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Faculteit Wetenschappen

Representation theory of Schur Algebras over rings

Proefschrift voorgelegd tot het behalen van de graad van doctor in de wetenschappen, groep wiskunde aan het Limburgs Universitair Centrum te verdedigen door

Lydia DELVAUX

Promotor : Prof. dr. E. Nauwelaerts

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Het is goed het einddoel
voor je reis te kennen;
maar het is de reis zelf
waar het uiteindelijk om gaat ...

(Ursula K. Le Guin)

Aan mijn drie beste vrienden,

Marcel, Rob en Tom.

Woord van dank

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Samenvatting

In 1933 ontwikkelde I. Schur een methode om een eindige groep G te onderzoeken die werkt op een verzameling Ω zodanig dat G een deelgroep H bevat die strikt transitief werkt op Ω. Zijn originele idee bestond erin om als "punten" geen willekeurige objecten of getallen van 1 tot n te nemen, maar groepelementen. H. Wielandt heeft deze methode van Schur voorgesteld op de meest eenvoudige manier door het begrip groepalgebra over $\mathbb{C}$ te gebruiken. Het idee is om aan bovenstaande situatie een actie van G op H te associëren, zodanig dat H op zichzelf werkt door linkse vermenigvuldiging. Beschouw dan de stabilisator L in G van het neutraal element en laat L werken op H. De sommen van de banen, beschouwd in de groepring $\mathbb{C}H$, brengen een deelmodule van $\mathbb{C}H$ voort, welke in dit geval wel degelijk een ring is; deze algebra wordt een Schuralgebra genoemd (deze wordt uitvoeriger besproken in 1.2.9).

Gebruikmakend van bovenstaande methode heeft men het volgende resultaat : G is dubbel transitief als en slechts als de geassocieerde Schuralgebra "triviaal" is, zie [Wie_2, Theorem 24.11]

Meer algemeen heeft H. Wielandt een Schuralgebra gedefinieerd als een deelalgebra van de groepring $\mathbb{C}G$ (G eindige groep) die geassocieerd is aan een bepaalde partitie van G, zie [Wie_2, Definition 23.1]. Een Schuralgebra over $\mathbb{C}$ is steeds halfeenvoudig, zie [Wie_1, p386, footnote]. Dit laatste probleem werd onderzocht door F. Roesler voor Schuralgebra's over een willekeurig veld, zie [R, Satz 1]. De studie van karakters van Schuralgebra's werd aangevat door O. Tamaschke en F. Roesler, zie [T_1] en [R]. In [T_2] beschouwt de auteur Schuralgebra's over $\mathbb{C}$ in de context van categorieën.

Het hoofddoel van deze thesis is de studie van onontbindbare modulen en karakters voor Schuralgebra's over een commutatieve ring R. Een Schuralgebra kan, ruwweg gesproken, gedefinieerd worden als een deelalgebra van een groepring RG geassocieerd aan een zekere partitie van G (G eindige groep), zie 1.2. In het eerste hoofdstuk beschrijven we twee belangrijke klassen van Schuralgebra's : dubbele

nevenklassen algebra's (en hun veralgemeningen) en fixringen van zekere automorfismengroepen.

In het tweede hoofdstuk ontwikkelen we een karaktertheorie voor Schuralgebra's. We tonen aan dat Schuralgebra's Frobeniusalgebra's zijn (onder een zekere voorwaarde). Daarom behandelen we het probleem in de meer algemene context van Frobeniusalgebra's over commutatieve ringen. In 2.1 verzamelen we algemeenheden over Frobeniusalgebra's en we geven een criterium voor de separabiliteit van een Frobeniusalgebra. Dan bestuderen we symmetrische functies op Frobeniusalgebra's en we tonen aan dat, onder zekere voorwaarden, deze functies worden voortgebracht door karakters. We drukken primitieve centrale idempotenten van een Frobeniusalgebra A uit in termen van karakters en we ontwikkelen orthogonaliteitsrelaties voor karakters op A. In het geval van Schuralgebra's introduceren we klassefuncties en we onderzoeken wanneer de verzameling van klassefuncties samenvalt met de verzameling van symmetrische functies. Deze laatste studie geeft feitelijk een analyse van het centrum van een Schuralgebra. Tenslotte berekenen we ook het karakter van een geïnduceerd moduul tussen twee Schuralgebra's.

In het derde hoofdstuk bestuderen we dubbele nevenklassen algebra's en hun veralgemeningen. We beschouwen de algemene situatie van Heckealgebra's : stel A een R-algebra en ε een idempotent ($\neq 0$) in A, dan wordt $\varepsilon A \varepsilon$ een Heckealgebra in A genoemd. In 3.1 bepalen we de primitieve centrale idempotenten van $\varepsilon A \varepsilon$ en we onderzoeken het verband tussen onontbindbare modulen over $\varepsilon A \varepsilon$ en onontbindbare modulen over A. In 3.2 wordt de corresponderende karaktertheorie beschreven. We tonen ook aan dat een *connected* ring R een splitsingsring is voor $\varepsilon A \varepsilon$ indien hij een splitsingsring is voor A.

In hoofdstuk 4 belichten we Schuralgebra's die fixringen zijn van bepaalde automorfismengroepen. We bestuderen hier het verband tussen onontbindbare modulen over een R-algebra A en over de fixring A^H, waarbij $H \to Aut_R(A)$ een groephomomorfisme is. In het geval van Schuralgebra's kunnen we de corresponderende karaktertheorie ontwikkelen.

Als speciaal geval wordt beschouwd: $\sigma : H \to Aut(G)$ met $\sigma_h(g) = hgh^{-1}$ waarbij H een deelgroep is van een eindige groep G. De fixring van RG geassocieerd aan deze

actie is ook de centralisator van RH in RG; hierdoor kunnen we deze Schuralgebra S ook bestuderen in het kader van centralisators waardoor meer relaties gevonden worden tussen de onontbindbare modulen over RH, RG en S, zie 5. Dit probleem wordt algemener opgevat. We beschouwen de centralisator S van B in A, waarbij B een deelalgebra is van een algebra A (in een separabele context) en we geven relaties tussen onontbindbare modulen over A, B en S, zie 5.2. In 5.4 passen we deze algemene resultaten toe op de centralisator S van $R *_\alpha H$ in de getwiste groepring $R *_\alpha G$ $(H < G)$ en we ontwikkelen de corresponderende karaktertheorie. (Hiervoor wordt een expliciete R-basis van S geconstrueerd, zie 5.3).
Tenslotte herzien we de theorie van Clifford voor normale deelgroepen en we belichten dat de karaktertheorie in 5.4 kan beschouwd worden als een veralgemening van de theorie van Clifford voor niet-normale deelgroepen.
Sommige resultaten worden geïllustreerd door middel van twee eenvoudige voorbeelden op het einde van dit hoofdstuk.

Introduction

In 1933 I. Schur developed a method to investigate a finite group G acting on a set Ω such that G contains a subgroup H acting strictly transitively on Ω. His fundamental idea consisted in taking as "points" not arbitrary objects or numbers from 1 to n, but group elements. H. Wielandt has presented Schur's method in the most simple way by using the concept of a group algebra over $\mathbb{C}$. The idea is to associate to the above situation an action of G on H, such that H acts on itself by left multiplication. Then consider the stabilizer L in G of the neutral element, and let L act on H. The sums of the orbits, considered in the group ring $\mathbb{C}H$, generate a submodule of $\mathbb{C}H$, which in this case is actually a ring; this algebra is called a Schur algebra (which is discussed in 1.2.9).
Using this method, one has : G is doubly transitive if and only if the associated Schur algebra is "trivial", see [Wie_2, Theorem 24.11].

More generally, H. Wielandt has defined a Schur algebra as a subalgebra of the group ring $\mathbb{C}G$ (G finite group) associated to a suitable partition of G, see [Wie_2, Definition 2.3.1]. A Schur algebra over $\mathbb{C}$ is semisimple, see [Wie_1, p386, footnote]. This latter problem is investigated by F. Roesler for a Schur algebra over an arbitrary field, see [R, Satz 1]. The investigation of the characters of a Schur algebra was set up by O. Tamaschke and F. Roesler, see [T_1] and [R]. In [T_2], the author started to study Schur algebras over $\mathbb{C}$ in a categorical context.

Our main objective is to study indecomposable modules and trace functions for Schur algebras over a commutative ring R. Roughly speaking, a Schur algebra is a subalgebra of a group ring RG associated to a suitable partition of G (G a finite group), see 1.2. In the first chapter we describe two important classes of Schur algebras : double coset algebras (and their generalizations) and fixed rings of certain automorphism groups.
In the second chapter, we develop a character theory for Schur algebras. We show that Schur algebras are Frobenius algebras (under a suitable condition). Therefore

we set up this problem in the more general context of Frobenius algebras over commutative rings. Some generalities on Frobenius algebras are collected in 2.1 and we give a criterion for the separability of a Frobenius algebra. We then study symmetric functions on Frobenius R-algebras and we show that, under certain conditions, they are generated over R by trace functions. We express primitive central idempotents of a Frobenius algebra A in terms of trace functions and we derive orthogonality relations for trace functions on A. In the case of Schur algebras we introduce class functions and we investigate when the set of class functions coincides with the set of symmetric functions. In fact, this latter study yields an analysis of the center of a Schur algebra. We also calculate the trace function of induced modules between two Schur algebras.

In chapter 3 we concentrate on double coset algebras and their generalizations. However, we consider the more general situation of Hecke algebras : if A is an R-algebra and ε a nonzero idempotent of A, then $\varepsilon A\varepsilon$ is called a Hecke algebra in A. In 3.1 we determine the primitive central idempotents of $\varepsilon A\varepsilon$ and we investigate the relationship between indecomposable modules over $\varepsilon A\varepsilon$ and indecomposable modules over A. In 3.2 we describe the corresponding character theory. We also prove that a connected ring R is a splitting ring for $\varepsilon A\varepsilon$ whenever it is a splitting ring for A.

In chapter 4 we focus on Schur algebras which are fixed rings of certain automorphism groups. In fact, we study the relationship between indecomposable modules over an R-algebra A and indecomposable modules over the fixed ring A^H, where $H \to Aut_R(A)$ is a group homomorphism. In the case of Schur algebras we develop the corresponding character theory.

As a special case we consider the following situation : $\sigma : H \to Aut(G)$ with $\sigma_h(g) = hgh^{-1}$ where H is a subgroup of a finite group G. The fixed ring of RG associated to this action is also the centralizer of RH in RG; this allows us to study this Schur algebra S in the framework of centralizers, which gives more relations between indecomposable modules over RH, RG and S, see 5. The problem is set up more general. We consider the centralizer S of B in A, where B is a subalgebra of an algebra A (in a separable context) and we give relations between indecomposable modules over A, B and S, see 5.2. In 5.4 we apply these general results on the centralizer S of $R *_\alpha H$ in the twisted group ring $R *_\alpha G$ ($H < G$) and we develop the corresponding character theory (we construct an explicit R-basis for S, see 5.3). Finally we review Clifford theory on normal subgroups and we see that the character theory in 5.4 can be viewed as a generalization of Clifford theory for non normal subgroups.

To end this chapter, we work out two easy examples to illustrate some of the results of the foregoing sections.

TABLE OF CONTENTS

Chapter 1

Preliminaries and the definition of a Schur algebra over a ring

1.1 Preliminaries

Throughout R is a commutative ring. A ring is said to be connected if 0 and 1 are the only idempotent elements. We begin with some useful facts about indecomposable modules.

Let A be an R-algebra and suppose that R is connected. We first remark that any left A-module, which is finitely generated and projective over R, is a finite direct sum of indecomposable left A-modules (use rank_R). However this decomposition is not necessarily unique. We shall give conditions (for a separable algebra) in order that the above decomposition into indecomposable modules is unique.

Now assume that A is finitely generated and projective as an R-module. Then there exist primitive central orthogonal nonzero idempotents $e_1, \ldots, e_q$ in A such that $1 = e_1 + \ldots + e_q$ (use rank_R). Moreover, each central nonzero idempotent of A is uniquely a sum of some e_i.

Let V be a left A-module, then $V = e_1V \oplus \ldots \oplus e_qV$ in A-mod. If V is an indecomposable left A-module, then there is a unique i such that $e_iV \neq 0$ (in fact $e_iv = v$ for all $v \in V$), and we say that V lies over e_i.
Further, if any two indecomposable left A-modules, being finitely generated projective as R-modules and lying over the same e_i, are isomorphic as A-modules,

then any left A-module V, which is finitely generated projective as R-module, is uniquely expressible as a finite direct sum of indecomposable left A-modules (up to isomorphism). Indeed, each nonzero e_iV is the direct sum of all indecomposable left A-modules lying over e_i and appearing in the decomposition of V, and use $\operatorname{rank}_R(e_iV)$. The above remark also holds if we replace projectivity over R by projectivity over A.

We now consider situations where the latter condition on indecomposable left A-modules is satisfied.

From now on we assume that A is finitely generated projective as an R-module and that A is a separable R-algebra, see [*DM-I*]. Then a left A-module is finitely generated projective as an R-module if and only if it is finitely generated projective as an A-module, use [*DM-I*, p48]. Furthermore we have the following result based on [*DM*, Theorem 1] :

1.1.1 Proposition. *Let R be semilocal and let A be a separable R-algebra, which is finitely generated and projective as R-module. Then any two indecomposable, finitely generated projective left A-modules lying over the same primitive central idempotent of A are isomorphic as A-modules.*

Proof. First we observe that A is semilocal too, that is $A/radA$ is a left (and right) Artinian ring, where $radA$ denotes the Jacobson radical. Indeed, since A is finitely generated over R, we have $(radR)A \subset radA$, whence $A/radA$ is a finitely generated $R/radR$-module. Since $R/radR$ is an Artinian ring, $A/radA$ is a left (and right) Artinian ring.

Further, since A is separable over R, it is separable over its center $Z(A)$. Moreover $Z(A)$ is semilocal too. For, since A is semilocal, A has only a finite number of maximal ideals. But then there are only finitely many maximal ideals in $Z(A)$, because A is separable over $Z(A)$, and so $Z(A)$ must be semilocal.

Now we may apply [*DM*, Theorem 1] and we obtain that any two indecomposable finitely generated projective left A-modules lying over the same primitive central idempotents of A are A-isomorphic. □

Note also that a separable R-algebra A, where R is a field, is classically separable and the dimension of A over R is finite [DM-I, p50]. In this case A is a semisimple ring. Recall that for any semisimple ring E, E-modules are projective over E, and indecomposable E-modules are simple (and conversely). In this case there is, up to isomorphic, only one simple E-module lying over a primitive central idempotent of E.

Next, a commutative extension L of R is said to be a *splitting ring* for A over R if $L \otimes_R A \cong \mathrm{End}_L(P_1) \oplus \ldots \mathrm{End}_L(P_t)$ as L-algebras where $P_1, \ldots, P_t$ are finitely generated projective faithful L-modules.

We now assume that R itself (which is connected) is a splitting ring for A (or A is split separable over R); that is, $A \cong \mathrm{End}_R(M_1) \oplus \ldots \oplus \mathrm{End}_R(M_q)$ as R-algebras, $M_1, \ldots, M_q$ being finitely generated projective (faithful) R-modules. Recall that finitely generated projective nonzero modules over connected commutative rings are always faithful, see [DM-I, p8]. Note also that the center of A is a free R-module of rank q.

Obviously M_i can be viewed as a left A-module by setting $(\varphi_1, \ldots, \varphi_q) \cdot m = \varphi_i(m)$, where $m \in M_i$ and $\varphi_j \in \mathrm{End}_R(M_j)$. Since R is connected, each M_i is an indecomposable left A-module, and they are not isomorphic as such. Now assume that M_i lies over the primitive central idempotent e_i of A. If finitely generated projective R-modules are free, for example, when R is semilocal or a principal ideal domain, then M_i is, up to isomorphism, the only indecomposable finitely generated projective left A-module lying over e_i; see [N_2-$v.O_2$, Proposition 1.8].

Note also that any semisimple $\mathbb{C}$-algebra is split separable over $\mathbb{C}$.

We now list some results on modules over split separable R-algebras. We need the following version of Frobenius reciprocity.

1.1.2 Note. Let $B \subset A$ be R-algebras with $1_A \in B$. Let V be a left B-module and W a left A-module. Then $Hom_B(V, W) \cong Hom_A(A \otimes_B V, W)$ as R-modules.

Indeed, it is easily seen that $Hom_B(V, W) \to Hom_A(A \otimes_B V, W) : \varphi \mapsto \psi$, with

$\psi(a \otimes v) = a\varphi(v)$ for $a \in A$, $v \in V$, is an isomorphism of R-modules.

1.1.3 Proposition. *Let A be an R-algebra and B a subalgebra of A with $1_A \in B$. Suppose R is connected, R is a splitting ring for B, and finitely generated projective R-modules are free. Let N, resp. M, be an indecomposable left B-module, resp. A-module, which is finitely generated projective over R.*

(1) Let V be a left B-module which is finitely generated projective over R, and let k be the multiplicity of N in a decomposition of V into indecomposable B-modules. If $k \neq 0$, then $Hom_B(N, V)$ and $Hom_B(V, N)$ are free R-modules of rank k. Otherwise they are zero.

(2) Let B be isomorphic to $End_R(N_1) \oplus \ldots \oplus End_R(N_t)$, N_i being finitely generated projective R-modules. Let V be a left B-module which is finitely generated projective as an R-module; thus $V \cong N_1^{k_1} \oplus \ldots \oplus N_t^{k_t}$ as left B-modules. Then $End_B(V)$ is a free R-module of rank $\sum\limits_{i=1}^{t} k_i^2$.

(3) Suppose R is also splitting ring for A. Then the multiplicity of N in M, viewed as B-module, is equal to the multiplicity of M in $A \otimes_B N$ (multiplicity in a decomposition into indecomposables).

Proof. (1) Let $V = L_1 \oplus \ldots \oplus L_n$ be a decomposition of V into indecomposable left B-modules L_i. Now $Hom_B(N, V) \cong \bigoplus\limits_{i=1}^{n} Hom_B(N, L_i)$ as R-modules. If L_i is not isomorphic to N in B-mod, then, by our hypotheses, L_i and N lie over distinct primitive central idempotents of B, whence $Hom_B(N, L_i) = 0$. Moreover $Hom_B(N, N) = RI$, see [N_2-$v.O_2$, 1.7].

(2) Let $V = L_1 \oplus \ldots \oplus L_n$ be a decomposition of V into indecomposable left B-modules L_i.

Consider the map $f : \mathrm{End}_R(V) \rightarrow \bigoplus\limits_{i,j=1}^{n} Hom_R(L_i, L_j) : \varphi \mapsto (\varphi_{ij})$; with $\varphi_{ij} \in Hom_R(L_i, L_j)$ given by $\varphi_{ij}(v_i) = \pi_j\varphi(v_i)$, where $v_i \in L_i$ and π_j is the projection of V onto L_j. Clearly f is an R-module isomorphism and it is easily seen that $\varphi \in \mathrm{End}_B(V)$ if and only if $\varphi_{ij} \in Hom_B(L_i, L_j)$ for all i and j.

For each pair of modules L_i, L_j with L_i isomorphic to L_j in B-mod, we choose a B-module isomorphism $\psi_{ij} : L_i \rightarrow L_j$. We will show that the n^2-tuples having a morphism ψ_{ij} at one place and zeros elsewhere form an R-basis for $f(\mathrm{End}_B(V))$, and then by the assumptions on R and B the assertion is proved. Clearly these n^2-tuples are linearly independent over R. For, if $r \in R$ with $r\psi_{ij} = 0$, then $rL_j = 0$. But since R is connected, L_j is a faithful R-module and thus $r = 0$. So it remains to show that they generate $f(\mathrm{End}_B(V))$. Let $\{f_1, \ldots, f_t\}$ be the set of primitive central nonzero idempotents of B and assume that N_i lies over f_i. Let $\varphi \in \mathrm{End}_B(V)$, hence $\varphi_{ij} \in \mathrm{End}_B(L_i, L_j)$ for all i and j. First consider a pair (i, j) for which L_i is not isomorphic to L_j in B-mod and assume that L_i lies over f_k. Then by our hypotheses $f_k L_j = 0$ and we have for any $v \in L_i$ that $\varphi_{ij}(v) = \varphi_{ij}(f_k v) = f_k \varphi_{ij}(v) = 0$, hence $\varphi_{ij} = 0$. Next, let (i, j) be a pair for which L_i is isomorphic to L_j in B-mod. Then $\psi_{ij}^{-1} \circ \varphi_{ij} \in Hom_B(L_i, L_i)$. But $L_i \cong N_k$ in B-mod, whence $\mathrm{End}_B(L_i) \cong \mathrm{End}_B(N_k)$ as R-algebras. Now $\mathrm{End}_B(N_k)$ coincides with the center of $\mathrm{End}_R(N_k)$ and thus $\mathrm{End}_B(N_k) = RI$. Therefore there is an $r \in R$ such that $\psi_{ij}^{-1} \circ \varphi_{ij} = rI$, hence $\varphi_{ij} = r\psi_{ij}$.

(3) Clearly $A \otimes_B N$ is projective over A and R. Combine assertion (1) and Note 1.1.2 $\square$

Furthermore, we recall the following basic facts about idempotents.

Remarks. Let R be connected and let A be an R-algebra which is finitely generated and projective as an R-module.

(i) For each nonzero idempotent a of A, Aa is a finitely generated projective R-module. Use $A = Aa \oplus A(1 - a)$.

(ii) Each nonzero idempotent a of A is a sum of primitive orthogonal idempotents of A. Use (i) and $\mathrm{rank}_R(Aa)$.

(iii) Let $\varepsilon \neq 0$ be a primitive idempotent of A. First note that $A\varepsilon$ is an indecomposable left A-module. So there is a unique primitive central nonzero idempotent e in A such that $e\varepsilon \neq 0$; in this case $e\varepsilon = \varepsilon$. Further, e is expressible as a sum of primitive orthogonal idempotents of A in such a way that one of the terms is ε. Indeed, $e = \varepsilon + (e - \varepsilon)$ with ε and $e - \varepsilon$ orthogonal idempotents.

Next, we give some basic facts about trace functions. Let A be an R-algebra and V a left A-module which is finitely generated projective over R. Let $\{v_1, \ldots, v_n\} \subset$

V, $\{\varphi_1, \ldots, \varphi_n\} \subset Hom_R(V, R)$ be an R-dual basis for V. The *trace function* (or character) from A to R afforded by V, notation t_V, is defined as follows : $t_V(a) = \sum_{i=1}^{n} \varphi_i(av_i)$, for all $a \in A$. It is easily seen that t_V does not depend on the choice of the dual basis. Further, $t_V(xy) = t_V(yx)$ for all $x, y \in A$ and if R is connected, then $t_V(1) = \text{rank}_R(V)1_R$, see [$N_2$-$v.O_2$, 2.5]. We need the following result.

1.1.4 Lemma. *Suppose R connected and $A \cong \bigoplus_{i=1}^{q} End_R(M_i)$ as R-algebras, where M_i are finitely generated projective R-modules. Let $\{e_1, \ldots, e_q\}$ be the set of primitive central orthogonal idempotents of A, and assume that M_i lies over e_i. Then $t_{Ae_i} = (rank_R M_i) t_{M_i}$ on A (Ae_i viewed as left A-module).*

Proof. Write M_i^* instead of $Hom_R(M_i, R)$. Since M_i is a finitely generated projective R-module, we know that $M_i^* \otimes_R M_i \cong \text{End}_R(M_i)$ as left $\text{End}_R(M_i)$-modules, where the left $\text{End}_R(M_i)$-module structure on $M_i^* \otimes_R M_i$ is induced by that on M_i. Clearly $Ae_i \cong \text{End}_R(M_i)$ as R-algebras and thus $Ae_i \cong M_i^* \otimes_R M_i$ as left A-modules. Moreover, M_i^* is finitely generated and projective over R. This implies that $t_{Ae_i} = t_{M_i^*}(1) t_{M_i} = (\text{rank}_R M_i^*) t_{M_i} = (\text{rank}_R M_i) t_{M_i}$ on A, see [N_2-$v.O_2$, Lemmas 2.2 and 2.5]. $\square$

To conclude this section, let us focus on group rings and twisted group rings. Let G be a finite group and consider the group ring RG. As R-module, RG is freely generated by symbols $\{u_g; g \in G\}$. Recall that in case $|G|^{-1} \in R$, RG is separable over R. Further, suppose R is connected and $|G|^{-1} \in R$. Let m be the exponent of G and let η be a primitive m-th root of unity. Then $R[\eta]$ is a splitting ring for RG, see [S]. Since an extension of a splitting ring is a splitting ring, we see that $R[\eta]$ is also a splitting ring for RH, where H is a subgroup of G.

We now recall some facts about twisted group rings. Let G be a finite group and α a 2-cocycle in $Z^2(G, U(R))$, where $U(R)$ is the group of units in R and G acts trivially on R. Thus $\alpha : G \times G \to U(R)$ satisfies : $\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z)$

for all $x, y, z \in G$. The corresponding twisted group ring will be denoted by $R *_\alpha G$. As R-module $R *_\alpha G$ is freely generated by symbols $\{u_g | g \in G\}$ and multiplication is defined by : $(au_x) \cdot (bu_y) = \alpha(x,y)abu_{xy}$ for $a, b \in R$, $x, y \in G$. In case $\alpha = 1$, we get the group ring RG.
One can check that for any $x \in G$, $\alpha(x,e) = \alpha(e,e) = \alpha(e,x)$ and $\alpha(x,x^{-1}) = \alpha(x^{-1},x)$, e being the neutral element of G. So $\alpha(e,e)^{-1}u_e$ is the unit element of $R *_\alpha G$. Moreover, there is a 2-cocycle β equivalent to α such that $\beta(e,e) = 1$.

In $[N_1\text{-}v.O_1]$ *α-regular* elements of G were studied. An element $g \in G$ is α-regular if $\alpha(g,x) = \alpha(x,g)$ for all $x \in C(g) = \{y \in G, yg = gy\}$. Obviously, an α-regular element will be β-regular for every 2-cocycle β equivalent to α.
Furthermore, if g is α-regular, then g^{-1} is α-regular too and xgx^{-1} is α-regular for all $x \in G$, cf. $[N_1\text{-}v.O_1$, Proposition 2.1]. To α one associates a map $f_\alpha : G \times G \to U(R) : (x,g) \to \alpha(x,g)\alpha^{-1}(xgx^{-1},x)$. Clearly, $u_x u_g (u_x)^{-1} = f_\alpha(x,g)u_{xgx^{-1}}$ for all $x, g \in G$.
Recall from $[N_1\text{-}v.O_1$, Proposition 2.3] that there is always a 2-cocycle β equivalent to α satisfying $\beta(e,e) = 1$ and $f_\beta(x,g) = 1$ for all β-regular g and all x in G. An *α-regular class* or *α-ray class* is a class of conjugated elements of G consisting of α-regular elements. Obviously an α-ray class will be a β-ray class for every 2-cocycle β equivalent to α. We recall from $[N_1\text{-}v.O_1$, Theorem 2.4].

1.1.5 Proposition. *Assume that $f_\alpha(x,g) = 1$ for all α-regular $g \in G$ and all $x \in G$. Then the α-ray class sums form an R-basis for the center of $R *_\alpha G$ in the following cases :*
(i) $\alpha = 1$, (ii) R is a domain, (iii) R is connected and $|G|^{-1} \in R$.

From $[N_2\text{-}v.O_2$, Proposition 3.3] we retain the following result on trace functions.

1.1.6 Proposition. *Let V be a left $R *_\alpha G$-module that is finitely generated and projective as an R-module.*
(1) Suppose that α has been modified such that $f_\alpha(x,g) = 1$ for α-regular g and

arbitrary x in G, then we have : $t_V(u_g) = t_V(u_{xgx^{-1}})$ for all α-regular g and all $x \in G$.

(2) If R is either a domain, or a connected ring such that $|G|^{-1} \in R$, then $t_V(u_g) = 0$ for each non α-regular g in G.

For later use, we also mention the following lemma on trace functions.

1.1.7 Lemma. *Let V be a left $R *_\alpha G$-module which is finitely generated projective over R. Then for any $a \in R *_\alpha G$ we have :*

$$\sum_{g\in G} t_V(au_g^{-1})u_g = \sum_{g\in G} t_V(u_g^{-1})u_g a = \sum_{g\in G} t_V(u_g^{-1})au_g.$$

Proof. Write $a = \sum_{k\in G} r_k u_k$ with $r_k \in R$. Then

$$\begin{aligned}
\sum_{g\in G} t_V(au_g^{-1})u_g &= \sum_{k\in G}\sum_{g\in G} r_k t_V(u_k u_g^{-1})u_g u_k^{-1} u_k \\
&= \sum_{k\in G}\sum_{g\in G} r_k t_V((u_{gk^{-1}})^{-1})u_{gk^{-1}} u_k \\
&= \sum_{k\in G}\sum_{x\in G} r_k t_V(u_x^{-1})u_x u_k \\
&= \sum_{x\in G} t_V(u_x^{-1})u_x a.
\end{aligned}$$

Further, $t_V(au_g^{-1}) = t_V(u_g^{-1}a)$ and, just as above, we obtain that $\sum_{g\in G} t_V(u_g^{-1}a)u_g = \sum_{g\in G} t_V(u_g^{-1})au_g$. □

To end, recall that in case $|G|^{-1} \in R$, $R *_\alpha G$ is separable over R. Further, one can construct a splitting ring for $R *_\alpha G$ in a "similar way" as for RG. In [N_1-$v.O_1$, Lemma 3.1 and Theorem 3.3], the authors established the following result for twisted group rings.

1.1.8 Theorem. *Suppose R is connected and $|G|^{-1} \in R$. For a given 2-cocycle α one may construct a normal separable commutative extension L of R which is a free R-module of finite rank and a connected ring such that L is a splitting ring for*

$R *_\alpha G$ *over* R.

Clearly if R is semilocal, then L is semilocal too. Furthermore, the splitting ring L for $R *_\alpha G$ constructed in [N_1-$v.O_1$] is also a splitting ring for $R *_\alpha H$, where H is a subgroup of G. Indeed, in the same way we may construct a splitting ring L' for $R *_\alpha H$ and it is easily seen that $L' \subset L$. Finally, note that for any connected splitting ring L of $R *_\alpha G$, the number of factors in the decomposition of $L *_\alpha G$ equals the number of α-regular classes in G, cf. [N_1-$v.O_1$, Corollary 2.5].

1.2 Definition Schur algebra and examples

Throughout this section R is a commutative ring and G is a finite group.

1.2.1 Definition. Let $\{E_g; g \in G\}$ be a partition of G such that $E_g^{-1} = E_{g^{-1}}$. Denote by G_0 a set of representatives of the distinct E_g. Now put $s_g = \sum\limits_{x \in E_g} u_x$ in RG. If $S = \bigoplus\limits_{g \in G_0} Rs_g$ is a subalgebra of RG with unit element 1_S, then S is said to be a *Schur algebra* in RG.

1.2.2 Remark. Keep the notation of 1.2.1 and suppose that $S = \bigoplus\limits_{g \in G_0} Rs_g$ is a subalgebra of RG with unit element. Then the following statement need not hold :

$$\forall g, h \in G : \quad E_g E_h = \bigcup_k E_k \text{ for some } k \in G \qquad (*)$$

However, if $\mathrm{char}(R) = 0$, then property $(*)$ follows from the ring structure of S. An example of a Schur algebra for which property $(*)$ does not hold is given in 1.2.13.

Of course, if $E_e = \{e\}$, then $s_e = 1_S$. Furthermore :

1.2.3 Lemma. *Let E_g, s_g be as in 1.2.1.*

(1) Suppose for all $g, h \in G$ we have $E_g E_h = \bigcup\limits_k E_k$ (some $k \in G$). Then E_e is a subgroup of G and $s_e s_g = s_g s_e = |E_e| s_g$ for all $g \in G$.

(2) Suppose that $S = \bigoplus\limits_{g \in G_0} Rs_g$ is a subalgebra of RG with unit element 1_S. Then $|E_e|$ is invertible in R. Moreover, if $|E_g|1_R \neq 0$ and $|E_g|1_R$ is not a zero divisor in R for each $g \in G$, then $1_S = |E_e|^{-1} s_e$.

Proof. (1) We shall prove that $xE_g \subset E_g$ for all $x \in E_e$. But then equality must hold, because $|xE_g| = |E_g|$. Analogously $E_g x = E_g$, and the assertions follow. Now take $y \in E_g$ and put $h = xy$. Then $E_h E_{g^{-1}} \cap E_e \neq \phi$, and thus by our hypothesis $E_e \subset E_h E_{g^{-1}}$. Therefore $E_h = E_g$.

(2) Write $1_S = \sum_{g \in G_0} r_g s_g$ with $r_g \in R$, and let $e \in G_0$. Then $s_t = \sum_{g \in G_0} r_g s_g s_t$ for all $t \in G$. Comparing coefficients of u_e, we obtain $1 = |E_e| r_e$ and $0 = |E_g| r_g$ for all $g \in G_0 \backslash \{e\}$. The result now follows. □

We also mention the following elementary fact.

1.2.4 Lemma. *(1) The map $\theta : RG \to RG : \sum_{g \in G} r_g u_g \mapsto \sum_{g \in G} r_g u_{g^{-1}}$ is an anti-isomorphism and $\theta \circ \theta = I$.*

(2) If S is a Schur algebra in RG, then $\theta(S) = S$.

We may consider the following componentwise multiplication on RG. Let $a, a' \in RG$, $a = \sum_{g \in G} r_g u_g$ and $a' = \sum_{g \in G} r'_g u_g$ with $r_g, r'_g \in R$. Then we define $a * a' = \sum_{g \in G} r_g r'_g u_g$. Note that $RG, *$ is a commutative R-algebra with $\sum_{g \in G} u_g$ as unit element. Evidently, every Schur algebra in RG is closed under this multiplication and contains $\sum_{g \in G} u_g$. On the other hand, we have :

1.2.5 Proposition. *Suppose R is a field.*

(1) Let S be an R-submodule of RG. If S is closed under the multiplication $*$ and $\sum_{g \in G} u_g \in S$, then there is a partition $\{E_g; g \in G\}$ of G such that $S = \bigoplus_{g \in G_0} R s_g$, where $s_g = \sum_{x \in E_g} u_x$ and G_0 denotes a set of representatives of the distinct E_g.

(2) Let S be an R-subalgebra of RG with unit element. If S satisfies the conditions in (1) and $\theta(S) \subset S$, then S is a Schur algebra in RG.

Proof. (1) We consider the R-algebra $S, *$. There exist orthogonal primitive nonzero idempotents in $S, *$, say $e_1, \ldots, e_m$, such that $\sum_{g \in G} u_g = e_1 + \ldots + e_m$. Clearly, $\{u_g; g \in G\}$ is the set of primitive idempotents of $RG, *$ and thus we have $e_1 = u_{g_1} + \ldots + u_{g_t}$ and so on. By the above remarks we obtain a partition of G, namely $E_{g_1} = \{g_1, \ldots, g_t\}$, etc.

Next, the multiplication $*$ makes Ru_g into a left S-module. Since $\dim_R(Ru_g) = 1$, Ru_g is a simple S-module. So RG is a semisimple left S-module and thus $S, *$ is a semisimple ring. But then $S * s_g \cong Ru_g$ as S-modules ($s_{g_1} = e_1$). Consequently $\dim_R(S * s_g) = 1$, and thus $Rs_g \subset S * s_g$ must be an equality.

(2) Let θ be as above. Clearly, $\theta : RG, * \to RG, *$ is an isomorphism of R-algebras and $\theta_\circ\theta = I$. Since $\theta(S) \subset S$, it follows that $\theta(s_g) = s_{g^{-1}}$ is a primitive idempotent of $S, *$. This proves our assertion. $\square$

We now describe two important classes of Schur algebras.

1.2.6 Double coset algebras Let H be a subgroup of G. Suppose that $|H|^{-1} \in R$ and consider the idempotent $\varepsilon = |H|^{-1} \sum\limits_{h \in H} u_h$ in RG. Then $\varepsilon RG \varepsilon$ is a Schur algebra, called a double coset algebra. Indeed, $H \times H$ acts on G as follows : $((h,k), g) \mapsto hgk^{-1}$, $h, k \in H$, $g \in G$, and $(HgH)^{-1} = Hg^{-1}H$. Furthermore, $|HgH|$ is invertible in R and $\sum\limits_{x \in HgH} u_x = |HgH| \varepsilon u_g \varepsilon$.

Clearly the partition $\{HgH; g \in G\}$ satisfies the property $(*)$ of Remark 1.2.2.

The following generalizes the above situation :

1.2.7 Proposition. *Let S be a Schur algebra in RG with associated partition $\{E_g; g \in G\}$. Let H be a subgroup of G such that $|H|^{-1} \in R$ and consider the idempotent $\varepsilon = |H|^{-1} \sum\limits_{h \in H} u_h$. If $\varepsilon \in S$ and $|E_g| 1_R \neq 0$ for all $g \in G$, then $\varepsilon S \varepsilon$ is a Schur algebra in RG with partition $\{HE_gH \; ; \; g \in G\}$. Moreover we have $m|H|^{-2}|HE_gH| 1_R = |E_g| 1_R$ with $m \in \mathbb{N}$.*

Proof. Put $s_g = \sum\limits_{x \in E_g} u_x$ and let G_0 denote a set of representatives of the distinct E_g. Now let $g \in G_0$. Clearly, $\varepsilon s_g \varepsilon = \sum\limits_{i=1}^{\ell} n_i |Hx_iH|^{-1} \underline{Hx_iH}$ with $n_i \in \mathbb{N}$, where $x_i \in E_g$ are representatives of the distinct HxH, $x \in E_g$, and $\underline{Hx_iH} = \sum\limits_{y \in Hx_iH} u_y$. Note that $n_1 + \ldots + n_\ell = |E_g|$. So there is some n_i such that $n_i 1_R \neq 0$, because $|E_g| 1_R \neq 0$. Since $\varepsilon \in S$, we have also $\varepsilon s_g \varepsilon = \sum\limits_{t \in G_0} r_t s_t$ with $r_t \in R$ ($r_t = m_t |H|^{-2} 1_R$ with $m_t \in \mathbb{N}$).

Comparing these expressions for $\varepsilon s_g \varepsilon$, we obtain $n_i |Hx_iH|^{-1} 1_R = r_g$ for $i = 1, \ldots, \ell$, whence also $r_t = r_g$ or $r_t = 0$. Moreover $r_g \neq 0$. Consequently, $\varepsilon s_g \varepsilon = r_g \underline{HE_gH}$ with $\underline{HE_gH} = \sum_{y \in HE_gH} u_y$. We also deduce that $\varepsilon s_g \varepsilon = r_g \sum_k s_k$ for some $k \in G_0$, and we conclude that $\underline{HE_gH} = \sum_k s_k$. Therefore $\underline{HE_gH} \in S \cap \varepsilon RG\varepsilon$, and this intersection is equal to $\varepsilon S \varepsilon$.

Next, the above discussion shows that for each $g \in G_0$, $HE_gH = \bigcup_k E_k$ for some $k \in G_0$. Using this, it is easily seen that sets of the form HE_gH coincide or are disjoint. Moreover $(HE_gH)^{-1} = HE_{g^{-1}}H$.

Finally, since $n_1 + \ldots + n_\ell = |E_g|$ and $n_i|Hx_iH|^{-1}1_R = r_g$, we have $|HE_gH|r_g = |E_g|1_R$, completing the proof. $\square$

1.2.8 Remarks (1) Proposition 1.2.7 remains valid if we replace the condition $|E_g|1_R \neq 0$ by the following condition : for any $g, h \in G$, $E_gE_h = \bigcup_\ell E_\ell$ for some $\ell \in G$. In this case, it follows at once from the hypotheses that $HE_gH = \bigcup_k E_k$ for some $k \in G$.

(2) If the partition $\{E_g\}$ associated to S satisfies the property $(*)$ of Remark 1.2.2, then so does the partition $\{HE_gH\}$ associated to $\varepsilon S \varepsilon$.

(3) The case where ε is in the center of S is discussed in 2.4.

In chapter 3, we study these algebras in a more general context. Namely, let A be an R-algebra and ε a nonzero idempotent of A. We shall be concerned with the algebra $\varepsilon A \varepsilon$, which is called a *Hecke algebra* in A.

1.2.9 The Schur algebra of Schur and Wielandt We give a description of the Schur algebra in [Wie_2-Chapter IV]. Let H be a subgroup of a finite group G. Let $\sigma : G \to S(H)$ be a homomorphism of groups and suppose that $\sigma_k(h) = kh$ for all $k, h \in H$. Put $L = Stab(e)$.

Take $g \in G$ and $h \in H$, then $\sigma_h(e) = h$ and $\sigma_g(h) = \sigma_g(\sigma_h(e)) = \sigma_{gh}(e)$. Thus for any $x \in H$, $x = \sigma_x(e)$ and $x = \sigma_g(h)$ if and only if $ghL = xL(*)$.

Consider the restriction $\sigma : L \to S(H)$; let E_h denote the orbit of this action. Note that $E_e = \{e\}$. Clearly, for $x \in H$, $x \in E_h$ if and only if $LxL = LhL$, use $(*)$. So $E_h = LhL \cap H$. Consequently, $E_{h^{-1}} = Lh^{-1}L \cap H = (LhL)^{-1} \cap H = (E_h)^{-1}$. We also observe that $LgL = L\sigma_g(e)L$ for any $g \in G$. Let H_0 denote a set of representatives of the distinct E_h. Put $s_h = \sum_{x\in E_h} u_x$ in RH with $R = \mathbb{C}$. Then $S = \bigoplus_{h\in H_0} Rs_h$ equals the centralizer in RH of the element $\sum_{\ell\in L} u_\ell$, see [Wie_2, Theorem 24.6]. It follows that S is a Schur algebra in RH.

Put $\varepsilon = |L|^{-1}\sum_{\ell\in L} u_\ell$. Then it is easily verified that $\psi : S \to \varepsilon RG\varepsilon : s \to \varepsilon s\varepsilon = s\varepsilon$ is an isomorphism of R-algebras.

1.2.10 Fixed rings of automorphism groups Let G and H be finite groups and let $\sigma : H \to \mathrm{Aut}(G)$ be a homomorphism of groups. The orbits $E_g = \{\sigma_h(g) \mid h \in H\}$, $g \in G$, form a partition of G; $E_g^{-1} = E_{g^{-1}}$ and $E_e = \{e\}$. Observe that this partition satisfies property $(*)$ of Remark 1.2.2.

Each σ_h extends to an R-algebra isomorphism of RG (again denoted by σ_h) as follows : $\sigma_h(\sum_g r_g u_g) = \sum_g r_g u_{\sigma_h(g)}$, with $g \in G$ and $r_g \in R$. Furthermore, $\sigma : H \to \mathrm{Aut}_R(RG) : h \mapsto \sigma_h$ is a homomorphism of groups.

Consider the fixed ring $RG^H = \{a \in RG \mid \forall h \in H : \sigma_h(a) = a\}$; we have :

1.2.11 Lemma. *Keep the above notation, put $s_g = \sum_{x\in E_g} u_x$ in RG, and let G_0 denote a set of representatives of the distinct E_g. Then $RG^H = \bigoplus_{g\in G_0} Rs_g$, i.e. RG^H is a Schur algebra in RG.*

Proof. Clearly $s_g \in RG^H$. Conversely, let $\sum_{g\in G} r_g u_g \in RG^H$, $r_g \in R$. Then for each $h \in H$ we have $\sum_{g\in G} r_g u_g = \sum_{g\in G} r_g u_{\sigma_h(g)}$, whence $r_{\sigma_h(g)} = r_g$ (for nonzero r_g). The result follows at once. $\square$

In chapter 4 we focus on fixed rings of automorphism groups for arbitrary R-algebras A.

1.2.12 Special case Let H be any subgroup of G. Then $\sigma : H \to Aut\ G : h \to \sigma_h$, with $\sigma_h(g) = hgh^{-1}$ for all $g \in G$ is a homomorphism of groups. The orbits of this action are called subclasses of H in G and the fixed ring $S = RG^H$ is called the subclass algebra of H in RG. This algebra has been studied when $R = \mathbb{C}$, see [K], [Tr]. It is clear that S is also the centralizer of RH in RG. This allows us to study the subclass algebra in the general context of centralizers, see chapter 5. Moreover, by using the subclass algebra S of H in RG, we shall develop a generalized Clifford theory.

1.2.13 Remark An example of a Schur algebra for which property $(*)$ of Remark 1.2.2 does not hold is given in [A-vd.B-v.O, example 2.7]. Namely, take $R = \mathbb{Z}_2$, $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and consider the partition

$$\begin{aligned} E_{(0,0)} &= \{(0,0),(0,1),(0,2),(1,0),(2,0)\}, \\ E_{(1,1)} &= \{(1,1),(1,2),(2,1),(2,2)\}. \end{aligned}$$

It is easily verified that $S = Rs_{(0,0)} \oplus Rs_{(1,1)}$ is a subalgebra of RG and $1_S = s_{(0,0)}$. Note that $E_{(0,0)}$ is not a subgroup of G. As a consequence, property $(*)$ does not hold, use 1.2.3 (1).

1.2.14 Note Let S be a Schur algebra in RG with associated partition $\{E_g; g \in G\}$. Assume R is connected, $|G|^{-1} \in R$, and consider the idempotent $\varepsilon = |G|^{-1} \sum\limits_{g \in G} u_g$. Clearly $\varepsilon \in S$ and $s_g\varepsilon = |E_g|\varepsilon = \varepsilon s_g$, with $s_g = \sum\limits_{x \in E_g} u_x$. Now $S\varepsilon = R\varepsilon$ is an indecomposable left S-module, and thus ε is a primitive idempotent of S. Moreover ε is an element of the center of S. Furthermore, $t_{S\varepsilon}(s_g) = |E_g| 1_R$. Of course, the above holds for $S = RG$.

Chapter 2

Character theory for Schur algebras

In this chapter we focus on the character theory for Schur algebras. However, we shall discuss this problem in the more general context of Frobenius algebras. In the first section, we collect some results on Frobenius algebras over rings. In the second section we study the symmetric functions on Frobenius algebras and we develop orthogonality relations. In the case of Schur algebras, we introduce class functions and we analyze when the center of a Schur algebra is a Schur algebra (see section 3).

To end, we calculate (under suitable conditions) the trace function of an induced module between two Schur algebras.

2.1 Frobenius algebras

Throughout, R is a commutative ring and A is a faithful R-algebra which is a finitely generated free R-module. Let $Z(A)$ denote the center of A. Recall that $A^* = \mathrm{Hom}_R(A, R)$ is a left A-module under the operation : $(a.f)(x) = f(xa)$ for $a, x \in A$, $f \in A^*$.

2.1.1 Remarks. An R-bilinear form on A is called associative if $b(xy, v) = b(x, yv)$ for all $x, y, v \in A$. As is well known, there is a one-to-one correspondence between associative R-bilinear forms $b : A \times A \to R$ and (left) A-linear maps $\beta : A \to A^*$, given by $b(x, y) = \beta(y)(x)$, $x, y \in A$.

On the other hand, an A-linear map $\beta : A \to A^*$ is completely determined by $\beta(1) = \tau$, and the above correspondence yields $b(x, y) = \tau(xy)$, $x, y \in A$.

2.1.2 Lemma. *Let b be an associative R-bilinear form on A, let $\beta : A \to A^*$ be the corresponding left A-linear map and $\tau = \beta(1)$. The following statements are equivalent :*

(1) There are R-bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$ in A such that $b(a_i, b_j)$ form an invertible matrix.

(2) For each R-basis $\{a_1, \ldots, a_n\}$ of A there exists an R-basis $\{b_1, \ldots, b_n\}$ of A with $b(a_i, b_j) = \delta_{ij}$.

(3) β is an isomorphism.

(4) For every $f \in A^*$ there is a unique $a \in A$ such that $f = a.\tau$.

Proof. (3) $\Rightarrow$ (2) : Let $\{\varphi_1, \ldots, \varphi_n\} \subset A^*$ be the dual basis of $\{a_1, \ldots, a_n\}$. If $\beta : A \to A^*$ is an isomorphism, then there is an R-basis $\{b_1, \ldots, b_n\}$ in A such that $\beta(b_j) = \varphi_j$. So $b(a_i, b_j) = \beta(b_j)(a_i) = \delta_{ij}$

(2) $\Rightarrow$ (1) : This is obvious.

(1) $\Rightarrow$ (3) : Again let $\{\varphi_k\}$ be the dual basis of $\{a_k\}$. Since $(b(a_i, b_j))_{ij}$ is the matrix of β with respect to the bases $\{b_k\}$ and $\{\varphi_k\}$, it follows that β is bijective.

(3) $\Leftrightarrow$ (4) : This is obvious. $\square$

Definition : A bilinear form satisfying property (2) in Lemma 2.1.2 is said to be nonsingular, and $\{a_k\}$, $\{b_k\}$ in (2) are called dual bases with respect to b. The R-algebra A is called a *Frobenius algebra* if there exists a nonsingular associative R-bilinear form on A.

Remarks. (1) Of course A^* is also a right A-module and a one-to-one correspondence between associative R-bilinear forms b on A and right A-linear maps $\beta' : A \to A^*$ is given by $b(x, y) = \beta'(x)(y)$, $x, y \in A$. The analogue of Lemma 2.1.2 holds.

(2) A nonsingular bilinear form is nondegenerate. When R is a field, the converse is true.

Furthermore we have :

2.1.3 Lemma. *Let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, and let $\beta : A \to A^*$ be the corresponding left A-linear map. Then $\beta^{-1} : A^* \to A$ is given by $\beta^{-1}(f) = \sum\limits_{i=1}^{n} f(a_i)b_i$.*

Proof. We have $\beta(\sum\limits_i f(a_i)b_i)(a_j) = b(a_j, \sum\limits_i f(a_i)b_i) = f(a_j)$. $\square$

Recall that $f \in A^*$ is said to be symmetric if $f(xy) = f(yx)$ for all $x, y \in A$. The set of all symmetric functions $f \in A^*$ will be denoted by $\mathrm{Sym}(A, R)$. The A-module structure on A^* makes $\mathrm{Sym}(A, R)$ into a $Z(A)$-module, where $Z(A)$ denotes the center of A.
Furthermore, we say that A is a *symmetric Frobenius algebra* if there exists a nonsingular associative R-bilinear form on A which is symmetric.

2.1.4 Proposition. *Let b be a nonsingular symmetric associative R-bilinear form on A, and let $\beta : A \to A^*$ be the corresponding left A-linear map. Then β induces an isomorphism of $Z(A)$-modules between $Z(A)$ and $Sym(A, R)$.*

Proof. Let $\tau = \beta(1)$; τ is symmetric. Obviously, if $a \in Z(A)$, then $\beta(a) = a.\tau$ is symmetric. Now let $f \in \mathrm{Sym}(A, R)$, hence $f = a.\tau$ for some $a \in A$. From $f(yx) = f(xy)$ it follows that $\tau(yxa) = \tau(xya) = \tau(yax)$, for all $x, y \in A$. Therefore $xa.\tau = ax.\tau$, whence $xa = ax$, for all $x \in A$. $\square$

The following lemma gives the relation between two bilinear forms, one of which is nonsingular.

2.1.5 Lemma. *Let b and b' be associative bilinear R-forms on A and suppose that b is nonsingular, then :*
(1) There is a unique $u \in A$ such that $b'(x, y) = b(x, yu)$ for all $x, y \in A$.
(2) b' is nonsingular if and only if u is invertible in A.
In this case : if $\{a_k\}$, $\{b_k\}$ are dual bases in A with respect to b, then $\{a_k\}$, $\{b_k u^{-1}\}$

are dual bases with respect to b'.

(3) If b is symmetric, then b' is symmetric if and only if u is a central element of A.

Proof. Let β resp. β' denote the left A-linear maps from A to A^* associated to b resp. b'.

(1) Since b is nonsingular there is a unique $u \in A$ such that $\beta'(1) = u \cdot \beta(1)$ see Lemma 2.1.2. By remark 2.1.1, we get the assertion in (1).

(2) If b' is nonsingular, then there is also a unique $u \in A$ such that $\beta(1) = u \cdot \beta'(1)$. So $\beta(1) = vu \cdot \beta(1)$, whence $vu = 1$. Similarly we get $uv = 1$. Conversely, suppose u is invertible in A, then $\beta'(1)$ is also a free generator of A^* viewed as a left A-module. By Lemma 2.1.2, we then conclude that b' is nonsingular.

(3) As $\beta'(1) = u \cdot \beta(1)$, the result follows from Proposition 2.1.4.

2.1.6 Examples. 1. Let G be a finite group and consider the twisted group ring $R *_\alpha G$ with R-basis $\{u_g | g \in G\}$. Consider the R-linear map $\tau : R *_\alpha G \to R : \sum_{g \in G} r_g u_g \mapsto r_e$. It is clear that τ defines a symmetric associative R-bilinear form on $R *_\alpha G$ with dual bases $\{u_g |\ g \in G\}$, $\{\alpha(g, g^{-1})^{-1} u_{g^{-1}} |\ g \in G\}$.

2. Let G be a finite group, let $\{E_g; g \in G\}$ be a partition of G such that $E_g^{-1} = E_{g^{-1}}$, and let G_0 denote a set of representatives of the distinct E_g. Put $s_g = \sum_{x \in E_g} u_x$ in RG and suppose that $S = \bigoplus_{g \in G_0} R s_g$ is a Schur algebra in RG.
Now consider $\tau : S \to R : \sum_{g \in G_0} r_g s_g \to r_e$. If each $|E_g|$ is invertible in R, then τ defines a symmetric associative R-bilinear form on S with dual bases $\{s_g |\ g \in G_0\}$, $\{|E_g|^{-1} s_{g^{-1}} |\ g \in G_0\}$.

3. Let $\mathcal{M} = M_{n_1}(R) \oplus \ldots \oplus M_{n_q}(R)$ be a direct sum of matrix algebras. We set $E_{ij}^{(k)} = (0, \ldots, 0, E_{ij}, 0, \ldots, 0) \in \mathcal{M}$ with E_{ij} at the k-th place, and the matrix E_{ij} has ij-entry equal to 1 and zeros elsewhere.
Consider the R-linear map $tr : \mathcal{M} \to R : (B_1, \ldots, B_q) \mapsto \sum_{i=1}^{q}$ trace (B_i). It is clear that tr defines a symmetric associative R-bilinear form on $\mathcal{M}$ with dual bases $\{E_{ij}^{(k)}\}$ and $\{E_{ji}^{(k)}\}$.

4. If A is a finite dimensional semisimple R-algebra, R being a field, then A is a symmetric Frobenius R-algebra, see [C-R, Proposition 9.8].

Let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$. We put $z = z_b = \sum_{i=1}^{n} a_i b_i$; this element has the following properties :

2.1.7 Lemma. *(1) Let t_A denote the trace function from A to R afforded by A viewed as left A-module. Then $t_A(x) = b(x, z)$ for all $x \in A$.*
(2) z is independent of the choice of the dual bases for b.
(3) If b is symmetric, then z is central and in this case $z = \sum_{i=1}^{n} b_i a_i$.

Proof. Let β be associated to b as in 2.1.1.
(1) The R-dual basis in A^* of $\{a_k\}$ is given by $\{\beta(b_k)\}$. Let $x \in A$, then $t_A(x) = \sum_{i=1}^{n} \beta(b_i)(xa_i) = \sum_{i=1}^{n} b(xa_i, b_i) = \sum_{i=1}^{n} b(x, a_i b_i) = b(x, z)$.
(2) The statement $t_A(x) = b(x, z)$ for all $x \in A$ is equivalent to $t_A = z \cdot \beta(1)$, and (2) follows.
(3) Follows from (1), (2) and Proposition 2.1.4. □

2.1.8 Remark. (1) Let b' be another nonsingular associative R-bilinear form on A. Then by Lemma 2.1.5(2), there is an invertible element $u \in A$ such that $z_b = z_{b'} u$.
(2) Since $t_A = z \cdot \beta(1)$, we have that t_A is a free generator of A^* viewed as left A-module if and only if z is invertible in A, see Lemma 2.1.5(2).

Keep the above notation. We now introduce the $Z(A)$-linear map $\zeta : A \to A : x \to \sum_{i=1}^{n} b_i x a_i$. We prove :

2.1.9 Proposition. *(1) $\zeta(x)$ is independent of the choice of the dual bases and $\zeta(A)$ is independent of the choice of the nonsingular associative bilinear form.*
(2) $\zeta(A)$ is an ideal of $Z(A)$, the center of A.
(3) If b is symmetric, then $\zeta(xy) = \zeta(yx)$ for all $x, y \in A$ and $\zeta(1) = z$.

Proof. (1) Let $\{a'_i\}$, $\{b'_i\}$ be another pair of dual bases with respect to b. If C and D are the matrices expressing $\{a'_i\}$ in terms of $\{a_i\}$ and $\{b'_i\}$ in terms of $\{b_i\}$ respectively, then $C^t D = I_n$. Thus also $DC^t = I_n$ and this yields $\sum_{i=1}^{n} b'_i x a'_i = \sum_{i=1}^{n} b_i x a_i$

for all $x \in A$.

Finally, from Lemma 2.1.5(2) it follows that $\zeta(A)$ is independent of the choice of the bilinear form b.

(2) For each $y \in A$, we have :

$$a_i y = \sum_{j=1}^{n} r_{ij} a_j \text{ implies } y b_i = \sum_{j=1}^{n} r_{ji} b_j,\ r_{ij} \in R \qquad (*)$$

Using these relations, we see that $\zeta(A)$ is contained in the center of A. It is also clear that $\zeta(A)$ is an ideal of the center.

(3) As b is symmetric, $\{b_i\}$, $\{a_i\}$ are dual bases with respect to b, i.e. $b(b_i, a_j) = \delta_{ij}$. Then by (1), $\zeta(xy) = \sum\limits_{i=1}^{n} a_i(xy)b_i$.

Using the relations $(*)$ in (2), we obtain $\zeta(xy) = \zeta(yx)$. □

In the last part of this section we give a necessary and sufficient condition for a Frobenius R-algebra to be separable over R and we investigate the invertibility of z.

2.1.10 Proposition. *If A is a Frobenius R-algebra such that $1 \in \zeta(A)$, then A is a separable R-algebra.*

Proof. Keep the above notation. By our assumption, there is an element $c \in A$ such that $\sum\limits_{i=1}^{n} b_i c a_i = 1$. Combining this relation with the relations $(*)$ in the proof of Proposition 2.1.9(2), we see that $\sum\limits_{i=1}^{n} b_i c \otimes a_i \in A \otimes_R A^\circ$ is a separability idempotent for A, cf. [DM-I,p40]. □

2.1.11 Remark. Let A be a symmetric Frobenius R-algebra. If b is symmetric and $z = \sum\limits_{i=1}^{n} a_i b_i$ is invertible in A, then $\zeta(z^{-1}) = 1$, whence A is separable over R.

In order to prove the converse of 2.1.10 we have to investigate the symmetric Frobenius algebra in 2.1.6(3).

Moreover, in this case there is a criterion for the invertibility of the element z.

2.1.12 Lemma. *Let $A \cong M_{n_1}(R) \oplus \ldots \oplus M_{n_q}(R)$, an R-algebras. Let b be any non-singular associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$*

in A. Then :

(1) The center of A coincides with $\zeta(A) = \{\sum_{i=1}^{n} b_i x a_i | x \in A\}$.

(2) $z_b = \sum_{i=1}^{n} a_i b_i$ is invertible in A if and only if $n_1, \ldots, n_q$ are invertible in R.

Proof. Put $\mathcal{M} = M_{n_1}(R) \oplus \ldots \oplus M_{n_q}(R)$ and, as in 2.1.6(3) consider the map tr. If we set $c = \sum_{k=1}^{q} E_{11}^{(k)}$, then we get $\sum_k \sum_{i,j} E_{ji}^{(k)} c E_{ij}^{(k)} = 1$. Furthermore, we have $z_{tr} = \sum_k \sum_{i,j} E_{i,j}^{(k)} E_{ji}^{(k)} = (n_1 I_{n_1}, \ldots, n_q I_{n_q})$.

Note also that b induces a nonsingular associative R-bilinear form $\tilde{b}$ on $\mathcal{M}$. We now prove the statements.

(1) This follows from Proposition 2.1.9(1).

(2) According to Remark 2.1.8(1), we can find on invertible element $u \in \mathcal{M}$ such that $z_{\tilde{b}} = z_{tr} u$. So $z_{\tilde{b}}$ is invertible if and only if z_{tr} is invertible in $\mathcal{M}$ and the assertion follows. □

Remark. In the special case $\mathcal{M} = M_3(\mathbb{Z}_3)$ we have $z = 0$. Thus $1 \in \varphi(A)$ or separability doesn't imply the invertibility of z.

We are now in a position to prove the converse of 2.1.10, more precisely :

2.1.13 Proposition. *Let A be a Frobenius R-algebra which is separable over R. Let b be an associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$. Then :*

(1) $\zeta(A)$ is equal to the center $Z(A)$ of A.

(2) If R is a field of characteristic zero, then $z = \sum_{i=1}^{n} a_i b_i$ invertible in A.

Proof. Recall that $\zeta(a) = \sum_{i=1}^{n} b_i a a_i$ for any $a \in A$.

(1) Step 1. Suppose that $R = K$ is a field. Then the algebraic closure $\overline{K}$ of K is a splitting field for $\overline{K} \otimes_K A$. Obviously, the form b can be extended to an associative $\overline{K}$-bilinear form $\overline{b}$ on $\overline{K} \otimes_K A$ with dual $\overline{K}$-bases $\{1 \otimes a_i\}$, $\{1 \otimes b_i\}$. By Lemma 2.1.12 there is an element $x \in \overline{K} \otimes_K A$ such that $\sum_{i=1}^{n} (1 \otimes b_i) x (1 \otimes a_i) = 1$. This gives a system of n linear equations with coefficients in K, having a solution in $\overline{K}^n$. But

then these equations must have a solution in K^n and therefore $1 \in \zeta(A)$.

Step 2. Let now R be an arbitrary commutative ring. First note that the separability of A implies that $Z(A)$ is a direct summand of A as R-module, see [DM-I.p.51 and p.55]. Hence $Z(A)$ is finitely generated as R-module, and thus $Z(A)$ is integral over R.

We now suppose that $1 \notin \zeta(A)$. Then the ideal $\zeta(A)$ is contained in some maximal ideal M of $Z(A)$. Since $Z(A)$ is integral over R, $m = M \cap R$ is a maximal ideal of R. [Co p.424]. Now, A/mA is a separable R/m-algebra. For $a \in A$, we set $\overline{a} = a + mA$. The form b defines an associative R/m-bilinear form $\widetilde{b}$, on A/mA as follows : $\widetilde{b}(\overline{x}, \overline{y}) = b(x,y) + m$ for all $x, y \in A$. Clearly $\{\overline{a}_i\}$, $\{\overline{b}_i\}$ are dual R/m-bases with respect to $\widetilde{b}$. By the first part of the proof, there is an element $x \in A$ such that $1 - \sum_{i=1}^{n} b_i x a_i \in mA$, whence $1 \in AM$. But $AM \cap Z(A) = M$, since A is separable. Consequently, $1 \in M$, a contradiction, and thus $1 \in \zeta(A)$.

(2) As in (1), reduce to the case of an algebraically closed field and apply Lemma 2.1.12(2). □

Remarks (1) Keep the hypotheses of 2.1.13 and suppose that A is commutative. Then z is invertible in A.

(2) Consider the twisted group ring $R *_\alpha G$ of 2.1.6(1). In this case $z = |G| u_e$, and it is easily seen that z is invertible if and only if $1 \in \zeta(R *_\alpha G)$. So we recover that $R *_\alpha G$ is separable over R if and only if $|G| 1_R$ is invertible in R.

2.1.14 Corollary. Let S be a Schur algebra in RG with associated partition $\{E_g\}$, $g \in G_0$. Assume $|G|^{-1} \in R$ and $|E_g|^{-1} \in R$, then $Z(S) = \{ \sum_{g \in G_0} |E_g|^{-1} s_g s s_{g^{-1}} | s \in S\}$.

Proof. By the hypotheses, S is separable over R, use [A-vdB-v.O, Proposition 4.1]. Now use 2.1.13(1).

To conclude, we give a criterion for the invertibility of z in terms of separability. Again let b be a nonsingular associative R-bilinear form on A and let z, ζ be as before.

We shall need the $Z(A)$-module $\ker\zeta$. Clearly $\ker\zeta$ is independent of the choice of the dual for b and, in case b is symmetric, kerζ is also independent of the choice of the nonsingular symmetric form, see 2.1.5.

2.1.15 Proposition. *Keep the above notation and assumptions and suppose that b is symmetric. Then the following statements are equivalent :*

(1) z is invertible in A.

(2) A is separable over R and $A = ker\zeta \oplus Z(A)$.

Proof. Note that $\zeta(c) = zc$ for all $c \in Z(A)$.

$(1) \Rightarrow (2)$: Clearly $\zeta(z^{-1}) = 1$, hence $\zeta(A) = Z(A)$ and A is separable over R, see 2.1.10. For each $x \in A$, we write $x = (x - \zeta(z^{-1}x)) + \zeta(z^{-1}x)$, and then it is easily checked that $A = \ker\zeta \oplus \zeta(A)$.

$(2) \Rightarrow (1)$: By the separability, we have $1 = \zeta(x)$ for some $x \in A$. There exist elements $y_1 \in \ker\zeta$, $y_2 \in Z(A)$ such that $x = y_1 + y_2$. Thus $1 = \zeta(y_2) = zy_2$. □

2.2 Symmetric functions on Frobenius algebras-orthogonality relations

Let R, A and $Z(A)$ be as in section 1, and write $Sym(A, R)$ for the set of all symmetric functions $f \in A^*$. If A is a symmetric Frobenius R-algebra, then there is an isomorphism of $Z(A)$-modules between $Z(A)$ and $Sym(A, R)$, see 2.1.4.

Now, we show that, under certain conditions, symmetric functions are determined by their values on the center. Again let A be a Frobenius algebra, let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_i\}$, $\{b_i\}$, and let ζ, z be as in section 1.

2.2.1 Proposition. *Assume that b is symmetric and that z is invertible in A. Given $f \in A^*$, the following conditions are equivalent :*

(1) $f \in Sym(A, R)$.

(2) $f(x) = f(\zeta(z^{-1}x))$ for all $x \in A$.

(3) $ker\zeta \subset ker f$.

Proof. (1) $\Rightarrow$ (2) : We have $f(\zeta(z^{-1}x)) = f(\sum_i b_i z^{-1} x a_i) = f(\sum_i a_i b_i z^{-1} x) = f(x)$.
(2) $\Rightarrow$ (3) : Note that $\zeta(z^{-1}x) = z^{-1}\zeta(x)$.
(3) $\Rightarrow$ (1) : For all $x, y \in A$, we have $\zeta(xy) = \zeta(yx)$, hence $xy - yx \in \ker\zeta \subset \ker f$. $\square$

2.2.2 Proposition. *Let b, ζ, z be as before and assume that b is symmetric. Then $\bigcap_f \ker f \subset \ker\zeta$, where f ranges over all elements of Sym(A, R). If z is invertible in A, then we get an equality.*

Proof. Let $\tau \in A^*$ be associated to b as in 2.1.1. Let $x \in A$ be such that $f(x) = 0$ for all $f \in \mathrm{Sym}(A, R)$. Then by Proposition 2.1.4, $\tau(xc) = 0$ for all $c \in Z(A)$. For each $y \in A$, we now have $\tau(y\zeta(x)) = \tau(\sum_i y b_i x a_i) = \tau(\sum_i a_i y b_i x) = \tau(\zeta(y)x) = 0$ using Proposition 2.1.9. Thus $\zeta(x).\tau = 0$, whence $\zeta(x) = 0$.
In case z is invertible, we may apply Proposition 2.2.1 and we obtain an equality. $\square$

For trace functions the result of 2.2.1(2) can be put into another form.

2.2.3 Proposition. *Let b, ζ, z be as before and assume that b is symmetric. Suppose R is connected and R is a splitting ring for $Z(A)$. Further, let $\{e_1, \ldots, e_q\}$ be the set of primitive central idempotents of A. Let now M be a left A-module, which is finitely generated projective over R, and assume that $e_k M = 0$ for $k \neq j$. Then we have for all $x \in A$:*
(1) $t_M(x)ze_j = rank_R(M)\zeta(x)e_j = t_M(\zeta(x))e_j$
(2) $t_M(z)t_M(x) = rank_R(M)t_M(\zeta(x))$

Proof. (1) By hypothesis, $Z(A) = Re_1 \oplus \ldots \oplus Re_q$. We may write $z = \sum_{i=1}^{q} \lambda_i e_i$ and $\zeta(x) = \sum_{i=1}^{q} \mu_i e_i$. Clearly, $t_M(\zeta(x)) = rank_R(M)\mu_j$. On the other hand, $t_M(\zeta(x)) = t_M(\sum_{i=1}^{n} b_i x a_i) = t_M(zx) = \lambda_j t_M(x)$. The formula then follows.
(2) Apply t_M to the formula in (1). $\square$

We now show that, under certain conditions, $\mathrm{Sym}(A, R)$ has an R-basis consisting of characters and we derive orthogonality relations for characters. Again let

A be a Frobenius R-algebra, let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, and put $z = \sum_{i=1}^{n} a_i b_i$. Moreover we assume that b is symmetric, although some results can be proved without this assumption (see 2.2.7(2)).

Further, suppose that R is connected and let $\{e_1, \ldots, e_q\}$ be the set of primitive central nonzero idempotents of A. Let now $M_1, \ldots, M_q$ be nonzero left A-modules which are finitely generated and projective over R, and assume that $e_k M_i = 0$ for $k \neq i$. Note that an indecomposable A-module P lies over exactly one e_i. Finally, we let rank stand for rank_R, and we recall that t_{M_i} denotes the trace function from A to R afforded by M_i.

2.2.4 Theorem. *Keep the above hypotheses and notation.*

(1) If R is a splitting ring for the center, that is, $Z(A) = Re_1 \oplus \ldots \oplus Re_q$, then

$$\begin{aligned} rank(M_j)e_j &= b(e_j, e_j) \sum_{i=1}^{n} t_{M_j}(a_i) b_i \\ t_{M_j}(z) e_j &= rank(Ae_j) \sum_{i=1}^{n} t_{M_j}(a_i) b_i \end{aligned}$$

(2) For $j \neq k$ we have $\sum_{i=1}^{n} t_{M_j}(a_i) t_{M_k}(b_i) = 0$.

(3) Let L_j be any nonzero left A-module which is finitely generated projective over R and has the property that $e_k L_j = 0$ for $k \neq j$ (special case : $L_j = M_j$). If R is a splitting ring for $Z(A)$, then

$$b(e_j, e_j) \sum_{i=1}^{n} t_{M_j}(a_i) t_{L_j}(b_i) = rank(M_j) rank(L_j) 1_R.$$

(4) With assumptions as in (3) we have

$$rank(M_j) t_{L_j} = rank(L_j) t_{M_j}.$$

(5) If $rank(M_i)1_R \neq 0$ and $rank(M_i)1_R$ is not a zero divisor in R for $i = 1, \ldots, q$, then $t_{M_1}, \ldots, t_{M_q}$ are linearly independent over R.

(6) If R is a splitting ring for $Z(A)$ and $rank(M_i)1_R$ is invertible in R for $i = 1, \ldots, q$, then $t_{M_1}, \ldots, t_{M_q}$ form an R-basis of $Sym(A, R)$.

(7) We have

$$ze_j = \sum_{i=1}^{n} t_{Ae_j}(a_i) b_i.$$

(8) If R is a splitting ring for $Z(A)$ and z is invertible in A, then $t_{Ae_1}, \ldots, t_{Ae_q}$ form an R-basis of $Sym(A,R)$ (Ae_i viewed as left A-module).

Proof. Let $\tau \in A^*$ be associated to b as in 2.1.1. For each t_{M_j} there is a unique $c_j \in A$ such that $t_{M_j} = c_j.\tau$. By Lemma 2.1.3, $c_j = \sum_{i=1}^{n} t_{M_j}(a_i)b_i$.
Further, it is easily seen that $e_k.t_{M_j} = 0$ for $k \neq j$. Consequently $(e_kc_j).\tau = 0$, whence $e_kc_j = 0$ for $k \neq j$. Therefore $c_j \in Ae_j$.
(1) Since b is symmetric, $c_j \in Z(A)$, see Proposition 2.1.4. Thus $c_j = r_je_j$ with $r_j \in R$. We now have $t_{M_j}(1) = \tau(c_j) = r_j\tau(e_j)$ and $t_{M_j}(1) = \operatorname{rank}(M_j)1_R$. Then $\operatorname{rank}(M_j)e_j = \tau(e_j)c_j$ and we obtain the first formula.
Further, we know that $t_A = z.\tau$. Using the fact that $t_A = \sum_{i=1}^{q} t_{Ae_i}$ on A, it is easily seen that $t_{Ae_j} = e_j.t_A$ (we view A and Ae_i as left A-modules). We thus obtain $t_{Ae_j} = (e_jz).\tau$. Since b is symmetric, z is central and thus $z = \sum_{i=1}^{q} \lambda_ie_i$ with $\lambda_i \in R$. Therefore $t_{Ae_j} = (\lambda_je_j).\tau$. As a consequence, we have $\operatorname{rank}(Ae_j)1_R = \lambda_j\tau(e_j)$.
On the other hand, $t_{M_j}(z) = \operatorname{rank}(M_j)\lambda_j$. We now have $t_{M_j}(z)e_j = \operatorname{rank}(M_j)\lambda_je_j = \lambda_j\tau(e_j)c_j = \operatorname{rank}(Ae_j)c_j$ and this gives the second formula.
(2) Apply t_{M_k}, $k \neq j$, to the expression $c_j = \sum_{i=1}^{n} t_{M_j}(a_i)b_i$.
(3) Apply t_{L_j} to the first formula in (1).
(4) There is a unique $c'_j \in A$ such that $t_{L_j} = c'_j.\tau$, and $c'_j = r'_je_j$ with $r'_j \in R$. Moreover, $\operatorname{rank}(L_j)1_R = r'_j\tau(e_j)$.
Let c_j, r_j be as above. Then we have $c_j.t_{L_j} = c'_j.t_{M_j}$ and thus $r_jt_{L_j} = r'_jt_{M_j}$. Multiplying by $\tau(e_j)$, we obtain the formula in (4).
(5) Suppose that $\sum_{i=1}^{q} \mu_it_{M_i} = 0$ with $\mu_i \in R$. Then $\sum_i \mu_it_{M_i}(e_k) = 0$ for $k = 1, \ldots, q$. We get $\operatorname{rank}(M_k)\mu_k = 0$, whence $\mu_k = 0$ for $k = 1, \ldots, q$.
(6) As before, we have $t_{M_j} = (r_je_j).\tau$ with $r_j \in R$. The invertibility of $\operatorname{rank}(M_j)$ in R implies the invertibility of r_j in R, because $\operatorname{rank}(M_j)1_R = r_j\tau(e_j)$. Now, $e_1, \ldots, e_q$ form an R-basis of $Z(A)$, and thus also $r_1e_1, \ldots, r_qe_q$. By Proposition 2.1.4, it follows that $t_{M_1}, \ldots, t_{M_q}$ form an R-basis of $\operatorname{Sym}(A,R)$.
(7) As in the proof of (1), $t_{Ae_j} = (ze_j).\tau$. The assertion follows from Proposition 2.1.3.
(8) We have $z = \sum_{i=1}^{q} \lambda_ie_i$ with $\lambda_i \in R$ and $t_{Ae_j} = (\lambda_je_j).\tau$. Since z is invertible

in A, each λ_i is invertible in R. We now proceed as in (6) in order to show that $t_{Ae_1}, \ldots, t_{Ae_q}$ form an R-basis of $\mathrm{Sym}(A, R)$. $\square$

2.2.5 Remarks. Keep the hypotheses and notation of Theorem 2.2.4 and assume that R is a splitting ring for $Z(A)$.

1. From the proof of 2.2.4 we retain that $\mathrm{rank}(M_j)1_R = r_j b(e_j, e_j)$ with $r_j \in R$. Further, $z = \sum_{i=1}^{q} \lambda_i e_i$ with $\lambda_i \in R$ and $t_{Ae_j} = (\lambda_j e_j).\tau$, in particular $\mathrm{rank}(Ae_j)1_R = \lambda_j b(e_j, e_j)$.
2. If $b(e_i, e_i)$ is invertible in R for $i = 1, \ldots, q$, then $b : Z(A) \times Z(A) \to R$ is nonsingular. The converse also holds.

As before, let b be a nonsingular symmetric associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, and put $z = \sum_{i=1}^{n} a_i b_i$. Suppose that R is connected and let $\{e_1, \ldots, e_q\}$ be the set of primitive central nonzero idempotents of A. We now assume that $A \cong \mathrm{End}_R(P_1) \oplus \ldots \oplus \mathrm{End}_R(P_q)$ as R-algebras, $P_1, \ldots, P_q$ being finitely generated projective R-modules.
Observe that $Z(A) = Re_1 \oplus \ldots \oplus Re_q$. We recall that each P_i is an indecomposable left A-module under the operation $(\varphi_1, \ldots, \varphi_q).p = \varphi_i(p)$, $p \in P_i$ and $\varphi_j \in \mathrm{End}_R(P_j)$, and we may assume that P_i lies over e_i.
Further, from the proof of Lemma 1.1.4 we retain that $t_{Ae_j} = \mathrm{rank}(P_j) t_{P_j}$ on A, in particular $\mathrm{rank}(Ae_j) = (\mathrm{rank} P_j)^2$.
Clearly we may apply Theorem 2.2.4 to t_{P_i}. Moreover the following holds true.

2.2.6 Proposition. *Keep the above hypotheses and notation. Then*
(1) We have

$$ze_j = \sum_{i=1}^{n} rank(P_j) t_{P_j}(a_i) b_i$$

$$t_{P_j}(z) = \sum_{i=1}^{n} rank(P_j) t_{P_j}(a_i) t_{P_j}(b_i)$$

(2) z is invertible in A if and only if all $rank(P_j)1_R$ are invertible in R. Moreover, $rank(P_j)1_R$ is invertible in R if and only if $t_{P_j}(z)$ is invertible in R.

Proof. (1) We have $t_{Ae_j} = \mathrm{rank}(P_j) t_{P_j}$. The first formula now follows from Theorem

2.2.4(7). Applying t_{P_j}, we obtain the second formula.

(2) Let $\tau \in A^*$ be associated to b. There is a unique $c_j \in A$ such that $t_{P_j} = c_j.\tau$ and $c_j \in Ae_j$. Then $t_{Ae_j} = \text{rank}(P_j)t_{P_j} = \text{rank}(P_j)c_j.\tau$. On the other hand, we know that $t_{Ae_j} = (ze_j).\tau$, see 2.2.4. Therefore $ze_j = \text{rank}(P_j)c_j$ and thus $z = (\sum_j \text{rank}(P_j)e_j)(\sum_j c_j)$. So the invertibility of z implies that all $\text{rank}(P_j)$ are invertible in R. To prove the converse, we write $z = \sum_{i=1}^{q} \lambda_i e_i$ with $\lambda_i \in R$ and we observe that $(\text{rank}P_j)^2 1_R = \text{rank}(Ae_j)1_R = \lambda_j b(e_j, e_j)$, see 2.2.5.

The last statement follows from $t_{P_j}(z) = \text{rank}(P_j)\lambda_j$ and the preceding formula. □

2.2.7 Remarks. (1) The case where R is a field of characteristic 0 and A is split separable over R was already treated in [*C-R*, Theorem 9.17]

(2) We do not need the assumption that the nonsingular associative R-bilinear form b is symmetric in the proofs of 2.2.4(2), (5), (7) and 2.2.6(1) and in the first part of the proof of 2.2.6(2) (the invertibility of z implies that all $rank(P_j)$ are invertible in R).

(3) Using 2.1.8(1), the result in 2.2.6(2) can be sharpened as follows. Suppose A is a symmetric Frobenius R-algebra but the form b is not necessarily symmetric and suppose all $rank_R(P_i)$ are invertible in R, then z_b is invertible in A. Compare with 2.1.12.

(4) From the proof of 2.2.6(2) we may deduce the following result. If $zx = 0$ implies $x = 0$ for all $x \in Z(A)$, then, for each i, $rank_R(P_i)1_R \neq 0$ and $rank_R(P_i)1_R$ is not a zero divisor in R. For a symmetric form b, the converse holds and the above property for $rank_R(P_i)1_R$ is equivalent to the analogous property for $tp_j(z)$.

(5) Keep the hypotheses of 2.2.6 and assume that z is invertible in A. Combining 2.2.3(1) and 2.2.6, $\zeta(x)e_j = t_{p_j}(x) \sum_{i=1}^{n} t_{p_j}(a_i)b_i$ for all $x \in A$.

Recall that a Schur algebra in RG (with associated partition $\{E_g; g \in G\}$) is a symmetric Frobenius R-algebra, whenever $|E_g|$ is invertible in R for all $g \in G$ (see 2.1.6(2)).

So we may apply 2.2.4 and 2.2.6. For (twisted) group rings we have :

2.2.8 Corollary *Let R be connected and let G be a finite group with $|G|^{-1} \in R$. Suppose that $R *_\alpha G \cong End_R(P_1) \oplus \ldots \oplus End_R(P_q)$ as R-algebra, $P_1, \ldots, P_q$ being finitely generated projective R-modulus. Let $\{e_1, \ldots, e_q\}$ be the set of primitive central nonzero idempotents of $R *_\alpha G$ are assume P_i lies over e_i. Then :*

(1) All $rank_R(P_i)$ are invertible in R.

(2) $\sum_{g\in G} \frac{1}{\alpha(g,g^{-1})} t_{P_j}(u_g) t_{P_k}(u_{g^{-1}}) = \delta_{jk}|G|\alpha(e,e)$

(3) $e_j = \frac{1}{|G|\alpha(e,e)} rank_R P_j \sum_{g\in G} \frac{1}{\alpha(g,g^{-1})} t_{P_j}(u_{g^{-1}}) u_g$

(4) $t_{P_1}, \ldots, t_{P_q}$ form an R-basis of $Sym(R *_\alpha G, R)$.

Proof. Put $A = R *_\alpha G$. As in example 2.1.6(1), we take the (symmetric) form associated to $\tau : A \to R : \sum_{g\in G} r_g u_g \to r_e$. In this case $\{u_g\}$, $\{\alpha(g,g^{-1})^{-1} u_{g^{-1}}\}$, $g \in G$, are dual bases and $z = |G|u_e = |G|\alpha(e,e)1_A$. Now apply 2.2.4 and 2.2.6.

□

2.2.9 Note. Let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, and let $\beta : A \to A^*$ be associated to b as in 2.1.1.

(1) Since β is bijective, β induces a ring structure on A^*. Explicitly, let $\varphi, \psi \in A^*$; $\varphi = \beta(s)$, $\psi = \beta(t)$. Then $\varphi \times \psi = \beta(st)$.

Now let $A = R *_\alpha G$ with bilinear form associated to $\tau : A \to R : \sum_{g\in G} r_g u_g \mapsto r_e$, as in example 2.1.6(1). By Lemma 2.1.3, we have

$$st = \sum_{k\in G} \varphi \times \psi(u_{k^{-1}})\ \alpha(k,k^{-1})^{-1} u_k.$$

On the other hand,

$$st = \sum_{g\in G}\sum_{h\in G} \varphi(u_{g^{-1}})\psi(u_{h^{-1}})\alpha(g,g^{-1})^{-1}\alpha(h,h^{-1})^{-1}\alpha(g,h)u_{gh}.$$

But

$$\begin{aligned} & \alpha(h,h^{-1})^{-1}\alpha(g,g^{-1})^{-1}\alpha(g,h)\alpha(gh,(gh)^{-1}) = \alpha(h,h^{-1})^{-1}\alpha(h,(gh)^{-1}) \\ = \ & \alpha(e,e)\alpha(h^{-1},g^{-1})^{-1}. \end{aligned}$$

Consequently,

$$st = \sum_{k\in G}\sum_{g\in G} \varphi(u_{g^{-1}})\psi(u_{k^{-1}g})\alpha(e,e)\alpha(k^{-1}g,g^{-1})^{-1}\alpha(k,k^{-1})^{-1}u_k.$$

So we obtain

$$\varphi \times \psi(u_k) = \sum_{g \in G} \varphi(u_{g^{-1}})\psi(u_{kg})\alpha(e,e)\alpha(kg,g^{-1})^{-1}.$$

(2) The map β also induces an R-bilinear form b^* on A^*. Explicitly, let $\varphi, \psi \in A^*$; $\varphi = \beta(s)$, $\psi = \beta(t)$. Then $b^*(\varphi, \psi) = b(s,t)$. Now let b be symmetric. Then we may write $s = \sum_i \varphi(b_i)a_i$ and $t = \sum_j \psi(a_j)b_j$. Consequently, $b^*(\varphi,\psi) = \sum_{i=1}^{n} \varphi(b_i)\psi(a_i)$. The formulas in 2.2.4(2)-(3) and in 2.2.6(1) can be rewritten using the R-bilinear form b^*. The basis in 2.2.4(6) and 2.2.4(8) is orthogonal relative to b^*.

(3) We also have the following multiplication on A^*. For $\varphi, \psi \in A^*$, define $\varphi * \psi(a_i) = \varphi(a_i)\psi(a_i)$ and extend by linearity. On the other hand, we may consider the following componentwise multiplication on A. Let $s, t \in A$, write $s = \sum_{i=1}^{n} r_i b_i$, $t = \sum_{i=1}^{n} r'_i b_i$ with $r_i, r'_i \in R$, and set $s * t = \sum_{i=1}^{n} r_i r'_i b_i$. Then $\beta(s * t) = \beta(s) * \beta(t)$, as is easily checked.

2.3 Class functions on Schur algebras

Throughout this section, R is a commutative ring, G is a finite group, and $\{E_g; g \in G\}$ is a partition of G such that $E_g^{-1} = E_{g^{-1}}$ and $|E_g|$ is invertible in R. Put $s_g = \sum_{x \in E_g} u_x$ in RG, $\widehat{s}_g = |E_g|^{-1} s_g$ and let G_0 denote a set of representatives of the distinct E_g. We assume that $S = \bigoplus_{g \in G_0} Rs_g$ is a subalgebra with unit element, i.e. S is a Schur algebra in RG. Note that $\widehat{s}_e = 1_S$, see 1.2.3(2).

Recall that $\tau : S \to R : \sum_{g \in G_0} r_g s_g \mapsto r_e$ defines a symmetric associative R-bilinear form b on S with dual bases $\{\widehat{s}_g\}$, $\{s_{g^{-1}}\}$. As in section 1, let $z = \sum_{g \in G_0} \widehat{s}_g s_{g^{-1}}$ and $\zeta : S \to Z(S) : s \mapsto \sum_{g \in G_0} \widehat{s}_g s s_{g^{-1}}$. Again, $Z(S)$ denotes the center of S.

2.3.1 Definition. We define an equivalence relation on G as follows : $g \sim h$ if and only if $f(\widehat{s}_g) = f(\widehat{s}_h)$ for all $f \in \mathrm{Sym}(S, R)$. In this case we say that g and h are *S-conjugated* (see also 2.3.12).

2.3.2 Proposition. *Let $g, h \in G$. If $g \sim h$, then $\zeta(\widehat{s}_g) = \zeta(\widehat{s}_h)$. In case z is invertible in S, the converse holds true.*

Proof. The result follows from Proposition 2.2.2. □

2.3.3 Remark. Suppose R is connected and $S \cong \mathrm{End}_R(P_1) \oplus \ldots \oplus \mathrm{End}_R(P_q)$ as R-algebras, $P_1, \ldots, P_q$ being finitely generated projective R-modules, and suppose that z is invertible in S. Then $g \sim h$ if and only if $t_{P_i}(\widehat{s}_g) = t_{P_i}(\widehat{s}_h)$ for $i = 1, \ldots, q$, see Theorem 2.2.4(6) and Proposition 2.2.6(2).

2.3.4 Lemma. *Let $g, h \in G$. If $g \sim h$, then $g^{-1} \sim h^{-1}$.*

Proof. Let $f \in \mathrm{Sym}(S, R)$. Take the map $\theta : RG \to RG : \sum_{g \in G} r_g u_g \mapsto \sum_{g \in G} r_g u_{g^{-1}}$ and consider the restriction to S. By Lemma 1.2.4, $f \circ \theta \in \mathrm{Sym}(S, R)$. Since $g \sim h$, we have $(f \circ \theta)(\widehat{s}_g) = (f \circ \theta)(\widehat{s}_h)$. The statement follows at once. □

For the remainder of this section, we fix the following notation. For $g \in G$, set $K_g = \{h \in G |\ g \sim h\}$. Obviously $\{K_g; g \in G\}$ is a partition of G and by Lemma 2.3.4, $K_{g^{-1}} = K_g^{-1}$. Put $v_g = \sum_{x \in K_g} u_x$ and let G_1 denote a set of representatives of the distinct K_g.

We observe that $K_g = E_g \cup \ldots \cup E_t$, in particular $v_g \in S$. Furthermore, $K_e = E_e$. Indeed, $\tau(\widehat{s}_e) = |E_e|^{-1} 1_R$ and $\tau(\widehat{s}_k) = 0$ for $k \notin E_e$.

2.3.5 Definition. Let $f \in S^*$. We say that f is a *class function* on S if $g \sim h$ in G implies that $f(\widehat{s}_g) = f(\widehat{s}_h)$. The set of all class functions forms an R-submodule of S^*, denoted by $Cf(S, R)$. Clearly $\mathrm{Sym}(S, R) \subset Cf(S, R)$.

2.3.6 Proposition. *(1)* $Z(S) \subset \bigoplus_{g \in G_1} Rv_g$.
(2) $Z(S) = \bigoplus_{g \in G_1} Rv_g$ *if and only if* $\mathrm{Sym}(S, R) = Cf(S, R)$.

Proof. Consider the left S-linear map $\beta : S \to S^*$ associated to τ as in 2.1.1. We know that β is bijective and $\beta(Z(S)) = \mathrm{Sym}(S, R)$, by Proposition 2.1.4. It suffices to show that $\beta(\oplus Rv_g) = Cf(S, R)$. We have $\beta(v_g)(\widehat{s}_k) = \tau(\widehat{s}_k v_g) = 1$ for $k \in K_{g^{-1}}$ and $\tau(\widehat{s}_k v_g) = 0$ for $k \notin K_{g^{-1}}$. Hence $\beta(\oplus Rv_g) \subset Cf(S, R)$. For the reverse inclusion, use Lemma 2.1.3. □

At the end of this section we give an example to show that the inclusion in 2.3.6(1) need not to be an equality. Our next objective is to analyze the equality $Z(S) = \oplus Rv_g$. We begin with a few remarks.

2.3.7 Remarks. 1. If $s_g \in Z(S)$, then $K_g = E_g$ by 2.3.6(1).
2. It is easily verified that $\zeta(v_g) = |K_g|\zeta(\widehat{s}_g)$. In particular, if $v_g \in Z(S)$, then $zv_g = |K_g|\zeta(\widehat{s}_g)$.
3. If $v_g \in Z(S)$ and z is invertible in S, then $|K_g|$ is invertible in R. Indeed, $v_g = |K_g|\zeta(\widehat{s}_g)z^{-1} = |K_g| \sum\limits_{k \in G_1} r_k v_k$ with $r_k \in R$, whence $1 = |K_g| r_g$.

2.3.8 Proposition. *Suppose that z is invertible in S. Then $Z(S) = \bigoplus\limits_{g \in G_1} Rv_g$ if and only if distinct $\zeta(\widehat{s}_k)$ are linearly independent over R.*

Proof. By Proposition 2.3.2, $\zeta(\widehat{s}_g)$, $g \in G_1$, are all distinct $\zeta(\widehat{s}_t)$. Suppose that $\zeta(\widehat{s}_g)$, $g \in G_1$, are linearly independent over R. Let $f \in Cf(S, R)$. It suffices to show that f is symmetric, see Proposition 2.3.6. Let $x \in S$ be such that $\zeta(x) = 0$ and write $x = \sum\limits_{k \in G_0} r_k \widehat{s}_k$, $r_k \in R$. So $0 = \zeta(x) = \sum\limits_{g \in G_1} (\sum\limits_{k \in J(g)} r_k)\zeta(\widehat{s}_g)$ with $J(g) = G_0 \cap K_g$, whence $\sum\limits_{k \in J(g)} r_k = 0$. It follows that $f(x) = 0$ and thus f is symmetric, see Proposition 2.2.1.
For the converse, use Remarks 2.3.7 (2) and (3). $\square$

As in section 1.2.5, we may consider the following componentwise multiplication on RG. Let $a, a' \in RG$, $a = \sum\limits_{g \in G} r_g u_g$ and $a' = \sum\limits_{g \in G} r'_g u_g$ with $r_g, r'_g \in R$. Then we define $a * a' = \sum\limits_{g \in G} r_g r'_g u_g$. Of course S is closed under this multiplication.

2.3.9 Proposition. *Suppose that R is a domain. If $Z(S)$ is closed under the above componentwise multiplication, then $Z(S) = \bigoplus\limits_{g \in G_1} Rv_g$.*

Proof. 1. We first assume that R is a field. Note that $\sum\limits_{g \in G} u_g = \sum\limits_{g \in G_0} s_g \in Z(RG) \cap S$, hence $\sum\limits_{g \in G} u_g \in Z(S)$. Then by Proposition 1.2.5, there is a partition $\{F_k; k \in G\}$ of

G such that $Z(S) = \bigoplus_{w_k} Rw_k$ with $w_k = \sum_{x \in F_k} u_x$. Since $Z(S) \subset \bigoplus_{g \in G_1} Rv_g$, each w_k is a sum of certain v_g. Fix w_k; say $w_k = v_{g_1} + \ldots + v_{g_m}$, $g_i \in G_1$. We now prove that $m = 1$.

Let $f \in \mathrm{Sym}(S, R)$. By 2.1.3 and 2.1.4, $c = \sum_{g \in G_0} f(\widehat{s}_{g^{-1}})s_g \in Z(S)$, and $c = \sum_{g \in G_1} f(\widehat{s}_{g^{-1}})v_g$. But $c * w_k = rw_k$ for some $r \in R$. Therefore $f(\widehat{s}_{g_1^{-1}}) = \ldots = f(\widehat{s}_{g_m^{-1}}) = r$. From this it follows that $g_1 \sim g_i$, $i = 1, \ldots, m$. Consequently, $m = 1$ and $w_k = v_{g_1}$. Then, using $\sum_{w_k} w_k = \sum_{g \in G_1} v_g$, we obtain $Z(S) = \bigoplus_{g \in G_1} Rv_g$.

2. Now let R be a domain with field of quotients L. Consider the Schur algebra $\overline{S} = \bigoplus_{g \in G_0} Ls_g$ in LG. We observe that $Z(S) = Z(\overline{S}) \cap S$. Then it is easily verified that $Z(\overline{S})$ is closed under componentwise multiplication in LG. Further, $g, h \in G$ are $\overline{S}$-conjugated if and only if they are S-conjugated. In order to prove this, one needs the following remarks. A map $f \in \mathrm{Sym}(S, R)$ can be extended to a map $\overline{f} \in \mathrm{Sym}(\overline{S}, L)$ by setting $\overline{f}(\sum_{g \in G_0} \ell_g s_g) = \sum_{g \in G_0} \ell_g f(s_g)$, $\ell_g \in L$. On the other hand, let $\varphi \in \mathrm{Sym}(\overline{S}, L)$. Then there exists $r \in R$ such that $r\varphi(s_g) \in R$ for all $g \in G_0$, and $r\varphi|_S \in \mathrm{Sym}(S, R)$. The above discussion yields the equality $Z(\overline{S}) = \bigoplus_{g \in G_1} Lv_g$. Consequently, $v_g \in Z(\overline{S}) \cap S = Z(S)$, as desired. $\square$

2.3.10 Remark. To the above defined componentwise multiplication on RG there corresponds a multiplication on $(RG)^*$, see 2.2.9(3). Namely, let $\varphi, \psi \in (RG)^*$, then $\varphi * \psi(u_{g^{-1}}) = \varphi(u_{g^{-1}})\psi(u_{g^{-1}})$, or equivalently, $\varphi * \psi(u_g) = \varphi(u_g)\psi(u_g)$ for all $g \in G$.

In the case where $Z(S) = \oplus Rv_g$ we can derive the second orthogonality relations.

2.3.11 Proposition. *Suppose R is connected and $S \cong End_R(P_1) \oplus \ldots \oplus End_R(P_q)$ as R-algebras, $P_1, \ldots, P_q$ being finitely generated projective R-modules, and suppose that z is invertible in S. If $Z(S) = \bigoplus_{g \in G_1} Rv_g$, then for $g, h \in G_1$ we have*

$$\sum_{i=1}^{q} |K_h| rank(P_i) t_{P_i}(z)^{-1} t_{P_i}(\widehat{s}_g) t_{P_i}(\widehat{s}_{h^{-1}}) = \delta_{gh}.$$

Proof. Note that $|G_1| = q$. By 2.2.4 and 2.2.6,

$$\sum_{g \in G_0} \operatorname{rank}(P_i) t_{P_i}(z)^{-1} t_{P_i}(\widehat{s}_g) t_{P_j}(s_{g^{-1}}) = \delta_{ij}.$$

This gives

$$\sum_{g \in G_1} \operatorname{rank}(P_i) t_{P_i}(z)^{-1} t_{P_i}(\widehat{s}_g) t_{P_j}(v_{g^{-1}}) = \delta_{ij}$$

and $t_{P_j}(v_{g^{-1}}) = |K_g| t_{P_j}(\widehat{s}_{g^{-1}})$. We can write this relation as $AB = I$; A, B being $q \times q$ matrices. Then $BA = I$, which implies the desired formula. □

2.3.12 Note. We discuss the case where $S = RG$. Here, $g, h \in G$ are RG-conjugated if and only if $h = tgt^{-1}$ for some $t \in G$. Indeed, suppose that $f(u_g) = f(u_h)$ for all $f \in \operatorname{Sym}(RG, R)$. In other words, $\tau(u_g c) = \tau(u_h c)$ for all $c \in Z(RG)$, see 2.1.4. Let s denote the sum in RG of all distinct conjugates $kg^{-1}k^{-1}$, $k \in G$. Clearly, $s \in Z(RG)$ and $\tau(u_g s) = 1$. Consequently, $\tau(u_h s) = 1$, whence $tg^{-1}t^{-1} = h^{-1}$ for some $t \in G$. The converse is obvious.
With notation as before, we have $v_g \in Z(RG)$ and $Z(RG) = \bigoplus_{g \in G_1} Rv_g$. Moreover, $\zeta(u_g) = |C_G(g)| v_g$.

Let H be a subgroup of G and consider the centralizer S of RH in RG (see 1.2.12). If g_1 and g_2 are S-conjugated, then they are RG-conjugated, use $Z(RG) \subset Z(S)$.

Let us now focus on the case where S is a double coset algebra. So let H be a subgroup of G with $|H|^{-1} \in R$, put $\varepsilon = |H|^{-1} \sum_{h \in H} u_h$ and consider $S = \varepsilon RG \varepsilon$, see also 1.2.6.
Let $Z(S)$ and τ be as before, and put $\widehat{s}_g = |HgH|^{-1} \sum_{x \in HgH} u_x$, for $g \in G$.
For RG-conjugacy we now set $C_k = \{tkt^{-1} \mid t \in G\}$ and $w_k = \sum_{x \in C_k} u_x$, with $k \in G$.

2.3.13 Proposition. *Consider $S = \varepsilon RG \varepsilon$ and let $g_1, g_2 \in G$.*
(1) If g_1 and g_2 are S-conjugated, then

$$|Hg_1H|^{-1} |Hg_1H \cap C_k| 1_R = |Hg_2H|^{-1} |Hg_2H \cap C_k| 1_R.$$

for any RG-conjugacy class C_k.

(2) If R is connected and R is a splitting ring for RG, then the converse of (1) holds.

Proof. Note that g_1 and g_2 are S-conjugated if and only if $\tau(\widehat{s}_{g_1}c) = \tau(\widehat{s}_{g_2}c)$ for all $c \in Z(S)$, see 2.1.4.

(1) Clearly $\varepsilon w_k \in Z(S)$. Further, $\tau(\widehat{s}_g \varepsilon w_k) = \tau(\widehat{s}_g w_k) = |HgH|^{-1}|HgH \cap C_{k^{-1}}|1_R$. The assertion now follows.

(2) It suffices to show that εw_k, $k \in G$, generate $Z(S)$ as R-module. Let $\{e_1, \dots, e_q\}$ be the set of primitive central nonzero idempotents of RG, and let $\varepsilon e_i \neq 0$ for $i = 1, \dots, m$. Take $a \in Z(S)$. By Theorem 3.1.5(1) - (3), we have $a = \sum\limits_{i=1}^{m} r_i \varepsilon e_i$ with $r_i \in R$. Moreover, $e_i = \sum r'_k w_k$ with $r'_k \in R$. $\square$

We conclude this section with a concrete example of the above situation, based on [Da]. This example shows that the inclusion in 2.3.6(1) need not to be an equality.

Example. Consider in $GL_3(\mathbb{Z}_3)$ the matrices

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $G = < a, b, d >$. We have the following relations : $a^3 = b^3 = I$, $d^2 = I$, $da = a^2 d$ and $db = b^2 d$. Further, $c = bab^{-1}a^{-1}$, $c^3 = I$ and c commutes with a, b, d. So each element of G can be expressed as $a^i b^j d^k c^\ell$ with $i, j, \ell = 0, 1, 2$ and $k = 0, 1$.

We need the following RG-conjugacy classes :

$$\begin{aligned} C_a &= \{a, ac, ac^2, a^2, a^2c, a^2c^2\} \\ C_d &= \{d, da, db, dabc^2, da^2, db^2, dab^2c, da^2bc, da^2b^2c^2\} \end{aligned}$$

Let $H = < d >$. We require :

$$\begin{aligned} HaH &= \{a, ad, da, a^2\} \\ HacH &= \{ac, da^2c, dac, a^2c\} \\ Hac^2H &= \{ac^2, dac^2, da^2c^2, a^2c^2\}. \end{aligned}$$

Now put $\varepsilon = \dfrac{1}{2}(u_I + u_d)$ in $\mathbb{C}G$ and consider $S = \varepsilon(\mathbb{C}G)\varepsilon$. Let $g \in K_a$, where K_a is the S-conjugacy class of a. By Proposition 2.3.13, $HgH \cap C_a \neq \phi$. Consequently, $HgH = HaH$ or $HgH = HacH$ or $HgH = Hac^2H$. Since $HaH \cap C_d \neq \phi$, we may exclude the last two possibilities (use 2.3.13). So we obtain $K_a = HaH$. However, $\underline{HaH} = \sum_{x \in HaH} u_x$ does not commute with $\underline{HbH}$, and thus $\underline{HaH} \notin Z(S)$.

2.4 Trace functions of induced modules

Throughout this section, R is a commutative ring, G is a finite group and H is a subgroup of G. Let S be a Schur algebra in RG with associated partition $\{E_g; g \in G\}$ and let B be a Schur algebra in RH with partition $\{F_h; h \in H\}$. Further, let G_0, resp. H_0, denote a set of representatives of the distinct E_g, resp. F_h. Put $s_g = \sum_{x \in E_g} u_x$ and $b_h = \sum_{x \in F_h} u_x$.

2.4.1 Definition. The Schur algebra B is called a Schur subalgebra of S if for each $h \in H$ we have $F_h = \cup E_g$, for some $g \in G$.

For the remainder of this section, we assume that B is a Schur subalgebra of S. We also assume $|E_g|^{-1} \in R$ and $|F_h|^{-1} \in R$ for all $g \in G$, $h \in H$. We set $\widehat{s}_g = |E_g|^{-1} s_g$, analogously $\widehat{b}_h$.

2.4.2 Definition. Let $f \in \mathrm{Hom}_R(B, R)$. We define $\widetilde{f} \in \mathrm{Hom}_R(S, R)$ as follows : $\widetilde{f}(\widehat{s}_g) = 0$ if $g \notin H$ and $\widetilde{f}(\widehat{s}_g) = f(\widehat{b}_g)$ if $g \in H$, and extend by linearity.
We observe that $\widetilde{f}|_B = f$.

Under certain conditions, we shall derive a formula for the trace function of an induced module. We set $z_S = \sum_{g \in G_0} \widehat{s}_g s_{g^{-1}}$ and $z_B = \sum_{h \in H_0} \widehat{b}_h b_{h^{-1}}$.

2.4.3 Proposition. *Assume that $F_e = E_e$ and that z_S is invertible in S. Suppose R is connected and finitely generated projective R-modules are free. Further, suppose $S \cong \bigoplus_{j=1}^{q} End_R(M_j)$ and $B \cong \bigoplus_{i=1}^{p} End_R(N_i)$ as R-algebras, where M_j, N_i are* *finitely generated projective R-modules. Set $N_i^S = S \otimes_B N_i$. Then*

$$t_{N_i}(z_B)t_{N_i^S} = rank_R(N_i)(\widetilde{t}_{N_i} \circ \zeta) \quad on\ S,$$

where $\zeta : S \to Z(S) : x \mapsto \sum_{g \in G_0} \widehat{s}_g x s_{g^{-1}}$.

Proof. Recall that N_i is an indecomposable left B-module (similar remark for M_j). From 2.2.7(4), we have for any $x \in S$

$$\zeta(x) = \sum_{j=1}^{q} t_{M_j}(x)(\sum_{g \in G_0} t_{M_j}(s_{g^{-1}})\widehat{s}_g).$$

Applying $\widetilde{t}_{N_i}$ to this expression yields

$$\begin{aligned} \widetilde{t}_{N_i}(\zeta(x)) &= \sum_{j=1}^{q} t_{M_j}(x) \left(\sum_{g \in G_0 \cap H} t_{M_j}(s_{g^{-1}}) t_{N_i}(\widehat{b}_g) \right) \\ &= \sum_{j=1}^{q} t_{M_j}(x) \left(\sum_{g \in H_0} t_{M_j}(b_{g^{-1}}) t_{N_i}(\widehat{b}_g) \right). \end{aligned}$$

By the hypothesis on R, we have $M_j|_B \cong \bigoplus_k N_k^{c_{kj}}$ as left B-modules, where $c_{kj} \in \mathbb{N}$. Thus $t_{M_j} = \sum_k c_{kj} t_{N_k}$ on B. Using the orthogonality relations 2.2.4(2) and 2.2.6(1), we then obtain

$$\operatorname{rank}_R(N_i)\widetilde{t}_{N_i}(\zeta(x)) = \sum_{j=1}^{q} t_{M_j}(x) c_{ij} t_{N_i}(z_B).$$

By the hypothesis on R, we can apply a version of Frobenius reciprocity, see [1.1.3(3)]. This gives $t_{N_i^S} = \sum_{j=1}^{q} c_{ij} t_{M_j}$, which completes the proof. □

2.4.4 Remarks. (1) The derived formula generalizes the result of [R-satz 10].
(2) If we take $x = \widehat{s_e}$ in the preceding formula, then we get $t_{N_i}(z_B)\operatorname{rank}_R(N_i^S) = \operatorname{rank}_R(N_i)\widetilde{t}_{N_i}(z_S)$.

2.4.5 Example. Let $S = RG$ and $B = RH$. In this case we have $z_S = |G|u_e$ and $z_B = |H|u_e$. With hypotheses and notation as in 2.4.3 (in particular $|G|^{-1} \in R$), we now obtain

$$|H| t_{N_i^S}(u_x) = \widetilde{t}_{N_i}(\sum_{g \in G} u_{gxg^{-1}}), \quad \text{for } x \in G$$

(use also 2.2.6(2)). Of course, this formula can be proved without any assumption. (N being a left RH-module, which is finitely generated and projective over R), see 5.5.6.

We prove

2.4.6 Proposition. *Keep the hypotheses and notation of 2.4.3. If* $rank_R(N_i) = rank_R(N_i^S)$, *then* $t_{N_i^S} = \widetilde{t}_{N_i}$ *on* S.

Proof. Write N instead of N_i. From the hypotheses and 1.1.3(3) it follows that N^S is an indecomposable left S-module.

Let $\{e_1, \ldots, e_q\}$, resp. $\{f_1, \ldots, f_p\}$, be the set of primitive central nonzero idempotents of S, resp. B. Assume N lies over f_i and N^S lies over e_j. Then we deduce that $f_i = e_j$. Indeed, we may write f_i as a sum of orthogonal primitive idempotents of B, say $f_i = \mu_1 + \ldots + \mu_r$. Since $N \cong B\mu_k$ in B-mod for $k : 1 \ldots r$, $N^S \cong S\mu_k$ in S-mod. Thus each μ_k is a primitive idempotent of S and $e_j\mu_k = \mu_k$. Now $e_j = e_jf_i + e_j(1-f_i) = \mu_1 + \ldots + \mu_r + e_j(1-f_i)$. But $r = rank_K(N) = rank_R(N^S)$ and thus $e_j(1-f_i) = 0$.

From Proposition 2.2.6(1), it follows that

$$t_N(z_B)f_i = (rank_R N)^2 \sum_{h \in H_0} t_N(\widehat{b}_{h^{-1}})b_h$$

and

$$t_{N^S}(z_S)e_j = (rank_R N)^2 \sum_{g \in G_0} t_{N^S}(\widehat{s}_{g^{-1}})s_g$$

Moreover, since z_S is invertible in S, $rank_R(N) = rank_R(N^S)$ is invertible in R, whence $t_N(z_B)$ is invertible in R (see 2.2.6(2)).

Expressing the equality $e_j = f_i$, we then obtain that $t_{N^S}(\widehat{s}_g) = 0$ for $g \notin H$ and $t_{N^S}(z_S)t_N(\widehat{b}_g) = t_N(z_B)t_{N^S}(\widehat{s}_g)$ for $g \in H$.

In particular $t_{N^S}(z_S)rank_R(N) = t_N(z_B)rank_R(N^S)$ (take $g = e$), whence $t_{N^S}(z_S) = t_N(z_B)$. We conclude that $t_{N^S} = \widetilde{t}_N$ on S. $\square$

Remark. Let H be a subgroup of a finite group G. Let S denote the centralizer of RH in RG and P is an indecomposable S-module. In example 5.6.1 one has

$rank_R P = rank_R(RG \otimes_S P)$ for some P.

Let H be a normal subgroup of G. In 5.5.8(3) sufficient conditions are found to have $rank_R P = rank_R(RG \otimes_S P)$.

Chapter 3

Hecke algebras

Throughout R is a commutative ring. Let G be a finite group and let H be a subgroup of G. Suppose that $|H|^{-1} \in R$ and consider the idempotent $e_H = |H|^{-1} \sum_{h \in H} u_h$ in RG. Then $e_H RG e_H$ is a Schur algebra, called a double coset algebra (see 1.2.6). More generally, let S be a Schur algebra in RG with associated partition $\{E_g\}$. If H is a subgroup of G such that $|H|^{-1} \in R$ and H is a union of some E_g, then, under suitable conditions, the algebra $e_H S e_H$ is a Schur algebra in RG with associated partition $\{HE_gH\}$ (see Proposition 1.2.7). We shall discuss relations between RG and $e_G RG e_H$ and between S and $e_H S e_H$ in a more general context.

Until further notice, A denotes an R-algebra and ε a nonzero idempotent in A. The algebra $\varepsilon A \varepsilon$ is called a Hecke algebra in A.
In the first section we investigate the relationship between indecomposable modules over $\varepsilon A \varepsilon$ and indecomposable A-modules. In the second section we focus on the character theory.

3.1 Indecomposable modules over Hecke algebras

Note that $(\mathrm{End}_A(A\varepsilon))^\circ \to \varepsilon A \varepsilon : \psi \to \psi(\varepsilon)$ is an isomorphism of R-algebras. Further, if A is finitely generated and projective as R-module, then so in $\varepsilon A \varepsilon$.

3.1.1 Proposition. *Suppose that A is finitely generated and projective as R-module and suppose that $\varepsilon A \varepsilon$ is a faithful R-algebra (this follows whenever R is connected). If A is separable over R, then so is $\varepsilon A \varepsilon$.*

Proof. Clearly $A\varepsilon$ is a finitely generated projective faithful R-module, whence $\mathrm{End}_R(A\varepsilon)$ is a central separable R-algebra. Further, observe that $\mathrm{End}_A(A\varepsilon)$ is the centralizer of $T(A)$ in $\mathrm{End}_R(A\varepsilon)$ where $T(A)$ is the algebra consisting of all left multiplications by elements of A. Since $T(A)$ is separable over R, it follows from [DM-I, Theorem 4.3] that $\mathrm{End}_A(A\varepsilon)$ is separable over R and so is $\varepsilon A\varepsilon$. $\square$

In particular, if $|G|$ is invertible in R, then every double coset algebra in RG is separable over R. From [C-R, 6.3] we may deduce :

3.1.2 Proposition. *Let $\mathcal{P}$ be the category of all finitely generated projective left $\varepsilon A\varepsilon$-modules and let $\mathcal{C} = \mathcal{C}(A\varepsilon)$ be the category of all left A-modules which are isomorphic to A-direct summands of $(A\varepsilon)^m$ for some m. Then the functors $A\varepsilon \otimes_{\varepsilon A\varepsilon} - : \mathcal{P} \to \mathcal{C}$ and $Hom_A(A\varepsilon, -) : \mathcal{C} \to \mathcal{P}$, denoted by F_1 resp. F_2, define an equivalence of categories between $\mathcal{P}$ and $\mathcal{C}$. Consequently, indecomposable modules in $\mathcal{P}$ correspond to indecomposable modules in $\mathcal{C}$ under F_1 and F_2.*

It is clear that $\mathrm{Hom}_A(A\varepsilon, M)$ is a right $\mathrm{End}_A(A\varepsilon)$-module, hence it is a left $\varepsilon A\varepsilon$-module (M being a left A-module). Moreover, $\mathrm{Hom}_A(A\varepsilon, M) \to \varepsilon M : \psi \mapsto \psi(\varepsilon)$ is an isomorphism of left $\varepsilon A\varepsilon$-modules.

Further, if $0 \neq M \in \mathcal{C}(A\varepsilon)$, then M is a finitely generated projective left A-module and $\varepsilon M \neq 0$. We now focus on central idempotents.

3.1.3 Remarks. Suppose that R is connected and that A is finitely generated and projective as R-module. Let $\{e_1, \ldots, e_q\}$, resp. $\{d_1, \ldots, d_m\}$, be the set of primitive central nonzero idempotents of A, resp. $\varepsilon A\varepsilon$.

1. Each nonzero εe_i is uniquely expressible as a sum of d_j's and each d_j appears in one and only one of the nonzero εe_i.
2. Let $P \in \mathcal{P}$ be indecomposable (notation as in 3.1.2). Then $A\varepsilon \otimes_{\varepsilon A\varepsilon} P$ lies over e_i if and only if $\varepsilon e_i \neq 0$ and P lies over some d_j appearing in the decomposition of εe_i.
3. We may write d_j as a sum of orthogonal primitive nonzero idempotents of $\varepsilon A\varepsilon$, say $d_j = \eta_1 + \ldots + \eta_k$ (use rank_R). It is clear that $\varepsilon A\varepsilon\eta_i$ is an indecomposable module in $\mathcal{P}$ lying over d_j.

3.1.4 Proposition. *Let R, A and $\{e_1, \ldots, e_q\}$ be as in 3.1.3. Suppose that any two indecomposable finitely generated projective left A-modules lying over the same e_i are isomorphic as A-modules, then :*

(1) The nonzero εe_i are precisely the distinct primitive central idempotents of $\varepsilon A\varepsilon$.

(2) Any two indecomposable finitely generated projective left $\varepsilon A\varepsilon$-modules lying over the same nonzero εe_i are isomorphic as $\varepsilon A\varepsilon$-modules.

(3) Let M be an indecomposable finitely generated projective left A-module lying over e_i. Then $\varepsilon M \neq 0$ if and only if $\varepsilon e_i \neq 0$, and this is equivalent to $M \in \mathcal{C}(A\varepsilon)$.

Proof. The result follows readily from 3.1.2 and 3.1.3 $\square$

3.1.5 Theorem. *Let R, A and $\{e_1, \ldots, e_q\}$ be as in 3.1.3 Suppose $A \cong End_R(M_1) \oplus \ldots \oplus End_R(M_q)$ as R-algebra, M_i being finitely generated projective R-modules, and assume that M_i lies over e_i. Then :*

(1) The nonzero εe_i are the primitive central idempotents of $\varepsilon A\varepsilon$.

(2) $\varepsilon M_i \neq 0$ if and only if $\varepsilon e_i \neq 0$, and this is equivalent to $M_i \in \mathcal{C}(A\varepsilon)$.

(3) Each nonzero εM_i is an indecomposable left $\varepsilon A\varepsilon$-module and $\varepsilon A\varepsilon \cong \bigoplus_i End_R(\varepsilon M_i)$ as R-algebras, where the sum is taken over the nonzero εM_i.

Proof. Recall that each M_i is an indecomposable left A-module under the operation $(\varphi_1, \ldots, \varphi_q) \cdot m = \varphi_i(m)$, $m \in M_i$ and $\varphi_j \in \mathrm{End}_R(M_j)$.

It is easily seen that each nonzero εM_i is a finitely generated projective R-module. Further, since A is separable over R, projectivity over R is equivalent to projectivity over A. The same remark holds for $\varepsilon A\varepsilon$.

(a) Obviously $\varepsilon M_i \neq 0$ yields $\varepsilon e_i \neq 0$. Now assume $\varepsilon e_i \neq 0$. Let $\{d_1, \ldots, d_m\}$ be as in 3.1.3; then εe_i is a sum of d_j's. Consider an indecomposable module $P \in \mathcal{P}$ which lies over some d_j, appearing in the decomposition of εe_i. We know that $Ae_i \cong \mathrm{End}_R(M_i)$, and $A\varepsilon \otimes_{\varepsilon A\varepsilon} P$ is a unitary left Ae_i-module. Therefore there is an R-module L such that $A\varepsilon \otimes_{\varepsilon A\varepsilon} P \cong L \otimes_R M_i$ as left A (or Ae_i)-modules, see e.g. [DM-I,p.19]. Then $P \cong L \otimes_R \varepsilon M_i$ as left $\varepsilon A\varepsilon$-modules. Consequently, $\varepsilon M_i \neq 0$ and $d_j \varepsilon M_i \neq 0$.

(b) Assume $\varepsilon M_i \neq 0$. We observe that $\varepsilon M_i \in \mathcal{P}$. Thus $A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon M_i \in \mathcal{C}(A\varepsilon)$ and it is a unitary left Ae_i-module. But then there is a finitely generated projective R-module N such that $A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon M_i \cong N \otimes_R M_i$ as left A (or Ae_i)-modules, see e.g. [DM-I, p.19 and 23]. Consequently, $\varepsilon M_i \cong N \otimes_R \varepsilon M_i$ as left $\varepsilon A\varepsilon$-modules. This implies that $\mathrm{rank}_R(\varepsilon M_i) = \mathrm{rank}_R(N)\mathrm{rank}_R(\varepsilon M_i)$, whence $\mathrm{rank}_R(N) = 1$. Therefore $\mathrm{End}_R(N) = RI \cong R$, see e.g. [DM-I, p.32]. Since we are dealing with equivalent categories, we have $\mathrm{End}_A(N \otimes_R M_i) \cong \mathrm{End}_R(N)$ and $\mathrm{End}_A(A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon M_i) \cong \mathrm{End}_{\varepsilon A\varepsilon}(\varepsilon M_i)$ as R-algebras, see e.g. [DM-I, p.17]. We conclude that $\mathrm{End}_{\varepsilon A\varepsilon}(\varepsilon M_i) = RI \cong R$. In particular, εM_i is an indecomposable left $\varepsilon A\varepsilon$-module, see [C-R, 6.4].

(c) Since each nonzero εM_i is indecomposable, it follows from (a) that each nonzero εe_i is a primitive central idempotent of $\varepsilon A\varepsilon$. Let $\varepsilon M_i \neq 0$. Since $\mathrm{End}_{\varepsilon A\varepsilon}(\varepsilon M_i) = RI$, we then obtain $\varepsilon A\varepsilon e_i \cong \mathrm{End}_R(\varepsilon M_i)$ as R-algebras, see [N_2-v.O_2, 1.7] (the isomorphism associates to $\varepsilon a\varepsilon e_i$ the left multiplication by $\varepsilon a\varepsilon e_i$). Now, $N \otimes_R \varepsilon M_i \cong \varepsilon M_i \cong R \otimes_R \varepsilon M_i$ as left $\varepsilon A\varepsilon$-modules, and thus $N \cong R$ (N as in (b)). Consequently, $M_i \cong A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon M_i \in \mathcal{C}(A\varepsilon)$, completing the proof. □

Remark. Keep the hypotheses of 3.1.5 and suppose that finitely generated projective R-modules are free. Then $\mathrm{rank}_R(\varepsilon M_i)$ is equal to the multiplicity of M_i in the decomposition of $A\varepsilon$ into indecomposable left A-modules, see 1.1.3(1).

Note. If E is a semisimple ring and ε a nonzero idempotent of E, then it is known that $\varepsilon E\varepsilon$ is semisimple too. In this case, indecomposable modules over E and $\varepsilon E\varepsilon$ are simple modules, and modules are projective.
The results in Proposition 3.1.4 remain true.

We now focus on the case where $\varepsilon \in Z(A)$, the center of A. This is equivalent to $\varepsilon A\varepsilon$ is an ideal of A, as is easily checked. Let A, R, $\{e_1, \dots, e_q\}$ be as in 3.1.3. In this case, ε is uniquely a sum of distinct primitive central idempotents of A, say $\varepsilon = e_1 + \ldots + e_t$ with $t \leq q$. So $\varepsilon A\varepsilon = Ae_1 \oplus \ldots \oplus Ae_t$ and $e_1, \dots, e_t$ are precisely the primitive central idempotents of $\varepsilon A\varepsilon$. A left $\varepsilon A\varepsilon$-module W becomes a left A-module by setting : $a.w = a\varepsilon w$, $a \in A$, $w \in W$ and we have at once :

3.1.6 Proposition. *Let $\varepsilon \in Z(A)$, then :*

(1) If W is an indecomposable left $\varepsilon A\varepsilon$-module, then it is also an indecomposable left A-module. Conversely, if M is an indecomposable left A-module such that $\varepsilon M \neq 0$, then $\varepsilon m = m$ for alle $m \in M$ and M is an indecomposable left $\varepsilon A\varepsilon$-module.

(2) If W is a finitely generated projective left $\varepsilon A\varepsilon$-module, then W is finitely generated and projective as A-module. If M is an finitely generated projective left A-module and $\varepsilon M \neq 0$, then εM is finitely generated and projective as $\varepsilon A\varepsilon$-module.

Proof. Straightforward. $\square$

3.1.7 Example Let H be a subgroup of a finite group G and assume $|H|^{-1} \in R$. Then $e_H = |H|^{-1} \sum_{h \in H} u_h \in Z(RG)$ if and only if H is a normal subgroup of G.
Note that in this case $e_H RG e_H$ is isomorphic to $R[G/H]$.

3.2 Trace functions on $\varepsilon A\varepsilon$

Throughout this section R is a connected commutative ring, A is an R-algebra which is finitely generated projective as R-module, and ε is a nonzero idempotent of A. Let $Z(A)$ denote the center of A. Further, let $\{e_1, \ldots, e_q\}$ be the set of primitive central nonzero idempotents of A. We first discuss the relationship between trace functions on A and on $\varepsilon A\varepsilon$.
Let M be a left A-module such that $\varepsilon M \neq 0$. If M is finitely generated projective over R, then so is εM. More precisely, if $f_i \in \mathrm{Hom}_R(M, R)$, $m_i \in M$ is an R-dual basis for M, then $\{f_i|_{\varepsilon M}\}$, $\{\varepsilon m_i\}$ is an R-dual basis for εM. Using R-dual bases for M and εM, we obtain $t_{\varepsilon M}(\varepsilon x\varepsilon) = t_M(\varepsilon x\varepsilon) = t_M(x\varepsilon)$ for all $x \in A$, in particular $t_M(\varepsilon) = \mathrm{rank}_R(\varepsilon M)1_R$. Furthermore :

3.2.1 Proposition. *Suppose that R is a splitting ring for $Z(A)$. Let M be a left A-module which is finitely generated projective over R, and assume M lies over only one primitive central idempotent. Suppose $\varepsilon M \neq 0$, then :*

$$rank_R(\varepsilon M)t_M(x) = rank_R(M)t_{\varepsilon M}(\varepsilon x \varepsilon) \qquad \textit{for all } x \in Z(A).$$

Proof. Assume M lies over e_k. By hypothesis, $Z(A) = Re_1 \oplus \ldots \oplus Re_q$, and thus $x = \sum_{j=1}^{q} r_j e_j$ with $r_j \in R$. We have $t_M(x) = \operatorname{rank}_R(M)r_k$ and $t_M(x\varepsilon) = r_k t_M(\varepsilon)$. From this the assertion follows. $\square$

3.2.2 Note. Keep the hypotheses of 3.2.1. In addition, suppose that A is a symmetric Frobenius R-algebra and let b be a symmetric associative bilinear form on A with dual bases $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$. Put $z = \sum_{i=1}^{n} a_i b_i$. Combining 2.1.19 and 3.2.1, we then obtain :

$$\operatorname{rank}_R(\varepsilon M)t_M(z)t_M(x) = (\operatorname{rank}_R M)^2 t_{\varepsilon M}\left(\sum_{i=1}^{n} \varepsilon b_i x a_i \varepsilon\right) \qquad \text{for all } x \in A.$$

3.2.3 Corollary. *Let $A = R *_\alpha G$ where G is a finite group such that $|G|^{-1} \in R$ and suppose that R is a splitting ring for A. Let M be an indecomposable left A-module which is finitely generated projective over R and assume $\varepsilon M \neq 0$. Further, modify α as in 1.1.8. Put $K_g = \{ygy^{-1} | \ y \in G\}$ and $v_g = \sum_{x \in K_g} u_x$ with $g \in G$. Then for any α-G-regular $g \in G$ we have :*

$$|K_g| rank_R(\varepsilon M)t_M(u_g) = rank_R(M)t_{\varepsilon M}(v_g \varepsilon).$$

Proof. By 1.1.6(1) we have $|K_g|t_M(u_g) = t_M(v_g)$. As $v_g \in Z(R *_\alpha G)$, see 1.1.5, we may apply proof 3.2.1. $\square$

In the case where $\varepsilon \in Z(A)$, we have the following. Let M be a left A-module which is finitely generated projective over R and assume that M lies over only one primitive central idempotent. Let $\varepsilon M \neq 0$, then $\varepsilon m = m$ for all $m \in M$ and $t_M(x) = t_M(\varepsilon x \varepsilon)$ for all $x \in A$. Furthermore :

3.2.4 Proposition. *Suppose that R is a splitting ring for A and that finitely generated projective R-modules are free. Let $M_1, \ldots, M_q$ be a basic set of indecomposable left A-modules which are finitely generated projective over R and let $\varepsilon M_i \neq 0$ for $i = 1, \ldots, t$. For each i, $1 \leq i \leq t$, suppose that either*

(i) $rank_R(\varepsilon M_i) = rank_R(M_i)$ *or*

(ii) $t_{M_i}(x) = t_{M_i}(\varepsilon x \varepsilon)$ *for all* $x \in A$.

Then $\varepsilon \in Z(A)$.

Proof. Let $1 \leq i \leq t$ and let M_i lie over e_i. Suppose $(1-\varepsilon)e_i \neq 0$. Using ranks, we may write εe_i, resp. $(1-\varepsilon)e_i$, as a sum of orthogonal primitive nonzero idempotents of A, say $\varepsilon e_i = \eta_1 + \ldots + \eta_l$ and $(1-\varepsilon)e_i = \mu_1 + \ldots + \mu_k$. Obviously $e_i = \varepsilon e_i + (1-\varepsilon)e_i$ and $\eta_s \mu_j = 0$ for $s = 1, \ldots, l$, $j = 1, \ldots, k$.

Case (i). The assumptions on R imply that $(l+k)\mathrm{rank}_R(M_i) = \mathrm{rank}_R(Ae_i) = (\mathrm{rank}_R M_i)^2$, whence $l + k = \mathrm{rank}_R(M_i)$. Clearly $\eta_1, \ldots, \eta_l$ are also primitive idempotents of $\varepsilon A \varepsilon$ and using Theorem 3.1.5, we deduce, just as above, that $l = \mathrm{rank}_R(\varepsilon M_i)$. Consequently, $(1-\varepsilon)e_i = 0$ or $\varepsilon e_i = e_i$. It follows that $\varepsilon = \sum_{i=1}^{t} \varepsilon e_i \in Z(A)$.

Case (ii). For $j = 1, \ldots, k$, we have $t_{M_i}(\mu_j) = t_{M_i}(\mu_j \varepsilon) = 0$. Now by [N$_2$-v.O$_2$, 1.7], $\mu_j A \mu_j \cong \mathrm{End}_A(A\mu_j)^{\circ} = RI$ as R-algebra, whence $\mu_j A \mu_j = R\mu_j$. Therefore $t_{M_i}(A\mu_j) = 0$. As $Ae_i \cong M_{n_i}(R)$, we know that the restriction of t_{M_i} to Ae_i is nondegenerate (see 2.1.6.(3)). So $\mu_j = 0$ and thus $(1-\varepsilon)e_i = 0$. Consequently, $\varepsilon \in Z(A)$. □

In chapter 2, we have developed a character theory for Frobenius algebras, in particular for Schur algebras. When A is a twisted group ring we may express primitive central idempotents of $\varepsilon A \varepsilon$ in terms of trace functions as follows :

3.2.5 Proposition. *Let $A = R *_\alpha G$ where G is a finite group such that $|G|^{-1} \in R$. Suppose $R *_\alpha G \cong End_R(M_1) \oplus \ldots \oplus End_R(M_q)$ as R-algebra, $M_1, \ldots, M_q$ being finitely generated projective R-modules. Assume that M_i lies over e_i and that $\varepsilon M_i \neq 0$ for $i = 1, \ldots, t$. Then for $1 \leq i, j \leq t$ we have :*

(1) $\varepsilon e_i = \frac{1}{|G|\alpha(e,e)}\, rank_R(M_i) \sum_{g \in G} \frac{1}{\alpha(g,g^{-1})} t_{\varepsilon M_i}(\varepsilon u_{g^{-1}} \varepsilon) \varepsilon u_g \varepsilon$

(2) $\sum\limits_{g\in G}\frac{1}{\alpha(g,g^{-1})}t_{\varepsilon M_i}(\varepsilon u_{g^{-1}}\varepsilon)t_{\varepsilon M_j}(\varepsilon u_g\varepsilon)=\delta_{ij}|G|rank_R(\varepsilon M_i)(rank_R M_i)^{-1}\alpha(e,e)$
Note that we may apply Theorem 3.1.5

Proof. By example 2.2.8 (3), $e_i=\frac{1}{|G|\alpha(e,e)}\ \mathrm{rank}_R M_i\sum\limits_{g\in G}\frac{1}{\alpha(g,g^{-1})}t_{M_i}(u_{g^{-1}})u_g$.
Using Lemma 1.1.7 and the fact that $t_{M_i}(u_{g^{-1}}\varepsilon)=t_{\varepsilon M_i}(\varepsilon u_{g^{-1}}\varepsilon)$, we obtain (1).
The second assertion follows by applying $t_{\varepsilon M_j}$ to the expression for εe_i. □

3.2.6 Note. Keep the above hypotheses. As in the proof of 3.2.5 we derive :

$$\varepsilon e_k=\frac{1}{|G|\alpha(e,e)}\mathrm{rank}_R(M_k)\sum_{g\in G}\frac{1}{\alpha(g,g^{-1})}t_{M_k}(u_{g^{-1}}\varepsilon)u_g \qquad \text{for } k=1,\ldots,q.$$

Note that $\varepsilon M_k\neq 0$ if and only if $\varepsilon e_k\neq 0$. Let now $\varepsilon=\sum\limits_{g\in G}r_g u_g$ with $r_g\in R$. Then it is easily seen that $|G|\alpha(e,e)r_e=\sum\limits_{i=1}^{t}\ \mathrm{rank}_R(M_i)\ \mathrm{rank}_R(\varepsilon M_i)1_R$.

To conclude we turn to the double coset algebra. Let G be a finite group, H a subgroup of G with $|H|^{-1}\in R$ and $A=RG$. Let $\varepsilon=|H|^{-1}\sum\limits_{h\in H}u_h$ and consider the double coset algebra $\varepsilon A\varepsilon$.
For any $g\in G$, $|HgH|$ is invertible in R and $|HgH|\varepsilon u_g\varepsilon=\underline{HgH}$, where $\underline{HgH}=\sum\limits_{x\in HgH}u_x$, see 1.2.6.
The distinct $\underline{HgH}$ form an R-basis for $\varepsilon A\varepsilon$. If we apply Proposition 3.2.5, then we obtain :

3.2.7 Proposition. *Keep the hypotheses and notation of 3.2.5 (with $\alpha=1$). Let $\{g_1,\ldots,g_m\}$ be a full set of double coset representatives for H in G. Then for $1\le i,j\le t$:*

(1) $\qquad \varepsilon e_i=|G|^{-1}rank_R(M_i)\sum\limits_{k=1}^{m}\dfrac{1}{|Hg_kH|}t_{\varepsilon M_i}\left(\underline{Hg_k^{-1}H}\right)\underline{Hg_kH}$

(2) $\qquad \sum\limits_{k=1}^{m}\dfrac{1}{|Hg_kH|}t_{\varepsilon M_i}(\underline{Hg_k^{-1}H})t_{\varepsilon M_j}(\underline{Hg_kH})=\delta_{ij}|G|rank_R(\varepsilon M_i)rank_R(M_i)^{-1}_{1_R}$

(3) $\qquad [G:H]1_R=\sum\limits_{i=1}^{t}rank_R(M_i)rank_R(\varepsilon M_i)1_R.$

We observe that $\varepsilon A\varepsilon$ is a symmetric Frobenius R-algebra. More precisely, let $g_1,\ldots,g_m$ be as above and $g_1=e$. Then $\tau:\varepsilon A\varepsilon\to R:\sum\limits_{k=1}^{m}r_k\ \underline{Hg_kH}\mapsto r_1$ defines a symmetric associative R-bilinear form on $\varepsilon A\varepsilon$ with dual R-bases $\left\{a_k = \underline{Hg_kH}\right\}$ and $\left\{b_k = \frac{1}{|Hg_kH|}\underline{Hg_k^{-1}H}\right\}$. So we may apply the results in chapter 2, in particular Proposition 2.2.6.

Now, keep the hypotheses and notation of 3.2.7. Comparing 3.2.7(1) and 2.2.6(1) we see that

$$z\varepsilon e_i = |G|\mathrm{rank}_R(\varepsilon M_i)\mathrm{rank}_R(M_i)^{-1}\varepsilon e_i,$$

with $z = \sum_{k=1}^{m} a_k b_k$ and $\varepsilon M_i \neq 0$.

Chapter 4

Fixed algebras of automorphism groups

In this chapter we study modules and characters over Schur algebras which are fixed rings of automorphism groups.

4.1 Indecomposable modules over fixed algebras

As in 1.2.10, R is a commutative ring, G and H are finite groups and $\sigma : H \to \mathrm{Aut}(G)$ is a homomorphism of groups. The orbits $E_g = \{\sigma_h(g) \mid h \in H\}$, $g \in G$, form a partition of G; $E_g^{-1} = E_{g^{-1}}$ and $E_e = \{e\}$. Each σ_h extends to an R-algebra isomorphism of RG (again denoted by σ_h) as follows : $\sigma_h(\sum\limits_g r_g u_g) = \sum\limits_g r_g u_{\sigma_h(g)}$, with $g \in G$ and $r_g \in R$. Furthermore,
$\sigma : H \to \mathrm{Aut}_R(RG) : h \mapsto \sigma_h$ is a homomorphism of groups.
Consider the fixed ring $RG^H = \{a \in RG \mid \forall h \in H : \sigma_h(a) = a\}$; we have :

4.1.1 Lemma *Keep the above notation, put $s_g = \sum\limits_{x \in E_g} u_x$ in RG, and let G_0 denote a set of representatives of the distinct E_g. Then $RG^H = \bigoplus\limits_{g \in G_0} Rs_g$, i.e. RG^H is a Schur algebra in RG.*

Proof. See 1.2.11.

4.1.2 Example Let G be a cyclic finite group and consider the action of $Aut(G)$ on G.

Let $g \in G$ be an element of order d. Then the orbit $O(g)$ of g consists of all elements of G having order d, in particular $\#O(g) = \varphi(d)$, where φ is the Euler function. Evidently, $\psi(g)$ has order d for any $\psi \in Aut(G)$. Now let $a \in G$ be an element of order d. Let $\#G = p_1^{r_1} \dots p_t^{r_t}$, p_i being prime and $r_i \in I\!N \backslash \{0\}$. We know that $G = S_1 S_2 \dots S_t$, where S_i is the Sylow p_i-subgroup of G. We write $g = g_1 g_2 \dots g_t$ and $a = a_1 a_2 \dots a_t$, $g_i, a_i \in S_i$. Then order $(g_i) =$ order $(a_i) = p_i^{k_i}$, $0 \leq k_i \leq r_i$. Since S_i is cyclic, $< g_i > = < a_i >$, whence $a_i = g_i^{m_i}$, $0 < m_i < p_i^{k_i}$ and m_i relatively prime to $p_i^{k_i}$. But then m_i is relatively prime to $p_i^{r_i}$. So $\psi_i : S_i \to S_i : x_i \mapsto x_i^{m_i}$ is an automorphism of S_i. Now consider $\psi : G \to G$ with $\psi(x) = \psi_1(x_1) \dots \psi_t(x_t)$, where $x_i \in S_i$ and $x = x_1 x_2 \dots x_t$. Then $\psi \in Aut(G)$ and $\psi(g) = a$.

We discuss the problem in a more general context. We recall a few facts about fixed rings of automorphism groups. Throughout A is an R-algebra, H a finite group and $\sigma : H \to \mathrm{Aut}_R(A)$ a homomorphism of groups.
For any $a \in A$, denote by $O(a)$ the orbit $\{\sigma_h(a) \mid h \in H\}$ and set $s(a) = \sum_{x \in O(a)} x$.
Clearly, $A^H = \{a \in A \mid \forall h \in H : \sigma_h(a) = a\}$ is an R-subalgebra of A containing 1_A.
Moreover, for any $a \in A$ we have $s(a) \in A^H$ as well as $\sum_{h \in H} \sigma_h(a) \in A^H$.
Further, the associated skew group ring is denoted by $A * H$. As a left A-module $A * H$ is freely generated by symbols $\{w_h | h \in H\}$ and multiplication is defined by $(aw_h) \cdot (bw_k) = a\sigma_h(b) w_{hk}$ for all $a, b \in A$, $h, k \in H$. Of course $A * H$ is also an R-algebra, where the R-module structure is inherited from A.
If $|H|^{-1} \in R$, then we may consider the idempotent $e_H = |H|^{-1} \sum_{h \in H} w_h$ in $A * H$.
From [M, Lemma 2.1] we retain :

4.1.3 Lemma *Assume $|H|^{-1} \in R$. Then $e_H(A * H)e_H = A^H e_H$, and $A^H e_H$ is isomorphic to A^H as R-algebra.*

Proof. Set $\varepsilon = e_H$, and observe that $(aw_e)v = av$ for all $a \in A$, $v \in A * H$.
For $a \in A$ and $k \in H$ we have $\varepsilon(aw_k) = |H|^{-1} \sum_{h \in H} \sigma_h(a) w_{hk}$. But $w_t \varepsilon = \varepsilon$. Therefore $\varepsilon(aw_k)\varepsilon = |H|^{-1} \sum_{h \in H} \sigma_h(a)\varepsilon$, and this shows that $\varepsilon(A * H)\varepsilon \subset A^H \varepsilon$. On the other hand, $a = |H|^{-1} \sum_{h \in H} \sigma_h(a)$ for all $a \in A^H$, and the equality follows.
Using the expressions given above, it is easily verified that $A^H \to A^H \varepsilon : a \mapsto a\varepsilon$ is

an isomorphism of R-algebras. $\square$

We may use the preceding lemma to prove :

4.1.4 Proposition *Let A, H, σ be as before and assume $|H|^{-1} \in R$.*
(1) If A is finitely generated and projective as R-module, then so is A^H.
(2) Suppose that A is finitely generated projective and faithful as R-module. If A is separable over R, then so is A^H.
(3) If A is a semisimple ring, then A^H is semisimple too.

Proof. (1) Let $\{a_1, \ldots, a_n\} \subset A$, $\{\varphi_1, \ldots, \varphi_n\} \subset \mathrm{Hom}_R(A, R)$ be a dual basis for A. Then it is easily checked that $\{|H|^{-1} \sum\limits_{h \in H} \sigma_h(a_i)\}$, $\{\varphi_i|_{A^H}\}$ is a dual basis for A^H.
(2) Let $\sum\limits_{i=1}^{m} x_i \otimes y_i \in A \otimes_R A^0$ be a separability idempotent for A. Then it is easily verified that $|H|^{-1} \sum\limits_{h \in H} \sum\limits_{i=1}^{m} (\sigma_h(x_i) w_h \otimes y_i w_{h^{-1}})$ is a separability idempotent for $A * H$. So $A * H$ is separable over R. Moreover, $A * H$ is finitely generated projective as R-module. We now apply Lemma 4.1.3 and Proposition 3.1.1.
(3) See [Mo-Theorem 1.15]. $\square$

Let us return to the case where $A = RG$ and H acts on G. Then $A * H$ is isomorphic to $R(G \times_\sigma H)$ as R-algebra, where $G \times_\sigma H$ is the semidirect product of G and H (i.e. $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \sigma_{h_1}(g_2), h_1 h_2)$ for $g_i \in G$, $h_i \in H$). The isomorphism maps $u_g w_h \in A * H$ onto (g, h) for any $g \in G$, $h \in H$.
In case $|H|^{-1} \in R$, the algebra RG^H is isomorphic to a double coset algebra in $R(G \times_\sigma H)$, see Lemma 4.1.3. Furthermore we have :

4.1.5 Proposition *(1) If $|H|$ and $|G|$ are invertible in R, then RG^H is separable over R.*
(2) Suppose R is connected, and $|H|$ and $|G|$ are invertible in R. If R is a splitting ring for $R(G \times_\sigma H)$, then R is a splitting ring for RG^H.
In particular, let m be the exponent of $G \times_\sigma H$ and η a primitive m-th root of unity, then $R[\eta]$ is a splitting ring for RG^H.

Proof. (1) We know that $|G|^{-1} \in R$ implies that RG is separable over R, and we may apply 4.1.4(2).

(2) The first statement follows from Lemma 4.1.3 and Proposition 3.1.5. The second part follows from [S]. $\square$

Next we deal with indecomposable modules. Connections between RG^H-modules and $R(G \times_\sigma H)$-modules are given by the theory of double coset algebras, developed chapter 3. We now investigate the relationship between indecomposable RG^H-modules and indecomposable RG-modules. We return to the general situation where A is an R-algebra, H a finite group and $\sigma : H \to \mathrm{Aut}_R(A)$ a homomorphism of groups. We require the following definition.

4.1.6 Definition Let M be a left A-module and let $h \in H$. We obtain a left A-module ${}^hM$ as follows : consider the underlying abelian group of M and let A act on it by setting $a \circ m = \sigma_h^{-1}(a)m$ for all $a \in A$, $m \in M$.

Observe that the induced R-module structure on ${}^hM$ coincides with that on M and ${}^hM \cong w_h A \otimes_A M$ as left A-modules.

4.1.7 Remarks 1. Let M, N be left A-modules and let $h, k \in H$. Then ${}^k({}^hM) = {}^{kh}M$ as A-modules, and $\mathrm{Hom}_A({}^hM, {}^hN) = \mathrm{Hom}_A(M, N)$.

2. Let M be a left A-module which is finitely generated and projective over R. For the trace functions we get : $t_{{}^hM}(a) = t_M(\sigma_h^{-1}(a))$ for all $a \in A$, $h \in H$.

3. If M is an indecomposable, resp. a finitely generated projective, left A-module, then so is ${}^hM$ for all $h \in H$. In particular, if $m_i \in M$, $f_i \in \mathrm{Hom}_A(M, A)$ is an A-dual basis for M. Then m_i, $\sigma_h \circ f_i$ is an A-dual basis for ${}^hM$.

4. Suppose that R is connected and that A is finitely generated and projective as R-module. Let $\{e_1, \ldots, e_q\}$, resp. $\{d_1, \ldots, d_m\}$, be the set of primitive central nonzero idempotents of A, resp. A^H (use rank_R). Then H acts on $\{e_1, \ldots, e_q\}$ by σ. Again, let $s(e_i)$ denote the sum of the idempotents in the orbit of e_i. Each $s(e_i)$ is uniquely expressible as a sum of d_j's, and each d_j appears in one and only one of the $s(e_i)$. Note also that d_j appears in $s(e_i)$ if and only if $d_j e_i \neq 0$.

5. Let R, A, e_i, d_j be as in (4), and let M be an indecomposable left A-module

lying over e_i. We observe that ${}^hM$ lies over $\sigma_h(e_i)$, $h \in H$.

Further, it is clear that $d_je_i = 0$ implies $d_jM = 0$. Moreover, if M is finitely generated projective over A and if any two indecomposable finitely generated projective left A-modules lying over the same primitive central idempotent are isomorphic as A-modules, then the converse is true. Indeed, suppose $d_jM = 0$ and write $e_i = \eta_1 + \ldots + \eta_t$, η_k being primitive idempotents of A. Then $d_jA\eta_k = 0$ for $k = 1, \ldots, t$, whence $d_je_i = 0$.

Note also that $M|_{A^H}$ is the direct sum of the nonzero d_jM.

4.1.8 Theorem *Suppose that R is connected and that A is finitely generated and projective as R-module. Let P be an indecomposable left A^H-module, and let e be a primitive central idempotent of A such that $e(A \otimes_{A^H} P) \neq 0$. Set $W = e(A \otimes_{A^H} P)$ and $F = \{h \in H \mid \sigma_h(e) = e\}$. Then*

(1) $A \otimes_{A^H} P \cong \bigoplus_{i=1}^{r} {}^{h_i}W$ as left A-modules, where $\{h_1, \ldots, h_r\}$ is a set of left coset representatives of F in H.

Moreover $F = \{h \in H \mid {}^hW \cong W$ as A-modules$\}$.

(2) If P is finitely generated and projective over A^H, then we may write $A \otimes_{A^H} P = M_1 \oplus \ldots \oplus M_s$ where each M_i is an indecomposable left A-module. In this case W is the direct sum of all M_i lying over e.

Proof. (1) Let $\{e = e_1, \ldots, e_t\}$ be the set of all primitive central idempotents of A for which $e_j(A \otimes_{A^H} P) \neq 0$, and set $W_j = e_j(A \otimes_{A^H} P)$. Then $A \otimes_{A^H} P = W_1 \oplus \ldots \oplus W_t$ ($W = W_1$).

Further, let d denote the primitive central idempotent of A^H for which $dP \neq 0$. Then $e_j(A \otimes_{A^H} P) \neq 0$ implies $e_jd \neq 0$. By Remark 4.1.7(4), it follows that $e_1, \ldots, e_t$ belong to the same orbit (of the action of H).

Now let $h \in H$. We observe that $A \otimes_{A^H} P \to {}^h(A \otimes_{A^H} P) : \sum_i a_i \otimes p_i \mapsto \sum_i \sigma_h^{-1}(a_i) \otimes p_i$ is an isomorphism of left A-modules. Thus $\sigma_h(e)(A \otimes_{A^H} P) \cong \sigma_h(e)_\circ {}^h(A \otimes_{A^H} P) = {}^hW \neq 0$ as A-modules. This yields $\sigma_h(e) = e_j$ for some $j \in \{1, \ldots, t\}$.

Moreover we obtain $W_j \cong {}^hW$. Furthermore, if $\sigma_h(e) = e$, then $W \cong {}^hW$. The converse follows from the fact that $eW = W$ and $\sigma_h(e)_\circ {}^hW = {}^hW$.

(2) It is clear that $A \otimes_{A^H} P$ is nonzero, finitely generated and projective over A,

hence also over R, and use rank_R. $\square$

4.1.9 Remark (1) From the proof of Theorem 4.1.8 it follows that $e(A \otimes_{A^H} P) \neq 0$ if and only if $A \otimes_{A^H} P \neq 0$ and $ed \neq 0$.
(2) Theorem 4.1.8 remains true if P is a left A^H-module which lies over only one primitive central idempotent of A^H.

As an immediate consequence of 4.1.8, we obtain :

4.1.10 Corollary *Keep the hypotheses and notation of Theorem 4.1.8(2), and suppose that any two indecomposable finitely generated projective left A-modules lying over the same primitive central idempotent are isomorphic as A-modules.*
Then $A \otimes_{A^H} P \cong \bigoplus_{i=1}^{r} ({}^{h_i}M)^k$ *as A-modules, where M is an indecomposable finitely generated projective left A-module lying over e and* $k \in \mathbb{N}$*. Moreover* $\sigma_h(e) = e$ *if and only if* ${}^hM \cong M$.

Note. In case A is a semisimple ring (then A^H is semisimple too), the statement in Corollary 4.1.10 remains true for a simple A^H-module P and a simple A-module M.

Let $\mathrm{Inn}(A)$ denote the group of inner automorphisms of A. As a special case we now obtain :

4.1.11 Corollary *Suppose that* $\sigma(H) \subset Inn(A)$. *Then we have* $A \otimes_{A^H} P = W$ *in Theorem 4.1.8, and we have* $A \otimes_{A^H} P \cong M^k$ *in Corollary 4.1.10.*

Note. Suppose that $\sigma(H) \subset \mathrm{Inn}(A)$. Let U denote the group of invertible elements of A and consider $j : U \to \mathrm{Inn}(A) : u \mapsto j_u$ with $j_u(a) = uau^{-1}$ for all $a \in A$. Take the subgroup $L = j^{-1}(\sigma(H))$ of U and restrict j to L. Then $A^H = A^L$ and A^L is the centralizer in A of the R-subalgebra generated by L.

4.2 Trace functions

We return to Schur algebras. So let G, H be finite groups, let $\sigma : H \to \mathrm{Aut}(G)$ be a homomorphism of groups. Again, for any $g \in G$, put $E_g = \{\sigma_h(g) | h \in H\}$ and $s_g = \sum_{x \in E_g} u_x$. Then $RG^H = \bigoplus_{g \in G_0} Rs_g$ where G_0 denotes a set of representatives of the distinct E_g. Suppose R is connected and let $\{e_1, \ldots, e_q\}$, resp. $\{d_1, \ldots, d_m\}$, be the set of primitive central nonzero idempotents of RG, resp. RG^H.

Under suitable conditions, the module relations in 4.1.10 and 4.1.11 can be translated into relations between trace functions on RG and on RG^H. Suppose finitely generated projective R-modules are free, $|G|^{-1} \in R$ and R is a splitting ring for RG and for RG^H (see e.g. 4.1.5(2)). Thus $RG \cong \mathrm{End}_R(M_1) \oplus \ldots \oplus \mathrm{End}_R(M_q)$ and $RG^H \cong \mathrm{End}_R(P_1) \oplus \ldots \oplus \mathrm{End}_R(P_m)$ as R-algebras, where the M_i and P_j are finitely generated projective R-modules. Assume that M_i lies over e_i and P_j over d_j. Combining Corollary 4.1.10, Proposition 2.4.3 and 4.1.7(2) we obtain for any $g \in G$:

(1) $kt_{P_j}(z) \sum_{\ell=1}^{r} t_{M_i}(u_{\sigma_{h_\ell}^{-1}(g)}) = \mathrm{rank}_R(P_j)\tilde{t}_{P_j}(\sum_{x \in G} u_{xgx^{-1}})$ with k, r, h_ℓ as in 4.1.10, $\tilde{t}_{p_j}$ as in 2.4.2 and $z = \sum_{g \in G_0} |E_g|^{-1} s_g s_{g^{-1}}$.
If P_j lies over d_j, then M_i lies over e_i with $e_i d_j \neq 0$, see 4.1.9(1). Moreover, by the hypotheses, $e_i d_j \neq 0$ if and only if $d_j M_i \neq 0$ (as in 4.1.7(5)).

If z is invertible in RG^H, then using relation (1) for $g = e$ and using 2.2.6(2), the relation (1) can be rewritten as :

(2) $|G| \mathrm{rank}_R(P_j) \sum_{\ell=1}^{r} t_{M_i}(u_{\sigma_{h_\ell}^{-1}(g)}) = r\mathrm{rank}_R(M_i)\tilde{t}_{P_j}(\sum_{x \in G} u_{xgx^{-1}})$

Now suppose $\sigma(H) \subset \mathrm{Inn}G$. Then in the relation (1) we have $r = 1$, $h_1 = e$ and $Z(RG) \subset RG^H$. As a consequence :

(3) $kt_{P_j}(z)t_{M_i}(u_g) = \mathrm{rank}_R(P_j)t_{P_j}(\sum_{x \in G} u_{xgx^{-1}})$.
In this case we have for each d_j a unique e_i such that $e_i d_j \neq 0$.

If z is invertible in RG^H, then the relation (3) can be rewritten as :

(4) $|G| \mathrm{rank}_R(P_j)t_{M_i}(u_g) = \mathrm{rank}_R(M_i)t_{P_j}(\sum_{x \in G} u_{xgx^{-1}})$. We shall obtain this formula in 5.4.10(∗) in a different context.

We now express primitive central idempotents of RG^H in terms of trace functions.

4.2.1 Proposition *Assume R connected, $|G|^{-1} \in R$ and $|H|^{-1} \in R$. Suppose that $RG \cong \bigoplus_{i=1}^{q} End_R(M_i)$ as R-algebras, M_i being a finitely generated projective R-module, and assume that M_i lies over e_i. Consider a primitive central idempotent $d = d_j$ of RG^H, and let e_i be such that $e_i d \neq 0$. Then :*

$$|F|d = |H||G|^{-1} rank_R(M_i) \sum_{g \in G_0} |E_g|^{-1} t_{dM_i}(s_{g^{-1}}) s_g$$

with $F = \{h \in H | \sigma_h(e_i) = e_i\}$. Moreover, $dM_i \neq 0$.

Proof. By 2.2.8, $e_i = |G|^{-1} \operatorname{rank}_R(M_i) \sum_{g \in G} t_{M_i}(u_{g^{-1}}) u_g$. Then, applying 1.1.7, $e_i d = |G|^{-1} \operatorname{rank}_R(M_i) \sum_{g \in G} t_{M_i}(u_{g^{-1}} d) u_g$. The same formula holds for $\sigma_h(e_i)$ and ${}^h M_i$, $h \in H$. Then adding up these results, and using 4.1.7(2) and $s(e_i)d = d$, we obtain :

$$|F|d = |H||G|^{-1} \operatorname{rank}_R(M_i) \sum_{g \in G} |E_g|^{-1} t_{M_i}(s_{g^{-1}} d) u_g.$$

Now the result follows, using $t_{M_i}(s_{g^{-1}} d) = t_{dM_i}(s_{g^{-1}})$. $\square$

4.2.2 Note Suppose that any two indecomposable finitely generated projective left RG^H-modules lying over the same primitive central idempotent d of RG^H are isomorphic as RG^H-modules (This follows for example when R is semilocal and $|G|^{-1}$, $|H|^{-1} \in R$).
Let P be an indecomposable left, finitely generated projective RG^H-module lying over d, then $dM_i \cong P^n$ as left RG^H-modules, $n \in \mathbb{N}$.

In chapter 2 we have developed a character theory for Frobenius algebras, in particular for $RG^H (|H|^{-1} \in R)$. Comparing the expressions of Proposition 4.2.1 and Proposition 2.2.6(1) we obtain :

4.2.3 Corollary *Keep the notation and hypotheses of 4.2.1. In addition, suppose that finitely generated projective R-modules are free and that R is a splitting ring for*

RG^H (see e.g. 4.1.5(2)). Let P be an indecomposable finitely generated projective left RG^H-module lying over d. Then $dM_i \cong P^n$ as RG^H-modules and $n\,rank_R(M_i)zd = |H|^{-1}|F|\ |G|rank_R(P)d$, with M_i, F as in 4.2.1 and $z = \sum_{g\in G_0} |E_g|^{-1}s_{g^{-1}}s_g$.

4.2.4 Remark If $\sigma(H) \subset \mathrm{Inn}(G)$, then $F = H$ in 4.2.1 and 4.2.3. In this case, we may get more information about d, P and n, see 5.2.5 - 5.2.6 - 5.2.8 - 5.2.13(1) - 5.4.4.

To conclude, we observe that in the case $\sigma(H) \subset \mathrm{Inn}(G)$, RG^H is always a centralizer of a group algebra RK in RG.
More precisely, consider $i : G \to \mathrm{Inn}(G) : g \mapsto i_g$ with $i_g(x) = gxg^{-1}$ for all $x \in G$. In this case, we take the subgroup $K = i^{-1}(\sigma(H))$ of G and we restrict i tot K. Extending to automorphisms of RG, we get $RG^H = RG^K$. Now, for any subgroup K of G and homomorphism $i : K \to \mathrm{Inn}(G)$, we see that RG^K is the centralizer of RK in RG. Further results on modules and trace functions over centralizers can be found in chapter 5.

Chapter 5

Centralizers

Let R be a commutative ring, G a finite group and $H < G$. Then $\sigma : H \to Aut(G) : h \to \sigma_h$, with $\sigma_h(g) = hgh^{-1}$, is a homomorphism of groups. The orbits $E_g = \{hgh^{-1} | h \in H\}$ are called subclasses of H in G. The subclass sums $s_g = \sum_{x \in E_g} u_x$ form an R-basis for the fixed ring $S = RG^H$ (see 1.2.11), which is called the subclass algebra of H in RG. In chapter 4 we studied modules and characters over Schur algebras, which are fixed rings of automorphism groups.
But the subclass algebra S is also the centralizer of RH in RG.
In this chapter we develop more relations between indecomposable modules over RG, RH and S. However we shall consider the more general context of centralizers in separable algebras (see sections 1-2).
In sections 3 and 4 we apply the results on centralizers to the twisted group rings $R *_\alpha H$ and $R *_\alpha G$ with $H < G$ and we develop a generalized Clifford theory. In section 5 we focus on the situation where $H \lhd G$.

5.1 The rank of a centralizer

Let B be a subalgebra of an R-algebra A. Under certain conditions, we will develop a formula relating the rank of the centralizer of B in A to the restriction to B of indecomposable left A-modules. Of course, the result can be applied to the case where $A = RG$ and $B = RH$, G being a finite group and $H < G$. Here the rank is equal to the number of subclasses. The latter extends a result of E.P. Wigner, see [W].

Now, let R be a connected commutative ring, and suppose that finitely generated projective R-modules are free. This occurs for example when R is a semilocal connected ring or a principal ideal domain. Further, A will be an R-algebra and B a subalgebra of A with $1_A \in B$ and we suppose that R is a splitting ring for A and B, that is, $A \cong End_R(M_1) \oplus \ldots \oplus End_R(M_s)$ and $B \cong End_R(N_1) \oplus \ldots \oplus End_R(N_t)$, where M_i and N_i are finitely generated projective R-modules. Recall that M_i is an indecomposable left A-module under the operation $(\varphi_1, \ldots, \varphi_s)m = \varphi_i(m)$; analogously N_i. By our hypotheses, each M_j, viewed as a left B-module, is uniquely expressible as a finite direct sum of N_i's, see [1.1.] and c_{ij} denotes the multiplicity of N_i in this decomposition of M_j. Note that c_{ij} may be equal to 0.

5.1.1 Remark. Keep the above notation and hypotheses. Then for each N_i there is some M_j such that N_i occurs in the decomposition into indecomposable left B-modules of M_j. Indeed, assume that N_i lies over the primitive central idempotent f_i of B. Since $A = Af_i \oplus A(1 - f_i)$, Af_i is a finitely generated projective R-module, and thus Af_i is isomorphic in A-mod to a finite direct sum of M_k's. Therefore there is some M_j such that $f_i M_j \neq 0$ and the statement follows.

Let S denote the centralizer of B in A, i.e. $S = \{a \in A | \forall b \in B : ab = ba\}$. We now prove that S is a free R-module of finite rank and we give an expression for the rank.

5.1.2 Proposition. *Keep the above notation and hypotheses. Let V be a finitely generated projective R-module and $T : A \to End_R(V)$ an R-algebra morphism. As a left A-module V is isomorphic to a direct sum of M_j's and we suppose that $M_1, \ldots, M_q$ occur in the decomposition (up to renumbering). Then the centralizer of $T(B)$ in $T(A)$ is a free R-module of rank $\sum_{j=1}^{q} \sum_{i=1}^{t} (c_{ij})^2$.*

Proof. Let $T_j : A \to End_R(M_j)$ be the R-algebra morphism corresponding to the left A-module structure of M_j, i.e. $T_j(a)(m) = am$ for all $m \in M_j$. Consider $f : T(A) \to \bigoplus_{j=1}^{q} T_j(A) : T(a) \mapsto (T_1(a), \ldots, T_q(a))$.
From $V \cong M_1^{k_1} \oplus \ldots \oplus M_q^{k_q}$ in A-mod it easily follows that f is well-defined and that f is injective. We now show that f is surjective. Let $\{e_1, \ldots, e_s\}$ be the set of

primitive central nonzero idempotents of A and assume that M_j lies over e_j. Consider $(T_1(a_1), \dots, T_q(a_q))$ with $a_1, \dots, a_q \in A$. Setting $a = a_1e_1 + \dots + a_qe_q$, we have $T_j(a) = T_j(a_je_j) = T_j(a_j)$ for each j. Also it is clear that f is an R-algebra morphism. Now it is easy to check that f induces an R-algebra isomorphism between the centralizer of $T(B)$ in $T(A)$ and $\bigoplus_{j=1}^{q} C_j$ where C_j is the centralizer of $T_j(B)$ in $T_j(A)$. But since R is a splitting ring for A, we have $T_j(A) = End_R(M_j)$ for each j and thus $C_j = End_B(M_j)$ for each j. Then we apply Proposition 1.1.3(2). □

5.1.3 Corollary. *With hypotheses as before, the centralizer S of B in A is isomorphic to $End_B(M_1) \oplus \dots \oplus End_B(M_s)$ as R-algebra and it is a free R-module of rank $\sum_{j=1}^{s} \sum_{i=1}^{t} (c_{ij})^2$.*

Proof. Consider the left regular representation of A, that is, $T : A \hookrightarrow End_R(A)$ given by $T(a)(x) = ax$ for all $x \in A$. It is easy to see that each M_j, $j = 1, \dots, s$, occurs in the decomposition of A into indecomposable left A-modules. Then the statement follows from Proposition 5.1.2 and its proof. □

If A and B are group rings, the Corollary 5.1.3 yields :

5.1.4 Proposition. *Let R be as before. Let G be a finite group with $|G|^{-1} \in R$, let H be a subgroup of G and suppose that R is a splitting ring for RG and RH. Further, let $M_1, \dots, M_s$, resp. $N_1, \dots, N_t$, be a basic set of indecomposable left RG-modules, resp. RH-modules, which are finitely generated and projective over R, and let c_{ij} be the multiplicity of N_i in M_j.*
Then $\sum_{j=1}^{s} \sum_{i=1}^{t} (c_{ij})^2 =$ number of subclasses.

Proof. This result is a consequence of 5.1.3 and 1.2.11.

5.1.5 Remark. Let R be any connected commutative ring and let G be a finite group such that $|G|^{-1} \in R$. If $m = exp(G)$ and η is a primitive m-the root of unity, then $R[\eta]$ is a splitting ring for the group ring RG over R, see [S]. Since an extension of a splitting ring is a splitting ring, we see that $R[\eta]$ is also a splitting ring for RH, where H is a subgroup of G.

To conclude this section, we mention the following result.

5.1.6 Proposition. *Let R be a commutative ring, let A be a separable R-algebra and let B be a separable R-subalgebra of A containing 1_A. Then the centralizer S of B in A is separable over R. Moreover, the centralizer of S in A is equal to $Z(A)B$, $Z(A)$ being the center of A.*

Proof. Consider $f : Z(A) \otimes_R B \to Z(A)B : \sum_i a_i \otimes b_i \to \sum_i a_i b_i$. Clearly f is a surjective R-algebra homomorphism, hence $Z(A)B \cong (Z(A) \otimes_R B)/ker\ f$.
Using [D.M-I, 1.7 p.44 and 1.11 p.46] we obtain that $Z(A)B$ is separable over $Z(A)$. Since A is separable over $Z(A)$, [DM-I, 3.8 p.55], we can use [D.M-I, 4.3 p.57] to conclude that the centralizer of $Z(A)B$ in A is separable over $Z(A)$. Clearly the centralizer of $Z(A)B$ in A is equal to S. Since $Z(A)$ is separable over R, we deduce that S is separable over R, see [D.M-I, 3.8 p.55 and 1.12 p.46].
The rest of the statement follows from [D.M-I, 4.3] □

5.2 Indecomposable modules over centralizers

Let B be a subalgebra of an R-algebra A. Our objective is to investigate the relations between indecomposable modules over A, B and the centralizer of B in A. Of course, the results can be applied to the case where A and B are group rings.

Throughout this section, R is a connected commutative ring and we suppose that finitely generated projective R-modules are free. Further, A will be an R-algebra and B a subalgebra of A with $1_A \in B$ and we suppose that R is a splitting ring for A and B, that is, $A \cong \mathrm{End}_R(M_1) \oplus \ldots \oplus \mathrm{End}_R(M_s)$ and $B \cong \mathrm{End}_R(N_1) \oplus \ldots \oplus \mathrm{End}_R(N_t)$, where the M_i and N_i are finitely generated projective R-modules. Let $\{e_1, \ldots, e_s\}$ respectively $\{f_1, \ldots, f_t\}$ be the set of primitive central nonzero idempotents of A respectively B and assume that M_i lies over e_i and N_i over f_i. Each M_j, viewed as a left B-module, is uniquely expressible as a finite direct sum of N_i's and c_{ij} denotes the multiplicity of N_i in this decomposition of M_j. Finally, S denotes the centralizer of B in A. From 5.1.6 and 5.1.3 we know that S is a separable R-algebra and a free R-module of finite rank.

Now each $Hom_B(N_i, M_j)$ is a left S-module under the operation $(s \cdot \varphi)(n) = s(\varphi(n))$ for $s \in S$, $\varphi \in Hom_B(N_i, M_j)$, $n \in N_i$.

5.2.1 Remarks. (1) The above left S-module structure arises from the following : $Hom_B(N_i, M_j)$ is a left $\mathrm{End}_B(M_j)$-module by composition of maps and so it is a left S-module by the algebra isomorphism given in 5.1.3.
(2) Let ε be a primitive idempotent of B such that $\varepsilon f_i \neq 0$, then by our hypotheses $N_i \cong B\varepsilon$ as left B-modules.
Further, $Hom_B(B\varepsilon, M_j) \to \varepsilon M_j : \varphi \mapsto \varphi(\varepsilon)$ is an S-module isomorphism.

5.2.2 Proposition. *If $c_{ij} = 0$, then $Hom_B(N_i, M_j) = 0$. Otherwise, $Hom_B(N_i, M_j)$ is a free R-module of rank c_{ij}.*

Proof. This statement follows from Proposition 1.1.3(1).

We now concentrate on the relation between $Hom_B(N_i, M_j)$ and primitive central idempotents. Later on we shall make use of these facts.

5.2.3 Remark. First note that the centers of A and B are contained in the center of S. Since $f_1, \ldots, f_t$ belong to the center of S, we know that each f_i is uniquely expressible as a sum of distinct primitive central idempotents of S. Moreover, since in the rings considered primitive central idempotents are orthogonal and their sum equals 1, we have that each primitive central nonzero idempotent of S appears in one and only one of the f_i's.
A similar observation holds for $e_1, \ldots, e_s$.

5.2.4 Lemma. *Let d be a primitive central nonzero idempotent of S. Then*

$dHom_B(N_i, M_j) \neq 0$ if and only if d appears in the decomposition of f_i and e_j (into primitive central idempotents of S). In particular, by 5.2.3, there is exactly one $Hom_B(N_i, M_j)$ such that $dHom_B(N_i, M_j) \neq 0$.

Proof. If $dHom_B(N_i, M_j) \neq 0$, then there exists $\varphi \in Hom_B(N_i, M_j)$, $n \in N_i$ such that $d\varphi(n) \neq 0$. Now $d\varphi(n) = d\varphi(f_i n) = df_i\varphi(n)$, whence $df_i \neq 0$, and $d\varphi(n) = de_j\varphi(n)$ implies $de_j \neq 0$. Consequently, d occurs in f_i and e_j.
Conversely, assume that d appears in the decomposition of f_i and e_j. Write e_j as a sum of primitive orthogonal idempotents of A. Since $de_j \neq 0$, we have $d\eta \neq 0$ for some primitive idempotent η of A appearing in this decomposition. Next we express f_i as a sum of primitive orthogonal idempotents of B, and $df_i\eta = d\eta \neq 0$ implies $d\varepsilon\eta \neq 0$ for some primitive idempotent ε of B appearing in the decomposition of f_i. Therefore $d\varepsilon A\eta \neq 0$. But by our hypotheses we have $B\varepsilon \cong N_i$ in B-mod and $A\eta \cong M_j$ in A-mod and, using Remark 5.2.1 we get $Hom_B(N_i, M_j) \cong \varepsilon A\eta$ in S-mod. Thus $dHom_B(N_i, M_j) \neq 0$ as required. □

5.2.5 Corollary. *(1) $Hom_B(N_i, M_j) \neq 0$ if and only if $f_ie_j \neq 0$.*
(2) The nonzero $Hom_B(N_i, M_j)$ are not isomorphic as left S-modules.
(3) If the nonzero $Hom_B(N_i, M_j)$ are indecomposable left S-modules, then the nonzero f_ie_j are precisely the distinct primitive central idempotents of S. Moreover $Hom_B(N_i, M_j)$ lies over f_ie_j.

Proof. Put $P_{ij} = Hom_B(N_i, M_j)$. Let $\varphi \in P_{ij}$; then $f_ie_j\varphi = \varphi$ and $f_ke_\ell\varphi = 0$ if $k \neq i$ or $\ell \neq j$.
(1) By the above observation, $P_{ij} \neq 0$ implies $f_ie_j \neq 0$. Conversely, if $f_ie_j \neq 0$, then f_ie_j is a sum of primitive central idempotents d of S. By Lemma 5.2.4, $dP_{ij} \neq 0$, whence $P_{ij} \neq 0$.
(2) Follows from the above observation.
(3) Let $f_ie_j \neq 0$. Then using Lemma 5.2.4 and the fact that $Hom_B(N_i, M_j)$ is indecomposable, we see that the decompositions of f_i and e_j into primitive central idempotents of S have one and only one element d in common, and thus $f_ie_j = d$. □

Our next objective is to investigate when the $Hom_B(N_i, M_j)$ are indecomposable left S-modules.

First we consider the case where we know that R is a splitting ring for S. For example, let R be a connected commutative ring, let G be a finite group with $|G|^{-1} \in R$, let $H < G$ and consider $A = RG$ and $B = RH$. Then the centralizer S of RH in RG is the fixed ring RG^H of RG under the action $\sigma : H \to Aut\ G$ with $\sigma_h(g) = hgh^{-1}$. Now let m be the exponent of $G \times_\sigma H$ and let η be a primitive m-th root of unity, then $T = R[\eta]$ is a splitting ring for S (see Proposition 4.1.5(2)). Note that T is also a splitting ring for RG and RH.

5.2.6 Proposition. *If R is a splitting ring for S, then the nonzero $Hom_B(N_i, M_j)$ are indecomposable left S-modules, and they are, up to isomorphism, the only indecomposable S-modules which are finitely generated projective.*

Proof. Let $S \cong \mathrm{End}_R(V_1) \oplus \ldots \oplus \mathrm{End}_R(V_q)$ as algebra, V_k being finitely generated projective R-modules. Consider a nonzero $Hom_B(N_i, M_j)$. Let d_k, $1 \leq k \leq n$, be the primitive central nonzero idempotents of S for which $d_k Hom_B(N_i, M_j) \neq 0$. Assume that V_k lies over d_k, $1 \leq k \leq n$. Then $Hom_B(N_i, M_j) \cong V_1^{m_1} \oplus \ldots \oplus V_n^{m_n}$ as left S-modules ($m_k \in \mathbb{N}$), see (1.1). As a consequence, $c_{ij} = m_1\ rank_R(V_1) \ldots + m_n rank_R(V_n)$. Thus

$$(rank\ V_1)^2 + \ldots + (rank\ V_n)^2 \leq c_{ij}^2 \qquad (*)$$

Now for each primitive central nonzero idempotent d of S, there is one and only one nonzero $Hom_B(N_k, M_\ell)$ such that $dHom_B(N_k, M_\ell) \neq 0$, see Lemma 5.2.4. On the other hand, Corollary 5.1.3 states that $rank_R(S) = \sum_{j=1}^{s} \sum_{i=1}^{t} c_{ij}^2$, and $rank_R(S) = (rankV_1)^2 + \ldots + (rankV_q)^2$. Combining these facts, we conclude that we have an equality in $(*)$.

But this implies $n = 1$ and $m_1 = 1$. The statement is now clear. $\square$

Remark. When $R = \mathbb{C}$, $A = RG$ and $B = RH$, Proposition 5.2.6 can be applied (use 5.1.6). In this case the irreducible modules of S are constructed in a different way by J. Karlof, see [K].

We now consider another situation in the S-modules $Hom_B(N_i, M_j)$ are indecomposable.

5.2.7. Theorem. *If R is a semilocal ring or a principal ideal domain, then each nonzero $Hom_B(N_i, M_j)$ is an indecomposable left S-module.*

Proof. We write $Hom_B(N_i, M_j)$ as a finite direct sum of indecomposable left S-modules. Let d_k, $1 \leq k \leq n$, be the primitive central nonzero idempotents of S for which $d_k Hom_B(N_i, M_j) \neq 0$ and choose for each k an indecomposable left S-module V_k lying over $d_:$ and appearing in the decomposition of $Hom_B(N_i, M_j)$. Note that each V_k is a finitely generated projective R-module. We first prove that $\text{rank}_R(Sd_k) \leq (\text{rank}_R(V_k))^2$ for each k.

Let us write d, V instead of d_k, V_k and rank instead of rank_R. Since S is separable and projective over R, the R-algebra morphism $T : S \to \text{End}_R(V)$, associating to $x \in S$ the left multiplication by x in $\text{End}(V)$, restricts to an injective R-algebra morphism $T : Sd \to \text{End}_R(V)$ mapping d to the identity, see [N_2-$v.O_2$, Proposition 1.6]. So when R is a principal ideal domain, it follows at once that $\text{rank}(Sd) \leq (\text{rank}V)^2 = \text{rank}\text{End}_R(V)$.

We now suppose that R is semilocal.

Express d as a sum of primitive orthogonal nonzero idempotents of S, say $d = \varepsilon_1 + \ldots + \varepsilon_m$. Since S is separable over R and R is semilocal, it follows from Proposition 1.1.1 that $S\varepsilon_i \cong V$ in S-mod, hence we have $\text{rank}(Sd) = m\text{rank}V$. It thus suffices to show that $m \leq \text{rank}V$. First observe that V is, up to isomorphism, the only indecomposable left $\text{End}_R(V)$-module, which is finitely generated and projective over R (under the operation : $\varphi.v = \varphi(v)$ for all $\varphi \in \text{End}_R(V), v \in V$). Therefore the number of primitive orthogonal nonzero idempotents of $\text{End}_R(V)$ appearing in a decomposition of the identity must be equal to $\text{rank}V$. But $I = T(d) = T(\varepsilon_1) + \ldots + T(\varepsilon_m)$ and $T(\varepsilon_1), \ldots, T(\varepsilon_m)$ are orthogonal nonzero idempotents of $\text{End}_R(V)$. Consequently, $m \leq \text{rank}V$, as desired.

So, using the fact that $\text{rank}_R(Hom_B(N_i, M_j)) = c_{ij}$, we obtain the following inequalities :

$$\mathrm{rank}(Sd_1) + \ldots + \mathrm{rank}(Sd_n) \leq (\mathrm{rank}V_1)^2 + \ldots + (\mathrm{rank}V_n)^2 \leq c_{ij}^2 \qquad (*)$$

Now for each primitive central nonzero idempotent d of S there is one and only one $Hom_B(N_k, M_\ell)$ such that $dHom_B(N_k, M_\ell) \neq 0$, see Lemma 5.2.4. On the other hand, Corollary 5.1.3 states that $\mathrm{rank}_R(S) = \sum\limits_{j=1}^{s}\sum\limits_{i=1}^{t}(c_{ij})^2$. Combining these facts, we conclude that we have equalities in $(*)$.
But $(\mathrm{rank}V_1)^2+\ldots+(\mathrm{rank}V_n)^2 = c_{ij}^2 = (\mathrm{rank}Hom_B(N_i, M_j))^2$ implies that $Hom_B(N_i, M_j)$ must be an indecomposable left S-module, which completes the proof. $\square$

5.2.8 Remark. Hypotheses as in 5.2.7. For later use, we deduce from 5.2.5(3) and the proof of Theorem 5.2.7 the following : if $Hom_B(N_i, M_j) \neq 0$, then $\mathrm{rank}_R(Sf_ie_j) = c_{ij}^2$.

Moreover, for semilocal rings we have :

5.2.9 Proposition. *If R is semilocal, then R is a splitting ring for S over R.*

Proof. Let d be a primitive central nonzero idempotent of S and let $V = Hom_B(N_i, M_j)$ lie over d ($d = f_ie_j$). By [N_2-$v.O_2$, Proposition 1.6 and Corollary 1.7], the left S-module structure of V induces an injective R-algebra homomorphism $T : Sd \to \mathrm{End}_R(V)$ mapping d to the identity, and T will be surjective if and only if $\mathrm{End}_S(V) = RI_V$. Express d as a sum of primitive orthogonal nonzero idempotents of S and let ε denote one of these terms. Since R is semilocal, we have $V \cong S\varepsilon$ in S-mod, see Proposition 1.1.1.
We first show that the nonzero idempotent $T(\varepsilon)$ is primitive in $\mathrm{End}_R(V)$. Let m denote the number of terms in the decomposition of d, then $\mathrm{rank}_R(Sd) = m\mathrm{rank}_R(V)$, because R is semilocal. But by Remark 5.2.8 $\mathrm{rank}_R(Sd) = c_{ij}^2$ and $c_{ij} = \mathrm{rank}_R(V)$, hence $m = \mathrm{rank}_R(V)$. Now, if $T(\varepsilon)$ is not primitive in $\mathrm{End}_R(V)$, then we can show that $m < \mathrm{rank}_R(V)$, as in the proof of Theorem 5.2.7 and this gives a contradiction. So $T(\varepsilon)$ is primitive.
We now prove that $\mathrm{End}_S(V) = RI$. We recall that $V \cong S\varepsilon$ in S-mod and we observe that $\mathrm{End}_S(S\varepsilon) \to \varepsilon S\varepsilon : \phi \mapsto \phi(\varepsilon)$ is an isomorphism of R-modules. So we have to

show that $\varepsilon S\varepsilon = R\varepsilon$. View V as indecomposable left $\mathrm{End}_R(V)$-module (under the operation : $\phi.v = \phi(v)$) and set $E = \mathrm{End}_R(V)$. We know that $V \cong ET(\varepsilon)$ in E-mod and, just as above, we have that $\mathrm{End}_E(V) \cong T(\varepsilon)ET(\varepsilon)$ in R-mod, mapping I at $T(\varepsilon)$. But $\mathrm{End}_E(V)$, being the center of $\mathrm{End}_R(V)$, is equal to RI. Therefore we obtain $T(\varepsilon S\varepsilon) \subset RT(\varepsilon)$, whence $\varepsilon S\varepsilon = R\varepsilon$ and this completes the proof. □

5.2.10 Note. If R is semilocal, then the nonzero $Hom_B(N_i, M_j)$ are, up to isomorphism, the only indecomposable left S-modules that are finitely generated projective.

Next, let us discuss the relationship between the centralizer S and certain Hecke algebras.

5.2.11 Proposition. *Suppose that R is a splitting ring for S (this follows whenever R is semilocal). Let ε be a primitive nonzero idempotent of B such that $f_i\varepsilon \neq 0$. Then $Sf_i \to \varepsilon A\varepsilon : sf_i \mapsto \varepsilon s\varepsilon$ is an isomorphism of R-algebras.*

Proof. Since $f_i\varepsilon \neq 0$, $N_i \cong B\varepsilon$ as left B-modules. Further, $Hom_B(N_i, M_j) \cong \varepsilon M_j$ as left S-modules, see 5.2.1(2). Now consider j such that $\varepsilon M_j \neq 0$. The latter is equivalent to $f_ie_j \neq 0$ and $Sf_ie_j \cong \mathrm{End}_R(\varepsilon M_j)$ as R-algebras, where the isomorphism associates to sf_ie_j the left multiplication by sf_ie_j; see 5.2.5 and 5.2.6. On the other hand, $\varepsilon M_j \neq 0$ equivalent to $\varepsilon e_j \neq 0$ and $\varepsilon A\varepsilon e_j \cong \mathrm{End}_R(\varepsilon M_j)$ as R-algebras, where the isomorphism associates to $\varepsilon a\varepsilon e_j$ the left multiplication by $\varepsilon a\varepsilon e_j$; see 3.1.5. Consequently, $Sf_i = \bigoplus\limits_j Sf_ie_j \cong \bigoplus\limits_j \mathrm{End}_R(\varepsilon M_j) \cong \varepsilon A\varepsilon$ where the sum is taken over the nonzero εM_j. Since $sf_ie_j\varepsilon = \varepsilon s\varepsilon e_j$, the above isomorphisms send sf_i to $\varepsilon s\varepsilon$, completing the proof. □

Remark. Let R be any connected commutative ring. Take $A = RG$, $B = RH$, $H < G$ with $|H|^{-1} \in R$ and consider $\varepsilon = |H|^{-1} \sum\limits_{h\in H} u_h$. Then ε is a primitive idempotent of B and ε is an element of $Z(B)$, thus $\varepsilon = f_i$ for some i. In this case, it is obvious that $\varepsilon A\varepsilon$ is a two-sided ideal in S.

We now investigate the relations between indecomposable modules over A, B and S. Put $P_{ij} = Hom_B(N_i, M_j)$ and let rank stand for rank_R.

5.2.12 Theorem. *(1) We have $M_j \cong \bigoplus_i P_{ij}^{rankN_i}$ as left S-modules, where the sum is taken over those i for which $c_{ij} \neq 0$.*

(2) If R is a splitting ring for S and $c_{ij} \neq 0$, then $A \otimes_S P_{ij} \cong M_j^{rankN_i}$ and $(A \otimes_B N_i) \otimes_S P_{ij} \cong M_j$ as left A-modules, where $A \otimes_B N_i$ is made into a right S-module by : $(a \otimes n)s = as \otimes n$ for $a \in A$, $n \in N_i$, $s \in S$.

(3) If R is a splitting ring for S and $c_{ij} \neq 0$, then $Hom_S(P_{ij}, M_j) \cong N_i$ as left B-modules, where $(b \cdot \varphi)(p) = b(\varphi(p))$ for $b = B$, $\varphi \in Hom_S(P_{ij}, M_j)$, $p \in P_{ij}$.

Proof. (1) Let i be such that $c_{ij} \neq 0$. Write f_i as a sum of primitive orthogonal nonzero idempotents of B, say $f_i = \varepsilon_1 + \ldots + \varepsilon_k$. By the hypotheses, $N_i \cong B\varepsilon_\ell$ in B-mod for $\ell = 1, \ldots, k$ and $k = \text{rank}_R(N_i)$. Now $f_i M_j = \varepsilon_1 M_j \oplus \ldots \oplus \varepsilon_k M_j$ and $\varepsilon_\ell M_j \cong P_{ij}$ in S-mod for $\ell = 1, \ldots, k$. Moreover, $f_i M_j \neq 0$ if and if $c_{ij} \neq 0$, and $M_j = \bigoplus_i f_i M_j$.

(2) The first statement follows from (1) and Proposition 1.1.3(3). We now prove the second statement.

Let ε be a primitive idempotent of B with $f_i\varepsilon \neq 0$. Then $N_i \cong B\varepsilon$ as left B-modules and $P_{ij} \cong \varepsilon M_j$ as left S-modules, see 5.2.1(2).

Since $c_{ij} \neq 0$, we have $P_{ij} \neq 0$ and thus $M_j \in \mathcal{C}(A\varepsilon)$, which is the category of all left A-modules which are isomorphic to A-direct summands of $(A\varepsilon)^m$ for some $m \in \mathbb{N}$, see 3.1.5(2). Therefore, $M_j \cong A\varepsilon \otimes_{\varepsilon A \varepsilon} \varepsilon M_j$ as left A-modules, see 3.1.2. Since R is a splitting ring for S, Proposition 5.2.11 yields $\varepsilon A \varepsilon \cong Sf_i$, and thus $M_j \cong A\varepsilon \otimes_S P_{ij}$ as left A-modules.

Clearly, $A \otimes_B B\varepsilon \to A\varepsilon : a \otimes \varepsilon \to a\varepsilon$ is an (A, S) bimodule isomorphism, where the right S-module structure of $A \otimes_B B\varepsilon$ is given by : $(a \otimes \varepsilon)s = as \otimes \varepsilon$ for $a \in A$, $s \in S$. So we obtain that $M_j \cong (A \otimes_B N_i) \otimes_S P_{ij}$ as left A-modules.

(3) From (1), 1.1.3(1) and 5.2.6 $Hom_S(P_{ij}, M_j)$ is a free R-module with rank equal to $\text{rank}_R(N_i)$. Moreover $f_\ell Hom_S(P_{ij}, M_j) \neq 0$ if and only if $\ell = i$, and the assertion follows. $\square$

5.2.13 Remarks. (1) Recall that R is a splitting ring for S whenever R is a semilocal ring (see 5.2.9).

(2) In the case that R is not a splitting ring for S we still have :

(a) If $c_{ij} \neq 0$, then we have $A \otimes_S Hom_B(N_i, M_j) \cong M_j^k$ as left A-modules. Indeed, it is easy to check that $V = A \otimes_S Hom_B(N_i, M_j)$ is nonzero and finitely generated projective over A, whence over R. Moreover $e_\ell V \neq 0$ if and only if $\ell = j$. (compare with 4.1.11)

(b) Note that $(A \otimes_B N_i) \otimes_S Hom_B(N_k, M_j) \neq 0$ implies that $k = i$ and $c_{ij} \neq 0$.

A last relation between the modules over A, B and S is given as follows :

5.2.14 Proposition. *Let $Z(A)$, resp. $Z(B)$ denote the center of A, resp. B. If $c_{ij} \neq 0$, then :*

$$(1)\ M_j^{c_{ij}} \cong P_{ij}^{rank_{M_j}} \qquad \text{as } Z(A)\text{-modules.}$$
$$(2)\ N_i^{c_{ij}} \cong P_{ij}^{rank_{N_i}} \qquad \text{as } Z(B)\text{-modules.}$$

Proof. (1) It is clear that $e_k P_{ij} \neq 0$ if and only if $k = j$. So the restriction of P_{ij} to $Z(A)$ is a finite sum of indecomposable $Z(A)$-modules lying over e_j. By the hypotheses, $Z(A)e_j = Re_j \cong R$ and $Z(A)e_j$ is, up to isomorphism, the only indecomposable $Z(A)$-module which is finitely generated projective as R-module and lies over e_j. Therefore $P_{ij} \cong (Z(A)e_j)^\ell$ as $Z(A)$-module, and comparing ranks with respect to R, we obtain $\ell = c_{ij}$.

Similarly, we may show that $M_j \cong (Z(A)e_j)^{\mathrm{rank}_{M_j}}$ as $Z(A)$-modules and the assertion (1) follows.

(2) Obviously $f_k P_{ij} \neq 0$ if and only if $k = i$. We now proceed as in (1). □

5.3 Centralizers in twisted group rings

The results of the preceding sections can be applied to the centralizer of RH in RG ($H < G$), as we have seen. In this section we concentrate on centralizers in twisted group rings, more precisely, on the centralizer S of $R *_\alpha H$ in $R *_\alpha G$ ($H < G$). Our main objective is to construct an R-basis for S. Therefore we introduce α-H-regular elements in G.

Throughout R is a commutative ring and G is a finite group. Let α be a 2-cocycle and consider the twisted group ring $R *_\alpha G$, with R-basis $\{u_g; g \in G\}$.

In section 1.1 we have summarized some basic facts about α-regular elements, studied in $[N_1\text{-}v.O_1]$. We now consider α-H regular elements, with $H < G$.

5.3.1 Definition. Let H be a subgroup of G. An element $g \in G$ is said to be α-H-regular if $\alpha(g,x) = \alpha(x,g)$ for all $x \in C_H(g) = \{y \in H | gy = yg\}$. Clearly, an α-H-regular element will be β-H-regular for every 2-cocycle β equivalent to α. Note that $g \in G$ is α-H-regular if and only if $u_g u_x = u_x y_g$ in $R *_\alpha G$ for all $x \in C_H(g)$. In case $H = G$, we get the definition of α-regular elements.

5.3.2 Lemma. *Let $g \in G$ be α-H-regular, then :*
(1) g^{-1} is an α-H-regular element.
(2) hgh^{-1} is α-H-regular for all $h \in H$.
(3) If $H \triangleleft G$, then ygy^{-1} is α-H-regular for all $y \in G$.

Proof. The proof is entirely similar to the proof of $[N_1\text{-}v.O_1, 2.1]$. □

If g is α-H-regular, then $E_g = \{hgh^{-1} | h \in H\}$ is said to be an α-H-regular subclass and $s_g = \sum_{x \in E_g} u_x$ is called an α-H-regular subclass sum in $R *_\alpha G$. In case $H = G$, we speak of α-regular classes or α-ray classes, cf. $[N_1\text{-}v.O_1]$.
Further, to α we associate a map $f_\alpha : G \times G \to U(R) : (x,g) \mapsto \alpha(x,g)\alpha^{-1}(xgx^{-1}, x)$, see also 1.1. In $R *_\alpha G$ we have for all $x, g \in G : u_x u_g (u_x)^{-1} = f_\alpha(x,g) u_{xgx^{-1}}$.
We now need $[N_1\text{-}v.O_1$, Lemma 2.2] in a slightly more general form.

5.3.3 Lemma. *Let $g \in G$ be α-H-regular. Then for $x, y \in H$, $xgx^{-1} = ygy^{-1}$ entails $f_\alpha(x,g) = f_\alpha(y,g)$.*

Proof. The proof is entirely similar to the proof of $[N_1\text{-}v.O_1,\ 2.2]$, where $H = G$. $\square$.

Using 5.3.3, we prove the following lemma which generalizes $[N_1\text{-}v.O_1, 2.3]$.

5.3.4 Lemma. *Given $\alpha \in Z^2(G, U(R))$ and $H < G$, then there is a 2-cocycle β equivalent to α satisfying $\beta(e,e) = 1$ and $f_\beta(x,g) = 1$ for all β-G-regular $g \in G$ and all $x \in G$ as well as for all β-H-regular $g \in G$ and all $x \in H$.*

Proof. First replace α by an equivalent 2-cocycle γ such that $\gamma(e,e) = 1$. Conjugation by elements of G gives an equivalence relation on the set of α-G-regular elements, and in every class we choose an element s_i. Furthermore, conjugation by elements of H defines an equivalence relation on the set consisting of elements which are α-H-regular but not α-G-regular, and we choose an element t_j in each of these classes.
We now define a map $\mu : G \to U(R)$ as follows :

$$\begin{array}{ll} \mu(g) = f_\gamma(y, s_i) & \text{if } g \text{ is } \alpha\text{-}G\text{-regular and } g = ys_iy^{-1} \text{ with } y \in G, \\ \mu(g) = f_\gamma(h, t_j) & \text{if } g \text{ is } \alpha\text{-}H\text{-regular but not } \alpha\text{-}G\text{-regular and } g = ht_jh^{-1} \text{ with } h \in H, \\ \mu(g) = 1 & \text{if } g \text{ is not } \alpha\text{-}H\text{-regular.} \end{array}$$

By Lemma 5.3.3, μ is well-defined. Put $\beta(a,b) = \gamma(a,b)\mu(a)\mu(b)\mu(ab)^{-1}$ for all $a, b \in G$. To show that β satisfies the required properties, we proceed as in the proof of $[N_1\text{-}v.O_1,\ 2.3]$. Clearly $\beta(e,e) = 1$ because $\mu(e) = 1$.
We now consider a β-H-regular element $g \in G$ which is not β-G-regular. Since g is also α-H-regular but not α-G-regular we have $g = ht_jh^{-1}$ for some t_j and some $h \in H$. For any $x \in H$, we calculate :

$$\begin{aligned} f_\beta(x,g) &= \beta(x,g)\beta^{-1}(xgx^{-1}, x) \\ &= \mu(g)\mu(xgx^{-1})^{-1}\gamma(x,g)\gamma^{-1}(xgx^{-1}, x) \end{aligned}$$

$$= \; f_\gamma(h,t_j)f_\gamma(xh,t_j)^{-1}f_\gamma(x,g).$$

Further in $R *_\gamma G$ we have :

$$\begin{aligned} f_\gamma(xh,t_j)u_{xgx^{-1}} &= u_{xh}u_{t_j}(u_{xh})^{-1} \\ &= u_x u_h u_{t_j}(u_h)^{-1}(u_x)^{-1} \\ &= u_x f_\gamma(h,t_j)u_g(u_x)^{-1} \\ &= f_\gamma(h,t_j)f_\gamma(x,g)u_{xgx^{-1}} \end{aligned}$$

So we obtain that $f_\gamma(xh,t_j) = f_\gamma(h,t_j)f_\gamma(x,g)$, whence $f_\beta(x,g) = 1$.
Note that if g is β-G-regular and $x \in G$, then $g = ys_iy^{-1}$ for some s_i and some $y \in G$ and, as above, one deduces that $f_\beta(x,g) = 1$. $\square$

Next, we consider the centralizer S of $R *_\alpha H$ in $R *_\alpha G$, where $H < G$. We prove :

5.3.5 Proposition. *Assume that $f_\alpha(h,g) = 1$ for all α-H-regular $g \in G$ and all $h \in H$ (see 5.3.4). Then the α-H-regular subclass sums s_g form an R-basis for the centralizer S in the following cases : (i) $\alpha = 1$, (ii) R is a domain, (iii) R is connected and $|G|^{-1} \in R$.*

Proof. The proof is similar to the proof of [N_1-$v.O_1$, 2.4], where $H = G$.
a) As before, for any α-H-regular g, let $E_g = \{hgh^{-1} | h \in H\}$ and put $s_g = \sum_{x\in E_g} u_x$ in $R *_\alpha G$. Then, for any $h \in H$, we have :

$$u_h s_g (u_h)^{-1} = \sum_{x\in E_g} f_\alpha(h,x)u_{hxh^{-1}} = s_g$$

and thus $s_g \in S$.
Moreover, it is clear that the distinct α-H-regular subclass sums are linearly independent over R.
b) Let now $w \in S$, $w = \sum_{g\in G} r_g u_g$ in $R *_g G$ with $r_g \in R$. For any $h \in H$, $u_h w (u_h)^{-1} = w$ leads to :

$$\sum_{g\in G} r_g f_\alpha(h,g)u_{hgh^{-1}} = \sum_{g\in G} r_g u_g \qquad (*)$$

If $r_g \neq 0$, then we will show that g is α-H-regular.
For any $h \in C_H(g)$ we get $r_g f_\alpha(h,g) = r_g$ by comparing the coefficients of u_g on both

sides of $(*)$. If R is a domain, then $f_\alpha(h,g)=1$ follows and thus g is α-H-regular. We now suppose that R is connected and that $|G|^{-1}\in R$; say $|G|=n$.
We first show that $f_\alpha(h,g)^n=1$. Define a map $\mu : G\to U(R)$ by $\mu(x)=\prod_{t\in G}\alpha(t,x)$. Then it is easily verified that $\mu(x)\mu(y)=\mu(xy)\alpha(x,y)^n$ for all $x,y\in G$. This entails $f_\alpha(x,y)^n=\mu(y)\mu(xyx^{-1})^{-1}$. Consequently, $f_\alpha(x,y)^n=1$ if $xy=yx$.
Fix $h\in C_H(g)$ and put $a=f_\alpha(h,g)$; hence $a^n=1$ and $r_ga=r_g$. It is clear that $e=n^{-1}(1+a+\ldots+a^{n-1})$ is an idempotent of R and so e is either 0 or 1.If $e=0$, then $0=r_ge=n^{-1}nr_g=r_g$, a contradiction. Therefore $e=1$. But then $n=1+a\ldots+a^{n-1}$ and thus $n(1-a)=1-a^n=0$, whence $a=1$. This proves that g is α-H-regular.
Finally we will show that w is an R-linear combination of α-H-regular subclass sums. If u_g has nonzero coefficients r_g in the decomposition of w, then it follows from $(*)$ that, for any $h\in H$, $u_{hgh^{-1}}$ appears in the decomposition of w with nonzero coefficient $r_gf_\alpha(h,g)$. Moreover g is α-H-regular and thus $f_\alpha(h,g)=1$ by the assumption on α, which completes the proof.
To conclude, note that the preceding discussion also proves the assertion in case $\alpha=1$. $\square$

Remark. If $H=G$, then S is the center of $R*_\alpha G$ and Proposition 5.3.5 states that the center is freely generated as an R-module by the α-regular class sums (under the hypotheses of 5.3.5); see also $[N_1$-$v.O_1$, Theorem 2.4].

Using 5.1 we now give a formula relating the number of α-H-regular subclasses to the restriction to $R*_\alpha H$ of the indecomposable left $R*_\alpha G$-modules $(H<G)$. Combining the results of 5.1.3, 5.3.4 and 5.3.5, we obtain :

5.3.6 Proposition. *Let R be connected and suppose that finitely generated projective R-modules are free. Let G be a finite group with $|G|^{-1}\in R$ and H a subgroup of G. Consider a 2-cocycle α and suppose that R is a splitting ring for $R*_\alpha G$ and $R*_\alpha H$. Further, let $M_1,\ldots,M_s$, resp. $N_1,\ldots,N_t$, be a basic set of indecomposable left $R*_\alpha G$-modules, resp. $R*_\alpha H$-modules, which are finitely generated and projec-*

tive over R, and let c_{ij} be the multiplicity of N_i in M_j. Then $\sum_{j=1}^{s}\sum_{i=1}^{t}(c_{ij})^2 =$ *number of α-H-regular subclasses.*

Remark. If we consider group rings, i.e. $\alpha = 1$, then 5.3.5 states that the subclass sums form an R-basis for the centralizer of RH in RG ($H < G$), as we already know (1.2.11). Moreover, in the case of group rings, 5.3.6 gives 5.1.4.

Of course, the results of section 5.2 can be applied to the centralizer of $R *_\alpha H$ in $R *_\alpha G$, see also example 5.6.2.

Recall from 1.1 (in particular 1.1.8) that one can construct a nice splitting ring for $R *_\alpha G$, which is also a splitting ring for $R *_\alpha H$. Note that $\sigma : H \to Aut_R(R *_\alpha G) : h \to \sigma_h$, with $\sigma_g(w) = u_h w(u_h)^{-1}$ is a homomorphism of groups ($w \in R *_\alpha G$). The fixed ring of this automorphism group is given by the centralizer of $R *_\alpha H$ in $R *_\alpha G$.

5.4 Projective group representation and centralizers : character theory

Consider the centralizer S of $R *_\alpha H$ in $R *_\alpha G(H < G)$. In this section we express primitive central idempotents of S in terms of trace functions in two different ways. We also derive orthogonality relations for trace functions on S. Furthermore we give formulas which relate the trace functions of indecomposable modules over $R *_\alpha G$, $R *_\alpha H$ and S.

Throughout this section R is a commutative ring and G is a finite group. Let α be a 2-cocycle, let H be a subgroup of G and let S be the centralizer of $R *_\alpha H$ in $R *_\alpha G$. Of course α may be trivial. If $\alpha \neq 1$, then we assume that α has been modified as in Lemma 5.3.4.

Further, for any α-H-regular element $g \in G$, we set $E_g = \{hgh^{-1} | h \in H\}$ and $s_g = \sum_{x \in E_g} u_x$ in $R *_\alpha G$. Let G_0 denote a set of representatives for the distinct α-H-regular subclasses E_g.

First we show that, under suitable conditions, S is a symmetric Frobenius R-algebra. We need the following lemma.

5.4.1. Lemma. *We have $\alpha(g, g^{-1}) = \alpha(xgx^{-1}, xg^{-1}x^{-1})$ for all α-G-regular $g \in G$ and all $x \in G$ as well as for all α-H-regular $g \in G$ and all $x \in H$.*

Proof. The proof for an α-G-regular $g \in G$ and $x \in G$ is in [N_2-$v.O_2$, Lemma 3.2]. The proof for an α-H-regular $g \in G$ and $x \in H$ is similar. Namely, from the 2-cocycle conditions we get : for all $g, x \in G$

1. $\alpha(x, g)\alpha(xg, g^{-1}) = \alpha(x, e)\alpha(g, g^{-1})$
2. $\alpha(xgx^{-1}, xg^{-1}x^{-1})\alpha(e, xg) = \alpha(xgx^{-1}, x)\alpha(xg^{-1}x^{-1}, xg)$

Now we restrict to $x \in H$ and an α-H-regular $g \in G$. Then $\alpha(x, g) = \alpha(xgx^{-1}, x)$ by the assumption on α. Since g^{-1} is α-H-regular too, we also have : $\alpha(x, g^{-1}) = \alpha(xg^{-1}x^{-1}, x)$. If we combine this latter result with the cocycle equality $\alpha(xg^{-1}x^{-1}, xg)\,\alpha(x, g^{-1}) = \alpha(xg^{-1}x^{-1}, x)\alpha(xg, g^{-1})$, then we obtain that $\alpha(xg^{-1}x^{-1}, xg) = \alpha(xg, g^{-1})$. If we substitute these equalities in (1), then we get :

$$\begin{aligned} \alpha(x, e)\alpha(g, g^{-1}) &= \alpha(xgx^{-1}, x)\alpha(xg^{-1}x^{-1}, xg) \\ &\overset{(2)}{=} \alpha(xgx^{-1}, xg^{-1}x^{-1})\alpha(e, xg) \\ \text{whence} \qquad \alpha(g, g^{-1}) &= \alpha(xgx^{-1}, xg^{-1}x^{-1}) \qquad \square \end{aligned}$$

5.4.2. Proposition. *Suppose $|G|^{-1} \in R$, and suppose that either R is connected or $\alpha = 1$. Then the R-linear map $\tau : S \to R : \sum r_g s_g \to r_e$ $(r_g \in R)$ defines a nonsingular associative R-bilinear form on S with dual bases $\{s_g | g \in G_0\}$ and $\{|E_g|^{-1}\alpha(g, g^{-1})^{-1}s_{g^{-1}} | g \in G_0\}$.*

Proof. Combine Proposition 5.3.5 and 5.4.1. $\square$

From now on (except for 5.4.6), we assume that R is connected and that finitely generated projective R-modules are free. Moreover suppose $|G|^{-1} \in R$ and suppose that R is a splitting for $R *_\alpha G$ and $R *_\alpha H$. Let $M_1, \ldots, M_s$, resp. $N_1, \ldots, N_t$, be a basic set of indecomposable left $R *_\alpha G$-modules, resp. $R *_\alpha H$-modules, which are finitely generated and projective over R. Each M_j, viewed as a left $R *_\alpha H$-module, is uniquely expressible as a finite direct sum of N_i's, and c_{ij} denotes the multiplicity of N_i in this decomposition of M_j. Further, let $\{e_1, \dots, e_s\}$ resp. $\{f_1, \dots, f_t\}$ be the set of primitive central nonzero idempotents of $R *_\alpha G$ resp. $R *_\alpha H$, and assume that M_i lies over e_i and N_i over f_i. Finally, put $P_{ij} = Hom_{R*_\alpha H}(N_i, M_j)$. If $c_{ij} = 0$, then $P_{ij} = 0$; otherwise P_{ij} is a left S-module and a free R-module of rank c_{ij} (see 5.2.2). Moreover, $P_{ij} \neq 0$ if and only if $f_i e_j \neq 0$ (see 5.2.5).

5.4.3. Note. If the nonzero P_{ij} are indecomposable left S-modules, then by 5.2.5 the nonzero $f_i e_j$ are precisely the distinct primitive central idempotents of S and P_{ij} lies over $f_i e_j$.
If R is also a splitting ring for S (this follows whenever R is semilocal), then by 5.2.6 the nonzero P_{ij} are indecomposable left S-modules. We also refer to Theorem 5.2.7.

Using the character theory of Frobenius algebras (section 2.2), we can express primitive central idempotents of S in terms of trace functions and we have orthogonality relations for trace functions on S, more precisely :

5.4.4. Proposition. *Keep the above hypotheses and notation, and suppose that R is a splitting ring for S. Put $z = \sum_{g \in G_0} \frac{1}{|E_g|\alpha(g, g^{-1})} s_g s_{g^{-1}}$. Then :*

(1) For nonzero P_{ij},

$$t_{P_{ij}}(z) f_i e_j = c_{ij}^2 \sum_{g \in G_0} \frac{1}{|E_g|\alpha(g, g^{-1})} t_{P_{ij}}(s_{g^{-1}}) s_g$$

$$z f_i e_j = c_{ij} \sum_{g \in G_0} \frac{1}{|E_g|\alpha(g, g^{-1})} t_{P_{ij}}(s_{g^{-1}}) s_g$$

(2) For nonzero P_{ij} and $P_{k\ell}$,

$$\sum_{g \in G_0} \tfrac{1}{|E_g|\alpha(g, g^{-1})} t_{P_{ij}}(s_g) t_{P_{k\ell}}(s_{g^{-1}}) = 0 \qquad whenever (i, j) \neq (k, l)$$

$$c_{ij} \sum_{g \in G_0} \tfrac{1}{|E_g|\alpha(g, g^{-1})} t_{P_{ij}}(s_g) t_{P_{ij}}(s_{g^{-1}}) = t_{P_{ij}}(z)$$

(3) z is invertible in S if and only if all nonzero c_{ij} are invertible in R, and for any $c_{ij} \neq 0$, the invertibility of c_{ij} in R is equivalent to the invertibility of $t_{P_{ij}}(z)$ in R.

Proof. Apply 2.2.4-2.2.6 to the bilinear form associated to $\tau : S \to R : \sum\limits_{g \in G_0} r_g s_g \mapsto r_e$. $\square$

5.4.5. Note. Keep the hypotheses of 5.4.4.

(1) If $|G|!$ is invertible in R, then all nonzero c_{ij} are invertible in R and thus z is invertible in S. Indeed, this is a direct consequence of $c_{ij} \leq rank_K(M_j) \leq |G|$.

(2) If $H \triangleleft G$, then all nonzero c_{ij} are invertible in R (this is shown in 5.5.8 (1)) and thus z is invertible in S.

5.4.6. Corollary. *We only suppose that either $\alpha = 1$ or R is connected (α has been modified). If either $|G|!$ is invertible in R or $H \triangleleft G$ and $|G|^{-1} \in R$, then $z = \sum\limits_{g \in G_0} \frac{1}{|E_g|\alpha(g,g^{-1})} s_g s_{g^{-1}}$ is invertible in S.*

Proof. Note that $|G|$ is invertible in R.

(i) Suppose that $R = K$ is a field. Then the algebraic closure $\overline{K}$ of K is a splitting field for $\overline{K} *_\alpha G$ and $\overline{K} *_\alpha H$, because $|G|^{-1} \in K$. In view of 5.3.5, $\overline{S} = \{ \sum\limits_{g \in G_0} \lambda_g s_g | \lambda_g \in \overline{K}\}$ is the centralizer of $\overline{K} *_\alpha H$ in $\overline{K} *_\alpha G$.
Since the hypotheses of 5.4.4 are satisfied for $\overline{K}$, $\overline{K} *_\alpha H$, $\overline{K} *_\alpha G$ and $\overline{S}$, we may apply Note 5.4.5, and we obtain that z is invertible in $\overline{S}$. This gives a system of n linear equations with coefficients in K, having a solution in $\overline{K}^n (n = |G_0|)$. But then these equations must have a solution in K^n, and therefore z is invertible in S.

(ii) Let now R be arbitrary. First note that S is separable over R, because $|G|^{-1} \in R$ (see 5.1.6). Suppose $1 \notin Z(S)z$, where $Z(S)$ is the center of S. We now proceed as in 2.1.13(1) in order to obtain a contradiction. More precisely, the ideal $Z(S)z$ is contained in some maximal ideal M of $Z(S)$. Since $Z(S)$ is integral over R, $m = M \cap R$ is a maximal ideal of R.
Now we need the following result. Define $\overline{\alpha} : G \times G \to U(R/m)$ by $\overline{\alpha}(x, y) = \alpha(x, y) + m$. Then $g \in G$ is $\overline{\alpha}$-H-regular if and only if g is α-H-regular. To show this, let g be $\overline{\alpha}$-H-regular, $h \in C_H(g)$ and put $a = \alpha(h, g)\alpha(g, h)^{-1}$. Thus we have $a - 1 \in m$. We know that $a^k = 1$, where $|G| = k$, see the

proof of 5.3.5. So $\varepsilon = k^{-1}(1 + a + \ldots + a^{k-1})$ is an idempotent of R and thus ε is either 0 or 1. If $\varepsilon = 0$, then $a - 1 \in m$ implies that $k1_R \in m$, a contradiction. Therefore $\varepsilon = 1$. But then $k(1-a) = 1 - a^k = 0$, whence $a = 1$, proving that g is α-H-regular. The converse is obvious. As a consequence, $\overline{\alpha}$ is modified as in 5.3.4. The centralizer of $R/m *_{\overline{\alpha}} H$ in $R/m *_{\overline{\alpha}} G$ is given by $\{ \sum_{g \in G_0} \overline{r_g} s_g | \overline{r_g} \in R/m\}$ (see 5.3.5), which is isomorphic to the R/m-algebra S/mS. By (i), there is an element $x \in S$ such that $1 - xz \in mS$, whence $1 \in SM$. But $SM \cap Z(S) = M$, since S is separable over R. Consequently $1 \in M$, a contradiction, and thus $1 \in Z(S)z$. $\square$

We now give another description of primitive central idempotents of S in terms of characters. But this formula depends not only on S, but also on $R *_\alpha G$ and $R *_\alpha H$.

5.4.7. Proposition. *Keep the hypotheses and notation of the discussion following 5.4.2. Then for nonzero P_{ij} and $P_{k\ell}$,*

$$(1) \quad f_i e_j = |G|^{-1} rank_R(N_i) rank_R(M_j) \sum_{g \in G_0} \frac{1}{|E_g| \alpha(g, g^{-1})} t_{P_{ij}}(s_{g^{-1}}) s_g$$

$$(2) \quad \sum_{g \in G_0} \frac{1}{|E_g| \alpha(g, g^{-1})} t_{P_{ij}}(s_{g^{-1}}) t_{P_{kl}}(s_g) = \delta_{ik} \delta_{jl} |G| c_{ij} (rank N_i)^{-1} (rank M_j)^{-1} 1_R$$

Proof. (1) By Corollary 2.2.8.(3), $e_j = |G|^{-1} rank_R(M_j) \sum_{g \in G} \frac{1}{\alpha(g, g^{-1})} t_{M_j}(u_{g^{-1}}) u_g$. Applying Lemma 1.1.7 yields

$$f_i e_j = |G|^{-1} rank_R(M_j) \sum_{g \in G} \frac{1}{\alpha(g, g^{-1})} t_{M_j}(f_i u_{g^{-1}}) u_g.$$

Clearly, if g is not α-H-regular, then the coefficient of u_g in the above decomposition must be zero, see 5.3.5. For an α-H-regular g and any $h \in H$, we have $t_{M_j}(f_i u_{g^{-1}}) = t_{M_j}(f_i u_{hg^{-1}h^{-1}})$ and thus $t_{M_j}(f_i u_{g^{-1}}) = \frac{1}{|E_g|} t_{M_j}(f_i s_{g^{-1}})$. So, by using Lemma 5.4.1

$$f_i e_j = |G|^{-1} rank_R(M_j) \sum_{g \in G_0} \frac{1}{|E_g| \alpha(g, g^{-1})} t_{M_j}(f_i s_{g^{-1}}) s_g.$$

But $t_{M_j}(f_i s_{g^{-1}}) = rank_R(N_i) t_{P_{ij}}(s_{g^{-1}})$, because of Theorem 5.2.12(1) and the fact that $t_{P_{kj}}(s_{g^{-1}} f_i) = 0$ whenever $k \neq i$ and $t_{P_{ij}}(s_{g^{-1}} f_i) = t_{P_{ij}}(s_{g^{-1}})$. Assertion (1) follows.

(2) Apply $t_{P_{k\ell}}$ to the expression for f_ie_j and use 2.2.8(1). $\square$

5.4.8 Remarks. (1) In addition, suppose that R is a splitting ring for S. Put $z = \sum_{g \in G_0} \frac{1}{|E_g|\alpha(g,g^{-1})} s_g s_{g^{-1}}$. Comparing Propositions 5.4.4(1) and 5.4.7(1), we obtain for nonzero P_{ij} : $|G|c_{ij}f_ie_j = rank_R(N_i)rank_R(M_j)zf_ie_j$. Moreover, from the preceding equality it follows that $rank_R(N_i)rank_R(M_j)t_{p_{ij}}(z) = |G|c_{ij}^2 1_R$, for nonzero P_{ij}.

(2) Compare with 4.2.1 and 4.2.3 (in the case of group rings).

We next derive formulas which relate the trace functions t_{M_j}, t_{N_i}, $t_{P_{ij}}$. We shall use the following notation : for any $g \in G$, $K_g = \{ygy^{-1}|y \in G\}$ and $v_g = \sum_{x \in K_g} u_x$.

5.4.9 Proposition. *We keep the hypotheses of the discussion following 5.4.2. Letting $c_{ij} \neq 0$, we have :*

(1) $c_{ij}t_{M_j}(u_g) = |K_g|^{-1} rank_R M_j t_{P_{ij}}(v_g)$ *for any α-G-regular $g \in G$.*

(2) $c_{ij}t_{N_i}(u_h) = |E_h|^{-1} rank_R N_i t_{P_{ij}}(s_h)$ *for any α-H-regular $h \in H$.*

Proof. Use Proposition 5.2.14 and 1.1.6(1). $\square$

5.4.10 Proposition. *Keep the hypotheses of the discussion following 5.4.2 and let $c_{ij} \neq 0$. Then for each α-G-regular $g \in G$ we have :*

$$c_{ij}t_{M_j}(u_g) = |G|^{-1} rank_R M_j \left[c_{ij}(rank_R N_i)^{-1} \sum_{x \in J} t_{N_i}(u_{xgx^{-1}}) + t_{P_{ij}} \left(\sum_{x \in G \setminus J} u_{xgx^{-1}} \right) \right]$$

where $J = \{x \in G | xgx^{-1} \in H\}$.

Moreover, $t_{P_{ij}} \left(\sum_{x \in G \setminus J} u_{xgx^{-1}} \right) = \sum_{x \in G \setminus J} |E_{xgx^{-1}}|^{-1} t_{P_{ij}}(s_{xgx^{-1}})$.

Proof. We first note that $rank_R(N_i)$ is invertible in R by 2.2.8(1). Further, for any algebra A, $Z(A)$ denotes the center of A.

For any $k \in H$, we have $u_k(\sum_{x \in J} u_{xgx^{-1}})(u_k)^{-1} = \sum_{x \in J} u_{kxg(kx)^{-1}}$, because xgx^{-1} is α-G-regular, whence $\sum_{x \in J} u_{xgx^{-1}} \in Z(R *_\alpha H)$. On the other hand $\sum_{x \in G} u_{xgx^{-1}} \in Z(R *_\alpha G)$

and thus $\sum_{x\in G\setminus J} u_{xgx^{-1}} \in Z(S)$. Now, $t_{M_j}(u_g) = |G|^{-1}t_{M_j}(\sum_{x\in G} u_{xgx^{-1}})$ because g is α-G-regular. Then, applying Proposition 5.2.14, we obtain :

$$\begin{aligned} c_{ij}t_{M_j}(u_g) &= |G|^{-1}rank_R M_j t_{P_{ij}}\left(\sum_{x\in G} u_{xgx^{-1}}\right) \qquad (*) \\ &= |G|^{-1}rank_R M_j\left[t_{P_{ij}}\left(\sum_{x\in J} u_{xgx^{-1}}\right) + t_{P_{ij}}\left(\sum_{x\in G\setminus J} u_{xgx^{-1}}\right)\right] \\ &= |G|^{-1}rank_R M_j\left[c_{ij}(rank_R N_i)^{-1}\sum_{x\in J} t_{N_i}(u_{xgx^{-1}}) + t_{P_{ij}}\left(\sum_{x\in G\setminus J} u_{xgx^{-1}}\right)\right] \end{aligned}$$

Finally, as $\sum_{x\in G\setminus J} u_{xgx^{-1}} \in S$, we have that $\sum_{x\in G\setminus J} u_{xgx^{-1}} = \sum_{x\in G\setminus J} |E_{xgx^{-1}}|^{-1}s_{xgx^{-1}}$.

5.4.11 Remarks. (1) Assume $H \lhd G$. If g is an α-G-regular element in H, then $J = G$ in 5.4.10. Moreover, if $H \lhd G$, then a nonzero c_{ij} is invertible in R, as we will show in 5.5.8(1).

(2) Let $\alpha = 1$ and suppose that R is a splitting ring for S, then we have that $M_j^{rank_R N_i} \cong RG \otimes_S P_{ij}$ as left RG-modules, see Theorem 5.2.12(2) ($c_{ij} \neq 0$). We may apply the theory of trace functions of induced modules, see 2.4 and in particular 4.2. Then, in case $\sum_{g\in G_0} |E_g|^{-1}s_g s_{g^{-1}}$ is invertible in S, we obtain formula $(*)$ of 5.4.10.

(3) Note also that $t_{M_j}(x) = \sum_i (rank_R N_i)t_{P_{ij}}(x)$ for all $x \in S$, where the sum is taken over those i for which $P_{ij} \neq 0$, see 5.2.12(1). So $(rank_R N_i)t_{P_{ij}}(x) = t_{M_j}(xf_i)$ for $x \in S$ and nonzero P_{ij}.

(4) For the case that $\alpha = 1$, $g \in H$ and $R = \mathbb{C}$, the formula of Proposition 5.4.10 was derived by J. Karlof in a different way, see [K, Corollary 3.6].

5.5 Review of Clifford's theorem for normal subgroups

In case the group G has a normal subgroup H, the analysis of indecomposable modules over $R *_\alpha G$ in terms of indecomposable modules over $R *_\alpha H$ is easier. In this section we review Clifford's theorem for normal subgroups. The original version deals with simple modules over group rings; see e.g. [C-R, p259], but here we are concerned with indecomposable modules over twisted group rings.

Throughout this section, R is a commutative ring and G is a finite group.

Let α be a 2-cocycle and consider the twisted group ring $R *_\alpha G$ with R-basis $\{u_g; g \in G\}$. Of course α may be trivial.

We first require some preliminary remarks.

5.5.1 Remarks. Let H be a normal subgroup of G and set $B = R *_\alpha H$.

1. Let N be a left B-module, let $g \in G$ and form $u_g B \otimes_B N = u_g \otimes N$. Clearly any element of this product is uniquely expressible as $u_g \otimes n$, $n \in N$, and $u_g \otimes N \cong N$ as R-modules. Since $H \lhd G$, there is a left B-module on $u_g \otimes N$, to be explicit, for any $h \in H$, $n \in N : u_h(u_g \otimes n) = u_g(u_g)^{-1}u_h u_g \otimes n = u_g \otimes f_\alpha(g^{-1}, h)u_{g^{-1}hg}n$ (f_α as in 5.3). Further, if N is an indecomposable left B-module, then so is $u_g \otimes N$ and conversely. If N is a B-submodule of $M_{|H}$ for some left $R*_\alpha G$-module M, then $u_g N$ is also a B-submodule of $M_{|H}$, and $u_g N \to u_g \otimes N : u_g n \to u_g \otimes n$ is an isomorphism of B-modules. $(M_{|H} = M_{|R*_\alpha H})$.

2. Keep the above notation. If N is finitely generated and projective over R with dual basis $\{n_1, \dots, n_k\} \subset N$, $\{\varphi_1, \dots, \varphi_k\} \subset Hom_R(N, R)$, then $\{u_g \otimes n_i\}, \{\widetilde{\varphi}_i\}$, with $\widetilde{\varphi}_i : u_g \otimes N \to R : u_g \otimes n \to \varphi_i(n)$, is a dual basis for $u_g \otimes N$. Using this, we have for any $h \in H : t_{u_g \otimes N}(u_h) = f_\alpha(g^{-1}, h)t_N(u_{g^{-1}hg})$.

3. Let f be a primitive central idempotent of $R *_\alpha H$. Then it is easily verified that, for any $g \in G, u_g f(u_g)^{-1}$ is also a primitive central idempotent of $R *_\alpha H$.

5.5.2 Proposition *Suppose R is connected. Let H be a normal subgroup of G and let M be an indecomposable left $R *_\alpha G$-module. Let f be any primitive central idempotent of $R*_\alpha H$ such that $fM \neq 0$. Set $W = fM$ and $F = \{g \in G | u_g W = W\}$. Then the following hold :*

(1) $M_{|H} = \bigoplus_{i=1}^{r} u_{g_i} W$ where $\{g_1, \dots, g_r\}$ is a set of left coset representatives of F in G. Moreover, $F = \{g \in G | u_g f(u_g)^{-1} = f\}$.

(2) $M \cong R *_\alpha G \otimes_{R*_\alpha F} W$ as left $R *_\alpha G$-module, and W is an indecomposable left $R *_\alpha F$-module.

Proof. (1) Let $\{f = f_1, \dots, f_m\}$ be the set of all primitive central idempotents in $R*_\alpha H$ for which $f_j M \neq 0$, and set $W_j = f_j M$ $(W = W_1)$. Then $M = W_1 \oplus \dots \oplus W_m$. Given $g \in G$ and W_j, we show that $u_g W_j = W_k$ for some $k \in \{1, \dots, m\}$. We have

$u_gW_j = u_gf_j(u_g)^{-1}u_gM = u_gf_j(u_g)^{-1}M$. Now $u_gf_j(u_g)^{-1}$ is a primitive central idempotent of $R *_\alpha H$ which doesn't annihilate M. Therefore $u_gf_j(u_g)^{-1} = f_k$ for some k and $u_gW_j = f_kM = W_k$. In fact, multiplication by u_g defines an action of G on $\{W_1, \ldots, W_m\}$. Consider the distinct G-orbits and let $T_1, \ldots, T_n$ denote the direct sums of their elements.

We have $M = W_1 \oplus \ldots \oplus W_m = T_1 \oplus \ldots \oplus T_n$. It is easy to see that each T_j is an $R *_\alpha G$-submodule of M. But M is indecomposable, hence $M = T_1 = \bigoplus\limits_{i=1}^{r} u_{gi}W_1$. Finally, since $u_gW = u_gf(u_g)^{-1}M$, $u_gW = W$ if and only if $u_gf(u_g)^{-1} = f$.

(2) Let $\{g_1, \ldots, g_r\}$ be as in (1). Obviously $R*_\alpha$ G is a free right $R*_\alpha F$-module with basis $\{u_{g_1}, \ldots, u_{g_r}\}$. Therefore any element of $R *_\alpha G \otimes_{R*_\alpha F} W$ is uniquely expressible as $\sum\limits_{i=1}^{r} u_{g_i} \otimes w_i$ with $w_i \in W$. Using (1), it then follows that $R *_\alpha G \bigotimes\limits_{R*_\alpha F} W \to M : \sum\limits_{i=1}^{r} u_{g_i} \otimes w_i \mapsto \sum\limits_{i=1}^{r} u_{g_i}w_i$ is an isomorphism of left $R *_\alpha G$-modules.

Furthermore, since $R *_\alpha G \bigotimes\limits_{R*_\alpha F} W$ is an indecomposable $R *_\alpha G$-module, W will be an indecomposable left $R *_\alpha F$-module. □

5.5.3. Note. If M is a left $R *_\alpha G$-module which is finitely generated projective as R-module, then we may write $M_{|H} = L_1 \oplus \ldots \oplus L_q$ where each L_i is an indecomposable left $R *_\alpha H$-module. In this case W is the direct sum of all L_i lying over f.

5.5.4. Proposition. *Keep the notation and hypotheses of 5.5.2. and let M be finitely generated projective over R. In addition, suppose that any two indecomposable left $R *_\alpha H$-modules, which are finitely generated projective over R and lie over the same primitive central idempotent of $R *_\alpha H$, are isomorphic as $R *_\alpha H$-modules. Let N be an indecomposable left $R *_\alpha H$-module lying over f and being finitely generated projective over R. Then we have :*

$M_{|H} \cong \bigoplus\limits_{i=1}^{r}(u_{g_i} \otimes_{R*_\alpha H} N)^k$ as left $R *_\alpha H$-modules, $k \in \mathbb{N}$, where $\{g_1, \ldots, g_r\}$ is a set of left coset representatives of F in G, and $F = \{g \in G \mid u_g \otimes N \cong N \text{ in } R*_\alpha H\text{-mod}\}$.

Proof. By the hypotheses, $W \cong N^k$ as $R *_\alpha H$-modules and $u_gW = W$ if and only if $u_g \otimes N \cong N$ in $R *_\alpha H$-mod. We now apply 5.5.2. □

Remarks : (1) As in 5.5.3, consider a decomposition of $M_{|H}$ into indecomposable $R *_\alpha H$-modules and take some L_i. Then in 5.5.4, we may choose N to be L_i.
(2) The additional hypotheses in 5.5.4 is satisfied if $|H|^{-1} \in R$ and if either R is semilocal or R is a splitting ring for $R *_\alpha H$ and finitely generated projective R-modules are free.

5.5.5 Corollary. *With the notation and hypotheses as in 5.5.4 we have :*
$rank_R M = k[G : F] rank_R N$.

Proof. Follows from Proposition 5.5.4 and the fact that $u_{g_i} \otimes N \cong N$ as R-modules.

Next, we focus on the corresponding relations for trace functions. We need the following fact about trace functions of induced modules.

5.5.6 Note. Let K be an arbitrary subgroup of G. Let V be a left $R *_\alpha K$-module which is finitely generated projective over R and set $V^G = R *_\alpha G \otimes_{R *_\alpha K} V$. Let $\{g_1, \dots, g_m\}$ be a set of left coset representatives of K in G. Obviously $R *_\alpha G$ is a free right $R *_\alpha K$-module with basis $\{u_{g_1}, \dots, u_{g_m}\}$. Therefore any element of V is uniquely expressible as $\sum\limits_{i=1}^{m} u_{g_i} \otimes w_i$ with $w_i \in V$. So if $\{v_1, \dots, v_l\} \subset V$, $\{\varphi_1, \dots, \varphi_l\} \subset Hom_R(V, R)$ is an R-dual basis for V, then $\{u_{g_i} \otimes v_j\}$, $\{\psi_{ij}\}$, with $\psi_{ij} : V^G \to R : \sum\limits_{i=1}^{m} u_{g_i} \otimes w_i \mapsto \varphi_j(w_i)$, is an R-dual basis for V^G.
Define $\widetilde{t}_V$ as follows : $\widetilde{t}_V(u_s) = t_V(u_s)$ if $s \in K$ and $\widetilde{t}_N(u_s) = 0$ if $s \notin K$. Using our dual bases, it is now easily checked that

$$t_{V^G}(u_g) = \sum_{i=1}^{m} f_\alpha(g_i^{-1}, g) \widetilde{t}_V(u_{g_i^{-1} g g_i}) \qquad \text{for any } g \in G$$

If we modify α such that $f_\alpha(x, g) = 1$ for all α-G-regular $g \in G$ and $x \in G$, then :

$$|K| t_{V^G}(u_g) = \sum_{x \in G} \widetilde{t}_V(u_{x^{-1} g x}) \qquad \text{for any } \alpha\text{-}G\text{-regular } g \in G.$$

Furthermore, $V^G \cong V^m$ as R-modules. In particular, when R is connected, we have : $rank_R(V^G) = [G : K] rank_R(V)$.

5.5.7 Corollary. *Assume that α has been modified such that $f_\alpha(x,g) = 1$ for all α-G-regular $g \in G$ and all $x \in G$.*

(1) Keep the hypotheses and notation of 5.5.2 and let M be finitely generated projective over R. Then for any α-G-regular $g \in G$ we have:

$$\begin{aligned} t_M(u_g) &= \sum_{i=1}^{r} \tilde{t}_W(u_{g_i^{-1}gg_i}) \\ |F| t_M(u_g) &= \sum_{x \in G} \tilde{t}_W(u_{x^{-1}gx}) \end{aligned}$$

where $\tilde{t}_W(u_y) = t_W(u_y)$ if $y \in F$ and $\tilde{t}_W(u_y) = 0$ if $y \notin F$.

(2) With the hypotheses and notation of 5.5.4, we have for an α-G-regular $h \in H$:

$$\begin{aligned} t_M(u_h) &= k \sum_{i=1}^{r} t_N(u_{g_i^{-1}hg_i}) \\ |F| t_M(u_h) &= k \sum_{x \in G} t_N(u_{x^{-1}hx}) \end{aligned}$$

Proof. (1) Apply Note 5.5.6. to Proposition 5.5.2. (2).
(2) Since $H \triangleleft G$, $x^{-1}hx \in H \subset F$ for all $x \in G$. Further, by the hypotheses, $W \cong N^k$ as $R *_\alpha H$-modules. We now apply (1). □

5.5.8 Remarks In the sections 3 and 4 we considered the following situation. Let H be an arbitrary subgroup of G. Let R be connected and suppose that finitely generated projective R-modules are free. Further suppose $|G|^{-1} \in R, R *_\alpha G \cong \mathrm{End}_R(M_1) \oplus \ldots \oplus End_R(M_s)$ and $R *_\alpha H \cong \mathrm{End}_R(N_1) \oplus \ldots \oplus \mathrm{End}_R(N_t)$, M_j and N_i being finitely generated projective R-modules. Each M_j, viewed as a left $R *_\alpha H$-module, is uniquely expressible as a finite sum of N_i's and c_{ij} denotes the multiplicity of N_i in this decomposition of M_j. We now asume that H is a normal subgroup of G. Note that the hypotheses of 5.5.4. are satisfied.
Then : (1) All c_{ij} are invertible in R. Indeed, by 2.2.8. (1), $\mathrm{rank}_R(M_j)$ is invertible in R, and use 5.5.5.
(2) We see that 5.5.7. (2) is a special case of formula 5.4.10. Use (1) and 5.5.5.
(3) If $\mathrm{rank}_R N_i = 1$ and $F = G$, then for any $c_{ij} \neq 0$ we have : $c_{ij} = \mathrm{rank}_R P_{ij} = \mathrm{rank}_R(R *_\alpha G \otimes_S P_{ij})$ where $P_{ij} = Hom_{R *_\alpha H}(N_i, M_j)$. Combine 5.2.2, 5.2.12(2) and 5.5.5.

To conclude, note that Theorem 5.2.12 and Proposition 5.4.9. may be useful for normal subgroup, for example when $F = G$ in 5.5.2. Compare with [C-R, Theorem 11.20] (Clifford).

5.6 Examples

In this section we work out two easy examples to illustrate some of the results of the foregoing sections. Throughout R denotes the field of the complex numbers $\mathbb{C}$. We consider the dihedral group D_6 of order 12. The group D_6 is generated by a and b such that $a^6 = e$; $b^{-1}ab^{-1} = a^{-1}$; $b^2 = e$.
The conjugacy classes are :

$$\begin{array}{lll} C_e = \{e\}; & C_a = \{a, a^5\}; & C_{a^2} = \{a^2, a^4\} \\ C_{a^3} = \{a^3\}; & C_b = \{b, a^2b, a^4b\}; & C_{ab} = \{ab, a^3b, a^5b\} \end{array}$$

We consider the subgroup $H = \{e, b\}$ (which is not normal). The subclasses of H in G are :

$$\begin{array}{llll} E_e = \{e\}; & E_a = \{a, a^5\}; & E_{a^2} = \{a^2, a^4\}; & E_{a^3} = \{a^3\}; \\ E_b = \{b\}; & E_{a^2b} = \{a^2b, a^4b\}; & E_{ab} = \{ab, a^5b\}; & E_{a^3b} = \{a^3b\} \end{array}$$

Note that each conjugacy class is a union of subclasses.

5.6.1 The simple left modules of RD_6

We construct these simple left modules out of the simple left modules of RH and the simple left modules of the centralizer S of RH in RG. An R-basis of S is given by the subclass-sums.

$$\begin{array}{llll} s_e = u_e; & s_a = u_a + u_{a^5}; & s_{a^2} = u_{a^2} + u_{a^4}; & s_{a^3} = u_{a^3}; \\ s_b = u_b; & s_{a^2b} = u_{a^2b} + u_{a^4b}; & s_{ab} = u_{ab} + u_{a^5b}; & s_{a^3b} = u_{a^3b}. \end{array}$$

As $|G|^{-1} \in R$ and R is algebraically closed, R is a splitting field for RG, RH.
Put $RG \cong \bigoplus_{j=1}^{6} \mathrm{End}_R(M_j)$ where each M_j denotes a vector space over $\mathbb{C}$. Then

$12 = \sum_{j=1}^{6} \dim^2 M_j$. This latter equality entails that (up to renumbering) $\dim M_1 = \dim M_2 = 2$ and $\dim M_i = 1$ for $i = 3, \ldots, 6$.

Put $RH \cong \bigoplus_{i=1}^{2} \mathrm{End}_R(N_i)$ where each N_i denotes a vector space over $\mathbb{C}$. Then $2 = \sum_{i=1}^{2} \dim^2 N_i$. Thus $\dim N_i = 1$ for $i = 1, 2$. Of course the RH-modules N_i are known.

Using Proposition 5.2.9. and Note 5.2.10, we have $S \cong \bigoplus_{c_{ij} \neq 0} \mathrm{End}_R(P_{ij})$ where $P_{ij} = Hom_{RH}(N_i, M_j)$. Moreover, $\dim P_{ij} = c_{ij}$ = multiplicity of N_i in the decomposition of M_j, viewed as left RH-module (5.2.2). Now $\dim S = 8 = \sum_{j=1}^{6} \sum_{i=1}^{2} c_{ij}^2$ and this sum has at least 6 nonzero terms ($M_{j|_{RH}} \neq 0$). So there is a basic set of 8 non-isomorphic simple left S-modules, each of dimension 1, which occur into the splitting of S. A basic set of non-isomorphic simple left S-modules is given by the table below. Note that each $P_k \cong \mathbb{C}$ as vector space.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$s_a \cdot 1$	-1	-1	2	2	1	1	-2	-2
$s_{a^3} \cdot 1$	1	1	1	1	-1	-1	-1	-1
$s_b \cdot 1$	1	-1	1	-1	1	-1	1	-1
$t_{P_k}(z)$	6	6	12	12	6	6	12	12

The element $z = \sum_{g \in G_0} |E_g|^{-1} s_g s_{g^{-1}} = 8s_e + 2s_{a^2}$.

Remark. To construct these simple left S-modules, we made use of some relations in S e.g. $(s_{a^3})^2 = s_e$; $(s_b)^2 = s_e$; $(s_a)^2 = s_a s_{a^3} + 2s_e$; $s_{a^2} = s_a \cdot s_{a^3}$; $s_{a^2b} = s_{a^2} s_b$; $s_{ab} = s_a s_b$; $s_{a^3b} = s_{a^3} s_b$.

We now construct a basic set of non-isomorphic simple left RG-modules.
As $\dim N_i = 1$ for $i = 1, 2$, $RG \otimes_S P_{ij} \cong M_j$ for each $P_{ij} \neq 0$, (see 5.2.12(2)). Moreover, $\dim M_j \cdot t_{Pij}(z) = 12$ (use 2.4.4(2)). Thus $\dim(RG \otimes_S P_k) = 2$ for $k = 1, 2, 5, 6$. As a consequence the S-modules P_k with $k = 1, 2, 5, 6$ may occur in the decompositions of the simple left RG-modules of dimension 2, viewed as left S-modules (see 5.2.12 (1)).

For $j = 1, 2 : M_j \cong P_{1j} \oplus P_{2j}$ in S-mod. Moreover, $M_j \cong P_{1j}^2 \cong P_{2j}^2$ as $Z(RG)$-modules (see 5.2.14).
Thus $t_{M_j}(u_a + u_{a^5}) = 2t_{P_{1j}}(s_a) = 2t_{P_{2j}}(s_a)$. We now set $M_1 \cong P_1 \oplus P_2$, $M_2 \cong P_5 \oplus P_6$ as left S-modules.

From here we determine the RG-module structure on M_j, $j = 1, 2$ as follows. As $\dim M_j = 2$, we set $M_j \cong \mathbb{C} \oplus \mathbb{C}$ as vector spaces.
Put $u_b(1,0) = (0,1)$; $u_b(0,1) = (1,0)$; $u_a(1,0) = (\varepsilon^i, 0)$ and $u_a(0,1) = (0, \varepsilon^j)$ where ε is a primitive 6-th of unity and $i, j \in \mathbb{N}$.
For the simple RG-module M_1 we have : $t_{M_1}(u_a) = t_{M_1}(u_{a^5}) = -1$. Thus $\varepsilon^i + \varepsilon^j = \varepsilon^{5i} + \varepsilon^{5j} = -1$ which means that $i = 2$ and $j = 4$ (or $i = 4$ and $j = 2$). For the simple RG-module M_2 we become : $t_{M_2}(u_a) = t_{M_2}(u_{a^5}) = 1$. Thus $\varepsilon^i + \varepsilon^j = \varepsilon^{5i} + \varepsilon^{5j} = 1$ which leads to $i = 1$ and $j = 5$ (or $i = 5$ and $j = 1$).

The construction of the simple RG modules of dimension 1 is rather easy.
For $j = 3, \ldots, 6 : M_j \cong P_{ij}$ as left S-modules. Thus $t_{M_j}(u_a) = \frac{1}{2} t_{M_j}(u_a + u_{a^5}) = \frac{1}{2} t_{P_{ij}}(s_a)$ and $t_{M_j}(u_b) = t_{P_{ij}}(s_b)$.
We set in S-mod $M_3 \cong P_3$; $M_4 \cong P_4$; $M_5 \cong P_7$ and $M_6 \cong P_8$. A basic set of simple RG-modules of dimension 1 is given by the following table.

	M_3	M_4	M_5	M_6
$u_a \cdot 1$	1	1	-1	-1
$u_b \cdot 1$	1	-1	1	-1

5.6.2 The simple modules of a twisted group ring $R *_\alpha D_6$

Let α be the 2-cocycle defined as $\alpha : D_6 \times D_6 \to \mathbb{C}_0 : \alpha(a^i, a^j b^k) = 1$ and $\alpha(a^i b, a^j b^k) = \varepsilon^j$ where ε denotes a primitive 6-th root if unity. First remark that the α-G-regular subclasses in G are :

$$C_e = \{e\}; \qquad C_a = \{a, a^5\}; \qquad C_{a^2} = \{a^2, a^4\}.$$

For $H = \{e, b\}$, the α-H- regular subclasses in G are :

$E_e = \{e\};$ $\quad E_a = \{a, a^5\};$ $\quad E_{a^2} = \{a^2, a^4\};$

$E_b = \{b\};$ $\quad E_{a^2b} = \{a^2b, a^4b\};$ $\quad E_{ab} = \{ab, a^5b\}.$

We first define a 2-cocycle β equivalent to α satisfying the conditions of Lemma 5.3.4. We proceed as in this latter lemma. Note that $\alpha(e,e) = 1$. Define $\mu : G \to \mathbb{C}_0$ as follows :

$\mu(e) = 1$; $\mu(a) = 1$; $\mu(a^5) = \varepsilon$; $\mu(a^2) = 1$; $\mu(a^4) = \varepsilon^2$.

$\mu(b) = 1$; $\mu(a^2b) = 1$; $\mu(a^4b) = \varepsilon^2$; $\mu(ab) = 1$; $\mu(a^5b) = \varepsilon$.

$\mu(a^3) = 1$; $\mu(a^3b) = 1$.

Now put $\beta(x,y) = \alpha(x,y)\mu(x)\mu(y)\mu(xy)^{-1}$ for all $x, y \in G$. The β-G-regular subclass sums form an R-basis for $Z(R *_\beta G)$ and the β-H-regular subclass sums form an R-basis for the centralizer S of $R *_\beta H$ in $R *_\beta G$ (see 5.3.5).

As $|G|^{-1} \in R$ and R is algebraically closed, R is a splitting field for $R *_\beta H$ and $R *_\beta G$. Put $R *_\beta G \cong \bigoplus_{j=1}^{3} \mathrm{End}_R(M_j)$ where each M_j denotes a vector space over $\mathbb{C}$. Clearly, $\dim M_j = 2$ for $j : 1, 2, 3$.

Put $R *_\beta H \cong \bigoplus_{i=1}^{2} \mathrm{End}_R(N_i)$ where each N_i is a vector space over $\mathbb{C}$ of dimension 1. Of course, the $R *_\alpha H$-modules N_i are known.

Using Proposition 5.2.9 and Note 5.2.10, we have $S \cong \bigoplus_{c_{ij} \neq 0} \mathrm{End}_R(P_{ij})$ where $P_{ij} = Hom_{R *_\beta H}(N_i, M_j)$. Moreover, $\dim P_{ij} = c_{ij} =$ multiplicity of N_i in the decomposition of M_j, viewed as left $R *_\beta H$-module (see 5.2.2).

Now $\dim S = 6 = \sum_{j=1}^{3} \sum_{i=1}^{2} c_{ij}^2$ and this sum has at least 3 nonzero terms ($M_{j|_{R *_\beta H}} \neq 0$). Moreover, $M_j \cong \bigoplus_{c_{ij} \neq 0} P_{ij}$ in S-mod (see 5.2.12(1)). Thus each nonzero $c_{ij} = 1$. A basic set of 6 non-isomorphic simple left S-modules is given by the following table. Note that $P_k \cong \mathbb{C}$ for $k : 1, \ldots, 6$.

	P_1	P_2	P_3	P_4	P_5	P_6
$s_a \cdot 1$	$1+\varepsilon$	$-(1+\varepsilon)$	0	$1+\varepsilon$	$-(1+\varepsilon)$	0
$s_b \cdot 1$	1	1	1	-1	-1	-1
ε denotes a primitive 6-th of unity.						

Remark. To construct these simple left S-modules, we made use of some relations in S e.g. $s_{ab} = s_a s_b$; $s_{a^2 b} = s_{a^2} s_b$; $s_{a^2} = (s_a)^2 - 2\varepsilon s_e$; $(s_a)^3 = 3\varepsilon s_a$; $(s_b)^2 = s_e$.

We now construct a basic set of non-isomorphic simple left $R *_\beta G$-modules.
For $j = 1, 2, 3 : M_j \cong P_{1j} \oplus P_{2j}$, in S-mod (see 5.2.12(1)). Moreover, $M_j \cong P_{1j}^2 \cong P_{2j}^2$ as $Z(R *_\beta G)$-modules (see 5.2.14). Thus $t_{M_j}(u_a + u_{a^5}) = 2t_{P_{1j}}(s_a) = 2t_{P_{2j}}(s_a)$. We now set $M_1 \cong P_1 \oplus P_4$, $M_2 \cong P_2 \oplus P_5$, $M_3 \cong P_3 \oplus P_6$ as left S-modules.

From here we determine the $R *_\beta G$-module structure on M_j, $j = 1, 2, 3$ as follows. As $\dim M_j = 2$, we set $M_j \cong \mathbb{C} \oplus \mathbb{C}$ as vector spaces. Put $u_b(1, 0) = (0, 1)$; $u_b(0, 1) = (1, 0)$; $u_a(1, 0) = (\varepsilon^i, 0)$ and $u_a(0, 1) = (0, \varepsilon^j)$ where ε is a primitive 6-th of unity and $i, j \in \mathbb{N}$.
For the simple $R *_\beta G$-module M_1 we have : $t_{M_1}(u_a) = t_{M_1}(u_{a^5}) = t_{P_1}(s_a) = 1 + \varepsilon$. Thus $\varepsilon^i + \varepsilon^j = \varepsilon \cdot \varepsilon^{5i} + \varepsilon \cdot \varepsilon^{5j} = 1 + \varepsilon$ which means that $i = 0$ and $j = 1$ (or $i = 1$ and $j = 0$).
For the simple $R *_\beta G$-module M_2 we become : $t_{M_2}(u_a) = t_{M_2}(u_{a^5}) = t_{P_2}(s_a) = -1 - \varepsilon$. Thus $\varepsilon^i + \varepsilon^j = \varepsilon \cdot \varepsilon^{5i} + \varepsilon \cdot \varepsilon^{5j} = -1 - \varepsilon$ which means that $i = 3$ and $j = 4$ (or $i = 4$ and $j = 3$). Note that $u_{a^5} = \varepsilon u_a^5$ in $R *_\beta G$.
To construct the simple left $R *_\beta G$-module M_3, we express that $s_a M_3 = 0$. This yields the following : $(\varepsilon^i, 0) + (\varepsilon \cdot \varepsilon^{5i}, 0) = (0, 0)$ and $(0, \varepsilon^j) + (0, \varepsilon \cdot \varepsilon^{5j}) = (0, 0)$. Note that $u_{a^5} = \varepsilon u_a^5$ in $R *_\beta G$). Now $i = 2$ and $j = 5$ (or $i = 5$ and $j = 2$).

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- E. Nauwelaerts, L. Delvaux. *Restriction of projective group representations to subgroups and centralizers;* J. Algebra, 157 (1993), 63-79.
- L. Delvaux, E. Nauwelaerts. *Projective group representations and centralizers : Character theory;* J. Algebra 168 (1994), 314-339.
- L. Delvaux, E. Nauwelaerts. *Applications of Frobenius algebras to representation theory of Schur algbras;* submitted.