Analysis of an interior penalty discontinuous Galerkin scheme for two phase flow in porous media with dynamic capillarity effects

KARPINSKI, Stefan & POP, Sorin (2017) Analysis of an interior penalty discontinuous Galerkin scheme for two phase flow in porous media with dynamic capillarity effects. In: NUMERISCHE MATHEMATIK, 136(1), p. 249-286.

DOI: 10.1007/s00211-016-0839-5
Handle: http://hdl.handle.net/1942/22553

Numerische Mathematik manuscript No.
(will be inserted by the editor)

Analysis of an interior penalty discontinuous Galerkin scheme for two phase flow in porous media with dynamic capillary effects

Stefan Karpinski · Iuliu Sorin Pop

Received: date / Accepted: date

Abstract We present an interior penalty discontinuous Galerkin scheme for a two-phase porous media flow model that incorporates dynamic effects in the capillary pressure. The approximation of the mass-conservation laws is performed in their original formulation, without introducing a global pressure. We prove the existence of a solution to the emerging fully discrete systems and the convergence of the scheme. Error-estimates are obtained for sufficiently smooth data.

Keywords Dynamic capillary pressure · Interior penalty discontinuous Galerkin method · Error estimates · h-p convergence

1 Introduction

Flow and transport processes in porous media are of high interest in many different fields of application. Examples in this sense are the geological CO_2-storage [39], reactive transport in porous media [40], designing of diapers [20], filters, etc. In view of their relevance, a proper understanding of such systems is essential. This can be achieved by means of experiments, which are, however, not always possible nor feasible. Alternatively, mathematical modeling and simulation tools, relying on mathematical and numerical analysis can provide relevant knowledge with minimal societal or environmental impact.

In this context, porous media flow models have been developed for describing such processes at various scales [8,32], and many different simulation and discretization techniques have been proposed in the literature. Since local mass conservation is an important feature of the porous media flow models, it is desirable that these numerical schemes have this feature as well. Important classes of methods sharing this property are finite volume methods [27,32], or mixed finite element methods [41,40,23], or discontinuous Galerkin methods [25,3,44,7].

Over the last couple of decades more and more interest has been paid to so-called non-standard effects like hysteresis and dynamic capillarity. This has lead to new modeling and discretization approaches. Typically appearing at smaller scales like the laboratory scale, such effects can explain experimental results like saturation overshoot [19] that are ruled out by standard models. At larger scales, it is common to neglect the capillary effects, and the resulting models are of hyperbolic type. However, defining the physically relevant (entropy) solution in this case still requires to use information emerging from smaller scales and thus to account for the non-equilibrium effects whenever appropriate [34,22].

Common models for two phase flow in porous media are assuming a nonlinear, algebraic relationship between the phase pressure difference and the saturation of one of the phases (say, wetting). Such relationships are obtained experimentally, but based on measurements that were made over long times so that the phases are at equilibrium [32,39]. Here we consider the case where the pressure difference - saturation relationship also involves a dynamic term, as proposed in [31,30]. In contrast to standard, equilibrium

Stefan Karpinski
ESPRiT Engineering GmbH
E-mail: stefan.karpinski@esprit-engineering.de

Iuliu Sorin Pop
Faculty of Sciences, Hasselt University,
Dept. of Mathematics, University of Bergen E-mail: sorin.pop@uhasselt.be

based porous media flow models, non-equilibrium models can explain effects like saturation overshoot or finger-pattern formation, which have been observed experimentally. The ability of non-equilibrium models to explain experimental results as mentioned above is has been proved by means of mathematical analysis. For example, the occurrence of non-monotonic travelling wave profiles depending on the magnitude of the dynamic capillarity effects has been analyzed rigorously in [22]. The existence and uniqueness of weak solutions for such types of models has been proved in [42,35,37,28,13], while appropriate numerical schemes are analyzed in [11,35,33].

In this paper we analyze a primal interior penalty discontinuous Galerkin (DG) discretization method for such non-equilibrium porous media flow models. Such methods have grown more popular in the last decades due to their versatility and easy adaptation to include heterogenities, parallelization, and hp-adaptivity. Although well developed for standard, equilibrium based two-phase flow problems [24], DG methods have not been implemented and analyzed yet for two phase flow with dynamic capillarity effects.

A common approach when dealing with such models is to employ the so-called global pressure, which allows rewriting the system in such a way that some nonlinear factors in the higher order terms become linear [14]. The advantage of this approach is that the a priori estimates can be obtained separately for each of the transformed pressures, which can then be used to estimate the saturation. This approach is followed in [24]. The drawback of this approach lies in the fact that the global pressure is not a physical quantity, and one needs to postprocess the results for extracting information that is relevant for the original application. Therefore, instead of reformulating the mass balance equations in terms of the global pressure, here only the original physical unknowns are used. This leads to a strong coupling of the mass balance equations, making impossible to obtain directly the a priori estimate for the pressure. Instead, one has to estimate both pressure and saturation simultaneously, as done in [27] and [35]. Noting that these two papers are considering finite volume and finite element approaches and that the former does not include dynamic capillarity effects, in this paper we provide the rigorous convergence proof for an interior penalty DG approximation of the two-phase flow model involving dynamic capillarity.

The paper is organized as follows. The mathematical model is presented in Section 2, the emphasis lying on the dynamic capillarity effects. The discretization is given in Section 3, together with the assumptions and the basic notations and for the interior penalty DG approximation. In Section 4, we give the main results of the paper: the existence of a solution to the nonlinear systems appearing after a complete discretization in space and time, and prove the convergence of the scheme by obtaining error estimates. Finally, Section 5 provides a numerical example confirming the theoretical estimates.

Notation In what follows we let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be an open bounded polygonal domain (the porous medium) with boundary Γ and $T > 0$ is a maximal, finite time. Both are considered dimensionless. The notations below are common in the functional analysis [1] and will be used in what follows. Whenever values on Γ are involved, these should be understood in the sense of traces Recalling the definitions of the traces [26].

- $L^p(\Omega)$ $(1 \leq p < \infty)$ is the usual space of functions that are p-Lebesgue integrable and $L^\infty(\Omega)$ is the space of functions that are essentially bounded in Ω. The elements of $W^{k,p}(\Omega)$ are the functions admitting weak derivatives up to order k that are again in L^p. For simplicity, we use the notation $H^k(\Omega)$ for $W^{k,2}(\Omega)$.
- For $1 \leq p \leq \infty$, $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ are the standard norms in $L^p(\Omega)$, respectively $W^{k,p}(\Omega)$. A simplified notation will be used for the norm in $W^{k,2}(\Omega)$, namely $\|\cdot\|_{\Omega,k}$.
- $H_0^k(\Omega)$ denotes the subspaces of $H^k(\Omega)$ taking the value 0 on the boundary (in the sense of traces).
- $L^q([0,T], W^{k,p}(\Omega))$ denotes the Bochner space of vector spaced valued functions $f : [0,T] \to W^{k,p}(\Omega)$ that are p-Bochner integrable on $[0,T]$.
- $H^1([0,T], L^2(\Omega))$ denotes the Bochner space of $L^2(\Omega)$ valued functions admitting a weak time-derivative in $L^2([0,T]; L^2(\Omega))$.

As for the domain Ω, the traces on Γ will lie in spaces like $L^p(\Gamma)$, $H^k(\Gamma)$, etc. In particular, by $H^{\frac{1}{2}}(\Gamma)$ we mean the traces on Γ of $H^1(\Omega)$ functions.

2 Mathematical model

We consider a (Darcy scale) model for the flow of two incompressible and immiscible fluids (wetting, respective non-wetting) through a porous medium. This is based on the following assumptions:

- All physical processes are isothermal.
- Gravitational forces are neglected.
- The flow velocities lie well within the Darcy regime.
- The porous matrix is rigid and has homogeneous characteristics.

We mention that neglecting gravitational effects is only for the ease of presentation. Including such effects can be done without any particular mathematical difficulty, but would lead to more complex calculations.

2.1 Governing equations

Under the assumptions stated above, the mathematical model [32,39] includes the mass conservation laws for each phase (the wetting and non-wetting, $\alpha = n$ or w):

$$\partial_t(S_\alpha \phi \rho_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha) = q_\alpha \ . \tag{1}$$

Here ϕ denotes the porosity of the medium, ρ_α the fluid phase densities, S_α the saturation of phase α, and q_α the volumetric sources or sinks. Further, $\mathbf{u}_\alpha$ is the Darcy velocity of the phase α, given by

$$\mathbf{u}_\alpha = -\lambda_\alpha(S_\alpha) K \nabla p_\alpha \ . \tag{2}$$

Here p_α is the pressure of the phase α, K the intrinsic permeability tensor, and $\lambda_\alpha = \dfrac{k_{r,\alpha}}{\mu_\alpha}$ is the phase α mobility function, with relative permeability $k_{r,\alpha}$ and dynamic viscosity μ_α. Observe that the model is assumed dimensionless, but the notations are referring directly to the corresponding dimensional quantities.

2.2 Closure relationships

As resulting from above, there are six unknown quantities (phase saturations, pressures and velocities) whereas only four equations are available. Observe that assuming that only two phases are present in the system one gets

$$S_w + S_n = 1. \tag{3}$$

The system is closed by the phase pressure difference - saturation relationship (for standard models, [32]), or its non-equilibrium version involving the time derivative of the saturation [31]

$$p_c := p_n - p_w = p_c(S_w, \partial_t S_w) \ . \tag{4}$$

2.3 Primary variables

The model above can be reduced to three equations one by choosing three primary unknowns. For example, letting these be the water saturation S_w, the non-wetting phase pressure p_n and the phase pressure difference $p_c = p_n - p_w$ one gets

$$\begin{aligned}
& -\partial_t S_w \phi - \nabla \cdot (\lambda_n(S_w) K \nabla p_n) = q_n, \\
& \partial_t S_w \phi - \nabla \cdot (\lambda_w(S_w) K \left(\nabla p_n - \nabla p_c\right)) = q_w, \\
& p_c = p_c(S_w, \partial_t S_w).
\end{aligned} \tag{5}$$

2.4 Constitutive relationships

Dynamic effects in the phase pressure difference As mentioned above, a common assumption in modelling the flow of two phases in porous media is that the phase pressure difference and the saturation are related through a nonlinear, algebraic relation. These are the so-called standard, equilibrium models: for a given medium and knowing that the wetting phase saturation has a certain value in a given location, the phase pressure difference has a fixed value depending only on the medium itself and the value of the saturation. This assumes a static distribution of the two phases inside the pores of the medium. In this context, several possible parameterizations relating p_c and S_w by using medium specific parameters have been proposed in the literature. Examples are the Brooks-Corey model [9], or the van Genuchten model [46, 38]. Such models are valid whenever the processes are very slow, so the dynamics of the flow, and in particular the redistribution of the phases inside pores before achieving equilibrium is disregarded.

Experimental results have proved the limitation of such equilibrium models. For example, the experiments in [19] show that non-monotonic saturation profiles (overshoots) can be obtained during infiltration processes in a dry porous medium, and that the amplitude of such overshoots depend on the flow velocity. Such results are ruled out for equilibrium models, which would predict monotonic profiles regardless of the chosen parameterization. Therefore alternative modelling theories were required, such as

$$p_c = p_{c,eq}(S_w) - \tau \partial_t S_w \ , \tag{6}$$

involving the time derivative of the saturation the one in [31]. Here $p_{c,eq}$ is the capillary pressure at equilibrium, and τ accounts for the dynamic effects. In this paper τ is assumed to be a positive constant.

Relative permeabilities The focus in this work is on the dynamic effects in the phase pressure difference. Therefore for the relative permeability functions, equilibrium models like Brooks-Corey [9] or van Genuchten [46] in conjunction with the Mualem and Burdine framework [38, 10] are assumed.

2.5 Initial and boundary conditions

The system is completed by the following initial and boundary conditions:

$$\begin{aligned} &\text{For all } x \in \Omega \text{ and at } t = 0, \\ &\qquad S_w(x,0) = s^0(x) \qquad \text{with, } s^0 \in H^1(\Omega). \qquad (7) \\ &\text{For all } x \in \Gamma \text{ and all } t \in [0,T], \\ &\qquad p_c(x,t) = p_c^D(x) \ , \quad p_n(x,t) = p_n^D(x) \qquad (8) \\ &\qquad \text{with } p_n^D \in H^{\frac{1}{2}}(\Gamma) \ , \ p_c^D \in H^{\frac{1}{2}}(\Gamma) \end{aligned}$$

where s^0, p_n^D and p_c^D are given functions. Note that the boundary value of S_w is defined implicitly by the Dirichlet conditions for p_c.

Remark 1 For the sake of clarity, here only Dirichlet boundary conditions are considered. The subsequent proofs can be extended towards other different boundary conditions ate the expense of technical calculations. Also, the boundary values are assume constant in time.

2.6 Weak formulation

The weak formulation of the model in (5)-(6) with the initial and boundary conditions (7)-(8) is

Problem 1 (Weak formulation) Find the triple (s_w, p_n, p_c) s.t. $s_w \in H^1([0,T], H^1(\Omega))$, $s_w = s^0$ at $t = 0$, $p_n - p_n^D \in L^2([0,T], H_0^1(\Omega))$, $p_c - p_c^D \in L^2([0,T], H_0^1(\Omega))$, and for all $\psi_p \in H_0^1(\Omega)$, $\psi_s \in H_0^1(\Omega)$, and almost every $t \in [0,T]$ it holds

$$\begin{aligned} -\int_\Omega \partial_t S_w \phi \psi_p + \int_\Omega \lambda_n(S_w) K (\nabla p_n) \cdot \nabla \psi_p &= \int_\Omega q_n \psi_p, \\ \int_\Omega \partial_t S_w \phi \psi_p + \int_\Omega \lambda_w(S_w) K (\nabla p_n - \nabla p_c) \cdot \nabla \psi_p &= \int_\Omega q_w \psi_p, \\ \int_\Omega p_c \psi_s = \int_\Omega p_{c,eq}(S_w) \psi_s - \int_\Omega \tau \partial_t S_w \psi_s&. \end{aligned} \tag{9}$$

Existence and uniqueness results for Problem 1 are obtained in [12,13,28,37,35].

3 Numerical scheme

Preliminaries Let $\mathcal{T}$ be a decomposition of the domain Ω into N non-degenerate elements T_i. We assume that $\mathcal{T}$ is admissible in the sense of the Definition 2.1 in [17]. Let $\mathcal{F}$ denote the union of all faces F_j, and let h be the maximal diameter of the elements.

Given $T_i \in \mathcal{T}$ and $F_i \in \mathcal{F}$, we define a set $F(T_i)$ of all the faces associated with the element T_i, s.t.,

$$F(T_i) := \left\{ \bigcup_{F_j \in \mathcal{F}} F_j : F_j \subset T_i \right\},$$

and, a set $T(F_i)$ of all the elements sharing the face F_i, s.t.,

$$T(F_i) := \left\{ \bigcup_{T_j \in \mathcal{T}} T_j : F_i \subset T_j \right\}.$$

In the conformiong case, $T(F_i)$ consists of exactly two elements.

With each face $F \in \mathcal{F}$ connecting element T_i and T_j, we associate a normal-vector $\mathbf{n}$ directed from T_i to T_j $(j > i)$.

Let $\Pi^k(T)$ denote the space of polynomials on T with degree $\leq k$. For the approximation of saturation S_w, we consider the broken Sobolev-Space with polynomials of order k_s , as,

$$V_h^s(\Omega) := \{v \in L^2(\Omega) : v|_T \in \Pi^{k_s}(T) \text{ for all } T \in \mathcal{T}\} , \tag{10}$$

and, for the approximation of the pressures p_n and p_c, we consider the broken Sobolev space with polynomials of order k_p , as,

$$V_h^p(\Omega) := \{v \in L^2(\Omega) : v|_T \in \Pi^{k_p}(T) \text{ for all } T \in \mathcal{T}\} . \tag{11}$$

Note that we represent a general broken Sobolev-Space with $V_h(\Omega)$ without specifying the polynomial order.

For $\psi^i, \psi^j \in V_h(\Omega)$, where, $\psi^i = (\psi|_{T^i})|_F$ is the trace of F on the side of the element T_i, and similarly, $\psi^j = (\psi|_{T^j})|_F$ is the trace of F on the side of the element T_j, we define the jump $[\![\cdot]\!]$ and the average $\{\cdot\}$ over the face F as,

$$\text{when } F \text{ is an interior face} \; : \quad [\![\psi]\!] = \left(\psi^i - \psi^j\right) \quad \text{and} \quad \{\psi\} = \frac{1}{2}\left(\psi^i + \psi^j\right) , \tag{12}$$

$$\text{when } F \text{ is a boundary face} \; : \quad [\![\psi]\!] = \psi^i \quad \text{and} \quad \{\psi\} = \psi^i. \tag{13}$$

where, the interior face connects elements T^i and T^j with $i < j$, and the boundary face has no element adjacent to T_i.

Next, we define the following norm on the broken Sobolev-Space,

$$\|v\|^2_{\Omega,DG} := \sum_{T_i \in \mathcal{T}} \|\nabla v\|^2_{T_i,0} + \sum_{F_i \in \mathcal{F}} \frac{1}{|F_i|} \|[\![v]\!]\|^2_{F_i,0} \tag{14}$$

and use the following lemma [17]:

Lemma 1 *Given a broken Sobolev-Space $V_h(\Omega)$, for any q such that,*

$$1 \leq q \leq \frac{2d}{d-2}, \ \text{if } d \geq 3$$
$$1 \leq q < \infty, \ \text{if } d = 2 \ ,$$

there exists a constant $\hat{C}$ depending on the polynomial degree, mesh-parameters and $|\Omega|$ such that, for all $v \in V_h(\Omega)$, the following inequality holds:

$$\|v\|_{L^q(\Omega)} \leq \hat{C}\|v\|_{\Omega,DG} \ . \tag{15}$$

Additionally, we use the following trace inequalities, which can be found in [47], [43], or [18]:

Lemma 2 *Let γ_0 denote the trace operator. There exists a constant C_t independent of the mesh size h, such that, for any $T \in \mathcal{T}$ with $F \in F(T)$ and for all $v \in H^k(T)$, the following holds:*

$$\|\gamma_0 v\|_{0,F} \leq C_t\sqrt{\frac{1}{|F|}}(\|v\|_{0,T} + |F|\|\nabla v\|_{0,T}) \tag{16}$$

For $v \in \Pi^k(T)$ and $f(k)$ which is a function of the polynomial degree k, the following holds:

$$\|\gamma_0 v\|_{0,F} \leq C_t\sqrt{\frac{f(k)}{|F|}}\|v\|_{0,T} \tag{17}$$

We also use the following elementary lemma [24]:

Lemma 3 *Let $\tilde{C}$ be the maximal number of elements sharing one face, and let $A : \mathcal{T} \to [0,\infty)$ be a function defined on the triangularization $\mathcal{T}$. Then, the following inequality holds:*

$$\sum_{F_i}\sum_{T(F_i)} A(T) \leq \tilde{C}\sum_{T_i} A(T_i)$$

Finally, we state the following well known (in-)equalities for $a, b \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$, which are used throughout the paper:

$$(a-b)\cdot a = \frac{1}{2}(a-b)^2 + \frac{1}{2}(a^2 - b^2) \tag{18}$$
$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2 \tag{19}$$

3.1 Discretization in space

The weak form (9) of the mathematical model governed by the Eqn-set (5) is discretized in space using an interior penalty discontinuous Galerkin numerical scheme.

Problem 2 (Spatial discretization) Given the penalty parameters $\sigma_n, \sigma_w \in \mathbb{R}^+$, the parameter $\theta \in \{-1, 0, 1\}$ and the function $f(\cdot)$ introduced in Lemma 2 depending on the polynomial order k_p, find

$s_w \in V_h^s(\Omega)$, $p_n \in V_h^p(\Omega)$ and $p_c \in V_h^p(\Omega)$, s.t., for all $\psi_s \in V_h^s(\Omega)$, $\psi_n \in V_h^p(\Omega)$ and $\psi_w \in V_h^p(\Omega)$ the following holds:

$$\begin{aligned}
\textbf{PDE-1}\text{:} \sum_{T_i \in \mathcal{T}} \int_{T_i} -\partial_t s_w \phi \psi_n + \sum_{T_i \in \mathcal{T}} \int_{T_i} \lambda_n(s_w) K \nabla p_n \nabla \psi_n \\
- \sum_{F_i \in \mathcal{F}} \int_{F_i} \{\lambda_n(s_w) K \nabla p_n \cdot \mathbf{n}\} [\![\psi_n]\!] + \theta \sum_{F_i \in \mathcal{F}} \int_{F_i} [\![p_n]\!] \{\lambda_n(s_w) K \nabla \psi_n\} \\
+\sigma_n \sum_{F_i \in \mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![p_n]\!] [\![\psi_n]\!] \\
= \theta \sum_{F_i \in \Gamma} \int_{F_i} [\![p_n^D]\!] \{\lambda_n(s^D) K \nabla \psi_n\} + \sigma_n \sum_{F_i \in \Gamma} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![p_n^D]\!] [\![\psi_n]\!]
\end{aligned} \tag{20}$$

$$\begin{aligned}
\textbf{PDE-2}\text{:} \sum_{T_i \in \mathcal{T}} \int_{T_i} \partial_t s_w \phi \psi_w + \sum_{T_i \in \mathcal{T}} \int_{T_i} \lambda_w(s_w) K \nabla (p_n - p_c) \nabla \psi_w \\
- \sum_{F_i \in \mathcal{F}} \int_{F_i} \{\lambda_w(s_w) K \nabla (p_n - p_c) \cdot \mathbf{n}\} [\![\psi_w]\!] \\
+\theta \sum_{F_i \in \mathcal{F}} \int_{F_i} \{\lambda_w(s_w) K \nabla \psi_w \cdot \mathbf{n}\} [\![p_n - p_c]\!] + \sigma_w \sum_{F_i \in \mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![p_n - p_c]\!] [\![\psi_w]\!] \\
= \theta \sum_{F_i \in \Gamma} \int_{F_i} \{\lambda_w(s^D) K \nabla \psi_w \cdot \mathbf{n}\} [\![p_n^D - p_c^D]\!] + \sigma_w \sum_{F_i \in \Gamma} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![p_n^D - p_c^D]\!] [\![\psi_w]\!]
\end{aligned} \tag{21}$$

$$\textbf{ODE-Pc}\text{:} \sum_{T_i \in \mathcal{T}} \int_{T_i} p_c \psi_s = \sum_{T_i \in \mathcal{T}} \int_{T_i} p_{c,eq}(s_w) \psi_s - \sum_{T_i \in \mathcal{T}} \int_{T_i} \tau \partial_t s_w \psi_s \tag{22}$$

The parameters σ_n and σ_w penalize discontinuities in the solutions (i.e., jumps) over the faces. The choice of $\theta = -1$ gives the non-symmetric- (NIPG) scheme, $\theta = 0$ gives the incomplete- (IIP) scheme, and $\theta = 1$ gives the symmetric-interior-penalty (SIPG) scheme.

3.2 Discretization in time

For the discretization in time, we use an implicit Euler scheme. We subdivide the time domain $[0, T]$ into N intervals of size $\Delta t > 0$ with $T = N \cdot \Delta t$. The i-th discrete time-step is denoted by t_i, s.t., $t_i = i \cdot \Delta t$.

Given a sufficiently smooth function $g(x, t)$, the time derivative of g is approximated by:

$$\partial^- g^{n+1} := \frac{g(t^{n+1}, x) - g(t^n, x)}{\Delta t} \tag{23}$$

3.3 Discrete system

Using Problem 2 and Eqn. (23), the fully-discrete scheme can be written as:

Problem 3 (Discrete problem at t^{n+1}) Let $P_n^n \in V_h^p(\Omega)$, $P_c^n \in V_h^p(\Omega)$ and $S_w^n \in V_h^s(\Omega)$, find $P_n^{n+1} \in V_h^p(\Omega)$, $P_c^{n+1} \in V_h^p(\Omega)$ and $S_w^{n+1} \in V_h^s(\Omega)$, s.t., for all $\psi_s \in V_h^s(\Omega)$, $\psi_n \in V_h^p(\Omega)$ and $\psi_w \in V_h^p(\Omega)$, the following holds:

$$\begin{aligned}
\textbf{PDE-1}\text{:} \sum_{T_i \in \mathcal{T}} \int_{T_i} -\partial^- S_w^{n+1} \phi \psi_n + \sum_{T_i \in \mathcal{T}} \int_{T_i} \lambda_n(S_w^{n+1}) K \nabla P_n^{n+1} \nabla \psi_n \\
- \sum_{F_i \in \mathcal{F}} \int_{F_i} \{\lambda_n(S_w^{n+1}) K \nabla P_n^{n+1} \cdot \mathbf{n}\} [\![\psi_n]\!]
\end{aligned}$$

$$+\,\theta \sum_{F_i\in\mathcal{F}} \int_{F_i} [\![P_n^{n+1}]\!]\{\lambda_n(S_w^{n+1})K\nabla\psi_n\} + \sigma_n \sum_{F_i\in\mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|}[\![P_n^{n+1}]\!][\![\psi_n]\!]$$

$$= \theta \sum_{F_i\in\Gamma} \int_{F_i} [\![p_n^D]\!]\{\lambda_n(s^D)K\nabla\psi_n\} + \sigma_n \sum_{F_i\in\Gamma} \int_{F_i} \frac{f(k_p)}{|F_i|}[\![p_n^D]\!][\![\psi_n]\!] \tag{24}$$

PDE-2:
$$\sum_{T_i\in\mathcal{T}} \int_{T_i} \partial^- S_w^{n+1}\phi\psi_w + \sum_{T_i\in\mathcal{T}} \int_{T_i} \lambda_w(S_w^{n+1})K\nabla(P_n^{n+1}-P_c^{n+1})\nabla\psi_w$$

$$-\sum_{F_i\in\mathcal{F}} \int_{F_i} \{\lambda_w(S_w^{n+1})K\nabla(P_n^{n+1}-P_c^{n+1})\cdot\mathbf{n}\}[\![\psi_w]\!]$$

$$+\,\theta \sum_{F_i\in\mathcal{F}} \int_{F_i} \{\lambda_w(S_w^{n+1})K\nabla\psi_w\cdot\mathbf{n}\}[\![P_n^{n+1}-P_c^{n+1}]\!]$$

$$+\,\sigma_w \sum_{F_i\in\mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|}[\![P_n^{n+1}-P_c^{n+1}]\!][\![\psi_w]\!]$$

$$= \theta \sum_{F_i\in\Gamma} \int_{F_i} \{\lambda_w(s^D)K\nabla\psi_w\cdot\mathbf{n}\}[\![p_n^D-p_c^D]\!]$$

$$+\,\sigma_w \sum_{F_i\in\Gamma} \int_{F_i} \frac{f(k_p)}{|F_i|}[\![p_n^D-p_c^D]\!][\![\psi_w]\!] \tag{25}$$

ODE-Pc:
$$\sum_{T_i\in\mathcal{T}} \int_{T_i} P_c^{n+1}\psi_s = \sum_{T_i\in\mathcal{T}} \int_{T_i} p_{c,eq}(S_w^{n+1})\psi_s - \sum_{T_i\in\mathcal{T}} \int_{T_i} \tau\partial^- S_w^{n+1}\psi_s \tag{26}$$

4 Numerical Analysis

In this section we prove that the scheme is well-posed and convergent. We first show the existence of a discrete solution using a fix-point argument, followed by the energy estimates for the discrete solutions. Finally, we show convergence of the scheme by proving some error estimates.

Preliminaries We make the following assumptions to prove existence and convergence of the numerical scheme:

(A1) The initial and boundary conditions in (7) and (8) are sufficiently smooth. Additionally, the initial condition is compatible with the boundary condition.

(A2) The permeability matrix $K \in \mathbb{R}^{d\times d}$ is symmetric and positive definite, i.e. there exist two constants $\overline{\kappa}$ and $\underline{\kappa}$, s.t., for any vector $x \in \mathbb{R}^d$, the following holds:

$$\underline{\kappa}\|x\|^2 \le x^T K x \le \overline{\kappa}\|x\|^2$$

(A3) The equilibrium capillary pressure function $p_{c,eq}(\cdot)$ is in $C^2(\mathbb{R})$, and is assumed to be positive, bounded and decreasing. Let $P_{c,eq}(\cdot)$ define the primitive, i.e.:

$$P_{c,eq}(S) := \begin{cases} \int_1^S p_{c,eq}(\xi)d\xi = \int_0^S p_{c,eq}(\xi)d\xi - \int_0^1 p_{c,eq}(\xi)d\xi & \text{for } S \le 1 \\ 0 & \text{otherwise} \end{cases}. \tag{27}$$

It can be inferred that $P_{c,eq}(S)$ is concave and negative.

(A4) The functions $\lambda_w(\cdot)$ and $\lambda_n(\cdot)$ are Lipschitz-continuous and bounded from above and below by the constants $0 < \overline{\lambda_\alpha} < \underline{\lambda_\alpha} < \infty$.

For the error analysis, let $s_w(t,x)$, $p_n(t,x)$ and $p_c(t,x)$ be the exact solutions of the problem. For simplicity, we will use $s_w^i = s_w(t_i,x)$, $p_n^i = p_n(t_i,x)$ and $p_c^i = p_c(t_i,x)$. We denote the approximations of $p_n(t)$, $p_c(t)$, and $s_w(t)$ for all $t \in [0,T]$ with, $\tilde{p}_n(t) \in V_h^p(\Omega)$, $\tilde{p}_c(t) \in V_h^p(\Omega)$, and $\tilde{s}_w(t) \in V_h^s(\Omega)$, respectively, and assume that $\tilde{p}_n(t) \in W^{1,\infty}(\Omega)$, $\tilde{p}_c(t) \in W^{1,\infty}(\Omega)$ and $\tilde{s}_w(t) \in W^{1,\infty}(\Omega)$ for all $t \in [0,T]$. We also assume that the solutions possess enough regularity, such that the the following approximation properties are fulfilled:

For all $t \in [0,T]$ and $T \in \mathcal{T}$, for $\tilde{p}_n(t) \in W^{1,\infty}$, $\tilde{p}_c(t) \in W^{1,\infty}$ and $\tilde{s}_w(t) \in W^{1,\infty}$ there exists a constant C independent of h, k_s, k_{p_n}, k_{p_c} and Δt s.t.,

$$\text{for} \quad 0 < q \le l_{p_n} \ , \quad \|p_n(t) - \tilde{p}_n(t)\|_{T,q} \le C \frac{h^{\min(k_{p_n}+1, l_{p_n})-q}}{k_{p_n}^{l_{p_n}-q}} \|p_n(t)\|_{T,l_{p_n}} \ , \tag{28}$$

$$\text{for} \quad 0 < q \le l_{p_c} \ , \quad \|p_c(t) - \tilde{p}_c(t)\|_{T,q} \le C \frac{h^{\min(k_{p_c}+1, l_{p_c})-q}}{k_{p_c}^{l_{p_c}-q}} \|p_c(t)\|_{T,l_{p_c}} \ , \tag{29}$$

$$\text{for} \quad 0 < q \le l_s \ , \quad \|s_n(t) - \tilde{s}_n(t)\|_{T,q} \le C \frac{h^{\min(k_s+1, l_s)-q}}{k_s^{l_s-q}} \|s_n(t)\|_{T,l_s} \ . \tag{30}$$

The proof for the results (28), (29) and (30) can be found in [2].

Further, we write the numerical errors for $i = 1, \dots, N$ as,

$$e^i_{s,h} = S^i - \tilde{s}^i_w,\ e^i_s = \tilde{s}^i_w - s^i,\ e^i_{p_\alpha,h} = P^i_\alpha - \tilde{p}^i_\alpha,\ e^i_{p_\alpha} = \tilde{p}^i_\alpha - p^i_\alpha \ .$$

4.1 Existence of a discrete solution

We now prove the existence of a discrete solution for the Problem 3.

For a given real numbers $P_{n,l} \in \mathbb{R}$, $P_{c,l} \in \mathbb{R}$ and $S_{w,k} \in \mathbb{R}$, we define $\tilde{P}_n, \tilde{P}_c \in V^p_h(\Omega)$ and $\tilde{S}_w \in V^s_h(\Omega)$ by,

$$\tilde{P}_n = \sum_{l=0}^{d_p} P_{n,l}\varphi^p_l \quad \tilde{P}_c = \sum_{l=0}^{d_p} P_{c,l}\varphi^p_l \quad \tilde{S}_w = \sum_{k=0}^{d_s} S_{w,k}\varphi^s_k \ , \tag{31}$$

where, φ^p_i and φ^s_k are elements of a basis for $V^p_h(\Omega)$ and $V^s_h(\Omega)$ and $d_p \in \mathbb{N}$ and $d_s \in \mathbb{N}$ denote the dimension. We define the coefficient vectors $\hat{P}_n, \hat{P}_c \in \mathbb{R}^{d_p}$ and $\hat{S}_w \in \mathbb{R}^{d_s}$ by:

$$\hat{P}_n = \left(P_{n,1},\, P_{n,2},\, \dots,\, P_{n,d_p}\right)^T \quad \hat{P}_c = \left(P_{c,1},\, P_{c,2},\, \dots,\, P_{c,d_p}\right)^T \quad \hat{S}_w = \left(S_{w,1},\, S_{w,2},\, \dots,\, S_{w,d_s}\right)^T \ . \tag{32}$$

Furthermore, for given real numbers $S^n_{w,k} \in \mathbb{R}$, we define $P_{w,l} \in \mathbb{R}$ for $l = 0, \dots, d_p$ and $dS_{w,k} \in \mathbb{R}$ for $k = 0, \dots, d_s$, with,

$$P_{w,l} := P_{n,l} - P_{c,l} \ , \quad \text{and} \quad dS_{w,k} = \frac{1}{\Delta t}(S_{w,k} - S^n_{w,k}) \ , \tag{33}$$

which gives us $\tilde{P}_w \in V^p_h(\Omega)$, $S^n_w \in V^s_h(\Omega)$ and $\tilde{dS}_w \in V^s_h(\Omega)$, s.t.,

$$\tilde{P}_w := \tilde{P}_n - \tilde{P}_c = \sum_{i=0}^{d_p} P_{w,i}\varphi^p_i \ , \quad \text{and} \quad \tilde{dS}_w := \frac{1}{\Delta t}(\tilde{S}_w - S^n_w) = \sum_{k=0}^{d_s} dS_{w,k}\varphi^s_k \ . \tag{34}$$

The coefficient vectors $\hat{P}_w \in \mathbb{R}^{d_p}$ and $\hat{dS}_w \in \mathbb{R}^{d_s}$ are defined analogous to (32).

Next, we define $\langle \cdot, \cdot \rangle_{\ell^2}$ as the ℓ^2-scalar product on $\mathbb{R}^{2d_p+d_s}$, and $\|\cdot\|_{\ell^2}$ as the induced ℓ^2-norm on $\mathbb{R}^{2d_p+d_s}$.

Note that for a coefficient vector $\hat{X} \in \mathbb{R}^{2d_p+d_s}$ and the induced vector $\tilde{X} \in V^p_h(\Omega) \times V^p_h(\Omega) \times V^s_h(\Omega)$ there exists constants $c > 0, c \in \mathbb{R}$ and $C > 0, C \in \mathbb{R}$ such that the following inequality holds:

$$c\|\tilde{X}\|_{\Omega,0} \le \|\hat{X}\|^2_{\ell^2} = \langle \hat{X}, \hat{X} \rangle_{\ell^2} \le C\|\tilde{X}\|_{\Omega,0} \ . \tag{35}$$

Using the definitions (31), (33) and (34) in (24)-(26), we define $F_i^{P_n}, F_i^{P_c}, F_k^S \in \mathbb{R}$ for $i = 0, 1, \ldots, d_p$ and $k = 0, 1, \ldots, d_s$, s.t.,

$$\begin{aligned} F_l^{P_n} :=& \sum_{T_i \in \mathcal{T}} \int_{T_i} -\frac{1}{\Delta t}(\tilde{S}_w - S_w^n)\phi\varphi_l^p + \sum_{T_i \in \mathcal{T}} \int_{T_i} \lambda_n(\tilde{S}_w) K \nabla \tilde{P}_n \nabla \varphi_l^p \\ &- \sum_{F_i \in \mathcal{F}} \int_{F_i} \{\lambda_n(\tilde{S}_w) K \nabla \tilde{P}_n \cdot \mathbf{n}\} [\![\varphi_l^p]\!] \\ &+ \theta \sum_{F_i \in \mathcal{F}} \int_{F_i} [\![\tilde{P}_n]\!] \{\lambda_n(\tilde{S}_w) K \nabla \varphi_l^p\} + \sigma_n \sum_{F_i \in \mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![\tilde{P}_n]\!][\![\varphi_l^p]\!] \\ &- \theta \sum_{F_i \in \Gamma} \int_{F_i} [\![p_n^D]\!] \{\lambda_n(s^D) K \nabla \varphi_l^p\} - \sigma_n \sum_{F_i \in \Gamma} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![p_n^D]\!][\![\varphi_l^p]\!] \ , \end{aligned} \tag{36}$$

$$\begin{aligned} F_l^{P_c} :=& \sum_{T_i \in \mathcal{T}} \int_{T_i} \frac{1}{\Delta t}(\tilde{S}_w - S_w^n)\phi\varphi_l^p + \sum_{T_i \in \mathcal{T}} \int_{T_i} \lambda_w(\tilde{S}_w) K \nabla (\tilde{P}_n - \tilde{P}_c) \nabla \varphi_l^p \\ &- \sum_{F_i \in \mathcal{F}} \int_{F_i} \{\lambda_w(\tilde{S}_w) K \nabla (\tilde{P}_n - \tilde{P}_c) \cdot \mathbf{n}\} [\![\varphi_l^p]\!] \\ &+ \theta \sum_{F_i \in \mathcal{F}} \int_{F_i} \{\lambda_w(\tilde{S}_w) K \nabla \varphi_l^p \cdot \mathbf{n}\} [\![\tilde{P}_n - \tilde{P}_c]\!] + \sigma_w \sum_{F_i \in \mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![\tilde{P}_n - \tilde{P}_c]\!][\![\varphi_l^p]\!] \\ &- \theta \sum_{F_i \in \Gamma} \int_{F_i} \{\lambda_w(s^D) K \nabla \varphi_l^p \cdot \mathbf{n}\} [\![p_n^D - p_c^D]\!] - \sigma_w \sum_{F_i \in \Gamma} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![p_n^D - p_c^D]\!][\![\varphi_l^p]\!] \ , \end{aligned} \tag{37}$$

$$F_k^S := \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi \tilde{P}_c \varphi_k^s - \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi p_{c,eq}(\tilde{S}_w) \varphi_k^s - \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi \tau \frac{1}{\Delta t}(\tilde{S}_w - S_w^n) \varphi_k^s \ . \tag{38}$$

As before, we define analogous to (32) the coefficient vectors $\hat{F}^{P_n}, \hat{F}^{P_c} \in \mathbb{R}^{d_p}$ and $\hat{F}^S \in \mathbb{R}^{d_s}$. Observe that, if $F_i^{P_n} = F_i^{P_c} = F_k^S = 0$ for all $i = 0, 1, \ldots, d_p$ and $k = 0, 1, \ldots, d_s$, then $\tilde{P}_n, \tilde{P}_c$ and $\tilde{S}_w$ are a solution to the Problem 3.

The definitions (31)-(38) define a continuous mapping $\mathcal{P} : \mathbb{R}^{2d_p + d_s} \to \mathbb{R}^{2d_p + d_s}$ by,

$$\mathcal{P}(\hat{P}_n, \hat{P}_w, \hat{dS}_w) = (\hat{F}^{P_n}, \hat{F}^{P_c}, \hat{F}^S) \ .$$

Existence To prove existence of a solution to our system, we use Lemma 1.4, p. 164, in [45]:

Lemma 4 *Let X be a finite dimensional Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $\mathcal{P}$ be a continuous mapping from X into itself such that,*

$$\langle \mathcal{P}(\xi), \xi \rangle > 0 \text{ for } \|\xi\| = k > 0 \ .$$

Then, there exists a $\xi \in X$, $\|\xi\| \le k$ s.t.,

$$\mathcal{P}(\xi) = 0 \ .$$

Another version of this lemma can be found in Chapter IV of [29].

To apply Lemma 4 we chose $\mathbb{R}^{2d_p + d_s}$ as the Hilbert space X and we use the scalar product $\langle \cdot, \cdot \rangle_{\ell^2}$ and the norm $\| \cdot \|_{\ell^2}$. Further, let $\left(\hat{P}_n \ \hat{P}_c \ \hat{S}_w\right) \in \mathbb{R}^{2d_p + d_s}$ and define $R > 0$ as

$$R := \langle \left(\hat{P}_n \ \hat{P}_c \ \hat{S}_w\right), \left(\hat{P}_n \ \hat{P}_c \ \hat{S}_w\right) \rangle = \langle \hat{P}_n, \hat{P}_n \rangle + \langle \hat{P}_w, \hat{P}_w \rangle + \langle \hat{dS}_w, \hat{dS}_w \rangle.$$

Specifically, we show that whenever

$$R > \sum_{T_i\in\mathcal{T}}\int_{T_i}\phi\frac{1}{\Delta t}|P_{c,eq}(S_w^n)| + \left(\frac{\sigma_n}{2}+\frac{\overline{\lambda_n}^2\theta^2C_t^2\tilde{C}}{\underline{\lambda_n}}\right)\sum_{F_i\in\Gamma}\frac{f(k_p)}{|F_i|}\|p_n^D\|_{F_i,0}^2 + \left(\frac{\sigma_w}{2}+\frac{\overline{\lambda_w}^2\theta^2C_t^2\tilde{C}}{\underline{\lambda_w}}\right)\sum_{F_i\in\Gamma}\frac{f(k_p)}{|F_i|}\|p_w^D\|_{F_i,0}^2,$$

then one gets

$$\langle\hat{F}^{P_n},\hat{P}_n\rangle+\langle\hat{F}^{P_c},\hat{P}_w\rangle+\langle\hat{F}^S,\hat{dS}_w\rangle=\sum_{i=0}^{d_p}P_{n,i}F_i^{P_n}+\sum_{i=0}^{d_p}(P_{n,i}-P_{c,i})F_i^{P_c}+\sum_{k=0}^{d_s}dS_{w,k}F_k^S>0\ , \tag{39}$$

which gives the following existence result:

Lemma 5 *For sufficiently large σ_n, σ_w , the Problem 3 has a solution.*

Proof We estimate the terms $(I) := \sum_{i=0}^{d_p} P_{n,i}F_i^{P_n}$, $(II) := \sum_{i=0}^{d_p} P_{w,i}F_i^{P_c}$, and $(III) := \sum_{k=0}^{d_s} dS_{w,k}F_k^S$ separately.

Estimate for (I) We start with:

$$\begin{aligned}(I) =& \sum_{T_i\in\mathcal{T}}\int_{T_i}\lambda_n(\tilde{S}_w)\left|K^{\frac12}\nabla\tilde{P}_n\right|^2+\sigma_n\sum_{F_i\in\mathcal{F}}\int_{F_i}\frac{f(k_p)}{|F_i|}[\![\tilde{P}_n]\!]^2\\ &-(1-\theta)\sum_{F_i\in\mathcal{F}}\int_{F_i}\{\lambda_n(\tilde{S}_w)K\nabla\tilde{P}_n\cdot\mathbf{n}\}[\![\tilde{P}_n]\!]-\sum_{T_i\in\mathcal{T}}\int_{T_i}\frac{1}{\Delta t}(\tilde{S}_w-S_w^n)\phi\tilde{P}_n\\ &-\theta\sum_{F_i\in\Gamma}\int_{F_i}[\![p_n^D]\!]\{\lambda_n(s^D)K\nabla\tilde{P}_n\}-\sigma_n\sum_{F_i\in\Gamma}\int_{F_i}\frac{f(k_p)}{|F_i|}[\![p_n^D]\!][\![\tilde{P}_n]\!]\\ =& P_1+P_2-P_3-P_4-P_5-P_6\ .\end{aligned}$$

Using the assumption (A4) for P_1+P_2, we get:

$$P_1+P_2\ \geq\underline{\lambda_n}\sum_{T_i\in\mathcal{T}}\left\|K^{\frac12}\nabla\tilde{P}_n\right\|_{T_i,0}^2+\sigma_n\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\|[\![\tilde{P}_n]\!]\|_{F_i,0}^2\ . \tag{40}$$

Using Cauchy-Schwarz inequality together with the assumption (A4), we get:

$$P_3\leq\overline{\lambda_n}(1-\theta)\sum_{F_i\in\mathcal{F}}\|\{K^{\frac12}\nabla\tilde{P}_n\}\|_{F_i,0}\|[\![\tilde{P}_n]\!]\|_{F_i,0}\ .$$

For a fixed face F_i, let $T_\pm$ be the adjacent elements. By the trace inequality (17), the following holds:

$$\begin{aligned}&\overline{\lambda_n}(1-\theta)\sum_{F_i\in\mathcal{F}}\|\{K^{\frac12}\nabla\tilde{P}_n\}\|_{F_i,0}\|[\![\tilde{P}_n]\!]\|_{F_i,0}\\ \leq&\overline{\lambda_n}(1-\theta)C_t\sqrt{\frac{f(k_p)}{|F_i|}}\frac12\sum_{F_i\mathcal{F}}\left(\left\|K^{\frac12}\nabla\tilde{P}_n\right\|_{T_+,0}+\left\|K^{\frac12}\nabla\tilde{P}_n\right\|_{T_-,0}\right)\|[\![\tilde{P}_n]\!]\|_{F_i,0}\ .\end{aligned}$$

Further, with Lemma 3 and Cauchy-Schwarz inequality we obtain:

$$\overline{\lambda_n}(1-\theta)C_t\sqrt{\frac{f(k_p)}{|F_i|}}\frac12\sum_{F_i\in\mathcal{F}}\left(\left\|K^{\frac12}\nabla\tilde{P}_n\right\|_{T_+,0}+\left\|K^{\frac12}\nabla\tilde{P}_n\right\|_{T_-,0}\right)\|[\![\tilde{P}_n]\!]\|_{F_i,0}$$

$$\leq \left(\sum_{T_i \in \mathcal{T}} \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|^2_{T_i,0} \right)^{\frac{1}{2}} \left(\overline{\lambda_n}^2 (1-\theta)^2 C_t^2 \tilde{C} \frac{f(k_p)}{|F_i|} \sum_{F_i \in \mathcal{F}} \| [\![\tilde{P}_n]\!] \|^2_{F_i,0} \right)^{\frac{1}{2}} ,$$

which, on using the scaled Young's inequality, leads to:

$$P_3 \leq \frac{\epsilon_1}{2} \sum_{T_i \in \mathcal{T}} \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|^2_{T_i,0} + \frac{1}{2\epsilon_1} \overline{\lambda_n}^2 (1-\theta)^2 C_t^2 \tilde{C} \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \| [\![\tilde{P}_n]\!] \|^2_{F_i,0} . \tag{41}$$

The term P_5 is estimated in a similar way as P_3 leading to:

$$P_5 \leq \frac{\epsilon_2}{2} \sum_{T_i \in \mathcal{T}} \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|^2_{T_i,0} + \frac{1}{2\epsilon_2} \overline{\lambda_n}^2 \theta^2 C_t^2 \tilde{C} \sum_{F_i \in \Gamma} \frac{f(k_p)}{|F_i|} \| p_n^D \|^2_{F_i,0} , \tag{42}$$

and, the term P_6 is estimated as:

$$P_6 \leq \frac{\epsilon_3}{2} \sum_{F_i \in \Gamma} \frac{f(k_p)}{|F_i|} \| \tilde{P}_n \|^2_{F_i,0} + \frac{\sigma_n^2}{2\epsilon_3} \sum_{F_i \in \Gamma} \frac{f(k_p)}{|F_i|} \| p_n^D \|^2_{F_i,0} . \tag{43}$$

Choosing $\epsilon_1 = \epsilon_2 = \frac{\underline{\lambda_n}}{2}$, and $\epsilon_3 = \sigma_n$ in (40), (41), (42) and (43), we get the following estimate for the term (I):

$$\begin{aligned} (I) \geq & \sum_{T_i \in \mathcal{T}} \int_{T_i} -\frac{1}{\Delta t} (\tilde{S}_w - S_w^n) \phi \tilde{P}_n + \sum_{T_i \in \mathcal{T}} \frac{\underline{\lambda_n}}{2} \| K^{\frac{1}{2}} \nabla \tilde{P}_n \|^2_{T_i,0} \\ & + \left(\frac{\sigma_n}{2} - \frac{(1-\theta)^2 \overline{\lambda_n}^2 C_t^2 \tilde{C}}{2\underline{\lambda_n}} \right) \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \| [\![\tilde{P}_n]\!] \|^2_{F_i,0} \\ & - \left(\frac{\sigma_n}{2} + \frac{\overline{\lambda_n}^2 \theta^2 C_t^2 \tilde{C}}{\underline{\lambda_n}} \right) \sum_{F_i \in \Gamma} \frac{f(k_p)}{|F_i|} \| p_n^D \|^2_{F_i,0} \end{aligned} \tag{44}$$

Estimate for (II) To estimate term (II), we follow the same steps as for term (I). We use the assumption (A4), trace inequalities from Lemma 2, Lemma 3, Cauchy-Schwarz and scaled Young's inequality, in that order, and with $\epsilon_4 = \epsilon_5 = \frac{\underline{\lambda_w}}{2}$ and $\epsilon_6 = \sigma_w$, we arrive at the following estimate:

$$\begin{aligned} (II) \geq & \sum_{T_i \in \mathcal{T}} \int_{T_i} \frac{1}{\Delta t} (\tilde{S}_w - S_w^n) \phi \tilde{P}_w + \sum_{T_i \in \mathcal{T}} \frac{\underline{\lambda_w}}{2} \| K^{\frac{1}{2}} \nabla \tilde{P}_w \|^2_{T_i,0} \\ & + \left(\frac{\sigma_w}{2} - \frac{(1-\theta)^2 \overline{\lambda_w}^2 C_t^2 \tilde{C}}{2\underline{\lambda_w}} \right) \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \| [\![\tilde{P}_w]\!] \|^2_{F_i,0} \\ & - \left(\frac{\sigma_w}{2} + \frac{\overline{\lambda_w}^2 \theta^2 C_t^2 \tilde{C}}{\underline{\lambda_w}} \right) \sum_{F_i \in \Gamma} \frac{f(k_p)}{|F_i|} \| p_w^D \|^2_{F_i,0} . \end{aligned} \tag{45}$$

Estimate for (III) We start with:

$$\begin{aligned} (III) = & \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi \tilde{P}_c \frac{1}{\Delta t} (\tilde{S}_w - S_w^n) - \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi p_{c,eq}(\tilde{S}_w) \frac{1}{\Delta t} (\tilde{S}_w - S_w^n) \\ & + \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi \tau \frac{1}{\Delta t^2} (\tilde{S}_w - S_w^n)^2 . \end{aligned}$$

Using the primitive defined in (27), we get the following estimate:

$$\begin{aligned} (III) \geq & \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi \tilde{P}_c \frac{1}{\Delta t} (\tilde{S}_w - S_w^n) + \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi \frac{1}{\Delta t} \left(|P_{c,eq}(\tilde{S}_w)| - |P_{c,eq}(S_w^n)| \right) \\ & + \sum_{T_i \in \mathcal{T}} \int_{T_i} \phi \tau \frac{1}{\Delta t^2} (\tilde{S}_w - S_w^n)^2 . \end{aligned} \tag{46}$$

Combined estimate For sufficiently large σ_n and σ_w, using (15) from Lemma 1 with $q = 2$, and summing the estimates (44), (45) and (46), we obtain:

$$\sum_{i=0}^{d_{p_n}} P_{n,i} F_i^{P_n} + \sum_{j=0}^{d_{p_c}} P_{w,j} F_j^{P_c} + \sum_{k=0}^{d_s} dS_{w,k} F_k^S \geq$$
$$C\|\tilde{P}_n\|^2_{\Omega,0} + C\|\tilde{P}_w\|^2_{\Omega,0} + C\|\tilde{dS}_w\|^2_{\Omega,0} + \sum_{T_i\in\mathcal{T}} \int_{T_i} \phi \frac{1}{\Delta t} |P_{c,eq}(\tilde{S}_w)| - \sum_{T_i\in\mathcal{T}} \int_{T_i} \phi \frac{1}{\Delta t} |P_{c,eq}(S_w^n)|$$
$$- \left(\frac{\sigma_n}{2} + \frac{\overline{\lambda_n}^2 \theta^2 C_t^2 \tilde{C}}{\underline{\lambda_n}} \right) \sum_{F_i\in\Gamma} \frac{f(k_p)}{|F_i|} \|p_n^D\|^2_{F_i,0} - \left(\frac{\sigma_w}{2} + \frac{\overline{\lambda_w}^2 \theta^2 C_t^2 \tilde{C}}{\underline{\lambda_w}} \right) \sum_{F_i\in\Gamma} \frac{f(k_p)}{|F_i|} \|p_w^D\|^2_{F_i,0} . \quad (47)$$

Observe that the positivity of the last but one terms in (44) and (45) is only guaranteed under restrictions on σ_n and σ_w. However, these restrictions do not depend on the time step or the argument in the mapping $\mathcal{P}$. Now, one can choose the radius R as announced above to guarantee that the right hand side in (47) is positive, and using (35) leads to the estimate (39), and the existence of a zero for $\mathcal{P}$ and hence of a solution to Problem 3 follows directly by Lemma 4.

4.2 Discrete energy estimate

Lemma 6 *For sufficiently large σ_n and σ_w, there exists a constant C independent of Δt, h and the polynomial degrees k_p and k_s, s.t., the following energy estimate holds:*

$$\Delta t \sum_{n=0}^{N} \sum_{T_i\in\mathcal{T}} \left\|\partial^- S_w^{n+1}\right\|^2_{T_i,0} + \Delta t \sum_{n=0}^{N} \sum_{T_i\in\mathcal{T}} \left\|K^{\frac{1}{2}} \nabla P_n^{n+1}\right\|^2_{T_i,0} + \Delta t \sum_{n=0}^{N} \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![P_n^{n+1}]\!]\|^2_{F_i,0}$$
$$+ \Delta t \sum_{n=0}^{N} \sum_{T_i\in\mathcal{T}} \|K^{\frac{1}{2}} \nabla P_w^{n+1}\|^2_{T_i,0} + \Delta t \sum_{n=0}^{N} \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![P_w^{n+1}]\!]\|^2_{F_i,0} + \sum_{T_i\in\mathcal{T}} \int_{T_i} |P_{c,eq}(S_w^{N+1})|$$
$$\leq C \sum_{T_i\in\mathcal{T}} \int_{T_i} |P_{c,eq}(S_w^0)| + C\Delta t \sum_{n=0}^{N} \sum_{F_i\in\Gamma} \frac{f(k_p)}{|F_i|} \|p_n^D\|^2_{F_i,0} + C\Delta t \sum_{n=0}^{N} \sum_{F_i\in\Gamma} \frac{f(k_p)}{|F_i|} \|p_w^D\|^2_{F_i,0} \quad (48)$$

Proof Starting with the discrete system at t^{n+1} (i.e. Problem 3), we test in (24) with P_n^{n+1}, in (25) with $P_w^{n+1} = P_n^{n+1} - P_c^{n+1}$ and in (26) with $\partial^- S_w^{n+1}$.

Note that we define a generic constant $C = C(\tau, \sigma_\alpha, \underline{\lambda_\alpha}, \overline{\lambda_\alpha}, \theta, C_t, \tilde{C})$ for $\alpha = w, n$. We proceed with the same steps as in the proof of Lemma 5 and obtain:

$$\sum_{T_i\in\mathcal{T}} \phi\tau \left\|\partial^- S_w^{n+1}\right\|^2_{T_i,0} + \frac{\underline{\lambda_n}}{2} \sum_{T_i\in\mathcal{T}} \left\|K^{\frac{1}{2}} \nabla P_n^{n+1}\right\|^2_{T_i,0}$$
$$+ (\sigma_n - \frac{1}{2\underline{\lambda_n}} \overline{\lambda_n}^2 (1-\theta)^2 C_t^2 \tilde{C}^2) \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![P_n^{n+1}]\!]\|^2_{F_i,0}$$
$$+ \frac{\underline{\lambda_w}}{2} \sum_{T_i\in\mathcal{T}} \|K^{\frac{1}{2}} \nabla P_w^{n+1}\|^2_{T_i,0} + (\sigma_w - \frac{1}{2\underline{\lambda_w}} \overline{\lambda_w}^2 (1-\theta)^2 C_t^2 \tilde{C}^2) \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![P_w^{n+1}]\!]\|^2_{F_i,0}$$
$$\leq \sum_{T_i\in\mathcal{T}} \int_{T_i} \frac{1}{\Delta t} \left(P_{c,eq}(S_w^{n+1}) - P_{c,eq}(S_w^n)\right) + \left(\frac{\sigma_n}{2} + \frac{\overline{\lambda_n}^2 \theta^2 C_t^2 \tilde{C}^2}{\underline{\lambda_n}} \right) \sum_{F_i\in\Gamma} \frac{f(k_p)}{|F_i|} \|p_n^D\|^2_{F_i,0}$$
$$+ \left(\frac{\sigma_w}{2} + \frac{\overline{\lambda_w}^2 \theta^2 C_t^2 \tilde{C}^2}{\underline{\lambda_w}} \right) \sum_{F_i\in\Gamma} \frac{f(k_p)}{|F_i|} \|p_w^D\|^2_{F_i,0}$$

The final estimate (48) is obtained by multiplying the above inequality by Δt and summing over all $n = 0 \ldots N$.

4.3 Error Estimates

After showing the existence of a discrete solution and deriving the general energy estimates, we now show a convergence result for the scheme.

4.3.1 Estimate for the non-wetting phase

Lemma 7 *For a suffcently large σ_n there exists a constant C independent of h, Δt, k_p and k_s such that the following estimate holds:*

$$\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[-\partial^- S_w^{n+1}+\partial_t s_w\right]\phi e_{p_n,h}^{n+1}+\sum_{T_i\in\mathcal{T}}\|K^{\frac{1}{2}}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2+\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2$$
$$\leq C\left(\frac{5}{2\underline{\lambda_n}}+\frac{3f(k_s)}{2\sigma_n f(k_p)}\right)\overline{\lambda_n'}\|K^{\frac{1}{2}}\nabla\tilde{p}_n^{n+1}\|_{\Omega,\infty}^2\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}\|_{T_i,0}^2+C\overline{\lambda_n'}\frac{5}{2\underline{\lambda_n}}\|K^{\frac{1}{2}}\nabla\tilde{p}_n^{n+1}\|_{\Omega,\infty}^2\|e_s^{n+1}\|_{\Omega,0}^2$$
$$+\frac{3f(k_s)}{2\sigma_n f(k_p)}\overline{\lambda_n'}\tilde{C}\|K^{\frac{1}{2}}\nabla\tilde{p}_n^{n+1}\|_{\Omega,\infty}(\|e_s^{n+1}\|_{\Omega,0}^2+h^2\|\nabla e_s^{n+1}\|_{\Omega,0}^2)$$
$$+C\frac{5\overline{\lambda_n}}{2\underline{\lambda_n}}\|K^{\frac{1}{2}}\nabla e_{p_n}^{n+1}\|_{\Omega,0}^2+C\left(\frac{3\sigma_n C_t^2\tilde{C}}{2}+\frac{5\theta\overline{\lambda_n}^2C_t^2\tilde{C}}{\underline{\lambda_n}}\right)(h^{-2}\|e_{p_n}^{n+1}\|_{\Omega,0}^2+\|\nabla e_{p_n}^{n+1}\|_{\Omega,0}^2)$$
$$+C\frac{3\overline{\lambda_n}^2C_t^2\tilde{C}}{2\sigma_n}\left(\left\|K^{\frac{1}{2}}\nabla e_{p_n}^{n+1}\right\|_{\Omega,0}^2+h^2\left\|K^{\frac{1}{2}}\nabla^2 e_{p_n}^{n+1}\right\|_{\Omega,0}^2\right)$$

Proof We subtract (20) and (24) and test with $e_{p_n,h}^{n+1}$ to get:

$$\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[-\partial^- S_w^{n+1}+\partial_t s_w\right]\phi e_{p_n,h}^{n+1}+\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[\lambda_n(S_w^{n+1})K\nabla P_n^{n+1}-\lambda_n(s_w)K\nabla p_n\right]\nabla e_{p_n,h}^{n+1}$$
$$+\sigma_n\sum_{F_i\in\mathcal{F}}\int_{F_i}\frac{f(k_p)}{|F_i|}\left[[\![P_n^{n+1}]\!]-[\![p_n]\!]\right][\![e_{p_n,h}^{n+1}]\!]$$
$$=\sum_{F_i\in\mathcal{F}}\int_{F_i}\left[\{\lambda_n(S_w^{n+1})K\nabla P_n^{n+1}\cdot\mathbf{n}\}-\{\lambda_n(s_w)K\nabla p_n\cdot\mathbf{n}\}\right][\![e_{p_n,h}^{n+1}]\!]$$
$$-\theta\sum_{F_i\in\mathcal{F}}\int_{F_i}\left[[\![P_n^{n+1}]\!]\{\lambda_n(S_w^{n+1})K\nabla e_{p_n,h}^{n+1}\}-[\![p_n]\!]\{\lambda_n(s_w)K\nabla e_{p_n,h}^{n+1}\}\right]$$

We rewrite this equation componentwise as:

$$P_1+P_2+P_3=P_4\ ,$$

and estimate each component individually.

We expand each term P_1 to P_4 by adding and subtracting $\tilde{p}_n$.

Estimate for P_2

$$P_2=\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[\lambda_n(S_w^{n+1})K\nabla e_{p_n,h}^{n+1}+(\lambda_n(S_w^{n+1})-\lambda_n(s_w))K\nabla\tilde{p}_n^{n+1}+\lambda_n(s_w)K\nabla e_{p_n}^{n+1}\right]\nabla e_{p_n,h}^{n+1}$$
$$=P_{2,1}+P_{2,2}+P_{2,3}$$

where, we estimate $P_{2,1}$, $P_{2,2}$ and $P_{2,3}$ as:

$$P_{2,1}\geq\sum_{T_i\in\mathcal{T}}\underline{\lambda_n}\|K^{\frac{1}{2}}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 \tag{49}$$

$$P_{2,2}\leq\sum_{T_i\in\mathcal{T}}\int_{T_i}\overline{\lambda_n'}(S_w^{n+1}-s_w^{n+1})K\nabla\tilde{p}_n^{n+1}\cdot\nabla e_{p_n,h}^{n+1}\leq\sum_{T_i\in\mathcal{T}}\int_{T_i}\overline{\lambda_n'}(e_{s,h}^{n+1}+e_s^{n+1})K\nabla\tilde{p}_n^{n+1}\cdot\nabla e_{p_n,h}^{n+1}$$

$$\leq \frac{\epsilon_{2,2}}{2} \sum_{T_i\in\mathcal{T}} \|K^{\frac{1}{2}}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 + \overline{\lambda_n'}^2 \frac{1}{2\epsilon_{2,2}} \|K^{\frac{1}{2}}\nabla \tilde{p}_n^{n+1}\|_{\Omega,\infty}^2 \sum_{T_i\in\mathcal{T}} \left(\|e_{s,h}^{n+1}\|_{T_i,0}^2 + \|e_s^{n+1}\|_{T_i,0}^2\right) \tag{50}$$

$$P_{2,3} \leq \frac{\epsilon_{2,3}}{2} \sum_{T_i\in\mathcal{T}} \|K^{\frac{1}{2}}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 + \frac{\overline{\lambda_n}^2}{2\epsilon_{2,3}} \sum_{T_i\in\mathcal{T}} \|K^{\frac{1}{2}}\nabla e_{p_n}^{n+1}\|_{T_i,0}^2 \tag{51}$$

Estimate for P_3

$$P_3 = \sigma_n \sum_{F_i\in\mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|} \left[[\![e_{p_n,h}^{n+1}]\!] + [\![e_{p_n}^{n+1}]\!] \right] [\![e_{p_n,h}^{n+1}]\!] = P_{3,1} + P_{3,2}$$

$$\text{where, } P_{3,1} = \sigma_n \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 \ , \tag{52}$$

$$\text{and, } P_{3,2} \leq \frac{\epsilon_{3,2}}{2} \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 + \frac{1}{2\epsilon_{3,2}} \sigma_n^2 C_t^2 \tilde{C} (h^{-2} \|e_{p_n}^{n+1}\|_{\Omega,0}^2 + \|\nabla e_{p_n}^{n+1}\|_{\Omega,0}^2) \tag{53}$$

Estimate for P_4

$$\begin{aligned} P_4 =& (1-\theta) \sum_{F_i\in\mathcal{F}} \int_{F_i} \left[\{\lambda_n(S_w^{n+1}) K \nabla e_{p_n,h}^{n+1} \cdot \mathbf{n}\} \right] [\![e_{p_n,h}^{n+1}]\!] \\ &+ \sum_{F_i\in\mathcal{F}} \int_{F_i} \left[\{(\lambda_n(S_w^{n+1}) - \lambda_n(s_w)) K \nabla \tilde{p}_n^{n+1} \cdot \mathbf{n} + \{\lambda_n(s_w) K \nabla e_{p_n}^{n+1} \cdot \mathbf{n}\} \right] [\![e_{p_n,h}^{n+1}]\!] \\ &- \theta \sum_{F_i\in\mathcal{F}} \int_{F_i} \left[[\![e_{p_n}^{n+1}]\!] \{\lambda_n(s_w) K \nabla e_{p_n,h}^{n+1}\} + [\![\tilde{p}_n^{n+1}]\!] \{(\lambda_n(S_w^{n+1}) - \lambda_n(s_w)) K \nabla e_{p_n,h}^{n+1}\} \right] \\ =& P_{4,1} + \cdots + P_{4,5} \end{aligned}$$

where, we estimate $P_{4,1}$ to $P_{4,4}$ separately, in the same way as Equation (41) in Lemma 5:

$$P_{4,1} \leq \frac{\epsilon_{4,1}}{2} \sum_{T_i\in\mathcal{T}} \left\| K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 + (1-\theta)^2 \frac{1}{2\epsilon_{4,1}} \overline{\lambda_n}^2 C_t^2 \tilde{C} \sum_{F_i} \frac{f(k_p)}{|F_i|} \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 \tag{54}$$

$$\begin{aligned} P_{4,2} \leq& \overline{\lambda_n'} \tilde{C} \|K^{\frac{1}{2}} \nabla \tilde{p}_n^{n+1}\|_{\Omega,\infty} \sum_{F_i\in\mathcal{F}} \int_{F_i} \{(e_{s,h}^{n+1} + e_s^{n+1})\} [\![e_{p_n,h}^{n+1}]\!] \\ \leq& \frac{\epsilon_{4,2}}{2} \sum_{F_i} \frac{f(k_s)}{|F_i|} \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 + \frac{1}{2\epsilon_{4,2}} \overline{\lambda_n'}^2 \tilde{C} \|K^{\frac{1}{2}} \nabla \tilde{p}_n^{n+1}\|_{\Omega,\infty}^2 \sum_{T_i\in\mathcal{T}}^{2} \|e_{s,h}^{n+1}\|_{T_i,0}^2 \\ &+ \frac{1}{2\epsilon_{4,2}} \overline{\lambda_n'}^2 \tilde{C} \|K^{\frac{1}{2}} \nabla \tilde{p}_n^{n+1}\|_{\Omega,\infty}^2 (\|e_s^{n+1}\|_{\Omega,0}^2 + h^2 \|\nabla e_s^{n+1}\|_{\Omega,0}^2) \end{aligned} \tag{55}$$

$$\begin{aligned} P_{4,3} \leq& \left(\overline{\lambda_n}^2 C_t^2 \tilde{C} \sum_{T_i\in\mathcal{T}} \left(\left\| K^{\frac{1}{2}} \nabla e_{p_n}^{n+1} \right\|_{T_i,0}^2 + |F_i|^2 \left\| K^{\frac{1}{2}} \nabla^2 e_{p_n}^{n+1} \right\|_{T_i,0}^2 \right) \right)^{\frac{1}{2}} \cdot \left(\sum_{F_i} \frac{f(k_p)}{|F_i|} \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 \right)^{\frac{1}{2}} \\ \leq& \overline{\lambda_n}^2 C_t^2 \tilde{C} \frac{1}{2\epsilon_{4,3}} \left(\left\| K^{\frac{1}{2}} \nabla e_{p_n}^{n+1} \right\|_{\Omega,0}^2 + h^2 \left\| K^{\frac{1}{2}} \nabla^2 e_{p_n}^{n+1} \right\|_{\Omega,0}^2 \right) + \frac{\epsilon_{4,3}}{2} \sum_{F_i} \frac{f(k_p)}{|F_i|} \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 \end{aligned} \tag{56}$$

$$P_{4,4} \leq \frac{\epsilon_{4,4}}{2} \sum_{T_i\in\mathcal{T}} \left\| K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 + \frac{1}{2\epsilon_{4,4}} \theta^2 \overline{\lambda_n}^2 C_t^2 \tilde{C} (h^{-2} \|e_{p_n}^{n+1}\|_{\Omega,0}^2 + \|\nabla e_{p_n}^{n+1}\|_{\Omega,0}^2) \tag{57}$$

If $\tilde{p}_n^{n+1}$ is continuous, the jump term in $P_{4,5}$ vanishes making $P_{4,5} = 0$. Otherwise, we proceed with the same steps as for $P_{4,4}$. We use the continuity of p_n to replace $[\![\tilde{p}_n^{n+1}]\!]$ by $[\![e_{p_n}^{n+1}]\!]$ and get the following estimate for $P_{4,5}$:

$$P_{4,5} \leq \frac{\epsilon_{4,5}}{2} \sum_{T_i\in\mathcal{T}} \left\| K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 + \frac{1}{2\epsilon_{4,5}} \theta^2 \overline{\lambda_n}^2 C_t^2 \tilde{C} (h^{-2} \|e_{p_n}^{n+1}\|_{\Omega,0}^2 + \|\nabla e_{p_n}^{n+1}\|_{\Omega,0}^2) \tag{58}$$

Combined estimate Putting the estimates (49) to (58) together, we get:

$$
\begin{aligned}
&\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[-\partial^- S_w^{n+1}+\partial_t s_w\right]\phi e_{p_n,h}^{n+1} \\
&+\left(\underline{\lambda_n}-\frac{\epsilon_{2,2}}{2}-\frac{\epsilon_{2,3}}{2}-\frac{\epsilon_{4,1}}{2}-\frac{\epsilon_{4,4}}{2}-\frac{\epsilon_{4,5}}{2}\right)\sum_{T_i\in\mathcal{T}}\|K^{\frac12}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 \\
&+\left(\sigma_n-\frac{f(k_p)\epsilon_{3,2}}{2}-\frac{f(k_s)\epsilon_{4,2}}{2}-\frac{f(k_p)\epsilon_{4,3}}{2}-(1-\theta)^2\frac{f(k_p)}{2\epsilon_{4,1}}\overline{\lambda_n}C_t\tilde{C}\right)\sum_{F_i\in\mathcal{F}}\frac{1}{|F_i|}\|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 \\
\le&\left(\frac{1}{2\epsilon_{2,2}}+\frac{1}{2\epsilon_{4,2}}\right)\overline{\lambda_n'}^2\|K^{\frac12}\nabla\tilde{p}_n^{n+1}\|_{\Omega,\infty}^2\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}\|_{T_i,0}^2 \\
&+\overline{\lambda_n'}^2\frac{1}{2\epsilon_{2,2}}\|K^{\frac12}\nabla\tilde{p}_n^{n+1}\|_{\Omega,\infty}^2\|e_s^{n+1}\|_{\Omega,0}^2+\frac{1}{2\epsilon_{4,2}}\overline{\lambda_n'}^2\tilde{C}\|K^{\frac12}\nabla\tilde{p}_n^{n+1}\|_{\Omega,\infty}^2(\|e_s^{n+1}\|_{\Omega,0}^2+h^2\|\nabla e_s^{n+1}\|_{\Omega,0}^2) \\
&+\frac{\overline{\lambda_n}^2}{2\epsilon_{2,3}}\sum_{T_i\in\mathcal{T}}\|K^{\frac12}\nabla e_{p_n}^{n+1}\|_{T_i,0}^2+\frac{1}{2\epsilon_{3,2}}\sigma_n^2C_t^2\tilde{C}\sum_{T_i\in\mathcal{T}}(|F_i|^{-2}\|e_{p_n}^{n+1}\|_{T_i,0}^2+\|\nabla e_{p_n}^{n+1}\|_{T_i,0}^2) \\
&+\overline{\lambda_n}^2C_t^2\tilde{C}\frac{1}{2\epsilon_{4,3}}\left(\left\|K^{\frac12}\nabla e_{p_n}^{n+1}\right\|_{\Omega,0}^2+h^2\left\|K^{\frac12}\nabla^2 e_{p_n}^{n+1}\right\|_{\Omega,0}^2\right) \\
&+\left(\frac{1}{2\epsilon_{4,4}}+\frac{1}{2\epsilon_{4,5}}\right)\theta^2\overline{\lambda_n}^2C_t^2\tilde{C}(h^{-2}\|e_{p_n}^{n+1}\|_{\Omega,0}^2+\|\nabla e_{p_n}^{n+1}\|_{\Omega,0}^2)
\end{aligned}
\tag{59}
$$

Choosing $\epsilon_{2,2}=\epsilon_{2,3}=\epsilon_{4,1}=\epsilon_{4,4}=\epsilon_{4,5}=\frac{\underline{\lambda_n}}{5}$, and $\epsilon_{3,2}=\frac{f(k_s)}{f(k_p)}\epsilon_{4,2}=\epsilon_{4,3}=\frac{\sigma_n}{3}$, we arrive at the desired estimate for the non-wetting phase.

4.3.2 Estimate for the wetting phase

Lemma 8 *For a sufficently large σ_w there exists a constant C independent of h, Δt, k_p and k_s such that the following estimate holds:*

$$
\begin{aligned}
&\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[-\partial^- S_w^{n+1}+\partial_t s_w\right]\phi e_{p_w,h}^{n+1}+\sum_{T_i\in\mathcal{T}}\|K^{\frac12}\nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2+\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2 \\
\le&C\left(\frac{5}{2\underline{\lambda_w}}+\frac{3f(k_s)}{2\sigma_w f(k_p)}\right)\overline{\lambda_w'}\|K^{\frac12}\nabla\tilde{p}_w^{n+1}\|_{\Omega,\infty}^2\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}\|_{T_i,0}^2+C\overline{\lambda_w'}\frac{5}{2\underline{\lambda_w}}\|K^{\frac12}\nabla\tilde{p}_w^{n+1}\|_{\Omega,\infty}^2\|e_s^{n+1}\|_{\Omega,0}^2 \\
&+\frac{3f(k_s)}{2\sigma_w f(k_p)}\overline{\lambda_w'}\tilde{C}\|K^{\frac12}\nabla\tilde{p}_w^{n+1}\|_{\Omega,\infty}(\|e_s^{n+1}\|_{\Omega,0}^2+h^2\|\nabla e_s^{n+1}\|_{\Omega,0}^2) \\
&+C\frac{5\overline{\lambda_w}}{2\underline{\lambda_w}}\|K^{\frac12}\nabla e_{p_w}^{n+1}\|_{\Omega,0}^2+C\left(\frac{3\sigma_w C_t^2\tilde{C}}{2}+\frac{5\theta\overline{\lambda_w}^2C_t^2\tilde{C}}{\underline{\lambda_w}}\right)(h^{-2}\|e_{p_w}^{n+1}\|_{\Omega,0}^2+\|\nabla e_{p_w}^{n+1}\|_{\Omega,0}^2) \\
&+C\frac{3\overline{\lambda_w}^2C_t^2\tilde{C}}{2\sigma_w}\left(\left\|K^{\frac12}\nabla e_{p_w}^{n+1}\right\|_{\Omega,0}^2+h^2\left\|K^{\frac12}\nabla^2 e_{p_w}^{n+1}\right\|_{\Omega,0}^2\right)
\end{aligned}
$$

Proof The proof is the same as for the non-wetting phase (Section 4.3.1) and is therefore left out.

4.3.3 Estimate for the capillary pressure

Lemma 9 *There exists a constant C independent of h, Δt, k_p and k_s such that the following estimate holds:*

$$\begin{aligned}
&\phi \sum_{T_i\in\mathcal{T}} \int_{T_i} e_{p_c,h}^{n+1}\partial^- e_{s,h}^{n+1} + \frac{|\underline{p'_{c,eq}}|\phi}{2} \sum_{T_i\in\mathcal{T}} \partial^- \|e_{s,h}^{n+1}\|_{T_i,0}^2 \\
&+\frac{|\underline{p'_{c,eq}}|\phi}{2} \sum_{T_i\in\mathcal{T}} \frac{1}{\Delta t}\|e_{s,h}^{n+1} - e_{s,h}^{n}\|_{T_i,0}^2 + \phi\tau \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 \\
&\le \frac{3\phi}{2\tau} \sum_{T_i\in\mathcal{T}} \int_{T_i} \|e_{p_c}^{n+1}\|_{T_i,0}^2 + \frac{|\underline{p'_{c,eq}}|\phi}{4} \sum_{T_i\in\mathcal{T}} \|e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{L_{p_c}^2\phi}{|\underline{p'_{c,eq}}|} \sum_{T_i\in\mathcal{T}} \|e_s^{n+1}\|_{T_i,0}^2 \\
&+\frac{\tau\phi}{2}\Delta t \sum_{T_i\in\mathcal{T}} \int_{t_n}^{t_{n+1}} \|\partial_{tt}\tilde{s}_w^{n+1}\|_{T_i,0}^2\, dt + \frac{3\tau\phi}{2} \sum_{T_i\in\mathcal{T}} \|\partial_t e_s^{n+1}\|_{T_i,0}^2
\end{aligned} \tag{60}$$

Proof We subtract (26) in Problem 3 from (22) in Problem 2, and use $\psi_{p_c} = \phi\partial^- e_{s,h}^{n+1}$ to get

$$\begin{aligned}
&\phi \sum_{T_i\in\mathcal{T}} \int_{T_i} e_{p_c,h}^{n+1}\partial^- e_{s,h}^{n+1} + \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} e_{p_c}^{n+1}\partial^- e_{s,h}^{n+1} \\
&- \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} (p_{c,eq}(S_w^{n+1}) - p_{c,eq}(\tilde{s}_w^{n+1}) + p_{c,eq}(\tilde{s}_w^{n+1}) - p_{c,eq}(s_w))\partial^- e_{s,h}^{n+1} \\
&+ \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} \tau(\partial^- e_{s,h}^{n+1} + (\partial^- - \partial_t)\tilde{s}_w^{n+1} + \partial_t e_s^{n+1})\partial^- e_{s,h}^{n+1} \\
&= \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} e_{p_c,h}^{n+1}\partial^- e_{s,h}^{n+1} + Pc_1 + Pc_2 + Pc_3 \\
&= 0\ .
\end{aligned} \tag{61}$$

Estimate for Pc_1 We use Hölder's and Young's inequality to obtain

$$Pc_1 \le \frac{\phi^2}{2\epsilon_{pc1}} \sum_{T_i\in\mathcal{T}} \int_{T_i} \|e_{p_c}^{n+1}\|_{T_i,0}^2 + \frac{\epsilon_{pc1}}{2} \sum_{T_i\in\mathcal{T}} \int_{T_i} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 \tag{62}$$

Estimate for Pc_2

$$\begin{aligned}
Pc_2 =& - \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} (p_{c,eq}(S_w^{n+1}) - p_{c,eq}(\tilde{s}_w^{n+1}))\partial^- e_{s,h}^{n+1} - \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} (p_{c,eq}(\tilde{s}_w^{n+1}) - p_{c,eq}(s_w))\partial^- e_{s,h}^{n+1} \\
=& Pc_{2,1} + Pc_{2,2}.
\end{aligned} \tag{63}$$

Here $Pc_{2,1}$ is estimated as

$$\begin{aligned}
Pc_{2,1} =& \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} |p'_{c,eq}(\xi)| e_{s,h}^{n+1}\partial^- e_{s,h}^{n+1} \ge \phi|\underline{p'_{c,eq}}| \sum_{T_i\in\mathcal{T}} \int_{T_i} e_{s,h}^{n+1}\partial^- e_{s,h}^{n+1} \\
=& \frac{|\underline{p'_{c,eq}}|\phi}{2} \sum_{T_i\in\mathcal{T}} \partial^- \|e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{|\underline{p'_{c,eq}}|\phi}{2} \sum_{T_i\in\mathcal{T}} \frac{1}{\Delta t}\|e_{s,h}^{n+1} - e_{s,h}^{n}\|_{T_i,0}^2.
\end{aligned}$$

For $Pc_{2,2}$, using Young's Inequality and the Lipschitz continuity one has

$$Pc_{2,2} \le \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} L_{p_c} e_s^{n+1} e_{s,h}^{n+1} \le \frac{\epsilon_{pc22}}{2} \sum_{T_i\in\mathcal{T}} \|e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{L_{p_c}^2\phi^2}{2\epsilon_{pc22}} \sum_{T_i\in\mathcal{T}} \|e_s^{n+1}\|_{T_i,0}^2$$

Estimate for Pc_3

$$
\begin{aligned}
Pc_3 =& \phi\tau \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 + \phi\tau \sum_{T_i\in\mathcal{T}} \int_{T_i} (\partial^- - \partial_t)\tilde{s}_w^{n+1} \partial^- e_{s,h}^{n+1} + \phi\tau \sum_{T_i\in\mathcal{T}} \int_{T_i} \partial_t e_s^{n+1} \partial^- e_{s,h}^{n+1} \\
=& Pc_{3,1} + Pc_{3,2} + Pc_{3,3}
\end{aligned} \tag{64}
$$

We approximate the consistency error in $Pc_{3,2}$ using a Taylor expansion

$$
\frac{1}{\Delta t}(\tilde{s}_w^{n+1} - \tilde{s}_w^n) - \partial_t \tilde{s}_w^{n+1} = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) \partial_{tt} \tilde{s}_w^{n+1} \, dt \ ,
$$

which leads to the following estimate for $Pc_{3,2}$

$$
\begin{aligned}
Pc_{3,2} \le & \frac{\tau^2\phi^2}{2\epsilon_{pc32}} \sum_{T_i\in\mathcal{T}} \|(\partial^- - \partial_t)\tilde{s}_w^{n+1}\|_{T_i,0}^2 + \frac{\epsilon_{pc32}}{2} \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 \\
\le & \frac{\epsilon_{pc32}}{2} \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{\tau^2\phi^2}{6\epsilon_{pc32}} \Delta t \sum_{T_i\in\mathcal{T}} \int_{t_n}^{t_{n+1}} \|\partial_{tt} \tilde{s}_w^{n+1}\|_{T_i,0}^2 \, dt \ .
\end{aligned}
$$

To estimate $Pc_{3,3}$, we use Young's inequality:

$$
Pc_{3,3} \le \frac{\epsilon_{pc33}}{2} \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{\tau^2\phi^2}{2\epsilon_{pc33}} \sum_{T_i\in\mathcal{T}} \|\partial_t e_s^{n+1}\|_{T_i,0}^2
$$

Combined estimate We substitute the estimates (62), (63) and (64) into (61) to get

$$
\begin{aligned}
& \phi \sum_{T_i\in\mathcal{T}} \int_{T_i} e_{p_c,h}^{n+1} \partial^- e_{s,h}^{n+1} + \frac{|\underline{p'_{c,eq}}|\phi}{2} \sum_{T_i\in\mathcal{T}} \partial^- \|e_{s,h}^{n+1}\|_{T_i,0}^2 \\
& + \frac{|\underline{p'_{c,eq}}|\phi}{2} \sum_{T_i\in\mathcal{T}} \frac{1}{\Delta t} \|e_{s,h}^{n+1} - e_{s,h}^n\|_{T_i,0}^2 + \phi\tau \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 \\
\le & \frac{\phi^2}{2\epsilon_{pc1}} \sum_{T_i\in\mathcal{T}} \int_{T_i} \|e_{p_c}^{n+1}\|_{T_i,0}^2 + \frac{\epsilon_{pc1}}{2} \sum_{T_i\in\mathcal{T}} \int_{T_i} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 \\
& + \frac{\epsilon_{pc22}}{2} \sum_{T_i\in\mathcal{T}} \|e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{L_{p_c}^2\phi^2}{2\epsilon_{pc22}} \sum_{T_i\in\mathcal{T}} \|e_s^{n+1}\|_{T_i,0}^2 \\
& + \frac{\epsilon_{pc32}}{2} \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{\tau^2\phi^2}{6\epsilon_{pc32}} \Delta t \sum_{T_i\in\mathcal{T}} \int_{t_n}^{t_{n+1}} \|\partial_{tt} \tilde{s}_w^{n+1}\|_{T_i,0}^2 \, dt \\
& + \frac{\epsilon_{pc33}}{2} \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{\tau^2\phi^2}{2\epsilon_{pc33}} \sum_{T_i\in\mathcal{T}} \|\partial_t e_s^{n+1}\|_{T_i,0}^2
\end{aligned}
$$

Setting $\epsilon_{pc1} = \epsilon_{pc32} = \epsilon_{pc33} = \dfrac{\phi\tau}{3}$ and, $\epsilon_{pc22} = \dfrac{|\underline{p'_{c,eq}}|\phi}{2}$, we get the desired estimate.

4.3.4 Convergence result

We are now in a position to deduce the following theorem about the convergence of the scheme:

Theorem 1 *For sufficiently large σ_n and σ_w, there exists a constant C independent of h and Δt, s.t., the following estimate holds:*

$$\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{N+1}\|_{T_i,0}^2+\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}-e_{s,h}^{n}\|_{T_i,0}^2+\Delta t\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+\Delta t\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\left(\|\nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2+\|\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2\right)+\Delta t\sum_{n=0}^{N}\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\left(\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2+\|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2\right)$$
$$\le C\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{0}\|_{T_i,0}^2+C\frac{h^{2\min(k_s+1,l_s)}}{k_s^{2l_s-2}}\left(1+\frac{1}{k_s^2}\right)\Delta t\sum_{n=0}^{N}\|s_w(t)\|_{\Omega,l_s}^2+C\Delta t^2$$
$$+C\frac{h^{2\min(k_s+1,l_s)}}{k_s^{2l_s}}\Delta t\sum_{n=0}^{N}\|\partial_t s_w(t)\|_{\Omega,l_s}^2+C\frac{h^{2\min(k_{p_n}+1,l_{p_n})-2}}{k_{p_n}^{2l_{p_n}-2}}\left(1+\frac{1}{k_{p_n}^2}+k_{p_n}^2\right)\Delta t\sum_{n=0}^{N}\|p_n(t)\|_{\Omega,l_{p_n}}^2$$
$$+C\frac{h^{2\min(k_{p_c}+1,l_{p_c})-2}}{k_{p_c}^{2l_{p_c}-2}}\left(1+\frac{1}{k_{p_c}^2}+k_{p_c}^2\right)\Delta t\sum_{n=0}^{N}\|p_c(t)\|_{\Omega,l_{p_c}}^2$$

Proof We add the results of Lemma 7, 8, and 9, and rearrange them to get:

$$\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[-\partial^- S_w^{n+1}+\partial_t s_w\right]\phi e_{p_n,h}^{n+1}+\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[\partial^- S_w^{n+1}-\partial_t s_w\right]\phi e_{p_w,h}^{n+1}$$
$$+\phi\sum_{T_i\in\mathcal{T}}\int_{T_i}e_{p_c,h}^{n+1}\partial^- e_{s,h}^{n+1}+\frac{\underline{|p'_{c,eq}|}\phi}{2}\sum_{T_i\in\mathcal{T}}\partial^-\|e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+\frac{\underline{|p'_{c,eq}|}\phi}{2}\sum_{T_i\in\mathcal{T}}\frac{1}{\Delta t}\|e_{s,h}^{n+1}-e_{s,h}^{n}\|_{T_i,0}^2+\phi\tau\sum_{T_i\in\mathcal{T}}\|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+\sum_{T_i\in\mathcal{T}}\left(\|K^{\frac12}\nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2+\|K^{\frac12}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2\right)+\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\left(\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2+\|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2\right)$$
$$\le\frac{3\phi}{2\tau}\sum_{T_i\in\mathcal{T}}\int_{T_i}\|e_{p_c}^{n+1}\|_{T_i,0}^2+\frac{\underline{|p'_{c,eq}|}\phi}{4}\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}\|_{T_i,0}^2+\frac{L_{p_c}^2\phi}{\underline{|p'_{c,eq}|}}\sum_{T_i\in\mathcal{T}}\|e_s^{n+1}\|_{T_i,0}^2$$
$$+\frac{\tau\phi}{2}\Delta t\sum_{T_i\in\mathcal{T}}\int_{t_n}^{t_{n+1}}\|\partial_{tt}\tilde{s}_w^{n+1}\|_{T_i,0}^2\,dt+\frac{3\tau\phi}{2}\sum_{T_i\in\mathcal{T}}\|\partial_t e_s^{n+1}\|_{T_i,0}^2$$
$$+\sum_{\alpha=w,n}\left[C\left(\frac{5}{2\underline{\lambda_\alpha}}+\frac{3f(k_s)}{2\sigma_\alpha f(k_p)}\right)\overline{\lambda'_\alpha}\|K^{\frac12}\nabla\tilde{p}_\alpha^{n+1}\|_{\Omega,\infty}^2\right]\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+\sum_{\alpha=w,n}\left[C\frac{5\overline{\lambda'_\alpha}}{2\underline{\lambda_\alpha}}\|K^{\frac12}\nabla\tilde{p}_\alpha^{n+1}\|_{\Omega,\infty}^2\|e_s^{n+1}\|_{\Omega,0}^2+\frac{3f(k_s)\overline{\lambda'_\alpha}\tilde{C}}{2\sigma_\alpha f(k_p)}\|K^{\frac12}\nabla\tilde{p}_\alpha^{n+1}\|_{\Omega,\infty}(\|e_s^{n+1}\|_{\Omega,0}^2+h^2\|\nabla e_s^{n+1}\|_{\Omega,0}^2)\right.$$
$$+C\frac{5\overline{\lambda_\alpha}}{2\underline{\lambda_\alpha}}\sum_{T_i\in\mathcal{T}}\|K^{\frac12}\nabla e_{p_\alpha}^{n+1}\|_{T_i,0}^2+C\left(\frac{3\sigma_\alpha C_t^2\tilde{C}}{2}+\frac{5\theta^2\overline{\lambda_\alpha}^2C_t^2\tilde{C}}{\underline{\lambda_\alpha}}\right)(h^{-2}\|e_{p_\alpha}^{n+1}\|_{\Omega,0}^2+\|\nabla e_{p_\alpha}^{n+1}\|_{\Omega,0}^2)$$
$$\left.+C\frac{3\overline{\lambda_\alpha}^2C_t^2\tilde{C}}{2\sigma_\alpha}\left(\left\|K^{\frac12}\nabla e_{p_\alpha}^{n+1}\right\|_{\Omega,0}^2+h^2\left\|K^{\frac12}\nabla^2 e_{p_\alpha}^{n+1}\right\|_{\Omega,0}^2\right)\right]\tag{65}$$

We combine the first three summation terms of (65) to get:

$$\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[-\partial^- S_w^{n+1}+\partial_t s_w\right]\phi e_{p_n,h}^{n+1}+\sum_{T_i\in\mathcal{T}}\int_{T_i}\left[\partial^- S_w^{n+1}-\partial_t s_w\right]\phi e_{p_w,h}^{n+1}+\phi\sum_{T_i\in\mathcal{T}}\int_{T_i}e_{p_c,h}^{n+1}\partial^- e_{s,h}^{n+1}$$
$$=\sum_{T_i\in\mathcal{T}}\int_{T_i}\phi\left[\partial^- e_{s,h}^{n+1}+(\partial^- -\partial_t)\tilde{s}_w^{n+1}+\partial_t e_s^{n+1}\right](e_{p_n,h}^{n+1}-e_{p_c,h}^{n+1}-e_{p_n,h}^{n+1})+\phi\sum_{T_i\in\mathcal{T}}\int_{T_i}e_{p_c,h}^{n+1}\partial^- e_{s,h}^{n+1}=$$

$$= \sum_{T_i\in\mathcal{T}} \int_{T_i} \phi \left[-(\partial^- - \partial_t)\tilde{s}_w^{n+1} e_{p_c,h}^{n+1} - \partial_t e_s^{n+1} e_{p_c,h}^{n+1} \right] =$$
$$= Ps_1 + Ps_2$$

Estimate for Ps_1

$$Ps_1 \le \frac{\epsilon_{ps1}}{2} \sum_{T_i\in\mathcal{T}} \|e_{p_c,h}^{n+1}\|_{T_i,0}^2 + \frac{\phi^2}{6\epsilon_{ps1}} \Delta t \sum_{T_i\in\mathcal{T}} \int_{t_n}^{t_{n+1}} \|\partial_{tt}\tilde{s}_w^{n+1}\|_{T_i,0}^2 \, dt \ . \tag{66}$$

Estimate for Ps_2

$$Ps_2 \le \frac{\epsilon_{ps2}}{2} \sum_{T_i\in\mathcal{T}} \|e_{p_c,h}^{n+1}\|_{T_i,0}^2 + \frac{\phi^2}{2\epsilon_{ps2}} \sum_{T_i\in\mathcal{T}} \|\partial_t e_s^{n+1}\|_{T_i,0}^2 \tag{67}$$

To absorb the error $\|e_{p_c,h}^{n+1}\|_{T_i,0}^2$, we use the triangle inequality together with Lemma 1 to get the following estimate:

$$\sum_{T_i\in\mathcal{T}} \|e_{p_c,h}^{n+1}\|_{T_i,0}^2$$
$$\le \sum_{T_i\in\mathcal{T}} \|\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 + \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 + \sum_{T_i\in\mathcal{T}} \|\nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2 + \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2 \ . \tag{68}$$

After substituting the estimates (66) and (67) together with the estimate (68) into the estimate (65), and choosing $\epsilon_{ps1} = \epsilon_{ps2} = \dfrac{1}{2}$ we get:

$$\frac{\underline{|p'_{c,eq}|}\phi}{2} \sum_{T_i\in\mathcal{T}} \partial^- \|e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{\underline{|p'_{c,eq}|}\phi}{2} \sum_{T_i\in\mathcal{T}} \frac{1}{\Delta t} \|e_{s,h}^{n+1} - e_{s,h}^{n}\|_{T_i,0}^2 + \frac{\phi\tau}{2} \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+ \frac{1}{2} \sum_{T_i\in\mathcal{T}} \left(\|K^{\frac{1}{2}} \nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2 + \|K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 \right) + \frac{1}{2} \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \left(\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2 + \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 \right)$$
$$\le \sum_{\alpha=w,n} \left[C \left(\frac{5}{2\underline{\lambda_\alpha}} + \frac{3f(k_s)}{2\sigma_\alpha f(k_p)} \right) \overline{\lambda'_\alpha} \|K^{\frac{1}{2}} \nabla \tilde{p}_\alpha^{n+1}\|_{\Omega,\infty}^2 + \frac{\underline{|p'_{c,eq}|}\phi}{4} \right] \sum_{T_i\in\mathcal{T}} \|e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+ \frac{L_{p_c}^2 \phi}{\underline{|p'_{c,eq}|}} \|e_s^{n+1}\|_{\Omega,0}^2 + \frac{3\phi}{2\tau} \|e_{p_c}^{n+1}\|_{\Omega,0}^2 + \left(\frac{\tau\phi}{2} + \frac{\phi^2}{3} \right) \Delta t \int_{t_n}^{t_{n+1}} \|\partial_{tt}\tilde{s}_w^{n+1}\|_{\Omega,0}^2 \, dt + \left(\frac{3\tau\phi}{2} + \phi^2 \right) \|\partial_t e_s^{n+1}\|_{\Omega,0}^2$$
$$+ \sum_{\alpha=w,n} \left[C \frac{\overline{\lambda'_\alpha} 5}{2\underline{\lambda_\alpha}} \|K^{\frac{1}{2}} \nabla \tilde{p}_\alpha^{n+1}\|_{\Omega,\infty}^2 \|e_s^{n+1}\|_{\Omega,0}^2 + \frac{3f(k_s)\overline{\lambda'_\alpha}\tilde{C}}{2\sigma_\alpha f(k_p)} \|K^{\frac{1}{2}} \nabla \tilde{p}_\alpha^{n+1}\|_{\Omega,\infty} (\|e_s^{n+1}\|_{\Omega,0}^2 + h^2 \|\nabla e_s^{n+1}\|_{\Omega,0}^2) \right.$$
$$+ C \frac{5\overline{\lambda_\alpha}}{2\underline{\lambda_\alpha}} \|K^{\frac{1}{2}} \nabla e_{p_\alpha}^{n+1}\|_{\Omega,0}^2 + C \left(\frac{3\sigma_\alpha C_t^2 \tilde{C}}{2} + C \frac{5\theta^2 \overline{\lambda_\alpha}^2 C_t^2 \tilde{C}}{\underline{\lambda_\alpha}} \right) (h^{-2} \|e_{p_\alpha}^{n+1}\|_{\Omega,0}^2 + \|\nabla e_{p_\alpha}^{n+1}\|_{\Omega,0}^2)$$
$$\left. + C \overline{\lambda_\alpha}^2 C_t^2 \tilde{C} \frac{3}{2\sigma_\alpha} \left(\left\| K^{\frac{1}{2}} \nabla e_{p_\alpha}^{n+1} \right\|_{\Omega,0}^2 + h^2 \left\| K^{\frac{1}{2}} \nabla^2 e_{p_\alpha}^{n+1} \right\|_{\Omega,0}^2 \right) \right] \tag{69}$$

Using a generic constant C, we rewrite (69) as:

$$\frac{\underline{|p'_{c,eq}|}\phi}{2} \sum_{T_i\in\mathcal{T}} \partial^- \|e_{s,h}^{n+1}\|_{T_i,0}^2 + \frac{\underline{|p'_{c,eq}|}\phi}{2} \sum_{T_i\in\mathcal{T}} \frac{1}{\Delta t} \|e_{s,h}^{n+1} - e_{s,h}^{n}\|_{T_i,0}^2 + \frac{\phi\tau}{2} \sum_{T_i\in\mathcal{T}} \|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+ \frac{1}{2} \sum_{T_i\in\mathcal{T}} \left(\|K^{\frac{1}{2}} \nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2 + \|K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 \right) + \frac{1}{2} \sum_{F_i\in\mathcal{F}} \frac{f(k_p)}{|F_i|} \left(\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2 + \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2 \right)$$
$$\le \left(C + \frac{\underline{|p'_{c,eq}|}\phi}{4} \right) \sum_{T_i\in\mathcal{T}} C \|e_{s,h}^{n+1}\|_{T_i,0}^2 + C \|e_s^{n+1}\|_{\Omega,0}^2 + C \|e_{p_c}^{n+1}\|_{\Omega,0}^2 + C\Delta t \int_{t_n}^{t_{n+1}} \|\partial_{tt}\tilde{s}_w^{n+1}\|_{\Omega,0}^2 \, dt$$
$$+ C \|\partial_t e_s^{n+1}\|_{\Omega,0}^2 + Ch^2 \|\nabla e_s^{n+1}\|_{\Omega,0}^2 + \sum_{\alpha=w,n} \left[C \|\nabla e_{p_\alpha}^{n+1}\|_{\Omega,0}^2 + Ch^{-2} \|e_{p_\alpha}^{n+1}\|_{\Omega,0}^2 + Ch^2 \left\| \nabla^2 e_{p_\alpha}^{n+1} \right\|_{\Omega,0}^2 \right]$$

Multiplying the above inequality by Δt, summing over $n = 0, \ldots, N$, and absorbing $\|e_{s,h}^{N+1}\|_{\Omega,0}^2$, we get:

$$\left(\frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2} - C\Delta t\right)\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{N+1}\|_{T_i,0}^2 + \frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2}\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}-e_{s,h}^{n}\|_{T_i,0}^2 + \frac{\phi\tau}{2}\Delta t\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+\frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\left(\|K^{\frac{1}{2}}\nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2 + \|K^{\frac{1}{2}}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2\right) + \frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\left(\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2 + \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2\right)$$
$$\leq\frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2}\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{0}\|_{T_i,0}^2 + \left(C + \frac{|\underline{\underline{p'_{c,eq}}}|\phi}{4}\right)\Delta t\sum_{n=0}^{N-1}\sum_{T_i\in\mathcal{T}}C\|e_{s,h}^{n+1}\|_{T_i,0}^2 + C\Delta t\sum_{n=0}^{N}\|e_{s}^{n+1}\|_{\Omega,0}^2$$
$$+C\Delta t\sum_{n=0}^{N}\|e_{p_c}^{n+1}\|_{\Omega,0}^2 + C\Delta t^2\int_0^T\|\partial_{tt}\tilde{s}_w^{n+1}\|_{\Omega,0}^2\,dt + C\Delta t\sum_{n=0}^{N}\|\partial_t e_s^{n+1}\|_{\Omega,0}^2 + Ch^2\Delta t\sum_{n=0}^{N}\|\nabla e_s^{n+1}\|_{\Omega,0}^2$$
$$+\sum_{\alpha=w,n}\left[C\Delta t\sum_{n=0}^{N}\|\nabla e_{p_\alpha}^{n+1}\|_{\Omega,0}^2 + Ch^{-2}\Delta t\sum_{n=0}^{N}\|e_{p_\alpha}^{n+1}\|_{\Omega,0}^2 + Ch^2\Delta t\sum_{n=0}^{N}\left\|\nabla^2 e_{p_\alpha}^{n+1}\right\|_{\Omega,0}^2\right]$$

For a sufficiently small Δt, we use Grönwall's inequality, and postulate that there exists a constant independent of Δt, h, $k_p$or k_s, s.t.:

$$\left(\frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2} - C\Delta t\right)\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{N+1}\|_{T_i,0}^2 + \frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2}\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}-e_{s,h}^{n}\|_{T_i,0}^2 + \frac{\phi\tau}{2}\Delta t\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+\frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\left(\|K^{\frac{1}{2}}\nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2 + \|K^{\frac{1}{2}}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2\right) + \frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\left(\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2 + \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2\right)$$
$$\leq\frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2}\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{0}\|_{T_i,0}^2 + C\Delta t\sum_{n=0}^{N}\|e_{s}^{n+1}\|_{\Omega,0}^2 + C\Delta t\sum_{n=0}^{N}\|e_{p_c}^{n+1}\|_{\Omega,0}^2 + C\Delta t^2\int_0^T\|\partial_{tt}\tilde{s}_w^{n+1}\|_{\Omega,0}^2\,dt$$
$$+C\Delta t\sum_{n=0}^{N}\|\partial_t e_s^{n+1}\|_{\Omega,0}^2 + Ch^2\Delta t\sum_{n=0}^{N}\|\nabla e_s^{n+1}\|_{\Omega,0}^2$$
$$+C\Delta t\sum_{n=0}^{N}\|\nabla e_{p_n}^{n+1}\|_{\Omega,0}^2 + Ch^{-2}\Delta t\sum_{n=0}^{N}\|e_{p_n}^{n+1}\|_{\Omega,0}^2 + Ch^2\Delta t\sum_{n=0}^{N}\left\|\nabla^2 e_{p_n}^{n+1}\right\|_{\Omega,0}^2$$
$$+C\Delta t\sum_{n=0}^{N}\|\nabla e_{p_w}^{n+1}\|_{\Omega,0}^2 + Ch^{-2}\Delta t\sum_{n=0}^{N}\|e_{p_w}^{n+1}\|_{\Omega,0}^2 + Ch^2\Delta t\sum_{n=0}^{N}\left\|\nabla^2 e_{p_w}^{n+1}\right\|_{\Omega,0}^2$$

Using the error estimates (28), (29) and (30), and the triangle inequality for the error terms in $p_w = p_n - p_c$, we can write:

$$\left(\frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2} - C\Delta t\right)\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{N+1}\|_{T_i,0}^2 + \frac{|\underline{\underline{p'_{c,eq}}}|\phi}{2}\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{n+1}-e_{s,h}^{n}\|_{T_i,0}^2 + \frac{\phi\tau}{2}\Delta t\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\|\partial^- e_{s,h}^{n+1}\|_{T_i,0}^2$$
$$+\frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{T_i\in\mathcal{T}}\left(\|K^{\frac{1}{2}}\nabla e_{p_w,h}^{n+1}\|_{T_i,0}^2 + \|K^{\frac{1}{2}}\nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2\right) + \frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{F_i\in\mathcal{F}}\frac{f(k_p)}{|F_i|}\left(\|[\![e_{p_w,h}^{n+1}]\!]\|_{F_i,0}^2 + \|[\![e_{p_n,h}^{n+1}]\!]\|_{F_i,0}^2\right)$$
$$\leq C\sum_{T_i\in\mathcal{T}}\|e_{s,h}^{0}\|_{T_i,0}^2 + C\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_s+1,l_s)}}{k_s^{2l_s}}\|s_w(t)\|_{\Omega,l_s}^2 + C\Delta t^2$$
$$+C\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_s+1,l_s)}}{k_s^{2l_s}}\|\partial_t s_w(t)\|_{\Omega,l_s}^2 + Ch^2\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_s+1,l_s)-2}}{k_s^{2l_s-2}}\|s_w(t)\|_{\Omega,l_s}^2$$
$$+C\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_{p_n}+1,l_{p_n})-2}}{k_{p_n}^{2l_{p_n}-2}}\|p_n(t)\|_{\Omega,l_{p_n}}^2 + Ch^{-2}\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_{p_n}+1,l_{p_n})}}{k_{p_n}^{2l_{p_n}}}\|p_n(t)\|_{\Omega,l_{p_n}}^2$$
$$+Ch^2\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_{p_n}+1,l_{p_n})-4}}{k_{p_n}^{2l_{p_n}-4}}\|p_n(t)\|_{\Omega,l_{p_n}}^2 + C\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_{p_c}+1,l_{p_c})-2}}{k_{p_c}^{2l_{p_c}-2}}\|p_c(t)\|_{\Omega,l_{p_c}}^2$$

$$+Ch^{-2}\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_{p_c}+1,l_{p_c})}}{k_{p_c}^{2l_{p_c}}}\|p_c(t)\|^2_{\Omega,l_{p_c}}+Ch^2\Delta t\sum_{n=0}^{N}\frac{h^{2\min(k_{p_c}+1,l_{p_c})-4}}{k_{p_c}^{2l_{p_c}-4}}\|p_c(t)\|^2_{\Omega,l_{p_c}}$$

from where, the stated estimate follows.

From the Theorem we can directly deduce the following Corollary:

Corollary 1 *For sufficiently smooth solutions $p_n \in L^2([0,T],H^{k_p+1}(\Omega))$, $p_c \in L^2([0,T],H^{k_p+1}(\Omega))$ and $s_w \in H^2([0,T],H^{k_s+1}(\Omega))$ and sufficiently large σ_n and σ_w, there exists a constant C independent of h and Δt, s.t., the following estimate holds:*

$$\begin{aligned}&\|e^{N+1}_{s,h}\|^2_{\Omega,0}+\Delta t\sum_{n=0}^{N}\|\partial^- e^{n+1}_{s,h}\|^2_{\Omega,0}+\Delta t\sum_{n=0}^{N}\left(\|e^{n+1}_{p_c,h}\|^2_{\Omega,DG}+\|e^{n+1}_{p_n,h}\|^2_{\Omega,DG}\right)\\ \le& C\Delta t^2+C\frac{h^{2k_s}}{k_s^{2k_s}}+C\frac{h^{2k_{p_n}}}{k_{p_n}^{2k_{p_n}-2}}+C\frac{h^{2k_{p_c}}}{k_{p_c}^{2k_{p_c}-2}}\end{aligned}$$

5 Numerical Experiments

In this section, we verify the convergence rates derived in Theorem 1 through numerical experiments. We consider an analytical solution to compute the L^2- and H^1-errors. We show the h and Δt dependence through successive refinement of the spatial mesh, respectively of the time step.

Problem definition We consider the domain $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ and $t \in [0,1]$. The properties of the phases and the porous medium are listed in Table 1.

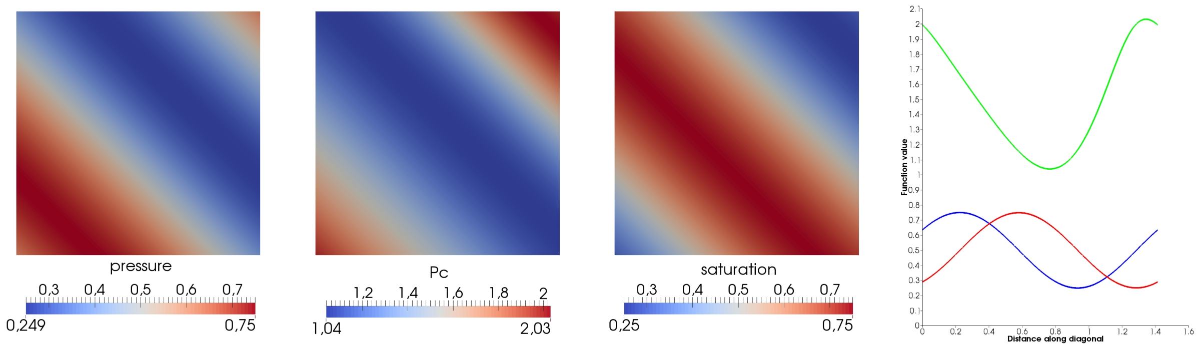
Table 1: Properties for Test problem 1

Phase Properties		
dyn. viscosity water	μ_w	1
dyn. viscosity oil	μ_n	1
density water	ρ_w	1
density oil	ρ_n	1
Hydraulic Properties		
abs. permeability	K	1
res. water saturation	S_{rw}	0
res. oil saturation	S_{rn}	0
porosity	φ	0.4
retardation coefficient	τ	1
Brooks-Correy Parameters		
entry pressure	p_d	1
pore size distr. index	λ	2

The right hand side in the equations are chosen such that the exact solution for $t \geq 0$ equals:

$$\begin{aligned}p_n(t,x,y) &= \frac{1}{4}\cos((x+y)\pi - t)+\frac{1}{2}\\ S_w(t,x,y) &= \frac{1}{4}\sin((x+y)\pi - t)+\frac{1}{2}\\ p_c(t,x,y) &= p_{c,eq}(S_w(t,x,y)) - \tau\partial_t S_w(t,x,y)\end{aligned}$$

Implementation We chose $\theta = 1$, which gives a NIPG scheme, and the penalty parameters as $\sigma_w = \sigma_n = 10$. We implement the numerical scheme in the C++ based DUNE-PDELab framework [4,6,5]. For linearization, we use the Newton-Raphson scheme with a line-search strategy [16]. We solve the resulting linear system with SuperLU solver [15].



(a) p_n **(b)** p_c **(c)** s_w **(d)** Plot over the diagonal $Y = X$, legend: S_w,p_n and p_c

Fig. 1: Simulation results at $t = 1$.

Simulation To show the spatial convergence ratese, we make five simulations each for polynomial orders 1 and 2, with the following mesh and time step refinements:

	no. of elements	p-order=1 time step size	p-order=2 time step size
Run-1:	2×2	$\Delta t = 1$	$\Delta t = 1$,
Run-2:	4×4	$\Delta t = 1/2$	$\Delta t = 1/4$,
Run-3:	8×8	$\Delta t = 1/4$	$\Delta t = 1/16$,
Run-4:	16×16	$\Delta t = 1/8$	$\Delta t = 1/64$,
Run-5:	32×32	$\Delta t = 1/16$	$\Delta t = 1/256$.

Additionally, to show the time convergence rates, make five simulations with polynomial order 2 and the following mesh and time step refinements:

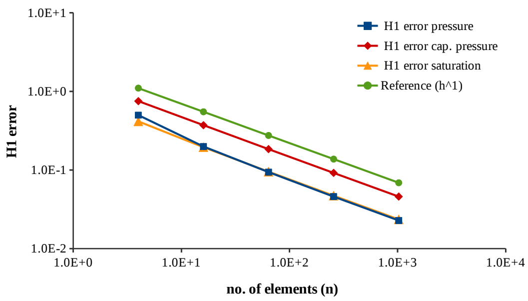
	no. of elements	p-order=2 time step size
Run-1:	2×2	$\Delta t = 1$,
Run-2:	4×4	$\Delta t = 1/2$,
Run-3:	8×8	$\Delta t = 1/4$,
Run-4:	16×16	$\Delta t = 1/8$,
Run-5:	32×32	$\Delta t = 1/16$.

In this case, the time steps are chosen such that the error due to time discretization is dominating.

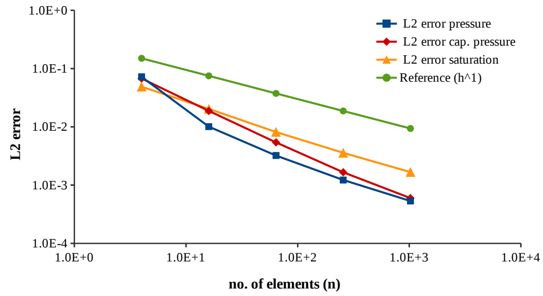
Results The solution of the problem at time $t = 1$ and with a refinement of 32×32 is shown in Figures 1a, 1b and 1c.

In Figure 2, we show the spatial convergence rates for the test problem. Figures 2a and 2b show the calculated error for piecewise linear polynomials for the non-wetting pressure p_n, capillary pressure p_c, and wetting saturation s_w. Figures 2c and 2d show the calculated error for piecewise quadratic polynomials for p_n, p_c, and s_w. In Figure 3, we show the temporal convergence rates for the test problem, with piecewise quadratic polynomials for p_n, p_c, and s_w.

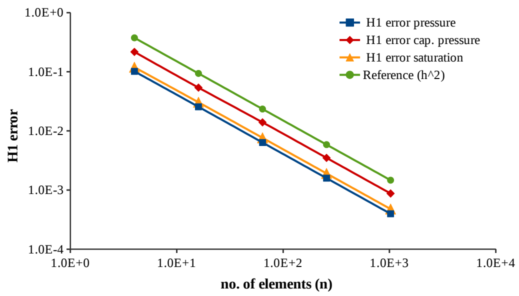
Observe the agreement with the theoretical convergence rates obtained in Theorem 1. In the first case, expected is a linear convergence rate. In the second case, using quadratic polynomials, we chose the time step $\frac{1}{4}$-th of the size of the spatial mesh. This prevents that the errors due to the time discretization dominate, affecting the convergence rates. To show the time convergence as in case two we choose quadratic polynomials. However, this time the time step is half of the size of the spatial discretization, so that the error due to time discretization becomes dominating. The expected convergence rates for each of the cases are plotted in green for reference.



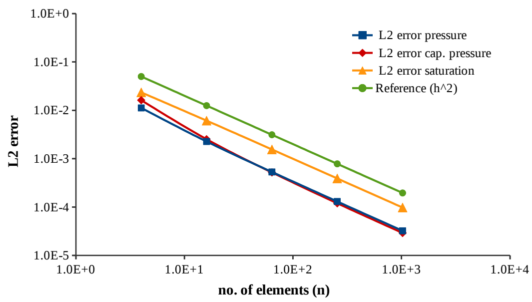
(a) H^1 error for piecewise linear polynomials



(b) L^2 error for piecewise linear polynomials



(c) H^1 error for piecewise quadratic polynomials



(d) L^2 error for piecewise quadratic polynomials

Fig. 2: Spatial convergence rates.

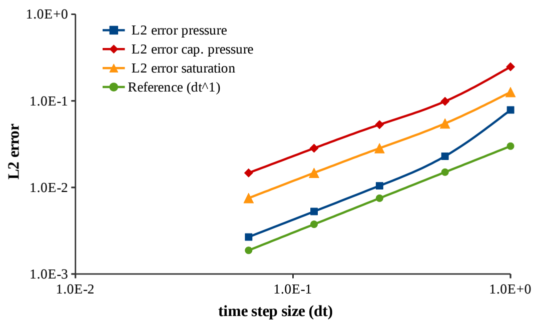


Fig. 3: Temporal convergence rates
L^2 error for piecewise quadratic polynomials.

6 Conclusions

We have presented a fully implicit interior penalty discontinuous Galerkin numerical scheme for a two-phase porous media flow model, where dynamic effects are incorporated in the capillary pressure. The proposed scheme is based on quantities that have a direct physical meaning (like saturation, or phase pressure) and avoids using concepts like Kirchhoff transform or global pressure. Building on a fixed point argument, we have proved the existence of a fully discrete solution. Further, we have shown the convergence of the scheme by obtaining a-priori error estimates, in dependence of the polynomial degree, the mesh-size, and the time-step-size.

Further aspects related to this model and the proposed discretization will be considered in the forthcoming research. Clearly, space-time adaptivity and domain decomposition schemes can increase the efficiency of the method discussed here. In this sense, we note that the emerging fully discrete systems are nonlinear, and therefore efficient linearization (iterative) techniques have to be developed and their convergence analyzed. A good starting point are the schemes discussed in [36]. Also, the possibility to extend such schemes towards models including hysteresis, or defined in heterogeneous domains with or without entry pressures [21,33].

Acknowledgements Stefan Karpinski wants to thank Prof. B.I. Wohlmuth for the introduction to this interesting topic and the guidance through his first years in academia. He would also like to express his gratitude to his employers, Dr. Roman Rojko and Dr. Stefan Zaprianov from ESPRiT Engineering GmbH for their mentorship and support. Special thanks also to Prof. F.A. Radu (Bergen) for the fruitful and productive discussions during his visit to the University of Bergen. This stay was supported by the International Research Training Group NUPUS, funded by the German Research Foundation DFG (GRK 1398), the Netherlands Organization for Scientific Research NWO (DN 81-754) and by the Research Council of Norway (215627).
Iuliu Sorin Pop acknowledges the support from the Akademia grant of Statoil and from the Shell-NWO/FOM CSER programme (project 14CSER016).

References

1. Adams, R., Fournier, J.: Sobolev Spaces. Pure and Applied Mathematics. Elsevier Science (2003)
2. Babuska, I., Suri, M.: The $h-p$ version of the finite element method with quasiuniform meshes. ESAIM: Mathematical Modelling and Numerical Analysis - Modlisation Mathmatique et Analyse Numrique **21**(2), 199–238 (1987). URL http://eudml.org/doc/193500
3. Bastian, P.: A fully-coupled discontinuous galerkin method for two-phase flow in porous media with discontinuous capillary pressure. Computational Geosciences **18**(5), 779–796 (2014). DOI 10.1007/s10596-014-9426-y. URL http://dx.doi.org/10.1007/s10596-014-9426-y
4. Bastian, P., Blatt, M., Dedner, A., Engwer, C., Fahlke, J., Gräser, C., Klöfkorn, R., Nolte, M., Ohlberger, M., Sander, O.: DUNE Web page (2011). Http://www.dune-project.org
5. Bastian, P., Blatt, M., Dedner, A., Engwer, C., Klöfkorn, R., Kornhuber, R., Ohlberger, M., Sander, O.: A generic grid interface for parallel and adaptive scientific computing. DFG Research Center MATHEON (2007)
6. Bastian, P., Heimann, F., Marnach, S.: Generic implementation of finite element methods in the distributed and unified numerics environment (dune). Kybernetika **46**(2), 294–315 (2010). URL http://eudml.org/doc/197255
7. Bastian, P., Riviere, B.: Superconvergence and h(div) projection for discontinuous galerkin methods. International Journal for Numerical Methods in Fluids **42**(10), 1043–1057 (2003). DOI 10.1002/fld.562. URL http://dx.doi.org/10.1002/fld.562
8. Bear, J.: Dynamics of Fluids in Porous Media. Dover Civil and Mechanical Engineering. Dover Publications (2013). URL https://books.google.de/books?id=fBMeVSZ_3u8C
9. Brooks R.H. Corey, A.: Hydraulic properties of porous media. Hydrology papers (Colorado State University) **3** (1964)
10. Burdine, N.: Relative permeability calculations from pore size distribution data. Journal of Petroleum Technology **5**(3), 71 – 78 (1953). DOI http://dx.doi.org/10.2118/225-G
11. Cao, X., Nemadjieu, S., Pop, I.S.: A multipoint flux approximation finite volume scheme for two phase porous media flow with dynamic capillarity. CASA-Report 15-33 (2015). URL http://www.win.tue.nl/analysis/reports/rana15-33.pdf
12. Cao, X., Pop, I.S.: Two-phase porous media flows with dynamic capillary effects and hysteresis : uniqueness of weak solutions. Computers and Mathematics with Applications **69(2015)**(7), 688–695 (2015). DOI http://dx.doi.org/10.1016/j.camwa.2015.02.009. URL http://purl.tue.nl/693256166388305
13. Cao, X., Pop, I.S.: Degenerate two-phase porous media flow model with dynamic capillarity. J. Differential Equations **260**(3), 2418–2456 (2016). DOI 10.1016/j.jde.2015.10.008. URL http://dx.doi.org/10.1016/j.jde.2015.10.008
14. Chavent, G., Jaffre, J.: Mathematical Models and Finite Elements for Reservoir Simulation Single Phase, Multiphase and Multicomponent Flows through Porous Media, *Studies in Mathematics and Its Applications*, vol. 17. Elsevier (1986). DOI http://dx.doi.org/10.1016/S0168-2024(08)70077-8. URL http://www.sciencedirect.com/science/article/pii/S0168202408700778
15. Demmel, J., Gilbert, J., Li, X.: Superlu users' guide. Technical Report LBNL-44289, Lawrence Berkeley National Laboratory (September 1999). URL http://crd-legacy.lbl.gov/~xiaoye/SuperLU/. Last update: September 2007
16. Deuflhard, P.: Newton methods for nonlinear problems : affine invariance and adaptive algorithms. Springer series in computational mathematics. Springer, Berlin, Heidelberg, New York (2004). URL http://opac.inria.fr/record=b1101121. Autre tirage : 2006
17. Di Pietro, D., Ern, A.: Discrete functional analysis tools for discontinuous galerkin methods with application to the incompressible navier-stokes equations. Mathematics of Computation **79**(271), 13031330 (2010). URL https://hal.archives-ouvertes.fr/hal-00278925
18. Di Pietro, D., Ern, A.: Mathematical Aspects of Discontinuous Galerkin Methods. Springer (2012)
19. DiCarlo, D.: Experimental measurements of saturation overshoot on infiltration. Water Resources Research **40**(4) (2004). DOI 10.1029/2003WR002670. URL http://dx.doi.org/10.1029/2003WR002670. W04215
20. Diersch, H.J., Clausnitzer, V., Myrnyy, V., Rosati, R., Schmidt, M., Beruda, H., Ehrnsperger, B., Virgilio, R.: Modeling unsaturated flow in absorbent swelling porous media: Part 1. theory. Transport in Porous Media **83**(3), 437–464 (2010). DOI 10.1007/s11242-009-9454-6. URL http://dx.doi.org/10.1007/s11242-009-9454-6
21. van Duijn, C., Cao, X., Pop, I.S.: Two-phase flow in porous media: dynamic capillarity and heterogeneous media. Transport in Porous Media **accepted** (2016)
22. van Duijn, C., Fan, Y., Peletier, L., Pop, I.S.: Travelling wave solutions for degenerate pseudo-parabolic equations modelling two-phase flow in porous media. Nonlinear Analysis: Real World Applications **14**(3), 1361 – 1383 (2013). DOI http://dx.doi.org/10.1016/j.nonrwa.2012.10.002. URL http://www.sciencedirect.com/science/article/pii/S1468121812002209
23. Durlofsky, L.: A triangle based mixed finite element–finite volume technique for modeling two phase flow through porous media. J. Comput. Phys. **105**(2), 252–266 (1993). DOI 10.1006/jcph.1993.1072. URL http://dx.doi.org/10.1006/jcph.1993.1072
24. Epshteyn, Y., Riviere, B.: Analysis of discontinuous galerkin methods for incompressible two-phase flow. Journal of Computational and Applied Mathematics **225**(2), 487 – 509 (2009). DOI http://dx.doi.org/10.1016/j.cam.2008.08.026. URL http://www.sciencedirect.com/science/article/pii/S0377042708004214

25. Ern, A., Mozolevski, I., Schuh, L.: Discontinuous galerkin approximation of two-phase flows in heterogeneous porous media with discontinuous capillary pressures. Computer Methods in Applied Mechanics and Engineering **199**(2324), 1491 – 1501 (2010). DOI http://dx.doi.org/10.1016/j.cma.2009.12.014. URL http://www.sciencedirect.com/science/article/pii/S0045782509004204
26. Evans, L.: Partial Differential Equations. Graduate studies in mathematics. American Mathematical Society (1998). URL https://books.google.de/books?id=5Pv4LVB_m8AC
27. Eymard, R., Herbin, R., Michel, A.: Mathematical study of a petroleum-engineering scheme. MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS **37**, 2003 (2003)
28. Fan, Y., Pop, I.S.: A class of pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the euler-implicit discretization. Math. Methods Appl. Sci. **34**, 23292339 (2011). DOI http://dx.doi.org/10.1002/mma.1537
29. Girault, V., Raviart, P.: Finite element methods for Navier-Stokes equations: theory and algorithms. Springer series in computational mathematics. Springer-Verlag (1986). URL https://books.google.de/books?id=BVfvAAAAMAAJ
30. Hassanizadeh, S., Beliaev, A.: A theoretical model of hysteresis and dynamic effects in the capillary relation for two-phase flow in porous media. Transport in Porous Media **43**(3), 487–510 (2001). DOI 10.1023/A:1010736108256. URL http://dx.doi.org/10.1023/A%3A1010736108256
31. Hassanizadeh, S., Gray, W.: Thermodynamic basis of capillary pressure in porous media. Water Resources Research **29**(10), 3389–3405 (1993). DOI 10.1029/93WR01495. URL http://dx.doi.org/10.1029/93WR01495
32. Helmig, R.: Multiphase Flow and Transport Processes in the Subsurface: A Contribution to the Modeling of Hydrosystems. Environmental engineering. Springer (1997). URL https://books.google.de/books?id=4ZVRAAAAMAAJ
33. Helmig, R., Weiss, A., Wohlmuth, B.: Dynamic capillary effects in heterogeneous porous media. Computational Geosciences **11**(3), 261–274 (2007). DOI 10.1007/s10596-007-9050-1. URL http://dx.doi.org/10.1007/s10596-007-9050-1
34. Kissling, F., Rohde, C.: The computation of nonclassical shock waves with a heterogeneous multiscale method. Netw. Heterog. Media **5**(3), 661–674 (2010). DOI 10.3934/nhm.2010.5.661. URL http://dx.doi.org/10.3934/nhm.2010.5.661
35. Koch, J., Rätz, A., Schweizer, B.: Two-phase flow equations with a dynamic capillary pressure. European Journal of Applied Mathematics **24**, 49–75 (2013). DOI 10.1017/S0956792512000307. URL http://journals.cambridge.org/article_S0956792512000307
36. List, F., Radu, F.A.: A study on iterative methods for solving richards equation. Computational Geosciences **accepted** (2016)
37. Mikelic, A.: A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure. Journal of Differential Equations **248**(6), 1561 – 1577 (2010). DOI http://dx.doi.org/10.1016/j.jde.2009.11.022. URL http://www.sciencedirect.com/science/article/pii/S0022039609004446
38. Mualem, Y.: A New Model for Predicting the Hydraulic Conductivity of Unsaturated Porous Media. Water Resources Research **12**, 513–522 (1976). DOI 10.1029/WR012i003p00513
39. Nordbotten, J., Celia, M.: Geological Storage of CO2: Modeling Approaches for Large-Scale Simulation. Wiley (2011). URL https://books.google.de/books?id=vMiFGYafu_wC
40. Radu, F., Pop, I.S.: Mixed finite element discretization and newton iteration for a reactive contaminant transport model with nonequilibrium sorption: convergence analysis and error estimates. Computational Geosciences **15**(3), 431–450 (2011)
41. Radu, F.A., Nordbotten, J.M., Pop, I.S., Kumar, K.: A robust linearization scheme for finite volume based discretizations for simulation of two-phase flow in porous media. Journal of Computational and Applied Mathematics **289**(0), 134 – 141 (2015). DOI http://dx.doi.org/10.1016/j.cam.2015.02.051. URL http://www.sciencedirect.com/science/article/pii/S0377042715001387. Sixth International Conference on Advanced Computational Methods in Engineering (ACOMEN 2014)
42. Rätz, A., Schweizer, B.: Hysteresis models and gravity fingering in porous media. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik **94**(7-8), 645–654 (2014)
43. Riviere, B., Wheeler, M., Girault, V.: A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. SIAM J. Numer. Anal. **39**(3), 902–931 (2001). DOI 10.1137/S003614290037174X. URL http://dx.doi.org/10.1137/S003614290037174X
44. Sun, S., Wheeler, M.: Symmetric and nonsymmetric discontinuous galerkin methods for reactive transport in porous media. SIAM Journal on Numerical Analysis **43**(1), 195–219 (2005). DOI 10.1137/S003614290241708X. URL http://dx.doi.org/10.1137/S003614290241708X
45. Temam, R.: Navier-Stokes equations: theory and numerical analysis, vol. 343. American Mathematical Soc. (2001)
46. Van Genuchten, M.: A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. Soil science society of America journal **44**(5), 892–898 (1980)
47. Warburton, T., Hesthaven, J.: On the constants in hp-finite element trace inverse inequalities. Computer Methods in Applied Mechanics and Engineering **192**(25), 2765 – 2773 (2003). DOI http://dx.doi.org/10.1016/S0045-7825(03)00294-9. URL http://www.sciencedirect.com/science/article/pii/S0045782503002949