^i_{\mathfrak{K}}(E, E) \cong \begin{cases} A & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

as left A-modules.

(S4) The composition $\operatorname{Hom}^j(F, E) \times \operatorname{Hom}^{n-j}(E, F) \to \operatorname{Hom}^n(E, E)$ is non-degenerate for all $j \in \mathbb{Z}$ and $F \in D^b\mathcal{A}$.

where $A = \operatorname{Hom}(E, E)$

If $\mathcal{A}$ is an Ext-finite abelian hereditary category with Serre duality, then the first two conditions are automatic. Furthermore, the third condition implies $E \cong \oplus_{i=1}^r E_i$ where $\operatorname{Hom}(E_i, E_j) \cong k$ if $i = j$ and 0 otherwise, and that for every i there is a unique j such that $\operatorname{Hom}^n(E_i, E_j) \cong k$ We have following lemma.

Lemma 1.66. *Let $\mathcal{A}$ be a an abelian category, and $E \in \operatorname{Ob} D^b\mathcal{A}$ satisfying (S1), (S2), and (S3). Then (S4) is equivalent to:*

(S4') for every i, there is a unique j such that $\operatorname{Hom}(E_i, F) \cong \operatorname{Hom}(F, E_j[n])^*$, natural in $F \in \operatorname{ind} D^b\mathcal{A}$.

Proof. We will denote the unique E_j corresponding to E_i as σE_i and the natural isomorphism $\operatorname{Hom}(E_i, F) \cong \operatorname{Hom}(F, \sigma E_i[n])^*$ will be denoted by $\eta_{E_i,F}$. Note that this, in particular, implies that $\mathrm{Hom}^n(E_i, \sigma E_i) \cong k$. Assume $(S4')$ holds. Consider the non-degenerate pairing given by:

$$\begin{array}{rcl} \mathrm{Hom}(F,E) \otimes_k \mathrm{Hom}^n(E,F) & \cong & \oplus_{i,j}(\mathrm{Hom}(F,E_i) \otimes_k \mathrm{Hom}^n(E_j,F)) \\ & \cong & \oplus_{i,j}(\mathrm{Hom}(F,E_i) \otimes_k \mathrm{Hom}(F,\sigma E_j)^*) \\ & \to & \oplus_i(\mathrm{Hom}(F,E_i) \otimes_k \mathrm{Hom}(F,E_i)^*) \\ & \to & \oplus_i k \end{array}$$

given by mapping an object $\oplus_{i,j}(f_i \otimes g_j) \in \oplus_{i,j}(\mathrm{Hom}(F,E_i) \otimes_k \mathrm{Hom}^n(E_j,F))$ to $\eta_{E_i,F}(g_i)(f_i)$. Due to the naturality, the latter is equal to $\eta_{E_i,\sigma E_i[n]}(f_i \circ g_i)(1)$.

The map $\mathrm{Hom}^n(E_i, \sigma E_i) \to k$, given by $h \mapsto \eta_{E_i,\sigma E_i[n]}(h)(1)$ defines an isomorphism, we may continue the pairing by

$$\begin{array}{rcl} \mathrm{Hom}(F,E) \otimes_k \mathrm{Hom}^n(E,F) & \to & \oplus_i k \\ & \cong & \oplus_i \mathrm{Hom}^n(E_i,\sigma E_i) \cong \mathrm{Hom}^n(E,E). \end{array}$$

Hence, the composition $\mathrm{Hom}(F,E) \otimes_k \mathrm{Hom}^n(E,F) \to \mathrm{Hom}^n(E,E)$ is a non-degenerate pairing, and (S4) is satisfied.

The other direction is easy. □

We now come to the main theorem of this section.

Theorem 1.67. *Let $\mathcal{A}$ be an Ext-finite abelian category, and let $E \in \mathrm{Ob}\,\mathcal{A}$ be an n-spherical object, then the twist functors T_E and T_E^* are quasi-inverses.*

Proof. This is just a reformulation of Proposition 1.64. □

In the rest of this thesis, we will be interested only in the case where $\mathcal{A}$ is an Ext-finite abelian hereditary category with Serre duality. In this case, the first two conditions are automatic and the map σ from Lemma 1.66 coincides with the Auslander-Reiten translate. The conditions (S3) and (S4') then are equivalent to: $E \cong \oplus_{i=1}^r \tau^i E_0$ where $\tau^{r+1}E_0 \cong E_0$, and $\mathrm{Hom}(\tau^i E_0, \tau^j E_0) \cong k$ if and only if $i \equiv j \pmod r$ and 0 otherwise.

In this case,

$$\begin{array}{rcl} E \otimes_A \mathrm{RHom}(E,X) & \cong & \oplus_{i=1}^r E_i \otimes_k \mathrm{RHom}(E_i,X) \\ E \otimes_A \mathrm{RHom}(X,E)^* & \cong & \oplus_{i=1}^r E_i \otimes_k \mathrm{RHom}(X,E_i)^*. \end{array}$$

These conditions are satisfied, for example, when E_0 is a peripheral object of a standard tube, thus an Auslander-Reiten component of the form $\mathbb{Z}A_\infty/\langle\tau^r\rangle$ for $r > 0$.

Chapter 2

Semi-hereditary additive categories

Let $\mathfrak{a}$ be a small preadditive category. We will say $\mathfrak{a}$ is semi-hereditary if and only if $\operatorname{mod} \mathfrak{a}$ is abelian and hereditary. Our main result of §2.1 is Proposition 2.1 where we prove a category is semi-hereditary if and only if every full subcategory with finitely many objects is semi-hereditary.

In section §2.2, we will discuss for which semi-hereditary categories $\mathfrak{a}$ the category $\operatorname{mod} \mathfrak{a}$ has Serre duality. We show in Theorem 2.9 that, if $\mathfrak{a}$ is a Karoubian category, then this is exactly the case when $\mathfrak{a}$ is a *dualizing k-variety*, or equivalently, when all indecomposable projectives have a cofinite injective presentation and vice versa (Proposition 2.7).

A description of such dualizing k-varieties will be given in terms of thread quivers. Intuitively, we may think of semi-hereditary dualizing k-varieties as strongly locally finite quivers Q without loops in which certain arrows have been replaced by linearly ordered locally discrete posets, thus by posets of the form $T \overrightarrow{\times} \mathbb{Z}$. As they will return in examples throughout this thesis, we will provide a way of sketching such a category by means a thread quivers.

In the same spirit, it is tempting to ask what would happen if a similar generalization would be possible for the category $\operatorname{nilp} \tilde{A}_n$, thus whether one can 'stretch' the quiver $\tilde{A}_n$ with cyclic orientation to an infinite variant, called a *big loop*. We will investigate this in §2.3, where it shown that the category of representations with a certain nilpotency condition does indeed form an abelian Ext-finite hereditary category with Serre duality, called a *big tube*. We have found no other references to this category in the available literature.

2.1 Semi-hereditary categories

We say that a small preadditive category $\mathfrak{a}$ is *semi-hereditary* if the finitely presented objects $\operatorname{mod}(\mathfrak{a})$ in $\operatorname{Mod}(\mathfrak{a})$ form an abelian and hereditary category. Following proposition shows that semi-hereditariness is a local property.

Proposition 2.1. *Let $\mathfrak{a}$ be a small preadditive category, then $\mathfrak{a}$ is semi-hereditary if and only if any full subcategory of $\mathfrak{a}$ with a finite number of objects is semi-hereditary.*

To show this result, we will need the following lemma.

Lemma 2.2. *Assume that f is fully faithful and assume that P is a summand of an object of the form $\bigoplus_{i=1}^n \mathfrak{b}(-,B_i)$ with B_i in the essential image of f. Then the canonical map $\mathfrak{b} \otimes_{\mathfrak{a}} P_{\mathfrak{a}} \to P$ is an isomorphism.*

Proof. P is given by an idempotent e in $\oplus_{i,j}\mathfrak{b}(B_j,B_i)$. Hence we may write P as the cokernel of

$$\bigoplus_{i=1}^{n} \mathfrak{b}(-,B_i) \xrightarrow{1-e} \bigoplus_{i=1}^{n} \mathfrak{b}(-,B_i)$$

The result now follows easily from the right exactness of $\mathfrak{b} \otimes_{\mathfrak{a}} -$. □

Proof of Proposition 2.1. First, assume every full subcategory of $\mathfrak{a}$ with a finite number of objects is semi-hereditary. We will show $\mathfrak{a}$ is semi-hereditary. As usual it is sufficient to prove that the kernel of a map between finitely generated projectives $p : P \to Q$ in $\operatorname{Mod}(\mathfrak{a})$ is a finitely generated projective and splits off.

Since a finitely generated projective $\mathfrak{a}$-module is a summand of an $\mathfrak{a}$-module of the form $\oplus_{i=1}^m \mathfrak{a}(-,A_i)$ we may without of loss of generality assume that p is a map of the form

$$p : \bigoplus_{i=1}^{m} \mathfrak{a}(-,A_i) \to \bigoplus_{j=1}^{n} \mathfrak{a}(-,B_j)$$

Such a map is given by a collection of maps $p_{ij} : A_i \to B_j$.

Let $\mathfrak{b}$ be the full subcategory of $\mathfrak{a}$ containing the objects $(A_i)_i$, $(B_j)_j$ and let $\mathcal{F}$ be the filtered collection of full subcategories of $\mathfrak{a}$ containing $\mathfrak{b}$ and having a finite number of objects.

For $\mathfrak{c} \in \mathcal{F}$ let $K_{\mathfrak{c}}$ be the kernel of the map

$$\bigoplus_{i=1}^{m} \mathfrak{c}(-,A_i) \to \bigoplus_{j=1}^{n} \mathfrak{c}(-,B_j)$$

given by the same $(p_{ji})_{ij}$. Put $K = \mathfrak{a} \otimes_{\mathfrak{b}} K_{\mathfrak{b}}$. By hypotheses $K_{\mathfrak{b}}$ is finitely generated and a summand of $\bigoplus_{i=1}^m \mathfrak{b}(-,A_i)$ and it follows that the analogous facts are true for K. So to prove the proposition it is sufficient to prove that K is the kernel of p.

That K is the kernel of p can be checked pointwise. Hence it is sufficient to show it for an arbitrary $\mathfrak{c} \in \mathcal{F}$. Since it is easy to see that $(\mathfrak{a} \otimes_{\mathfrak{b}} K_{\mathfrak{b}})_{\mathfrak{c}} = \mathfrak{c} \otimes_{\mathfrak{b}} K_{\mathfrak{b}}$ we need to show that the canonical map $\mathfrak{c} \otimes_{\mathfrak{b}} K_{\mathfrak{b}} \to K_{\mathfrak{c}}$ is an isomorphism. Since $K_{\mathfrak{c}}$ is a summand of $\bigoplus_{i=1}^{m} \mathfrak{c}(-, A_i)$ and obviously $K_{\mathfrak{b}} = (K_{\mathfrak{c}})_{\mathfrak{b}}$ this follows from lemma 2.2.

The other implication is trivial. □

Remark 2.3. Proposition 2.1 is false with semi-hereditary replaced by hereditary.

Remark 2.4. It is not true that a filtered direct limit of semi-hereditary rings is semi-hereditary. A counterexample is given in [9].

Lemma 2.5. *Let $\mathfrak{b} \to \mathfrak{c}$ be a full embedding of preadditive categories. Then $\mathfrak{c} \otimes_{\mathfrak{b}} - : \mathrm{mod}(\mathfrak{b}) \to \mathrm{mod}(\mathfrak{c})$ is fully faithful.*

Proof. This may be checked on objects of the form $\mathfrak{b}(-, B)$ where it is clear. □

Since $(\mathfrak{c} \otimes_{\mathfrak{b}} -, (-)_{\mathfrak{b}})$ is an adjoint pair, if follows from this lemma that $(\mathfrak{c} \otimes_{\mathfrak{b}} -)_{\mathfrak{b}}$ is naturally isomorphic to the identity functor on $\mathrm{mod}(\mathfrak{b})$.

Lemma 2.6. *Let $\mathfrak{b} \to \mathfrak{c}$ be a full embedding of semi-hereditary categories. Then the (fully faithful) functor $\mathfrak{c} \otimes_{\mathfrak{b}} - : \mathrm{mod}(\mathfrak{b}) \to \mathrm{mod}(\mathfrak{c})$ is exact.*

Proof. Let $M \in \mathrm{mod}(\mathfrak{b})$ and consider a projective resolution $0 \to P_1 \xrightarrow{\theta} P_0 \to M \to 0$ in $\mathrm{mod}(\mathfrak{b})$. Then P_1 is a direct summand of some $\oplus_{i=1}^{n} \mathfrak{b}(-, B_i)$. Put $K = \ker(\mathfrak{c} \otimes_{\mathfrak{b}} \theta)$. Then K is a direct summand of $\mathfrak{c} \otimes_{\mathfrak{b}} P_1$ (since $\mathfrak{c}$ is semi-hereditary) and $K_{\mathfrak{b}} = 0$ (since θ is injective).

Assume K is non-zero. Since K is a direct summand of $\mathfrak{c} \otimes_{\mathfrak{b}} P_1$ we obtain a non-zero map $\mathfrak{c} \otimes_{\mathfrak{b}} P_1 \to K$ and hence a non-zero map $\oplus_{i=1}^{n} \mathfrak{c}(-, B_i) \to K$ and thus ultimately a non-zero element of some $K(B_i)$, contradicting the fact that $K_{\mathfrak{b}} = 0$.

Thus $K = 0$. If we denote the left satellites of $\mathfrak{c} \otimes_{\mathfrak{b}} -$ by $\mathrm{Tor}_i^{\mathfrak{b}}(\mathfrak{c}, -)$ then we have just shown that $\mathrm{Tor}_i^{\mathfrak{b}}(\mathfrak{c}, -) = 0$ for $i > 0$. Hence $\mathfrak{c} \otimes_{\mathfrak{b}} -$ is exact. □

2.2 Dualizing k-varieties

In this section, based on joint work with Carl Fredrik Berg ([7]), we show that so called dualizing k-varieties are a good generalization for strongly locally finite quivers in our setting (the category of finitely presented modules will have Serre duality). We will introduce thread quivers in order to sketch semi-hereditary dualizing k-varieties and every such dualizing k-variety is uniquely determined, up to equivalence, by its thread quiver.

2.2.1 Representations of dualizing k-varieties

We now recall some definitions from [4]. A Hom-finite additive k-linear category where idempotents split is called a *finite k-variety*. Note that a finite k-variety is always Krull-Schmidt.

Let $\mathfrak{a}$ be a finite k-variety. There is a functor $D\colon \mathrm{Fun}(\mathfrak{a}, \mathrm{mod}\, k) \to \mathrm{Fun}(\mathfrak{a}^{\circ}, \mathrm{mod}\, k)$ given by sending a module $M\colon \mathfrak{a} \to \mathrm{mod}\, k$ to the dual $D(M)$ where $D(M)(x) =$

$\mathrm{Hom}_k(M(x), k)$ for all $x \in \mathfrak{a}$. If this functor induces a duality $D\colon \operatorname{mod}\mathfrak{a} \to \operatorname{mod}\mathfrak{a}^{\circ}$ by restricting to the finitely presented objects in $\mathrm{Fun}(\mathfrak{a}, \operatorname{mod} k) \subset \operatorname{Mod}\mathfrak{a}$, then we will say that $\mathfrak{a}$ is a *dualizing k-variety*.

We have the following easy proposition.

Proposition 2.7. *Let $\mathfrak{a}$ be a finite k-variety, then $\mathfrak{a}$ is a dualizing k-variety if and only if all standard projectives are cofinitely presented and all standard injectives are finitely presented.*

Corollary 2.8. *A dualizing k-variety is coherent.*

Proof. By Proposition 2.7 the category of finitely presented representations and cofinitely presented representations coincide as subcategories of $\operatorname{Mod}\mathfrak{a}$. The former is closed under cokernels, the latter under kernels. This shows that $\operatorname{mod}\mathfrak{a}$ is an abelian subcategory of $\operatorname{Mod}\mathfrak{a}$. □

The next theorem is the main result of this section.

Theorem 2.9. *Let $\mathfrak{a}$ be a finite k-variety such that every object in* $\operatorname{mod}\mathfrak{a}$ *has finite projective dimension, then $\mathcal{A} = \operatorname{mod}\mathfrak{a}$ has Serre duality if and only if $\mathfrak{a}$ is a dualizing k-variety.*

Proof. First, we assume $\operatorname{mod}\mathfrak{a}$ has Serre duality. We prove that all standard injectives are finitely presented and all standard projectives cofinitely presented. We start with the former.

If $P \in \operatorname{Ob}\mathcal{A}$ is a projective, then we find $\mathrm{Hom}_{D^b\mathcal{A}}(P, Y) \cong \mathrm{Hom}_{\mathcal{A}}(P, H^0 Y)$.

Let $f \in \mathfrak{a}(A_i, A_j)$ with $A_i, A_j \in \operatorname{Ob}\mathfrak{a}$. The commutative diagram

$$\begin{tikzcd}
\mathfrak{a}(A, A_j)^* \arrow[r] & \mathfrak{a}(A, A_i)^* \\
\mathrm{Hom}(\mathfrak{a}(-, A), \mathfrak{a}(-, A_j))^* \arrow[r] \arrow[u, "\sim"] & \mathrm{Hom}(\mathfrak{a}(-, A), \mathfrak{a}(-, A_i))^* \arrow[u, "\sim"'] \\
\mathrm{Hom}(\mathfrak{a}(-, A_j), F\mathfrak{a}(-, A)) \arrow[r] \arrow[u, "\sim"] & \mathrm{Hom}(\mathfrak{a}(-, A_i), F\mathfrak{a}(-, A)) \arrow[u, "\sim"'] \\
H^0(F\mathfrak{a}(-, A))(A_j) \arrow[r] \arrow[u, "\sim"] & H^0(F\mathfrak{a}(-, A))(A_i) \arrow[u, "\sim"']
\end{tikzcd}$$

where the upper and the lower commuting squares are given by the Yoneda lemma, shows that $H^0(F\mathfrak{a}(-, A)) \cong \mathfrak{a}(A, -)^*$. Since $H^0(F\mathfrak{a}(-, A)) \in \operatorname{Ob}\mathcal{A}$, all standard injectives are finitely presented.

Denote by $\mathcal{P}$ and $\mathcal{I}$ the full subcategories spanned by (standard) projective and injective objects in $\operatorname{mod}\mathfrak{a}$, respectively. Note that for every $A, B \in \operatorname{Ob}\mathfrak{a}$, there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(\mathfrak{a}(-,A),\mathfrak{a}(-,B)) \cong \operatorname{Hom}_{\mathcal{A}}(\mathfrak{a}(A,-)^*,\mathfrak{a}(B,-)^*)$$

since they are, by Yoneda's lemma, both naturally isomorphic to $\mathfrak{a}(A,B)$, such that the Nakayama functor $N:\mathcal{P}\to\mathcal{I}$ given by $N\mathfrak{a}(-,A)=\mathfrak{a}(A,-)^*$ is an equivalence of categories. This lifts to an equivalence $N: K^b\mathcal{P}\to K^b\mathcal{I}$. The composition

$$D^b\mathcal{A}\cong K^b\mathcal{P}\xrightarrow{N} K^b\mathcal{I}\to D^b\mathcal{A}$$

gives the Serre functor, and is thus an equivalence. This yields $K^b\mathcal{I}\to D^b\mathcal{A}$ is an equivalence and, as such, every object of $\mathcal{A}$ is cofinitely presented.

For the other direction, assume $\mathfrak{a}$ is a dualizing k-variety. Again, denote by $\mathcal{P}$ and $\mathcal{I}$ the full subcategories spanned by (standard) projective and injective objects in $\operatorname{mod}\mathfrak{a}$, and by $N: K^b\mathcal{P}\to K^b\mathcal{I}$ the equivalence induced by the Nakayama functor. We denote the composition

$$D^b\operatorname{mod}\mathfrak{a}\cong K^b\mathcal{P}\xrightarrow{N} K^b\mathcal{I}\cong D^b\operatorname{mod}\mathfrak{a}$$

by F. We claim F is a Serre functor; the proof is taken from [40, Lemma II.1.2].

Since every $X,Y\in D^b\operatorname{mod}\mathfrak{a}$ correspond to bounded complexes of projectives in $K^b\mathcal{P}$, we may reduce to the case where $X\cong\mathfrak{a}(-,A)$ and $Y\cong\mathfrak{a}(-,B)$. The required isomorphism

$$\operatorname{Hom}(\mathfrak{a}(-,A),\mathfrak{a}(-,B))\cong\operatorname{Hom}(\mathfrak{a}(-,B),\mathfrak{a}(A,-)^*)^*$$

follows from Yoneda's lemma.

□

2.2.2 Thread quivers

We will say a finite k-variety $\mathfrak{a}$ is *locally discrete* if no indecomposable object is an accumulation point, i.e. for every $A\in\operatorname{ind}\mathfrak{a}$ there are sets $M,N\subseteq\operatorname{ind}\mathfrak{a}$ such that for all $B\in\operatorname{ind}\mathfrak{a}$ a non-invertible map $A\to B$ or $B\to A$ factor nontrivially through $\oplus_i M_i$ or $\oplus_i N_i$, respectively, where $M_i\in M$ and $N_i\in N$.

If additionally M and N may be chosen to be finite sets, then we will say $\mathfrak{a}$ is *locally finite*. Thus if $\mathfrak{a}$ is locally discrete and every object has only a finite number of direct predecessors and successors.

In particular, a finite k-variety $\mathfrak{a}$ is locally finite and locally discrete if, for all $A\in\operatorname{ind}\mathfrak{a}$, there is a right almost split map $A\to M$ and a left almost split map $N\to A$, for certain objects $M,N\in\operatorname{Ob}\mathfrak{a}$.

Example 2.10. Let $\mathcal{P}$ be the poset $\mathbb{N}\cdot\{+\infty\}$. We may draw the Auslander-Reiten quiver of $k\mathcal{P}$ as

$$0\longrightarrow 1\longrightarrow 2\longrightarrow 3 \cdots\cdots +\infty$$

It is clear that $k\mathcal{P}$ is not locally discrete.

We now give an equivalent formulation of these properties.

Proposition 2.11. *A finite k-variety is locally finite and locally discrete if and only if all standard simples of $\mathfrak{a}$ are finitely presented and cofinitely presented.*

Proof. Assume $\mathfrak{a}$ is locally finite and locally discrete. For an indecomposable $A \in \operatorname{ind} \mathfrak{a}$, let $N \to A$ be a left almost split map which gives rise to a map in $\operatorname{mod} \mathfrak{a}$

$$\mathfrak{a}(-, N) \longrightarrow \mathfrak{a}(-, A).$$

It is straightforward to see that the cokernel is the standard simple S_A. Dually, one shows all standard simples are cofinitely presented.

Next, assume all standard simples are finitely and cofinitely presented. We prove that every indecomposable $A \in \operatorname{ind} \mathfrak{a}$ admits a left almost split map $N \to A$, for a certain object $N \in \operatorname{Ob} \mathfrak{a}$. Consider a presentation of S_A

$$Q \longrightarrow \mathfrak{a}(-, A) \longrightarrow S_A \longrightarrow 0.$$

We may write the projective Q as $\mathfrak{a}(-, N)$, and the induced map $N \to A$ is easily seen to be left almost split.

Dually, one proves a A admits a right almost split map $A \to M$. □

If $\mathfrak{a}$ is a dualizing k-variety, a locally finite and locally discrete object $X \in \operatorname{ind} \mathfrak{a}$ will be called a *thread object* if X has a unique direct predecessor and a unique direct successor, or equivalently, there is a right almost split map $A \to M$ and a left almost split map $N \to A$ where M and N are indecomposable. An indecomposable that is not a thread object is called a *non-thread object*.

In the following, an *interval* $[X, Y]$ in a finite k-variety is a full subcategory given by

$$[X, Y] = \{A \in \operatorname{ind} \mathfrak{a} \mid \mathfrak{a}(X, A) \neq 0 \text{ and } \mathfrak{a}(A, Y) \neq 0\}.$$

Proposition 2.12. *Let $\mathfrak{a}$ be a semi-hereditary dualizing k-variety, then $\mathfrak{a}$ is locally finite and locally discrete, and for all $X, Y \in \operatorname{ind} \mathfrak{a}$, the interval $[X, Y]$ has only finitely many non-thread objects. If $\mathfrak{a}$ is connected, then $\mathfrak{a}$ has only countably many sinks or sources.*

Proof. We consider the Yoneda embedding $i : \mathfrak{a} \longrightarrow \operatorname{mod} \mathfrak{a}$. For all $A \in \operatorname{ind} \mathfrak{a}$, we may obtain the corresponding standard simple S_A as the image of a nonzero map $\mathfrak{a}(-, A) \longrightarrow \mathfrak{a}(A, -)^*$ the corresponding projective, hence S_A finite and cofinitely presented. By Proposition 2.11 we know $\mathfrak{a}$ is locally finite and locally discrete.

That every interval $[X, Y]$ has only finitely many non-thread object follows from the fact that $\mathfrak{a}(-, X)$ is cofinitely presented by standard injectives and that $\mathfrak{a}(Y, -)^*$ is finitely presented by standard injectives.

Let $\mathfrak{a}$ be connected, and let X be a source of $\mathfrak{a}$. Since $\mathfrak{a}(X, -)^*$ is finitely presented, X maps non-zero to only a finite number of sinks. Likewise, for every sink Y there are only finitely many sources mapping to Y. We see that the number of sinks and sources is countable. □

It follows from previous proposition that the Auslander-Reiten quiver Q of $\mathfrak{a}$ is a locally finite quiver. However, Q does not have to be connected, even if $\mathfrak{a}$ is. To remedy this, we will replace infinite intervals $[X,Y]$ by a dashed arrow $X \dashrightarrow Y$ when $[X,Y]$ consists only of thread objects, and omit all objects in $]X,Y[$.

Such a quiver in which we distinguish between full (normal) arrows $\bullet \longrightarrow \bullet$ and dashed arrows (called *thread arrows*) $\bullet \dashrightarrow \bullet$ will be called a *thread quiver*. We will refer to the thread quiver associated with $\mathfrak{a}$ by $\Gamma_t\mathfrak{a}$.

An nonempty interval $[X,Y] \subseteq \operatorname{ind}\mathfrak{a}$ consisting only of thread objects will be called a *thread*.

Proposition 2.13. *Let $\mathfrak{a}$ be a semi-hereditary dualizing k-variety. If $X,Y,Z \in \operatorname{ind}\mathfrak{a}$ such that $[X,Y]$ and $[X,Z]$ are threads, then $[X,Y] \subseteq [X,Z]$ or $[X,Y] \subseteq [X,Z]$. Moreover,* $\dim \mathfrak{a}(X,Y) = 1$.

Proof. Assume $Y \notin [X,Z]$ and $Z \notin [X,Y]$. The situation is as depicted below.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & & \arrow[r] & Z \arrow[r] & {} \arrow[r] & {} \\
\arrow[r] & \arrow[r] & X \arrow[r] & {} \arrow[ur, dotted, no head] \arrow[dr, dotted, no head] & & & & \\
 & & & & \arrow[r] & Y \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}$$

It is clear that the standard injective $\mathfrak{a}(X,-)^*$ is not finitely presented, and that the standard projectives $\mathfrak{a}(-,Y)$ and $\mathfrak{a}(-,Z)$ are not cofinitely presented.

Likewise one shows that, if $[X,Y]$ is a thread, $\mathfrak{a}(X,Y) = 1$ since otherwise $\mathfrak{a}(-,Y)$ is not cofinitely presented. □

As a consequence of this proposition, a thread is necessarily linearly ordered. The objects of $\operatorname{ind}\mathfrak{a}$ in $[X,Y]$ fall into three classes: one part contains X and is poset isomorphic to $\mathbb{N}$, one part contains Y and is poset isomorphic to $-\mathbb{N}$, and one part does not have a minimal nor maximal element and is of the form $\mathbb{Z} \vec{\times} \mathcal{P}$, for a linearly ordered poset $\mathcal{P}$.

We shall sometimes endow a thread arrow in $[X,Y]$ with the poset $\mathbb{Z} \vec{\times} \mathcal{P}$.

Example 2.14. Let $\mathcal{P}$ be a linearly ordered poset and let $\mathfrak{a}$ be the semi-hereditary category associated to the poset $\mathbb{N} \cdot \mathbb{Z} \vec{\times} \mathcal{P} \cdot (-\mathbb{N})$. Then thread quiver of $\mathfrak{a}$ is given by

$$\longrightarrow \overset{\mathbb{Z}\vec{\times}\mathcal{P}}{\dashrightarrow} \longrightarrow$$

Proposition 2.15. *Let $\mathfrak{a}$ be a semi-hereditary dualizing k-variety. Then the associated thread quiver $\Gamma_t\mathfrak{a}$ is a strongly locally finite quiver in which every vertex is incident with at most one thread arrow.*

Proof. That the thread quiver is a strongly locally finite quiver follows from Proposition 2.7. If $[X,Y]$ is a thread represented by an arrow $X \dashrightarrow Y$, then X and Y are thread objects. The other claims follow from this. □

Conversely, with every thread quiver Q we may associate a semi-hereditary preadditive path category kQ as follows: the objects are a disjunct union of vertices and the objects associated with each thread arrow of Q, and the Hom-spaces are the obvious ones. The composition is given by composing paths.

We will denote by $\operatorname{add} kQ$ the closure of kQ under finite direct sums. If there are only finitely many arrows between vertices of Q, then $\operatorname{add} kQ$ is a finite k-variety. If Q is strongly locally finite, then Proposition 2.7 implies $\operatorname{add} kQ$ is a semi-hereditary dualizing k-variety.

The following result is now easy to see.

Proposition 2.16. *Let $\mathfrak{a}$ be a semi-hereditary dualizing k-variety, then $\Gamma_t\mathfrak{a}$ is a strongly locally finite thread quiver and $\mathfrak{a}$ is equivalent to* $\operatorname{add} k\Gamma_t\mathfrak{a}$.

2.3 Big tubes

In this section, we will define a new class of hereditary categories with Serre duality. In contrast to the above constructions these new hereditary categories are realized as representations of certain semi-hereditary categories which are *not* Hom-finite.

We start with the definition of a big loop. If $\mathcal{L}$ is a linearly ordered set, then we may define a (small) category $\mathcal{L}^\bullet$ where the object set is given by elements of $\mathcal{L}$, the morphisms by

$$\operatorname{Hom}_{\mathcal{L}^\bullet}(i,j) = \begin{cases} \mathbb{N} & \text{if } i \leq j \\ \mathbb{N}\setminus\{0\} & \text{if } i > j \end{cases}$$

and where the composition is given by addition. Note that the identity morphism in $\operatorname{Hom}_{\mathcal{L}^\bullet}(i,i)$ is given by the zero in $\mathbb{N}$. Also, the category $\mathcal{L}^\bullet$ is not k-linear.

The linearization $k\mathcal{L}^\bullet$ of the above category may be described as follows: the objects are the objects of $k\mathcal{L}^\bullet$ and the morphisms are given by

$$\operatorname{Hom}_{k\mathcal{L}^\bullet}(i,j) = \begin{cases} k[x] & \text{if } i \leq j \\ xk[x] & \text{if } i > j \end{cases}$$

The objects of the path completed category $\widehat{k\mathcal{L}^\bullet}$ are the same as those of $k\mathcal{L}^\bullet$, while the morphisms are given by

$$\operatorname{Hom}_{\widehat{k\mathcal{L}^\bullet}}(i,j) = \begin{cases} k[[x]] & \text{if } i \leq j \\ xk[[x]] & \text{if } i > j \end{cases}$$

If $\mathcal{L}$ is locally discrete without a minumum or a maximum, thus if every element in $\mathcal{L}$ has a direct predecessor and successor, then $\widehat{k\mathcal{L}^\bullet}$ is called a *big loop*.

Note that, with this definition, $\widehat{k\tilde{A}_n}$ is not a big loop.

Recall that, for a preadditive category $\mathfrak{a}$, we will denote by $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}$ the full subcategory of $\operatorname{Mod} \mathfrak{a}$ given by all right $\mathfrak{a}$-modules which have a finite presentation by standard projective and a cofinite presentation by standard injectives. If $\mathfrak{a}$ is a

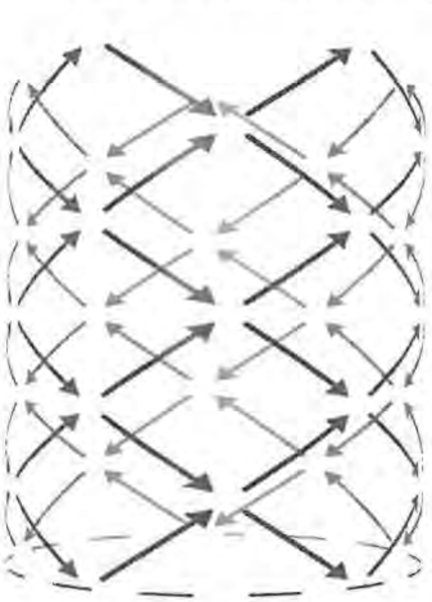


Figure 2.1: A standard tube

semi-hereditary category, then $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}$ is abelian and hereditary (see Proposition 2.1).

If $\mathfrak{a}$ is a big loop, then we will call the category $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}$ a *big tube*.

We will now discuss the objects and morphisms occurring in such a big tube. Since every object in $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}$ is finitely generated, it suffices to discuss the objects and morphisms of $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}'$ for a well-chosen additive subcategory $\mathfrak{a}'$ of $\mathfrak{a}$ with finitely many indecomposables. In this case, $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}' \cong \operatorname{nilp} \tilde{A}_n$.

The indecomposable objects of $\operatorname{nilp} \tilde{A}_n$ are easily understood. The simple objects are the standard simples, thus with every vertex X of $\tilde{A}_n$ we associate the simple representation S by $S(X) \cong k$, $S(Y) \cong 0$ when $X \neq Y$, and $X(\alpha) = 0$ for every arrow α.

An indecomposable nilpotent module M is uniquely determined by a simple top T, a simple socle S, and a *winding number* $n \in \mathbb{N}$ where $n = \dim \operatorname{Hom}(M, M) - 1$.

The Auslander-Reiten quiver of $\operatorname{nilp} \tilde{A}_n$ is of the form $\mathbb{Z}A_\infty/\langle \tau^{n+1} \rangle$ as in Figure 2.1, where the peripheral objects correspond to the simple representations.

Likewise, in $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}$, the simple representations are given by the standard simples, and every $M \in \operatorname{Ob} \operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}$ is uniquely determined by a simple top T, a simple socle S, and a *winding number* $n \in \mathbb{N}$ where $n = \dim \operatorname{Hom}(M, M) - 1$.

A module M with above properties will be written as $M(T, S; n)$.

The category $\operatorname{mod}^{\mathrm{cfp}} \mathfrak{a}$ has Serre duality. Indeed, since it has no projectives or injectives, it suffices to check it has almost split sequences. Let $M(S_X, S_Y; n)$ be an indecomposable module where S_X and S_Y are the standard simples corresponding to

$X, Y \in \mathfrak{a}$, respectively. It is straightforward to check that

$$\tau M(S_X, S_Y; n) = M(S_{X-1}, S_{Y-1}; n)$$

does indeed define an Auslander-Reiten translate, where $X - 1$ and $Y - 1$ are the direct predecessors of X and Y, respectively.

All irreducible maps are one of the following form

$$\begin{array}{rcll} M(S_X, S_Y; n) & \to & M(S_{X+1}, S_Y; n) & \text{with } X+1 \neq Y \\ M(S_X, S_Y; n) & \to & M(S_X, S_{Y+1}; n) & \text{with } X \neq Y+1 \\ M(S_{X-1}, S_X; n) & \to & M(S_X, S_X; n+1) & \\ M(S_X, S_{X-1}; n+1) & \to & M(S_X, S_X; n) & \end{array}$$

where $X, Y \in \operatorname{ind} \mathfrak{a}$ and $n \in \mathbb{N}$. Every indecomposable, not of the form $M(S_X, S_X; 0)$ has thus two direct successors and two direct predecessors; modules of the form $M(S_X, S_X; 0)$ are peripheral and simple. The Auslander-Reiten components are thus of the form $\mathbb{Z}A_\infty$ if the component has peripheral objects, and of the form $\mathbb{Z}A_\infty^\infty$ otherwise. We may sketch this situation as in Figure 2.2 where, as usual, triangles and squares are used to represent components of the form $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$, respectively.

Remark 2.17. A big tube does not correspond to a connected component in the Auslander-Reiten quiver, whereas a (small) tube does correspond with such a connected component.

A big tube may also occur as a subcategory of an abelian Ext-finite hereditary category with Serre duality.

Example 2.18. Let $\mathfrak{a}$ be the semi-hereditary category given by the thread quiver

$$\begin{tikzcd} & a_1 \arrow[r] & \cdots \arrow[r] & b_1 \arrow[dr] & \\ a_0 \arrow[ur] \arrow[rrrr] & & & & b_0 \end{tikzcd}$$

The full subcategory $\mathcal{A}$ of $\operatorname{mod} \mathfrak{a}$ given by all representations where the arrow $a_0 \to b_0$ is an isomorphism, is an exact subcategory, closed under extensions, direct summands, and τ-shifts.

It is readily verified that this subcategory is the essential image of an embedding $\operatorname{mod}^{\mathrm{cfp}} \widehat{k\mathbb{Z}^\bullet} \to \operatorname{mod} \mathfrak{a}$, hence $\mathcal{A}$ corresponds to a big tube.

Remark 2.19. The previous example may be interpreted as (being derived equivalent to) a weighted projective line of weight type (∞), thus there is one point with weight ∞; the other points have weight one. This notation is not unambiguous as one could replace $\mathbb{Z}$ by any other (larger) linearly ordered locally discrete set.

Figure 2.2: Sketch of the Auslander-Reiten component of a big tube

Chapter 3

Directed abelian hereditary categories

3.1 Introduction

The material in this chapter is based [53]. We will classify, up to derived equivalence, all abelian Ext-finite directed hereditary categories satisfying Serre duality. Our classification will be stated in terms of the representation theory of certain partially ordered sets.

Below $\mathcal{L}$ will be a totally ordered ordered set in which every element has an immediate successor and an immediate predecessor. These partially ordered sets are easily seen to be of the form $\mathcal{T} \overrightarrow{\times} \mathbb{Z}$ where $\mathcal{T}$ is totally ordered and $\overrightarrow{\times}$ denotes the lexicographically ordered product. We may draw $\mathcal{L}$ as

$$\cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots$$

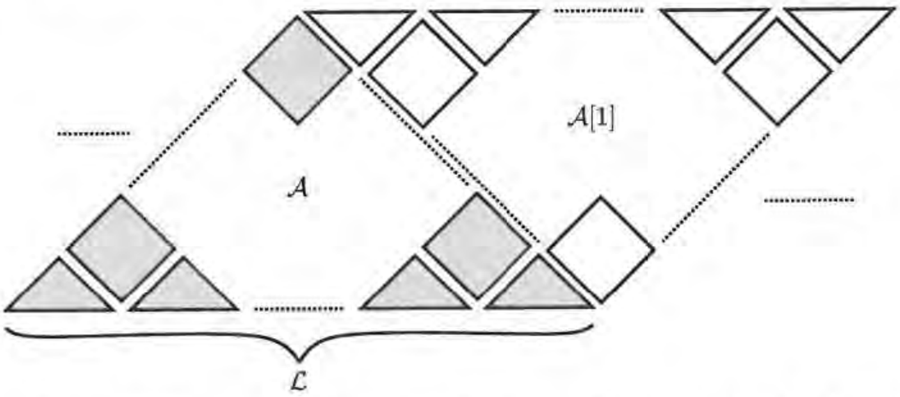
If $\mathcal{L} = \mathbb{Z}$ then $\mathcal{L}$ is a A_∞^∞ quiver. By analogy, we will write $A_\mathcal{L}$ for $\mathcal{L}$. We define $D_\mathcal{L}$ as the union of $A_\mathcal{L}$ with two distinguished objects which are strictly larger than the elements of $A_\mathcal{L}$ but incomparable with each other. We may represent this graphically as

$$\begin{tikzcd}[row sep=small] & \bullet \\ \cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots[\cdots \to \bullet \to \bullet \to \bullet \to \cdots]\cdots \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd}$$

The following is our main result (Theorem 3.44 in the text).

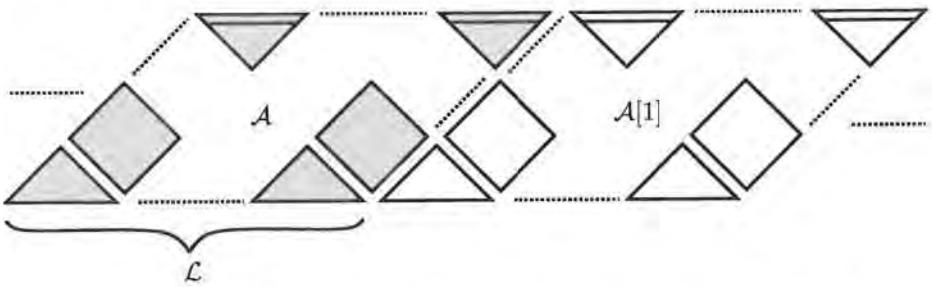
Theorem 3.1. *A connected directed hereditary category $\mathcal{A}$ satisfying Serre duality is derived equivalent to* $\operatorname{mod}^{\mathrm{cfp}} k\mathcal{P}$ *where $\mathcal{P}$ is either a Dynkin quiver, $A_\mathcal{L}$ or $D_\mathcal{L}$.*

The categories occurring in Theorem 3.1 have rather attractive Auslander-Reiten quivers. If $\mathcal{P} = A_{\mathcal{L}}$ then the Auslander-Reiten quivers of $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ and its derived category have the form:



In this picture the triangles and squares are symbolic representations for $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$-components, respectively (see below).

If $\mathcal{P} = D_{\mathcal{L}}$ then the Auslander-Reiten quivers of $\operatorname{mod}^{\mathrm{cfp}} kD_{\mathcal{L}}$ and its derived category have the form:



In this picture the triangles with a double base are symbolic representation for $\mathbb{Z}D_\infty$-components (see below).

The proof of Theorem 3.1 is quite involved and consists of a number of steps which we now briefly sketch.

Step 1. The following result (see Theorem 1.34) is used a various places. Since it does not depend on Serre duality it may be of independent interest.

Theorem 3.2. *Let $\mathcal{A}$ be a directed abelian* Ext*-finite hereditary category. Then* $\dim \operatorname{Ext}^i_{\mathcal{A}}(X,Y) \leq 6$ *for all indecomposable $X, Y \in \mathcal{A}$ and $i = 0, 1$. Furthermore if either of the vector spaces* $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ *or* $\operatorname{Ext}_{\mathcal{A}}(Y,X)$ *is zero then the other is at most 1-dimensional.*

Step 2. For the rest of the proof we assume that $\mathcal{A}$ is a connected directed hereditary category satisfying Serre duality and we put $\mathcal{C} = D^b(\mathcal{A})$. Our first aim is to identify the shapes of the connected components of the Auslander-Reiten quiver of $\mathcal{C}$. Since such a component is a stable translation quiver it must be of the form $\mathbb{Z}B/G$ where B is an oriented tree and G is acting on $\mathbb{Z}B$ ([43]). Using an appropriate generalization of the theory of sectional paths ([5]) and directedness we deduce:

1. All components are *standard*, i.e. all maps are linear combinations of compositions of irreducible ones. In particular all relations can be obtained from the mesh relations.

2. $|G| = 1$ and furthermore one of the following is true: B is Dynkin, $B = A_\infty$, $B = A_\infty^\infty$ or $B = D_\infty$ (these may be characterized as the trees not containing non-Dynkin diagrams).

If $\mathcal{C}$ has a component $\mathbb{Z}B$ with B Dynkin then from connectedness it follows easily that $\mathcal{C} \cong D^b \operatorname{mod} kB$. So below we exclude this case.

Step 3. The next step is to understand the maps between different components. Let $\mathcal{K}$ be an Auslander-Reiten component of $\mathcal{C}$. Since we know all morphisms in $\mathcal{K}$ we may select a *partial tilting set* (§1.9) in $\mathcal{K}$ which generates $\mathcal{K}$. In this way we construct a partially ordered set $\mathcal{P}$ together with an exact embedding $D^b \operatorname{mod} k\mathcal{P} \to \mathcal{C}$ whose essential image contains $\mathcal{K}$ (and its shifts). The fact that this essential image usually also contains other components allows us to obtain information on the interaction between different components.

Step 4. Now we develop the *probing* technique (§3.4.1). Let us say that an indecomposable object S in a $\mathbb{Z}A_\infty$-component is *quasi-simple* if the middle term of the right Auslander-Reiten triangle built on S is indecomposable. Using the technique developed in Step 3 we prove that most indecomposable objects in $\mathcal{C}$ have precisely two distinct quasi-simples mapping to it and these quasi-simples identify the object uniquely.

Step 5. Our next observation is that we if we have a morphism $X \to Y$ in $\mathcal{C}$ we can often determine the quasi-simples mapping to its cone by knowing the quasi-simples mapping to X and Y. This gives us a hold on the triangulated structure of $\mathcal{C}$.

Step 6. Finally we use the information gathered in the previous steps to construct a tilting set in $\mathcal{C}$. For example if $\mathcal{C}$ has no $\mathbb{Z}D_\infty$-components then this tilting set is

$$\{X \in \operatorname{ind}(\mathcal{C}) \mid \operatorname{Hom}_{\mathcal{C}}(S, X) \neq 0\}$$

where S is an arbitrarily chosen quasi-simple in $\mathcal{C}$. The structure of this tilting set allows us finally to complete the proof of Theorem 3.1.

3.2 Preliminary results

Recall that a Krull-Schmidt category $\mathcal{C}$ is *directeddirected category* if all indecomposable objects are *directing*, thus if for all indecomposable $X \in \mathrm{Ob}(\mathcal{C})$, there is no path $X \cong X_0 \to X_1 \to X_2 \to \cdots \to X_n \cong X$ of indecomposable objects X_i, $0 \leq i \leq n$, with $\mathrm{rad}(X_{i-1}, X_i) \neq 0$ for all $1 \leq i \leq n$. As a result, we may conclude that $\mathrm{rad}(X,X) = 0$ and thus $\mathrm{Hom}(X,X) = k$.

Since $\mathrm{Hom}_{D^b\mathcal{A}}(X, Y[-n]) = 0$ for $n > 0$ and $X, Y \in \mathrm{Ob}(\mathcal{A})$, it is easy to check that the category $D^b\mathcal{A}$ is directed as well.

We shall formulate certain restrictions on the Hom-sets of directed categories as a lemma.

Lemma 3.3. *Let $\mathcal{A}$ be a hereditary directed abelian category. Consider two indecomposable objects $X, Y \in \mathrm{Ob}\, D^b\mathcal{A}$. If $\mathrm{Hom}(X,Y) \neq 0$ then $\mathrm{Hom}(X, Y[z]) = 0$ and $\mathrm{Hom}(Y, X[z+1]) = 0$ for all $z \in \mathbb{Z}_0$.*

Proof. Since $\mathcal{A}$ is hereditary and X is indecomposable, we know that X is contained in $\mathcal{A}[z]$ considered as full subcategory of $D^b\mathcal{A}$, for a certain $z \in \mathbb{Z}$. Without loss of generality, assume X to be contained in $\mathcal{A}[0]$. Since $\mathrm{Hom}(X,Y) \neq 0$ and $\mathcal{A}$ is hereditary, we may assume either $Y \in \mathcal{A}[0]$ or $Y \in \mathcal{A}[1]$. If $Y \in \mathcal{A}[0]$, we have $\mathrm{Hom}(X, Y[z]) = 0$ for all $z \in \mathbb{Z}_0$. Indeed, the case $z < 0$ is clear and the case $z > 1$ follows from heredity. Thus assume $\mathrm{Hom}(X, Y[1]) \neq 0$. The triangle $Y \to M \to X \to Y[1]$ built on a nonzero morphism $X \to Y[1]$ yields a path from Y to X, contradicting directedness.

We will continue by proving $\mathrm{Hom}(Y, X[z+1]) = 0$ for all $z \in \mathbb{Z}_0$. This is clear for $z < -1$, follows from directedness for $z = -1$ and from hereditary when $z > -1$.

Note that in the case $X, Y \in \mathcal{A}[0]$ we have only used that there exists a path from X to Y to prove that $\mathrm{Hom}(X, Y[z]) = 0$ for $z \in \mathbb{Z}_0$. The case where $Y \in \mathcal{A}[1]$ is analogous. Then $Y[-1] \in \mathcal{A}$, and the triangle $Y[-1] \to M \to X \to Y$ gives a path from $Y[-1]$ to X. Due to the first part of the proof, this suffices to conclude $\mathrm{Hom}(Y[-1], X[z]) = 0$ and $\mathrm{Hom}(X, Y[-1][z+1]) = 0$ or equivalently, $\mathrm{Hom}(Y, X[z+1]) = 0$ and $\mathrm{Hom}(X, Y[z]) = 0$. □

Next, let $\mathcal{A}$ be a directed k-linear abelian hereditary Ext-finite category, not necessarily satisfying Serre duality. Given a pair of indecomposable objects, X and Y, in $D^b\mathcal{A}$ we will use Theorem 1.34 to find a full and exact subcategory $\mathcal{B}$ of $D^b\mathcal{A}$, such that $X, Y \in \mathcal{B}$ and $\mathcal{B} \cong D^b(\mathrm{mod}\, A)$ for a certain finite dimensional k-algebra A.

Lemma 3.4. *Let $\mathcal{A}$ be a directed Ext-finite hereditary k-linear category and let $X, Y \in \mathrm{ind}\, D^b\mathcal{A}$ such that $\mathrm{Hom}(X,Y) \neq 0$. Consider the triangle*

$$E \to X \otimes Hom(X,Y) \to Y \to E[1] \tag{3.1}$$

built on the canonical map $X \otimes \mathrm{Hom}(X,Y) \to Y$, then $(\mathrm{ind}\, E) \cup \{X\}$ is a partial tilting set.

Proof. In order to ease notation, write $\mathrm{Hom}(X,Y) = V$. We will prove that $(\mathrm{ind}\, E) \cup \{X\}$ is a partial tilting set by applying Hom-functors to triangle (3.1). Out of the long exact sequence given by $\mathrm{Hom}(X,-)$ and directedness we deduce that $\mathrm{Hom}(X, E[z]) = 0$ for all $z \neq 0, 1$. For $z = 1$, consider the following exact sequence

$$\mathrm{Hom}(X, X \otimes V) \to \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X, E[1]) \to \mathrm{Hom}(X, (X \otimes V)[1]).$$

Since the map $\mathrm{Hom}(X, X \otimes V) \to \mathrm{Hom}(X,Y)$ is an isomorphism and since due to directedness $\mathrm{Hom}(X, (X \otimes V)[1]) = 0$, we have $\mathrm{Hom}(X, E[1]) = 0$. Thus, $\mathrm{Hom}(X, E[z]) = 0$, for all $z \in \mathbb{Z}_0$.

Next, the functor $\mathrm{Hom}(-, X)$ yields a long exact sequence from which we easily deduce that $\mathrm{Hom}(E, X[z]) = 0$, for all $z \in \mathbb{Z}_0$.

Finally, we are left to prove that $\mathrm{Hom}(E, E[z]) = 0$, for all $z \in \mathbb{Z}_0$. To this end, we first apply the functor $\mathrm{Hom}(Y, -)$ to triangle (3.1). Using Lemma 3.3 to see that $\mathrm{Hom}(Y, Y[z]) = 0$ and $\mathrm{Hom}(Y, X[z+1]) = 0$ for all $z \in \mathbb{Z}_0$, we may deduce that $\mathrm{Hom}(Y, E[z+1]) = 0$ for all $z \in \mathbb{Z}_0$. One may now readily see that the long exact sequence one acquires by applying $\mathrm{Hom}(-, E)$ to triangle (3.1) yields $\mathrm{Hom}(E, E[z]) = 0$ for all $z \in \mathbb{Z}_0$. This proves the assertion. □

Theorem 3.5. *Let $\mathcal{A}$ be a directed abelian Ext-finite k-linear hereditary category and let $X, Y \in \mathrm{ind}\,\mathcal{A}$, then $\dim \mathrm{Hom}(X,Y) \leq 6$ and $\dim \mathrm{Ext}(X,Y) \leq 6$. If $\mathrm{Ext}(Y,X) = 0$, respectively $\mathrm{Hom}(Y,X) = 0$, then $\dim \mathrm{Hom}(X,Y) \leq 1$, respectively $\dim \mathrm{Ext}(X,Y) \leq 1$.*

Proof. We will work on the derived category $D^b\mathcal{A}$. Possibly by renaming $Y[1]$ to Y, it suffices to prove that $\dim \mathrm{Hom}_{D^b\mathcal{A}}(X,Y) \leq 6$ and $\dim \mathrm{Hom}_{D^b\mathcal{A}}(X,Y) \leq 1$ if $\mathrm{Hom}_{D^b\mathcal{A}}(Y, X[1]) = 0$.

We may assume $\mathrm{Hom}(X,Y) \neq 0$. Lemma 3.3 then yields that $\mathrm{Hom}(X, Y[z]) = 0$ and $\mathrm{Hom}(Y, X[z+1]) = 0$ for all $z \in \mathbb{Z}_0$. If furthermore $\mathrm{Hom}(Y, X[1]) = 0$, then $\{X, Y\}$ is a partial tilting set and, due to Theorem 1.34, we know $A = \mathrm{End}(X \oplus Y)$ is a representation-directed algebra, i.e. $\mathrm{mod}\, A$ is a directed category. From this we may deduce that $\dim \mathrm{Hom}(X,Y) = 1$.

If $\mathrm{Hom}(Y, X[1]) \neq 0$, then we turn our attention to the triangle $E \to X \otimes \mathrm{Hom}(X,Y) \to Y \to E[1]$. Lemma 3.4 yields that $(\mathrm{ind}\, E) \cup \{X\}$ is a partial tilting set. Denote the algebra $\mathrm{End}(E \oplus X)$ by A. Theorem 1.34 then gives a full and exact embedding $i : D^b \mathrm{mod}\, A \to D^b\mathcal{A}$. This shows that A is a representation-directed algebra. Let P and Q be the projective objects of $\mathrm{mod}\, A$ corresponding to E and X, respectively, under i. Since i is exact, we know $R = \mathrm{cone}(P \to Q \otimes \mathrm{Hom}(X,Y))$ corresponds to Y under i.

By applying the functor $\mathrm{Hom}(-, Y)$ to the triangle $E \to X \otimes \mathrm{Hom}(X,Y) \to Y \to E[1]$ one sees that $\mathrm{Hom}(E \oplus X, Y[z]) = 0$ for all $z \in \mathbb{Z}_0$. Indeed, by Lemma 3.3 we have $\mathrm{Hom}(X, Y[z]) = 0$ and then the long exact sequence yields $\mathrm{Hom}(E, Y[z]) = 0$.

Since $\mathrm{Hom}(E \oplus X, Y[z]) = 0$ for all $z \in \mathbb{Z}_0$, we deduce that $\mathrm{Hom}(P \oplus Q, R[z]) = 0$ for all $z \in \mathbb{Z}_0$ and hence we may interpret R as an A-module. Thus $\dim \mathrm{Hom}(Q, R)$

is the number of times the top of Q occurs in the Jordan-Hölder decomposition of R. This, and thus also $\dim \operatorname{Hom}(X, Y)$ is bounded by 6, by [44, 2.4 (9")] $\square$

3.3 Examples of hereditary directed categories

3.3.1 Notations

In this chapter, we will mainly be interested in the category of finitely presented and cofinitely presented representations of a poset $\mathcal{P}$, which we will denote by $\operatorname{mod}^{\mathrm{cfp}} k\mathcal{P}$.

Note that if, in $\operatorname{Mod} k\mathcal{P}$, the finitely generated projectives are cofinitely presented, we have $\operatorname{mod}^{\mathrm{cfp}} k\mathcal{P} \cong \operatorname{mod} k\mathcal{P}$.

We will say that a poset is a *forest* if for all $i, j \in \mathcal{P}$ such that $i < j$ the *interval*

$$[i, j] = \{k \in \mathcal{P} \mid i \leq k \leq j\}$$

is totally ordered. It is clear that a subposet of a forest is a forest.

Now, assume that Q is a poset which is a forest. Then any finite subposet Q_0 of Q is still a forest and hence $\operatorname{Mod} kQ_0$ is hereditary. It follows from Proposition 2.1 that $\operatorname{mod} kQ$ and also $\operatorname{mod}^{\mathrm{cfp}} kQ$ are hereditary abelian categories.

We will now proceed to define two posets of special interest. Recall that a poset is *locally discrete* if no element is an accumulation point. Thus a linearly ordered poset is locally discrete if and only if for each non-maximal element i there exists an immediate successor $i+1$ and for each non-minimal element i there exists an immediate predecessor $i-1$. If $\mathcal{L}$ is a linearly ordered poset, we will denote by $D_{\mathcal{L}}$ the set $\{Q_1, Q_2\} \dot{\cup} \mathcal{L}$ endowed with a poset structure induced by the relation

$$X < Y \Leftrightarrow \begin{cases} X, Y \in \mathcal{L} \text{ and } X <_{\mathcal{L}} Y, \text{ or} \\ X \in \{Q_1, Q_2\} \text{ and } Y \in \mathcal{L} \end{cases}$$

In analogy with the notation used for Dynkin quivers, we will also write $A_{\mathcal{L}}$ for $\mathcal{L}$. For the rest of this section, we will assume $\mathcal{L}$ to be a locally discrete linearly ordered set with no extremal elements, thus not having a maximal nor a minimal element.

3.3.2 The category $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$

These categories have already been considered in [45]. In this section we will recall some results. Note that the closure of the preadditive category $A_{\mathcal{L}}$ under finite direct sums is not a dualizing k-variety, such that we may not apply the results about dualizing k-varieties from Chapter 2.

For all $i, j \in A_{\mathcal{L}}$ with $i \leq j$ we will write

$$A_{i,j} = \operatorname{coker}((kA_{\mathcal{L}})(-, i-1) \to (kA_{\mathcal{L}})(-, j)).$$

It is easily seen that $A_{i,j}$ is cofinitely presented, thus it is an indecomposable object of $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$. Following lemma will classify all objects of $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$.

Lemma 3.6. *The objects of* $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ *are all isomorphic to finite direct sums of modules of the form* $A_{i,j}$.

Proof. We first prove that $A_{i,j}$ is indecomposable. It is easy to see that the number of indecomposable summands of $A_{i,j}$ is at most $\dim \operatorname{Hom}((kA_{\mathcal{L}})(-,j), A_{i,j})$. Applying the functor $(kA_{\mathcal{L}})(-,j)$ to the exact sequence

$$0 \to (kA_{\mathcal{L}})(-,i-1) \to (kA_{\mathcal{L}})(-,j) \to A_{i,j} \to 0$$

yields $\dim \operatorname{Hom}((kA_{\mathcal{L}})(-,j), A_{i,j}) = 1$.

Conversely, let X be an indecomposable object of $\operatorname{mod} kA_{\mathcal{L}}$. Since X is finitely presented in $\operatorname{Mod} kA_{\mathcal{L}}$, we may choose finitely many projectives generating a full subcategory $\mathcal{A}$ of $\operatorname{Mod} kA_{\mathcal{L}}$ containing X, such that the embedding $i : \mathcal{A} \to \operatorname{Mod} kA_{\mathcal{L}}$ is right exact. This subcategory $\mathcal{A}$ is equivalent to $\operatorname{mod} kA_n$ for a certain $n \in \mathbb{N}$, hence

$$X = \operatorname{coker}(f : (kA_n)(-,i) \to (kA_n)(-,j))$$

for certain $i, j \in A_n$. We may assume $f \neq 0$, since otherwise X would be projective in $\mathcal{A}$ and in $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$, and hence will not have a cofinite presentation in $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$. □

Proposition 3.7. *Let* $\mathcal{L}$ *be a locally discrete linearly ordered poset without extremal elements, then the category* $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ *is a connected directed hereditary abelian Ext-finite k-linear category satisfying Serre duality.*

Proof. It follows from the proof of Proposition 2.1 and its dual that $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ is abelian and hereditary since it is a full and exact subcategory of $\operatorname{Mod} kA_{\mathcal{L}}$.

It is easily seen that $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ is connected. We need only check that $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ is directed and satisfies Serre duality.

First we show that $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ is directed. Assume there is a cycle $X \cong X_0 \to X_1 \to X_2 \to \cdots \to X_n \cong X$ of indecomposable objects X_i, $0 \leq i \leq n$, with $\operatorname{rad}(X_{i-1}, X_i) \neq 0$ for all $1 \leq i \leq n$. Each X_i has a finite presentation in $\operatorname{Mod} kA_{\mathcal{L}}$, thus we may choose finitely many (finitely generated) projectives generating a full subcategory $\mathcal{A}$ of $\operatorname{Mod} kA_{\mathcal{L}}$ containing every X_i. Since $\mathcal{A}$ is equivalent to the directed category $\operatorname{mod} kA_n$ for a certain $n \in \mathbb{N}$, this gives the required contradiction.

Since $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ has neither projectives nor injectives (for every indecomposable $A_{i,j}$ there is a non-split epimorphism $A_{i,j+1} \to A_{i,j}$, and a non-split monomorphism $A_{i,j} \to A_{i-1,j}$), we know the existence of a Serre functor on $D^b \operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ is equivalent to the existence of Auslander-Reiten sequences in $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$. Let $A_{i,j}$ be an indecomposable object of $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$. If $i \neq j$, we claim that the exact sequence

$$0 \to A_{i-1,j-1} \to A_{i-1,j} \oplus A_{i,j-1} \to A_{i,j} \to 0 \tag{3.2}$$

is an Auslander-Reiten sequence. To illustrate this, let Y be an indecomposable object of $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ and choose finitely many projectives generating a full subcategory $\mathcal{A}$ of $\operatorname{Mod} kA_{\mathcal{L}}$ containing Y and the exact sequence (3.2). It is clear that $\mathcal{A}$ is equivalent

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \\
& A_{(t,z-2),(t,z)} \arrow[ur] \arrow[dr] & & A_{(t,z+1),(t,z-1)} \arrow[ur] \arrow[dr] & & A_{(t,z),(t,z-2)} \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & A_{(t,z-1),(t,z)} \arrow[ur] \arrow[dr] & & A_{(t,z),(t,z+1)} \arrow[ur] \arrow[dr] & & \cdots \\
& A_{(t,z-1),(t,z-1)} \arrow[ur] & & A_{(t,z),(t,z)} \arrow[ur] & & A_{(t,z+1),(t,z+1)} \arrow[ur] &
\end{tikzcd}$$

Figure 3.1: A $\mathbb{Z}A_\infty$-component of $\operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$

to the category $\operatorname{mod} kA_n$ for a certain $n \in \mathbb{N}$ and that the short exact sequence (3.2) is an almost split exact sequence in $\mathcal{A}$. Hence all morphisms $Y \to A_{i,j}$ and $A_{i-1,j-1} \to Y$ factor through the middle term. This shows that the exact sequence (3.2) is an Auslander-Reiten sequence in $\operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$.

If $i = j$, then one checks the short exact sequence

$$0 \to A_{i-1,i-1} \to A_{i-1,i} \to A_{i,j} \to 0$$

is almost split. $\square$

Finally, we will give the Auslander-Reiten quiver of $\operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$. Therefore, let $\mathcal{T}$ be any linearly ordered set and consider the poset $\mathcal{L} = \mathcal{T} \vec{\times} \mathbb{Z}$ defined by endowing $\mathcal{T} \times \mathbb{Z}$ with the lexicographical ordering. Thus, for all $t, t' \in \mathcal{T}$ and $z, z' \in \mathbb{Z}$, we have

$$(t,z) \leq (t',z') \Leftrightarrow \begin{cases} t < t', \text{ or} \\ t = t' \text{ and } z \leq z' \end{cases}$$

It is readily seen that $\mathcal{L}$ is a locally discrete linearly ordered set with no extremal elements and, conversely, that every such ordered set is constructed in this way.

For every $t \in \mathcal{T}$ the Auslander-Reiten quiver of $\operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$ has a $\mathbb{Z}A_\infty$-component as given in Figure 3.1. With two distinct elements $t < t'$ correspond a $\mathbb{Z}A_\infty^\infty$-component as given in Figure 3.2.

3.3.3 The category $\operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$

This section closely parallels the previous one, although some arguments are slightly more elaborate. For all $i, j \in D_\mathcal{L}$ with $i \leq j$ we will write

$$\begin{aligned}
A_{i,j} &= \operatorname{coker}((kD_\mathcal{L})(-,j) \to (kD_\mathcal{L})(-,i-1)) \\
A_j^1 &= \operatorname{coker}((kD_\mathcal{L})(-,j-1) \to (kD_\mathcal{L})(-,Q_1)) \\
A_j^2 &= \operatorname{coker}((kD_\mathcal{L})(-,j-1) \to (kD_\mathcal{L})(-,Q_2)) \\
B_{i,j} &= \operatorname{coker}((kD_\mathcal{L})(-,Q_1) \oplus (kD_\mathcal{L})(-,Q_2) \to (kD_\mathcal{L})(-,i-1) \oplus (kD_\mathcal{L})(-,j-1))
\end{aligned}$$

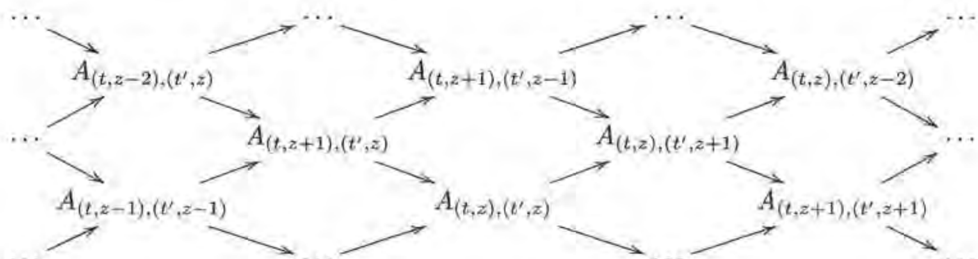


Figure 3.2: A $\mathbb{Z}A_\infty^\infty$-component of $\operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$

where in the definition of $B_{i,j}$ we assume $i \neq j$.

It is easy to see that $A_{i,j}, A_j^1, A_j^2$ and $B_{i,j}$ are also cofinitely presented, hence they are objects of $\operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$. In following lemma we prove that these are all indecomposable objects.

Lemma 3.8. *The objects of* $\operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$ *are all isomorphic to finite direct sums of modules of the form* $A_{i,j}, A_j^1, A_j^2$ *or* $B_{i,j}$.

Proof. Analogue to the proof of Lemma 3.6 □

Proposition 3.9. *Let $\mathcal{L}$ be a locally discrete linearly ordered poset without extremal elements, then the category* $\operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$ *is a connected directed hereditary abelian Ext-finite category satisfying Serre duality.*

Proof. Analogue to the proof of Proposition 3.7. □

As in the case of $\operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$, we will give a description of the Auslander-Reiten components of $\operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$. Let $\mathcal{L} = \mathcal{T} \overrightarrow{\times} \mathbb{Z}$, for a certain linearly ordered poset $\mathcal{T}$. For every $t \in \mathcal{T}$ the Auslander-Reiten quiver of $\operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$ has a $\mathbb{Z}A_\infty$-component and $\mathbb{Z}D_\infty$-component as given in Figures 3.3 and 3.4. With two distinct elements $t < t'$ correspond two $\mathbb{Z}A_\infty^\infty$-component as given in Figures 3.5 and 3.6.

3.4 Auslander-Reiten components of directed categories

In this section, let $\mathcal{A}$ be a connected directed hereditary abelian k-linear Ext-finite category satisfying Serre duality, and write $\mathcal{C} = D^b\mathcal{A}$. Since $\mathcal{A}$ satisfies Serre duality, the Auslander-Reiten quiver of $\mathcal{C}$ is stable and thus, due to [43], we may state that the only possible components of the Auslander-Reiten quivers are of the form $\mathbb{Z}B/G$, where G is an admissible subgroup of $\operatorname{Aut}(\mathbb{Z}B)$ and B is an oriented tree. We wish to

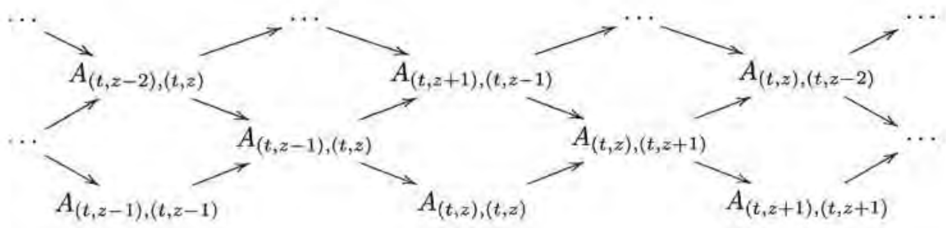


Figure 3.3: A $\mathbb{Z}A_\infty$-component of $\operatorname{mod}^{\text{cfp}} kD_{\mathcal{L}}$

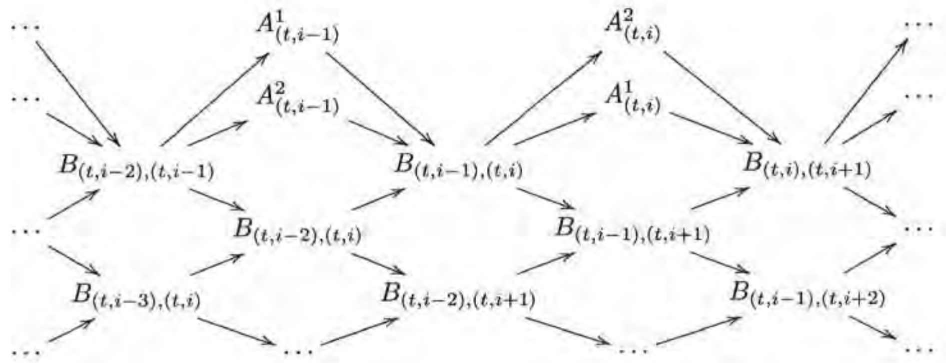


Figure 3.4: A $\mathbb{Z}D_\infty$-component of $\operatorname{mod}^{\text{cfp}} kD_{\mathcal{L}}$

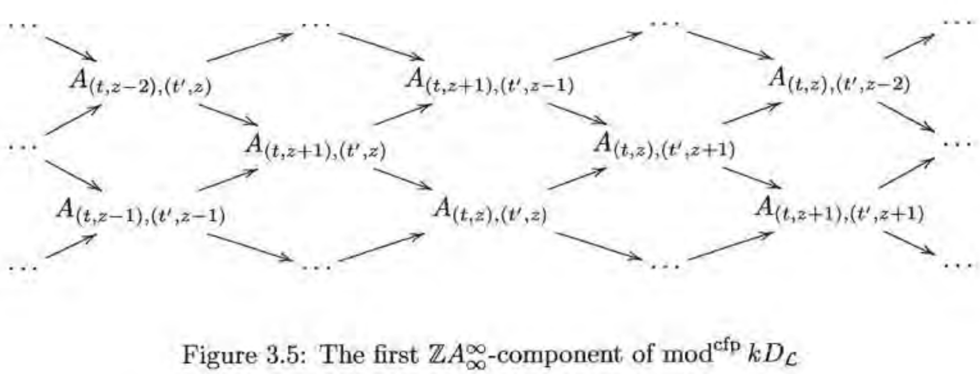


Figure 3.5: The first $\mathbb{Z}A^\infty_\infty$-component of $\operatorname{mod}^{\text{cfp}} kD_{\mathcal{L}}$

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \\
& B_{(t,z),(t',z-2)} \arrow[ur] \arrow[dr] & & B_{(t,z+1),(t',z-1)} \arrow[ur] \arrow[dr] & & B_{(t,z+2),(t',z)} \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & B_{(t,z),(t',z-1)} \arrow[ur] \arrow[dr] & & B_{(t,z+1),(t',z)} \arrow[ur] \arrow[dr] & & \cdots \\
& B_{(t,z-1),(t',z-1)} \arrow[ur] \arrow[dr] & & B_{(t,z),(t',z)} \arrow[ur] \arrow[dr] & & B_{(t,z+1),(t',z+1)} \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] & & \cdots \arrow[ur] & & \cdots \arrow[ur] & & \cdots
\end{tikzcd}
$$

Figure 3.6: The second $\mathbb{Z}A_\infty^\infty$-component of $\operatorname{mod}^{\text{cfp}} kD_{\mathcal{L}}$

show that, in our context, the only possible components are $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$, or $\mathbb{Z}Q$ where Q is Dynkin, and that all such components are *standard* in the sense of §1.10.2, i.e. $\operatorname{rad}^\infty(X,Y) = 0$ when X and Y are contained in the same Auslander-Reiten component. We will need some preliminary results.

Lemma 3.10. *Let A and B be indecomposables of $\mathcal{C}$. If there exists a sectional path $A = A_0 \to \cdots \to A_n = B$, then* $\dim \operatorname{Hom}(A,B) = 1$.

Proof. We start by noting that $\operatorname{Hom}(A,B) \neq 0$ since the composition of irreducible maps from the same sectional sequence is non-zero (Proposition 1.53). Next, we will prove by induction on n that $\operatorname{Hom}(A \oplus A_n, (A \oplus A_n)[z]) = 0$ for all $z \in \mathbb{Z}_0$ and thus, by Lemma 3.5, that $\dim \operatorname{Hom}(A,B) = 1$.

First, assume $n = 1$, or equivalently, $A_1 = B$. There is an irreducible morphism $A \to B$ and hence also a morphism $\tau B \to A$. Therefore, by directedness, $\operatorname{Hom}(A, \tau B) = 0$ and by Serre duality, $\operatorname{Hom}(B, A[1]) = 0$. Also, since $\operatorname{Hom}(A,B) \neq 0$, Lemma 3.3 yields $\operatorname{Hom}(A, B[z]) = 0$ and $\operatorname{Hom}(B, A[z+1]) = 0$ for all $z \in \mathbb{Z}_0$. Combining those two facts gives $\operatorname{Hom}(A \oplus B, (A \oplus B)[z]) = 0$ for all $z \in \mathbb{Z}_0$. Applying Theorem 3.5 then yields the assertion in case $n = 1$.

Next, assume the assertion has been proven for $n \in \{1, 2, \ldots, k-1\}$, we wish to prove the case $n = k$. First of all note that since $\operatorname{Hom}(A \oplus A_{k-1}, (A \oplus A_{k-1})[z]) = 0$ for all $z \in \mathbb{Z}_0$ we have, using Serre duality, $\dim \operatorname{Hom}(A_{k-1}, A[1]) = \dim \operatorname{Hom}(A, \tau A_{k-1}) = 0$. Considering the Auslander-Reiten triangle $\tau A_{k-1} \to M \to A_{k-1} \to \tau A_{k-1}[1]$, and using the fact that $\dim \operatorname{Hom}(A, A_{k-1}) = 1$, one has $\dim \operatorname{Hom}(A, M) = 1$. Since M has both A_{k-2} and τA_k as directs summands and it has already been proven that $\dim \operatorname{Hom}(A, A_{k-2}) = 1$, it follows easily that $\operatorname{Hom}(A, \tau A_k) = 0$, and thus $\operatorname{Hom}(A_k, A[1]) = 0$. It has already been noted that $\operatorname{Hom}(A, A_k) \neq 0$ since they lie in the same sectional sequence and thus, by Lemma 3.3, that $\operatorname{Hom}(A_k, A[z]) = 0$ and $\operatorname{Hom}(A, A_k[z+1]) = 0$ for all $z \in \mathbb{Z}_0$. Combining this with the earlier proven $\operatorname{Hom}(A_k, A[1]) = 0$, we arrive at $\operatorname{Hom}(A \oplus A_k, (A \oplus A_k)[z]) = 0$ for all $z \in \mathbb{Z}_0$. Finally, we may invoke Theorem 3.5 to conclude that $\dim \operatorname{Hom}(A, A_k) = 1$. $\square$

Proposition 3.11. *Let A and B be indecomposable objects of $\mathcal{C}$. If there exists a sectional path $A = A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} A_n = B$, then* $\dim \operatorname{Hom}(A,B) = 1$ *and* $\operatorname{rad}^\infty(A,B) = 0$. *Also, there is only one sectional path from A to B in the Auslander-Reiten quiver.*

Proof. It has already been proven in Lemma 3.10 that $\dim \operatorname{Hom}(A,B) = 1$. Now, if $\operatorname{rad}^\infty(A,B) \neq 0$, then $\dim \operatorname{rad}^\infty(A,B) = 1$. Let $\tau B \to M \to B \to \tau B[1]$ be the Auslander-Reiten triangle built on B. A non-zero morphism $f \in \operatorname{rad}^\infty(A,B)$ should factor through M or, more precisely, through the direct summand A_{n-1} of M. Indeed, since Lemma 3.10 implies $\operatorname{Hom}(A, \tau B) = 0$ we easily obtain $\dim \operatorname{Hom}(A,M) = 1$ and since $\dim \operatorname{Hom}(A, A_{n-1}) = 1$ we may write $f = \alpha_{n-1} \circ f_{n-1}$, with $f_{n-1} \in \operatorname{rad}^\infty(A, A_{n-1})$. By iteration, one has that $f = \alpha_{n-1} \circ \ldots \circ \alpha_1 \circ f_1$, with $f_1 \in \operatorname{rad}^\infty(A, A_1)$, clearly a contradiction. Thus $\operatorname{rad}^\infty(A,B) = 0$.

It follows from Proposition 1.53 that there can be at most one sectional path between two indecomposables. □

We will now discuss the form of the components that can occur in the Auslander-Reiten quiver of the category $\mathcal{C}$. Recall from [43] that a stable component $\mathcal{K}$ from the Auslander-Reiten quiver of $\mathcal{C}$ is covered by $\pi : \mathbb{Z}B \to \mathcal{K}$, where B is defined as follows: fix a vertex X from $\mathcal{K}$, then the vertices of B are defined to be all (finite, non-trivial) sectional paths of $\mathcal{K}$ starting at X, and there is an arrow in B from the sectional path $X \to \cdots \to Y$ to the sectional path $X \to \cdots Y \to Z$. With these definitions, it is clear that B is a tree with a unique source. There also is a morphism $f : B \to \mathcal{K}$ by mapping a sectional path $X \to \cdots \to Y$ to Y. This morphism f extends to the covering $\pi : \mathbb{Z}B \to \mathcal{K}$ of translation quivers given by

$$(z, X \to \cdots \to Y) \mapsto \tau^{-z} Y.$$

In the following lemma, we will prove that the map π is injective, such that $\mathbb{Z}B \cong \mathcal{K}$.

Lemma 3.12. *Every component of the Auslander-Reiten quiver of $\mathcal{C}$ is isomorphic to $\mathbb{Z}B$, as stable translation quivers, for a certain oriented tree B with a unique source.*

Proof. As stated before, we need only to prove that the map

$$\pi : \mathbb{Z}B \to \mathcal{K} : (z, X \to \cdots \to Y) \mapsto \tau^{-z} Y$$

is injective. Consider $(z, X \to \cdots \to Y), (z', X \to \cdots \to Y') \in \mathbb{Z}B$. Seeking a contradiction, assume that $(z, X \to \cdots \to Y) \neq (z', X \to \cdots \to Y')$ and $\pi(z, X \to \cdots \to Y) = \pi(z', X \to \cdots \to Y')$, thus $\tau^{-z} Y = \tau^{-z'} Y'$. Thus we assume there to be in $\mathcal{K}$ two sectional paths starting in the same vertex, and ending in the same τ-orbit.

We will consider the sectional paths

$$X = A_0 \to A_1 \to \cdots \to A_{n-1} \to A_n = Y$$

and

$$X = B_0 \to B_1 \to \cdots \to B_{m-1} \to B_m = Y' = \tau^{z'-z} Y$$

where we may without loss of generality assume that $A_i \neq \tau^k B_j$ for $1 \leq i \leq n-1$, for $1 \leq j \leq m-1$ and for all $k \in \mathbb{Z}$.

We will consider two separate cases. First, assume that $z' - z \geq n$. In that case, we have a path from $\tau^{z'-z}Y$ to $\tau^n Y$ and a path

$$\tau^n Y \to \tau^{n-1}A_{n-1} \to \tau^{n-2}A_{n-2} \to \cdots \to A_0 = B_0 \to B_m = Y' = \tau^{z'-z}Y$$

contradicting directedness.

If $z' - z < n$, then we find two different sectional paths

$$X = B_0 \to \cdots \to B_m = \tau^{z'-z}Y \to \tau^{z'-z-1}A_{n-1} \to \cdots \to \tau A_{n-(z'-z)+1} \to A_{n-(z'-z)}$$

and

$$X = A_0 \to A_1 \to \cdots \to A_{n-(z'-z)}$$

from X to $A_{n-(z-z')}$ contradicting Proposition 3.11. □

We are now ready to prove the main theorem of this section.

Theorem 3.13. *Let $\mathcal{A}$ be a directed hereditary abelian k-linear Ext-finite category satisfying Serre duality. Each component of the Auslander-Reiten quiver of $\mathcal{C} = D^b\mathcal{A}$ is both*

1. *standard, and*

2. *of the form $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$ or $\mathbb{Z}Q$, where Q is a quiver of Dynkin type.*

We will split the proof of this theorem over the next two lemmas. In the next lemma, we will denote by $d(a,b)$ the usual graph-theoretical distance between vertices a and b.

Lemma 3.14. *Each component of the Auslander-Reiten quiver of $\mathcal{C}$ is standard.*

Proof. Let X and Y be two indecomposable objects in an Auslander-Reiten component $\mathcal{K}$ of $\mathcal{C}$. In order to prove $\operatorname{rad}^\infty(X,Y) = 0$ we will write $\mathcal{K}$ as $\mathbb{Z}B$ such that X corresponds to $(0,b)$ where b is the source of the tree B. We will consider two cases.

The first case is where Y has coordinates (n, v_Y) with $n \geq 0$ and $v_Y \in B$. If $n = 0$ then $\operatorname{rad}^\infty(X,Y) = 0$ as a consequence of Proposition 3.11. Now assume $n > 0$. If $\operatorname{rad}^\infty(X,Y) \neq 0$, then consider the Auslander-Reiten triangle $\tau Y \to M \to Y \to \tau Y[1]$. There is at least one indecomposable summand Y' of M such that $\operatorname{rad}^\infty(X,Y') \neq 0$. The coordinates of Y' are either $(n, v_{Y'})$ where $d(b, v_{Y'}) = d(b, v_Y) - 1$ or $(n-1, v_{Y'})$ where $d(b, v_{Y'}) = d(b, v_Y) + 1$. Since $d(b, v_Y)$ is finite, iteration gives $\operatorname{rad}^\infty(X,Z) \neq 0$ for a certain Z with coordinates $(0, v_Z)$. This contradicts Proposition 3.11.

The last case is where Y has coordinates $(-n, v_Y)$ with $n > 0$. We will proceed by induction on n to prove that $\operatorname{Hom}(X,Y) = 0$. First, we consider $n = 1$, thus let Y have coordinates $(-1, v_Y)$. We will use a second induction argument on $d = d(b, v_Y)$.

If $d = 0$, then $b = v_Y$ and there is a path from Y to X. Directedness then implies $\mathrm{Hom}(X,Y) = 0$. Now, assume $d \geq 1$. Choose a direct predecessor v_Z of v_Y. Let $Z \in \mathcal{K}$ be the indecomposable object corresponding to the coordinates $(-1, v_Z)$, then Y is a direct summand of M where M is defined by the Auslander-Reiten triangle $Z \to M \to \tau^{-1}Z \to Z[1]$.

By the induction on d above we know that $\mathrm{Hom}(X,Z) = 0$ and by Proposition 3.11 that $\dim \mathrm{Hom}(X, \tau^{-1}Z) = 1$ so we infer that $\dim \mathrm{Hom}(X,M) = 1$ if $X \ncong \tau^{-1}Z$, and $\dim \mathrm{Hom}(X,M) = 0$ if $X \cong \tau^{-1}Z$. Thus there is at most one indecomposable direct summand X' of M such that $\mathrm{Hom}(X,X') \neq 0$. This needs to be an indecomposable object lying on a sectional path from X to $\tau^{-1}Z$, hence $X' \ncong Y$ since Y has coordinates $(-1, v_Y)$. We conclude $\mathrm{Hom}(X,Y) = 0$.

We continue with the induction on n, thus let Y have coordinates $(-n, v_Y)$ with $n > 1$. Also in this case, we will use a second induction argument on $d = d(b, v_Y)$. As above, if $d = 0$, then $b = v_Y$ and there is a path from Y to X. Directedness then implies $\mathrm{Hom}(X,Y) = 0$. Thus we may assume $d \geq 1$. Let v_Z be a direct predecessor of v_Y in the tree B and let $Z \in \mathcal{K}$ be the indecomposable object corresponding to the coordinates $(-n, v_Z)$. Again, Y is a direct summand of M where M is defined by the Auslander-Reiten triangle $Z \to M \to \tau^{-1}Z \to Z[1]$.

By induction on d and n we know $\mathrm{Hom}(X,Z) = 0$ and $\mathrm{Hom}(X, \tau^{-1}Z) = 0$, respectively, and hence also $\mathrm{Hom}(X,M) = 0$. We conclude $\mathrm{Hom}(X,Y) = 0$. □

Lemma 3.15. *Each component of the Auslander-Reiten quiver of $\mathcal{C}$ is of the form $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$ or $\mathbb{Z}Q$, where Q is a quiver of Dynkin type.*

Proof. This is an immediate consequence of Proposition 3.16. □

Proposition 3.16. *Let $\mathcal{K}$ be a component of the Auslander-Reiten quiver of $\mathcal{C}$, and let Q be a section of $\mathcal{K}$. Then the vertices of Q form a partial tilting set.*

Proof. Let X and Y be two indecomposable objects from the section Q. We must show that $\mathrm{Hom}(X, Y[z]) = 0$ for $z \in \mathbb{Z}_0$. Without loss of generality, assume $X \in \mathrm{Ob}\,\mathcal{A}[0]$. It is clear that there are $i, j \in \mathbb{N}$ such that there is a sectional path from $\tau^i Y$ to X and from X to $\tau^{-j}Y$ and thus, by Proposition 3.11 and the fact that $\mathcal{A}$ is hereditary, we may conclude $\tau^i Y \in \mathrm{Ob}\,\mathcal{A}[-1]$ or $\tau^i Y \in \mathrm{Ob}\,\mathcal{A}[0]$, and $\tau^{-j}Y \in \mathrm{Ob}\,\mathcal{A}[0]$ or $\tau^{-j}Y \in \mathrm{Ob}\,\mathcal{A}[1]$. Since there are paths from $\tau^{-j}Y$ to Y and from Y to $\tau^i Y$, we may infer that $Y \in \mathcal{A}[-1]$, $Y \in \mathcal{A}[0]$, or $Y \in \mathcal{A}[1]$. Hence $\mathrm{Hom}(X, Y[z]) = 0$ for $z < -1$ and $z > 2$. We will proceed to show that $\mathrm{Hom}(X, Y[z]) = 0$ for $z \in \{-1, 1, 2\}$.

Therefore, we will show there exists an indecomposable $Z \in \mathcal{K}$ such that there are sectional paths from Z to both X and Y. Indeed, let $n \in \mathbb{N}$ be the smallest natural number such that there is a path

$$\tau^n Y = A_0 \to A_1 \to \cdots \to A_m = X.$$

Note that such a path is necessarily sectional and $n \leq m$. By turning the first n arrows one gets a path from A_n to Y and a path from A_n to X which are sectional by minimality of n. Hence let $Z = A_n$.

First, we will prove $\mathrm{Hom}(X, Y[-1]) = 0$. Considering the non-split triangle $Y[-1] \to M \to Z \to Y$ we see there is a path from $Y[-1]$ to Z. If $\mathrm{Hom}(X, Y[-1]) \neq 0$, then there would be a path $Z \to X \to Y[-1]$, contradicting directedness.

Next we will consider $\mathrm{Hom}(X, Y[1]) = 0$. We have $\mathrm{Hom}(X, Y[1]) \cong \mathrm{Hom}(Y, \tau X)^*$ where the last is shown to be 0 as in the proof of Lemma 3.14.

Finally, we will prove that $\mathrm{Hom}(X, Y[2]) = 0$. If $\mathrm{Hom}(X, Y[2]) \neq 0$, then also $\mathrm{Hom}(Y[1], \tau X) \neq 0$. Since Z admits sectional paths to both X and Y, we know that $\mathrm{Hom}(Z, X)^* \cong \mathrm{Hom}(\tau X, \tau^2 Z[1]) \neq 0$ and $\mathrm{Hom}(Z, Y) \cong \mathrm{Hom}(Z[1], Y[1]) \neq 0$. This gives morphisms $Z[1] \to Y[1] \to \tau X \to \tau^2 Z[1]$, contradicting directedness. □

Having proved Theorem 3.13, we now turn our attention to the possible shapes of the Auslander-Reiten components. First we will discuss a tool we will be developing and using in the next sections.

3.4.1 Probing

In this section, we shall show it is possible to understand objects through the collection of quasi-simple object mapping to them. We will refer to this method as *probing*. The main results are Propositions 3.17 and 3.18 and are being proved on a case by case study of the different form of the Auslander-Reiten components; this will be done in §3.4.2, §3.4.3, §3.4.4, and §3.4.5.

As usual, $\mathcal{A}$ is a connected directed abelian hereditary k-linear Ext-finite category satisfying Serre duality, and we write $\mathcal{C} = D^b\mathcal{A}$. We have proven in Theorem 3.13 that the only occurring Auslander-Reiten quivers are of the form $\mathbb{Z}Q$ where Q is either A_∞, A_∞^∞, D_∞ or a Dynkin quiver. In Proposition 3.21 will be proven that if $\mathcal{C}$ has an Auslander-Reiten component $\mathbb{Z}Q$ where Q is a Dynkin quiver, then this is the only component of $\mathcal{C}$. Since we are interested in the connection between different components, we will exclude such Auslander-Reiten components from this section.

We will start our discussion with a definition.

Let $\mathcal{U}$ and $\mathcal{U}'$ be Auslander-Reiten components. We will say $\mathcal{U}$ *maps to* $\mathcal{U}'$ if there is an object $X \in \mathcal{U}$ and $Y \in \mathcal{U}'$ such that $\mathrm{Hom}(X, Y) \neq 0$.

It will turn out that the $\mathbb{Z}A_\infty$-components, also called *wings*, are the building blocks of the category $\mathcal{C}$. We consider the following map.

$$\begin{array}{rcl} \phi^{\text{comp}} : \{\text{components of } \mathcal{C}\} & \to & \{\text{sets of wings of } \mathcal{C}\} \\ \mathcal{U} & \mapsto & \{\mathcal{W} \mid \mathcal{W} \text{ is a wing that maps to } \mathcal{U}\} \end{array}$$

We now prove some properties of ϕ^{comp}.

Proposition 3.17. *The map ϕ^{comp} is injective. Also*

- *if $\mathcal{U}$ is a $\mathbb{Z}A_\infty$-component, then $\phi^{comp}(\mathcal{U}) = \{\mathcal{U}[-1], \mathcal{U}\}$,*
- *if $\mathcal{U}$ is a $\mathbb{Z}A_\infty^\infty$-component, then $\phi^{comp}(\mathcal{U}) = \{\mathcal{V}, \mathcal{W}\}$, with $\mathcal{V} \neq \mathcal{W}[z]$ for all $z \in \mathbb{Z}$,*

- *if $\mathcal{U}$ is a $\mathbb{Z}D_\infty$-component, then $\phi^{comp}(\mathcal{U})$ consists of a single wing.*

Proof. If $\mathcal{U}$ is a $\mathbb{Z}A_\infty$-component, a $\mathbb{Z}A_\infty^\infty$-component, or a $\mathbb{Z}D_\infty$-component, then the form of ϕ^{comp} is a direct consequence of Propositions 3.23 and 3.25, Propositions 3.26 and 3.30, or Propositions 3.32 and 3.36, respectively. Injectivity of ϕ^{comp} follows from Propositions 3.25, 3.31 and 3.37. □

We now turn our attention from the components to the objects. Again, we start with a definition.

Quasi-simple objects will be used to, in a certain sense, give coordinates to objects of $\mathcal{C}$ much like wings can be used as coordinates for components. We define the function

$$\begin{aligned} \phi^{\mathrm{obj}} : \{\text{indecomposables of } \mathcal{C}\} &\to \{\text{sets of quasi-simples of } \mathcal{C}\} \\ X &\mapsto \{S \mid S \text{ is a quasi-simple that maps non-zero to } X\} \end{aligned}$$

Proposition 3.18. *Let X be an indecomposable object lying in an Auslander-Reiten component $\mathcal{U}$. We have the following properties.*

1. *For all $\mathcal{W} \in \phi^{comp}(\mathcal{U})$, there is an $S \in \phi^{obj}(X)$ such that $S \in \mathcal{W}$.*
2. *The set $\phi^{obj}(X)$ consists of two elements, except if X is a peripheral object from a $\mathbb{Z}D_\infty$-component, then $\phi^{obj}(X)$ has only one element.*
3. *The fiber of $\phi^{obj}(X)$ consists of one element, except when X is a peripheral object in a $\mathbb{Z}D_\infty$-component, then the fiber of $\phi^{obj}(X)$ consists of two elements.*
4. *If $S \in \phi^{obj}(X)$ then* $\dim \mathrm{Hom}(S, X) = 1$.
5. *If S is a quasi-simple and $f : S \to X$ a non-zero non-invertible morphism, then the map g in the triangle $S \xrightarrow{f} X \xrightarrow{g} C \longrightarrow S[1]$ is irreducible, except if X is a peripheral object from a $\mathbb{Z}D_\infty$-component.*

Proof. 1. This follows from Propositions 3.23, 3.26 and 3.32.

2. First assume that X is not a peripheral object from a $\mathbb{Z}D_\infty$-component. Propositions 3.23, 3.26 and 3.32 yield that there are at least two different quasi-simple objects mapping non-zero to X. Propositions 3.25, 3.30 and 3.36 imply that these are unique.

   If X is a peripheral object from a $\mathbb{Z}D_\infty$-component, then Proposition 3.32 yields that there is at least one quasi-simple object mapping non-zero to X. Finally, Proposition 3.36 then shows this quasi-simple is unique.

3. Again, first assume X is not a peripheral object of a $\mathbb{Z}D_\infty$-component.

   If $\mathcal{U}$ is a $\mathbb{Z}A_\infty$-component, then Proposition 3.17 yields $\phi^{\mathrm{comp}}(\mathcal{U}) = \{\mathcal{U}[-1], \mathcal{U}\}$. We may infer from (1) and (2) that $\phi^{\mathrm{obj}}(X) = \{S, T\}$ with $S \in \mathcal{U}[-1]$ and

$T \in \mathcal{U}$. Proposition 3.23 now yields that the restricted function $\phi^{\mathrm{obj}}|_{\mathbb{Z}A_\infty}$ is injective.

If $\mathcal{U}$ is a $\mathbb{Z}A_\infty^\infty$-component, then Proposition 3.17 yields $\phi^{\mathrm{comp}}(\mathcal{U}) = \{\mathcal{V}, \mathcal{W}\}$, with $\mathcal{V} \neq \mathcal{W}[z]$ for all $z \in \mathbb{Z}$. Now, (1) and (2) yield that $\phi^{\mathrm{obj}}(X) = \{S, T\}$ with $S \in \mathcal{V}$ and $T \in \mathcal{W}$. By Proposition 3.26 we see that the restricted function $\phi^{\mathrm{obj}}|_{\mathbb{Z}A_\infty^\infty}$ is injective.

If $\mathcal{U}$ is a $\mathbb{Z}D_\infty$-component, then Proposition 3.17 yields $\phi^{\mathrm{comp}}(\mathcal{U}) = \{\mathcal{V}\}$ and (1) and (2) imply that $\phi^{\mathrm{obj}}(X) = \{S, T\}$ with $S, T \in \mathcal{V}$. We may now use Proposition 3.32 to see that the restricted function $\phi^{\mathrm{obj}}|_{\mathbb{Z}D_\infty}$ is injective.

We may now conclude that the fiber of $\phi^{\mathrm{obj}}(X)$ consists of only one object, X.

Now, assume X is a peripheral object of a $\mathbb{Z}D_\infty$-component $\mathcal{U}$. We have already shown that there is a unique quasi-simple object, S, such that $\mathrm{Hom}(S, X) \neq 0$ and $S \in \mathcal{W}$ where $\phi^{\mathrm{comp}}(\mathcal{U}) = \{\mathcal{W}\}$. Since ϕ^{comp} is injective, $\mathcal{U}$ is the only $\mathbb{Z}D_\infty$-component where $\mathcal{W}$ maps to. Proposition 3.32 now yields that the fiber of $\phi^{\mathrm{obj}}(X) = S$ consists of exactly two objects, both peripheral objects of $\mathcal{U}$.

(4 & 5) These are immediate consequences of Propositions 3.23, 3.26 and 3.32.

□

Proposition 3.18(5) will be used in combination with following lemma from [5], adapted to the triangulated case.

Lemma 3.19. *Consider the triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$ *and a morphism* $h : Z' \to Z$. *If* f *is irreducible, then* g *factors through* h *or vice versa, thus there exists a* $t : Z' \to Y$ *such that* $h = gt$ *or an* $s : Y \to Z'$ *such that* $g = hs$.

Proof. Consider the morphism of triangles

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & C \arrow[r, "v"] \arrow[d, "w"] & Z' \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r] & X[1]
\end{tikzcd}$$

Because f is irreducible, we know that u is split mono (and thus v split epi) or w split epi. In the former case there exists a $t : Z' \to Y$ such that $h = gt$ while in the latter there is a morphism $s : Y \to Z'$ such that $g = hs$. □

Example 3.20. Let $\mathcal{L} = \mathbb{Z}$ and consider the category $\mathcal{A} = \mathrm{mod}^{\mathrm{cfp}}\, k(A_\mathcal{L})$ as in §3.3.2. As usual, we write $\mathcal{C} = D^b\mathcal{A}$. Consider a non-zero $f \in \mathrm{Ext}(A_{0,3}, A_{-1,1})$ and the corresponding triangle $A_{-1,1} \to M \to A_{0,3} \xrightarrow{f} A_{-1,1}[1]$. We will probe M to identify the direct summands of M.

One has $\phi^{\mathrm{obj}}(A_{-1,1}) = \{A_{-1,-1}, A_{2,2}[-1]\}$ and $\phi^{\mathrm{obj}}(A_{0,3}) = \{A_{0,0}, A_{4,4}[-1]\}$ and may easily verify that the triangle extended with all quasi-simple objects is

$$\begin{tikzcd}
A_{0,0}[-1] \arrow[d] & A_{-1,-1} \arrow[r, equal] \arrow[d] & A_{-1,-1} \arrow[dr] & & A_{0,0} \arrow[r, equal] \arrow[dl] & A_{0,0} \arrow[d] & A_{-1,-1}[1] \arrow[d] \\
A_{0,3}[-1] \arrow[r] & A_{-1,1} \arrow[rr] & & M \arrow[rr] & & A_{0,3} \arrow[r, "f"] & A_{-1,1}[1] \\
A_{4,4}[-2] \arrow[u] & A_{2,2}[-1] \arrow[r, equal] \arrow[u] & A_{2,2}[-1] \arrow[ur] & & A_{4,4}[-1] \arrow[r, equal] \arrow[ul] & A_{4,4}[-1] \arrow[u] & A_{2,2} \arrow[u]
\end{tikzcd}$$

We know that $\mathcal{C}$ does not have any $\mathbb{Z}D_\infty$-components, and thus, by Proposition 3.18(2), that all objects have exactly two quasi-simples mapping non-zero to them. Since there are exactly four quasi-simples mapping to M, we may conclude that M has exactly two direct summands, M_1 and M_2.

Using Proposition 3.18(1), we may infer that there are two possibilities, either $\phi^{\mathrm{obj}}(M_1) = \{A_{-1,-1}, A_{2,2}[-1]\}$ and $\phi^{\mathrm{obj}}(M_2) = \{A_{0,0}, A_{4,4}[-1]\}$, or $\phi^{\mathrm{obj}}(M_1) = \{A_{0,0}, A_{2,2}[-1]\}$ and $\phi^{\mathrm{obj}}(M_2) = \{A_{-1,-1}, A_{4,4}[-1]\}$.

In the former case, Proposition 3.18(3) yields $M_1 \cong A_{-1,1}$ and $M_2 \cong A_{0,3}$. Lemma 1.23 then implies that there exists a non-zero morphism from $A_{-1,1}$ to M_1. This morphism is necessarily an isomorphism; we conclude that $A_{-1,1} \to M$ is a split monomorphism and hence that $f = 0$. A contradiction.

In the latter case, 3.18(3) yields $M_1 \cong A_{0,1}$ and $M_2 \cong A_{-1,3}$.

3.4.2 A $\mathbb{Z}Q$-component with Q a Dynkin quiver

We first consider a category $\mathcal{C} = D^b\mathcal{A}$ whose Auslander-Reiten quiver has a $\mathbb{Z}Q$-component where Q is a Dynkin quiver. Note that the categories $\operatorname{mod} kQ$ and $\operatorname{mod}^{\mathrm{cfp}} kQ$ are equivalent.

Following proposition shows we may exclude these components from our further discussion of the other components.

Proposition 3.21. *Let $\mathcal{A}$ be a connected directed hereditary abelian k-linear Ext-finite category satisfying Serre duality. Assume the Auslander-Reiten quiver of $\mathcal{C} = D^b\mathcal{A}$ has a $\mathbb{Z}Q$-component with Q a Dynkin quiver, then $\mathcal{C} \cong D^b \operatorname{mod} kQ$.*

Proof. Proposition 3.16 yields that the section Q in the $\mathbb{Z}Q$-component $\mathcal{K}$ is a partial tilting set. Using the exactness of the Serre functor F, it is easily seen that the full and exact embedding $\Delta : D^b \operatorname{mod} kQ \to \mathcal{C}$ given by Theorem 1.34 commutes with Serre duality. Thus those indecomposable objects in the essential image of Δ are exactly those whose isomorphism class lie in $\mathcal{K}$. We claim that for all X in the essential image of Δ and for all $Y \in \operatorname{Ob}\mathcal{C}$, we have $\operatorname{rad}^\infty(X, Y) = 0$. Indeed, if $\operatorname{rad}^\infty(X, Y) \neq 0$ then, for all $n \in \mathbb{N}$, there would be an $X_n \in \mathcal{K}$ such that $\operatorname{rad}^n(X, X_n) \neq 0$. Yet, this is not true in $D^b \operatorname{mod} kQ$.

Hence, since $\mathcal{A}$ is connected, $\mathcal{C}$ can consist of only one component. We conclude $\mathcal{C} \cong D^b \operatorname{mod} kQ$. □

We give a further result in this context.

Proposition 3.22. *Let $\mathcal{A}$ be a hereditary category. If $D^b\mathcal{A} \cong D^b \operatorname{mod} kQ$ where Q is a Dynkin quiver, then $\mathcal{A} \cong \operatorname{mod} kQ'$ where the quiver Q' is a tilt of Q.*

Proof. First note that $D^b(\operatorname{mod}^{\mathrm{cfp}} kQ)$ and thus also $D^b\mathcal{A}$ are directed and Ext-finite. Since $\mathcal{A}$ and $\operatorname{mod}^{\mathrm{cfp}} kQ$ are hereditary, we have

$$\begin{aligned}
\operatorname{card}(\operatorname{ind} \mathcal{A}) &= \operatorname{card}\{T\text{-orbits in } \operatorname{ind} D^b\mathcal{A}\} \\
&= \operatorname{card}\{T\text{-orbits in } \operatorname{ind} D^b(\operatorname{mod}^{\mathrm{cfp}} Q)\} \\
&= \operatorname{card}(\operatorname{ind} \operatorname{mod}^{\mathrm{cfp}}(Q))
\end{aligned}$$

Hence $\operatorname{ind} \mathcal{A}$ is finite. We will now show this implies that $\mathcal{A}$ has enough projectives.

Indeed, let $X \in \operatorname{Ob}(\mathcal{A})$. If X is not projective, there exists an object $M = \oplus_i M_i$ where M_i is indecomposable for all i, such that there is a non-split epimorphism $M \to X$. Since $\mathcal{A}$ is directed and $\operatorname{ind} \mathcal{A}$ is finite, every sequence of non-split epimorphisms $\cdots \to M \to X$ needs to be finite and we deduce the existence of a projective object P that admits a non-split epimorphism $P \to X$.

Consider the object $P = \oplus_j P_j$ where P_j ranges through all projectives of $\operatorname{ind} \mathcal{A}$. We see that P is a generator, and hence $\mathcal{A} \cong \operatorname{mod}(A)$ with $A = \operatorname{End}(P)$.

Since A is hereditary and of finite representation type, A needs to be Morita equivalent to the path algebra of a Dynkin quiver Q'.

Finally, note that the Auslander-Reiten quiver of $D^b\mathcal{A}$ and $D^b \operatorname{mod}^{\mathrm{cfp}}(Q)$, are equal to $\mathbb{Z}Q'$ and $\mathbb{Z}Q$, respectively, hence Q' is a tilt of Q. □

3.4.3 A $\mathbb{Z}A_\infty$-component

Proposition 3.23. *Let Q be the quiver*

$$\begin{tikzcd}
0 & & 2 & & 4 & \\
& 1 \arrow[ul] \arrow[ur] & & 3 \arrow[ul] \arrow[ur] & & {} \arrow[ul, dotted, no head]
\end{tikzcd}$$

and let $\mathcal{K}$ be a $\mathbb{Z}A_\infty$-component of $\mathcal{C}$. The smallest full and exact subcategory of $\mathcal{C}$ containing $\mathcal{K}$ is equivalent to $D^b \operatorname{mod} kQ$, and the embedding $\Delta : D^b \operatorname{mod} kQ \to \mathcal{C}$ commutes with Serre duality. Hence Δ maps Auslander-Reiten components to Auslander-Reiten components.

Proof. Consider within $\mathcal{K}$ the quiver Q as in Figure 3.7. We will denote the indecomposable corresponding to the vertex i of Q by P_i.

Invoking Proposition 3.16 and Theorem 1.34, we may consider a full and exact embedding $\Delta : D^b \operatorname{mod} kQ \to \mathcal{C}$ which we claim to commute with the Serre functor.

Considering the exactness of Δ, and the connection between the Auslander-Reiten translation τ and the Serre functor F, it is easy to see $\Delta F P_i \cong F \Delta P_i$ for all $i \in \mathbb{N}$.

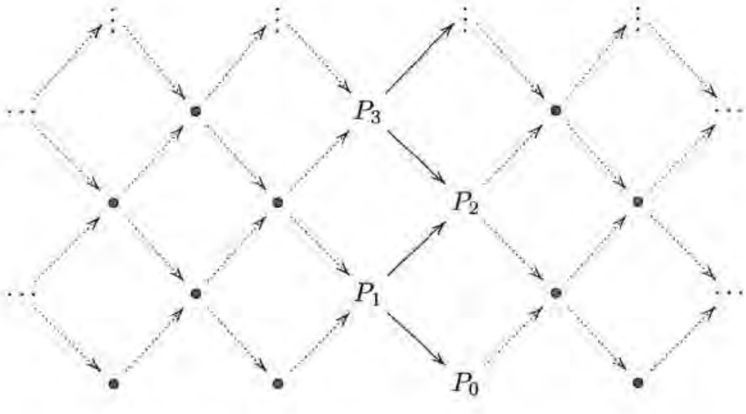


Figure 3.7: The quiver Q in a $\mathbb{Z}A_\infty$-component

Since the Serre functor is exact and commutes with Δ on generators of $D^b \operatorname{mod} kQ$, it will commute with Δ.

We still need to check whether Δ maps Auslander-Reiten components to Auslander-Reiten components essentially surjective, i.e. if an indecomposable of a component is in the essential image of Δ, then so is every indecomposable of that component. To this end, consider an indecomposable object C in $\mathcal{C}$ in the essential image of Δ such that there is an irreducible $D \to C$ where D is not in the essential image of Δ (the dual case where there is an irreducible $C \to D$ is completely analogue). If C is in the essential image of Δ, then so is τC since Δ commutes with F. We may consider the Auslander-Reiten triangles

$$\begin{tikzcd}
\tau M \arrow[r] \arrow[d, dotted, "\Delta"] & N' \arrow[r] \arrow[d, dotted, "\Delta"] & M \arrow[r] \arrow[d, dotted, "\Delta"] & \tau M[1] \arrow[d, dotted, "\Delta"] \\
\tau C \arrow[r] & N \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}$$

Since Δ is full, faithful and exact, we know that $\operatorname{End}(N) \cong \operatorname{End}(N')$, hence N and N' consists of the same number of indecomposable summands and there must be an indecomposable direct summand D' of N' such that $\Delta(D') \cong D$. $\square$

Remark 3.24. The category $\operatorname{mod} kQ$ occurring in the proof has been described in [40]. We may sketch the bounded derived category $D^b \operatorname{mod} kQ$ as shown in Figure 3.8 where we have marked the abelian subcategory $\operatorname{mod} kQ$ with gray. Note that $D^b \operatorname{mod} kQ \cong D^b \operatorname{mod}^{\mathrm{cfp}} k\mathbb{Z}$.

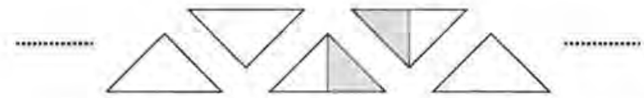


Figure 3.8: A sketch of $D^b \operatorname{mod} kQ$ occurring in Proposition 3.23

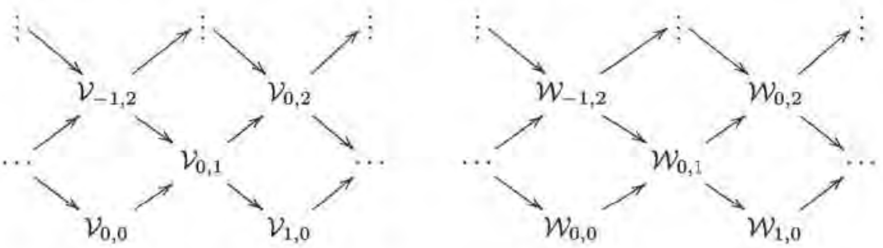


Figure 3.9: Labeling the vertices in the components $\mathcal{V}$ and $\mathcal{W}$ from Proposition 3.25

Proposition 3.25. *Let $\mathcal{V}$ and $\mathcal{W}$ be wings. If $\mathcal{V}$ maps to $\mathcal{W}$, then $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{V}[1]$.*

Proof. It is clear that $\mathcal{V}$ maps to $\mathcal{W}$ if $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{V}[1]$.

To prove that $\mathcal{V}$ does not map to $\mathcal{W}$ otherwise, we start by fixing a notation. Let Q be the quiver

$$0 \to 1 \to 2 \to 3 \to \cdots$$

Since $\mathcal{V} \cong \mathbb{Z}Q \cong \mathcal{W}$ as stable translation quivers, we may label the vertices of $\mathcal{V}$ by $\mathcal{V}_{m,n}$ and the vertices of $\mathcal{W}$ by $\mathcal{W}_{m,n}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ as is illustrated in Figure 3.9.

One sees easily that

$$\dim \operatorname{Hom}(\mathcal{V}_{m,n}, \mathcal{W}_{i,1}) = \dim \operatorname{Hom}(\mathcal{V}_{m,n}, \mathcal{W}_{i,0}) + \dim \operatorname{Hom}(\mathcal{V}_{m,n}, \mathcal{W}_{i+1,0})$$

and, by induction, that

$$\dim \operatorname{Hom}(\mathcal{V}_{m,n}, \mathcal{W}_{i,j}) = \sum_{k=0}^{j} \dim \operatorname{Hom}(\mathcal{V}_{m,n}, \mathcal{W}_{i+k,0})$$

Dually, one has

$$\dim \operatorname{Hom}(\mathcal{V}_{m,n}, \mathcal{W}_{i,j}) = \sum_{l=0}^{n} \dim \operatorname{Hom}(\mathcal{V}_{m+l,0}, \mathcal{W}_{i,j}).$$

Assume $\operatorname{Hom}(\mathcal{V}_{m,n}, \mathcal{W}_{i,j}) \neq 0$, then there are p and q in $\mathbb{N}$ with $m \leq p \leq m+n$ and $i \leq q \leq i+j$ such that $\operatorname{Hom}(\mathcal{V}_{p,0}, \mathcal{W}_{q,0}) \neq 0$, and hence $\operatorname{Hom}(\tau^k \mathcal{V}_{p,0}, \tau^k \mathcal{W}_{q,0}) \cong$

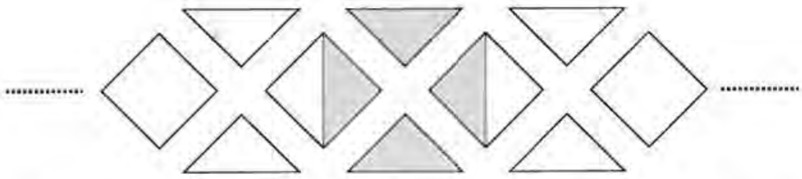


Figure 3.10: A sketch of $D^b \operatorname{mod} kQ$ occurring in Proposition 3.26

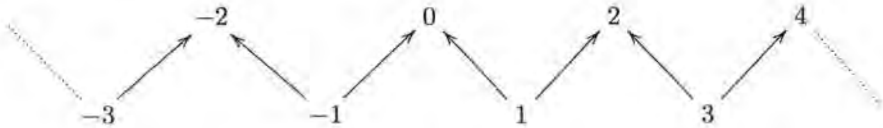
$\operatorname{Hom}(\mathcal{V}_{p-k,0}, \mathcal{W}_{q-k,0}) \neq 0$ for all $k \in \mathbb{Z}$. Finally, we obtain

$$\begin{aligned}
\dim \operatorname{Hom}(\mathcal{V}_{p,6}, \mathcal{W}_{q,6}) &= \sum_{k=0}^{6} \dim \operatorname{Hom}(\mathcal{V}_{p,6}, \mathcal{W}_{q+k,0}) \\
&= \sum_{k,l=0}^{6} \dim \operatorname{Hom}(\mathcal{V}_{p+l,0}, \mathcal{W}_{q+k,0}) \\
&\geq \sum_{k=0}^{6} \dim \operatorname{Hom}(\mathcal{V}_{p+k,0}, \mathcal{W}_{q+k,0}) \\
&\geq 7
\end{aligned}$$

contradicting Theorem 3.5. □

3.4.4 A $\mathbb{Z}A_\infty^\infty$-component

Proposition 3.26. *Let Q be the quiver*



and let $\mathcal{K}$ be a $\mathbb{Z}A_\infty^\infty$-component of $\mathcal{C}$, then the smallest full and exact subcategory of $\mathcal{C}$ containing $\mathcal{K}$ is equivalent to $D^b \operatorname{mod} kQ$, and the embedding $\Delta : D^b \operatorname{mod} kQ \to \mathcal{C}$ commutes with Serre duality. Hence Δ maps Auslander-Reiten components to Auslander-Reiten components.

Proof. The construction of the functor Δ is similar to the construction in the proof of Proposition 3.23; one now finds the quiver Q within the $\mathbb{Z}A_\infty^\infty$-component $\mathcal{K}$. □

Remark 3.27. The category $\operatorname{mod} kQ$ occurring in the proof has been discussed in [40]. We may sketch the derived category $D^b \operatorname{mod} kQ$ as in Figure 3.10 where we have filled the abelian subcategory $\operatorname{mod} kQ$ with grey. Note that the category $\operatorname{mod}^{\mathrm{cfp}} k(\{0,1\} \overrightarrow{\times} \mathbb{Z})$ is derived equivalent to $\operatorname{mod} kQ$.

Since the embedding $\Delta : D^b \operatorname{mod} kQ \to \mathcal{C}$ maps Auslander-Reiten components to Auslander-Reiten components, we may define the map

$$\phi_{\mathbb{Z}A_\infty^\infty} : \{\mathbb{Z}A_\infty^\infty\text{-components of } \mathcal{C}\} \to \{\text{pairs of } \mathbb{Z}A_\infty\text{-components of } \mathcal{C}\}$$

by mapping a $\mathbb{Z}A_\infty^\infty$-components $\mathcal{K}$ to the pair of wings within the essential image of Δ that map non-zero to $\mathcal{K}$.

Remark 3.28. Note that the map $\phi_{\mathbb{Z}A_\infty^\infty}$ does not depend on the choice of the partial tilting set in the proof of Proposition 3.26.

Remark 3.29. It will follow from the following Propositions that $\phi_{\mathbb{Z}A_\infty^\infty} = \phi^{\text{comp}}|_{\mathbb{Z}A_\infty^\infty}$.

Proposition 3.30. *If a wing $\mathcal{W}$ maps to a $\mathbb{Z}A_\infty^\infty$-component $\mathcal{K}$, then $\mathcal{W} \in \phi_{\mathbb{Z}A_\infty^\infty}(\mathcal{K})$.*

Proof. We start by fixing an $X \in \mathcal{K}$. An argument analogous to the proof of Proposition 3.25 shows that there are only finitely many quasi-simple objects of $\mathcal{W}$ that map non-zero to X. We may choose a quasi-simple S from $\mathcal{W}$ such that $\operatorname{Hom}(\tau^{-1}S, X) = 0$ and $\operatorname{Hom}(S, X) \neq 0$. Now, consider the triangle $S \to X \to C \to S[1]$. Applying the functor $\operatorname{Hom}(X, -)$, we may conclude easily that $\dim \operatorname{Hom}(X, C) = 1$.

Next, consider the Auslander-Reiten triangle $X \to Y \oplus Y' \to \tau^{-1}X \to X[1]$. Since the morphism $X \to C$ factors through $Y \oplus Y'$, we may assume there exists a morphism $X \to Y$ such that the composition $X \to Y \to C$ is non-zero. This gives rise to the following morphism of triangles

$$\begin{tikzcd}
S \arrow[r] & X \arrow[r] & C \arrow[r] & S[1] \\
T \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, equal] & Y \arrow[r] \arrow[u] & T[1] \arrow[u]
\end{tikzcd}$$

Since $X \to Y$ is an irreducible morphism between indecomposable objects, Proposition 3.26 yields that T is a quasi-simple object from a wing $\mathcal{V} \in \phi_{\mathbb{Z}A_\infty^\infty}(\mathcal{K})$. The induced morphism $T \to S$ is easily seen to be non-zero. Proposition 3.25 yields $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{V}[1]$. By Proposition 3.26 we may exclude the latter. We conclude $\mathcal{W} \in \phi_{\mathbb{Z}A_\infty^\infty}(\mathcal{K})$. □

Proposition 3.31. *The map*

$$\phi_{\mathbb{Z}A_\infty^\infty} : \{\mathbb{Z}A_\infty^\infty\text{-components of } \mathcal{C}\} \to \{\text{pairs of } \mathbb{Z}A_\infty\text{-components of } \mathcal{C}\}$$

is injective.

Proof. Let $\mathcal{K}$ en $\mathcal{K}'$ be $\mathbb{Z}A_\infty^\infty$-components such that $\phi_{\mathbb{Z}A_\infty^\infty}(\mathcal{K}) = \{\mathcal{V}, \mathcal{W}\} = \phi_{\mathbb{Z}A_\infty^\infty}(\mathcal{K}')$. Fix a quasi-simple object S from the component $\mathcal{V}$ and a quasi-simple object T from the component $\mathcal{W}$. Proposition 3.26 yields a unique indecomposable object X from $\mathcal{K}$ and a unique indecomposable object X' from $\mathcal{K}'$ such that $\operatorname{Hom}(S, X)$, $\operatorname{Hom}(T, X)$,

$\mathrm{Hom}(S, X')$, and $\mathrm{Hom}(T, X')$ are all non-zero. Furthermore, it follows from Proposition 3.26 that all these Hom-spaces are 1-dimensional and from Proposition 3.30 that these are the only quasi-simples mapping to X or X'. We wish to prove that $X \cong X'$.

Since by Proposition 3.26 the map $X \to Y_1$ occurring in the triangle $S \to X \to Y_1 \to S[1]$ is irreducible, we may use Lemma 3.19 to see there is a morphism $X \to X'$ or $X' \to X$. Thus without loss of generality, we may conclude there is a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, equal] & X' \\ S \arrow[r] & X \arrow[u] \end{tikzcd}$$

Analogously, considering the triangle $T \to X \to Y_2 \to T[1]$ and directedness gives a morphism $X \to X'$ and we obtain the commuting diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, equal] & X' \\ T \arrow[r] & X \arrow[u] \end{tikzcd}$$

It easily follows there is a morphism $f : X \to X'$ such that both compositions $S \to X \to X'$ and $T \to X \to X'$ are non-zero.

In order to prove that f is an isomorphism, we will use quasi-simples to *probe* the object $M = \mathrm{cone}(f : X \to X')$, i.e. we will look which quasi-simple objects map to M. The triangle built on f, extended with all the quasi-simple objects mapping to each of its objects looks like

$$\begin{tikzcd} S \arrow[r, equal] \arrow[d] & S \arrow[d] & & S[1] \arrow[r, equal] \arrow[d] & S[1] \arrow[d] \\ X \arrow[r, "f"] & X' \arrow[r, "g"] & M \arrow[r, "h"] & X[1] \arrow[r, "f[1]"] & X'[1] \\ T \arrow[r, equal] \arrow[u] & T \arrow[u] & & T[1] \arrow[r, equal] \arrow[u] & T[1] \arrow[u] \end{tikzcd}$$

We will show that no quasi-simple object U may map to M. Seeking a contradiction, assume that $f' : U \to M$ is such a non-zero morphism. We will first consider the case where the composition $U \to M \xrightarrow{h} X[1]$ is zero. In this case the map $U \to M$ factors though $g : X' \to M$, and U would be isomorphic to either S or T. But since then $\dim \mathrm{Hom}(U, X') = 1$, we may further conclude that $U \to X'$ factors through $f : X \to X'$ so that the composition $U \to X \xrightarrow{f} X' \xrightarrow{g} M$ is non-zero, a contradiction.

Analogous, if the map $U \to M \xrightarrow{h} X[1]$ is non-zero, then U would map non-zero to $X[1]$ and would hence be isomorphic to either $S[1]$ or $T[1]$. Again, since then

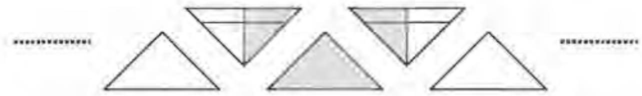


Figure 3.11: A sketch of $D^b \operatorname{mod} kQ$ occurring in Proposition 3.32

$\dim \operatorname{Hom}(U, X[1]) = 1$, we may conclude that the composition $U \to M \xrightarrow{h} X[1] \xrightarrow{f[1]} X'[1]$ is non-zero, a contradiction.

Using Propositions 3.23 and 3.26, and already using Proposition 3.32 from the next section, we see that every non-zero object of $\mathcal{C}$ has at least one quasi-simple object mapping to it. The cone M thus has to be the zero object, establishing the fact that X and X' are isomorphic, and thus that $\mathcal{K} = \mathcal{K}'$. □

3.4.5 A $\mathbb{Z}D_\infty$-component

In this part, we will discuss the $\mathbb{Z}D_\infty$-component within the directed category. Mostly, the proofs are analogous to the case of a $\mathbb{Z}A_\infty^\infty$-component.

Proposition 3.32. *Let Q be the quiver*

$$\begin{tikzcd}
 & 2 \arrow[dl] \arrow[d] \arrow[dr] & & 4 \arrow[dl] \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & & 8 \arrow[dl] \arrow[dr, dotted, no head] & \\
0 & 1 & 3 & & 5 & & 7 & & {}
\end{tikzcd}$$

and let $\mathcal{K}$ be a $\mathbb{Z}D_\infty$-component of $\mathcal{C}$, then the smallest full and exact subcategory of $\mathcal{C}$ containing $\mathcal{K}$ is equivalent to $D^b \operatorname{mod} kQ$, and the embedding $\Delta : D^b \operatorname{mod} kQ \to \mathcal{C}$ commutes with Serre duality. Hence Δ maps Auslander-Reiten components to Auslander-Reiten components.

Proof. The proof is analogue to the proof of Proposition 3.23. One finds the quiver Q within the $\mathbb{Z}D_\infty$-component $\mathcal{K}$. □

Remark 3.33. The category $\operatorname{mod} kQ$ occurring in Proposition 3.32 has been discussed in [40]. We may sketch the derived category $D^b \operatorname{mod} kQ$ as in Figure 3.11 where we have filled the abelian subcategory $\operatorname{mod} kQ$ with gray. Note that the category $\operatorname{mod}^{\mathrm{cfp}} kD_{\mathbb{Z}}$ is derived equivalent to the category $\operatorname{mod} kQ$.

As in discussion of the $\mathbb{Z}A_\infty^\infty$-component, the embedding $\Delta : D^b \operatorname{mod} kQ \to \mathcal{C}$ maps Auslander-Reiten components to Auslander-Reiten components. Hence, we may define the map

$$\phi_{\mathbb{Z}D_\infty} : \{\mathbb{Z}D_\infty\text{-components of } \mathcal{C}\} \to \{\text{singletons of } \mathbb{Z}A_\infty\text{-components of } \mathcal{C}\}$$

by mapping a $\mathbb{Z}D_\infty$-components $\mathcal{K}$ to the set of wings within the essential image of Δ that map non-zero to $\mathcal{K}$.

Remark 3.34. Note that the map $\phi_{\mathbb{Z}D_\infty}$ does not depend on the choice of the partial tilting set in the proof of Proposition 3.32.

Remark 3.35. It will follow from the following Propositions that $\phi_{\mathbb{Z}D_\infty} = \phi^{\text{comp}}|_{\mathbb{Z}D_\infty}$.

The proofs of the following propositions are analogous to the proofs of the corresponding properties in our discussion of the $\mathbb{Z}A_\infty^\infty$-component.

Proposition 3.36. *If there exists a non-zero morphism from a $\mathbb{Z}A_\infty$-component $\mathcal{W}$ to a $\mathbb{Z}D_\infty$-component $\mathcal{K}$, then $\phi_{\mathbb{Z}D_\infty}(\mathcal{K}) = \{\mathcal{W}\}$.*

Proposition 3.37. *The map*

$$\phi_{\mathbb{Z}D_\infty} : \{\mathbb{Z}D_\infty\text{-components of } \mathcal{C}\} \to \{\text{singletons of } \mathbb{Z}A_\infty\text{-components of } \mathcal{C}\}$$

is injective.

3.5 Classification

Let $\mathcal{A}$ be a connected directed hereditary abelian k-linear Ext-finite category satisfying Serre duality, and write $\mathcal{C} = D^b\mathcal{A}$. In this section we will complete the proof of the classification of these categories as follows. Associated with a quasi-simple object $S \in D^b\mathcal{A}$, we will consider a set $S_\rightarrow$ as defined below. This is a partial tilting set and as such gives a full and exact embedding $D^b \operatorname{mod} kS_\rightarrow \to D^b\mathcal{A}$ which we will show is actually an equivalence of triangulated categories. Finally, the classification will follow from the shape of the poset $S_\rightarrow$.

Choose a quasi-simple object S in a wing $\mathcal{W}$. We will consider two cases. First, assume that S does not map to two peripheral objects Q_S^1 and Q_S^2 of a $\mathbb{Z}D_\infty$-component or, equivalently, there is no $\mathbb{Z}D_\infty$-component $\mathcal{K}$ such that $\mathcal{W} \in \phi_{\mathbb{Z}D_\infty}(\mathcal{K})$. In this case just let $S_\rightarrow$ be the set of indecomposable objects X such that there exists a map from S to X, thus

$$S_\rightarrow = \{X \in \operatorname{ind}\mathcal{C} \mid \operatorname{Hom}(S,X) \neq 0\}.$$

Secondly, assume S does map to two peripheral objects Q_S^1 and Q_S^2 of a $\mathbb{Z}D_\infty$-component or, equivalently, there is a $\mathbb{Z}D_\infty$-component $\mathcal{K}$ such that $\mathcal{W} \in \phi_{\mathbb{Z}D_\infty}(\mathcal{K})$. Then, let $S_\rightarrow$ be the set of indecomposable objects X such that there exists a map from S to X and a map from X to Q_S^1 or Q_S^2, thus

$$S_\rightarrow = \{X \in \operatorname{ind}\mathcal{C} \mid \operatorname{Hom}(S,X) \neq 0 \text{ and } \operatorname{Hom}(X, Q_S^1 \oplus Q_S^2) \neq 0\}.$$

This defines the full preadditive subcategory $S_\rightarrow$ of $\mathcal{C}$. We will define a poset structure by

$$X \leq Y \Leftrightarrow \operatorname{Hom}(X,Y) \neq 0.$$

Before proving in Lemma 3.40 that this does indeed define a poset structure, we will give two examples.

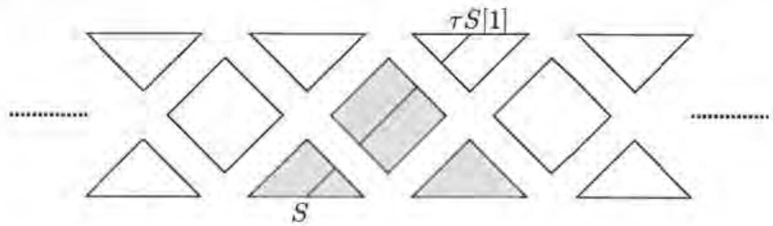


Figure 3.12: The set $S_{\rightarrow}$ in $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$

Figure 3.13: The set $S_{\rightarrow}$ in $\operatorname{mod}^{\mathrm{cfp}} kD_{\mathcal{L}}$

Therefore, we will fix following notation. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be posets. The poset $\mathcal{P}_1 \cdot \mathcal{P}_2$ has $\mathcal{P}_1 \dot{\cup} \mathcal{P}_2$ as underlying set and

$$X \leq Y \Leftrightarrow \begin{cases} X \leq Y \text{ in } \mathcal{P}_1, \text{ or} \\ X \leq Y \text{ in } \mathcal{P}_2, \text{ or} \\ X \in \mathcal{P}_1 \text{ and } Y \in \mathcal{P}_2. \end{cases}$$

Example 3.38. Let $\mathcal{L}$ be the poset $\{0,1\} \overrightarrow{\times} \mathbb{Z}$. The abelian category $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ consists of two $\mathbb{Z}A_\infty$-components, one containing the indecomposables of the form $A_{(0,i),(0,j)}$ and one containing the indecomposable objects of the form $A_{(1,i),(1,j)}$, and a $\mathbb{Z}A_\infty^\infty$-component wherein all the indecomposable objects $A_{(0,i),(1,j)}$ lie, for all $i,j \in \mathbb{Z}$. Now, let $S = A_{(1,0),(1,0)}$. We may then describe the set $S_{\rightarrow}$ as

$$S_{\rightarrow} = \{A_{(1,-n),(1,0)} | n \in \mathbb{N}\} \cdot \{A_{(0,z),(1,0)} | z \in \mathbb{Z}\} \cdot \{A_{(1,1),(1,n)}[1] | n \in \mathbb{N} \text{ and } n \geq 1\}.$$

We may draw $S_{\rightarrow}$ within $D^b \operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ as in Figure 3.12 where, as usual, the abelian category $\operatorname{mod}^{\mathrm{cfp}} kA_{\mathcal{L}}$ has been filled with gray.

Example 3.39. In this example, we will consider $D_{\mathcal{L}}$ where $\mathcal{L} = \mathbb{Z}$. The abelian category $\operatorname{mod}^{\mathrm{cfp}} kD_{\mathcal{L}}$ consists of a $\mathbb{Z}A_\infty$-component containing the indecomposables of the form $A_{i,j}$ and a $\mathbb{Z}D_\infty$-component containing the indecomposables of the form $B_{i,j}$, A_i^1, and A_i^2. If $S = A_{0,0}$ then we may describe the set $S_{\rightarrow}$ as

$$S_{\rightarrow} = \{A_{-n,0} | n \in \mathbb{N}\} \cdot \{B_{0,n+1} | n \in \mathbb{N}\} \cdot \{B_0^1, B_0^2\}.$$

Graphically, we may represent $S_{\rightarrow}$ within $D^b \operatorname{mod}^{\mathrm{cfp}} k\mathcal{L}$ as in Figure 3.13.

Note that in Examples 3.38 and 3.39 the set $S_{\rightarrow}$ is of the form $A_{\mathcal{L}}$ or $D_{\mathcal{L}}$ where $\mathcal{L}$ is a *bounded* locally discrete linearly ordered set, i.e. a locally discrete linearly ordered set with both a minimal and a maximal element. It is easily seen that for such linearly ordered posets we have $\mathcal{L} \cong (\mathbb{N}) \cdot (\mathcal{T} \overrightarrow{\times} \mathbb{Z}) \cdot (-\mathbb{N})$ for a certain linearly ordered set $\mathcal{T}$.

Following lemma will classify all possible posets that may occur as $S_\rightarrow$. For $X, Z \in S_\rightarrow$ we will write

$$[X, Z] = \{Y \in S_\rightarrow \mid X \leq Y \leq Z\}.$$

Lemma 3.40. *The set $S_\rightarrow$ is a poset of the form $A_\mathcal{L}$ or $D_\mathcal{L}$ where $\mathcal{L}$ is a bounded locally discrete linearly ordered set.*

Proof. We start with the case where S does not map to peripheral objects of a $\mathbb{Z}D_\infty$-component and wish to prove that $S_\rightarrow \cong A_\mathcal{L}$ where $\mathcal{L}$ is a bounded locally discrete linearly ordered set.

The relation $\leq$ in $S_\rightarrow$ is obviously reflexive. The fact that it is antisymmetric follows from directedness. In order to prove transitivity and linearly ordered, it suffices to prove that $\operatorname{Hom}(X, Y) \neq 0$ or $\operatorname{Hom}(Y, X) \neq 0$. For all $X, Y \in S_\rightarrow$, we may consider the commutative diagram

$$\begin{tikzcd} Y & & & \\ S \arrow[u] \arrow[r, "f"] & X \arrow[r, "g"] & C \arrow[r] & S[1] \end{tikzcd}$$

where the bottom line is a triangle and $g : X \to C$ is irreducible as is shown in Proposition 3.18(5). Lemma 3.19 now yields that $\operatorname{Hom}(X, Y) \neq 0$ or $\operatorname{Hom}(Y, X) \neq 0$, thus $S_\rightarrow$ is a linearly ordered poset. Even more so, if $X \leq Y$ we may assume that the composition $S \to X \to Y$ is non-zero.

Using Serre duality, it is easily seen that $S_\rightarrow = [S, \tau S[1]]$, thus $S_\rightarrow$ is bounded.

To prove that $S_\rightarrow$ is locally discrete, assume $Z \in S_\rightarrow$ is a non-minimal element. We need to prove there exists a finite set $A \subseteq S_\rightarrow$ such that, for all $X \in S_\rightarrow$ with $X < Z$, there is a $Y \in A$ with $X \leq Y < Z$.

Therefore, consider the Auslander-Reiten triangle $\tau Z \to M \to Z \to \tau Z[1]$. Write $M = \oplus_i M_i$ where M_i is indecomposable. We have already proven that there exists a non-zero morphism from X to Z such that the composition $S \to X \to Z$ is non-zero. Since $X \to Z$ factors through M, it is clear that there exists an $M_i \in \operatorname{ind} M$ such that $M_i \in S_\rightarrow$ and $X \leq M_i < Z$, thus we have shown $A = S_\rightarrow \cap \operatorname{ind} M$.

The case where Z is a non-maximal element is analogous. Hence $S_\rightarrow$ is locally discrete.

We now turn our attention to the case where S does map to two peripheral objects of a $\mathbb{Z}D_\infty$-component and wish to prove that $S_\rightarrow \cong D_\mathcal{L}$ where $\mathcal{L}$ is a bounded locally discrete linearly ordered set.

Note that the left Auslander-Reiten triangles built on Q_S^1 and Q_S^2 both have the same indecomposable middle term N. It is straightforward to check that $S_\rightarrow \cong [S, N] \cdot \{Q_S^1, Q_S^2\}$. Analogous to the first part of the proof, one shows that $[S, N]$ is a bounded locally discrete linearly ordered set, thus $S_\rightarrow \cong D_\mathcal{L}$ where $\mathcal{L} = [S, N]$. $\square$

We will now prove most technical results needed to prove our main result, Theorem 3.44. In particular, we will construct an embedding $D^b \operatorname{mod} S_\rightarrow \to \mathcal{C}$ and prove that it is an equivalence of categories.

Lemma 3.41. *The set $S_\rightarrow$ is a partial tilting set.*

Proof. We need to prove that $\operatorname{Hom}(X, Y[z]) = 0$ for all $z \in \mathbb{Z}_0$, and all $X, Y \in S_\rightarrow$. If X and Y are the peripheral objects Q_S^1 and Q_S^2 then the assertion follows easily. Indeed, both are then contained within the same $\mathbb{Z}D_\infty$-component, and all the maps within such component are known from Proposition 3.32.

Since we may now assume that either X or Y is not isomorphic to Q_S^1 or to Q_S^2, Lemma 3.40 yields that we may assume $\operatorname{Hom}(X, Y) \neq 0$ and hence, by Lemma 3.3, we need only to prove that $\operatorname{Hom}(Y, X[1]) = 0$.

Let $f \subset \operatorname{Hom}(Y, X[1])$ and consider the triangle

$$X[-1] \longrightarrow Y \longrightarrow M \longrightarrow X \xrightarrow{f} Y[1] \qquad (3.3)$$

We will now probe M to prove $M \cong X \oplus Y$ and hence $f = 0$.

First, consider the case where $X = Q_S^1$ and $Y \neq Q_S^2$.

Since $Y \in S_\rightarrow$, we have $\operatorname{Hom}(S, Y) \neq 0$, and consequently $\operatorname{Hom}(S, Y[1]) = 0$. We conclude that the composition $S \to X \xrightarrow{f} Y[1]$ needs to be zero.

Let S and T' be the quasi-simples mapping to Y, thus $\phi^{\text{obj}}(Y) = \{S, T'\}$. Since $\phi^{\text{obj}}(X[-1]) = \{S[-1]\}$, we have that $\operatorname{Hom}(S, X[-1]) = 0$ and $\operatorname{Hom}(T', X[-1]) = 0$.

If S' is a quasi-simple such that $\operatorname{Hom}(S', M) \neq 0$, then either $\operatorname{Hom}(S', X) \neq 0$ or $\operatorname{Hom}(S', Y) \neq 0$, hence $S' \cong S$ or $S' \cong T'$.

Thus triangle (3.3) enriched with all the quasi-simples mapping to each of its components is

$$\begin{tikzcd}
S[-1] \arrow[d] & S \arrow[r, equal] \arrow[d] & S \arrow[dr] & & S \arrow[dl] \arrow[r, equal] & S \arrow[d] & S[1] \arrow[d] \\
X[-1] \arrow[r] & Y \arrow[rr] & & M \arrow[rr] & & X \arrow[r, "f"] & Y[1] \\
& T' \arrow[u] \arrow[rr, equal] & & T' \arrow[u] & & & T'[1] \arrow[u]
\end{tikzcd}$$

Using Proposition 3.18 it is easy to see that either Q_S^1 or Q_S^2 have to be direct summands of M. However, since all non-zero morphisms in $\operatorname{Hom}(Q_S^1, Q_S^1)$ are isomorphisms and $\operatorname{Hom}(Q_S^2, Q_S^1) = 0$, Lemma 1.23 implies $f = 0$.

We may now assume that neither X nor Y are the peripheral objects Q_S^1 or Q_S^2. Proposition 3.18(2) yields there are two quasi-simple objects, S and T, mapping to X and two quasi-simple objects, S and T', mapping to Y. We will have to consider two cases, namely one where the composition $T \longrightarrow X \xrightarrow{f} Y[1]$ is zero and one where it is non-zero. We start with the former.

We will probe M to proof $M \cong X \oplus Y$, so that $f = 0$. Therefore, we wish to find all quasi-simples that admit a non-zero map to M.

Since $\operatorname{Hom}(S, Y) \neq 0$, we know by directedness that $\operatorname{Hom}(S, Y[1]) = 0$ and hence that the composition $S \longrightarrow X \xrightarrow{f} Y[1]$ is zero.

If S' is a quasi-simple such that $\operatorname{Hom}(S', M) \neq 0$, then either $\operatorname{Hom}(S', X) \neq 0$ or $\operatorname{Hom}(S', Y) \neq 0$, hence $S' \cong S$, $S' \cong T$, or $S' \cong T'$.

Thus triangle (3.3) enriched with all the quasi-simples mapping to each of its components is

$$\begin{tikzcd}
S[-1] \arrow[d] & S \arrow[r, equal] \arrow[d] & S \arrow[dr] & & S \arrow[r, equal] \arrow[dl] & S \arrow[d] & S[1] \arrow[d] \\
X[-1] \arrow[r] & Y \arrow[rr] & & M \arrow[rr] & & X \arrow[r, "f"] & Y[1] \\
T[-1] \arrow[u] & T' \arrow[r, equal] \arrow[u] & T' \arrow[ur] & & T \arrow[r, equal] \arrow[ul] & T \arrow[u] & T'[1] \arrow[u]
\end{tikzcd}$$

Since $X \in S_{\rightarrow}$ and X is not isomorphic to either Q^1_S or Q^2_S, Lemma 3.40 implies neither Q^1_S nor Q^2_S can map to X, and thus cannot be direct summands of M (using Lemma 1.23 to see that this would indeed induce a non-zero map from Q^1_S or Q^2_S to X) it follows from Proposition 3.18(2) that M is the direct sum of exactly two indecomposable objects, M_1 and M_2.

It is now easy to see that we may assume $\phi^{\text{comp}}(M_1) = \{S, T\}$ and $\phi^{\text{comp}}(M_1) = \{S, T'\}$. Proposition 3.18(3) now yields $M_1 \cong X$ and $M_2 \cong Y$, hence $M = X \oplus Y$, and thus $f = 0$.

We now consider the latter case where neither X nor Y are the peripheral objects Q^1_S or Q^2_S and the composition $T \longrightarrow X \xrightarrow{f} Y[1]$ is non-zero. This yields

$$\begin{tikzcd}
S[-1] \arrow[d] & S \arrow[r, equal] \arrow[d] & S \arrow[dr] & & S \arrow[r, equal] \arrow[dl] & S \arrow[d] & S[1] \arrow[d] \\
X[-1] \arrow[r] & Y \arrow[rr] & & M \arrow[rr] & & X \arrow[r, "f"] & Y[1] \\
T[-1] \arrow[r, equal] \arrow[u] & T' \arrow[u] & & & & T \arrow[r, equal] \arrow[u] & T'[1] \arrow[u]
\end{tikzcd}$$

which is easily seen to be false since this would imply that either Q^1_S or Q^2_S would map to X. Indeed, these are the only elements that do not have two different quasi-simples mapping to them and as such the only possible direct summands of M. $\square$

Lemma 3.42. *If S is a quasi-simple object of $\mathcal{C}$, then $\mathcal{C} \cong D^b \operatorname{mod} S_{\rightarrow}$.*

Proof. First, due to Lemma 3.41 and Theorem 1.34, we may consider a full and exact embedding $D^b \operatorname{mod} S_{\rightarrow} \to \mathcal{C}$. To show this is an equivalence of triangulated categories, we need to check that it is essentially surjective. We will proceed in three steps.

1. We start by showing that every indecomposable Y with $\operatorname{Hom}(S, Y) \neq 0$ lies within the subcategory $D^b \operatorname{mod} S_{\rightarrow}$ of $\mathcal{C}$. If S does not map to a $\mathbb{Z}D_\infty$-component, this follows directly from the definition of $S_{\rightarrow}$. If S does map to peripheral objects, Q_S^1 and Q_S^2, of a $\mathbb{Z}D_\infty$-component, then we show there are non-zero morphisms $f_1 : Q_S^1 \to Y$ and $f_2 : Q_S^2 \to Y$. First note that there is no map $Y \to Q_S^1$ since otherwise $Y \in S_{\rightarrow}$. Now, the existence of f follows from the diagram

$$\begin{tikzcd} Q_S^1 \arrow[dr, "f_1"] & & & \\ S \arrow[u] \arrow[r] & Y \arrow[r, "h"] & C \arrow[r] & S[1] \end{tikzcd}$$

where the bottom line is a triangle, $h : Y \to C$, is irreducible and the morphism $f_1 : Q_S^1 \to Y$ is given by Lemma 3.19. Analogously, one proves the existence of $f_2 : Q_S^2 \to Y$.

Consider the following triangle, enriched with the quasi-simples mapping to each of its entries

$$\begin{tikzcd} S[-1] \arrow[d] & S \arrow[d] & S \arrow[dr] & & S \arrow[dl] & S \arrow[d] & S[1] \arrow[d] \\ Y[-1] \arrow[r] & X \arrow[rr, "\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}"'] & & Q_S^1 \oplus Q_S^2 \arrow[rr, "(f_1 \quad f_2)"'] & & Y \arrow[r] & X[1] \\ T[-1] \arrow[u] & T[-1] \arrow[u] & & & & T \arrow[u] & T \arrow[u] \end{tikzcd}$$

Since neither f_1 nor f_2 are zero, Lemma 1.23 yields that g and g' are no isomorphisms. In particular, X cannot contain Q_S^1 or Q_S^2 as direct summands. Due to the quasi-simple objects mapping to X, we may easily deduce that X is an indecomposable object and $X \in S_{\rightarrow}$. Since $D^b \operatorname{mod} S_{\rightarrow}$ is an exact subcategory of $\mathcal{C}$, we may conclude that $Y \in D^b \operatorname{mod} S_{\rightarrow}$.

2. We will now consider the more general case where Y is an indecomposable object of $\mathcal{C}$ such that $\operatorname{Hom}(S, Y[z]) = 0$ for all $z \in \mathbb{Z}$. Since the category $\mathcal{C}$ is connected we may assume, without loss of generality, the existence of at least one indecomposable object X of $D^b \operatorname{mod} S_{\rightarrow}$ such that $\operatorname{Hom}(X, Y) \neq 0$ or $\operatorname{Hom}(Y, X) \neq 0$. First, assume the former. Since $D^b \operatorname{mod} S_{\rightarrow}$ is generated by elements of $S_{\rightarrow}$ by taking finitely many cones and shifts, we may assume the existence of an indecomposable object $P \in S_{\rightarrow}$ such that $\operatorname{Hom}(P, Y) \neq 0$.

Consider the triangle

$$E \to P \otimes \mathrm{Hom}(P, Y) \to Y \to E[1]$$

as in Lemma 3.4 where it has been proved that $\mathrm{add}\, E \cup \{P\}$ is a partial tilting set and Y lies within the subcategory of $\mathcal{C}$ generated by $E \oplus P$. Let E_1 be any indecomposable direct summand of E and consider the following triangle

$$E_1 \to P \to C_1 \to E_1[1].$$

Due to Lemma 3.43 we may conclude that C_1 is indecomposable. Applying the functor $\mathrm{Hom}(S, -)$ shows that either $\mathrm{Hom}(S, E_1) \neq 0$ or $\mathrm{Hom}(S, C_1) \neq 0$, and as such, either $E_1 \in D^b \mathrm{mod}\, S_\to$ or $C_1 \in D^b \mathrm{mod}\, S_\to$. In both cases, since $D^b \mathrm{mod}\, S_\to$ is an exact subcategory of $\mathcal{C}$, we may conclude that $E_1 \in D^b \mathrm{mod}\, S_\to$, and thus that E and hence also Y lie within $D^b \mathrm{mod}\, S_\to$.

3. Finally, if $\mathrm{Hom}(Y, X) \neq 0$ where X is an indecomposable of $D^b \mathrm{mod}\, S_\to$, consider the triangle $Y \to X \to C \to Y[1]$. Due to Lemma 1.23 we know there to be non-zero morphisms from X to every direct summand of C. In this case, it has been established in the second part of this proof that every indecomposable summand of C lies in $\mathrm{Ob}\, D^b \mathrm{mod}\, S_\to$ and hence also that $C \in \mathrm{Ob}\, D^b \mathrm{mod}\, S_\to$. Due to the fact that $D^b \mathrm{mod}\, S_\to$ is an exact subcategory of $\mathcal{C}$, we may conclude $Y \in D^b \mathrm{mod}\, S_\to$. This proves the assertion.

□

In previous lemma, we have used this easy lemma.

Lemma 3.43. *Let $X, Y \in \mathrm{Ob}\, \mathcal{C}$ be indecomposables objects, and let $X \to Y \to Z \to X[1]$ be the triangle built on a non-zero morphism $X \to Y$. If $\mathrm{Hom}(Y, X[1]) = 0$ then Z is indecomposable.*

Proof. Using Lemma 1.23 it should be clear that Z has at most $\dim \mathrm{Hom}(Y, Z)$ direct summands. Applying the functor $\mathrm{Hom}(Y, -)$ to the triangle $X \to Y \to Z \to X[1]$, and using that $\dim \mathrm{Hom}(Y, Y) = 1$ and $\mathrm{Hom}(Y, X[1]) = 0$, it follows easily that $\dim \mathrm{Hom}(Y, Z) = 1$ and thus that Z is indecomposable. □

We now prove our main result.

Theorem 3.44. *A connected directed hereditary abelian k-linear Ext-finite category $\mathcal{A}$ satisfying Serre duality is derived equivalent to* $\mathrm{mod}^{\mathrm{cfp}}\, k\mathcal{P}$ *where $\mathcal{P}$ is either a Dynkin quiver, $A_\mathcal{L}$, or $D_\mathcal{L}$ where $\mathcal{L}$ is a locally discrete linearly ordered set without maximum or minimum.*

Proof. From Theorem 3.13 we know that the only components of the Auslander-Reiten quiver of $D^b\mathcal{A}$ are of the form $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$ or $\mathbb{Z}Q$, where Q is a quiver of Dynkin type.

First, assume $\mathcal{C}$ has an Auslander-Reiten component of the form $\mathbb{Z}Q$ where Q is a Dynkin quiver. It has been proven in Proposition 3.21 that $D^b\mathcal{A} \cong D^b \operatorname{mod}^{\mathrm{cfp}} kQ$

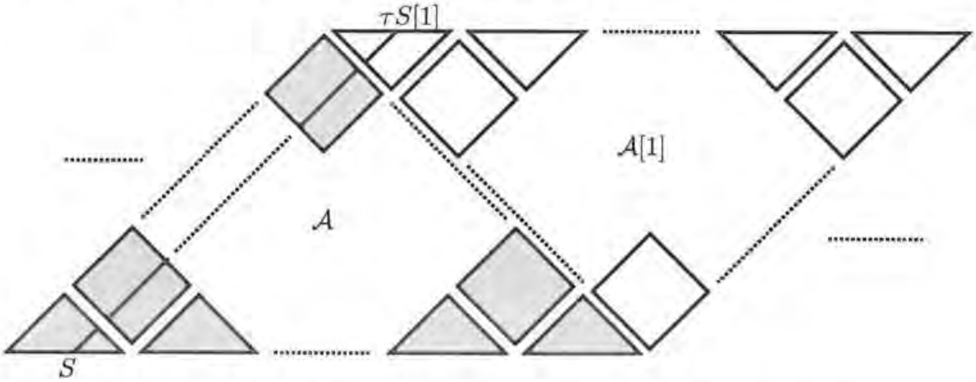
Thus we may now assume that the Auslander-Reiten quiver of $D^b\mathcal{A}$ does not have a component of the form $\mathbb{Z}Q$ where Q is a Dynkin quiver. The only possible components then are of the form $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$ or $\mathbb{Z}D_\infty$. It follows from Proposition 3.17 that there is at least one wing $\mathcal{W}$. Fix a quasi-simple $S \in \mathcal{W}$ and consider the set $S_\rightarrow$. Lemma 3.42 yields that $D^b\mathcal{A} \cong D^b \operatorname{mod} S_\rightarrow$.

From Lemma 3.40 we know that $S_\rightarrow \cong A_{\mathcal{L}'}$ or $S_\rightarrow \cong D_{\mathcal{L}'}$ where $\mathcal{L}' = (\mathbb{N}) \cdot (\mathcal{T}' \overrightarrow{\times} \mathbb{Z}) \cdot (-\mathbb{N})$ for a certain linearly ordered set $\mathcal{T}'$. In the first case, we will give a poset $A_\mathcal{L}$ such that $D^b\mathcal{A}$ is equivalent to $D^b \operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$; in the second case we will give a poset $D_\mathcal{L}$ such that $D^b\mathcal{A}$ is equivalent to $D^b \operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$.

Thus, first assume $S_\rightarrow \cong A_{\mathcal{L}'}$. We will now consider the category $D^b \operatorname{mod} kA_\mathcal{L}$ where $\mathcal{L} = \mathcal{T} \overrightarrow{\times} \mathbb{Z}$ and $\mathcal{T} = \mathcal{T}' \cdot \{*\}$. This category has already been discussed in §3.3.2. Fix the quasi-simple $T = A_{(*,0),(*,0)}$ of $D^b \operatorname{mod} A_\mathcal{L}$. We may characterize $T_\rightarrow$ as

$$T_\rightarrow = \{A_{(*,-n),(*,0)} \mid n \in \mathbb{N}\} \cdot \{A_{(t,z),(*,0)} \mid (t,z) \in \mathcal{T}' \overrightarrow{\times} \mathbb{Z}\} \cdot \{A_{(*,1),(*,n+1)}[1] \mid n \in \mathbb{N}\}$$

and thus $T_\rightarrow \cong A^\circ_{\mathcal{L}'}$ as drawn in the following figure.

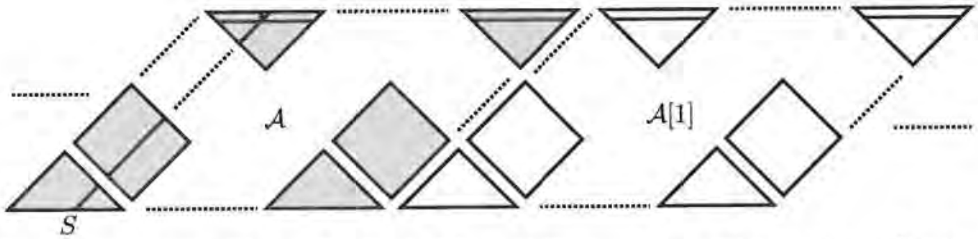


By Lemma 3.42 we have that $D^b \operatorname{mod} kA_\mathcal{L} \cong D^b \operatorname{mod} kA_{\mathcal{L}'}$, thus $D^b\mathcal{A} \cong D^b \operatorname{mod}^{\mathrm{cfp}} kA_\mathcal{L}$.

We will now consider the second case where $S_\rightarrow \cong D_{\mathcal{L}'}$. Consider $D^b \operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$ where $\mathcal{L} = \mathcal{T} \overrightarrow{\times} \mathbb{Z}$ for $\mathcal{T} = \mathcal{T}' \cdot \{*\}$. This category has already been discussed in §3.3.3. Fix the quasi-simple $T = A_{(*,0),(*,0)}$ of $D^b \operatorname{mod}^{\mathrm{cfp}} kD_\mathcal{L}$. We may write $T_\rightarrow$ as

$$\begin{aligned} T_\rightarrow \quad = \quad & \{A_{(*,-n),(*,0)} \mid n \in \mathbb{N}\} \\ & \cdot \{A_{(t,z),(*,0)} \mid (t,z) \in \mathcal{T}' \overrightarrow{\times} \mathbb{Z}\} \\ & \cdot \{B_{(*,0),(*,n+1)} \mid n \in \mathbb{N}\} \cdot \{Q_T^1, Q_T^2\} \end{aligned}$$

and thus $T_\rightarrow \cong D^\circ_{\mathcal{L}'}$ with $\mathcal{L}' = \mathcal{T}' \overrightarrow{\times} \mathbb{Z}$ as in the following figure.



Lemma 3.42 yields $D^b \operatorname{mod}^{\mathrm{cfp}} kD_{\mathcal{L}} \cong D^b \operatorname{mod} kD_{\mathcal{L}'}$, and thus $D^b\mathcal{A} \cong D^b \operatorname{mod}^{\mathrm{cfp}} kD_{\mathcal{L}}$. $\square$

Remark 3.45. It follows from the proof of Theorem 3.44 that $\mathcal{L}$ can also be chosen to be a *bounded* linearly ordered set (i.e. $\mathcal{L}$ has a maximal and a minimal element). It then follows in particular that $\mathcal{A}$ is derived equivalent to a hereditary category which has both enough projectives and injectives.

Remark 3.46. It is proven in Proposition 3.22 that if, in the statement of Theorem 3.44, $\mathcal{P}$ is a Dynkin quiver, then $\mathcal{A}$ is equivalent to $\operatorname{mod} kQ$ for a certain Dynkin quiver Q.

Chapter 4

Uniserial categories

We will say an abelian category $\mathcal{A}$ is uniserial if, for every $X \in \operatorname{ind} \mathcal{A}$ the subobjects of X are linearly ordered. If $\mathcal{A}$ is a hereditary uniserial length category with finitely many simples, then it is known from [18] that $\mathcal{A}$ is equivalent to either $\operatorname{nilp} \tilde{A}_n$ or $\operatorname{mod} kA_n$ with $n \in \mathbb{N}$. In this chapter, we prove a generalization of this result.

Theorem 4.1. *Let $\mathcal{A}$ be a connected Ext-finite abelian hereditary category with Serre duality. If $\mathcal{A}$ is uniserial, then $\mathcal{A}$ is equivalent to one of the following:*

1. $\operatorname{mod} A_n$,
2. $\operatorname{nilp} \tilde{A}_n$,
3. *a big tube,*
4. $\operatorname{mod}^{\mathrm{cfp}} A_{\mathcal{L}}$ *where $\mathcal{L}$ is a linearly ordered locally discrete set which has either a maximum and minimum, or no maximum and no minimum.*

Furthermore, we will turn our attention to *tubes*, components of the Auslander-Reiten quiver of the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ where $n > 0$. Most notably, we will show that every component with a finite τ-orbit is a tube (Theorem 4.5), and that tubes are directing in the following sense: if there is an oriented cycle containing an object of a tube, then every object of this path lies in this tube (see Theorem 4.6). These results lie at the basis of our classification of the hereditary (fractional) Calabi-Yau categories in Chapter 5.

4.1 Classification of uniserial categories

In this section, we shall establish some useful facts about uniserial categories. Recall that an abelian category $\mathcal{A}$ is said to be *uniserial* if, for every $X \in \operatorname{ind} \mathcal{A}$, the subobjects of X are linearly ordered. This property is self-dual, thus the dual of an abelian uniserial category is again an abelian uniserial category.

The following proposition contains most of the information we will need to classify all hereditary uniserial categories with Serre duality.

It will be convenient to extend the notion of a peripheral object to an abelian category $\mathcal{A}$ in the following way: an object $S \in \operatorname{ind} \mathcal{A}$ is called *peripheral* if there is an Auslander-Reiten sequence starting or ending at S has an indecomposable middle term. In particular, projective-injective objects are never peripheral.

If $\mathcal{A}$ is hereditary with Serre duality, then a peripheral object X in $\mathcal{A}$ will give a peripheral object X in $D^b\mathcal{A}$. The other direction is true if X is not projective-injective in $\mathcal{A}$.

Proposition 4.2. *Let $\mathcal{A}$ be an Ext-finite abelian hereditary uniserial category with Serre duality. Every indecomposable object has a simple top and simple socle, and the middle term of every Auslander-Reiten sequence has at most two direct summands. Furthermore, if $\mathcal{A}$ is connected and not equivalent to* $\operatorname{mod} k$, *then the simple objects are exactly the peripheral objects.*

Proof. Let X be an indecomposable object of $\mathcal{A}$, and assume X is not projective. We show that the middle term M of Auslander-Reiten sequence $0 \to \tau X \to M \to X \to 0$ has at most two direct summands.

Assume M has at least two direct summands, M_1 and M_2. We claim that the corresponding irreducible morphisms $\alpha_1 : M_1 \to X$ and $\alpha_2 : M_2 \to X$ may not be both epimorphisms or monomorphisms.

It follows directly from the definition of a uniserial object and irreducible morphisms that α_1 and α_2 may not both be monomorphisms.

If α_1 and α_2 are both epimorphisms, then $\ker \alpha_1 \oplus \ker \alpha_2$ is a subobject of τX. Since τX is uniserial, every subobject must be indecomposable. A contradiction.

We have shown that the middle term M of Auslander-Reiten sequence $0 \to \tau X \to M \to X \to 0$ where X is not projective has at most two direct summands. If $M \cong M_1 \oplus M_2$, then one of the corresponding maps $M_1 \to X$ and $M_2 \to X$ is a monomorphism and the other is an epimorphism.

Let $f : X \to Y$ be an irreducible morphism between indecomposable objects. If f is an epimorphism then we claim $\ker f$ is simple. Dually, if f is a monomorphism then $\operatorname{coker} f$ is simple.

Let us prove the claim in the case that f is an epimorphism. Let $K = \ker f$ and S a subobject K. We find following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & X \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. Since X is uniserial, the quotient object C is indecomposable, thus either $X \to C$ or $C \to Y$ is an isomorphism. We conclude that $S \cong K$ or $S \cong 0$, thus K is simple.

Now we show that every indecomposable object has simple top and simple socle (and that X is even simple if it is peripheral).

If X is projective or injective, then X has a simple top or a simple socle, respectively, as any indecomposable projective or injective object does.

If X is non-peripheral and non-projective, then we may consider the Auslander-Reiten sequence $0 \to \tau X \to M \to X \to 0$ where M is not indecomposable. There are thus irreducible morphisms $\alpha_1 : M_1 \to X$ and $\alpha_2 : M_2 \to X$. As shown before, one is a monomorphism of which the cokernel is the simple top of X.

The dual reasoning implies that if X is non-peripheral and non-injective, then X has a simple socle.

Finally, assume $\mathcal{A}$ is connected and $\mathcal{A} \not\cong \operatorname{mod} k$, and let X be a peripheral object. In this case, X is either the kernel of an irreducible epimorphism or the cokernel of an irreducible monomorphism and, as such, a simple object. If X is simple, then any irreducible $X \to Y$ or $Y \to X$ is a monomorphism (if X is not injective) or an epimorphism (if X is not projective), respectively. As shown before, the Auslander-Reiten sequence starting or ending in X has an indecomposable middle term, hence X is peripheral.

□

Proposition 4.3. *Let $\mathcal{A}$ be a connected abelian Ext-finite hereditary uniserial category with Serre duality. Let $S, T \in \operatorname{ind} \mathcal{A}$ be peripheral objects, then there is an $X \in \operatorname{ind} \mathcal{A}$ such that $\operatorname{Hom}(S, X) \neq 0$ or $\operatorname{Hom}(X, T) \neq 0$, or vice versa.*

Proof. The case where $\mathcal{A}$ is semi-simple is trivial, so by Proposition 4.2, we may assume the peripheral object are exactly the simple ones.

By Proposition 1.26, it suffices to show that, given two peripheral objects $S, T \in \operatorname{ind} \mathcal{A}$, there is always a path from S to T or vice versa.

To this end, let S, T_1, and T_2 be peripheral objects and assume there are paths from S to both T_1 and T_2. By Proposition 1.26 we know there is an object $X_1 \in \operatorname{ind} \mathcal{A}$ such that $\operatorname{Hom}(S, X_1) \neq 0$ and $\operatorname{Hom}(X_1, T_1) \neq 0$, and an object $X_1 \in \operatorname{ind} \mathcal{A}$ such that $\operatorname{Hom}(S, X_2) \neq 0$ and $\operatorname{Hom}(X_2, T_2) \neq 0$.

From the proof of Proposition 4.2, we know S is the kernel of an irreducible map $X_1 \to X_1'$, and thus Lemma 3.19 yields that the map $S \to X_1$ factors through X_2 or vice versa. We will assume the former.

In this case X_2 is a subobject of X_1. We may assume the quotient X_1/X_2 is not zero, and thus has as simple top T_1 and as simple socle $\tau^{-1}T_2$. We find a path from T_2 to T_1 via $\tau^{-1}T_2$. □

To complete the classification of the abelian uniserial categories with Serre duality in Theorem 4.1, it will be convenient to know the directed ones.

Lemma 4.4. *A connected directed category is uniserial if and only if it is equivalent to either*

1. $\operatorname{mod} A_n$,

2. $\operatorname{mod}^{\mathrm{cfp}} A_{\mathcal{L}}$ *where* $\mathcal{L}$ *is a linearly ordered locally discrete set without minima or maxima.*

3. $\operatorname{mod}^{\mathrm{cfp}} A_{\mathcal{L}}$ *where* $\mathcal{L}$ *is a linearly ordered locally discrete set with a minimum and a maximum.*

Proof. We may assume $\mathcal{A} \not\cong \operatorname{mod} k$ as this case is trivially dealt with. So assume $\mathcal{A}$ is not semi-simple.

Proposition 4.2 yields every middle term of an Auslander-Reiten triangle in $D^b\mathcal{A}$ has at most two indecomposable direct summands. It follows from Theorem 3.44 that $\mathcal{A}$ is derived equivalent to one of the above categories. If $\mathcal{A}$ has finitely many simple objects, then this result is well-known.

Let $\mathcal{L}$ be the set consisting of all peripheral of $\mathcal{A}$. Proposition 4.2 yields $\mathcal{L}$ consists of all simple objects. It follows from Proposition 4.3 that for every two object $S, T \in \mathcal{L}$, there is an $X \in \mathcal{A}$ such that $\operatorname{Hom}(S, X) \neq 0$ and $\operatorname{Hom}(X, T) \neq 0$ or vice versa. Since $\mathcal{A}$ is directing, this induces a linear ordering on the elements of $\mathcal{L}$. It is clear that $\mathcal{A} \cong \operatorname{mod}^{\mathrm{cfp}} \mathcal{L}$. □

We now come to the main result of this section.

proof of Theorem 4.1. It follows from Proposition 4.2 that the set $\mathcal{S} \subseteq \operatorname{ind} \mathcal{A}$ consisting of all simple objects, is not empty.

If $\mathcal{A}$ has only finitely many simple objects, then $\mathcal{A}$ is a length category and it is well-known that $\mathcal{A}$ is equivalent to either $\operatorname{mod} A_n$, or $\operatorname{nilp} \tilde{A}_n$.

Assume $\mathcal{A}$ has infinitely many nonisomorphic simples. For any finite $\Sigma \subseteq \mathcal{S}$, we consider the full subcategory $\mathcal{A}_\Sigma$ of $\mathcal{A}$ spanned by

$$\{X \in \operatorname{Ob} \mathcal{A} \mid \operatorname{Hom}(X, Y) = 0 = \operatorname{Ext}(X, Y), \forall Y \in \mathcal{S} \setminus \Sigma\}.$$

It is clear that $\mathcal{A}_\Sigma$ is abelian, hereditary, uniserial, where the simples are given by Σ. Since $\mathcal{A}_\Sigma$ is uniserial and has finitely many simples, it is a length category, hence it is equivalent to either $\operatorname{mod} A_n$, or $\operatorname{nilp} \tilde{A}_n$.

Let $\Sigma \subseteq \mathcal{S}$ be a finite subset such that $\mathcal{A}_\Sigma \cong \operatorname{nilp} \tilde{A}_n$. It is clear that $\mathcal{A}_{\Sigma'} \cong \operatorname{nilp} \tilde{A}_{n'}$ when $\Sigma' \subseteq \mathcal{S}$ is a nonempty finite set with $\Sigma' \subseteq \Sigma$ or $\Sigma \subseteq \Sigma'$. Thus for every finite $\Sigma \subseteq \mathcal{S}$, there is an $n \in \mathbb{N}$ such that the category $\mathcal{A}_\Sigma$ is equivalent to the nilpotent representations over a quiver of the form $\tilde{A}_n$.

We have shown that every connected uniserial hereditary category with Serre duality is a 2-colimit of categories equivalent to $\operatorname{mod} A_n$, or $\operatorname{nilp} \tilde{A}_n$. In the first case, $\mathcal{A}$ is directed and the classification follows from Lemma 4.4, while in the second case, the classification follows from a strengthened version of [17] (see [51]). □

4.2 Tubes

Throughout, let $\mathcal{A}$ be an Ext-finite hereditary abelian category with Serre duality. We will be interested in the stable components $\mathcal{K}$ of the Auslander-Reiten quiver of

$\mathcal{A}$ or $D^b\mathcal{A}$ of the form $\mathbb{Z}A_\infty/\langle\tau^r\rangle$, called a *tube* and we will refer to r as the *rank* of the tube. If $r = 1$, then $\mathcal{K}$ is called a *homogeneous tube*.

These standard tubes occur, for example, in the finite dimensional representations of tame algebras and in the category of coherent sheaves on a smooth projective curve. Categories consisting only of tubes will be discussed in Chapter 5.

The following theorems are the main results of this section. Recall that a t-structure on $D^b\mathcal{A}$ is called *τ-invariant* if the heart $\mathcal{H}[0] \subset D^b\mathcal{A}$ is invariant under τ. If $\mathcal{H}$ is hereditary, then $\mathcal{H}$ does not have nonzero projective or injective objects.

Theorem 4.5. *An Auslander-Reiten component in $D^b\mathcal{A}$ is a tube if and only if it contains an indecomposable object X such that $\tau^r X \cong X$, for $r \geq 1$.*

Theorem 4.6. *Let $\mathcal{K}$ be a tube in $D^b\mathcal{A}$. Then*

1. *$\mathcal{K}$ is standard,*
2. *$\mathcal{K}$ is directing in the sense that if there is a path $X_0 \to \cdots \to X_n$ in $D^b\mathcal{A}$ with $X_0, X_n \in \mathcal{K}$, then $X_i \in \mathcal{K}$ for all i.*
3. *There exists a τ-invariant t-structure on $D^b\mathcal{A}$ with hereditary heart $\mathcal{H} \supseteq \mathcal{K}$ such that the peripheral objects of $\mathcal{K}$ are simple in $\mathcal{H}$.*

A tube as in Theorem 4.6(3) where the peripheral objects are all simple will be called a *simple tube*. It is readily verified that a simple tube is automatically standard.

For the proof of Theorem 4.6 we will first show (2) and (3) hold under the assumption that $\mathcal{K}$ is standard, and conclude by proving (1).

Proof of Theorem 4.6(3) (assuming $\mathcal{K}$ is standard). Consider the τ-invariant t-structure given by

$$\begin{aligned} \operatorname{ind}\mathcal{D}^{\leq 0} &= \{X \in \operatorname{ind} D^b\mathcal{A} \mid \text{There is a path from } \mathcal{K} \text{ to } X\} \\ \operatorname{ind}\mathcal{D}^{\geq 1} &= \{X \in \operatorname{ind} D^b\mathcal{A} \mid \text{There is no path from } \mathcal{K} \text{ to } X.\} \end{aligned}$$

We denote $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. Note that since $\mathcal{D}^{\leq 0}$ satisfies the conditions of Theorem 1.30, the heart $\mathcal{H}$ is an abelian and hereditary category derived equivalent to $\mathcal{A}$.

Let $E_0 \in \operatorname{ind}\mathcal{K}$ be a peripheral object and denote $E = \oplus_{i=1}^r E_0$

With $\mathcal{K}$ we may associate a twist functor $T_E : D^b\mathcal{A} \longrightarrow D^b\mathcal{A}$, where $A = \operatorname{End}(E)$ such that

$$T_E(X) \cong \operatorname{cone}(E \otimes_A \operatorname{RHom}(E, X) \longrightarrow X).$$

Since E is 1-spherical, this functor is an autoequivalence of $D^b\mathcal{A}$.

To prove that E is semi-simple in $\mathcal{H}$, we shall show that the canonical map ϵ : $E \otimes \operatorname{Hom}(E, X) \longrightarrow X$ in $\mathcal{H}$ is a monomorphism for all indecomposable $X \in \operatorname{Ob}\mathcal{H}$.

Consider the following exact sequence in $\mathcal{H}$

$$0 \longrightarrow H^{-1}T_E(X) \longrightarrow E \otimes_A \operatorname{Hom}(E, X) \longrightarrow X \longrightarrow H^0 T_E(X) \longrightarrow E \otimes_A \operatorname{Ext}(E, X) \longrightarrow 0.$$

Since $\mathcal{H}$ is hereditary and $T_E(X)$ is indecomposable, either $H^{-1}T_E(X)$ or $H^0T_E(X)$ is zero. Seeking a contradiction, we shall assume ϵ is an epimorphism and hence $H^{-1}T_E(X) \in \operatorname{ind}\mathcal{H}$. In particular, there is a path from E to $H^{-1}T_E(X)$.

Since T_E is an autoequivalence and there is a path from E to $T_E(X)[-1]$, there must also be a path from $T_E^{-1}(E) \cong E$ to $X[-1]$, a contradiction. □

Let $\mathcal{K}$ be a standard tube of rank r with peripheral objects $\tau^i E_0$, for $0 \leq i \leq r-1$. Such a tube corresponds with an abelian subcategory of $\mathcal{A}$, equivalent to the category of nilpotent representations of $\tilde{A}_r$ with cyclic orientation. Hence, any object A in $\mathcal{K}$ is a finite number of extensions of these peripheral objects. The number of peripheral objects occurring in the composition series of A will be denoted $l_{\mathcal{K}}A$; this corresponds to the normal length of A in $\operatorname{nilp}\tilde{A}_r$.

If $\mathcal{K}$ is a simple tube, then $l_{\mathcal{K}}A$ is equal to the normal length of A in $\mathcal{A}$.

Lemma 4.7. *Let $\mathcal{K}$ be a simple tube and write E for the direct sum of all the peripheral (simple) objects in $\mathcal{K}$. Let X be an indecomposable object of $\mathcal{A}$ that does not lie in $\mathcal{K}$. If $\operatorname{Hom}(E, X) \neq 0$ for a peripheral object E of $\mathcal{K}$, then for any $l \in \mathbb{N}$, X has a subobject Y with $l_{\mathcal{K}}Y = l$.*

Proof. Applying the twist functor T to X gives an exact sequence

$$0 \longrightarrow E \otimes_A \operatorname{Hom}(E, X) \longrightarrow X \longrightarrow T_E(X) \longrightarrow E \otimes_A \operatorname{Ext}(E, X) \longrightarrow 0$$

which we may splice as

$$0 \longrightarrow E \otimes_A \operatorname{Hom}(E, X) \longrightarrow X \longrightarrow X_1 \longrightarrow 0$$

and

$$0 \longrightarrow X_1 \longrightarrow T_E(X) \longrightarrow E \otimes_A \operatorname{Ext}(E, X) \longrightarrow 0.$$

Note that X_1 is non-zero because X does not lie in the standard tube $\mathcal{K}$. Since T_E is indecomposable it follows readily from the second exact sequence that $\operatorname{Hom}(E, X_1) \neq 0$

Iteration shows we may find a subobject Y of X with $l_{\mathcal{K}}Y = l$ for any $l \in \mathbb{N}$. □

Lemma 4.8. *Let $\mathcal{K}$ be a standard tube of rank r, and let $A, B \in \operatorname{ind}\mathcal{K}$. If $l = \min\{l_{\mathcal{K}}A, l_{\mathcal{K}}B\}$, then there is a $k \in \mathbb{N}$ such that* $\dim \operatorname{Hom}(\tau^k A, B) \geq \frac{l}{r}$.

Proof. Recall that $\mathcal{K}$ corresponds to the category of nilpotent representations of the quiver $\tilde{A}_n$ with cyclic orientation. We will work in this last category.

If $l_{\mathcal{K}}A \leq l_{\mathcal{K}}B$, we will choose k such that the simple socle of A is isomorphic to the simple socle of B. In this case, $\dim \operatorname{Hom}(\tau^k A, B) \geq \frac{l}{r}$.

If $l_{\mathcal{K}}A > l_{\mathcal{K}}B$, then we choose k such that the simple top of A is isomorphic to the simple top of B. We find again $\dim \operatorname{Hom}(\tau^k A, B) \geq \frac{l}{r}$. □

Lemma 4.9. *Let $\mathcal{K}$ be a standard tube. If there are Auslander-Reiten components $\mathcal{K}_1$ and $\mathcal{K}_2$ different from $\mathcal{K}$ with maps from $\mathcal{K}_1$ to $\mathcal{K}$ and from $\mathcal{K}$ to $\mathcal{K}_2$, then $\mathcal{K}$ is not a simple tube.*

Proof. Seeking a contradiction, assume $\mathcal{K}$ does contain a simple object.

Let $X_1 \in \mathcal{K}_1$ and $X_2 \in \mathcal{K}_2$ be objects mapping nonzero to and from an object $Y \in \mathcal{K}$, respectively. We obtain from Lemma 4.7 and its dual that for every $l \in \mathbb{N}$, there is a quotient object Y_1 of X_1 and a subobject Y_2 of X_2, both lying in $\mathcal{K}$, with $l_{\mathcal{K}} Y_1 = l_{\mathcal{K}} Y_2 = l$.

If we write r for the rank of $\mathcal{K}$, then Lemma 4.8 yields $\dim \operatorname{Hom}(\tau^k Y_i, Y_j) \geq \frac{l}{r}$, for a certain $k \in \mathbb{Z}$. Since l may be chosen arbitrarily large, and since $\dim \operatorname{Hom}(\tau^k X_i, X_j) \geq \dim \operatorname{Hom}(\tau^k Y_i, Y_j)$, we find the required contradiction. □

Proof of Theorem 4.6(2) (assuming $\mathcal{K}$ is standard). We choose a tilt $\mathcal{H}$ of $\mathcal{A}$ as in Theorem 4.6(3), thus $\mathcal{K}$ corresponds to a simple tube in $\mathcal{H}$. It is clear that, if $X_0 \to X_1 \to \cdots \to X_n$ is a path in the original abelian category, then it is a path in the tilted category $\mathcal{H}$.

It follows from Lemma 4.9 that every object in this path lies in $\mathcal{K}$. □

This final result states that an Auslander-Reiten component is a standard tube if and only it contains a finite τ-orbit, and finishes the proof of both Theorems 4.6 and 4.5.

Lemma 4.10. *Let $\mathcal{K}$ be a component of the Auslander-Reiten quiver such that there is an $X \in \operatorname{ind} \mathcal{K}$ with $\tau^r X \cong X$ for $r > 0$, then $\mathcal{K}$ is a standard tube.*

Proof. First, we use X to fix a τ-invariant t-structure in $D^b\mathcal{A}$ in the usual way:

$$\begin{aligned} \operatorname{ind} \mathcal{D}^{\leq 0} &= \{Y \in \operatorname{ind} D^b\mathcal{A} \mid \text{There is a path from } \tau^n X \text{ to } Y, \text{ for a certain } n \in \mathbb{N}\} \\ \operatorname{ind} \mathcal{D}^{\geq 1} &= \operatorname{ind} D^b\mathcal{A} \setminus \mathcal{D}^{\leq 0} \end{aligned}$$

The translation τ thus defines an exact autoequivalence $\mathcal{H} \xrightarrow{\sim} \mathcal{H}$. We may assume r is the smallest natural number such that $\tau^r X \cong X$.

We claim there is a peripheral object S lying in a standard tube, such that there is a path from X to S and vice versa. It would then follows from Theorem 4.6(2) that X lies in the same Auslander-Reiten component as S and hence that $\mathcal{K}$ is a standard tube.

To show whether S is such an object, we need only to verify that the τ-period of S is finite and that $\sum_{i=0}^{s-1} \dim \operatorname{Hom}(S, \tau^i S) = 1$ where $s > 0$ is the smallest natural number such that $S \cong \tau^s S$. We will prove the existance of such an object S by induction on $d = \sum_{i=0}^{r-1} \dim \operatorname{Hom}(X, \tau^i X)$.

The case $d = 1$ is trivial, thus assume $d > 1$. In this case, there is a j with $0 \leq j < r$ such that $\operatorname{rad}(X, \tau^j X) \neq 0$. Fixing isomorphisms $X \cong \tau^r X$ and $X \cong \tau^{r+j} X$, the functor τ^r induces an automorphism of $\operatorname{Hom}(X, \tau^j X)$.

Let $f \in \operatorname{rad}(X, \tau^j X)$ be an eigenvector of this automorphism and denote $X_1 = \operatorname{im} f$. It is clear $\tau^r X_1 \cong X_1$. Furthermore, since X_1 is a quotient object of X and a subobject of $\tau^j X$, there are paths from X to X_1 and vice versa. Furthermore, there are monomorphisms

$$\operatorname{Hom}(X_1, \tau^i X_1) \hookrightarrow \operatorname{Hom}(X, \tau^{i+j} X) \text{ for } 0 \leq i < r,$$

such that, using that f is a radical map, we find

$$\sum_{i=0}^{r-1} \dim \operatorname{Hom}(X_1, \tau^i X_1) \leq \sum_{i=0}^{r-1} \dim \operatorname{Hom}(X, \tau^{i+j} X) - 1,$$

as $1_X \in \operatorname{Hom}(X, X)$ is easily seen to not factor through X_1.

Changing r in the left hand side to the smallest $r_1 > 0$ such that $X_1 \cong \tau^{r_1} X_1$, concludes the proof. $\square$

Chapter 5

Hereditary Calabi-Yau Categories

Let $\mathcal{A}$ be an abelian category with Serre duality. If the Serre functor $F : D^b\mathcal{A} \to D^b\mathcal{A}$ is naturally equivalent to $[n]$, then we will say $\mathcal{A}$ is Calabi-Yau of dimension n, or that $\mathcal{A}$ is n-Calabi-Yau. It is well-known that the global dimension of an abelian Calabi-Yau category is equal to its global dimension (See Proposition 5.5). Since our main interest lies with hereditary categories, we will only consider 1-Calabi-Yau categories.

Our main result (see [52]) concerning abelian 1-Calabi-Yau categories (reformulated in the body of the text as Theorem 5.13) is the following.

Theorem 5.1. *Let $\mathcal{A}$ be a connected abelian 1-Calabi-Yau category. Then $\mathcal{A}$ is* derived *equivalent to one of the following two categories.*

1. *Finite dimensional representations of $k[[t]]$.*
2. *The category of coherent sheaves on an elliptic curve.*

We mention the following particular application of this theorem. Recently Polishchuk and Schwartz [38] constructed a category $\mathcal{C}$ of holomorphic vector bundles on a *non-commutative 2-torus*. Polishchuk subsequently showed that $\mathcal{C}$ is derived equivalent to the category of coherent sheaves on an elliptic curve [38]. Part of Polishchuk's proof amounts to establishing the highly non-trivial fact that $\mathcal{C}$ is 1-Calabi-Yau [38, Cor 2.12]. Once one knows this, one could now finish the proof by simply invoking Theorem 5.1 (with $\mathcal{A}$ being a suitable abelian hull of $\mathcal{C}$).

In the proof of Theorem 5.13, we do not use the 1-Calabi-Yau property to its full extend, but only that $FX \cong X[1]$ for all $X \in \operatorname{Ob} D^b\mathcal{A}$. From the above theorem, we then obtain that $\mathcal{A}$ is 1-Calabi-Yau if and only if $\mathcal{A}$ satisfies the 1-Calabi-Yau property on objects alone.

As an extension of the definition of Calabi-Yau, we will follow [27] and say an abelian $\mathcal{A}$ with Serre duality is *fractionally Calabi-Yau* of dimension $\frac{m}{n}$ if $F^n \cong$

$[m]$ where $n > 0$. Note that a 1-Calabi-Yau category is fractionally Calabi-Yau of dimension 1, but the converse need not to be true (see Example 5.18 below).

For hereditary categories, we obtain following classification (Theorem 5.40 in the text).

Theorem 5.2. *Let $\mathcal{A}$ be a connected abelian hereditary category which is fractionally Calabi-Yau, but not 1-Calabi-Yau, then $\mathcal{A}$ is derived equivalent to either*

1. *the category of finite presented modules* $\operatorname{mod} Q$ *over a Dynkin quiver Q, or*
2. *the category of nilpotent representations* $\operatorname{nilp} \tilde{A}_n$ *where $n > 1$ and $\tilde{A}_n$ has cyclic orientation, or*
3. *the category of coherent sheaves* $\operatorname{coh} \mathbb{X}$ *over a weighted projective line of tubular type*

Here, the only categories with fractional Calabi-Yau dimension different from 1 are equivalent to $\operatorname{mod} Q$ over a Dynkin quiver Q. Once one deals with this case, the fractional Calabi-Yau and the 1-Calabi-Yau case are handled in a rather similar fashion.

Again, in the proof of Theorem 5.2, we only use that $F^n X \cong X[m]$, and not the full force of the fractionally Calabi-Yau property, namely that $F^n \cong [m]$. Apparently, it suffices to check the fractionally Calabi-Yau property on objects alone to see whether a hereditary category is fractionally Calabi-Yau.

If $\mathcal{A}$ is Calabi-Yau or fractionally Calabi-Yau of dimension 1, then every Auslander-Reiten component is a standard tube (see §4.2). We may assume that the Auslander-Reiten quiver of $\mathcal{A}$ has at least two different tubes, as the remaining case is easily disposed with. Using connectedness and results from §4.2 we may in fact select two peripheral objects L and S lying in different tubes such that $\operatorname{Hom}(L, S) \neq 0$ and let E be the direct sum of all peripheral object lying in the same tube as S. After doing so we consider the sequence of objects $\mathcal{E} = (T_E^n L)_{n \in \mathbb{Z}}$ in $D^b\mathcal{A}$, where T_E is a twist functor (see §1.12). We construct a certain associated t-structure on $D^b(\mathcal{A})$ with heart $\mathcal{H}$ such that $\mathcal{E}$ is a coherent sequence in the sense of [37] in $\mathcal{H}$.

If $\mathcal{A}$ is 1-Calabi-Yau, then $\mathcal{E}$ is even ample so that $\mathcal{E}$ defines a finitely presented graded coherent algebra A such that $\mathcal{H}$ is equivalent to the category $\operatorname{qgr}(A)$ of finitely presented graded A-modules modulo the finite dimensional ones.

We then show that A is a domain of Gelfand-Kirillov dimension two and we invoke the celebrated Artin and Stafford classification theorem [2] which shows that $\operatorname{qgr}(A)$ is of the form $\operatorname{coh}(X)$ for a projective curve X. Since $\mathcal{H}$ is 1-Calabi-Yau this implies that X must be an elliptic curve, finishing the proof.

It is not hard to describe the abelian 1-Calabi-Yau categories that occur within the derived equivalence classes in Theorem 5.1 (see e.g. [16]). We discuss this using the language of this paper in §5.2.4.

If $\mathcal{A}$ is not 1-Calabi-Yau but fractionally Calabi-Yau of dimension 1, then we will use the coherent sequence $\mathcal{E}$ to construct a tilting object in $D^b\mathcal{H}$. A well-known result of Happel's ([23]) then yields that $\mathcal{H}$ is equivalent to the category of coherent sheaves on a weighted projective line $\mathbb{X}$.

5.1 Ample sequences

For the benefit of the reader, we will recall some definitions and results from [37] which will be used in the rest of this chapter. Throughout, let $\mathcal{H}$ be a Hom-finite abelian category.

We begin with the definition of an ample sequence.

1. A sequence $\mathcal{E} = (L_i)_{i\in\mathbb{Z}}$ is called *projective* if for every epimorphism $X \to Y$ in $\mathcal{H}$ there is an $n \in \mathbb{Z}$ such that $\operatorname{Hom}(L_i, X) \to \operatorname{Hom}(L_i, Y)$ is surjective for $i < n$.

2. A projective sequence $\mathcal{E} = (L_i)_{i\in\mathbb{Z}}$ is called *coherent* if for every $X \in \operatorname{Ob}\mathcal{H}$ and $n \in \mathbb{Z}$, there are integers $i_1, \ldots, i_s \leq n$ such that the canonical map
$$\bigoplus_{j=1}^{s} \operatorname{Hom}(L_i, L_{i_j}) \otimes \operatorname{Hom}(L_{i_j}, X) \longrightarrow \operatorname{Hom}(L_i, X)$$
is surjective for $i \ll 0$.

3. A coherent sequence $\mathcal{E} = (L_i)_{i\in\mathbb{Z}}$ is *ample* if for all $X \in \mathcal{H}$ the map $\operatorname{Hom}(L_i, X) \neq 0$ for $i \ll 0$.

Let $A_{ij} = \operatorname{Hom}(L_i, L_j)$ for $i \leq j$. We may define an algebra $A = A(\mathcal{E}) = \oplus_{i\leq j} A_{ij}$ in a natural way. If $A_{ii} \cong k$, then A is a *coherent*[1] *$\mathbb{Z}$-algebra* in the sense of [37] (see [37, Proposition 2.3]).

We will refer to the right A-modules having a resolution by finitely generated projectives as *coherent modules.* These modules form an abelian category, $\operatorname{coh} A$, and the finite dimensional modules form a Serre subcategory denoted by $\operatorname{coh}^b A$. We define the quotient
$$\operatorname{cohproj} A \cong \operatorname{coh} A/\operatorname{coh}^b A.$$

Finally, let $\mathcal{E} = (L_i)_{i\in\mathbb{Z}}$ be a sequence. We will denote by $\mathcal{H}_0$ the full subcategory of $\mathcal{H}$ spanned by the objects $X \in \operatorname{Ob}\mathcal{H}$ with the property that $\operatorname{Hom}(L_i, X) = 0$ for $i \ll 0$. If $\mathcal{E}$ is projective, then $\mathcal{H}_0$ is a Serre subcategory of $\mathcal{H}$; if $\mathcal{E}$ is ample then $\mathcal{H}_0 = 0$.

We may use this to give a description of Ext-finite abelian categories with an ample sequence.

Theorem 5.3. *[37, Theorem 2.4] Let $\mathcal{E} = (L_i)$ be a coherent sequence, $A = A(\mathcal{E})$ the corresponding $\mathbb{Z}$-algebra, then there is a equivalence of categories $\mathcal{H}/\mathcal{H}_0 \cong \operatorname{cohproj} A$.*

[1] The formalism of $\mathbb{Z}$-algebras is equivalent to the study of preadditive categories whose object set is equal to $\mathbb{Z}$. In particular, the notion of coherence (see §1.2.2) in the two setting corresponds to each other.

We will be interested in the special case where there is an automorphism $t : D^b\mathcal{H} \longrightarrow D^b\mathcal{H}$ such that $L_i \cong t^iL$. We let $R = R(\mathcal{E}) = \oplus_{i\in\mathbb{N}} R_i$ where $R_i = \mathrm{Hom}(L, t^iL)$ and make it into a $\mathbb{Z}$-graded algebra in an obvious way.

If R is Noetherian then the coherent R-modules correspond to the finitely generated ones and $\mathrm{cohproj}\, R$ corresponds to $\mathrm{qgr}\, R$, the category of finitely generated modules modulo the finite dimensional ones, introduced in [1].

We will use following corollary of Theorem 5.3.

Corollary 5.4. *Let $\mathcal{A}$ be a Hom-finite abelian category, t be an autoequivalence of $\mathcal{A}$ and L an object of $\mathcal{A}$. If $\mathcal{E} = (t^iL)$ is a coherent sequence and the corresponding graded algebra $R = R(\mathcal{E})$ is noetherian, then $\mathcal{H}/\mathcal{H}_0 \cong \mathrm{qgr}\, R$.*

5.2 Abelian 1-Calabi-Yau categories

5.2.1 Preliminary results

Let $\mathcal{A}$ be an Ext-finite abelian category with Serre duality. We will say that $\mathcal{A}$ is *Calabi-Yau of dimension n* or *n-Calabi-Yau* if the Serre functor on $D^b\mathcal{A}$ is naturally equivalent to $[n]$, thus if for every $X, Y \in \mathrm{Ob}\, D^b\mathcal{A}$, there are isomorphisms

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, X[n])^*$$

natural in X and Y, and where $(-)^*$ is the vector space dual.

The following proposition relates the Calabi-Yau dimension and the global dimension of the abelian category $\mathcal{A}$.

Proposition 5.5. *Let $\mathcal{A}$ be an abelian n-Calabi-Yau category, then the global dimension of $\mathcal{A}$ is n.*

Proof. If follows from the n-Calabi-Yau property that for every $X \in \mathrm{Ob}\,\mathcal{A}$, the functor $\mathrm{Ext}^n(-, X) \cong \mathrm{Hom}(X, -)^*$ is right exact and not naturally isomorphic to 0, hence the global dimension of $\mathcal{A}$ is n. □

Since the Serre functor F is naturally isomorphic to $\tau[1]$, the category $\mathcal{A}$ will be n-Calabi-Yau if and only if $\tau \cong [n-1]$.

Restricting ourselves to the case where $\mathcal{A}$ is a 1-Calabi-Yau categories, we see that $\mathcal{A}$ is hereditary and that τ is naturally isomorphic to the identity functor.

Consequently, $\tau X \cong X$, for every $X \in \mathrm{ind}\,\mathcal{A}$, and Theorem 4.5 yields that every Auslander-Reiten component is a standard homogeneous tube, i.e. it is of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$, cf. Figure 5.1, were the bottom element is endo-simple. In particular, every object is a finite extension of endo-simple objects.

Since endo-simple objects will play an important role in the discussion of abelian 1-Calabi-Yau categories, we give an easy consequence of Theorem 4.6.

Corollary 5.6. *Let $\mathcal{A}$ be an abelian 1-Calabi-Yau category. Every cycle $X_0 \to X_1 \to \cdots \to X_n \to X_0$ of non-zero morphisms between indecomposable objects contains at most one isomorphism class of endo-simple objects.*

Figure 5.1: A homogeneous tube.

5.2.2 Elliptic curves

For the benefit of the reader, we recall certain properties of the category of coherent sheaves on an elliptic curve X. This category has first been described in [3]; a more recent treatment may be found in [15].

An elliptic curve is a curve of genus 1 and thus, in particular, $\mathcal{A} = \operatorname{coh} X$ is a 1-Calabi-Yau category.

Let $\mathcal{O}$ be the structure sheaf and, for a point P, let $k(P)$ be the corresponding torsion sheaf of length one. For a coherent sheaf $\mathcal{F}$ the *degree* and *rank* may be defined as

$$\begin{aligned} \deg \mathcal{E} &= \chi(\mathcal{O}, \mathcal{E}) \in \mathbb{Z}, \\ \operatorname{rk} \mathcal{E} &= \chi(\mathcal{E}, k(P)) \in \mathbb{Z}, \end{aligned}$$

respectively, and where $\chi(X,Y) = \dim \operatorname{Hom}(X,Y) - \dim \operatorname{Ext}(X,Y)$ is the Euler form. It follows from the Riemann-Roch theorem that

$$\chi(\mathcal{E}, \mathcal{F}) = \deg \mathcal{F} \operatorname{rk} \mathcal{E} - \deg \mathcal{E} \operatorname{rk} \mathcal{F}. \tag{5.1}$$

Furthermore, the *slope* of $\mathcal{E}$ is defined as $\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}} \in \mathbb{Q} \cup \{\infty\}$. A coherent sheaf $\mathcal{F}$ is called *stable* or *semi-stable* if for every short exact sequence $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ we have $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$ or $\mu(\mathcal{E}) < \mu(\mathcal{F})$, respectively.

It is well-known that all the indecomposable coherent sheaves are semi-stable. For stable sheaves, we have the following equivalent conditions

1. $\mathcal{E}$ is stable,

2. $\mathcal{E}$ is endo-simple, thus $\operatorname{End} \mathcal{E} \cong k$,

3. $\operatorname{rk} \mathcal{E}$ and $\deg \mathcal{E}$ are coprime.

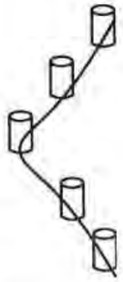


Figure 5.2: Sketch of $\mathcal{A}_\infty$

Every indecomposable (semi-stable) sheaf is a finite extension of an endo-simple one with itself. We may visualize this via the Auslander-Reiten quiver of $\operatorname{coh} X$. All Auslander-Reiten components are homogeneous tubes, i.e. components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$, cf. Figure 5.1, where the bottom element is a stable sheaf.

Every such tube corresponds to an abelian subcategory of $\operatorname{coh} X$ equivalent to $\operatorname{Mod}^{\mathrm{fd}} k[[t]]$ and all indecomposable objects in the same homogeneous tube have the same slope. Thus the full subcategory of $\operatorname{coh} X$ spanned by all indecomposable objects of a given slope θ is an abelian subcategory $\mathcal{A}_\theta$ of $\operatorname{coh} X$ and is of the form $\oplus \operatorname{Mod}^{\mathrm{fd}} k[[t]]$, where the sum is indexed by the isomorphism classes of stable objects with the given slope. For all $\theta, \theta' \in \mathbb{Q} \cup \{\infty\}$, the categories $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are equivalent. This equivalence is not unique.

It follows directly from (5.1) that, for non-isomorphic stable sheaves, $\mathcal{E}$ and $\mathcal{F}$, we have $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$ if and only if $\mu(\mathcal{E}) < \mu(\mathcal{F})$. Thus for semi-stable sheaves $\mathcal{E}'$ and $\mathcal{F}'$ we have $\operatorname{Hom}(\mathcal{E}', \mathcal{F}') \neq 0$ if and only if $\mu(\mathcal{E}') < \mu(\mathcal{F}')$ or $\mathcal{E}'$ and $\mathcal{F}'$ lie in the same tube.

Finally, we will give a global sketch of the Auslander-Reiten quiver of $\operatorname{coh} X$. There is a 1-1-correspondence between the points P of X and the torsion sheaves $k(P)$ of $\operatorname{coh} X$. With every torsion sheaf, there corresponds a unique homogeneous tube, and every indecomposable with infinite slope is lies in one of these tubes. The abelian category $\mathcal{A}_\infty$ of all semi-stable objects of infinite slope is thus a direct sum of homogeneous tubes, parametrized by the points of the elliptic curve. This invokes the image of Figure 5.2.

Since for all $\theta, \theta' \in \mathbb{Q} \cup \{\infty\}$ the categories $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are equivalent, we will sketch the Auslander-Reiten quiver of $\operatorname{coh} X$ as in Figure 5.3.

5.2.3 Classification

Let $\mathcal{A}$ be a connected k-linear abelian Ext-finite 1-Calabi-Yau category. In this section, we wish to classify all such categories up to derived equivalence. If every two endo-simples of $\mathcal{A}$ are isomorphic, then $\mathcal{A}$ is equivalent to the finite dimensional nilpotent representations of the one loop quiver.

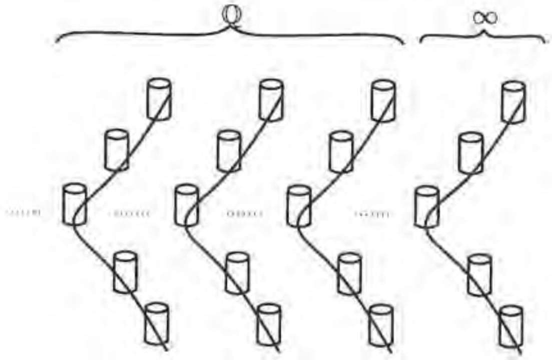


Figure 5.3: Sketch of the coherent sheaves on an elliptic curve.

So, assume there are at least two non-isomorphic endo-simples, L and B. By connectedness, we may assume $\operatorname{Hom}(L,B) \neq 0$. First, we will find a t-structure in $D^b\mathcal{A}$ such that the heart $\mathcal{H}$ admits an ample sequence $\mathcal{E}$. Then we will use Theorem 5.3 to show $\mathcal{A} \cong \operatorname{qgr} R(\mathcal{E})$. A discussion of $R(\mathcal{E})$ will then complete the classification of abelian 1-Calabi-Yau categories up to derived equivalence.

From here on, we will always denote $\operatorname{Hom}(L,B)$ by V and its dimension by d.

The sequence $\mathcal{E}$ and a t-structure in $D^b\mathcal{A}$

With L and B as above, associate the autoequivalence $t = T_B : D^b\mathcal{A} \longrightarrow D^b\mathcal{A}$ and the sequence $\mathcal{E} = (L_i)$ where $L_i = t^i L$.

The following will define a t-structure in $D^b\mathcal{A}$; the heart will be denoted by $\mathcal{H}$.

$$\begin{aligned} \operatorname{ind} D^{\leq 0} &= \{X \in \operatorname{ind} D^b\mathcal{A} \mid \text{there is a path from } L_i \text{ to } X, \text{ for an } i \in \mathbb{Z}\} \\ \operatorname{ind} D^{\geq 1} &= \operatorname{ind} D^b\mathcal{A} \setminus \operatorname{ind} D^{\leq 0} \end{aligned}$$

If follows directly from this definition that t restricts to an autoequivalence on $\mathcal{H}$, which we will also denote by t. Note that this implies $L_i \in \operatorname{Ob}\mathcal{H}$, for all $i \in \mathbb{Z}$. Also, since $\operatorname{Hom}(B[-1], L_i) \neq 0$, there is no path from L_i to $B[-1]$ and hence we have $B \in \operatorname{Ob}\mathcal{H}$.

It follows from Theorem 1.30 that $\mathcal{H}$ is hereditary and $D^b\mathcal{H} \cong D^b\mathcal{A}$. Since $\mathcal{H}$ is a 1-Calabi-Yau category, the results we have proved about $\mathcal{A}$ apply to $\mathcal{H}$ as well.

Note that, since $t^i B \cong B$, we find there is a natural isomorphism $\operatorname{Hom}(L,B) \cong \operatorname{Hom}(L_i, B)$ and as such, we get triangles of the form $B[-1] \otimes V^* \longrightarrow L_{i-1} \longrightarrow L_i \longrightarrow B \otimes V^*$. Such a triangle in $D^b\mathcal{A}$ gives rise to an exact sequence

$$0 \longrightarrow L_{i-1} \longrightarrow L_i \longrightarrow B \otimes V^* \longrightarrow 0$$

in $\mathcal{H}$, which is the universal extension of L_{i-1} with B and all these exact sequences are transformed into each other by t.

Lemma 5.7. *Let $\mathcal{E} = (L_i)_{i\in I}$ and B as above, then*

1. $\mathrm{Hom}(L_i, L_j) = \mathrm{Ext}(L_j, L_i) = 0$ *for* $i > j$,
2. $\mathrm{Hom}(B, L_i) = \mathrm{Ext}(L_i, B) = 0$ *for all* $i \in I$.

Proof. Recall that L and B are nonisomorphic endo-simples, hence they lie in different tubes. Since there is a map $L \to B$, Corollary 5.6 yields the required result. □

If $\mathcal{H}$ is of the form $\mathrm{coh}\, X$ for an elliptic curve X (which we will show below to be the case) one may verify that L corresponds to a stable vector bundle of rank $\dim V$ and B to the structure sheaf $k(P)$ of a point P. The L_i are equal to $L(-iP)$.

$\mathcal{E}$ is an ample sequence in $\mathcal{H}$

We now wish to show the sequence $\mathcal{E} = (L_i)_{i\in\mathbb{Z}}$ is ample. The following lemma will be useful.

Lemma 5.8. *If* $\mathrm{Hom}(L_i, X) \neq 0$, *then* $\mathrm{Hom}(L_j, X) \neq 0$ *for all* $j \leq i$.

Proof. It suffices to show that $\dim \mathrm{Hom}(L_{i-1}, X) \geq \dim \mathrm{Hom}(L_i, X)$. Apply the functor $\mathrm{Hom}(-, X)$ to the exact sequence

$$0 \longrightarrow L_{i-1} \longrightarrow L_i \longrightarrow B \otimes V^* \longrightarrow 0.$$

If $\mathrm{Hom}(L_i, X) \neq 0$ and $\mathrm{Hom}(L_{i-1}, X) = 0$ then $\mathrm{Hom}(B \otimes V^*, X) \neq 0$. Since there is an epimorphism $L_j \to B \otimes V^*$, we find $\mathrm{Hom}(L_j, X) \neq 0$. □

Proposition 5.9. *In $\mathcal{H}$ the sequence $\mathcal{E} = (L_i)$ is ample.*

Proof. First, we will show $\mathcal{E}$ is projective. Therefore, let $X \to Y$ be an epimorphism and let K be the kernel. By the construction of $\mathcal{H}$ in §5.2.3, we know there are paths from the sequence $\mathcal{E}$ to every direct summand of K. Hence, by Corollary 5.6, we know $\mathrm{Hom}(K, L_i) = 0$ for $i \ll 0$ and, by the Calabi-Yau property, $\mathrm{Ext}(L_i, K) = 0$. Thus $\mathrm{Hom}(L_i, X) \to \mathrm{Hom}(L_i, Y)$ is surjective for $i \ll 0$.

Next, we will show $\mathcal{E}$ is coherent. Thus we consider an indecomposable object $X \in \mathcal{H}$. If $\mathrm{Hom}(L_j, X) = 0$ for all $j \in \mathbb{Z}$, then the statement is empty, so we may assume there is a $j \in \mathbb{Z}$ such that $\mathrm{Hom}(L_{j+2}, X) \neq 0$, and hence by Lemma 5.8, that $\mathrm{Hom}(L_i, X) \neq 0$ for all $i < j+2$. Fix an $i < j$, we will prove that $f : L_{i-1} \longrightarrow X$ factors through $L_i \oplus L_j$. Iteration then implies f factors through a number of copies of $L_{j-1} \oplus L_j$, and hence $\mathcal{E}$ is coherent.

To prove previous claim, it will be convenient to work in the derived category. The following two triangles in $D^b\mathcal{H}$ will be used

$$B \otimes V^*[-1] \xrightarrow{\theta} L_{i-1} \longrightarrow L_i \longrightarrow B \otimes V^* \tag{5.2}$$

and

$$B \otimes V^*[-1] \xrightarrow{\varphi} L_j \longrightarrow L_{j+1} \longrightarrow B \otimes V^* \tag{5.3}$$

where $V = \mathrm{Hom}(L_i, B) \cong \mathrm{Hom}(L_{j+1}, B)$. We may assume $f : L_{i-1} \longrightarrow X$ does not factor though L_i, hence from triangle (5.2) it follows that the composition $f \circ \theta \neq 0$.

Note that, since $\mathrm{Hom}(L_{j+1}, X) \neq 0$, we may use Corollary 5.6 to see $\mathrm{Hom}(X, L_{j+1}) = 0$, and hence also $\mathrm{Ext}(L_{j+1}, X) = 0$.

Applying the functor $\mathrm{Hom}(-, X)$ on the triangle (5.3) and using $\mathrm{Ext}(L_{j+1}, X) = 0$, shows that every map $B \otimes V^*[-1] \longrightarrow X$ factors though φ. Hence there is a morphism $g : L_j \longrightarrow X$ such that the following diagram commutes.

$$\begin{tikzcd} B \otimes V^*[-1] \arrow[r, "\theta"] \arrow[d, "\varphi"'] & L_{i-1} \arrow[d, "f"] \\ L_j \arrow[r, "g"] & X \end{tikzcd}$$

Furthermore, applying $\mathrm{Hom}(-, L_j)$ to triangle (5.2) yields (using Lemma 5.7) that φ factors through θ, hence there is a map $h : L_{i-1} \longrightarrow L_j$ such that $g \circ h \circ \theta = f \circ \theta$, or $(g \circ h - f) \circ \theta = 0$.

Summarizing, $f = g \circ h + f'$, where $f' : L_{i-1} \longrightarrow X$ lies in $\ker(\theta, X)$ and as such factors through L_i. The map f factors though $L_i \oplus L_j$ and we may conclude the sequence $\mathcal{E}$ is coherent.

Finally, we show the sequence $\mathcal{E}$ is ample. Let X be an indecomposable object. Due to the construction of $\mathcal{H}$, we know that there is an (oriented) path from L_n to X, for a certain $n \in \mathbb{Z}$. Thus it suffices to prove that if $\mathrm{Hom}(L_n, X) \neq 0$, then there is a finite set $I \subset \mathbb{Z}$ such that

$$\bigoplus_{i \in I} L_i \otimes \mathrm{Hom}(L_i, X) \longrightarrow X$$

is an epimorphism.

Let $i_1, \ldots, i_m \in \mathbb{Z}$ be as in the definition of coherence. Consider the map

$$\theta : \bigoplus_{j=1}^{m} L_{i_j} \otimes \mathrm{Hom}(L_{i_j}, X) \longrightarrow X \tag{5.4}$$

and let $C = \operatorname{coker} \theta$. To ease notation, we will write $M = \bigoplus_{j=1}^{m} L_{i_j} \otimes \mathrm{Hom}(L_{i_j}, X)$.

There is an exact sequence $0 \longrightarrow \operatorname{im} \theta \longrightarrow X \longrightarrow C \longrightarrow 0$. Using the Calabi-Yau property, we see $\mathrm{Hom}(\operatorname{im} \theta, C) \neq 0$, and since $\operatorname{im} \theta$ is a quotient object of M, this yields $\mathrm{Hom}(M, C) \neq 0$. Hence we may assume there is an i_j such that $\mathrm{Hom}(L_{i_j}, C) \neq 0$.

Since $\mathcal{E}$ is projective, there is an $l \ll 0$ such that the induced map in $\mathrm{Hom}(L_l, C)$ lifts to a map in $\mathrm{Hom}(L_l, X)$. Again using coherence, this map should factor through M. We may conclude $C = 0$, and hence θ is an epimorphism. □

Description of $R = R(\mathcal{E})$

Having shown in Proposition 5.9 that $\mathcal{E}$ is an ample sequence, we may invoke Proposition 5.3 to see the that $\mathcal{H} \cong \operatorname{cohproj} A(\mathcal{E})$.

We will now proceed to discuss the graded algebra $R = R(\mathcal{E})$. In particular, we wish to show R is a finitely generated domain of Gelfand-Kirillov dimension 2 which admits a Veronese subalgebra generated in degree one. It would then follow from [2] that R is noetherian and that $\operatorname{qgr} R$ is equivalent to $\operatorname{coh} X$ where X is a curve, while it would follow from Corollary 5.4 that $\mathcal{H} \cong \operatorname{qgr} R$.

We start by showing $\operatorname{GKdim} R = 2$.

Lemma 5.10. *Let $\mathcal{E} = (L_i)_{i \in I}$ and B be as before. If $j > i$, then*

$$\dim \operatorname{Hom}(L_i, L_j) = (j - i)d^2$$

where $d = \dim \operatorname{Hom}(L_0, B)$.

Proof. We apply $\operatorname{Hom}(L_i, -)$ to the short exact sequence

$$0 \longrightarrow L_{j-1} \longrightarrow L_j \longrightarrow B \otimes \operatorname{Hom}(L_0, B)^* \longrightarrow 0.$$

We will proceed by induction on $j > i$. Note that $\dim \operatorname{Hom}(L_i, B) = \dim \operatorname{Hom}(L_0, B)^* = d$ and Lemma 5.7 implies that $\operatorname{Ext}(L_i, L_j) = 0$.

If $j = i + 1$, then it follows from $\dim \operatorname{Hom}(L_i, L_i) = \dim \operatorname{Ext}(L_i, L_i) = 1$ that $\dim \operatorname{Hom}(L_i, L_j) = d^2$. For higher j, we find by induction $\dim \operatorname{Hom}(L_i, L_j) = (j - i)d^2$. □

Lemma 5.11. *Assume L and B are non-isomorphic endo-simple objects of $D^b\mathcal{A}$ chosen such that $d = \dim \operatorname{Hom}_{D^b\mathcal{A}}(L, B)$ is minimal and $d \neq 0$. Then R is a domain.*

Proof. It suffices to show every non-zero non-isomorphism $f : L_0 \longrightarrow L_i$ is a monomorphism. We will prove this by induction on i. The case $i = 0$ is trivial. So let $i \geq 1$.

Since $\operatorname{im} f$ is a quotient object of L_0 and $\dim \operatorname{Hom}(L, B) = d$, it follows that $\dim \operatorname{Hom}(\operatorname{im} f, B) \leq d$, and due to the minimality of d, we know that either $\dim \operatorname{Hom}(\operatorname{im} f, B)$ 0, or $\dim \operatorname{Hom}(\operatorname{im} f, B) = d$, and that $\operatorname{im} f$ is an endo-simple object.

If $\dim \operatorname{Hom}(\operatorname{im} f, B) = 0$, the inclusion $\operatorname{im} f \hookrightarrow L_i$ has to factor through a map $j : \operatorname{im} f \longrightarrow L_{i-1}$.

$$\begin{tikzcd}
 & & L_0 \arrow[d] & & \\
 & & \operatorname{im} f \arrow[dl, "\exists j"'] \arrow[d, hook] & & \\
0 \arrow[r] & L_{i-1} \arrow[r] & L_i \arrow[r] & B \otimes \operatorname{Hom}(L_i, B)^* \arrow[r] & 0
\end{tikzcd}$$

Composition gives a non-zero map $L_0 \longrightarrow L_{i-1}$ which is a monomorphism by the induction hypothesis. We conclude that f is a monomorphism.

We are left with $\dim \operatorname{Hom}(\operatorname{im} f, B) = d$. By the minimality of d, we find that the monomorphism $\operatorname{Hom}(\operatorname{im} f, B) \to \operatorname{Hom}(L_0, B)$ is an isomorphism, and hence $\dim \operatorname{Hom}(K, B)$

0 where $K = \ker f$. With $\mathcal{E}$ being ample, we may assume there is a $k \in \mathbb{Z}$, such that L_k maps non-zero to every direct summand of K. Using the exact sequence $0 \to L_k \to L_{k+1} \to B \otimes \mathrm{Hom}(L_{k+1}, B)^* \to 0$ and $\mathrm{Hom}(K, B) = 0$, we find that $\dim \mathrm{Hom}(L_k, K) = \dim \mathrm{Hom}(L_l, K)$, for all $l \in \mathbb{Z}$, hence by Lemma 5.7 $\dim \mathrm{Hom}(K, L_l) = 0$.

Since K is a subobject of L_0, this shows $K \cong 0$. We conclude that f is a monomorphism. □

In general, however, R will not be generated in degree 1. We show that the Veronese subalgebra $R^{(3)} = \oplus_k R_{3k}$ of R is generated in degree 1.

Lemma 5.12. *The sequence $\mathcal{E}^{(3)} = (L_{3k})_{k\in\mathbb{Z}}$ is an ample sequence. Furthermore $R^{(3)} = R(\mathcal{E}^{(3)})$ is generated in degree 1.*

Proof. The sequence $\mathcal{E}^{(3)}$ is projective and ample since $\mathcal{E}$ is. Coherence of $\mathcal{E}^{(3)}$ may be shown as in the proof of Proposition 5.9.

Next, we prove $R^{(3)}$ is generated in degree one. Therefore, it suffices to show that for every $k > 1$ every map $L_0 \to L_{3k}$ factors through the canonical map $\theta : L_0 \to L_3 \otimes \mathrm{Hom}(L_0, L_3)^*$. It follows from the proof of Lemma 5.11 that θ is a monomorphism. Writing $V = \mathrm{Hom}(L_0, L_3)$, we have the following short exact sequence

$$0 \longrightarrow L_0 \xrightarrow{\theta} L_3 \otimes V^* \longrightarrow Q \longrightarrow 0$$

where $Q[0] = T_{L_3} L_0$ is an endo-simple object since T_{L_3} is an automorphism in $D^b\mathcal{A}$. Applying the functor $\mathrm{Hom}(-, L_{3k})$ to the short exact sequence above gives the exact sequence

$$0 \to \mathrm{Hom}(Q, L_{3k}) \to \mathrm{Hom}(L_3 \otimes V^*, L_{3k}) \to \mathrm{Hom}(L_0, L_{3k}) \to \mathrm{Ext}(Q, L_{3k}) \to 0.$$

We now consider the dimensions of these vector spaces. Since

$$\dim \mathrm{Hom}(L_0, L_{3k}) = (3k)d^2 < \dim \mathrm{Hom}(L_3 \otimes V^*, L_{3k}) = 9(k-1)d^4$$

we may see $\mathrm{Hom}(Q, L_{3k}) \neq 0$ and $\dim \mathrm{Ext}(Q, L_{3k}) \neq \dim \mathrm{Hom}(Q, L_{3k})$, hence $L_{3k} \not\cong Q$.

Using Corollary 5.6, we obtain $\mathrm{Ext}(Q, L_{3k}) = 0$, hence every map $L_0 \longrightarrow L_{3k}$ lifts through θ and the algebra $R^{(3)}$ is generated in degree one. □

Classification up to derived equivalence

We are now ready to prove the main result of this article.

Theorem 5.13. *Let $\mathcal{A}$ be a connected k-linear abelian Ext-finite 1-Calabi-Yau category, then $\mathcal{A}$ is derived equivalent to either*

1. *the category of finite dimensional representations of $k[[t]]$, or*

2. the category of coherent sheaves on an elliptic curve X.

Proof. First, assume $\mathcal{A}$ consists of only one (homogeneous ans standard) tube; in this case, we easily see that $\mathcal{A}$ is equivalent to $\operatorname{Mod}^{\mathrm{fd}} k[[t]]$.

Next, assume there are at least two tubes. Since $\mathcal{A}$ is connected and we may choose two endo-simples, L and B, such that $\operatorname{Hom}(L, B) \neq 0$, yet with a minimal dimension. Let $\mathcal{H}$ be the abelian category constructed in §5.2.3.

By Lemmas 5.11 and 5.12, we know $R^{(3)} = R(\mathcal{E}^{(3)})$ is a domain of GK-dimension 2 which is finitely generated by elements of degree one, hence by [2] we find that $R^{(3)}$ is noetherian and $\operatorname{qgr} R^{(3)}$ is equivalent to the coherent sheaves on a curve X.

Since R is noetherian, it follows from 5.3 that $\mathcal{H}$ is equivalent to $\operatorname{qgr} R^{(3)}$.

The structure sheaf $\mathcal{O}_X$ of X is an endo-simple object. Since the genus of X is $\dim H^1(\mathcal{O}_X) = \dim \operatorname{Ext}(\mathcal{O}_X, \mathcal{O}_X) = \dim \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) = 1$, we know X is an elliptic curve. □

Remark 5.14. In the proof of Theorem 5.13 we do not use $F \cong [1]$, but only $FX \cong X[1]$, for all $X \in \operatorname{Ob} D^b\mathcal{A}$. In other words, we only use that $\mathcal{A}$ satisfies the 1-Calabi-Yau property *on objects*. It then follows from Theorem 5.13 that $\mathcal{A}$ is 1-Calabi-Yau.

5.2.4 Classification of abelian 1-Calabi-Yau categories

We will now combine Theorem 5.13 with [16, Proposition 5.1] to obtain a description of all abelian 1-Calabi-Yau categories. First, we recall some results from [24].

Let $\mathcal{A}$ be any hereditary abelian category. A *torsion theory* on $\mathcal{A}$, $(\mathcal{F}, \mathcal{T})$, is a pair of full additive subcategories of $\mathcal{A}$, such that $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and having the additional property that for every $X \in \operatorname{Ob} \mathcal{A}$ there is a short exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $F \in \mathcal{F}$ and $T \in \mathcal{T}$.

We will say the torsion theory $(\mathcal{F}, \mathcal{T})$ is *split* if $\operatorname{Ext}(\mathcal{F}, \mathcal{T}) = 0$. In case of a split torsion theory we obtain, by *tilting*, a hereditary category $\mathcal{H}$ derived equivalent to $\mathcal{A}$ with an induced split torsion theory $(\mathcal{T}, \mathcal{F}[1])$. Furthermore, the category $\mathcal{H}$ will only be hereditary if and only if $(\mathcal{F}, \mathcal{T})$ is a split torsion theory.

We now discuss all possible torsion theories when $\mathcal{A}$ is equivalent to $\operatorname{coh} X$. Note that, since $\mathcal{H}$ will be 1-Calabi-Yau and hence hereditary, all torsion theories on $\mathcal{A}$ will be split.

Let $(\mathcal{F}, \mathcal{T})$ be a torsion theory on $\mathcal{A}$, and let $\mathcal{E}$ be an indecomposable of $\mathcal{T}$. Then every indecomposable $\mathcal{F}$ with slope strictly larger than $\mu(\mathcal{E})$ has to be in $\mathcal{T}$ since $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$. Furthermore, if $\mathcal{E}$ is in $\mathcal{T}$ and there is a path from $\mathcal{E}$ to an indecomposable $\mathcal{E}'$, then $\mathcal{E}' \in \operatorname{ind} \mathcal{T}$.

We may now give a characterization of all possible torsion theories.

Theorem 5.15. *[16] Let X be an elliptic curve. Every category $\mathcal{H}$ derived equivalent to $\mathcal{A} = \operatorname{coh} X$ may be obtained by tilting with respect to a torsion theory. Moreover, all* *torsion theories on* $\operatorname{coh} X$ *are split and may be described as follows. Let* $\theta \in \mathbb{R} \cup \{\infty\}$. *Denote by* $\mathcal{A}_{>\theta}$ *and* $\mathcal{A}_{\geq\theta}$ *the subcategory of* $\mathcal{A}$ *generated by all indecomposables* $\mathcal{E}$ *with* $\mu(\mathcal{E}) > \theta$ *and* $\mu(\mathcal{E}) \geq \theta$, *respectively. All full subcategories* $\mathcal{T}$ *of* $\mathcal{A}$ *with* $\mathcal{A}_{\geq\theta} \subseteq \mathcal{T} \subseteq \mathcal{A}_{>\theta} \subseteq \mathcal{A}$ *give rise to a torsion theory* $(\mathcal{F}, \mathcal{T})$, *with* $\operatorname{ind} \mathcal{F} = \operatorname{ind} \mathcal{A} \setminus \operatorname{ind} \mathcal{T}$.

Proof. That these are all possible torsion theories, follows from the above discussion. That all categories $\mathcal{H}$ may be obtained in this way, is shown in [16, Proposition 5.1]. Alternatively, it is straightforward to check these torsion theories generate all bounded t-structures on $D^b\mathcal{A}$ up to shifts. □

Example 5.16. We give some examples of torsion theories. In here $\mathcal{H}$ always stands for the category tilted with respect to the described torsion theory.

1. If $\theta \in \mathbb{Q} \cup \{\infty\}$ and $\mathcal{T} = \mathcal{A}_{>\theta}$, then the tilted category $\mathcal{H}$ is equivalent to $\operatorname{coh} X$. Indeed, it follows from the proof of Theorem 5.13 that $\mathcal{H}$ is equivalent to $\operatorname{coh} Y$ for an elliptic curve Y, and then from [26] that X and Y are isomorphic.
2. If $\mathcal{T} = \mathcal{A}_{\geq\theta}$, then $\mathcal{H}$ is dual to $\mathcal{A}$. This follows from Grothendieck duality.
3. If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathcal{T} = \mathcal{A}_{>\theta} = \mathcal{A}_{\geq\theta}$ then $\mathcal{H}$ is equivalent to the category of holomorphic bundles on a noncommutative two-torus ([36]).

If $\mathcal{A}$ consists of exactly one tube, then $\mathcal{A}$ is not only derived equivalent to the category of finite dimensional representations of $k[[t]]$, but also equivalent.

5.3 Fractionally Calabi-Yau categories

5.3.1 Definitions and examples

Let $\mathcal{A}$ be an Ext-finite abelian category. Recall that $\mathcal{A}$ is Calabi-Yau of dimension n if $D^b\mathcal{A}$ has a Serre functor $F : D^b\mathcal{A} \to D^b\mathcal{A}$, and $F \cong [n]$.

By extension, if $F^n \cong [m]$ for some $n > 0$, then we will say $\mathcal{A}$ is *fractionally Calabi-Yau of dimension* $\frac{m}{n}$.

Example 5.17. [27] Let Q be a Dynkin quiver, and let h be its Coxeter number. In $D^b \operatorname{mod} Q$ we find

$$F^h = (\tau[1])^h = \tau^h[h] = [h-2]$$

and hence $D^b \operatorname{mod} Q$ is Calabi-Yau of fractional dimension $\frac{h-2}{h}$.

Example 5.18. Let Q be the quiver $\tilde{A}_n$ with cyclic orientation. In the category $D^b \operatorname{nilp} Q$, we have $F^n = [n]$, hence $\operatorname{nilp} Q$ is Calabi-Yau with fractional dimension 1, yet it will only be 1-Calabi-Yau if Q is the one-loop quiver.

There is no strong connection between the fractional Calabi-Yau dimension and the global dimension of an abelian category as in the case of n-Calabi-Yau categories (see Proposition 5.5). We do however have the following proposition.

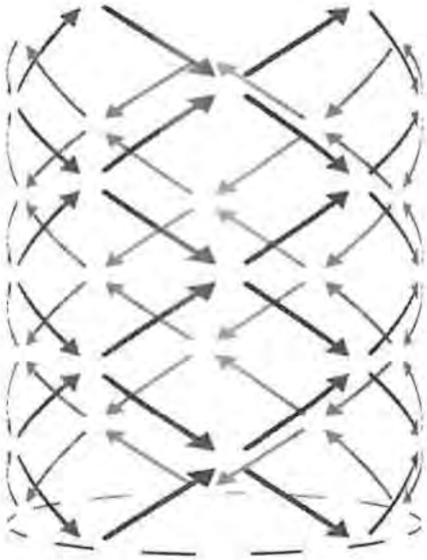


Figure 5.4: Auslander-Reiten quiver of nilp $\tilde{A}_4$

Proposition 5.19. *Let $\mathcal{A}$ be a hereditary (but not semi-simple) abelian category which is fractionally Calabi-Yau of dimension d, then $0 \leq d \leq 1$.*

Proof. Let $F^n \cong [m]$ for some $n > 0$, thus $d = \frac{m}{n}$. We find $\tau^n \cong [m-n]$. Since for every $X \in \operatorname{ind} D^b\mathcal{A}$ there is a path from $\tau^n X \cong X[m-n]$ to X, we have $m - n \leq 0$, thus $d \leq 1$.

For the lower bound of d, let $X \in \operatorname{ind} \mathcal{A}[0]$. If X is projective in $\mathcal{A}[0]$, then $\tau X \in \operatorname{ind} \mathcal{A}[-1]$, otherwise $\tau X \in \operatorname{ind} \mathcal{A}[0]$. It now follows from iteration $\tau^a X \in \mathcal{A}[b]$ where $b \leq -a$, hence $n \leq n - m$ and thus $d \geq 0$.

Finally, if $d = 0$ then the above reasoning shows that every object of $\mathcal{A}$ is projective-injective, and thus $\mathcal{A}$ is a Frobenius category. Since $\mathcal{A}$ is not semi-simple, it cannot be hereditary. □

Not every abelian category with fractional Calabi-Yau dimension d where $0 < d \leq 1$ is hereditary as following example shows. We believe, however, that any such category is derived equivalent to a hereditary one.

Example 5.20. Let Q be the quiver $\cdot \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot$ with relation $\beta\alpha = 0$, then $\operatorname{mod} Q$ is not hereditary, yet it is derived equivalent to $\operatorname{mod} A_2$, and as such is Calabi-Yau of fractional dimension $\frac{1}{3}$.

5.3.2 Representations of Dynkin quivers

As already seen in Example 5.17, the category of finite dimensional representations of Dynkin quivers gives examples of fractionally Calabi-Yau categories. We will show these are the only hereditary fractional Calabi-Yau categories with dimension strictly smaller than one.

We start with following lemma.

Lemma 5.21. *Let $\mathcal{A}$ be a connected Ext-finite abelian hereditary category with Serre duality. Then $\mathcal{A} \cong \operatorname{mod} kQ$ where Q is a Dynkin quiver if and only if for all $X \in \operatorname{ind} \mathcal{A}$ we have $d^\bullet(X[1], X) \neq \infty$.*

Proof. Since $\mathcal{A}$ is hereditary, there is no path from $X[1]$ to X thus $d^\bullet(X[1], X) \geq 0$.

If $\mathcal{A} \cong \operatorname{mod} Q$ where Q is Dynkin, then it is well-known that $d^\bullet(X[1], X) \neq \infty$, for every $X \in \operatorname{ind} D^b\mathcal{A}$. Indeed, if h is the Coxeter number of Q, then $\tau^h \cong [-2]$ in $D^b \operatorname{mod} Q$ such that

$$\begin{aligned} d^\bullet(X[1], X) &= d^\bullet(X[2], X[1]) \\ &\leq d^\bullet(X[2], X) + d^\bullet(X, X[1]) \\ &\leq h \end{aligned}$$

where we have used $0 \geq d^\bullet(X, X[1])$ since there is a path from X to $X[1]$, hence $d^\bullet(X[1], X) \neq \infty$.

For the other implication, let $X \in \operatorname{ind} D^b\mathcal{A}$. We know that $d^\bullet(X[1], X) \neq \infty$, hence Corollary 6.12 yields that X is directing. We may now use Theorem 3.44 to see $\mathcal{A}$ is equivalent to $\operatorname{mod} kQ$ for a Dynkin quiver Q (see Remark 3.46).

□

The classification of abelian hereditary categories with fractional Calabi-Yau dimension $d < 1$ follows directly from following proposition, which might be of independent interest.

Proposition 5.22. *Every connected Ext-finite abelian hereditary category $\mathcal{A}$ with Serre duality which is not equivalent to $\operatorname{mod} Q$ with Q a Dynkin quiver, is derived equivalent to a hereditary category $\mathcal{H}$ without any nonzero projectives and injectives.*

Proof. Let $A \in \operatorname{ind} D^b\mathcal{A}$, and let $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$ define a t-structure, where

$$\begin{aligned} \operatorname{ind} \mathcal{D}^{\geq 1} &= \{X \in \operatorname{ind} D^b\mathcal{A} \mid \text{there is no path from } \tau^n A \text{ to } X \text{ for an } n \in \mathbb{N}\} \\ \operatorname{ind} \mathcal{D}^{\leq 0} &= \{X \in \operatorname{ind} D^b\mathcal{A} \mid \text{there is a path from } \tau^n A \text{ to } X \text{ for an } n \in \mathbb{N}\} \end{aligned}$$

It follows from Lemma 5.21 that this is a bounded t-structure whenever $\mathcal{A}$ is not equivalent to $\operatorname{mod} Q$ with Q a Dynkin quiver. □

Proposition 5.23. *Let $\mathcal{A}$ be a connected abelian hereditary category which is fractional Calabi-Yau of dimension $d < 1$, then $\mathcal{A}$ is equivalent to* $\operatorname{mod} Q$ *where Q is a Dynkin quiver.*

Proof. Let $F^n = [m]$ such that the fractional Calabi-Yau dimension of $\mathcal{A}$ is $\frac{m}{n} < 1$, then it follows from $F \cong \tau[1]$ that $\tau^n \cong [m-n]$ where $m-n < 0$, hence $\mathcal{A}$ has nonzero projectives and injectives.

If $\mathcal{A}$ is fractional Calabi-Yau of dimension $d < 1$, then so is every category $\mathcal{H}$ derived equivalent to $\mathcal{A}$. Hence $\mathcal{H}$ must have nonzero projectives and injectives, contradicting Proposition 5.22. □

5.3.3 Weighted projective lines

Weighted projective lines were first introduced in [19] as lines in a weighted projective plane. They may be seen as generalizations of projective lines were finitely many points x have been given a weight $p(x) \in \mathbb{N}$ strictly larger than 1.

Following more recent treatments, we will define a weighted projective line $\mathbb{X}$ through the attached abelian category of coherent sheaves $\operatorname{coh}\mathbb{X}$.

Definition 5.24. [30, Theorem 1] A connected Ext-finite abelian hereditary noetherian category $\mathcal{A}$ with a tilting complex and no nonzero projectives is said to be a *category of coherent sheaves* $\operatorname{coh}\mathbb{X}$ *over a weighted projective line* $\mathbb{X}$.

In our classification of fractional Calabi-Yau categories, we will use following famous characterization of hereditary categories with a tilting object.

Theorem 5.25. *[23] Let $\mathcal{H}$ be a connected Ext-finite abelian hereditary category which admits a tilting complex. Then $\mathcal{A}$ is derived equivalent to either*

1. $\operatorname{mod} A$, *where A is a finite dimensional hereditary algebra, or*
2. $\operatorname{coh}\mathbb{X}$ *where $\mathbb{X}$ is a weighted projective line.*

In the case of a weighted projective line, the tilting complex corresponds to the quiver in Figure 5.5 where $t \geq 2$ and with relations $f_i^{p_i} = f_2^{p_2} - \lambda_i f_1^{p_1}$, for all $2 \leq i \leq t$, where $\lambda_i \neq \lambda_j$ for $i \neq j$. These are exactly the canonical algebras introduced by Ringel in [44].

The canonical algebra Λ given by an t-tuple of weights $\mathbf{p} = (p_1, p_2, \ldots, p_t)$, and a $(t-2)$-tuple $\lambda = (\lambda_3, \ldots, \lambda_t)$ of pairwise distinct elements of k, will be denoted by $\Lambda(\mathbf{p}, \lambda)$. We will call $\mathbf{p}$ the *weight type* of Λ and of the weighted projective line $\mathbb{X}$.

Let $\mathbb{X}$ be a weighted projective line of weight type $\mathbf{p} = (p_1, p_2, \ldots, p_t)$. We will call

$$\chi_{\mathcal{H}} = 2 - \sum_{i=1}^{t} \left(1 - \frac{1}{p_i}\right)$$

the *Euler characteristic* of $\mathcal{H} = \operatorname{coh}\mathbb{X}$.

$$\begin{tikzcd}
 & L_1^{(1)} \arrow[r, "f_1"] & L_2^{(1)} \arrow[r, "f_1"] & \cdots \arrow[r, "f_1"] & L_{p_1-1}^{(1)} \arrow[ddr, "f_1"] & \\
 & L_1^{(2)} \arrow[r, "f_2"] & L_2^{(2)} \arrow[r, "f_2"] & \cdots \arrow[r, "f_2"] & L_{p_2-1}^{(2)} \arrow[dr, "f_2"] & \\
L \arrow[uur] \arrow[ur] \arrow[dr] & \vdots & & & \vdots & L_1 \\
 & L_1^{(t)} \arrow[r, "f_t"] & L_2^{(t)} \arrow[r, "f_t"] & \cdots \arrow[r, "f_t"] & L_{p_t-1}^{(t)} \arrow[ur, "f_t"] &
\end{tikzcd}$$

Figure 5.5: The tilting complex of a weighted projective line.

We will only be interested in the case where $\chi_{\mathcal{H}} = 0$. Such a category $\mathcal{H}$ will be called *tubular*; every Auslander-Reiten component is a tube and the category $\mathcal{H}$ is fractional 1-Calabi-Yau ([29]).

Note that $\chi_{\mathcal{H}} = 0$ implies that the weight type of $\mathbb{X}$ is one of the following: $(2,2,2,2), (3,3,3), (2,4,4)$, or $(2,3,6)$, thus the corresponding canonical algebra is as in Figure 5.6. By removing the final vertex from any of these quivers, one obtains an extended Dynkin quiver.

If $\chi_{\mathcal{H}} < 0$, then $\mathcal{H}$ is derived equivalent to the category of finitely presented modules over a tame hereditary algebra. If $\chi_{\mathcal{H}} > 0$, then $\mathcal{H}$ is wild. In neither of these two cases, $\mathcal{H}$ is fractional Calabi-Yau.

For a more comprehensive treatment of weighted projective lines, we refer to [31] and the references therein.

5.3.4 Categories with fractional Calabi-Yau dimension 1

Let $\mathcal{A}$ be a connected hereditary category with fractional Calabi-Yau dimension 1. The proof of the classification will consist of two parts: one case where the Auslander-Reiten component of $\mathcal{A}$ has only one component, and one case where there are at least two. The former case is easily dealt with; for discussion the latter case, it will be pivotal to have elements L and S as in the following lemma.

Lemma 5.26. *Let $\mathcal{A}$ be a fractional 1-Calabi-Yau category, then every Auslander-Reiten component is a standard tube. If $\mathcal{A}$ is connected and not equivalent to* $\operatorname{mod} \tilde{A}_n$*, then there are two peripheral objects $L, S \in \operatorname{ind} D^b\mathcal{A}$ lying in different Auslander-Reiten components with*

1. $\operatorname{Ext}(L,L) = 0$,

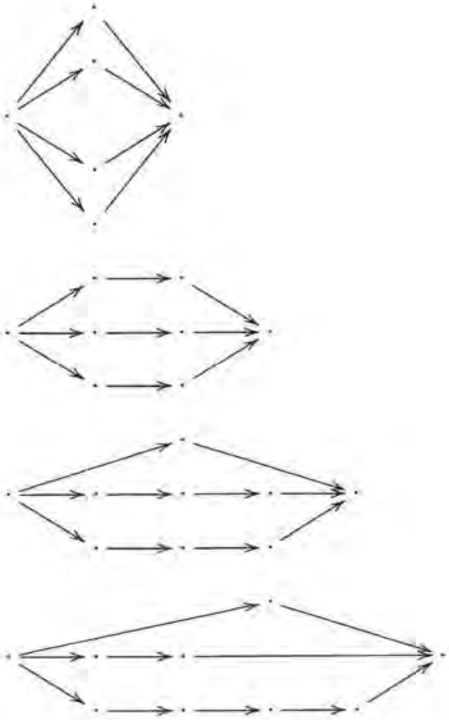


Figure 5.6: Canonical algebras with weight type $(2,2,2,2), (3,3,3), (2,4,4)$, and $(2,3,6)$, respectively.

2. $\operatorname{Hom}(L, S) \neq 0$,

3. $\operatorname{Hom}(L, \tau^{-i} S) = 0$ *for* $0 < i < s$,

where s is the rank of the Auslander-Reiten component containing S.

Proof. Since $\mathcal{A}$ is a fractional 1-Calabi-Yau category, there is an $n \in \mathbb{N}$ such that $F^n \cong [n]$, and hence $\tau^n \cong 1_{\mathcal{A}}$. It follows from Theorem 4.5 that every Auslander-Reiten component is a standard tube of rank at most n.

As stated in Remark 5.14, we may assume τ is not the identity on objects, hence there is a peripheral object with $L \not\cong \tau L$. In particular, $\dim \operatorname{Ext}(L, L) = 0$.

If $\mathcal{A}$ is connected and all objects lie in the same Auslander-Reiten component, then $\mathcal{A}$ is equivalent to $\operatorname{nilp} \tilde{A}_n$. We will therefore assume that not all objects lie in the same Auslander-Reiten component, hence there is another peripheral object $S \in \operatorname{ind} D^b\mathcal{A}$ such that $\operatorname{Hom}(L, S) \neq 0$.

We may assume $\operatorname{Hom}(L, \tau^{-i}S) = 0$ when $0 < i < s$. Indeed, if $s \neq 1$ then let $0 < i < s$ be the smallest integer such that $\operatorname{Hom}(L, \tau^{-i}S) \neq 0$. It follows readily from the following triangle

$$T^*_{\tau^{-i}S}L \to L \to \tau^{-i}S \otimes_k \operatorname{Hom}(L, \tau^{-i}S)^* \to T^*_{\tau^{-i}S}L[1]$$

that $\operatorname{Hom}(T^*_{\tau^{-i}S}L, \tau^{-j}S) = 0$ when $0 < j \leq i$, and that $\operatorname{Hom}(T^*_{\tau^{-i}S}L, S) = 0$. Iteration shows there is an object L' such that L' maps nonzero to S, but admits no nonzero maps to τ-shifts of S not isomorphic to S. $\square$

We will assume L and S have been chosen as above. We will denote by s the rank of the tube containing, S, and will write $d = \dim \operatorname{Hom}(L, S) = \dim \operatorname{Hom}(L, E)$, where $E = \oplus_{i=1}^{s} t^{-i}S$.

The sequence $\mathcal{E}$ and a t-structure in $D^b\mathcal{A}$

As in the 1-Calabi-Yau case, we will consider the autoequivalence $t = T_E : D^b\mathcal{A} \to D^b\mathcal{A}$. Note that if L is a peripheral object not lying in the same tube as S, then tL will lie in a different tube as L.

For every $i \in \mathbb{Z}$, we have following triangle

$$t^iL \to t^{i+1}L \to E \otimes_A \operatorname{Hom}(t^iL, E)^* \to t^iL[1]. \tag{5.5}$$

In general, the A-modules $\operatorname{Hom}(t^iL, E)$ and $\operatorname{Hom}(t^jL, E)$ will not be isomorphic. However, since $t^sS \cong S$, we have $\operatorname{Hom}(t^iL, E) \cong \operatorname{Hom}(t^jL, E)$ when $i - j$ is a multiple of s.

We will consider the sequence $\mathcal{E} = (L_i)_{i\in\mathbb{Z}}$ where $L_i = t^{is}L$. Theorem 1.30 yields the following will define a t-structure with a hereditary heart $\mathcal{H}$:

$$\begin{aligned} \operatorname{ind} D^{\leq 0} &= \{X \in \operatorname{ind}\mathcal{C} \mid \text{there is a path from } L_i \text{ to } X, \text{ for an } i \in \mathbb{Z}\} \\ \operatorname{ind} D^{\geq 1} &= \operatorname{ind}\mathcal{C} \setminus \operatorname{ind} D^{\leq 0} \end{aligned}$$

This does indeed define a bounded t-structure, as there are paths from $E[-1]$ to L_i and thus by Theorem 4.6, we see that no direct summand of $E[-1]$ lies in $\operatorname{ind} D^{\leq 0}$. It is now easy to see that $E \in \mathcal{H}$ and $L_i \in \mathcal{H}$, for all $i \in \mathbb{Z}$.

It is easy to see that the functor t restricts to an autoequivalence on $\mathcal{H}$ which we will also denote by t.

By the choice of this t-structure, the triangles (5.5) correspond to short exact sequences

$$0 \to t^iL \to t^{i+1}L \to E \otimes_A \operatorname{Hom}(t^iL, E)^* \to 0.$$

We find following relation in $\mathcal{H}$ between L_i and L_{i+1}, for all $i \in \mathbb{Z}$

$$0 \to L_i \to L_{i+1} \to B \to 0$$

where B is an object of which every direct summand lies in the Auslander-Reiten component containing S. It is easy to see that $t^s B \cong B$ and that hence the short exact sequences above are related to each other through the autoequivalence t^s of $\mathcal{H}$.

Furthermore, it follows from Lemma 5.26 that $\dim \operatorname{Hom}(L, B) = \dim \operatorname{Hom}(L, E \otimes_A \operatorname{Hom}(L, E)^*) = d^2$.

The proof of the following lemma is similar to that of Lemma 5.7.

Lemma 5.27. *Let $\mathcal{E} = (L_i)_{i \in \mathbb{Z}}$ and E as above, then*

1. $\operatorname{Hom}(L_i, L_j) = 0$ *for* $i > j$,
2. $\operatorname{Ext}(L_j, L_i) = 0$ *for* $i > j$,
3. $\operatorname{Hom}(E, L_i) = \operatorname{Ext}(L_i, E) = 0$ *for all* $i \in \mathbb{Z}$.

To show $\mathcal{E}$ is a coherent sequence, we will also use following lemma.

Lemma 5.28. *If* $\operatorname{Hom}(L_i, X) \neq 0$, *then* $\operatorname{Hom}(L_j, X) \neq 0$ *for all* $j \leq i$.

Proof. As in the proof of Lemma 5.8. □

Proposition 5.29. *The sequence $\mathcal{E}$ is coherent in $\mathcal{H}$, and for any $X \in \operatorname{ind} \mathcal{H}$, there is an n such that* $\operatorname{Hom}(\tau^n L_i, X) \neq 0$, *for* $i \ll 0$.

Proof. The proof that $\mathcal{E}$ is projective and coherent, is analogous to the proof of Proposition 5.9, and uses Lemma 5.27.

For the sequence $\tau^n \mathcal{E} = (\tau^n L_i)_{i \in \mathbb{Z}}$, let $i_{n,1}, i_{n,2}, \ldots, i_{n,m_n}$ as in the definition of coherence. As in the proof of Proposition 5.9, one may show that the natural map

$$\theta : \bigoplus_{n=1}^{r} \bigoplus_{j=1}^{m_n} L_{i_{n,j}} \otimes \operatorname{Hom}(L_{i_j}, X) \longrightarrow X, \tag{5.6}$$

where r is the rank of the tube containing L, is surjective. This yields the last claim. □

In general, the sequence $\mathcal{E}$ will not be ample. We will write $\mathcal{H}_0$ for the full subcategory of $\mathcal{H}$ spanned by all objects X such that $\operatorname{Hom}(L_i, X) = 0$ for $i \ll 0$. It follows from Lemma 5.28 that $\operatorname{Hom}(L_i, X) = 0$ for all $i \in \mathbb{Z}$. Recall that, since $\mathcal{E}$ is projective, $\mathcal{H}_0$ is a Serre subcategory.

Description of $\mathcal{H}/\mathcal{H}_0$

We will now prove that the category $\mathcal{H}/\mathcal{H}_0$ is equivalent to the category $\operatorname{coh} \mathbb{Y}$ of coherent sheaves on a smooth projective curve $\mathbb{Y}$. The proof is analogous to the proof presented in §5.2.3, namely, we will show that $R = R(\mathcal{E})$ is a domain generated in degree one, with $\operatorname{GKdim} R = 2$. We may then infer from [2] that R is noetherian and that $\operatorname{qgr} R \cong \operatorname{coh} \mathbb{Y}$ for a smooth projective curve $\mathbb{Y}$. Corollary 5.4 then yields $\mathcal{H}/\mathcal{H}_0 \cong \operatorname{coh} \mathbb{Y}$.

Lemma 5.30. *Let $\mathcal{E} = (L_i)_{i\in I}$ and $E = \oplus_{i=1}^{s} \tau^i S$ be as before. If $j \geq i$, then*

$$\dim \operatorname{Hom}(L_i, L_j) = 1 + (j-i)d^2$$

where $d = \dim \operatorname{Hom}(L, E)$.

Proof. This immediately follows from the short exact sequence

$$0 \longrightarrow L_i \longrightarrow L_{i+1} \longrightarrow B \longrightarrow 0$$

together with $\dim \operatorname{Hom}(L_i, L_i) = 1$, $\dim \operatorname{Hom}(L_i, B) = d^2$, and Lemma 5.27. □

Lemma 5.31. *Assume L and S are chosen as above, with that additional property that $d = \dim \operatorname{Hom}_{D^b\mathcal{A}}(L, S)$ is minimal and $d \neq 0$. Then R is a domain.*

Proof. The proof follows that of Lemma 5.11 closely. We will use induction on i to show every non-zero non-isomorphism $f : L_0 \longrightarrow L_i$ is a monomorphism. The case $i = 0$ is trivial. So let $i \geq 1$.

Since $\operatorname{im} f$ is a quotient object of L_0 and $\dim \operatorname{Hom}(L, B) = d^2$, we see that $\dim \operatorname{Hom}(\operatorname{im} f, B) \leq d^2$. By $\operatorname{Ext}(L_0, L_i) = 0$, we easily find $\operatorname{Ext}(\operatorname{im} f, \operatorname{im} f) = 0$ hence, by the minimality of d, either $\dim \operatorname{Hom}(\operatorname{im} f, B) = 0$ or $\dim \operatorname{Hom}(\operatorname{im} f, B) = d^2$ holds and $\operatorname{im} f$ is an endo-simple object.

The rest of this proof is identical to the proof of Lemma 5.11. □

Lemma 5.32. *The algebra $R = R(\mathcal{E})$ is generated in degree 1.*

Proof. Analogous to the proof of Lemma 5.12. □

Proposition 5.33. *The category $\mathcal{H}/\mathcal{H}_0$ is equivalent to the category of coherent sheaves on a projective curve $\mathbb{Y}$.*

Proof. We have shown that $R = R(\mathcal{E})$ is a domain generated in degree one, with $\operatorname{GKdim} R = 2$, thus we may invoke [2] to show R is noetherian and that $\operatorname{qgr} R$ is equivalent to the category of coherent sheaves on a projective curve $\mathbb{Y}$. It follows from Corollary 5.4 that the quotient category $\mathcal{H}/\mathcal{H}_0$ is equivalent to $\operatorname{coh} \mathbb{Y} \cong \operatorname{qgr} R$. □

Remark 5.34. As a direct consequence of previous proposition, we see that $\dim \operatorname{Hom}(L_i, B) = 1$ and thus also $\dim \operatorname{Hom}(L_i, E) = 1$.

Objects of finite length

In order to construct a tilting set as in §5.3.3, we will need more information about the simple tubes in $\mathcal{H}$. To do this, we will discuss the subcategory $\mathcal{H}_f$ of $\mathcal{H}$ consisting of all objects of finite length, thus $\mathcal{H}_f$ is the full subcategory of $\mathcal{H}$ consisting of all simple tubes of $\mathcal{H}$ (see Lemma 4.9).

Following lemma will

Lemma 5.35. *For all $X \in \operatorname{Ob}\mathcal{H}$ and all $i \in \mathbb{Z}$, we have* $\dim \operatorname{Hom}_{\mathcal{H}}(L_i, X) \leq \dim \operatorname{Hom}_{\mathcal{H}/\mathcal{H}_0}(\pi(L_i), \pi(X))$.

Proof. We know that

$$\operatorname{Hom}_{\mathcal{H}/\mathcal{H}_0}(\pi(L_i), \pi(X)) = \varinjlim(\operatorname{Hom}_{\mathcal{H}}(L_i', X/X'))$$

where the direct limit is taken over all subobjects L_i' of L_i and all subobjects X' of X such that $L_i/L_i', X' \in \mathcal{H}_0$. By the definition of $\mathcal{H}_0$, we have $L_i' = L_i$. From $X' \in \mathcal{H}_0$ follows that $\operatorname{Hom}_{\mathcal{H}}(L_i, X)$ is always a subspace of $\operatorname{Hom}_{\mathcal{H}}(L_i, X/X')$. The required inequality follows easily. □

Proposition 5.36. *Every nonzero object $X \in \operatorname{Ob}\mathcal{H}$ maps nonzero to a simple object of $\mathcal{H}$.*

Proof. If X has finite length, then the statement is trivial. So assume X has infinite length, such that there is a sequence of epimorphisms

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_k \to \cdots$$

where the kernels $K_k = \ker(X_k \to X_{k+1})$ are nonzero. By Proposition 5.29, there is an $n \in \mathbb{N}$ such that the sequence $(\tau^n L_i)_{i\in\mathbb{Z}}$ maps nonzero to K_k for an infinite number of k's. We may choose notations such that this sequence is $(\tau^n L_i)_{i\in\mathbb{Z}}$.

Thus in $\mathcal{H}_0/\mathcal{H}$, there is an infinite sequence of epimorphisms

$$\pi X = \pi X_0 \to \pi X_1 \to \pi X_2 \to \cdots \to \pi X_k \to \cdots$$

where infinitely many morphisms are not invertible. Indeed, the functor π is exact and infinitely many kernels are nonzero. This shows $\pi(X)$ has infinite length.

Since π is exact and E semi-simple, πE is semi-simple in $\mathcal{H}/\mathcal{H}_0$ as well. In $\mathcal{H}/\mathcal{H}_0 \cong \operatorname{coh}\mathbb{Y}$, every indecomposable object of infinite length maps nonzero to every simple object. We find that $\operatorname{Hom}_{\mathcal{H}/\mathcal{H}_0}(\pi X, \pi E) \neq 0$ and claim this will imply $\operatorname{Hom}_{\mathcal{H}}(X, E) \neq 0$.

By the definition of a quotient category, we find a subobject X' of X such that $\operatorname{Hom}_{\mathcal{H}}(X', E) \neq 0$, thus either $\operatorname{Hom}_{\mathcal{H}}(X, E) \neq 0$, and we are done, or $\operatorname{Hom}_{\mathcal{H}}(E, X/X') \cong \operatorname{Ext}_{\mathcal{H}}(X/X', E)^* \neq 0$. In the latter case, since E is semi-simple, it follows from Lemma 4.9 that $\operatorname{Hom}_{\mathcal{H}}(X/X', E) \neq 0$ and hence also $\operatorname{Hom}_{\mathcal{H}}(X, E) \neq 0$. □

Proposition 5.37. *Every simple tube of $\mathcal{H}$ has exactly one peripheral object S such that* $\operatorname{Hom}(L, S) \neq 0$.

Proof. Let $\mathcal{K}$ be a simple tube in $\mathcal{H}$. It follows from Proposition 5.29 that there is at least one peripheral, and hence simple, object S such that $\pi S \not\cong 0$. In this case, πS is also simple, hence is a peripheral object in a simple tube in $\mathcal{H}/\mathcal{H}_0 \cong \operatorname{coh}\mathbb{X}$.

Next, we show $\mathcal{K}$ has only one such peripheral object. Let $S' \in \operatorname{ind}\mathcal{H}$ be a peripheral object of $\mathcal{K}$ with $S \not\cong S'$ and $\pi S' \not\cong 0$. Since $\mathcal{K}$ corresponds to an abelian subcategory of $\mathcal{H}$ equivalent with $\operatorname{mod} \tilde{A}_n$ where $\tilde{A}_n$ has cyclic orientation, there is an object $X \in \operatorname{ind} \mathcal{H}$ with simple top S, simple socle S', and $\dim \operatorname{End}(X) = 1$.

We now readily verify that $\operatorname{Hom}(\pi X, \pi S) \neq 0$, $\operatorname{Hom}(\pi S', \pi X) \neq 0$, and $\dim \operatorname{End}(\pi X) = 1$. The last statement shows πX is indecomposable. We have shown there is a path from $\pi S'$ to πS, and thus $\pi S'$ and πS lie in the same simple tube of $\mathcal{H}/\mathcal{H}_0$, hence $\pi S \cong \pi S'$. This implies $\operatorname{Hom}(S, S') \neq 0$ and hence $S \cong S'$. □

A tilting object

In this final step in the classification of connected abelian hereditary categories which are fractionally Calabi-Yau of dimension 1 but not 1-Calabi-Yau, we will use the previous results to construct a tilting complex in the derived category.

Let $\mathcal{A}$ be such a category. We may choose objects L and S as in Lemma 5.26 and use these to find a hereditary category $\mathcal{H}$ derived equivalent to $\mathcal{A}$. This object L is exceptional, it will be our starting point of the titling object.

Let $\mathbb{X}$ be a set parameterizing the simple tubes of $\mathcal{H}$, thus with every $x \in \mathbb{X}$, there corresponds a unique $\mathcal{K}_x$. Proposition 5.37 yields that such a tube has a unique simple object S_x such that $\operatorname{Hom}(L, S_x) \neq 0$. Denote by r_x the rank of the tube $\mathcal{K}_x$ and write $E_x = \oplus_{i=i}^{s_x} \tau^{-i} S_x$. Proposition 5.29 yields $\dim \operatorname{Hom}(L, E_x) \neq 0$.

With every tube $\mathcal{K}_x$, there corresponds a twist functor $T_{E_x} : D^b\mathcal{H} \to D^b\mathcal{H}$, which restricts to an autoequivalence $\mathcal{H} \to \mathcal{H}$, also denoted by T_{E_x}. We will write $T^i_{E_x} L$ as $L^{(x)}_i$.

Lemma 5.38. *The set $\mathcal{L} = \{L^{(x)}_i \mid x \in \mathbb{X}, 0 \leq i \leq s_x\}$ forms a tilting set.*

Proof. This follows easily from the exact sequences

$$0 \to L^{(x)}_i \to L^{(x)}_{i+1} \to E_x \otimes \operatorname{Hom}(L^{(x)}_i, E_x)^* \to 0,$$

together with Proposition 5.36. □

Since there are nonzero maps from $L^{(x)}_{s_x}$ to $L^{(y)}_{s_y}$ for all $x, y \in \mathbb{X}$, we see that $L^{(x)}_{s_x} \cong L^{(y)}_{s_y}$. We may sketch the partial tilting set as in Figure 5.3.4 where $\dim \operatorname{Hom}(L, L_1) = 2$, and with relations $f^{s_{x_i}}_{x_i} = f^{s_{x_2}}_{x_2} - \lambda_{x_i} f^{s_{x_1}}_{x_1}$, for all $x_i \in \mathbb{X}$. Note that since the cokernel of $f^{s_x}_x$ lies in the tube $\mathcal{K}_x$, we see that $\lambda_x \neq \lambda_y$ for distinct $x, y \in \mathbb{X}$.

In order to show this is a tilting object in $D^b\mathcal{A}$, we need to show $\mathcal{L}$ has only finitely many objects, or equivalently, there are only finitely many simple tubes which are not homogeneous. Therefore, let $\mathcal{L}' = \mathcal{L} \setminus \{L_1\}$. It is clear that the associated additive category is $\mathfrak{a}'$ semi-hereditary.

Let $\mathcal{K}$ be the standard tube containing L_1 as peripheral object. Since L_1 is exceptional, the rank of $\mathcal{K}$ is at least two and the Auslander-Reiten component of $\mathcal{K} \cap L_1^{\perp}$ is again a tube. All peripheral objects here are generated by the partial tiling set $\mathfrak{a}'$, hence there is a full additive Karoubian subcategory $\mathfrak{a}_f$ generated by finitely many indecomposables of $\mathfrak{a}'$ generating those peripheral objects in $\mathcal{K} \cap L_1^{\perp}$.

$$\begin{tikzcd}
 & L_1^{(x_1)} \arrow[r, "f_{x_1}"] & L_2^{(x_1)} \arrow[r, "f_{x_1}"] & \cdots \arrow[r, "f_{x_1}"] & L^{(x_1)}_{s_{x_1}-1} \arrow[ddr, "f_{x_1}"] & \\
 & L_1^{(x_2)} \arrow[r, "f_{x_2}"'] & L_2^{(x_2)} \arrow[r, "f_{x_2}"'] & \cdots \arrow[r, "f_{x_2}"'] & L^{(x_2)}_{s_{x_2}-1} \arrow[dr, "f_{x_2}"'] & \\
L \arrow[uur, "f_{x_1}"] \arrow[ur, "f_{x_2}"'] \arrow[dr, "f_{x_t}"'] & \vdots & & & \vdots & L_1 \\
 & L_1^{(x_t)} \arrow[r, "f_{x_t}"'] & L_2^{(x_t)} \arrow[r, "f_{x_t}"'] & \cdots \arrow[r, "f_{x_t}"'] & L^{(x_t)}_{s_{x_t}-1} \arrow[ur, "f_{x_t}"'] & \\
 & & \vdots & \vdots & &
\end{tikzcd}$$

Figure 5.7: The partial tilting set $\mathcal{L}$

We see that every full additive Karoubian subcategory $\mathfrak{b}$ of $\mathfrak{a}'$ containing $\mathfrak{a}_f$ must also generate the tube given by $\mathcal{K} \cap L_1^{\perp}$. Hence, $\mathfrak{b}$ corresponds to a tame algebra and its quiver is an extended Dynkin quiver. Since the order of a vertex in such a quiver is bounded by 4, we infer that the order of L_0 in $\mathfrak{a}'$ is at most 4. The set $\mathcal{L}$ thus is finite and corresponds to a tilting object in $D^b\mathcal{H}$.

We will gather these results in following proposition.

Proposition 5.39. *Let $\mathcal{A}$ be a connected abelian hereditary category which is fractional Calabi-Yau of dimension 1 but not 1-Calabi-Yau. If $\mathcal{A}$ consists of more than one tube, than $D^b\mathcal{A}$ admits a tilting object.*

5.3.5 Proof of classification

Theorem 5.40. *Let $\mathcal{A}$ be a connected abelian hereditary category which is fractional Calabi-Yau of dimension d, but not 1-Calabi-Yau then $\mathcal{A}$ is derived equivalent to either*

1. *the category of finite presented modules* $\operatorname{mod} Q$ *over a Dynkin quiver Q, or*
2. *the category of nilpotent representations* $\operatorname{nilp} \tilde{A}_n$ *where $\tilde{A}_n$ has cyclic orientation and $n > 1$, or*
3. *the category of coherent sheaves* $\operatorname{coh} \mathbb{X}$ *over a weighted projective line of tubular type.*

Proof. By Proposition 5.19, we need only to consider $0 < d \leq 1$. If $d \neq 1$, then it follows from Proposition 5.23 that $\mathcal{A}$ is equivalent to the category of finite presented modules $\operatorname{mod} Q$ over a Dynkin quiver Q.

We are left with the case where $d = 1$. In this case, every Auslander-Reiten component of $\mathcal{A}$ is a standard tube (see Lemma 5.26). If there is only one such tube, $\mathcal{A}$ is equivalent to category of nilpotent representations $\operatorname{nilp} \tilde{A}_n$ where $\tilde{A}_n$ has cyclic orientation. This category is 1-Calabi-Yau if and only if $n = 1$.

In case there is more than one tube, connectedness implies there are two tubes with a nonzero morphism between them. Proposition 5.39 yields $\mathcal{A}$ admits a tilting complex.

Invoking Theorem 5.25 shows $\mathcal{A}$ is either derived equivalent to $\operatorname{mod} A$ for a finite dimensional algebra A, or to $\operatorname{coh} \mathbb{X}$ for a weighted projective line $\mathbb{X}$. Since $\mathcal{A}$ every Auslander-Reiten component of $\mathcal{A}$ is a standard tube, we are in the latter case where $\mathbb{X}$ is of tubular type. □

Remark 5.41. Instead of invoking Theorem 5.25, we might also use the shape of the constructed tilting object to infer $\mathcal{A}$ is derived equivalent to $\operatorname{coh} \mathbb{X}$ for a weighted projective line $\mathbb{X}$ of tubular type.

Remark 5.42. In the same spirit as Remark 5.14, we note that we do not use $F^n \cong [m]$, but only $F^n X \cong X[m]$, for all $X \in \operatorname{Ob} D^b\mathcal{A}$, thus that $\mathcal{A}$ satisfies a fractional Calabi-Yau property *on objects* alone. From Theorem 5.40 we infer that $\mathcal{A}$ is fractional Calabi-Yau.

Chapter 6

Connecting subcategories

In this chapter, based on joint work with Carl Fredrik Berg [7], we will generalize the concepts and results of §1.10. We strongly advise the reader to familiarize himself with the concepts and methods used therein before proceeding.

6.1 Introduction and overview

Let $\mathcal{A}$ be an abelian Ext-finite hereditary category with Serre duality. The full additive subcategory consisting of all projectives is called the *category of projectives* and we will denote this category by $\mathcal{Q}_{\mathcal{A}}$.

Objects of the form $\oplus_i \tau^{n_i} P_i$ where the P_i's are indecomposable projectives and $n_i \geq 0$, are called *preprojective*.

Recall that an Auslander-Reiten component which contains an indecomposable projective is called a *preprojective component*. The following result implies an Auslander-Reiten component is preprojective if every indecomposable is a preprojective object. (see for example [5, Corollary VIII.1.10]).

Proposition 6.1. *Let $\mathcal{K}$ be an Auslander-Reiten component of $\mathcal{A}$, and let $X \to Y$ be an arrow in $\mathcal{K}$. If X or Y are projective, then the other is preprojective.*

Proof. Since there is an arrow $X \to Y$, there is an irreducible morphism $f : X \to Y$ which is necessarily is a monomorphism or an epimorphism.

If Y is projective, then it follows that f is a monomorphism and thus, from heredity, that X is projective.

If X is projective, we may assume Y is not projective. In this case $\operatorname{Irr}(\tau Y, X) \neq 0$ such that the first part of the proof implies that τY is projective. We see that Y is preprojective. □

Preinjective objects and components are defined similarly.

The preprojective and preinjective components are the only components of the Auslander-Reiten quiver of $\mathcal{A}$ which are not stable, thus the Auslander-Reiten translate τ is not defined on every vertex, or it is not invertible.

In the derived category $D^b\mathcal{A}$, the preprojective and the -1^{th} shift of the preinjective components are "glued" together, such that, if $\operatorname{ind}\mathcal{A}$ is not finite, then the resulting components are stable (see Proposition 5.22).

The definition of a hereditary section (see below) is chosen to mimic the properties of $\mathcal{Q}_{\mathcal{A}}$ in $D^b\mathcal{A}$. In fact, given a hereditary section $\mathcal{Q}_{\mathcal{H}}$ in $D^b\mathcal{A}$, one might consider an associated (split) t-structure of $D^b\mathcal{A}$ and obtain a hereditary category $\mathcal{H}$ derived equivalent with $\mathcal{A}$ such that the category of projectives of $\mathcal{H}$ corresponds to $\mathcal{Q}_{\mathcal{H}}$ (see Theorem 6.21). However, the category $\mathcal{H}$ is not unique with this property, even when $\mathcal{Q}_{\mathcal{H}}$ is not trivial (see Example 6.23).

Let $\mathfrak{a}$ be an additive subcategory of $D^b\mathcal{A}$. We will denote by $\mathbb{Z}\mathfrak{a}$ the full additive subcategory of $D^b\mathcal{A}$ spanned by all objects of the form $\oplus_i \tau^{n_i} A_i$ where the $A_i \in \operatorname{ind}\mathfrak{a}$ and $n_i \in \mathbb{Z}$. If $\mathfrak{a} = \mathcal{Q}_{\mathcal{H}}$ then we will call this subcategory $\mathbb{Z}\mathcal{Q}_{\mathcal{H}}$ the *connecting subcategory*.

We wish to stress the difference between $\mathbb{Z}Q$ where Q is a quiver (see §1.11.2), and $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ where $\mathcal{Q}_{\mathcal{A}}$ is a hereditary section. In the former case, $\mathbb{Z}Q$ is a stable translation quiver, while in the latter case $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ is an additive subcategory of $D^b\mathcal{A}$. Also, $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ is not completely determined by $\mathcal{Q}_{\mathcal{A}}$ without knowledge of the embedding into the ambient category $D^b\mathcal{A}$ (see Example 6.24). In particular, it may even happen that $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ is connected even if $\mathcal{Q}_{\mathcal{A}}$ is not (see Example 6.38)!

If $\mathcal{Q}_{\mathcal{H}}$ is a hereditary section of $D^b\mathcal{A}$ such that $\mathbb{Z}\mathcal{Q}_{\mathcal{H}} \cong \mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ as subcategories of $D^b\mathcal{A}$, then we will say $\mathcal{Q}_{\mathcal{H}}$ is a *tilt* of $\mathcal{Q}_{\mathcal{A}}$.

Our main goal is to prove a statement similar to Theorem 1.55, but then extended to cover the case where the connecting subcategory does may admit infinite radicals. It appears there is an extra (rather technical) condition needed, which we will refer to as condition (*) (see §6.4.2).

Theorem 6.2. *Let $\mathcal{A}$ be a connected abelian hereditary category satisfying Serre duality with category of projectives $\mathcal{Q}_{\mathcal{A}}$. If $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ is connected and satisfies (*), then there exists a tilt $\mathcal{Q}_{\mathcal{H}}$ of $\mathcal{Q}_{\mathcal{A}}$ such that $\mathcal{Q}_{\mathcal{H}}$ is a dualizing k-variety and the natural embedding $i : D^b \operatorname{mod} \mathcal{Q}_{\mathcal{H}} \to D^b\mathcal{A}$ commutes with Serre duality.*

Hereditary sections not satisfying condition (*) seem to be rather artificial, yet this condition is not an artifact of our method but an intrinsic requirement; the statement may be shown to be false if (*) is not satisfied. For examples and more information, we refer to §6.4.2.

If, however, condition (*) is not satisfied, then we may extend $\mathcal{Q}_{\mathcal{A}}$ to a larger partial tilting set, which we might similarly "tilt" to obtain a dualizing k-variety. We have then following theorem (Corollary 6.64 in the text).

Theorem 6.3. *Let $\mathcal{A}$ be an abelian hereditary category satisfying Serre duality with category of projectives $\mathcal{Q}_{\mathcal{A}}$. If $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ is connected, then there exists a full additive* *subcategory $\mathcal{Q}$ of $D^b\mathcal{A}$ such that* $\operatorname{ind}\mathcal{Q}$ *is a dualizing k-variety, and the natural embedding* $i : D^b \operatorname{mod}\mathcal{Q}_{\mathcal{H}} \to D^b\mathcal{A}$ *commutes with Serre duality. Moreover,* $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ *lies in the essential image of* i.

In particular, this solves a problem posed by Idun Reiten in [41] to classify all abelian hereditary categories satisfying Serre duality and generated by preprojectives.

Corollary 6.4. *Let $\mathcal{A}$ be an abelian hereditary category satisfying Serre duality. If $\mathcal{A}$ is generated by preprojectives, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod}\mathcal{Q}$, *where $\mathcal{Q}$ is a semi-hereditary dualizing k-variety.*

In general we conjecture that, given a nontrivial hereditary section $\mathcal{Q}_{\mathcal{A}}$, or just a directing object, one might always find a hereditary section $\mathcal{Q}_{\mathcal{H}}$ and an equivalence $D^b \operatorname{mod}\mathcal{Q}_{\mathcal{H}} \to D^b\mathcal{A}$ (see Conjecture 6.65). There seem to be no counterexamples known, and we know this conjecture holds for several special cases, namely when $\mathcal{A}$ is noetherian or artinian ([40] and its dual), saturated (Theorem 6.67), or directing (Theorem 3.44), or when $\operatorname{Ob}\mathcal{Q}_{\mathcal{A}}$ is a finite set (Proposition 6.68).

We will start with a definition of a hereditary section which works intrinsically on $D^b\mathcal{A}$, and then show (Theorem 6.21) that every hereditary section corresponds to the category of projectives of a well-chosen hereditary category derived equivalent to $\mathcal{A}$.

6.2 Round trip distance and light cone distance

To specify a split t-structure $(D^{\geq 0}, D^{\leq 0})$ on $D^b\mathcal{A}$ where $\mathcal{A}$ is an abelian category, it suffices to give a nontrivial additive subcategory $D^{\leq 0} \subset D^b\mathcal{A}$ closed under successors, in the sense that if there is a path from $X \in \operatorname{ind} D^{\leq 0}$ to $Y \in \operatorname{ind} D^b\mathcal{A}$, then $Y \in \operatorname{ind} D^{\leq 0}$ (see Theorem 1.30).

Closely related to additive subcategories closed under successors, is the concept of a light cone distance. We will reintroduce the light cone distance and the round trip distance from §1.11.2 in a slightly different (more general) context; they coincide when all considered maps are obtainable by a finite number of compositions of irreducible maps.

Although we will only be interested in the case where $\mathcal{A}$ is an abelian hereditary Ext-finite category with Serre duality and $\mathcal{C} = D^b\mathcal{A}$, the definitions and results of this section hold whenever $\mathcal{C}$ is an Ext-finite triangulated Krull-Schmidt category satisfying Serre duality.

6.2.1 Light cone distance

For all $X, Y \in \operatorname{ind}\mathcal{C}$, we define the *(right) light cone distance* as

$$d^{\bullet}(X,Y) = \inf\{n \in \mathbb{Z} \mid \text{there is a path from } X \text{ to } \tau^{-n}Y\}.$$

In particular, $d^{\bullet}(X,Y) \in \mathbb{Z} \cup \{\pm\infty\}$.

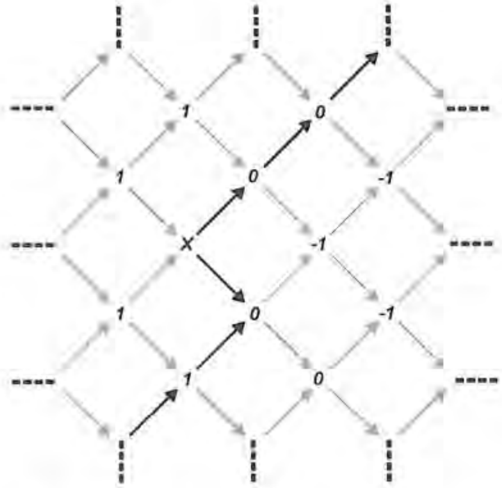


Figure 6.1: The connecting subcategory as stable translation quiver.

Remark 6.5. Even when X and Y lie in the same Auslander-Reiten component, the right light cone distance does not need to coincide with the one given in §1.11.2, as the following example illustrates.

Example 6.6. Let $\mathfrak{a}$ be the semi-hereditary dualizing k-variety whose thread quiver is

The Auslander-Reiten quiver of the connecting subcategory of $D^b \operatorname{mod} \mathfrak{a}$ is of the form $\mathbb{Z}A_\infty^\infty$. In Figure 6.1 we have labeled the vertices with the right light cone distance $d^\bullet(X,-)$ as a stable translation quiver, while in Figure 6.2 we have used the definition of right light cone distance given in this chapter.

Lemma 1.41 stays valid with the altered definition of a right light cone distance.

Lemma 6.7. *For all $X, Y \in \operatorname{ind} \mathcal{C}$, we have $d^\bullet(X, \tau^n Y) = d^\bullet(X,Y) + n$.*

Note that the function $d^\bullet$ is not symmetric. However, Lemma 1.42 and its proof remain valid in our current context.

Proposition 6.8. *For all $X, Y, Z \in \operatorname{ind} \mathcal{C}$, we have*

$$d^\bullet(X,Z) \leq d^\bullet(X,Y) + d^\bullet(Y,Z),$$

whenever this sum is defined.

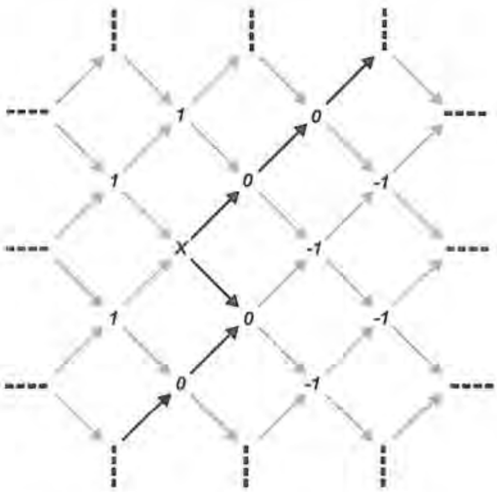


Figure 6.2: The connecting subcategory.

For a subsets $\mathcal{T}_1, \mathcal{T}_2 \subseteq \operatorname{ind}\mathcal{C}$, we define the right light cone distance in an obvious way:

$$d^\bullet(\mathcal{T}_1, \mathcal{T}_2) = \min_{\substack{T_1 \in \mathcal{T}_1 \\ T_2 \in \mathcal{T}_2}} d^\bullet(T_1, T_2).$$

Corollary 6.9. *Let $X, Y \in \operatorname{ind}\mathcal{C}$ and $\mathcal{T} \subseteq \operatorname{ind}\mathcal{C}$, we have*

$$d^\bullet(X, \mathcal{T}) \leq d^\bullet(X, Y) + d^\bullet(Y, \mathcal{T}),$$

whenever this sum is defined.

Proof. Let $T \in \mathcal{T}$ such that $d^\bullet(Y, \mathcal{T}) = d^\bullet(Y, T)$. We find

$$\begin{aligned} d^\bullet(X, Y) + d^\bullet(Y, \mathcal{T}) &= d^\bullet(X, Y) + d^\bullet(Y, T) \\ &\geq d^\bullet(X, T) \geq d^\bullet(X, \mathcal{T}). \end{aligned}$$

□

As in §1.11.2, we may define a *right and left light cone distance sphere* by

$$S^\bullet(X, n) = \{Y \in \operatorname{ind} D^b\mathcal{A} \mid d^\bullet(X, Y) = n\}$$

and

$$S_\bullet(X, n) = \{Y \in \operatorname{ind} D^b\mathcal{A} \mid d^\bullet(Y, X) = n\},$$

respectively, for any $n \in \mathbb{Z}$ and $X \in \operatorname{Ob}\mathcal{C}$. Furthermore, we will denote $S^\bullet_{\mathcal{Q}}(X, n) = S^\bullet(X, n) \cap \operatorname{ind}\mathcal{Q}$ and $S^{\mathcal{Q}}_\bullet(X, n) = S_\bullet(X, n) \cap \operatorname{ind}\mathcal{Q}$.

Finally, let

$$B^\bullet(X, n) = \{Y \in \operatorname{ind} D^b\mathcal{A} \mid d^\bullet(X, Y) \leq n\}.$$

6.2.2 Light cone distance and directedness

Lemma 6.10. *Let $X, Y, Z \in \operatorname{ind}\mathcal{C}$ such that $d^\bullet(X,Z) = 0$. For all non-zero $f \in \operatorname{Hom}(X,Y)$ and $g \in \operatorname{Hom}(Y,Z)$ we have that gf is non-zero.*

Proof. Without loss of generality, we may assume g is not an isomorphism, and hence $C = \operatorname{cone}(g : Y \to Z)$ is nonzero. It follows from Lemma 1.23 that $\operatorname{Hom}(Z, C_i) \neq 0$ for every direct summand C_i of C. Using Serre duality we find $\operatorname{Hom}(C_i[-1], \tau Z) \neq 0$, and therefore $d^\bullet(C_i[-1], Z) \leq -1$. Triangle inequality then gives $d^\bullet(X, C_i[-1]) \geq 1$ and hence $\operatorname{Hom}(X, C[-1]) = 0$. We deduce that $f : X \to Y$ does not factor through $C[-1]$ and hence gf is non-zero. □

Proposition 6.11. *An object $X \in \operatorname{ind}\mathcal{C}$ is directing if and only if $d^\bullet(X,X) = 0$, or equivalently, X is non-directing if and only if $d^\bullet(X,X) = -\infty$.*

Proof. It is clear that directing implies $d^\bullet(X,X) = 0$.

To prove the other implication, assume there is a non-trivial path

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X.$$

Since $d^\bullet(X,X) = 0$, triangle inequality yields $d^\bullet(X_i, X_j) = 0$ for all $i, j \in \{0, \ldots, n\}$. Lemma 6.10 now gives that $f = f_n \ldots f_1 f_0$ is non-zero.

Since X is indecomposable, $\operatorname{End} X$ is a finite dimensional local algebra every element is either nilpotent or invertible. Lemma 6.10 yields f is not nilpotent, hence it is invertible, a contradiction.

□

Corollary 6.12. *Let $X, Y \in \operatorname{ind}\mathcal{C}$ such that $d^\bullet(X,Y) \in \mathbb{Z}$, then both X and Y are directing.*

Proof. Using triangle inequality, we have $d^\bullet(X,Y) \leq d^\bullet(X,X) + d^\bullet(X,Y)$, and hence $0 \leq d^\bullet(X,X)$. We always have $d^\bullet(X,X) \leq 0$, so we get $d^\bullet(X,X) = 0$. Proposition 6.11 shows X is directing. Showing Y is directing is similar. □

Corollary 6.13. *Let $X \in \operatorname{ind}\mathcal{C}$. If X is directing, then so is every indecomposable Y in the Auslander-Reiten component of X.*

Proof. Since Y lies in the same Auslander-Reiten component as X, we know $d^\bullet(X,Y) < \infty$. Then by Proposition 6.11 and triangle inequality, $0 = d^\bullet(X,X) \leq d^\bullet(X,Y) + d^\bullet(Y,X)$, and hence $d^\bullet(Y,X) > -\infty$. Invoking Corollary 6.12 completes the proof.

□

6.2.3 Round trip distance

For $X, Y \in \operatorname{ind}\mathcal{C}$, we define the *round trip distance* $d(X,Y)$ as the symmetrization of the right light cone distance

$$d(X,Y) = d^\bullet(X,Y) + d^\bullet(Y,X),$$

whenever this is well-defined. It is easy to see that $d(X,Y)$ depends only on the τ-orbit of X and Y, thus $d(X,Y) = d(\tau^n X, \tau^m Y)$, for all $m, n \in \mathbb{Z}$ (compare with Lemma 6.7).

Note that, since we restrict ourselves to indecomposables of $\mathbb{Z}\mathcal{Q}$, where $\mathcal{Q}$ is the category of projectives of a hereditary category $\mathcal{A}$ with Serre duality, then we know that both $d^\bullet(X,Y)$ and $d^\bullet(Y,X)$ will be in $\mathbb{Z} \cup \{\infty\}$, hence $d(X,Y)$ is well-defined.

Following proposition shows d defines a pseudometric.

Proposition 6.14. *Let $\mathbb{Z}\mathcal{Q}$ as above. For all $X, Y, Z \in \operatorname{ind} \mathbb{Z}\mathcal{Q}$ we have*

1. $d(X,Y) \geq 0$
2. $d(X,X) = 0$
3. $d(X,Y) = d(Y,X)$
4. $d(X,Z) \leq d(X,Y) + d(Y,Z)$

Proof. The claims (2), (3), and (4) follow from Proposition 6.11, the definition, and Proposition 6.8, respectively. Since then $0 = d(X,X) \leq d(X,Y) + d(Y,X) = 2d(X,Y)$, the first claim holds as well. □

A *round trip distance sphere* is defined in an obvious way.

6.3 Hereditary sections and threads

We now come to the discussion of hereditary sections. Throughout, let $\mathcal{A}$ be a connected abelian Ext-finite hereditary category with Serre duality. We will write $\mathcal{C} = D^b\mathcal{A}$.

6.3.1 Hereditary sections

Before defining a hereditary section, we need following concept.

Definition 6.15. We will say a full additive Karoubian subcategory $\mathcal{Q}$ of $\mathcal{C}$ is *convex* if for any path $X \to X_1 \to \cdots \to X_n \to Y$ it follows from $X, Y \in \mathcal{Q}$ that $X_i \in \mathcal{Q}$, for all i.

Definition 6.16. A full additive Karoubian subcategory $\mathcal{Q}$ of $\mathcal{C}$ is *τ-convex* if for all $X \in \operatorname{ind} \mathcal{C}$ the following condition hold: if there are paths from $\mathcal{Q}$ to the τ-orbit of X and vice versa, then $\operatorname{ind} \mathcal{Q}$ contains an object of the τ-orbit X^τ of X.

In what follows, $\mathcal{Q}$ will consists only of directing objects. In this case we may give an alternative formulation of τ-convex: $\mathcal{Q}$ will be τ-convex if and only if for every $X \in \operatorname{ind} \mathcal{C}$, the condition $d(\mathcal{Q}, X) \neq \infty$ implies that $\mathcal{Q}$ meets the τ-orbit of X.

Definition 6.17. A *hereditary section* is a full, convex, τ-convex and Karoubian additive subcategory $\mathcal{Q}$ of $\mathcal{C}$ such that $\operatorname{ind}\mathcal{Q}$ meets every τ-orbit at most once.

Example 6.18. If $\mathcal{A}$ is a hereditary abelian Ext-finite category with Serre duality with $\mathcal{Q}_\mathcal{A}$ as category of projectives, then $\mathcal{Q}_\mathcal{A}$ is a hereditary section in $D^b\mathcal{A}$. In Theorem 6.21 the converse of this statement will be shown.

Proposition 6.19. *The subcategory $\mathcal{Q}$ of an Ext-finite category is a hereditary section if and only if it is a full and τ-convex additive Karoubian subcategory of $\mathcal{C}$ such that $d^\bullet(X,Y) \geq 0$ for all $X, Y \in \operatorname{ind}\mathcal{Q}$.*

Proof. Assume $\mathcal{Q}$ is a hereditary section. If $d^\bullet(X,Y) < 0$, then there is a path from X to $\tau^n Y$ for an $n > 0$. But since there is also a path a path from $\tau^n Y$ to Y and $\mathcal{Q}$ is convex we see that $\tau^n Y \in \operatorname{Ob}\mathcal{Q}$, a contradiction. This proves one direction.

Assume $\mathcal{Q}$ is a full and τ-convex additive Karoubian subcategory $\mathcal{Q}$ of $\operatorname{ind}\mathcal{C}$ such that $d^\bullet(X,Y) \geq 0$ for all $X, Y \in \mathcal{Q}$. Since $d^\bullet(X, \tau^{-n}X) < 0$ for all $n > 0$, $\mathcal{Q}$ contains at most one object from each τ-orbit.

Assume $X, Y \in \mathcal{Q}$ with paths from X to Z and form Z to Y, thus in particular $d^\bullet(X,Z) \leq 0$ and $d^\bullet(Z,Y) \leq 0$. Since $\mathcal{Q}$ is τ-convex, $\mathcal{Q}$ contains an object in Z^τ. Using triangle inequality and we find $d^\bullet(X,Y) \leq d^\bullet(X,Z) + d^\bullet(Z,Y) \leq 0$. Since we have assumed $d^\bullet(X,Y) \geq 0$, we see $d^\bullet(X,Z) = 0$ and $d^\bullet(Z,Y) = 0$. Thus the object $\mathcal{Q}$ contains from Z^τ must be Z. Hence $\mathcal{Q}$ is convex. $\square$

Next corollary will be used to construct certain types of hereditary sections.

Corollary 6.20. *Let $\mathcal{A}$ be an abelian hereditary Ext-finite k-linear category satisfying Serre duality and denote by $\mathcal{Q}_\mathcal{A}$ its category of projectives. Any full and τ-convex additive Karoubian subcategory $\mathcal{Q}'$ of $\mathcal{Q}$ is a hereditary section in $D^b\mathcal{A}$.*

We now come to the main result of this section.

Theorem 6.21. *Let $\mathcal{A}$ be a connected Ext-finite abelian category with Serre duality and let $\mathcal{Q}$ be a hereditary section of $D^b\mathcal{A}$, then there exists an Ext-finite abelian hereditary category $\mathcal{H}$ with Serre duality, such that $\mathcal{A}$ is derived equivalent to $\mathcal{H}$ and the category of projectives of $\mathcal{H}$ is given by $\mathcal{Q}$.*

Proof. We will define a t-structure on $\mathcal{C}$ as follows.

$$\begin{aligned}
\operatorname{ind}\mathcal{D}^{\geq 1} &= \left\{X \in \operatorname{ind}\mathcal{C} \mid X \in \tau^{\mathbb{N}_0}\mathcal{Q} \text{ or there is no path from } \mathcal{Q} \text{ to } X^\tau\right\} \\
\operatorname{ind}\mathcal{D}^{\leq 0} &= \left\{X \in \operatorname{ind}\mathcal{C} \mid X \notin \tau^{\mathbb{N}_0}\mathcal{Q} \text{ and there is a path from } \mathcal{Q} \text{ to } X^\tau\right\}
\end{aligned}$$

It is easy to see that this satisfies the conditions of Theorem 1.30 such that $(D^{\geq 0}, D^{\leq 0})$ does indeed define a bounded and split t-structure. The heart $\mathcal{H} = D^{\geq 0} \cap D^{\leq 0}$ of this t-structure is hereditary and $D^b\mathcal{H} \cong D^b\mathcal{A}$ as triangulated categories. Furthermore, $\mathcal{H}$ has Serre duality since $\mathcal{A}$ has.

Note that, by construction, $\mathcal{Q}$ is contained in $\mathcal{H}$. We will now show $\mathcal{Q}$ is the category of projectives of $\mathcal{H}$. Therefore, let Y be an object of $\operatorname{ind}\mathcal{Q}$, then by definition $\tau Y \notin \operatorname{Ob}\mathcal{H}$ and hence Y is projective in $\mathcal{H}$.

Conversely, let Y be a projective indecomposable of $\mathcal{H}$. Since $\tau Y \notin \operatorname{Ob}\mathcal{H}$ and the only condition on the objects of $\mathcal{D}^{\geq 0}$ and $\mathcal{D}^{\leq 0}$ not closed under τ is $Y \notin \tau^{\mathbb{N}_0}\mathcal{Q}$. We conclude $Y \in \mathcal{Q}$. □

Corollary 6.22. *Every hereditary section $\mathcal{Q}$ of $\mathcal{C}$ is semi-hereditary, a partial tilting set, locally discrete and locally finite, and all indecomposables of $\mathcal{Q}$ are directing.*

The abelian category constructed in the proof of Theorem 6.21 is not unique as subcategory of $D^b\mathcal{A}$ with the required properties, as following example shows.

Example 6.23. Let $\mathfrak{a}$ be the dualizing k-variety given by the thread quiver

$$A_0 \longrightarrow \overset{\mathbb{Z}}{\cdots\cdots} \longrightarrow C_0$$

The Auslander-Reiten quiver of the category $D^b \operatorname{mod}\mathfrak{a}$ is sketched in the uppermost part of Figure 6.3 where, as usual, the abelian subcategory $\operatorname{mod}\mathfrak{a} \subset D^b \operatorname{mod}\mathfrak{a}$ has been filled with grey.

We will consider the hereditary section $\mathcal{Q}$ spanned by all objects of $\mathfrak{a} \subset D^b \operatorname{mod}\mathfrak{a}$ of the form $\mathfrak{a}(-, A_i)$ for $i \in \mathbb{N}$. The corresponding abelian category given in the proof of Theorem 6.21 is marked in grey in the middle sketch.

However, it is easily verified that there is another t-structure on $D^b \operatorname{mod}\mathfrak{a}$, whose heart is marked in the third sketch of Figure 6.3, of which the category of projectives correspond with $\mathcal{Q}$.

Recall from §6.1 that given a hereditary section $\mathcal{Q}$, we will denote by $\mathbb{Z}\mathcal{Q}$ the full additive subcategory of $D^b\mathcal{A}$ spanned by all objects of the form $\oplus_i \tau^{n_i} P_i$ where the $P_i \in \operatorname{ind}\mathcal{Q}_{\mathcal{A}}$ and $n_i \in \mathbb{Z}$. Note that if we denote by Q the Auslander-Reiten quiver of $\mathcal{Q}$, then the Auslander-Reiten components of $D^b\mathcal{A}$ containing $\mathcal{Q}$ are stable translation quivers of the form $\mathbb{Z}Q$.

Note that the hereditary section $\mathcal{Q}$ alone is not sufficient to determine $\mathbb{Z}\mathcal{Q}$ completely as may be seen from the following example.

Example 6.24. Let $\mathfrak{a}$ be semi-hereditary dualizing k-variety given by

The Auslander-Reiten quiver of the connecting subcategory where all vertices have been labeled with $d^\bullet(X, -)$ is given in the first part of Figure 6.4. We will also consider a different hereditary of $D^b\mathcal{A}$, as indicated by the second part of Figure 6.4.

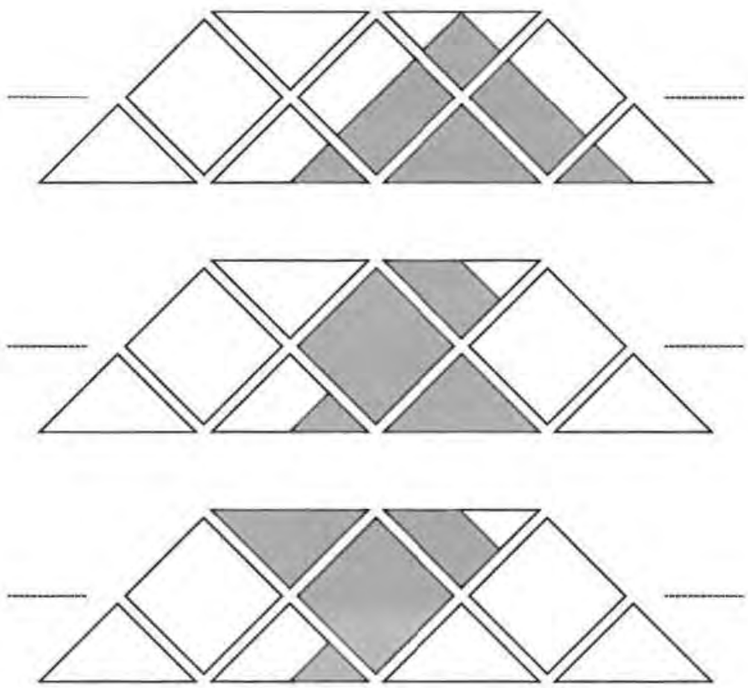


Figure 6.3: Illustration of Example 6.23

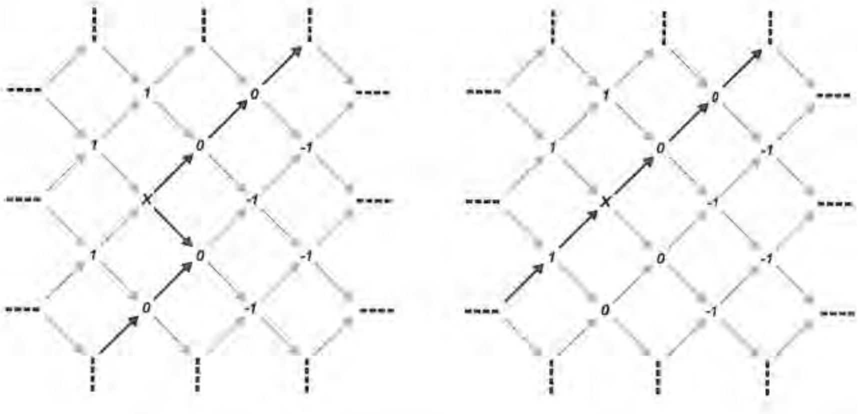


Figure 6.4: The hereditary section generated by the projectives and an alternative hereditary section.

Next, we let $\mathfrak{b}$ be the semi-hereditary k-variety given by

We will consider the Auslander-Reiten quiver of $\mathbb{Z}Q$ and label all vertices with $d^\bullet(X, -)$, and again choose a different hereditary section in Figure 6.5.
It is clear the hereditary sections chosen in $D^b \operatorname{mod} \mathfrak{a}$ and $D^b \operatorname{mod} \mathfrak{b}$ are equivalent, yet the right light cone distance is different.

It may even occur that Q is not connected, but $\mathbb{Z}Q$ is.

Example 6.25. Let $\mathfrak{a}$ be the dualizing k-variety given by the thread quiver

The Auslander-Reiten quiver of $D^b \operatorname{mod} \mathfrak{a}$ may be sketched as in the top of Figure 6.6, where the $\mathbb{Z}A_\infty$-components and the $\mathbb{Z}A_\infty^\infty$-components are represented by triangles and squares, respectively, and where as usual the abelian subcategory $\operatorname{mod} \mathfrak{a}$ has been marked with grey. The category $\operatorname{mod} \mathfrak{a}$ is clearly connected.

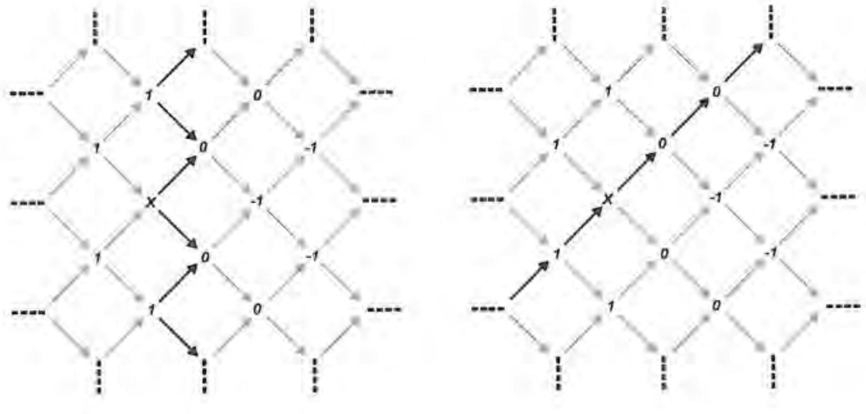


Figure 6.5: The hereditary section generated by the projectives and an alternative hereditary section.

Taking as a hereditary section $\mathcal{Q}$ the full subcategory of the hereditary section $\mathfrak{a} \subseteq D^b \operatorname{mod} \mathfrak{a}$ lying in $\mathbb{Z}A_\infty$-components (See Corollary 6.20), gives an abelian category $\mathcal{H}$ as in Theorem 6.21, drawn in the second part of Figure 6.6. While clearly $\mathbb{Z}\mathcal{Q}$ is not connected, the corresponding abelian category $\mathcal{H}$ is.

The following proposition shows that a tilt of a hereditary section is again a hereditary section if all right light cone distances are nonnegative.

Proposition 6.26. *Let $\mathcal{A}$ be an abelian hereditary Ext-finite k-linear category satisfying Serre duality and let $\mathcal{Q}$ be the category of projectives of $\mathcal{A}$. Let $\mathcal{Q}'$ be a full additive Karoubian subcategory of $D^b\mathcal{A}$ such that $\mathbb{Z}\mathcal{Q} = \mathbb{Z}\mathcal{Q}'$ and $d^\bullet(X,Y) \geq 0$ for all $X, Y \in \operatorname{ind}\mathcal{Q}'$, then $\mathcal{Q}'$ is a hereditary section in $D^b\mathcal{A}$.*

Proof. Since $\mathbb{Z}\mathcal{Q} = \mathbb{Z}\mathcal{Q}'$ it is clear that every indecomposable of $\mathcal{Q}'$ is directing and that $\mathcal{Q}'$ is τ-convex. Since all light cone distances must be positive, we find that at most one object of a τ-orbit may be in $\operatorname{ind}\mathcal{Q}'$.

To show that $\mathcal{Q}'$ is convex, consider $X, Z \in \operatorname{ind}\mathcal{Q}'$ and $Y \in \operatorname{ind} D^b\mathcal{A}$ such that there are paths from X to Y and from Y to Z, hence $d^\bullet(X,Y) \leq 0$ and $d^\bullet(Y,Z) \leq 0$. Triangle inequality yields $d^\bullet(X,Z) \leq 0$; we may conclude $d^\bullet(X,Z) = 0$, and therefore $d^\bullet(X,Y) = 0$ and $d^\bullet(Y,Z) = 0$.

It follows from τ-convexity that at least one object of the τ-orbit of Y is in $\operatorname{ind}\mathcal{Q}'$ and since light cone distances must be positive, we see that $Y \in \operatorname{ind}\mathcal{Q}'$. $\square$

We will use following lemma.

Lemma 6.27. *Let $\mathcal{Q}$ be a full additive Karoubian subcategory of $D^b\mathcal{A}$, maximal with respect to the property that $d^\bullet(X,Y) \geq 0$ for all $X, Y \in \operatorname{ind}\mathcal{Q}$, then $\mathcal{Q}$ is a hereditary section.*

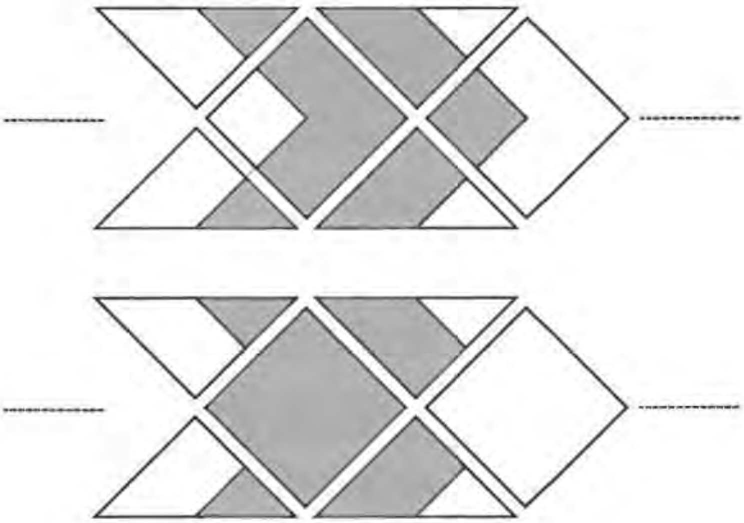


Figure 6.6: Illustration by Example 6.25

Proof. Due to Proposition 6.19, we only need to show $\mathcal{Q}$ is τ-convex. Let $Z \in D^b\mathcal{A}$ be an object such that there is a path from $\mathcal{Q}$ to the τ-orbit of Z and a path from the τ-orbit of Z to $\mathcal{Q}$. Let $X, Y \in \operatorname{ind} \mathcal{Q}$ be such that $d^\bullet(X, Z)$ and $d^\bullet(Z, Y)$ are minimal.

We may choose Z such that $d^\bullet(X, Z) = 0$; it follows from triangle inequality that $d^\bullet(Z, Y) \geq 0$. Since $\mathcal{Q}$ is maximal, we have $Z \in \operatorname{ind} \mathcal{Q}$, hence $\mathcal{Q}$ is a hereditary section. □

6.3.2 Light Cone Tilt

In this section, we will give some properties of a special tilt already used implicitly by Ringel in [47]. As usual, let $\mathcal{A}$ be a connected abelian Ext-finite hereditary category with Serre duality. Let X be any indecomposable directing object of $D^b\mathcal{A}$ and define the additive full subcategory $\mathcal{Q} \subset D^b\mathcal{A}$ by $\operatorname{ind} \mathcal{Q} = S^\bullet(X, 0)$, thus $\mathcal{Q}$ is the full additive subcategory of $D^b\mathcal{A}$ generated by those indecomposable objects Y such that X admits a path to Y, but no path to τY.

Theorem 6.21 yields there is a hereditary category $\mathcal{H}$ derived equivalent to $\mathcal{A}$ such that the category of projectives is given by $\mathcal{Q}$. We will refer to $\mathcal{H}$ as the *light cone tilt* centered on X.

Lemma 6.28. *In the light cone tilt centered on X, we have* $\operatorname{Hom}(X, P) \neq 0$, *for all projectives P.*

Proof. By construction, we know that $d^\bullet(X, P_i) = 0$, for every direct summand P_i of P. The result follows directly from Lemma 6.10. □

Lemma 6.29. *In the light cone tilt centered on X, all standard projectives are cofinitely presented.*

Proof. Let P be a standard projective and consider the canonical map $P \to FX \otimes \operatorname{Hom}(P, FX)$ with kernel K. Since P is projective, the kernel needs to be projective as well.

It is straightforward to check that $\operatorname{Hom}(X, K) = 0$, hence $K = 0$ and the canonical map is a monomorphism. $\square$

Proposition 6.30. *In a light cone tilt, all preprojectives are finitely presented and cofinitely presented.*

Proof. We will prove this using induction on the light cone distance of an indecomposable preprojective.

If Y is an indecomposable preprojective object with $d^\bullet(X, Y) = 0$, then the statement is Lemma 6.29.

So, assume now that $d^\bullet(X, Y) = n$. In this case, the preinjective object $\tau^{-n}Y$ is cofinitely presented. From the exact sequence $0 \to \tau^{-n}Y \to FX \otimes V \to J \to 0$, we deduce the short exact sequence $0 \to \tau^{-n+1}X \otimes V \to \tau^{-n+1}F^{-1}J \to Y \to 0$, showing that Y is generated by objects with strictly smaller light cone distance that Y. Using induction now shows that Y is finitely presented.

Since all preprojectives are finitely presented and all projectives cofinitely presented, it follows that all preprojectives are cofinitely presented. $\square$

6.3.3 Subcategories of hereditary sections

Let $\mathcal{A}$ be a connected abelian Ext-finite hereditary category satisfying Serre duality. In this section, we will discuss the structure of a hereditary section $\mathcal{Q}$ of $D^b\mathcal{A}$. The main idea we will pursue is that the shape of a hereditary section should be very close to that of a dualizing k-variety. Following proposition proves to be a useful tools in this.

For the statement of this proposition, we will define an interval $[X, Y]$ where $X, Y \in \operatorname{ind} \mathcal{Q}$ as follows

$$[X, Y] = \{Z \in \operatorname{ind} \mathcal{Q} \mid \mathcal{Q}(X, Z) \neq 0 \neq \mathcal{Q}(Z, Y)\}.$$

The interval $]X, Y[$ is defined in a similar fashion.

Proposition 6.31. *Let $\mathcal{Q}$ be a hereditary section and $[X, Y]$ be an interval. The full additive subcategory $\mathcal{S}$ of $\mathcal{Q}$ spanned by all objects in $\operatorname{ind} \mathcal{Q}$ not in $]X, Y[$ is locally discrete and locally finite.*

Proof. We know that, by Theorem 6.21, there is a hereditary category $\mathcal{H}$ with Serre duality, derived equivalent to $\mathcal{A}$, such that $\mathcal{Q}$ is the quiver of projectives of $\mathcal{H}$. Hence there is a full and exact embedding $i : \operatorname{mod} \mathcal{Q} \to \mathcal{H}$. Also we know, by Proposition 6.22, that all standard simples of $\operatorname{mod} \mathcal{Q}$ are finitely presented and copresented.

Since $\mathcal{S}$ consists of projective objects of $\mathcal{H}$, we know there is a full and exact embedding $j : \operatorname{mod}\mathcal{S} \to \mathcal{H}$ extending the natural embedding $\mathcal{S} \to \mathcal{H}$.

We will prove the standard simple $S_Z \in \operatorname{Mod}\mathcal{S}$ is finitely presented and copresented, for all $Z \in \operatorname{ind}\mathcal{S}$.

Consider the corresponding standard simple $S_Z \in \operatorname{mod}\mathcal{Q}$ as object in $\mathcal{H}$ by the embedding $i : \operatorname{mod}\mathcal{Q} \to \mathcal{H}$. It is clear S_Z has a finite presentation

$$P \to Z \to S_Z \to 0$$

in $\mathcal{H}$, where $P \in \operatorname{Ob}\mathcal{Q}$. We may decompose $P = P_{]X,Y[} \oplus P_{\mathcal{S}}$ where $P_{]X,Y[} \in \operatorname{Ob}]X,Y[$ and $P_{\mathcal{S}} \in \operatorname{Ob}\mathcal{S}$.

Next, consider canonical map $Y \to F_{\mathcal{H}}X \otimes \operatorname{Hom}(Y, F_{\mathcal{H}}X)^*$ in $\mathcal{H}$ and let K be the kernel. Note that K is projective and that $\dim\operatorname{Hom}(X,K) = \dim\operatorname{Hom}(K, F_{\mathcal{H}}X) = 0$, hence $K \in \operatorname{Ob}\mathcal{S}$.

It is clear that the cokernel C of the map

$$K \oplus P_{\mathcal{S}} \oplus (X \otimes \operatorname{Hom}(X,Z)) \to Z$$

lies in the essential image of j. We claim that C is isomorphic to $j(S_Z)$. Therefore, it suffices to show that for every $A \in \operatorname{ind}\mathcal{S}$, such that $A \not\cong Z$, we have $\operatorname{Hom}_{\mathcal{H}}(A, C) = 0$. Equivalently, we need to prove that a map $f : A \to Z$ in $\mathcal{H}$ factors through $K \oplus P_{\mathcal{S}} \oplus (X \otimes \operatorname{Hom}(X,Z))$.

In $\mathcal{H}$ the map $f : A \to Z$ would factor as

$$A \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} P_{\mathcal{S}} \oplus P_{]X,Y[} \longrightarrow Z.$$

We will show that g_2 factors through $K \oplus (X \otimes \operatorname{Hom}(X,Z))$.

The composition $A \to P_{]X,Y[} \to Y$ is non-zero. There are two possibilities, either the composition $P \to Y \to FX \otimes \operatorname{Hom}(Y, FX)^*$ is zero and then $P \to Y$ factors through K, or the composition is non-zero and $A \in [X,Y]$. Yet, A is chosen in $\operatorname{ind}\mathcal{S}$, hence either $g_2 = 0$ or $A \cong X$. It is clear that in both cases g_2 factors through $X \otimes \operatorname{Hom}(X,Z)$.

Showing that S_Z is cofinitely presented by standard injectives in $\operatorname{Mod}\mathcal{S}$ is dual. □

We will denote $\operatorname{add}(\operatorname{ind}\mathcal{Q}\setminus]X,Y[)$ by $\mathcal{Q}_{[X,Y]}$. Note that this is again a Krull-Schmidt additive category. The previous proposition shows that is locally finite and locally discrete.

For easy reference, we state following lemma.

Lemma 6.32. *Let $\mathcal{Q}$ be a hereditary section and $X, Y, Z \in \operatorname{ind}\mathcal{Q}$. If there is an irreducible morphism $X \to Z$ in $\mathcal{Q}_{[X,Y]}$, then there is an irreducible map $A \to C$ with $C \in [X,Y]$.*

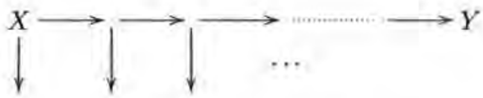
We now use Proposition 6.31 to give two example of locally finite locally discrete k-varieties (cf. Chapter 2) that cannot occur as hereditary sections.

Example 6.33. Let $\mathcal{Q}$ be the category

The Auslander-Reiten quiver of $\mathcal{Q}_{[X,Y]}$ is given by

which is clearly not locally discrete; the simple representation $S_X \in \operatorname{Ob}\operatorname{Mod}\mathcal{Q}_{[X,Y]}$ is not cofinitely presented.

Example 6.34. Let $\mathcal{Q}$ be the category



We draw the Auslander-Reiten quiver of $\mathcal{Q}_{[X,Y]}$ as

which is not locally finite. Indeed the simple representation S_X is not cofinitely presented.

6.3.4 Threads and Thread Objects

As with dualizing k-varieties, the concepts of threads will be paramount in our discussion of hereditary sections. However, a major difference between dualizing k-varieties and hereditary sections is that in the latter one might encounter so-called *broken threads.*

Definition 6.35. Let $\mathcal{Q}$ be a hereditary section. A *thread object* is an object $X \in \operatorname{ind}\mathcal{Q}$ which has a unique direct predecessor and a unique direct successor in $\operatorname{ind}\mathcal{Q}$.

Let $X, Y \in \operatorname{ind}\mathcal{Q}$, we define

$$\Xi(X,Y) = \{Z \in \operatorname{ind}\mathcal{Q} \mid d^\bullet(X,Z) + d^\bullet(Z,Y) = d^\bullet(X,Y)\}.$$

If $\Xi(X,Y)$ consists of only thread objects, then we call $\Xi(X,Y)$ a *thread*. If furthermore $d^\bullet(X,Y) > 0$ or $d^\bullet(X,Y) = 0$, then we call $\Xi(X,Y)$ a *broken thread* or an *unbroken thread*, respectively.

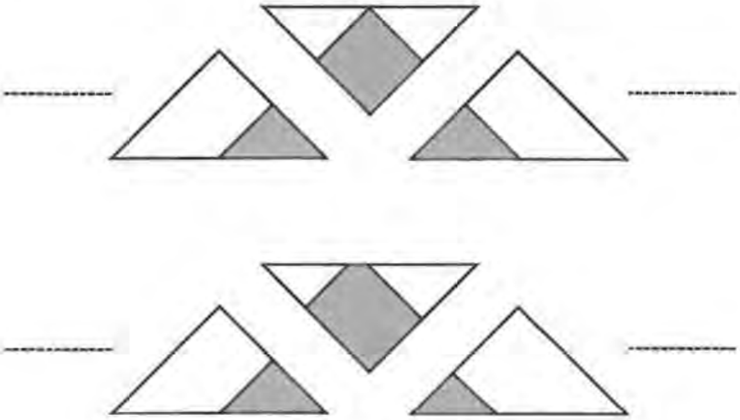


Figure 6.7: Illustration of Example 6.38

Remark 6.36. If $d^\bullet(X,Y)=0$, then $\Xi(X,Y)=[X,Y]$.

Remark 6.37. If $\Xi(X,Y)$ is a thread, then so are $\Xi(X,Z)$ and $\Xi(Z,Y)$, for every $Z\in\Xi(X,Y)$.

Example 6.38. Let $\mathfrak{a}$ be the dualizing k-variety given by

$$A_0 \longrightarrow A_1 \longrightarrow \cdots\cdots \longrightarrow B_1 \longrightarrow B_0$$

The Auslander-Reiten quiver of $D^b \operatorname{mod}\mathfrak{a}$ by be sketched as in the upper part of Figure 6.7 where the triangles represent $\mathbb{Z}A_\infty$-components, and where the category $\operatorname{mod}\mathfrak{a}$ has been marked with grey.

We will denote by $\mathcal{Q}$ the hereditary section in $D^b \operatorname{mod}\mathfrak{a}$ corresponding to the projectives of $\operatorname{mod}\mathfrak{a}$. The interval $[\mathfrak{a}(-,A_1),\mathfrak{a}(-,B_1)] = \Xi(\mathfrak{a}(-,A_1),\mathfrak{a}(-,B_1)) \subset \operatorname{ind}\mathcal{Q}$ is an (unbroken) thread.

We will now consider the hereditary section $\mathcal{Q}'\subseteq D^b \operatorname{mod}\mathfrak{a}$ spanned by all objects of the form $\mathfrak{a}(-,A_i)$ and $\tau\mathfrak{a}(-,B_i)$ as in the lower part of Figure 6.7. Now $\Xi(\mathfrak{a}(-,A_1),\tau\mathfrak{a}(-,B_1))\subset \operatorname{ind}\mathcal{Q}'$ is a broken thread.

Apart from the broken threads, another major difference between dualizing k-varieties and hereditary sections is the occurrence of so-called half-open threads in hereditary sections.

Definition 6.39. We define

$$\begin{aligned}\Xi(X,\rightarrow) &= \{Z\in\operatorname{ind}\mathcal{Q}\mid \Xi(X,Z)\text{ is a thread}\}\\ \Xi(\leftarrow,X) &= \{Z\in\operatorname{ind}\mathcal{Q}\mid \Xi(Z,X)\text{ is a thread}\}\end{aligned}$$

If $\Xi(X,\rightarrow)$ or $\Xi(\leftarrow,X)$ is not a thread, then we will say it is a *right-open thread* or *left-open thread*, respectively. We will define *left/right-open broken/unbroken threads.*

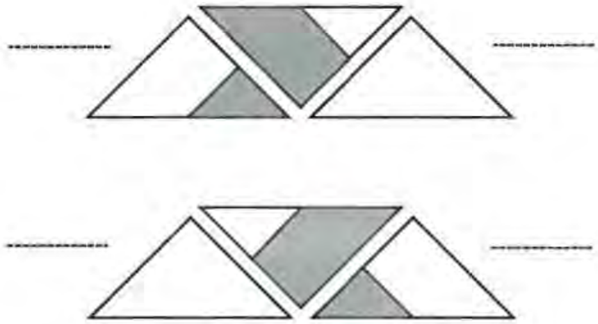


Figure 6.8: Illustration of Example 6.40

Example 6.40. As in Example 6.38, let $\mathfrak{a}$ be the dualizing k-variety given by

$$A_0 \longrightarrow A_1 \longrightarrow \cdots\cdots \longrightarrow B_1 \longrightarrow B_0$$

The hereditary section spanned by all objects of the form $\mathfrak{a}(-, A_i)$ is a right-open (unbroken) thread $\Xi(\mathfrak{a}(-, A_0), \rightarrow)$. Likewise, the hereditary section spanned by all objects of the form $\mathfrak{a}(-, B_i)$ is a left-open (unbroken) thread $\Xi(\leftarrow, \mathfrak{a}(-, B_0))$.

The corresponding abelian categories have been filled with grey in Figure 6.8.

Example 6.41. Let $\mathfrak{a}$ be the dualizing k-variety given by

$$A_0 \longrightarrow A_1 \longrightarrow \cdots\cdots \longrightarrow B_{-1} \longrightarrow B_0 \longrightarrow B_1 \longrightarrow \cdots\cdots \longrightarrow C_{-1} \longrightarrow C_0$$

or, drawn according to the conventions of §2.2.2,

$$A_0 \longrightarrow \xrightarrow{\mathbb{Z}} \longrightarrow C_0$$

The category $D^b \operatorname{mod} \mathfrak{a}$ may be sketched as the first part of Figure 6.9 where the triangles represent $\mathbb{Z}A_\infty$-components, and the squares represent $\mathbb{Z}A_\infty^\infty$-components. As usual, the abelian category $\operatorname{mod} \mathfrak{a} \subset D^b \operatorname{mod} \mathfrak{a}$ has been marked with grey.

Choosing a hereditary section $\mathcal{Q}$ spanned by objects of the form $\mathfrak{a}(-, A_i)$ and $\tau\mathfrak{a}(-, B_i)$ gives rise to an abelian category as sketched in the second part of Figure 6.9. Here $\operatorname{ind} \mathcal{Q} = \Xi(\mathfrak{a}(-, A_0), \rightarrow)$ is a right-open (broken) thread.

Likewise, choosing a hereditary section $\mathcal{Q}$ spanned by objects of the form $\tau^{-1}\mathfrak{a}(-, B_i)$ and $\tau\mathfrak{a}(-, C_i)$ gives rise to an abelian category as sketched in the last part of Figure 6.9. Here $\operatorname{ind} \mathcal{Q} = \Xi(\leftarrow, \mathfrak{a}(-, C_0))$ is again a left-open (broken) thread.

The following proposition gives most results we will need about threads. We start with a lemma.

Lemma 6.42. *Let $\Xi(X, Y')$ be a unbroken thread in $D^b\mathcal{A}$, then*

$$\operatorname{cone}(\tau X' \to X) \cong \operatorname{cone}(\tau Y' \to Y)$$

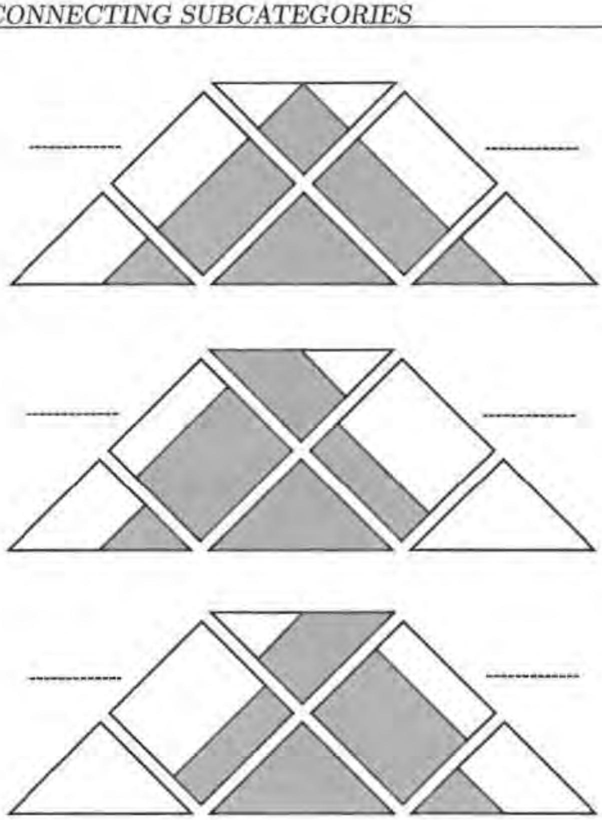


Figure 6.9: Illustration of Example 6.41

where X' is the unique direct successor of X and Y the unique direct predecessor of Y'.

Proof. We will work in the light cone tilt centered on $\tau X'$, thus $\tau X'$, $\tau Y'$, and X are indecomposable projectives. Since Y is a preprojective, it follows from Proposition 6.30 that Y is finitely presented, and from the proof of that proposition that a presentation is given by $0 \longrightarrow \tau X' \longrightarrow X \oplus \tau Y' \longrightarrow Y \longrightarrow 0$. In particular, Y is the push-out of the diagram

$$\begin{tikzcd} \tau Y' \arrow[r, dashed] & Y \\ \tau X' \arrow[u] \arrow[r] & X \arrow[u, dashed] \end{tikzcd}$$

Hence $\operatorname{coker}(\tau Y' \longrightarrow Y) \cong \operatorname{coker}(\tau X' \longrightarrow X)$. Both morphisms are irreducible and thus either a monomorphism or an epimorphism. Since $\tau X' \longrightarrow X$ is a monomorphism, we see that $\tau Y' \longrightarrow Y$ is a monomorphism as well.

We conclude $\operatorname{cone}(\tau X' \to X) \cong \operatorname{cone}(\tau Y' \to Y)$. □

Proposition 6.43. *Let $\mathcal{Q}$ be a hereditary section in $D^b\mathcal{A}$ and let $X, Y, Z \in \operatorname{ind} \mathbb{Z}\mathcal{Q}$.*

1. *If $\Xi(X,Y)$ and $\Xi(X,Z)$ are threads, then $\Xi(X,Y) \subseteq \Xi(X,Z)$ or $\Xi(X,Z) \subseteq \Xi(X,Y)$.*

2. *If $\Xi(X,\rightarrow)$ is a half-open thread, then there is no broken thread $\Xi(X,Z)$ such that $\Xi(X,\rightarrow) \subseteq \Xi(X,Z)$.*

3. *If $\Xi(X,\rightarrow)$ is a right-open unbroken thread and $Y \in \mathbb{Z}\mathcal{Q}$ then there is a $Z \in \Xi(X,\rightarrow)$ such that no object of the right-open broken thread $\Xi(Z,\rightarrow)$ maps nonzero to Y.*

4. *Let $\Xi(X,\rightarrow)$ be a right-open unbroken thread and let $X' \in \operatorname{ind} \mathcal{Q}$ be the unique direct successor of X. Then there is a triangle $X \to C \to FX' \to X[1]$ where C is indecomposable, and $C \notin \operatorname{ind} \mathcal{Q}$. Furthermore, if $\operatorname{Hom}(Q,C) \neq 0$, then either $\operatorname{Hom}(X,Q) \neq 0$ or $\operatorname{Hom}(Q,X) \neq 0$.*

Proof. 1. In the light cone tilt centered on X we find both $\Xi(X, \tau^{-d^\bullet(X,Z)}Z)$ and $\Xi(X, \tau^{-d^\bullet(X,Y)}Y)$ are unbroken threads. Example 6.33 now yields the result.

2. This is trivial.

3. Consider a light cone tilt centered on X. Proposition 6.30 yields the preprojective Y is finitely presented. The assertion follows easily.

4. Let $Y \in R$ but $Y \not\cong X$; in particular $\operatorname{Hom}(X',Y) \neq 0$. Applying $\operatorname{Hom}(Y,-)$ to the triangle above, and using $\operatorname{Hom}(Y, X[1]) = 0$, shows $\operatorname{Hom}(Y,C) \neq 0$. Since R is a right-open thread, it follows that $C \notin \operatorname{ind} \mathcal{Q}$.

   The last claim follows readily from Lemma 3.19.

□

6.3.5 Thread quivers

In this subsection, we will give some properties the component $\mathbb{Z}\mathcal{Q}$.

Lemma 6.44. *For all X, Y ind $\mathcal{Q}$, there may only be finitely many non-thread objects in $[X,Y]$.*

Proof. Every non-thread object in $]X,Y[$ gives either an extra arrow from X or an extra arrow to Y in the Auslander-Reiten quiver of $\mathcal{Q}_{[X,Y]}$. Proposition 6.31 now yields the result. □

Proposition 6.45. *Let $\mathcal{Q}$ be a hereditary section. The full subcategory $\mathcal{S}$ consisting of all non-thread objects and their neighbours is locally finite and locally discrete, and every interval $[Y,Z] \subseteq \operatorname{ind} \mathcal{S}$ is finite.*

Proof. First note that every non-thread object in $\mathcal{Q}$ corresponds to a non-thread object in $\mathcal{S}$. Since all neighbours are in $\mathcal{S}$, it is clear that a non-thread object in locally finite and locally discrete.

Let $X \in \operatorname{ind} \mathcal{Q}$ be a thread object. By construction, X is adjacent to at most two indecomposables of $\mathcal{S}$, of which at least one is a non-thread object in $\mathcal{Q}$. Due to Lemma 6.44, any interval $[Y, Z] \subseteq \operatorname{ind} \mathcal{Q}$ may contain only finitely many non-thread objects, and thus the corresponding interval $[Y, Z] \subseteq \operatorname{ind} \mathcal{S}$ is finite. Hence every object of $\mathcal{S}$ is locally finite and discrete. This finishes the proof. □

Since $\mathcal{Q}$ is semi-hereditary (Corollary 6.22), so is $\mathcal{S}$. It now follows from Proposition 6.45 that $\mathcal{S}$ may be seen as a free k-linear category associated to a quiver. There is however quite some loss of information in passing from $\mathbb{Z}\mathcal{Q}$ to $\mathcal{S}$, as broken and unbounded threads are not represented in $\mathcal{S}$. The next example shows $\mathcal{S}$ does not need to be connected, even if $\mathbb{Z}\mathcal{Q}$ is.

Example 6.46. We will consider the abelian category constructed in Example 6.38. Since there are no nonzero maps from $\mathfrak{a}(-, A_i)$ to $\tau\mathfrak{a}(-, B_i) \cong \mathfrak{a}(B_i, -)^*[-1]$, it is clear $\mathcal{Q}$ is not connected. However, the connecting subcategory $\mathbb{Z}\mathcal{Q}$ is connected.

A *thread quiver* is a quiver $Q = (Q_0, Q_1, h, t)$, where the vertices Q_0 are divided into black vertices $\bullet$ and white vertices $\circ$, and where the arrows are divided into dashed arrows $\dashrightarrow$ and full arrows $\longrightarrow$.

We will consider following thread quiver, associated with a hereditary section $\mathcal{Q}$ in $D^b\mathcal{A}$ where $\mathcal{A}$ is an abelian hereditary Ext-finite category with Serre duality.

The black vertices are given by $\operatorname{ind} \mathcal{S}$ and the full arrows between these vertices are induced by the Auslander-Reiten quiver of $\mathcal{Q}$. If there is a thread $\Xi(X, Y)$ where $X, Y \in \operatorname{ind} \mathcal{S}$, then we will draw a dashed arrow $X \dashrightarrow Y$. We will also label it with $d^\bullet(X, Y)$.

If there is an half-open thread $\Xi(X, \rightarrow)$ or $\Xi(\leftarrow, X)$, then we will add a white vertex and a dashed arrow $X \dashrightarrow \circ$ or $\circ \dashrightarrow X$ and label it with $\sup_{Z \in \Xi(X, \rightarrow)} d^\bullet(X, Z)$ or $\sup_{Z \in \Xi(\leftarrow, X)} d^\bullet(Z, X)$, respectively.

By construction, every black vertex in Q has the same number of neighbours in Q as the corresponding object in $\mathcal{Q}$. Every white vertex has exactly one neighbour. In particular, Q is locally finite.

Note that the valuation of the dashed arrows depends, not only on $\mathcal{Q}$, but on the embedding of $\mathcal{Q}$ in $D^b\mathcal{A}$. Also note that $\mathbb{Z}\mathcal{Q}$ is connected if and only if the associated thread quiver is connected.

Following result is now easy.

Proposition 6.47. *Let $\mathcal{A}$ be an abelian hereditary category satisfying Serre duality and let $\mathcal{Q}$ be a hereditary section in $D^b\mathcal{A}$. If $\mathbb{Z}\mathcal{Q}$ is connected, then there are only a countable amount of non-thread objects and maximal unbounded broken threads in $\mathcal{Q}$.*

Proof. Every non-thread object or maximal unbounded thread in $\mathcal{Q}$ corresponds to a black vertex or white vertex in Q. Since Q is a connected and locally finite quiver, there may be only countably many objects. □

6.4 Tilting hereditary sections to dualizing k-varieties

In this section, we will examine when a hereditary section has a tilt which is a dualizing k-variety. In this case, the tilted hereditary section $\mathcal{Q}_{\mathcal{H}}$ will admit a full and exact embedding $D^b \operatorname{mod} \mathcal{Q} \to D^b\mathcal{A}$ which commutes with Serre duality.

It will however, not always be possible to find such a tilt. The necessary extra condition (*) will be discussed in §6.4.2.

We will start with a criterium for a hereditary section to be a dualizing k-variety, and some useful lemmas in the same context.

6.4.1 Hereditary sections as dualizing k-varieties

Definition 6.48. We will say a set $S \subseteq \operatorname{ind} \mathfrak{a}$, for a small additive category $\mathfrak{a}$, is *boundedbounded* if there is a finite set $C \subseteq S$ such that $\forall B \in S, \exists C_1, C_2 \in C$: $\operatorname{Hom}(B, C_1) \neq 0$ and $\operatorname{Hom}(C_2, B) \neq 0$.

Proposition 6.49. *Let $\mathcal{A}$ be an abelian hereditary category satisfying Serre duality. A hereditary section $\mathcal{Q}$ in $D^b\mathcal{A}$ is a dualizing k-variety if and only if the sets $S^{\bullet}_{\mathcal{Q}}(A,0)$ and $S^{\mathcal{Q}}_{\bullet}(A,0)$ are bounded for all $A \in \operatorname{ind} \mathcal{Q}$.*

Proof. By Corollary 6.22 we know $\mathcal{Q}$ is semi-hereditary and by Proposition 2.7 it suffices to show that, for all $A \in \operatorname{ind} \mathcal{Q}$, the functor $\mathcal{Q}(-, A)$ is cofinitely presented and the functor $\mathcal{Q}(A, -)^*$ is finitely presented.

By Theorem 6.21 we know there is a hereditary category $\mathcal{H}$ with Serre duality, derived equivalent to $\mathcal{A}$, such that $\mathcal{Q}$ is the category of projectives and we may consider the natural embedding $i : \operatorname{mod} \mathcal{Q} \to \mathcal{H}$. We may thus reduce to proving $\mathcal{Q}(-, A)$ is cofinitely generated and the functor $\mathcal{Q}(A, -)^*$ is finitely generated.

We will prove $\mathcal{Q}(A, -)^*$ is finitely generated; the other statement is dual. To prove $\mathcal{Q}(A, -)^*$ is finitely generated, we will show FA is finitely generated in $\mathcal{H}$. This is indeed sufficient since then FA lies in the essential image of i and for all $M \in \operatorname{mod} \mathcal{Q}$, we have

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{H}}(iM, FA) &\cong \operatorname{Hom}_{\mathcal{H}}(A, iM)^* \\
&\cong \operatorname{Hom}_{\operatorname{mod} \mathcal{Q}}(\mathcal{Q}(-, A), M)^* \\
&\cong M(A)^* \\
&\cong \operatorname{Hom}_{\operatorname{mod} \mathcal{Q}}(M, \mathcal{Q}(A, -)^*) \\
&\cong \operatorname{Hom}_{\mathcal{H}}(iM, i\mathcal{Q}(A, -)^*)
\end{aligned}$$

such that $i\mathcal{Q}(A, -)^* \cong FA$.

To prove a map $\xi : P \to FA$ is an epimorphism, it suffices to show that every map $B \to FA$, where $B \in \mathcal{Q}$ or more specifically $B \in S^{\mathcal{Q}}_{\bullet}(A, 0)$, factors through ξ. Indeed, since FA is injective, the cokernel I of ξ is injective and $\operatorname{Hom}(F^{-1}I, I) \neq 0$ where $F^{-1}I \in \mathcal{Q}$.

By assumption we know the set $S^\bullet_\mathcal{Q}(A,0)$ is bounded, hence there is a –possibly decomposable– object C such that $\operatorname{Hom}(B,C) \neq 0$, for all $B \in S^\mathcal{Q}_\bullet(A,0)$. Since $\mathcal{Q}$ is semi-hereditary, this gives a monomorphism $\mathcal{Q}(A,B) \to \mathcal{Q}(A,C)$, for all $B \in \operatorname{Ob}\mathcal{Q}$. Using the embedding i and Serre duality in $\mathcal{H}$, we find

$$\begin{tikzcd} \operatorname{Hom}(A,C)^* \arrow[r, "\sim"] \arrow[d] & \operatorname{Hom}(C,FA) \arrow[d] \\ \operatorname{Hom}(A,B)^* \arrow[r, "\sim"] & \operatorname{Hom}(B,FA) \end{tikzcd}$$

and hence every map $B \to FA$ factors through a map $C \to FA$. Since every map $C \to FA$ factors through the canonical map $C \otimes \operatorname{Hom}(C,FA) \to FA$, we conclude by setting $P = C \otimes \operatorname{Hom}(C,FA)$ and letting $\xi : C \otimes \operatorname{Hom}(C,FA) \to FA$ be the canonical map.

Since F is an exact functor commuting with i on the generators, it is easily seen to commute with i for every object. □

The following lemma may be useful in proving the sets $S^\bullet_\mathcal{Q}(A,0)$ and $S^\mathcal{Q}_\bullet(A,0)$ are bounded. The first part is a consequence of Proposition 6.30; the second part follows from Proposition 6.31.

Lemma 6.50. *Let $\mathcal{A}$ be an abelian hereditary category with Serre duality and denote the category of projectives by $\mathcal{Q}$.*

1. *Let $X, Y \in \operatorname{ind}\mathbb{Z}\mathcal{Q}$, then the set $S = S^\bullet(X,0) \cap S_\bullet(Y,0)$ is bounded.*
2. *Let $S \subseteq S^\bullet(X,0)$ be convex and bounded. Then, for any $Y \in \operatorname{ind}\mathbb{Z}\mathcal{Q}$, the set $S \setminus B^\bullet(Y,0)$ is convex and bounded.*

Proof. 1. Note that if $d^\bullet(X,Y) > 0$, it would follow from triangle inequality that S is empty; we may thus assume $d^\bullet(X,Y) \leq 0$.

Consider a light cone tilt $\mathcal{H}_X$ of $\mathcal{A}$ centered on X. Since $d^\bullet(X,Y) \leq 0$, we know Y is a preprojective object in $\mathcal{H}_X$ and hence by Proposition 6.30 there exists a minimal projective presentation

$$0 \longrightarrow Q \longrightarrow P \longrightarrow Y \longrightarrow 0$$

where P, Q are projective in $\mathcal{H}_X$. Let P' be a maximal direct summand of P such that every indecomposable lies in S.

We will show every object $A \in S$ maps non-zero to P'. Since $d^\bullet(X,A) = 0$, the object A corresponds to a projective object in $\mathcal{H}_X$ and since $d^\bullet(A,Y) = 0$, Lemma 6.10 yields $\operatorname{Hom}(A,Y) \neq 0$. We easily infer $\operatorname{Hom}(A,P) \neq 0$.

Every direct summand P_i of P maps nonzero to Y, yielding $d^\bullet(P_i,Y) \leq 0$. Since $d^\bullet(A,Y) = 0$, triangle inequality shows that a non-zero map $A \longrightarrow Y$ may only factor through those direct summands P_i of P with $d^\bullet(P_i, Y) \geq 0$. Hence every object $A \in S$ maps non-zero to P'.

Dually, we obtain an object I such that $\operatorname{ind} I \subseteq S$ and $\operatorname{Hom}(I, A) \neq 0$, for all object $A \in S$.

2. It follows from triangle inequality that $S \setminus B^\bullet(Y, 0)$ is convex. We will prove the set $S' = S \setminus B^\bullet(Y, 0)$ is bounded. Assume S is bounded by a finite set $C \subseteq S$. If no element of C lies in $B^\bullet(Y, 0)$, then $S' = S$ and hence bounded.

   Thus let A be an indecomposable direct summand of C in $B^\bullet(Y, 0)$, say $d^\bullet(Y, A) = -n \leq 0$. Next, consider the co-light cone tilt $\mathcal{H}^A$ centered on A with category of injectives $\mathcal{Q}^A$. It is clear that both X and $\tau^{-n}Y$ lie in $\mathcal{Q}^A$, as well as all objects in S mapping non-zero to A.

   Due to Proposition 6.31 we know the category $\mathcal{Q}^A_{[\tau^{-n}Y, A]}$ is locally finite and locally discrete, thus there is a finite set of indecomposable direct predecessors $\{N_i\}$ of A lying in S. We obtain

$$\begin{aligned} d^\bullet(Y, N_i) &\leq d^\bullet(Y, \tau A) + d^\bullet(\tau A, N_i) \\ &= -n + 1 + d^\bullet(\tau A, N_i) \leq -n + 1 \end{aligned}$$

   We replace $A \in C$ by $\{N_i\}_i$. Iterating this process gives an object C' bounding S'.

□

6.4.2 The condition (*)

Let $\mathcal{A}$ be a connected abelian hereditary Ext-finite category satisfying Serre duality and denote the category of projectives by $\mathcal{Q}$. We will assume $\mathbb{Z}\mathcal{Q}$ is connected.

If $\mathcal{Q}$ is a dualizing k-variety, then by Proposition 2.7 we know $\mathcal{Q}(-, A)$ is cofinitely presented. This means that at least one source S maps non-zero to A, hence $d^\bullet(S, A) = 0$. Dually we find that A maps non-zero to at least one sink T, such that $d^\bullet(A, T) = 0$.

Proposition 6.47 yields there are only a countable amount of sinks and sources, hence $\mathcal{Q}$ satisfies following property : there is a countable subset $\mathcal{T} \subseteq \operatorname{ind} \mathcal{Q}$ such that $d(\mathcal{T}, X) = 0$, for all $X \in \operatorname{ind} \mathcal{Q}$.

We will weaken this property to :

(*) : there is a countable subset $\mathcal{T} \subseteq \operatorname{ind} \mathbb{Z}\mathcal{Q}$ such that $d(\mathcal{T}, X) < \infty$, for all $X \in \operatorname{ind} \mathbb{Z}\mathcal{Q}$.

It is thus clear (*) needs to be satisfied when $\mathcal{Q}$ is a dualizing k-variety. Before starting to discuss (*) we recall following definitions.

Definition 6.51. Let $\mathcal{Q}$ be a poset. The subset $\mathcal{T} \subseteq \mathcal{P}$ is said to be *cofinal* such that for every $X \in \mathcal{Q}$ there is a $Y \in \mathcal{T}$ such that $X \leq Y$. The least cardinality of the cofinal subsets of $\mathcal{Q}$ is called the *cofinality* of $\mathcal{Q}$ and is denote by $\operatorname{cofin} \mathcal{Q}$.

Dually, one defines a *coinitial* subset of $\mathcal{Q}$ and the *coinitiality* of $\mathcal{Q}$ is denoted by $\operatorname{coinit} \mathcal{Q}$.

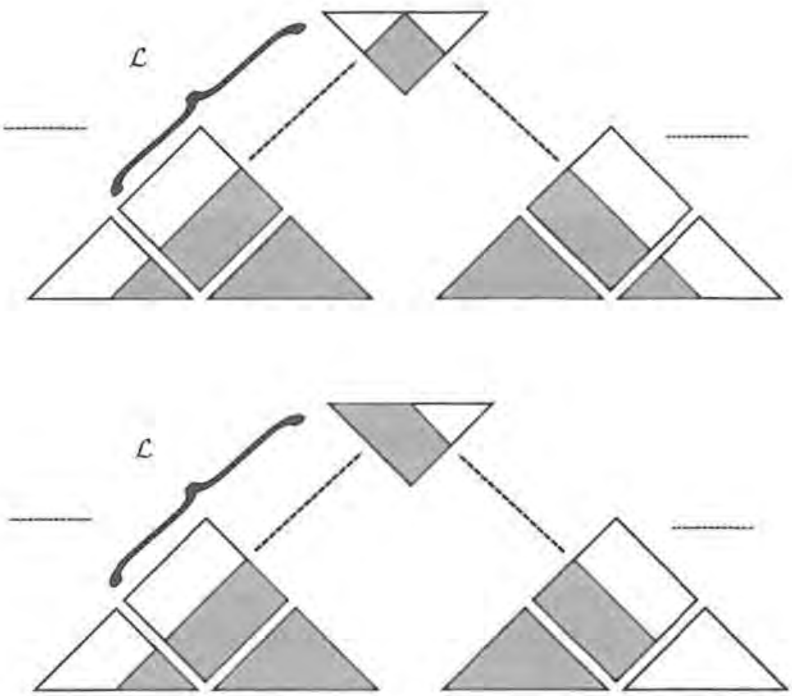


Figure 6.10: Illustration of Example 6.52

Next example shows (*) is not always satisfied.

Example 6.52. Let $\mathcal{L}$ be a linearly ordered and locally discrete set such that $\operatorname{cofin}\mathcal{L} > \aleph_0$. For example, if $\mathcal{L}'$ is a linearly ordered set with $\operatorname{cofin}\mathcal{L}' > \aleph_0$ we may define the poset $\mathcal{L} = \mathcal{L}' \overset{\rightarrow}{\times} \mathbb{Z}$.

Let $\mathcal{P}$ be the poset $\mathbb{N} \cdot \mathcal{L} \cdot (-\mathbb{N})$, thus $k\mathcal{P}$ is the semi-hereditary dualizing k-variety given by the thread quiver

$$\longrightarrow \overset{\mathcal{L}}{\cdots\cdots} \longrightarrow$$

. We may sketch the category as the upper part of Figure 6.10.

In $\operatorname{mod} k\mathcal{P}$, we consider a new hereditary category $\mathcal{H}$ by choosing a hereditary section $\mathcal{Q}$ in $\operatorname{mod} k\mathcal{P}$ generated by all standard projectives of the form $\mathcal{P}(-, A)$ where $A \in \mathbb{N}$ or $A \in \mathcal{L}$. The category $\mathcal{H}$ is marked with grey in Figure 6.10.

The new category $\mathcal{H}$ has category of projectives $\mathcal{Q}$ and $\mathbb{Z}\mathcal{Q}$ does not satisfy (*).

The following proposition states that the only case in which condition (*) fails, are akin to Example 6.52.

Proposition 6.53. *Let $\mathcal{A}$ be an abelian hereditary Ext-finite category satisfying Serre duality and denote the category of projectives by $\mathcal{Q}$. Assume $\mathbb{Z}\mathcal{Q}$ is connected. If $\mathbb{Z}\mathcal{Q}$ does not satisfy (*), then $\mathcal{Q}$ contains a right-open unbroken thread $R = \Xi(X, \rightarrow)$ with* $\operatorname{cofin} R > \aleph_0$, *or a left-open unbroken thread $R = \Xi(\leftarrow, X)$ with* $\operatorname{coinit} R > \aleph_0$.

Proof. Let $\mathcal{T}$ be the set of all non-thread objects. It follows from Proposition 6.47 that $\mathcal{T}$ is countable. Since $\mathbb{Z}\mathcal{Q}$ does not satisfy (*) and there is only a countable number of maximal left-open or right-open threads (Proposition 6.47), one of these threads R must satisfy following condition : there is no countable subset $\mathcal{T}' \subseteq R$ such that $d(\mathcal{T}', Y) < \infty$ for all $Y \in R$.

Assume $R = \Xi(X, \rightarrow)$. For every $n \in \mathbb{N}$ we may consider the following subset.

$$R_n = \{Y \in R \mid d^\bullet(X, Y) = n\}.$$

If R_n is non-empty for arbitrarily large n, then it is easily seen, by letting $\mathcal{T}'$ contain X and an object from every nonempty R_n, that $d(\mathcal{T}', Y) \leq \infty$ for all $Y \in R$. A contradiction.

Thus there is a largest n such that R_n is non-empty. We need to prove that $\operatorname{cofin} R_n > \aleph_0$. If $\mathcal{T}'$ were a countable cofinal subset of R_n, then $d(\mathcal{T}' \cup \{X\}, Y) \leq n$, for every $Y \in R$, where $\mathcal{T}' \cup \{X\}$ is countable. A contradiction.

The case where $R = \Xi(\leftarrow, X)$ is dual. □

We now come to a case where where (*) is automatically satisfied.

Proposition 6.54. *If $\mathcal{A}$ is generated by preprojectives and cogenerated by preinjectives, then the category of projectives $\mathcal{Q}$ satisfies (*).*

Proof. Seeking a contradiction, we will assume $\mathbb{Z}\mathcal{Q}$ does not satisfy condition (*). By Proposition 6.53, we may assume has a right-open unbroken thread $\mathcal{R} = \Xi(X, \rightarrow)$ with $\operatorname{cofin} \mathcal{R} > \aleph_0$ or a left-open unbroken thread $\mathcal{R} = \Xi(\leftarrow, X)$ with $\operatorname{coinit} \mathcal{R} > \aleph_0$. We will assume the former, the latter is dual.

Let X' be the direct successor of X in $\mathcal{Q}$ and consider the exact sequence

$$0 \longrightarrow X \xrightarrow{f} C \longrightarrow FX' \longrightarrow 0.$$

from Proposition 6.43.

Since $\mathcal{R}$ is right-open and $\tau^{-1}Q$ is preprojective, Proposition 6.43 yields that not every object of $\mathcal{R}$ admits a map to $\tau^{-1}Q$. Let $Y \in \mathcal{R}$ be such an object and Y' be its unique direct successor.

By Lemma 6.42 we know there is an exact sequence

$$0 \longrightarrow Y \xrightarrow{f'} C \longrightarrow FY' \longrightarrow 0.$$

Since $\tau^{-1}Q$ does not contain any projective direct summands, we see $\operatorname{Hom}(Y, \tau^{-1}Q) = 0$. Together with $\operatorname{Hom}(\tau^{-1}Q, Y) = 0$, this gives a contradiction to Proposition 6.43

We conclude that $\operatorname{cofin}(\mathcal{R}) \leq \aleph_0$. □

6.4.3 Main theorem

In this section, we will investigate in which connected abelian hereditary categories $\mathcal{A}$ satisfying Serre duality we may 'tilt' the category of projectives $\mathcal{Q}$ to a dualizing k-variety. By §6.4.2 we will need to assume $\mathbb{Z}\mathcal{Q}$ satisfies condition $(*)$. Our main result is Theorem 6.61 where we show (*) is also sufficient.

Our plan of proof follows the proof of Theorem 1.51. The main difference, however, is that choosing one object X will not suffice to define a useful round trip distance on every object of $\operatorname{ind}\mathcal{Q}$. To remedy this, we will construct a set $\mathcal{T}$ in such a way that $d(\mathcal{T},Y)\neq\infty$ for all $Y\in\operatorname{ind}\mathcal{Q}$. With some additional conditions on $\mathcal{T}$, we may show that the associated hereditary section, is a dualizing k-variety.

We start by constructing the set $\mathcal{T}$ and the associated hereditary section, $\mathcal{Q}_{\mathcal{H}}$.

Construction 6.55. By (*) we may assume there is a countable subset $\mathcal{T}=\{T_i\}_{i\in\mathbb{N}}\subset\operatorname{ind}\mathbb{Z}\mathcal{Q}$ such that $d(\mathcal{T},X)<\infty$ for all $X\in\mathbb{Z}\mathcal{Q}$.

Without loss of generality, we may assume $d(T_i,T_j)=\infty$ for $i\neq j$. Indeed, if $d(T_i,T_j)\neq\infty$, we may consider the set $\mathcal{T}'=\mathcal{T}\setminus\{T_j\}$ and use triangle inequality to show $d(\mathcal{T}',X)<\infty$ for all $X\in\mathbb{Z}$.

Thus, let $\mathcal{T}$ be chosen such that $d(T_i,T_j)=\infty$ for $i\neq j$. We may furthermore choose T_i such that $d^\bullet(T_i,T_j)\geq i$ for all $i>j$.

Associated to $\mathcal{T}$, we will consider the full subcategory $\mathcal{Q}_{\mathcal{H}}$ of $D^b\mathcal{A}$ as follows: for every $X\in\operatorname{ind}\mathbb{Z}\mathcal{Q}$, we fix a τ-shift of X such that

$$d^\bullet(\mathcal{T},X)=\left\lfloor\frac{d(\mathcal{T},X)}{2}\right\rfloor.$$

Example 6.56. Let $\mathfrak{a}$ be the dualizing k-variety given by the thread quiver

$$A_0\longrightarrow\overset{\mathbb{Z}\cdot\mathbb{Z}}{\cdots\cdots>}\longrightarrow D_0,$$

thus $\mathfrak{a}$ corresponds to the poset $\mathbb{N}\cdot\mathbb{Z}\cdot\mathbb{Z}\cdot\mathbb{N}$. The Auslander-Reiten quiver of $D^b\operatorname{mod}\mathfrak{a}$ is as sketched in the upper part of Figure 6.11. We will consider the hereditary section $\mathcal{Q}$ spanned by all objects of $\mathfrak{a}\subset D^b\operatorname{mod}\mathfrak{a}$ lying in a $\mathbb{Z}A_\infty^\infty$. The corresponding hereditary category $\mathcal{A}$ is as given by the lower part of Figure 6.11.

We choose a set $\mathcal{T}=\{T_0,T_1\}$ as in Figure 6.12, satisfying the conditions $d(T_0,T_1)=\infty$ and $d^\bullet(T_0,T_1)\geq 0$ from Construction 6.55. In Figure 6.12, the light cones $S^\bullet(\mathcal{T},0)$ and $S_\bullet(\mathcal{T},0)$ have been marked by black arrows, and the corresponding full subcategory $\mathcal{Q}_{\mathcal{H}}$ of $D^b\mathcal{A}$ has been indicated by '$\bullet$'.

We first verify that $\mathcal{Q}_{\mathcal{H}}$ defined above is indeed a hereditary section.

Proposition 6.57. *The subcategory $\mathcal{Q}$ defined in Construction 6.55 is a hereditary section.*

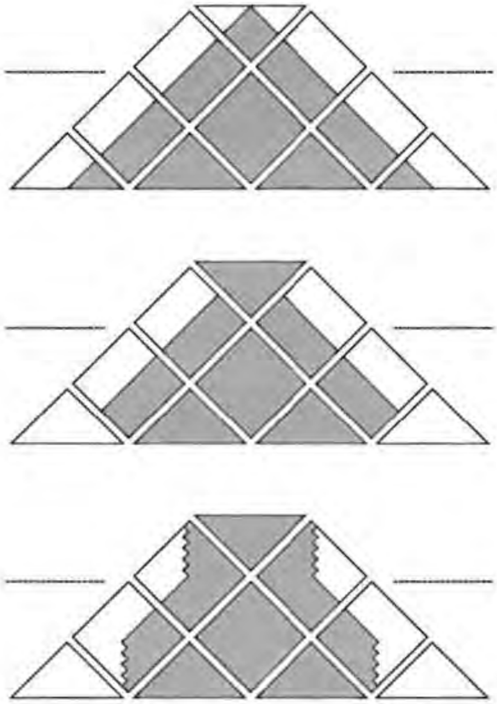


Figure 6.11: Illustrations for mod $\mathfrak{a}$, $\mathcal{A}$, and $\mathcal{H}$ of Example 6.56

Proof. Using Proposition 6.26, we only need to check $d^\bullet(Y,Z) \geq 0$, for all $Y, Z \in \operatorname{ind} \mathcal{Q}_\mathcal{H}$. Using triangle inequality, we find

$$\begin{aligned} d^\bullet(Y,Z) &\geq d^\bullet(\mathcal{T},Z) - d^\bullet(\mathcal{T},Y) \\ &= \left\lfloor \frac{d(\mathcal{T},Z)}{2} \right\rfloor - \left\lfloor \frac{d(\mathcal{T},Y)}{2} \right\rfloor \geq 0 \end{aligned}$$

if $d(\mathcal{T},Y) \leq d(\mathcal{T},Z)$, and

$$\begin{aligned} d^\bullet(Y,Z) &\geq d^\bullet(Y,\mathcal{T}) - d^\bullet(Z,\mathcal{T}) \\ &= \left\lceil \frac{d(\mathcal{T},Y)}{2} \right\rceil - \left\lceil \frac{d(\mathcal{T},Z)}{2} \right\rceil \geq 0. \end{aligned}$$

if $d(\mathcal{T},Z) \leq d(\mathcal{T},Y)$. □

The tilt in Construction 6.55 has been chosen such that 'in the neighbourhood' of an object $A \in \mathcal{Q}$, the round trip distance $d(\mathcal{T}, -)$ is determined by a finite subset

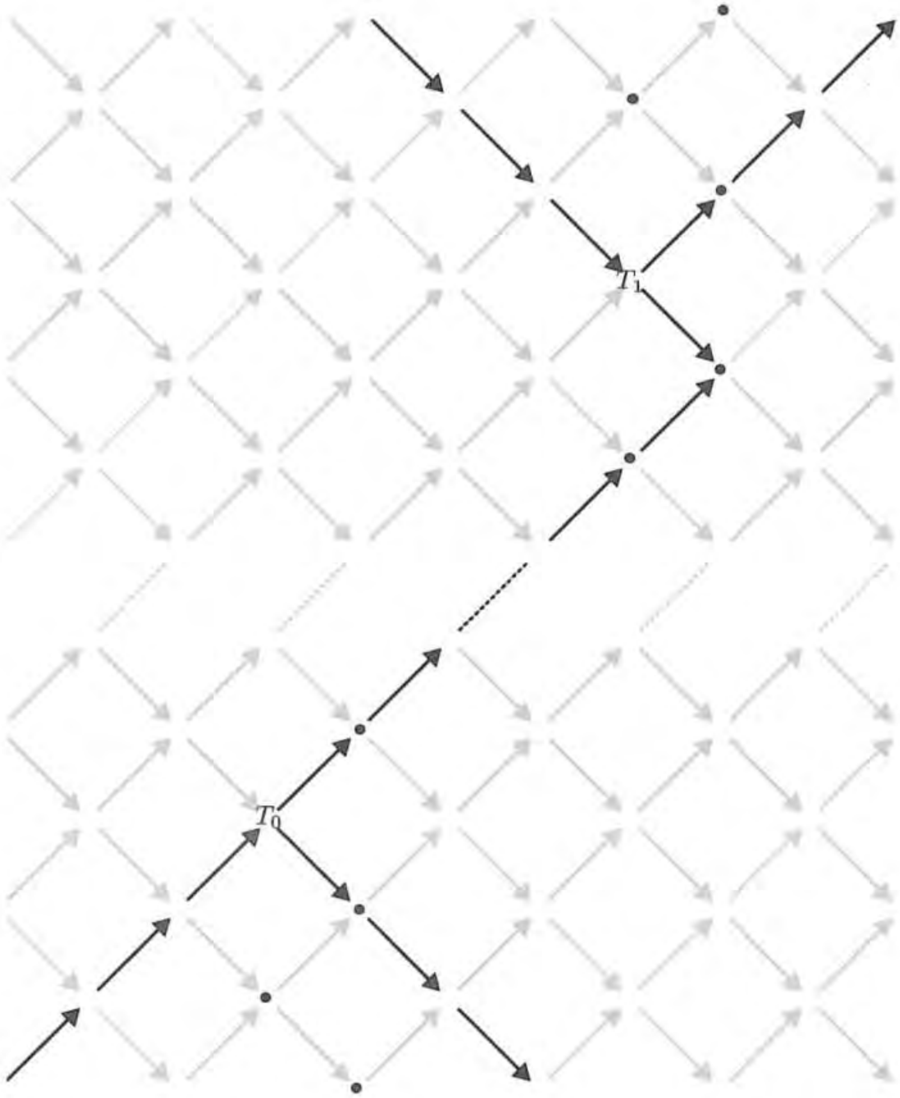


Figure 6.12: The light cones and chosen hereditary section of Example 6.56

$\mathcal{T}' \subseteq \mathcal{T}$. Next lemma expresses this. We will write

$$\begin{aligned} T_A^\bullet &= \{T \in \mathcal{T} \mid d^\bullet(T, A) = d^\bullet(\mathcal{T}, A)\} \\ T_\bullet^A &= \{T \in \mathcal{T} \mid d^\bullet(A, T) = d^\bullet(A, \mathcal{T})\} \end{aligned}$$

Lemma 6.58. *The sets $T_A^\bullet$ and $T_\bullet^A$ are finite.*

Proof. For every $T_i \in T_A^\bullet$ and $T_j \in T_\bullet^A$ we have

$$d(\mathcal{T}, A) = d^\bullet(T_i, A) + d^\bullet(A, T_j) \geq d^\bullet(T_i, T_j).$$

Since by construction $d^\bullet(X_i, X_j) \geq \max\{i, j\}$ if $i \neq j$, we see that the sets $T_A^\bullet$ and $T_\bullet^A$ have to be finite. □

Following Proposition 6.49, to prove $\mathcal{Q}_\mathcal{H}$ is a dualizing k-variety, it suffices to show the sets $S_{\mathcal{Q}_\mathcal{H}}^\bullet(A, 0)$ and $S_\bullet^{\mathcal{Q}_\mathcal{H}}(A, 0)$ are bounded for all $A \in \operatorname{ind} \mathcal{Q}_\mathcal{H}$. To this end, we will take a closer look at what maps in $\mathcal{Q}_\mathcal{H}$ learn us about the round trip distance and the right light cone distance.

Lemma 6.59. *Let $A, B \in \operatorname{ind} \mathcal{Q}_\mathcal{H}$ with $\operatorname{Hom}(A, B) \neq 0$, then*

1. $d(\mathcal{T}, A) - 1 \leq d(\mathcal{T}, B) \leq d(\mathcal{T}, A) + 1$,
2. (a) $d^\bullet(\mathcal{T}, A) - 1 \leq d^\bullet(\mathcal{T}, B) \leq d^\bullet(\mathcal{T}, A) + 1$
   (b) $d^\bullet(A, \mathcal{T}) - 1 \leq d^\bullet(B, \mathcal{T}) \leq d^\bullet(A, \mathcal{T}) + 1$

Proof. One may easily derive (2) from (1), while (1) follows from the inequalities

$$\begin{aligned} 0 &= d^\bullet(A, B) \\ &= d^\bullet(\mathcal{T}, B) - d^\bullet(\mathcal{T}, A) \\ &= \left\lfloor \frac{d(\mathcal{T}, B)}{2} \right\rfloor - \left\lfloor \frac{d(\mathcal{T}, A)}{2} \right\rfloor \end{aligned}$$

and

$$\begin{aligned} 0 &= d^\bullet(A, B) \\ &= d^\bullet(A, \mathcal{T}) - d^\bullet(B, \mathcal{T}) \\ &= \left\lceil \frac{d(\mathcal{T}, B)}{2} \right\rceil - \left\lceil \frac{d(\mathcal{T}, A)}{2} \right\rceil. \end{aligned}$$

□

Before coming to the proof of the main theorem, it will be convenient to prove next lemma. In here, we have following notations

$$\begin{aligned} T_S^\bullet &= \bigcup_{B \in S} T_B^\bullet \\ T_\bullet^S &= \bigcup_{B \in S} T_\bullet^B \end{aligned}$$

where S is a subset of $\operatorname{ind} \mathbb{Z}\mathcal{Q}$.

Lemma 6.60. *Let $S = S^{\bullet}_{\mathcal{Q}_{\mathcal{H}}}(A, 0)$ or $S = S^{\mathcal{Q}_{\mathcal{H}}}_{\bullet}(A, 0)$ where $A \in \operatorname{ind} \mathcal{Q}$. Then the sets $T^{\bullet}_S$ and $T^S_{\bullet}$ are finite.*

Proof. We will only prove the case where $S = S^{\bullet}_{\mathcal{Q}_{\mathcal{H}}}(A, 0)$; the other case is dual.

Seeking a contradiction, assume $T^S_{\bullet}$ is infinite. It follows from Lemma 6.58 that $T^{\bullet}_A$ is finite; let $T_b \in T^S_{\bullet} \setminus T^{\bullet}_A$ and choose a $B \in S$ be such that $T_b \in T^B_{\bullet}$. Using $d^{\bullet}(A, B) = 0$ and triangle inequality one finds for all $T_a \in T^{\bullet}_A$

$$\begin{aligned} d^{\bullet}(\mathcal{T}, A) + d^{\bullet}(B, \mathcal{T}) &= d^{\bullet}(T_a, A) + d^{\bullet}(B, T_b) \\ &= d^{\bullet}(T_a, A) + d^{\bullet}(A, B) + d^{\bullet}(B, T_b) \\ &\geq d^{\bullet}(T_a, T_b) \geq \max\{a, b\}. \end{aligned}$$

Lemma 6.59 yields that $d^{\bullet}(B, \mathcal{T}) \leq d^{\bullet}(A, \mathcal{T}) + 1$, such that $d^{\bullet}(\mathcal{T}, A) + d^{\bullet}(A, \mathcal{T}) - 1 \geq \max\{a, b\}$. This implies $T^S_{\bullet}$ is finite.

Again seeking a contradiction, we will now assume $T^{\bullet}_S$ is infinite. Let $T_i \in T^{\bullet}_S \setminus T^S_{\bullet}$, and let $B \in S$ such that $T_i \in T^{\bullet}_B$. For all $T_b \in T^B_{\bullet}$ we have, using Lemma 6.59,

$$\begin{aligned} d(\mathcal{T}, A) + 1 &\geq d(\mathcal{T}, B) \\ &= d^{\bullet}(\mathcal{T}, B) + d^{\bullet}(B, \mathcal{T}) \\ &= d^{\bullet}(T_i, B) + d^{\bullet}(B, T_b) \\ &\geq d^{\bullet}(T_i, T_b) \geq \max\{i, b\} \end{aligned}$$

Since we have assumed $T^{\bullet}_S$ is infinite and have shown $T^S_{\bullet}$ is finite, we may choose i arbitrarily large. A contradiction. □

Theorem 6.61. *Let $\mathcal{A}$ be a connected abelian hereditary category satisfying Serre duality with category of projectives $\mathcal{Q}_{\mathcal{A}}$. If $\mathbb{Z}\mathcal{Q}_{\mathcal{A}}$ is connected and satisfies (*), then $\mathcal{A}$ is derived equivalent to an abelian hereditary category $\mathcal{H}$ satisfying Serre duality and with category of projectives $\mathcal{Q}_{\mathcal{H}}$ such that $\mathcal{Q}_{\mathcal{H}}$ is a dualizing k-variety and the natural embedding $i : D^b \operatorname{mod} \mathcal{Q}_{\mathcal{H}} \to D^b \mathcal{A}$ commutes with Serre duality.*

Proof. Let $\mathcal{T}$ and $\mathcal{Q}_{\mathcal{H}}$ be as in Construction 6.55. By Proposition 6.49, it suffices to show both $S^{\bullet}_{\mathcal{Q}_{\mathcal{H}}}(A, 0)$ and $S^{\mathcal{Q}_{\mathcal{H}}}_{\bullet}(A, 0)$ are bounded, for all $A \in \mathcal{Q}_{\mathcal{H}}$.

We start by proving the former. To ease notation, we will write $S = S^{\bullet}_{\mathcal{Q}_{\mathcal{H}}}(A, 0)$; Lemma 6.60 shows $T^{\bullet}_S$ and $T^S_{\bullet}$ are finite.

Due to Lemma 6.59 we know

$$d^{\bullet}(\mathcal{T}, A) - 1 \leq d^{\bullet}(\mathcal{T}, B) \leq d^{\bullet}(\mathcal{T}, A) + 1$$

and

$$d^{\bullet}(A, \mathcal{T}) - 1 \leq d^{\bullet}(B, \mathcal{T}) \leq d^{\bullet}(A, \mathcal{T}) + 1$$

for all $B \in S$. In particular, there are only finitely many possibilities for $d^{\bullet}(\mathcal{T}, B)$ and $d^{\bullet}(B, \mathcal{T})$.

Let $n, m \in \mathbb{N}$ such that $d^\bullet(\mathcal{T}, A) - 1 \le n \le d^\bullet(\mathcal{T}, A) + 1$ and $d^\bullet(A, \mathcal{T}) - 1 \le m \le d^\bullet(A, \mathcal{T}) + 1$. It suffices to show the set $S \cap S^\bullet(\mathcal{T}, n) \cap S_\bullet(\mathcal{T}, m)$ is bounded. To prove this, we need only to verify that

$$S_{i,j} = S \cap S^\bullet(T_i, n) \cap S_\bullet(T_j, m) = \{X \in S \mid d^\bullet(T_i, X) = n \text{ and } d^\bullet(X, T_j) = m\}$$

is bounded, for all $T_i \in T_S^\bullet$ and $T_j \in T_\bullet^S$. We may assume $S_{i,j} \neq \emptyset$.

Consider the set

$$\Sigma_{i,j} = S^\bullet(T_i, n) \cap S_\bullet(T_j, m) = \{X \in \operatorname{ind} \mathbb{Z}\mathcal{Q} \mid d^\bullet(T_i, X) = n \text{ and } d^\bullet(X, T_j) = m\}.$$

It follows form Lemma 6.50(1) that $\Sigma_{i,j}$ is bounded, say by a non-zero object C. In general, however, C does not have to lie in $\mathcal{Q}_{\mathcal{H}}$.

Let C' be a direct summand of C. We will show $d^\bullet(C', T_j) = d^\bullet(C', \mathcal{T}) = m$. Since we are constructing a bound for $S_{i,j}$, we may assume there is a $B \in S_{i,j}$ such that $\operatorname{Hom}(B, C') \neq 0$.

By triangle inequality we find $d^\bullet(B, C') + d^\bullet(C', \mathcal{T}) \ge d^\bullet(B, \mathcal{T})$. Since $d^\bullet(B, C') \le 0$ and $d^\bullet(B, \mathcal{T}) = m$, we also have $d^\bullet(C', \mathcal{T}) \ge m$. Yet, $d^\bullet(C', \mathcal{T}) \le d^\bullet(C', T_j) = m$, thus indeed $d^\bullet(C', \mathcal{T}) = m$.

However, there may be an object $T_k \in T$, such that $d^\bullet(\mathcal{T}, C') = d^\bullet(T_k, C') = n_k < n$. In this case, we may use Lemma 6.50(2) with $X = \tau^n T_i$, and $Y = \tau^{n_k} T_k$. Iteration gives a new object C_1 such that for every indecomposable direct summand C_1' of C_1 we have $d^\bullet(T_i, C_1') = d^\bullet(\mathcal{T}, C_1') = n$ and $d^\bullet(C_1', \mathcal{T}) = m$, thus $C_1 \in \mathcal{Q}$.

Moreover, $d^\bullet(A, C_1') \le 0$ since we may assume there is a path from A to C_1'. Proposition 6.19 yields $d^\bullet(A, C_1') \ge 0$, thus $C_1' \in S^\bullet_{\mathcal{Q}_{\mathcal{H}}}(A, 0)$ and hence also $C_1' \in S_{i,j}$. We conclude $S_{i,j}$ is bounded. $\square$

6.5 Categories generated by $\mathbb{Z}\mathcal{Q}$

In this section, we will consider categories $\mathcal{A}$ such that $D^b\mathcal{A}$ is generated by the connecting subcategory $\mathbb{Z}\mathcal{Q}$. Examples of such categories $\mathcal{A}$ are those who are generated by preprojectives.

Theorem 6.62. *Let $\mathcal{A}$ be an abelian hereditary Ext-finite category with Serre duality and denote by $\mathcal{Q}$ the category of projectives. If $D^b\mathcal{A}$ is generated by $\mathbb{Z}\mathcal{Q}$, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} \mathcal{Q}_{\mathcal{H}}$ *for a certain dualizing k-variety $\mathcal{Q}_{\mathcal{H}}$.*

Proof. If $\mathcal{Q}$ satisfies property (*), then the statement follows from Theorem 6.61. So, assume $\mathcal{Q}$ does not satisfy (*). We extend $\mathcal{Q}$ to a maximal hereditary section $\mathcal{Q}'$ as in Lemma 6.27, and claim $\mathcal{Q}'$ satisfies (*).

By Proposition 6.53, it suffices to prove there are no right-open unbroken threads $\mathcal{R} = \Xi(X, \rightarrow)$ or left-open unbroken threads $\mathcal{R} = \Xi(\leftarrow, X)$. We will only consider the first case.

Let X_1 be the unique direct successor of X and let $C = \operatorname{cone}(\tau X_1 \to X_0)$, as in Proposition 6.43.

Seeking a contradiction, assume $d^\bullet(Z,C) < 0$, for a certain $Z \in \mathcal{Q}'$, thus in particular there is a path $Z \to Z_1 \to \cdots \to Z_n \to \tau^{-1}C$.

According to Proposition 6.43, we assume X has been chosen such that $d^\bullet(X,Z) = \infty$. From triangle inequality, it follows $d^\bullet(X_i, Z) = \infty$, for all $X_i \in \mathcal{R}$.

Let $\Sigma \subset \mathbb{Z}\mathcal{Q}$ be a finite set, containing $X, \tau X_1$ and Z and generating $\tau^{-1}C$ and every object Z_i occurring in the path from Z to $\tau^{-1}C$. It follows from Corollary 6.20 that the full subcategory $\mathcal{Q}' \subseteq \mathcal{Q}$ of all objects Y such that $d(\Sigma, Y) < \infty$ is a hereditary section in $D^b\mathcal{A}$.

We will now consider a set $\mathcal{T} \subset \mathbb{Z}\mathcal{Q}'$ as in Construction 6.55 such that $\mathcal{T}$ contains $X, \tau X_1$ and Z. Using Theorem 6.61, we may use $\mathcal{T}$ to find a hereditary section $\mathcal{Q}'_\mathcal{T}$ which is a dualizing k-variety.

Consider the full additive subcategory of $\mathcal{Q}_f \subseteq \mathcal{Q}'_\mathcal{T}$ spanned of only those indecomposables needed to generate $X, X_1, \tau^{-1}C, Z$ and every object Z_i occurring in the path from Z to $\tau^{-1}C$. The triangulated subcategory generated by $\mathcal{Q}_f$ is equivalent to $D^b \operatorname{mod} \mathcal{Q}_f$. We claim $C \in D^b \operatorname{mod} \mathcal{Q}_f$ lies in the connecting subcategory $\mathbb{Z}\mathcal{Q}_f$ of $D^b \operatorname{mod} \mathcal{Q}_f$.

Using Lemma 6.42, one easily checks that the C and R_n, where n has been chosen maximal, are in the same τ-orbit. The situation is as sketched in Figure 6.13.

Since Z corresponds to a preprojective object in $\operatorname{mod} \mathcal{Q}_f$ and there is a path from Z to τC, it must lie somewhere in the marked area, and hence $d^\bullet(R_i, Z) \leq 0$ for a certain R_i. A contradiction, since we have chosen X such that $d^\bullet(X_i, Z) = \infty$, for all $X_i \in \mathcal{R}$.

□

As a corollary of this theorem, we have following result which provides a classification suggested in [41].

Corollary 6.63. *Let $\mathcal{A}$ be an abelian hereditary Ext-finite category with Serre duality. If $\mathcal{A}$ is generated by preprojectives, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} \mathcal{Q}$ *for a semi-hereditary dualizing k-variety $\mathcal{Q}$.*

Corollary 6.64. *Let $\mathcal{A}$ be an abelian hereditary Ext-finite category with Serre duality and denote the category of projectives with $\mathcal{Q}_\mathcal{A}$. There is a full subcategory $\mathcal{Q}$ which is a dualizing k-variety and such that* $\operatorname{ind} \mathcal{Q}$ *is a partial tilting set in $D^b\mathcal{A}$. The embedding $D^b \operatorname{mod} \mathcal{Q} \to D^b\mathcal{A}$ commutes with Serre duality and $\mathbb{Z}\mathcal{Q}_A$ lies in the essential image of i.*

6.6 Categories with a directing object

Let $\mathcal{A}$ be a connected abelian hereditary Ext-finite category with Serre duality and a directing object X. In $D^b\mathcal{A}$, we may consider the light cone tilt centered on X and obtain an abelian category $\mathcal{A}'$ with nonzero projectives.

Let $\mathcal{Q}$ and $i : D^b \operatorname{mod} \mathcal{Q} \to D^b\mathcal{A}'$ be as given in Corollary 6.64. Since $\mathcal{Q}$ is a spanning class for $D^b \operatorname{mod} \mathcal{Q}$, Theorem 1.33 yields that i is an equivalence if and only

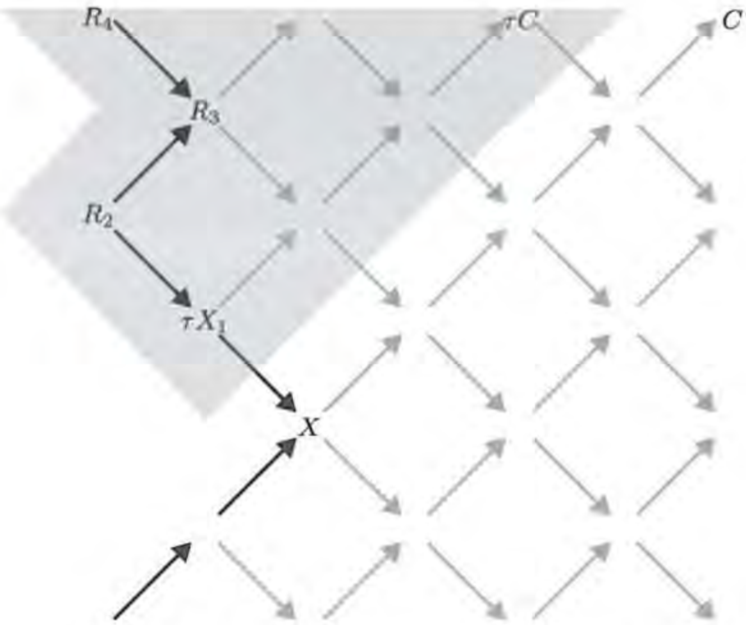


Figure 6.13: Part of the preprojective component of $\operatorname{mod} \mathcal{Q}_f$

if i admits a left (or right) adjoint. We conjecture such an adjoint always exists when $\mathcal{A}'$ is a light cone tilt (as above). For example, it is known to be true for noetherian and dually artinian categories ([40]), and directed categories (see Chapter 3).

Conjecture 6.65. *Let $\mathcal{A}$ be a connected abelian hereditary Ext-finite category with Serre duality. If $\mathcal{A}$ has a directing object, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} \mathcal{Q}$ *for a semi-hereditary dualizing k-variety $\mathcal{Q}$.*

We will now consider two special cases where we can prove this conjecture.

6.6.1 Saturated categories

We will day a homological functor $H : D^b\mathcal{A} \to \operatorname{mod} k$ is of *finite type* if, for every $A \in D^b\mathcal{A}$ only a finite number of $H(A[n])$ is nonzero.

An Ext-finite abelian category $\mathcal{A}$ is said to be *saturated* if every homological H functor of finite type is representable, thus $H \cong \operatorname{Hom}_{D^b\mathcal{A}}(A, -)$.

It has been shown in [12] that $\operatorname{mod} \Lambda$, where Λ is a finite dimensional k-algebra, is saturated, as is the category $\operatorname{coh} X$ of coherent sheaves over a non-singular projective curve.

In [10], it was shown that $\operatorname{coh} \mathcal{O}$, where $\mathcal{O}$ is a sheaf of hereditary $\mathcal{O}_X$-orders over a non-singular projective curve X.

If $\mathcal{A}$ is hereditary and noetherian, then these are all connected and saturated categories ([40]). We shall show in Theorem 6.67 that if, instead of noetherianness, we require $\mathcal{A}$ to have a directing object, then $\mathcal{A}$ is derived equivalent to $\operatorname{mod}\Lambda$, where Λ is a finite dimensional hereditary k-algebra. First, we show that representations of saturated dualizing k-varieties do not give new examples of saturated categories.

Lemma 6.66. *Let $\mathfrak{a}$ be a semi-hereditary finite k-variety. The category* $\operatorname{mod}\mathfrak{a}$ *is saturated if and only if* $\operatorname{ind}\mathfrak{a}$ *is finite.*

Proof. We need only to consider to show that, if $\operatorname{ind}\mathfrak{a}$ is infinite, then $\operatorname{mod}\mathfrak{a}$ is not saturated. By Theorem 2.9, we may assume $\mathfrak{a}$ is a dualizing k-variety.

If every standard projective in $\operatorname{mod}\mathfrak{a}$ has finite length, then so has every object in $\operatorname{mod}\mathfrak{a}$ and [40, Lemma V.1.1] yields the required result.

Thus assume there is a projective object with infinite length. Using Proposition 2.12, we see there must be an infinite thread $[A, B]$ for $A, B \in \operatorname{ind}\mathfrak{a}$. Let

$$A \to A_1 \to A_2 \to \cdots$$

be the first part of this thread. We define a functor

$$G_j = \operatorname{Hom}_{D^b \operatorname{mod}\mathfrak{a}}(-, \mathfrak{a}(-, A_j)) : D^b \operatorname{mod}\mathfrak{a} \to \operatorname{mod} k$$

and consider the colimit

$$G = \varinjlim \operatorname{Hom}_{D^b \operatorname{mod}\mathfrak{a}}(-, \mathfrak{a}(-, A_j)) : D^b \operatorname{mod}\mathfrak{a} \to \operatorname{Mod} k$$

For any $X \in \operatorname{Ob} D^b \operatorname{mod}\mathfrak{a}$, it follows easily from the triangle

$$\oplus_i \mathfrak{a}(-, C_i)[i] \to \oplus_i \mathfrak{a}(-, D_i)[i] \to X \to \oplus_i \mathfrak{a}(-, C_i)[i+1]$$

that the map $G_j(X) \to G_k(X)$ is an isomorphism for $k \geq j$ and $j \gg 0$.

From this follows that G takes values in $\operatorname{mod} k$, is homological, and of finite type. However, it is straightforward to verify that G is not representable. We conclude that $\operatorname{mod}\mathfrak{a}$ is not saturated. □

Theorem 6.67. *Let $\mathcal{A}$ be a connected saturated abelian Ext-finite hereditary category with a directing object. Then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} A$ *for a finite dimensional hereditary algebra A.*

Proof. Since $\mathcal{A}$ is saturated, the embedding $i : D^b \operatorname{mod} \mathcal{Q}_{\mathcal{H}} \to D^b\mathcal{A}$ has a left adjoint. Theorem 1.33 shows that i is an equivalence. The following Lemma 6.66 yields the required result. □

6.6.2 A directing object in a $\mathbb{Z}Q$-component with Q finite

We will now discuss second case where the embedding $i : D^b\mathcal{Q}_{\mathcal{H}} \to D^b\mathcal{A}$ has an adjoint.

Proposition 6.68. *Let $\mathcal{A}$ be a connected abelian Ext-finite hereditary category with Serre duality. If $D^b\mathcal{A}$ has a directing object X lying in a component of the form $\mathbb{Z}Q$ where Q is a finite quiver, then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod} A$ *for a finite dimensional hereditary algebra A.*

Proof. Consider the light cone $S(X,0) \in D^b\mathcal{A}$ and denote the corresponding hereditary section with $\mathcal{Q}$. It is clear that $\operatorname{ind}\mathcal{Q}$ is finite, thus there is an embedding $i : D^b \operatorname{mod}\mathcal{Q} \to D^b\mathcal{A}$, commuting with the Serre functor, which is easily seen to have a left and a right adjoint. Theorem 1.33 yields the required result. □

Chapter 7

The reduced Grothendieck group

In this final chapter, we will discuss abelian hereditary categories with Serre duality, with some additional assumptions on the reduced Grothendieck group $K_0^{red}\mathcal{A}$ of $\mathcal{A}$. In particular, we will classify those categories where the rank of $K_0^{red}\mathcal{A}$ is zero or one.

The interesting cases, however, appear to be those where the rank is at least two, as the category of coherent sheaves on a smooth projective curve falls into this class. The main obstacle to the classification in the rank two case appears to be finding an object with is either exceptional, or 1-spherical. Such an object should conjecturally always exist. We may prove the existence if we assume there is an object with few self-extensions (Lemma 7.4). However, this result does not seem strong enough to handle the general case.

7.1 Reduced Grothendieck group and Euler form

Definition 7.1. Let $\mathcal{A}$ be an abelian (or more generally, an exact) category. Let $F(\mathcal{A})$ be the free group generated by the isomorphism classes of objects of $\mathcal{A}$ and let $R(\mathcal{A})$ be the subgroup of $F(\mathcal{A})$ generated by expressions $[A] - [B] + [C]$ for every exact sequence $0 \to A \to B \to C \to 0$. The *Grothendieck group* $K_0(\mathcal{A})$ is the quotient group $F(\mathcal{A})/R(\mathcal{A})$.

We may also define the Grothendieck group of a triangulated category by replacing the short exact sequence $0 \to A \to B \to C \to 0$ by a triangle $A \to B \to C \to A[1]$ in the definition above.

It is shown in [22] that the natural embedding $\mathcal{A} \to D^b\mathcal{A}$ induces an isomorphism on the corresponding Grothendieck groups.

Definition 7.2. Let $\mathcal{A}$ be an Ext-finite abelian category with finite global dimension.

The *Euler form* $\chi : \mathcal{A} \times \mathcal{A} \to \mathbb{Z}$ is given by

$$\chi(A, B) = \sum_{i \in \mathbb{N}} (-1)^i \operatorname{Ext}^i(A, B).$$

It follows directly from the definition that $\chi(X, A) - \chi(X, B) + \chi(X, C) = 0$ and $\chi(A, X) - \chi(B, X) + \chi(C, X) = 0$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence. Hence the Euler form induces a bilinear form on the Grothendieck group $K_0(\mathcal{A})$, which we will also denote with χ.

Now, assume $\mathcal{A}$ is hereditary and satisfies Serre duality. In this case, $\chi(A, B) = \chi(A, FB)$ for all $A, B \in \operatorname{Ob} D^b\mathcal{A}$.

We will define the *radical of the Euler form* as

$$\operatorname{rad} \chi = \{X \in K_0(\mathcal{A}) \mid \chi(X, -) = 0\}.$$

Since $\mathcal{A}$ has Serre duality, $\chi(X, -) = 0$ if and only if $\chi(-, X) = 0$. The *reduced Grothendieck group* $K_0^{red}(\mathcal{A})$ of $\mathcal{A}$ is defined as $K_0(\mathcal{A})/\operatorname{rad}\chi$. It is a free group, and we denote its rank by $\operatorname{Num}\mathcal{A}$.

Associated with a basis of $K_0^{red}\mathcal{A}$, the corresponds a matrix with the bilinear form $\chi(-,-)$ and with the $\mathbb{Z}$-linear transformation $[A] \mapsto [\tau A]$, called the *cartan matrix* and the *coxeter matrix*, respectively. There is the following connection.

Proposition 7.3. *Let $\mathcal{A}$ be a hereditary Ext-finite category with Serre duality. If $\operatorname{Num}\mathcal{A}$ is finite, we have $C = A^{-1}A^t$ where A and C are the Cartan matrix and Coxeter matrix, respectively.*

Similarly, we define an Euler form on $D^b\mathcal{A}$, $K_0(D^b\mathcal{A})$, and $K_0^{red}(D^b\mathcal{A})$, all denoted by χ. This should give no confusion.

Following lemma will be used later. We suspect that it also holds without the given object X.

Lemma 7.4. *Let $X \in \operatorname{Ob}\mathcal{A}$ with* $\dim \operatorname{Ext}(X, X) = 1$, *then there is an object $Y \in \operatorname{Ob}\mathcal{A}$ such that Y is either exceptional or 1-spherical.*

Proof. Let E be a endo-simple subobject and quotient object of X. Since $\mathcal{A}$ is hereditary, we have $d = \dim \operatorname{Ext}(E, E) \leq \dim \operatorname{Ext}(E, X) \leq \dim \operatorname{Ext}(X, X)$.

If $d = 0$, then E is exceptional and we are done. Thus let $d = 1$. If E is not 1-spherical, thus if $E \not\cong \tau E$, then consider a nonzero morphism $f : E \to \tau E$. We know either $\ker f$ or $\operatorname{coker} f$ is not zero.

If $\ker f$ is not zero, then by applying $\operatorname{Hom}(E, -)$ to the short exact sequence

$$0 \to \ker f \to E \to \operatorname{im} f \to 0$$

we find, using $\dim \operatorname{Ext}(E, \operatorname{im} f) = \dim \operatorname{Hom}(\operatorname{im} f, \tau E) = 1$,

$$\begin{aligned} \dim \operatorname{Ext}(E, \ker f) &= \dim \operatorname{Hom}(E, E) - \dim \operatorname{Hom}(E, \operatorname{im} f) \\ &\quad - \dim \operatorname{Ext}(E, E) + \dim \operatorname{Ext}(E, \operatorname{im} f) \\ &= 1 - 1 - 1 + 1 = 0. \end{aligned}$$

Since $\dim \operatorname{Ext}(\ker f, \ker f) \leq \dim \operatorname{Ext}(E, \ker f)$, we find that every indecomposable direct summand of $\ker f$ is exceptional.

A dual reasoning shows that, if $\operatorname{coker} f \not\cong 0$, every direct summand of $\operatorname{coker} f$ is exceptional. This completes the proof. $\square$

Before proceeding, it will be convenient to have the next lemma.

Lemma 7.5. *Assume $\mathcal{A}$ is connected and E is a 1-spherical object in $\mathcal{A}$. If $[E]=0$ in $K_0^{red}(\mathcal{A})$, then $\mathcal{A} \cong \operatorname{Mod}^{fd} k[[t]]$.*

Proof. The object E lies in a homogeneous tube. Let $X \in \operatorname{ind} \mathcal{A}$ such that $\operatorname{Hom}(X,E) \neq 0$, or $\operatorname{Hom}(E,X) \neq 0$. Assume the former, the latter is dual.

Since $[E]=0$, we know $\chi(X,E)$, such that $\dim \operatorname{Ext}(X,E) = \dim \operatorname{Hom}(E,X) \neq 0$. By Theorem 4.6, this shows X lies in the same component as E. We see that $\mathcal{A} \cong \operatorname{Mod}^{\mathrm{fd}} k[[t]]$. $\square$

7.2 The case $\operatorname{Num} \mathcal{A} = 0$

We will first discuss the case where $\mathcal{A}$ is an Ext-finite abelian hereditary category satisfying Serre duality with $\operatorname{Num} \mathcal{A} = 0$.

Lemma 7.6. *Let $E \in \operatorname{Ob} \mathcal{A}$ be an endo-simple object, then E is simple.*

Proof. Assume E is not simple, then there is a short exact sequence

$$0 \to A \to E \to B \to 0$$

such that $\operatorname{Ext}(B,A) \neq 0$. Since $\operatorname{Num} \mathcal{A} = 0$, we know $\chi(B,A) = 0$ holds, thus $\operatorname{Hom}(B,A) \neq 0$.

The composition $E \to B \to A \to E$ is nonzero and is not an isomorphism, a contradiction, since E is endo-simple. $\square$

Proposition 7.7. *Let $\mathcal{A}$ be a connected Ext-finite abelian hereditary category with $\operatorname{Num} \mathcal{A} = 0$, then $\mathcal{A}$ is equivalent to $\operatorname{Mod}^{fd} k[[t]]$.*

Proof. Let E be an endo-simple object in $\mathcal{A}$. It follows from Lemma 7.6 that E is simple. Since $\dim \operatorname{Hom}(E, \tau E) = 1$, Schur's Lemma yields $E \cong \tau E$, hence E is 1-spherical. Lemma 7.5 yields the required result. $\square$

7.3 The case $\operatorname{Num} \mathcal{A} = 1$

Lemma 7.8. *For any $X, Y \in \operatorname{Ob} \mathcal{A}$, we have $\chi(X,Y) = \chi(Y,X)$.*

Proof. Let A be the Cartan matrix of $K_0\mathcal{A}$ with respect to a chosen basis, then one easily calculates the Coxeter matrix as $C = (-1)$. $\square$

Lemma 7.9. *There is an endo-simple object $X \in \operatorname{Ob} D^b\mathcal{A}$ with* $\dim \operatorname{Ext}(X,X)$ *equal to 0 or 1.*

Proof. Let X be an endo-simple object such that $\chi(X,X)$ is maximal. Seeking a contradiction, assume $\chi(X,X) < 0$. In this case $\dim \operatorname{Hom}(X,\tau X) > 1$ and hence $X \not\cong \tau X$.

Using Lemma 7.8 we have

$$\chi(\tau X, X) = \chi(X, \tau X) = -\chi(X,X) > 0,$$

thus $\dim \operatorname{Hom}(X,\tau X) \neq 0$ and $\dim \operatorname{Hom}(\tau X, X) \neq 0$. Since X and τX are endo-simple, no map from $X \to \tau X$ may be an epimorphism or a monomorphism.

Let $f : X \to \tau X$ be a nonzero morphism with endo-simple image I, kernel K, and cokernel C. From the short exact sequence $0 \to K \to X \to I \to 0$, we obtain $\chi(I,K) + \chi(I,I) = \chi(I,X)$. Since X is endo-simple we find $\dim \operatorname{Hom}(I,K) = 0$. Together with $\dim \operatorname{Ext}(I,K) \neq 0$ this shows $\chi(I,K) < 0$, hence $\chi(I,I) > \chi(I,X)$.

Using the same short exact sequence, we find $\chi(X,K) + \chi(X,I) = \chi(X,X)$. Since X is endo-simple, $\dim \operatorname{Hom}(X,K) = 0$, hence $\chi(X,K) \leq 0$, and thus $\chi(X,I) \geq \chi(X,X)$.

Combining these two inequalities with Lemma 7.8, we find $\chi(I,I) > \chi(X,X)$, a contradiction. Hence $\chi(X,X) \geq 0$. Because $\dim \operatorname{Hom}(X,X) = 1$, this implies $\dim \operatorname{Ext}(X,X)$ is either 0 or 1. □

Proposition 7.10. *Let $\mathcal{A}$ be a connected Ext-finite abelian hereditary category with Serre duality and* $\operatorname{Num} \mathcal{A} = 1$, *then $\mathcal{A}$ is either equivalent to*

1. *the category* $\operatorname{mod} k$ *of finite dimensional k-vector spaces, or*
2. *the category* $\operatorname{nilp} k\tilde{A}_1$ *where $\tilde{A}_1$ has cyclic orientation.*

Proof. First, we will show there is an exceptional object in $\mathcal{A}$. Let X be an endo-simple object of $\mathcal{A}$. Using Lemma 7.9, we see that X is either exceptional, or that $\dim \operatorname{Ext}(X,X) = 1$. In the latter case, Lemma 7.4 yields an exceptional or 1-spherical object E.

If E is 1-spherical, then E is a peripheral object of a homogeneous tube. Also, using $\chi(E,E) = 0$ and $\operatorname{Num} \mathcal{A} = 1$, we find $[E] = 0$ in $K_0^{\text{red}}\mathcal{A}$. It follows from Lemma 7.5 that $\mathcal{A} \cong \operatorname{Mod}^{\text{fd}} k[[t]]$, a contradiction since $\operatorname{Num} \mathcal{A} = 1$.

We may thus assume $D^b\mathcal{A}$ has an exceptional object X. We claim this X is simple. Seeking a contradiction, let $0 \to A \to X \to B \to 0$ be an exact sequence with $A \not\cong 0 \not\cong B$.

Since $\chi(X,X) = 1$, we know that $\chi(X,A) \geq 1$ or $\chi(X,B) \geq 1$. In the first case, $\operatorname{Hom}(X,A) \neq 0$, a contradiction since X is endo-simple. In the second case, Lemma 7.8 implies $\chi(B,X) > 0$, such that $\operatorname{Hom}(B,X) \neq 0$. Again, this is a contradiction. We conclude X is simple.

If $\mathcal{A}$ has a projective object, this is the unique projective object up to isomorphism, and hence $\mathcal{A}$ is equivalent to $\operatorname{mod} k$.

Thus, assume $\mathcal{A}$ has no nonzero projectives; in particular, τ is an equivalence. Using Lemma 7.8 we find

$$\begin{aligned} 1 &= \chi(X,X) \\ &= -\chi(X,\tau X) \\ &= -\chi(\tau X,X) = \chi(X,\tau^2 X), \end{aligned}$$

such that $\operatorname{Hom}(X,\tau^2 X) \neq 0$, and thus $X \cong \tau^2 X$ since X and $\tau^2 X$ are simple. It follows from Theorem 4.5 that the Auslander-Reiten component containing X is a standard tube of rank 2, thus X is contained in a subcategory of $\mathcal{A}$ equivalent to $\operatorname{nilp} kQ$ where Q is the quiver $\tilde{A}_1$ with cyclic orientation.

It now follows easily from connectedness that $\mathcal{A}$ is equivalent to $\operatorname{nilp} kQ$. □

7.4 The case $\operatorname{Num}\mathcal{A} = 2$

In this section, our main result will be the following theorem. We will say a quiver Q is a *generalized Kronecker quiver* if Q_0 has two elements, say a and b, such that there are $n > 0$ arrows from a to b and no arrows from b to a.

Theorem 7.11. *Let $\mathcal{A}$ be a connected abelian hereditary Ext-finite category with Serre duality. If $\operatorname{Num}\mathcal{A} = 2$ and $\mathcal{A}$ has an object X with $\dim \operatorname{Ext}(X,X) < 1$, then $\mathcal{A}$ is derived equivalent to one of the following:*

1. $\operatorname{nilp} k\tilde{A}_2$ *where $\tilde{A}_2$ has cyclic orientation,*

2. $\operatorname{mod} kQ$, *where Q is a generalized Kronecker quiver,*

3. $\operatorname{coh} X$, *for a smooth projective curve X.*

Remark 7.12. The condition that such an object X exists seems to be fulfilled in all known examples. We conjecture it holds for every abelian hereditary category with Serre duality.

First note that it follows from Lemma 7.4 that $\mathcal{A}$ has an exceptional or a 1-spherical object. The proof of Theorem 7.11 will be split in two parts, accordingly.

In the first part we will assume $\mathcal{A}$ has an exceptional object E, and use this object to construct a (partial) tilting set. Using that exceptional objects are uniquely determined by their class in the reduced Grothendieck group (Proposition 7.13), we will complete the classification.

In the second part of the proof, we will assume the existence of a 1-spherical object which we will use to construct a t-structure resembling the ones used in §5.2.3 and §5.3.4. We will show the heart of this t-structure is noetherian and use the classification of these categories in [40] to complete the proof.

7.4.1 $\mathcal{A}$ has an exceptional object

Let $\mathcal{A}$ be a connected abelian hereditary Ext-finite category with Serre duality and $\operatorname{Num}\mathcal{A} = 2$. We will furthermore assume $\mathcal{A}$ has an exceptional object E, thus E is indecomposable and $\dim \operatorname{Ext}(E,E) = 0$. It follows from Proposition 1.16 that $\dim \operatorname{Hom}(E,E) = 1$.

For exceptional objects, we have the following result, which is a slight modification of [31, Proposition 5.3].

Proposition 7.13. *Each exceptional object E of $\mathcal{A}$ is completely determined by its class $[E]$ in the reduced Grothendieck group $K_0^{red}\mathcal{A}$.*

With the exceptional object E, we will associate the perpendicular category $E^\perp$ in the usual way, namely $E^\perp$ is the full subcategory of $D^b\mathcal{A}$ spanned by all objects Y with $\operatorname{Hom}(E[i],Y) = 0$, for all $i \in \mathbb{Z}$. It is straightforward to check $E^\perp$ is a triangulated category.

Moreover, the standard t-structure of $D^b\mathcal{A}$ induces a t-structure on $E^\perp$ yielding a hereditary heart $\mathcal{B}$. This heart is the orthogonal category on E, taken in $\mathcal{A}$. In particular, $E^\perp \subseteq D^b\mathcal{A}$ is a derived category.

The natural embedding $i : E^\perp \to D^b\mathcal{A}$ is full and exact. It is straightforward to check that the twist functor T_E induces a left adjoint $L : D^b\mathcal{A} \to E^\perp$ of i and the twist functor T^*_{FE} induces a right adjoint $R : D^b\mathcal{A} \to E^\perp$.

Moreover, the inclusion i induces an isomorphism $K_0^{red}(D^b\mathcal{A}) \cong E^\perp \oplus \mathbb{Z}[E]$ such that $\operatorname{Num} E^\perp = \operatorname{Num}\mathcal{A} - 1$.

Following proposition implies $E^\perp$ has Serre duality. (cf. [27, Lemma 1]).

Proposition 7.14. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be triangulted categories. Assume $\mathcal{C}_2$ has a Serre functor F and that there is an fully faithful functor $G : \mathcal{C}_1 \to \mathcal{C}_2$ which admits a left adjoint L and a right adjoint R, then $\mathcal{C}_1$ has Serre duality.*

Proof. For all $X, Y \in \mathcal{C}_1$ we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}_1}(X,Y) &\cong \operatorname{Hom}_{\mathcal{C}_2}(GX,GY) \\ &\cong \operatorname{Hom}_{\mathcal{C}_2}(GY,FGX)^* \cong \operatorname{Hom}(Y,RFGX)^* \end{aligned}$$

such that $R \circ F \circ G$ is a right Serre functor. Likewise, one shows that $L \circ F \circ G$ is a left Serre functor. It now follows from [40, Lemma I.1.4] that $\mathcal{C}_1$ has Serre duality. □

Since $\operatorname{Num} E^\perp = 1$ and $E^\perp$ satisfies Serre duality, it follows from Proposition 7.10 that $E^\perp$ has an exceptional object X. We thus know $\operatorname{Ext}(X,X) = \operatorname{Ext}(E,X) = \operatorname{Ext}(E,E) = 0$, $\operatorname{Hom}(E,X) = 0$, and $\operatorname{Hom}(E,E) \cong k \cong \operatorname{Hom}(X,X)$. In general however, $\operatorname{Ext}(X,E) \neq 0$. Instead of X, we will consider $E_1 = T^*_E X$. It follows from the triangle

$$E[-1] \otimes RHom(X,E)^* \to T^*_E X \to X \to E \otimes RHom(X,E)^*$$

that $\operatorname{Ext}(E_2, E_2) = \operatorname{Ext}(E_1, E_2) = \operatorname{Ext}(E_1, E_1) = \operatorname{Ext}(E_2, E_1) = 0$, $\operatorname{Hom}(E_2, E_1) = 0$ and $\operatorname{Hom}(E_1, E_1) \cong k \cong \operatorname{Hom}(E_2, E_2)$ where $E_2 = E$.

If we choose E_1, E_2 such that $d = \dim \operatorname{Hom}(E_1, E_2)$ is minimal, then E_1, E_2 forms a basis of $K_0^{red}(D^b\mathcal{A})$. The Cartan and Coxeter matrices with respect to this basis are given by

$$A = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & -d \\ d & d^2-1 \end{pmatrix},$$

respectively.

The existence of such an exceptional sequence suffices to classify these kind of categories. We will use following lemma.

Proposition 7.15. *Let $\mathcal{A}$ be a connected abelian hereditary Ext-finite category with Serre duality and $\operatorname{Num}\mathcal{A} = 2$. If $\mathcal{A}$ has an exceptional object, then $\mathcal{A}$ is derived equivalent to one of the following*

1. $\operatorname{nilp} k\tilde{A}_2$ *where $\tilde{A}_2$ has cyclic orientation,*
2. $\operatorname{mod} kQ$ *where $Q_0 = \{1, 2\}$ and where there are $d > 1$ arrows $a \to 2$.*

Proof. Let E_2, E_1 be an exceptional sequence as above. We will consider two cases, namely $d = 1$ and $d > 1$.

1. Assume $d = 1$. In this case, $C^3 = -I$ and thus, by Proposition 7.13, $\tau^3 E_2 \cong E_2[-i]$, for an $i \in \mathbb{N}$. If $i = 0$, then it follows from Theorem 4.5 that $\mathcal{A} \cong \operatorname{nilp} k\tilde{A}_2$. If $i \neq 0$, then Lemma 5.21 shows $D^b\mathcal{A} \cong D^b \operatorname{mod} kA_2$.

2. We are left with the case $d > 1$. Let Q be the quiver with $Q_0 = \{a, b\}$ and where there are $d > 1$ arrows $1 \to 2$. Denote the standard projectives corresponding with the vertices 1 or 2 by P_1 or P_2, respectively.

   Let $i : D^b \operatorname{mod} kQ \to D^b\mathcal{A}$ be the embedding given by Theorem 1.34; thus $iP_1 = E_1$ and $iP_b = E_2$. We shall show i commutes with the Serre functor on a spanning set.

   Choose P_1, P_2 as a basis for $K_0^{red}(D^b \operatorname{mod} kQ)$; it is clear that $[iP_1] = [E_1]$ and $[i\tau P_1] = [\tau E_1]$. If $\tau E_1 \in \mathcal{A}[-1]$, then so is $i\tau P_1$. Indeed, this follows easily from $iP_1 \cong E_1$ and $\chi(i\tau P_1, \tau E_1) = 1$. In this case, we may apply Proposition 7.13 to see $i\tau P_1 \cong \tau E_1$, or thus $i\tau P_1 \cong \tau i P_1$

   Likewise, if $i\tau P_1 \in \mathcal{A}[0]$, we find $\tau E_1 \in \mathcal{A}[0]$ such that Proposition 7.13 yields $i\tau P_1 \cong \tau i P_1$.

   We are left with the case $i\tau P_1 \in \mathcal{A}[-1]$ and $\tau E_2 \in \mathcal{A}[0]$. In this case, one shows as above that $i\tau^2 P_1 \in \mathcal{A}[-1]$, such that $i\tau^2 P_1 \cong \tau i \tau P_1$.

   In any case, we see that $i\tau P_1 \cong \tau i P_1$ or $i\tau^2 P_1 \cong \tau i \tau P_1$. Likewise, one shows $i\tau P_2 \cong \tau i P_2$ or $i\tau^2 P_2 \cong \tau i \tau P_2$, such that τ, and thus also the Serre functor, commutes with i on a spanning set. We may now invoke Theorem 1.33 to conclude i is an equivalence.

□

7.4.2 $\mathcal{A}$ has an 1-spherical object

We will now assume $D^b\mathcal{A}$ has a 1-spherical object E. Following Theorem 4.5, E is the peripheral object of a standard homogeneous tube $\mathcal{K}_E$.

Since $\operatorname{Num}\mathcal{A} \neq 0$, we know that $\mathcal{A} \not\cong \operatorname{Mod}^{\mathrm{fd}} k[[t]]$, thus choose an $L \in \operatorname{ind} D^b\mathcal{A}$, not lying in the tube $\mathcal{K}_E$, and such that $d = \dim \operatorname{Hom}(L, E)$ is nonzero and minimal. We claim $[L], [E]$ form a basis for $K_0^{red}(D^b\mathcal{A})$.

It follows from Theorem 4.6 that $\operatorname{Ext}(L, E) = 0$, thus $\chi(L, E) = d$. Since $\chi(E, E) = 0$, we see that $[E]$ and $[L]$ are linearly independent. It follows from the minimality of d that they generated $K_0^{red} D^b\mathcal{A}$, hence they form a basis.

It will be convenient to write down the Cartan matrix A with respect to the basis $[L], [E]$:

$$A = \begin{pmatrix} m & d \\ -d & 0 \end{pmatrix}$$

where $m = \chi(L, L)$.

Next, let $T_E^* : D^b\mathcal{A} \to D^b\mathcal{A}$ be a twist functor associated with the 1-spherical object E; for notational simplicity, we will write $t = T_E^*$. We may define a t-structure on $D^b\mathcal{A}$ as follows

$$\begin{aligned} \operatorname{ind} \mathcal{D}^{\leq 0} &= \{Y \in \operatorname{ind} D^b\mathcal{A} \mid \text{there is a path from } t^i L \text{ to } Y, \text{ for an } i \in \mathbb{Z}\} \\ \operatorname{ind} \mathcal{D}^{\geq 1} &= \operatorname{ind} D^b\mathcal{A} \setminus \operatorname{ind} \mathcal{D}^{\leq 0} \end{aligned}$$

We will write $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ for the heart of this t-structure. It has been shown in Theorem 1.30 that $\mathcal{H}$ is hereditary and derived equivalent to $\mathcal{A}$.

The autoequivalence $t : D^b\mathcal{A} \to D^b\mathcal{A}$ restricts to an autoequivalence $\mathcal{H} \to \mathcal{H}$, also denoted by t. The triangles

$$t^{i+1}X \longrightarrow t^i X \longrightarrow E \otimes \operatorname{RHom}(X, E)^* \longrightarrow t^{i+1}X[1]$$

induce exact sequences

$$0 \longrightarrow t^{i+1}X \longrightarrow t^i X \longrightarrow E \otimes \operatorname{Hom}(X, E)^* \longrightarrow 0 \tag{7.1}$$

in $\mathcal{H}$. Note that the coordinates of $t^i L$ in $K_0^{red}\mathcal{H}$ with respect to the basis $[L], [E]$ are $(1, -i)$.

The proof of following lemma is the same as that of Lemma 5.8.

Lemma 7.16. *For all $X \in \operatorname{Ob}\mathcal{H}$, we have* $\dim \operatorname{Hom}(t^i L, X) \leq \dim \operatorname{Hom}(t^{i+1} L, X)$.

Our plan is to show the category $\mathcal{H}$ is noetherian. To this end, the next lemma will be useful.

Lemma 7.17. *With respect to the basis $[L], [E]$ of $K_0^{red}\mathcal{H}$, every object X has coordinates (a, b) where $a \geq 0$. Furthermore, if $a = 0$, then $b > 0$ and* $\operatorname{Hom}(X, L) = 0$.

Proof. We may assume X is indecomposable. It follows from Theorem 4.6 that $\chi(X,E) = \dim \operatorname{Hom}(X,E) \geq 0$ and from the Cartan matrix that $\chi(X,E) = ad$, such that $a \geq 0$.

Assume $a = 0$; we will prove $b \geq 0$. If $b < 0$, we use the Cartan matrix to see $\chi(X, t^i L) > 0$, for all $i \in \mathbb{Z}$. In particular, $\operatorname{Hom}(X, t^i L) \neq 0$, and it follows from Theorem 4.6 that there is no path from $\tau^i L$ to X. A contradiction.

To show $(a,b) \neq (0,0)$, let $\mathcal{H}_0$ be full subcategory spanned by the objects $X \in \operatorname{Ob}\mathcal{H}$ with $[X] = 0$ in $K_0^{red}\mathcal{H}$. This is easily checked to be an abelian subcategory, closed under extensions and τ. It follows from Proposition 7.7 that $\mathcal{H}_0$ consists entirely of homogeneous tubes; a contradiction by Lemma 7.5.

Lastly, let X have coordinates $(0,1)$. In this case, X lies in $E^\perp$. Note that $E^\perp$ is an abelian hereditary category, closed under τ and consists entirely of objects Y with coordinates $(0,n)$ such that $\operatorname{Num} E^\perp$ is at most 1. Since $\tau E \cong E$, we see the coordinates of Y are also $(0,n)$. From Propositions 7.7 and 7.10, we see that $\operatorname{Num} E^\perp = 0$ and that $E^\perp$ is a direct sum of tubes. Since $E^\perp$ is closed under τ, every object Y lies in a homogeneous tube of $\mathcal{H}$.

By the definition of $\mathcal{H}$, there is a path from $t^i L$ to X such that $\operatorname{Hom}(X, t^i L) = 0$ by Theorem 4.6. We may now invoke Lemma 7.16 to see $\operatorname{Hom}(X,L) = 0$. □

Proposition 7.18. *The category $\mathcal{H}$ is equivalent to the category of coherent sheaves* $\operatorname{coh} X$ *on a smooth projective curve X.*

Proof. It is well-known that the subcategory $\mathcal{H}_N$ of $\mathcal{H}$ consisting of noetherian objects is abelian and hereditary (see for example [39, Proposition 5.7.2]).

We will first check E is simple in $\mathcal{H}$, such that $E \in \operatorname{Ob}\mathcal{H}_N$. It suffices to show that E does not map to any Auslander-Reiten component, save $\mathcal{K}_E$. Assume $\operatorname{Hom}(E,X) \neq 0$, for an $X \in \operatorname{ind}\mathcal{H}$. By applying $\operatorname{Hom}(-,X)$ to triangle (7.1), and using Lemma 7.16, we find $\operatorname{Ext}(E,X) \neq 0$. Theorem 4.6 yields that X lies in the tube $\mathcal{K}_E$. We conclude that E is simple.

Next, we shall show L is noetherian. Therefore, consider a sequence of subobjects

$$M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow L$$

and write (a_i, b_i) for the coordinates of M_i with respect to the basis $[L], [E]$ of $K_0^{red}\mathcal{H}$. Since M_i and M_{i+1}/M_i must satisfy the conditions posed in Lemma 7.17, we see that $a_1 = 1$ and $b_i < b_{i+1}$, for all $i \in \mathbb{N}$. This shows L is noetherian.

Note that, by the definition of $\mathcal{H}$, the functor τ is an autoequivalence, such that it restricts to an autoequivalence of $\mathcal{H}_N$. We see that $\mathcal{H}_N$ has Auslander-Reiten sequences, and hence Serre duality. Such categories were classified in [40]; we see that $\mathcal{H}_N$ is equivalent to $\operatorname{coh} X$ for a smooth projective curve X and, as such, is saturated.

The embedding $i : \mathcal{H}_N \to \mathcal{H}$ commutes with τ, and thus the induced embedding $i : D^b\mathcal{H}_N \to D^b\mathcal{H}$ commutes with the Serre functor. Since $\mathcal{H}_N$ is saturated, the embedding $i : D^b\mathcal{H}_N \to D^b\mathcal{H}$ admits a left adjoint. We may now invoke Theorem 1.33 to conclude $\mathcal{H}_N \cong \mathcal{H}$. □

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Index