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An efficient linear solver for the hybridized discontinuous Galerkin method

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Discretizing partial differential equations by an implicit solving technique ultimately leads to a linear system of equations that has to be solved. The number of globally coupled unknowns is especially large for discontinuous Galerkin (DG) methods. It can be reduced by using hybridized discontinuous Galerkin (HDG) methods, but still efficient linear solvers are needed. It has been shown that, if hierarchical basis functions are used, a hierarchical scale separation (HSS) ansatz can be an efficient solver. In this work, we couple the HDG method with an HSS solver to solve a scalar nonlinear problem. It is validated by comparing the results with results obtained by GMRES with ILU(3) preconditioning as linear solver.

1 Introduction

The discontinuous Galerkin (DG) methods [2, 5] make up a class of popular high-order methods for computational fluid dynamics (CFD) applications. However, these methods need more degrees of freedom to represent numerical solutions than the classical finite element and finite volume methods; the computational expenses due to global coupling of all degrees of freedom can become particularly significant when time-implicit solvers are used. In [1], it has been shown that the arising linear system can be solved efficiently by applying a hierarchical scale separation (HSS) approach. Even greater reductions in the linear system size can be achieved by utilizing a hybridized discontinuous Galerkin (HDG) approach [3]. In this work we extend the work of Schütz and Aizinger [6] to a nonlinear convection-diffusion equation.

2 Numerical method

In this work, we aim to solve the viscous Burgers equation

$$0.5\,\nabla \cdot (w^2, w^2)^T - \varepsilon \Delta w = s(w, \nabla w) \quad \forall (x,t) \in \Omega$$

on a two-dimensional domain $\Omega = (0,1)^2$ with homogeneous boundary conditions. w is the unknown and the diffusion constant ε is non-negative. An additional source term $s(w, \nabla w)$ is added to obtain the same boundary layer as in [4]. We mentioned in the introduction that we solve this by spatially discretizing the equation using the hybridized discontinuous Galerkin method. Therefore, Ω is partitioned into N_e elements Ω_k such that $\Omega = \bigcup_k \Omega_k$ The unknown solution is then represented by polynomials of degree p that have only local support on each element Ω_k, i.e. the solution may have jumps between elements. The relaxed continuity leads to a large number of globally coupled unknowns when implicit solving techniques are employed which is common for steady-state computations. This number can be greatly reduced by hybridization. For this purpose, additional unknowns on the edges of elements are introduced that take over the global coupling from the element unknowns. In most cases this method leads to significant savings in terms of memory requirements and computing time [7]. The possibly nonlinear system of equations is solved using Newton's method. Then, the arising system of linear equations has to be solved by a suitable linear solver.

3 A hierarchical scale separation approach

In [1], it has been shown that the arising linear system of a DG discretization can be solved efficiently by applying a hierarchical scale separation (HSS) approach. For a DG method using a hierarchical basis, the linear system can be trivially split into two parts: A coarse scale system that corresponds to the unknowns for low-order polynomial modes and a fine scale system that contains the remaining modes. The coarse-scale system is much smaller than the full system - especially for large values of approximation space order p - and therefore can be solved *efficiently* using black-box linear solvers. Using a simple smoother, the fine-scale system computes a high-order correction to the low-order modes. Overall, this yields an iterative scheme. This is especially attractive for the HDG method where the linear system is already smaller than for classical DG methods. First results of the HSS approach with the HDG method can be found in [6].

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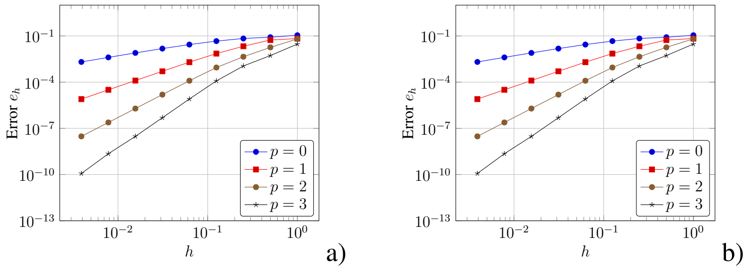

Fig. 1: The errors obtained when using **a)** ILU(3)+GMRES and **b)** HSS as linear solver for $\varepsilon = 0.1$ on different meshes.

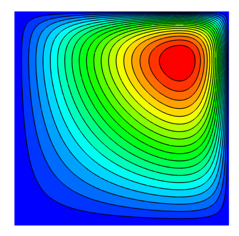

Fig. 2: The solution for $\varepsilon = 0.1$ on a mesh with $N_e = 512$ elements and $p = 3$.

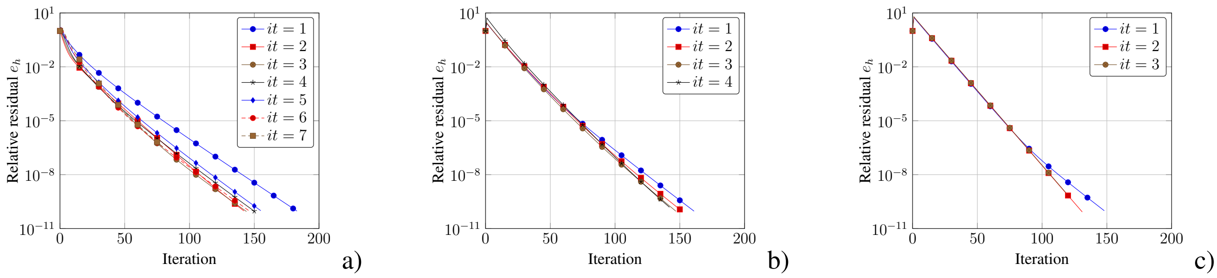

Fig. 3: The convergence behavior of the HSS solver for polynomial ansatz functions of degree $p = 2$ and $N_e = 2048$ elements. The residual history for **a)** $\varepsilon = 0.01$, **b)** $\varepsilon = 0.1$ and **c)** $\varepsilon = 1.0$ is shown. It shows the convergence behavior of the HSS solver in each Newton iteration.

4 Numerical results

We verify the HSS solver against GMRES with ILU(3) as preconditioner. Figure 1 shows that for both solvers, we achieve the same error levels on uniformly refined meshes. Here, $h = \frac{1}{\sqrt{0.5 N_e}}$ is the mesh size of the triangular mesh. The solution on $N_e = 512$ elements and $p = 3$ is depicted in Figure 2. In Figure 3, we display the convergence history of the Newton-HSS solver on a mesh with $N_e = 2048$ elements and polynomials with $p = 2$. The computations have been carried out for different diffusion constants. For decreasing ε the boundary layer is more pronounced, thus making the system of equations harder to solve. While this leads to an increasing number of Newton iterations, the convergence behavior of the HSS solver depends only slightly on the diffusion constant. This is in agreement with the findings for linear problems in [6].

5 Conclusion and outlook

We have applied a new iterative linear solver to a nonlinear PDE that has been discretized using a hybridized discontinuous Galerkin method. The results show that it is applicable to the 2D viscous Burgers equation, i.e. a nonlinear convection-diffusion equation and produces the same results as a standard linear solver.

Future work will focus on the extension to *systems* of nonlinear partial differential equations such as Euler or Navier-Stokes equations. Additionally, an efficient optimization that allows a fair comparison to more standard methods is needed.

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