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## CLEFT EXTENSIONS OF KOSZUL TWISTED CALABI-YAU ALGEBRAS

XIAOLAN YU, FRED VAN OYSTAEYEN, AND YINHUO ZHANG

ABSTRACT. Let H be a twisted Calabi-Yau (CY) algebra and  $\sigma$  a 2-cocycle on H. Let A be an N-Koszul twisted CY algebra such that A is a graded  $H^{\sigma}$ -module algebra. We show that the cleft extension  $A \#_{\sigma} H$  is also a twisted CY algebra. This result has two consequences. Firstly, the smash product of an N-Koszul twisted CY algebra with a twisted CY Hopf algebra is still a twisted CY algebra. Secondly, the cleft objects of a twisted CY Hopf algebra are all twisted CY algebras. As an application of this property, we determine which cleft objects of  $U(\mathcal{D}, \lambda)$ , a class of pointed Hopf algebras introduced by Andruskiewitsch and Schneider, are Calabi-Yau algebras.

#### INTRODUCTION

We work over a fix a field k. Without otherwise stated, all vector spaces, algebras are over k. Given a 2-cocycle  $\sigma$  on a Hopf algebra H (Definition 1.3), we can construct the algebras  $H^{\sigma}$  and  $_{\sigma}H$ . Their products are deformed from the product of H by

$$x * y = \sigma(x_1, y_1) x_2 y_2 \sigma^{-1}(x_3, y_3)$$
$$x \cdot_{\sigma} y = \sigma(x_1, y_1) x_2 y_2,$$

for any  $x, y \in H$  respectively. The algebra  $H^{\sigma}$  together with its original coalgebra structure form a Hopf algebra, called a cocycle deformation of H. On the one hand, the algebra  ${}_{\sigma}H$  together with the original regular coaction  ${}_{\sigma}H \to {}_{\sigma}H \otimes H$  form a right H-cleft extension over the field  $\Bbbk$ . It is called a right cleft object. On the other hand,  ${}_{\sigma}H$  is a left  $H^{\sigma}$ -cleft object with respect to the original coalgebra  ${}_{\sigma}H \to H^{\sigma} \otimes {}_{\sigma}H$ . Therefore,  ${}_{\sigma}H$  is an  $(H^{\sigma}, H)$ -bicleft object. The Hopf algebra  $H^{\sigma}$  is characterized as the Hopf algebra L such that  ${}_{\sigma}H$  is an (L, H)-biGalois object ([33]).

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In [28], Masuoka studied cocycle deformations and cleft objects of a class of pointed Hopf algebras. This class of algebras includes the pointed Hopf algebras  $U(\mathcal{D}, \lambda)$  of finite Cartan type introduced by Andruskiewitsch and Schneider ([5]). The Hopf algebras  $U(\mathcal{D}, \lambda)$  consists of pointed Hopf algebras with finite Gelfand-Kirillov dimension, which are domains with finitely generated abelian groups of group-like elements, and generic infinitesimal braiding ([1]). By results in [28], we know that a pointed Hopf algebra  $U(D, \lambda)$  and its associated graded Hopf algebra U(D, 0) are cocycle deformations of each other.

The Calabi-Yau (CY for short) property of the algebras  $U(\mathcal{D}, \lambda)$  are discussed in [39]. CY algebras were introduced by Ginzburg [19] in 2006. They were studied in recent years because of their applications in algebraic geometry and mathematical physics. More general than CY algebras are so-called twisted CY algebras, which form a large class of algebras possessing the similar homological properties as the CY algebras and include CY algebras as a subclass. Associated to a twisted CY algebra, there exists a so-called Nakayama automorphism. This automorphism is unique up to an inner automorphism. A twisted CY algebra is CY if and only if its Nakayama automorphism is an inner automorphism.

For the Hopf algebra  $U(\mathcal{D}, \lambda)$ , both  $U(\mathcal{D}, \lambda)$  itself and its associated graded Hopf algebra  $U(\mathcal{D}, 0)$  are twisted CY algebras ([39, Theorem 3.9]). A more interesting phenomenon is that the CY property of  $U(D, \lambda)$  is dependent only on the CY property of U(D, 0). In other words, if U(D, 0) is CY, then any lifting  $U(D, \lambda)$  is CY. Note that  $U(D, \lambda)$  is a cocycle deformation of U(D, 0). This raises a natural question whether a cocycle deformation of a graded pointed (twisted) CY Hopf algebra is still a (twisted) CY algebra. For a Hopf algebra H and its cocycle deformation  $H^{\sigma}$ , the algebra  $_{\sigma}H$  can be viewed as the "connection" between H and  $H^{\sigma}$  as it defines a Morita tensor equivalence between the comodule categories over the two Hopf algebras. To understand the relation between the twisted CY property of H and that of  $H^{\sigma}$ , we shall first answer the question whether  $_{\sigma}H$  is a twisted CY algebra when H is.

The algebra  ${}_{\sigma}H$  can be viewed as the crossed product  $\Bbbk \#_{\sigma}H$  (the definition of a crossed product will be reviewed in Section 1.2). More generally, one could ask whether the crossed product  $A \#_{\sigma}H$  will be a twisted CY algebra when both A and H are twisted CY algebras. In this paper, we are able to answer the question when A is a graded N-Koszul algebra. We note here that to form an algebra  $A \#_{\sigma}H$ , it is only required that  $\sigma$  is an invertible map in  $\operatorname{Hom}(H \otimes H, A)$  satisfying the cocycle condition and A is a twisted H-module. When A is a graded N-Koszul algebra, the assumption that  $\sigma$  has its image in k is necessary to make sure that the obtained crossed product  $A\#_{\sigma}H$  is still a graded algebra. In this case  $\sigma$  is just a 2-cocycle on H and A is a left graded  $H^{\sigma}$ -module algebra. Here A is a left graded  $H^{\sigma}$ -module algebra means that A is a left  $H^{\sigma}$ -module algebra such that each graded piece  $A_i$  is a left  $H^{\sigma}$ -module. The following theorem is our main result (see Theorem 2.18):

**Theorem 0.1.** Let H be a twisted CY Hopf algebra with homological integral  $\int_{H}^{l} = \mathbb{k}_{\xi}$ , where  $\xi : H \to \mathbb{k}$  is an algebra homomorphism and  $\sigma$  a 2-cocycle on H. Let A be a N-Koszul graded twisted CY algebra with Nakayama automorphism  $\mu$  such that A is a left graded  $H^{\sigma}$ -module algebra. Then  $A \#_{\sigma} H$  is a twisted CY algebra with Nakayama automorphism  $\rho$  defined by  $\rho(a \# h) = \mu(a) \# \operatorname{hdet}_{H^{\sigma}}(h_1)(S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(h_2)))\xi(h_3)$  for all  $a \# h \in A \#_{\sigma} H$ .

Here,  $hdet_{H^{\sigma}}$  denotes the homological determinant of the  $H^{\sigma}$ -action. The homological integral of a twisted CY Hopf algebra will be given in Section 2. The notion  $S_{\sigma,\tau}$  will be recalled in Section 1.1. Examples of Theorem 0.1 will be provided in Section 4.

Theorem 0.1 has two consequences. Firstly, in Theorem 0.1, if we let the cocycle  $\sigma$  be trivial, then the crossed product  $A \#_{\sigma} H$  is just the smash product A # H. Therefore, we obtain the following result on smash products.

**Theorem 0.2.** Let H be a twisted CY Hopf algebra with homological integral  $\int_{H}^{l} = \mathbb{k}_{\xi}$ , where  $\xi : H \to \mathbb{k}$  is an algebra homomorphism and A an N-Koszul graded twisted CY algebra with Nakayama automorphism  $\mu$  such that A is a left graded H-module algebra. Then A#H is a twisted CY algebra with Nakayama automorphism  $\rho$  defined by  $\rho(a\#h) = \mu(a)\# \operatorname{hdet}_{H}(h_{1})(S^{-2}(h_{2}))\xi(h_{3})$ , for any  $a\#h \in A\#H$ .

This generalizes the results in [23] and [32]. The smash products of CY algebras has been studied quite broadly. For instance, see [16], [20], [23], [38], [32]. The results in [23] and [32] are probably two of the most general results in this direction. [23] states that when H is an involutory Hopf CY algebra and Ais an N-Koszul CY algebra, the smash product A#H is CY if and only if the homological determinant of the H-action on A is trivial. One of the main results in [32] states that the smash product A#H is a twisted CY algebra when A is a graded twisted CY algebra and H a finite dimensional Hopf algebra acting on A. The Nakayama automorphism of A#H is determined by the ones of A and H, along with the homological determinant of the H-action.

Secondly, in Theorem 0.1, if we let the algebra A be  $\Bbbk$ , we obtain the following description of the twisted CY property of cleft objects.

**Theorem 0.3.** Let H be a twisted CY Hopf algebra with  $\int_{H}^{l} = \xi \mathbb{k}$ , and  $\sigma$  a 2-cocycle on H. Then the right cleft object  $\sigma H$  is a twisted CY algebra with Nakayama automorphism  $\mu$  defined by

$$\mu(x) = S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(x_1))\xi S(x_2)$$

for any  $x \in {}_{\sigma}H$ .

As an application of Theorem 0.3, we study the CY property of the cleft objects of the Hopf algebras  $U(\mathcal{D}, \lambda)$  in Section 3. It turns out that all cleft objects of the algebra  $U(\mathcal{D}, \lambda)$  are twisted CY algebras. Their Nakayama automorphisms are given explicitly in Proposition 3.7. Hence we are able to characterize when a clefts object is CY. It is interesting that a cleft object of  $U(\mathcal{D}, \lambda)$  could be a CY algebra even when  $U(\mathcal{D}, \lambda)$  itself is not. We give such an example at the end of Section 3.

Our motivating examples are the algebras of the form  $A \#_{\sigma} \Bbbk G$ , where A is a polynomial algebra, G is a finite group acting on A, and  $\sigma : G \times G \rightarrow$  $\mathbb{C}^{\times}$  is a 2-cocycle on G. Such crossed products are of interest in geometry due to their relationship with corresponding orbifolds (for e.g., see [2], [12], [36]). In Section 4, we show that these crossed products are all twisted CY algebras. PBW deformations of the crossed product  $A \#_{\sigma} \Bbbk G$  are the twisted Drinfeld Hecke algebras defined in [37]. If the cocycle is trivial, then  $A \# \Bbbk G$ , the skew group algebra, is just the Drinfeld Hecke algebras defined by V. Drinfeld [14]. They have been studied by many authors, for example [15], [6], [25]. Quantum Drinfeld Hecke algebras are anther generalizations of Drinfeld Hecke algebras by replacing polynomial algebras by quantum polynomial algebras [27], [31]. More generally, Naidu defined twisted quantum Drinfeld Hecke algebras in [30]. A twisted quantum Drinfeld Hecke algebra is an algebra of the form  $A \#_{\sigma} \Bbbk G$ , where A is a quantum polynomial algebra, G is a finite group acting on A, and  $\sigma$  is a 2-cocycle on G. Twisted quantum Drinfeld Hecke algebras are generalizations of both twisted Drinfeld Hecke algebras and quantum Drinfeld Hecke algebras. A quantum polynomial algebra is a Koszul algebra. If PBW deformations of the algebra  $A \#_{\sigma} H$  in Theorem 0.1 are still twisted CY algebras, then twisted quantum Drinfeld Hecke algebras will all be twisted CY algebras. We will discuss this problem in our upcoming paper.

#### 1. Preliminaries

Throughout this paper, the unadorned tensor  $\otimes$  means  $\otimes_{\Bbbk}$  and Hom means  $\operatorname{Hom}_{\Bbbk}$ .

Given an algebra A, we write  $A^{op}$  for the opposite algebra of A and  $A^e$  for the enveloping algebra  $A \otimes A^{op}$ . An A-bimodule can be identified with a left  $A^e$ -module or a right  $A^e$ -module.

For an A-bimodule M and two algebra automorphisms  $\mu$  and  $\nu$ , we let  ${}^{\mu}M^{\nu}$  denote the A-bimodule such that  ${}^{\mu}M^{\nu} \cong M$  as vector spaces, and the bimodule structure is given by

$$a \cdot m \cdot b = \mu(a)m\nu(b),$$

for all  $a, b \in A$  and  $m \in M$ . If one of the automorphisms is the identity, we will omit it. It is well-known that  $A^{\mu} \cong {}^{\mu^{-1}}A$  as A-A-bimodules.  $A^{\mu} \cong A$  as A-A-bimodules if and only if  $\mu$  is an inner automorphism.

We assume that the Hopf algebras considered in this paper have bijective antipodes. For a Hopf algebra H, we use Sweedler's (sumless) notation for the comultiplication and coaction of H.

## 1.1. Cogroupoid.

**Definition 1.1.** a *cocategory* C consists of:

- A set of objects  $ob(\mathcal{C})$ .
- For any  $X, Y \in ob(\mathcal{C})$ , an algebra  $\mathcal{C}(X, Y)$ .
- For any  $X, Y, Z \in ob(\mathcal{C})$ , algebra homomorphisms

$$\Delta_{XY}^{\mathbb{Z}}: \mathcal{C}(X,Y) \to \mathcal{C}(X,Z) \otimes \mathcal{C}(Z,Y) \text{ and } \varepsilon_X: \mathcal{C}(X,X) \to \Bbbk$$

such that for any  $X, Y, Z, T \in ob(\mathcal{C})$ , the following diagrams commute:

$$\begin{array}{cccc} \mathcal{C}(X,Y) & \xrightarrow{\Delta^{Z}_{X,Y}} & \mathcal{C}(X,Z) \otimes \mathcal{C}(Z,Y) \\ & & & & & \\ \Delta^{T}_{X,Y} & & & & & \\ \mathcal{C}(X,T) \otimes \mathcal{C}(T,Y) & \xrightarrow{1 \otimes \Delta^{Z}_{T,Y}} & \mathcal{C}(X,T) \otimes \mathcal{C}(T,Z) \otimes \mathcal{C}(Z,Y) \\ & & & & & \\ \mathcal{C}(X,Y) & & & & & \\ & & & & & \\ \mathcal{C}(X,Y) & & & & & \\ \mathcal{C}(X,Y) & & & & & \\ \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Y) & \xrightarrow{1 \otimes \varepsilon_{Y}} \mathcal{C}(X,Y) & \mathcal{C}(X,X) \otimes \mathcal{C}(X,Y) & \xrightarrow{\varepsilon_{X} \otimes 1} \mathcal{C}(X,Y). \end{array}$$

Thus a cocategory with one object is just a bialgebra.

A cocategory  $\mathcal{C}$  is said to be *connected* if  $\mathcal{C}(X, Y)$  is a non zero algebra for any  $X, Y \in ob(\mathcal{C})$ .

**Definition 1.2.** A cogroupoid C consists of a cocategory C together with, for any  $X, Y \in ob(C)$ , linear maps

$$S_{X,Y}: \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(Y,X)$$

such that for any  $X, Y \in \mathcal{C}$ , the following diagrams commute:

$$\begin{array}{c|c} \mathcal{C}(X,X) & \xrightarrow{\varepsilon_X} & \Bbbk & \xrightarrow{u} & \mathcal{C}(X,Y) \\ & \Delta_{X,X}^Y & & & m \uparrow \\ \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,X) & \xrightarrow{1 \otimes S_{Y,X}} & \mathcal{C}(X,Y) \otimes \mathcal{C}(X,Y) \\ & \mathcal{C}(X,X) & \xrightarrow{\varepsilon_X} & \Bbbk & \xrightarrow{u} & \mathcal{C}(Y,X) \\ & \Delta_{X,X}^Y & & & m \uparrow \\ \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,X) & \xrightarrow{S_{X,Y} \otimes 1} & \mathcal{C}(Y,X) \otimes \mathcal{C}(Y,X) \end{array}$$

We refer to [8] for basic properties of cogroupoids.

In this paper, we are mainly concerned with the 2-cocycle cogroupoid of a Hopf algebra.

**Definition 1.3.** Let H be a Hopf algebra. A *(right) 2-cocycle* on H is a convolution invertible linear map  $\sigma : H \otimes H \to \Bbbk$  satisfying

(1) 
$$\sigma(h_1, k_1)\sigma(h_2k_2, l) = \sigma(k_1, l_1)\sigma(h, k_2l_2)$$

(2) 
$$\sigma(h,1) = \sigma(1,h) = \varepsilon(h)$$

for all  $h, k, l \in H$ . The set of 2-cocycles on H is denoted  $Z^2(H)$ .

The convolution inverse of  $\sigma$ , denote  $\sigma^{-1}$ , satisfies

(3) 
$$\sigma^{-1}(h_1k_1, l)\sigma^{-1}(h_2, k_2) = \sigma^{-1}(h, k_1l_1)\sigma^{-1}(k_2, l_2)$$

(4) 
$$\sigma^{-1}(h,1) = \sigma^{-1}(1,h) = \varepsilon(h)$$

for all  $h, k, l \in H$ . Such a convolution invertible map is called a *left 2-cocycle* on H. Conversely, the convolution inverse of a left 2-cocycle is just a right 2-cocycle.

The set of 2-cocycles defines the 2-cocycle cogroupoid of H.

Let  $\sigma, \tau \in Z^2(H)$ . The algebra  $H(\sigma, \tau)$  is defined to be the vector space H together with the multiplication given by

(5) 
$$h \cdot k = \sigma(h_1, k_1) h_2 k_2 \tau^{-1}(h_3, k_3),$$

for any  $h, k \in H$ .

Now we recall the necessary structural maps for the 2-cocycle cogroupoid on H. For any  $\sigma, \tau, \omega \in Z^2(H)$ , define the following maps:

(6) 
$$\Delta_{\sigma,\tau}^{\omega} = \Delta : H(\sigma,\tau) \longrightarrow H(\sigma,\omega) \otimes H(\omega,\tau)$$
$$h \longmapsto h_1 \otimes h_2.$$

(7) 
$$\varepsilon_{\sigma} = \varepsilon : H(\sigma, \sigma) \longrightarrow \Bbbk.$$

(8) 
$$S_{\sigma,\tau}: H(\sigma,\tau) \longrightarrow H(\tau,\sigma)$$
$$h \longmapsto \sigma(h_1, S(h_2))S(h_3)\tau^{-1}(S(h_4), h_5).$$

It is routine to check that the inverse of  $S_{\sigma,\tau}$  is given as follows:

(9) 
$$S_{\sigma,\tau}^{-1}: H(\tau,\sigma) \longrightarrow H(\sigma,\tau)$$
  
 $h \longmapsto \sigma^{-1}(h_5, S^{-1}(h_4))S^{-1}(h_3)\tau(S^{-1}(h_2), h_1).$ 

The 2-cocycle cogroupoid of H, denoted by  $\underline{H}$ , is the cogroupoid defined as follows:

- (i)  $ob(\underline{H}) = Z^2(H)$ .
- (ii) For  $\sigma, \tau \in Z^2(H)$ , the algebra  $\underline{H}(\sigma, \tau)$  is the algebra  $H(\sigma, \tau)$  defined in (5).
- (iii) The structural maps  $\Delta_{\bullet,\bullet}^{\bullet}$ ,  $\varepsilon_{\bullet}$  and  $S_{\bullet,\bullet}$  are defined in (6), (7) and (8) respectively.

[8, Lemma 3.13] shows that the maps  $\Delta_{\bullet,\bullet}^{\bullet}$ ,  $\varepsilon_{\bullet}$  and  $S_{\bullet,\bullet}$  indeed satisfy the conditions required for a cogroupoid. It is clear that a 2-cocycle cogroupoid is connected. The following lemma follows from basis properties of cogroupoids.

**Lemma 1.4.** [8, Proposition 2.13] Let  $\underline{H}$  be the 2-cocycle cogroupoid, and let  $\sigma, \tau \in ob(\underline{H})$ .

- (i)  $S_{\sigma,\tau}: H(\sigma,\tau) \to H(\tau,\sigma)^{op}$  is an algebra homomorphism.
- (ii) For any  $\omega \in ob(\underline{H})$  and  $h \in H$ , we have

$$\Delta^{\omega}_{\tau,\sigma}(S_{\sigma,\tau}(h)) = S_{\omega,\tau}(h_1) \otimes S_{\sigma,\omega}(h_2).$$

The Hopf algebra H(1,1) (where 1 stands for  $\varepsilon \otimes \varepsilon$ ) is just the Hopf algebra Hitself. Let  $\sigma$  be a 2-cocycle. We write  ${}_{\sigma}H$  for the algebra  $H(\sigma,1)$ . Similarly, we write  $H_{\sigma^{-1}}$  for the algebra  $H(1,\sigma)$ . To make the presentation clear, we let  ${}_{\sigma}$  and  ${}_{\sigma^{-1}}$  denote the multiplications in  ${}_{\sigma}H$  and  $H_{\sigma^{-1}}$  respectively.

The Hopf algebra  $H(\sigma, \sigma)$  is just the *cocycle deformation*  $H^{\sigma}$  of H defined by Doi in [13]. The comultiplication of  $H^{\sigma}$  is the same as the comultiplication of H. However, the multiplication and the antipode are deformed:

$$h * k = \sigma(h_1, k_1)h_2k_2\sigma^{-1}(h_3, k_3),$$

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$$S_{\sigma,\sigma}(h) = \sigma(h_1, S(h_2))S(h_3)\sigma^{-1}(S(h_4), h_5)$$

for any  $h, k \in H^{\sigma}$ . In the following,  $S_{\sigma,\sigma}$  is denoted by  $S^{\sigma}$  for simplicity.

1.2. Cleft extensions. A Hopf algebra H is said to *measure* an algebra A if there is a k-linear map  $H \otimes A \to A$ , given by  $h \otimes a \mapsto h \cdot a$ , such that  $h \cdot 1 = \varepsilon(h)$ and  $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$  for all  $h \in H$ ,  $a, b \in A$ .

**Definition 1.5.** Let H be a Hopf algebra and A an algebra. Assume that H measures A and that  $\sigma$  is an invertible map in Hom $(H \otimes H, A)$ . The crossed product  $A \#_{\sigma} H$  of A with H is defined on the vector space  $A \otimes H$  with multiplication given by

$$(a\#h)(b\#k) = a(h_1 \cdot b)\sigma(h_2, k_1)\#h_3k_2$$

for all  $h, k \in H$ ,  $a, b \in A$ . Here we write a # h for the tensor product  $a \otimes h$ .

The following lemma is well-known (cf. [29, Lemma 7.1.2]).

**Lemma 1.6.**  $A \#_{\sigma} H$  is an associative algebra with identity element 1 # 1 if and only if the following two conditions are satisfied:

(i) A is a twisted H-module. That is,  $1 \cdot a = a$  for all  $a \in A$ , and

$$h \cdot (k \cdot a) = \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3),$$

for all  $h, k \in H$ ,  $a \in A$ .

(ii)  $\sigma$  is a cocycle. That is,  $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1$  for all  $h \in H$  and

$$[h_1 \cdot \sigma(k_1, m_1)]\sigma(h_2, k_2 m_2) = \sigma(h_1, k_1)\sigma(h_2 k_2, m)$$

for all  $h, k, m \in H$ .

Note that if  $\sigma$  is trivial, that is,  $\sigma(h,k) = \varepsilon(h)\varepsilon(k)1$ , for all  $h,k \in H$ . Then the crossed product  $A \#_{\sigma} H$  is just the smash product A # H.

**Remark 1.7.** Let  $A \#_{\sigma} H$  be a crossed product and  $\sigma$  an invertible map in  $\operatorname{Hom}(H \otimes H, \mathbb{k})$ . Then  $A \#_{\sigma} H$  is an associative algebra if and only if  $\sigma$  is a 2-cocycle and A is an  $H^{\sigma}$ -module algebra.

**Definition 1.8.** Let  $A \subseteq B$  be an extension of algebras, and H a Hopf algebra.

- (i)  $A \subseteq B$  is called a *(right) H*-extension if *B* is a right *H*-comodule algebra such that  $B^{coH} = A$ .
- (ii) The *H*-extension  $A \subseteq B$  is said to be *H*-cleft if there exists a right *H*-comodule morphism  $\gamma : H \to B$  which is (convolution) invertible. Note that this  $\gamma$  can be chosen such that  $\gamma(1) = 1$ .

If  $\Bbbk \subseteq B$  is *H*-cleft, then *B* is called a *(right) cleft object*. Left cleft extensions and left cleft objects can be defined similarly.

**Lemma 1.9.** [29, Theorem 7.2.2, Proposition 7.2.3, Proposition 7.2.7] Let Hbe a Hopf algebra. An H-extension  $A \subseteq B$  is H-cleft with right convolution invertible H-comodule morphism  $\gamma : H \to B$  if and only if  $B \cong A \#_{\sigma} H$  as algebras with a convolution invertible map  $\sigma : H \otimes H \to A$ . The twisted Hmodule action on A is given by

$$h \cdot a = \gamma(h_1) a \gamma^{-1}(h_2),$$

for all  $a \in A$ ,  $h \in H$ . Moreover,  $\gamma$  and  $\sigma$  are constructed each other by

$$\sigma(h,k) = \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2)$$

and

$$\gamma(h) = 1 \# h, \quad \gamma^{-1}(h) = \sigma^{-1}(Sh_2, h_3) \# Sh_1$$

for all  $h, k \in H$ ,  $a \in A$ .

From this lemma, we see that right cleft objects of a Hopf algebra H are just the algebras  $_{\sigma}H$ , where  $\sigma$  is a 2-cocycle on H.

1.3. **AS-Gorenstein algebras.** In this paper, unless otherwise stated, a graded algebra will always mean an N-graded algebra. An N-graded algebra  $A = \bigoplus_{i \ge 0} A_i$  is called connected if  $A_0 = \Bbbk$ .

**Definition 1.10.** A connected graded algebra A is called *AS-Gorenstein* if the following conditions hold:

- (i) A has finite injective dimension d on both sides,
- (ii)  $\operatorname{Ext}_{A}^{i}({}_{A}\mathbb{k}, {}_{A}A) \cong \begin{cases} 0, & i \neq d; \\ \mathbb{k}(l), & i = d, \end{cases}$  where *l* is an integer,
- (iii) The right version of (ii) holds.

If, in addition,

(iv) A is of finite global dimension d, then A is called AS-regular.

Noe that an AS-Gorenstein (regular) algebra can be defined on an augmented algebra in general, see [10]. For an algebra A, if the injective dimension of  $_AA$  and  $A_A$  are both finite, then these two integers are equal by [40, Lemma A]. We call this common value the *injective dimension* of A. The left global dimension and the right global dimension of a Noetherian algebra are equal. When the global dimension is finite, then it is equal to the injective dimension.

**Definition 1.11.** (cf. [10, defn. 1.2]). Let A be a Noetherian algebra with a fixed augmentation map  $\varepsilon : A \to \mathbb{k}$ .

- (i) The algebra A is said to be AS-Gorenstein, if (a) injdim  $_{A}A = d < \infty$ , (b) dim  $\operatorname{Ext}_{A}^{i}(_{A}\mathbb{k}, _{A}A) = \begin{cases} 0, & i \neq d; \\ 1, & i = d, \\ (c) \text{ the right versions of (a) and (b) hold.} \end{cases}$
- (ii) If, in addition, the global dimension of A is finite, then A is called AS-regular.

The concept of a homological integral for an AS-Gorenstein Hopf algebra was introduced by Lu, Wu and Zhang in [24] to study infinite dimensional Noetherian Hopf algebras. It is a generalization of the concept of an integral of a finite dimensional Hopf algebra. It turns out that homological integrals are useful in describing homological properties of Hopf algebras (see e.g. [18, Theorem 2.3]).

**Definition 1.12.** Let A be an AS-Gorenstein algebra with injective dimension d. Then  $\operatorname{Ext}_{A}^{d}({}_{A}\mathbb{k}, {}_{A}A)$  is a 1-dimensional right A-module. Any nonzero element in  $\operatorname{Ext}_{A}^{d}({}_{A}\mathbb{k}, {}_{A}A)$  is called a *left homological integral* of A. We write  $\int_{A}^{l}$  for  $\operatorname{Ext}_{A}^{d}({}_{A}\mathbb{k}, {}_{A}A)$ . Similarly,  $\operatorname{Ext}_{A}^{d}({}_{K}A, {}_{A}A)$  is a 1-dimensional left A-module. Any nonzero element in  $\operatorname{Ext}_{A}^{d}({}_{K}A, {}_{A}A)$  is called a *right homological integral* of A. Write  $\int_{A}^{r}$  for  $\operatorname{Ext}_{A}^{d}({}_{K}A, {}_{A}A)$ .

 $\int_A^l$  and  $\int_A^r$  are called *left and right homological integral modules* of A respectively.

The left integral module  $\int_A^l$  is a 1-dimensional right *A*-module. Thus  $\int_A^l \cong \Bbbk_{\xi}$  for some algebra homomorphism  $\xi : A \to \Bbbk$ . Similarly,  $\int_A^r \cong {}_{\eta}\Bbbk$  for some algebra homomorphism  $\eta$ .

1.4. *N*-Koszul algebras. Let *V* be a finite dimensional vector space, and  $T(V) = \mathbb{k} \otimes V \otimes V^{\otimes 2} \otimes \cdots$  be the tensor algebra with the usual grading. A graded algebra  $T(V)/\langle R \rangle$  is called *N*-homogenous if *R* is a subspace of  $V^{\otimes N}$ . Let  $V^*$  be the dual space Hom $(V, \mathbb{k})$ . The algebra  $A^! = T(V^*)/\langle R^{\perp} \rangle$  is called the homogeneous dual of *A*, where  $R^{\perp}$  is the orthogonal subspace of *R* in  $(V^*)^{\otimes N}$ .

**Remark 1.13.** Let  $\phi$  be the map defined as follows:

$$\phi: \qquad (V^*)^{\otimes n} \quad \to \quad (V^{\otimes n})^* \\ f_n \otimes f_{n-1} \otimes \cdots \otimes f_1 \quad \mapsto \quad \phi(f_n \otimes f_{n-1} \otimes \cdots \otimes f_1),$$

where  $\phi(f_n \otimes f_{n-1} \otimes \cdots \otimes f_1)(x_1 \otimes \cdots x_{n-1} \otimes x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$ , for any  $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in V^{\otimes n}$ . This map  $\phi$  is a bijection. Throughout, we identify  $(V^*)^{\otimes n}$  with  $(V^{\otimes n})^*$  via this bijection.

Let  $\mathbf{n}: \mathbb{N} \to \mathbb{N}$  be the function defined by

$$\mathbf{n}(i) = \begin{cases} Nk, & i = 2k \\ Nk+1, & i = 2k+1. \end{cases}$$

An N-homogenous algebra A is called N-Koszul if the trivial module  $_A$ k admits a graded projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to {}_A \Bbbk \to 0$$

such that  $P_i$  is generated in degree  $\mathbf{n}(i)$  for all  $i \ge 0$ . A Koszul algebra is a just 2-Koszul algebra.

The Koszul bimodule complex of a Koszul algebra is constructed by Van den Bergh in [35]. This complex was generalized to N-Koszul case in [7]. Now let  $A = T(V)/\langle R \rangle$  be an N-Koszul algebra. Let  $\{e_i\}_{i=1,2,\dots,n}$  be a basis of V and  $\{e_i^*\}_{i=1,2,\dots,n}$  the dual basis. Define two N-differentials

$$d_l, d_r: A \otimes (A_p^!)^* \otimes A \to A \otimes (A_{p-1}^!)^* \otimes A$$

as follows:

$$d_l(x \otimes \omega \otimes y) = \sum_{i=1}^n x e_i \otimes e_i^* \cdot \omega \otimes y$$
$$d_r(x \otimes \omega \otimes y) = \sum_{i=1}^n x \otimes \omega \cdot e_i^* \otimes e_i y,$$

for  $x \otimes \omega \otimes y \in A \otimes (A_p^!)^* \otimes A$ . The left action  $e_i^* \cdot \omega$  is defined by  $[e_i^* \cdot \omega](\alpha) = \omega(\alpha e_i^*)$  for any  $\alpha \in (A_{p-1}^!)^*$ . The right action  $\omega \cdot e_i^*$  is defined similarly. One can check that  $d_l$  and  $d_r$  commute. Fix a primitive N-th root of unity q. Define  $d : A \otimes (A_p^!)^* \otimes A \to A \otimes (A_{p-1}^!)^* \otimes A$  by  $d = d_l - q^{p-1}d_r$ . We obtain the following N-complex:

$$\mathbf{K}_{\mathbf{l}-\mathbf{r}}(\mathbf{A}):\cdots\xrightarrow{d_l-d_r}A\otimes (A_N^!)^*\otimes A\xrightarrow{d_l-q^{N-1}d_r}\cdots\xrightarrow{d_l-qd_r}A\otimes V\otimes A\xrightarrow{d_l-d_r}A\otimes A\to 0.$$

The bimodule Koszul complex  $\mathbf{K}_{\mathbf{b}}(\mathbf{A})$  is a contraction of  $\mathbf{K}_{\mathbf{l}-\mathbf{r}}(\mathbf{A})$ . It is obtained by keeping the arrow  $A \otimes V \otimes A \xrightarrow{d_l - d_r} A \otimes A$  at the far right, then putting together the N - 1 consecutive ones, and continuing alternately:

$$\mathbf{K}_{\mathbf{b}}(\mathbf{A}): \cdots \xrightarrow{d^{N-1}} A \otimes (A_{N+1}^{!})^{*} \otimes A \xrightarrow{d} A \otimes (A_{N}^{!})^{*} \otimes A \xrightarrow{d^{N-1}} A \otimes V \otimes A \xrightarrow{d} A \otimes A \to 0.$$
  
Here  $d = d_{l} - d_{r}$  and  $d^{N-1} = d_{l}^{N-1} + d_{l}^{N-2} d_{r} + \cdots + d_{l} d_{r}^{N-2} + d_{r}^{N-1}.$ 

An *N*-homogenous algebra is *N*-Koszul if and only if the complex  $\mathbf{K}_{\mathbf{b}}(\mathbf{A}) \rightarrow A \rightarrow 0$  is exact via the multiplication  $A \otimes A \rightarrow A$  [7, Theorem 4.4]. Moreover, in such a case,  $\mathbf{K}_{\mathbf{b}}(\mathbf{A}) \rightarrow A \rightarrow 0$  is a minimal bimodule free resolution of A.

#### 1.5. Calabi-Yau algebras.

**Definition 1.14.** An algebra A is called a *twisted Calabi-Yau algebra of dimension d* if

- (i) A is homologically smooth, that is, A has a bounded resolution of finitely generated projective A<sup>e</sup>-modules;
- (ii) There is an automorphism  $\mu$  of A such that

(10) 
$$\operatorname{Ext}_{A^{e}}^{i}(A, A^{e}) \cong \begin{cases} 0, & i \neq d \\ A^{\mu}, & i = d \end{cases}$$

as  $A^e$ -modules.

If such an automorphism  $\mu$  exists, it is unique up to an inner automorphism and is called the *Nakayama automorphism* of *A*. A *Calabi-Yau algebra* is a twisted Calabi-Yau algebra whose Nakayama automorphism is an inner automorphism.

A Graded twisted CY algebra can be defined in a similar way. That is, we should consider the category of graded modules and condition (10) should be replaced by

$$\operatorname{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A_{\mu}(l), & i = d, \end{cases}$$

where l is an integer and  $A_{\mu}(l)$  is the shift of  $A_{\mu}$  by degree l.

We end this section with the following lemma, which shows that AS-regular Hopf algebras are just twisted CY Hopf algebras.

**Lemma 1.15.** Let A be a Noetherian AS-regular Hopf algebra with  $\int_{A}^{l} = \mathbb{k}_{\xi}$ , where  $\xi : A \to \mathbb{k}$  is an algebra homomorphism. The followings hold:

- (i) [32, Lemma 1.3] The algebra A is twisted CY with Nakayama automorphism μ defined by μ(x) = S<sup>-2</sup>(x<sub>1</sub>)ξ(x<sub>2</sub>) for any x ∈ A. (Alternatively, the algebra automorphism ν defined by ν(x) = ξ(x<sub>1</sub>)S<sup>2</sup>(x<sub>2</sub>) is also a Nakayama automorphism of A).
- (ii) [18, Theorem 2.3] The algebra A is CY if and only if  $\xi = \varepsilon$ , and  $S^2$  is an inner automorphism.

## 2. The CY property of Cleft extension

Let H be a Hopf algebra,  $\sigma$  a 2-cocycle on H and A an N-Koszul  $H^{\sigma}$ -module algebra. Then the crossed product  $A \#_{\sigma} H$  is an associative algebra. In this section we show that  $A \#_{\sigma} H$  is a twisted CY algebra if both A and H are

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twisted CY algebras. This generalizes [23, Theorem 2.12] and [32, Theorem 0.2],

The following definition is inspired by " $H_{S^i}$ -equivariant A-bimodule" introduced in [32, Definition 2.2], where H is a Hopf algebra and i is an even integer.

**Definition 2.1.** Let H be a Hopf algebra and A a left H-module algebra. For a given even integer i, we define an algebra  $A^e \rtimes_{S^i} H$ . As vector spaces,  $A^e \rtimes_{S^i} H = A \otimes A \otimes H$ . The multiplication is given by

$$(a \otimes b \otimes g)(a' \otimes b' \otimes h) = a(S^{i}g_{1} \cdot a') \otimes (g_{3} \cdot b')b \otimes g_{2}h,$$

for any  $a \otimes b \otimes g, a' \otimes b' \otimes h \in A \otimes A \otimes H$ .

- **Remark 2.2.** (i) When i = 0,  $A^e \rtimes_{S^i} H$  is just the algebra  $A^e \rtimes H$  introduced by Kaygun [21].
  - (ii) An  $A^e \rtimes_{S^i} H$ -module M is a vector space such that it is both an  $A^e$ module and an H-module satisfying

(11) 
$$h \cdot (amb) = ((S^{i}h_{1}) \cdot a)(h_{2} \cdot m)(h_{3} \cdot b),$$

for any  $h \in H$ ,  $a, b \in A$  and  $m \in M$ .

**Lemma 2.3.** Let M be an  $A^e \rtimes_{S^i} H$ -module and N an  $(A \# H)^e$ -module.

 (i) The space Hom<sub>A<sup>e</sup></sub>(M, N) is a left H-module with the H-action defined by

(12) 
$$(h \to f)(m) = (S^i h_3) f[(S^{-1} h_2) \cdot m](S^{-1} h_1)$$

for any  $h \in H$ ,  $f \in \operatorname{Hom}_{A^e}(M, N)$  and  $m \in M$ .

(ii) The space  $M \otimes_{A^e} N$  is a left H-module with the H-action given by

(13) 
$$h \cdot (m \otimes n) = h_2 \cdot m \otimes h_3 n(S^{i+1}h_1)$$

for any  $h \in H$  and  $m \otimes n \in M \otimes N$ .

*Proof.* The proof is routine and quite similar to the proofs of Lemma 1.8 and Lemma 1.9 in [23].

**Remark 2.4.** Keep the notations as in Lemma 2.3,  $\operatorname{Hom}_{A^e}(M, N)$  can be made into a right *H*-module by defining  $f \leftarrow h = Sh \rightharpoonup f$  for any  $h \in H$  and  $f \in \operatorname{Hom}_{A^e}(M, N)$ . That is,

(14) 
$$(f \leftarrow h)(m) = S^{i+1}h_1f(h_2 \cdot m)h_3.$$

Since A is a left H-module algebra, the algebra  $A^e$  is an  $(A#H)^e$ -module with the following module structure:

(15) 
$$(a\#h) \cdot (x \otimes y) = a(h \cdot x) \otimes y, \ (x \otimes y) \cdot (b\#g) = x \otimes (S^{-1}g) \cdot (yb)$$

for any  $x \otimes y \in A^e$  and a # h,  $b \# g \in A \# H$ .

By Lemma 2.3,  $\operatorname{Hom}_{A^e}(M, A^e)$  is a left *H*-module for any  $A^e \rtimes H$ -module M. Furthermore, the  $A^e$ -bimodule structure of  $A^e$  induces a left  $A^e$ -module structure on  $\operatorname{Hom}_{A^e}(M, A^e)$ . That is,

(16) 
$$[(a \otimes b) \cdot f](x) = f(x)(b \otimes a),$$

for any  $a \otimes b \in A^e$ ,  $f \in \operatorname{Hom}_{A^e}(M, A^e)$  and  $x \in M$ .

In [23] the authors showed that if H is involutory, then  $\operatorname{Hom}_{A^e}(M, A^e)$  is again an  $A^e \rtimes H$ -module for any  $A^e \rtimes H$ -module M. In general, we have the following.

**Lemma 2.5.** Let M be an  $A^e \rtimes H$ -module. Then  $\operatorname{Hom}_{A^e}(M, A^e)$  is an  $A^e \rtimes_{S^{-2}} H$ -module.

In [23, Theorem 2.4] the Van den Bergh duality was generalized to algebras with a Hopf action from an involutory Hopf algebra. In fact, we can drop the condition "involutory".

**Proposition 2.6.** Let H be a Hopf algebra and A a left H-module algebra. Assume that A admits a finitely generated  $A^e$ -projective resolution of finite length such that it is a complex of  $A^e \rtimes H$ -modules. Suppose there exists an integer d such that

$$\operatorname{Ext}_{A^e}^i(A, A^e) = \begin{cases} 0, & i \neq d; \\ U, & i = d, \end{cases}$$

where U is an invertible  $A^e$ -module. Then for any  $(A \# H)^e$ -module N, we have

$$\operatorname{HH}^{i}(A, N) \cong {}_{S^{-2}}\operatorname{HH}_{d-i}(A, U \otimes_{A} N)$$

as left H-modules.

Proof. Suppose that P is an  $A^e \rtimes H$ -module such that it is finitely generated and projective as an  $A^e$ -module, and N is an  $(A\#H)^e$ -module. By Lemma 2.5,  $\operatorname{Hom}_{A^e}(P, A^e)$  is an  $A^e \rtimes_{S^{-2}} H$ -module. So  $\operatorname{Hom}_{A^e}(P, A^e) \otimes_{A^e} N$  is an H-module with the module structure given by (13). Moreover, the equation (12) defines an H-module structure on  $\operatorname{Hom}_{A^e}(P, N)$ . With these H-actions, one can check that the canonical isomorphism

$$\Psi : \operatorname{Hom}_{A^e}(P, A^e) \otimes_{A^e} N \to \operatorname{Hom}_{A^e}(P, N)$$

is also an *H*-isomorphism. Therefore, the proof of [23, Theorem 2.4] works for non-involutory Hopf algebras. But for a non-involutory Hopf algebra *H*, the module *U* is an  $A^e \rtimes_{S^{-2}} H$ -module by Lemma 2.5. Thus,  $U \otimes_A N$  is an  $(A \# H)^e$ -module with module structure defined by

$$(17) \ (a\#h) \cdot (u \otimes n) = a((S^2h_1) \cdot u) \otimes (S^2h_2) \cdot n, \ (u \otimes n) \cdot (b\#g) = u \otimes n \cdot (b\#g),$$

for any a # h,  $b \# g \in A \# H$  and  $u \# n \in U \otimes N$ . Consequently, we have the following *H*-isomorphisms:

$$\begin{aligned} \operatorname{HH}^{i}(A,N) &\cong \operatorname{Ext}_{A^{e}}^{i}(A,N) \\ &\cong \operatorname{H}^{i}(\operatorname{RHom}_{A^{e}}(A,N)) \\ &\cong \operatorname{H}^{i}(\operatorname{RHom}_{A^{e}}(A,A^{e})^{L} \otimes_{A^{e}} N) \\ &\cong \operatorname{H}^{i}(U[-d]^{L} \otimes_{A^{e}} N) \\ &\cong \operatorname{H}^{i-d}(U^{L} \otimes_{A^{e}} N) \\ &\cong \operatorname{H}^{i-d}(_{S^{-2}}[A \otimes_{A^{e}} (U^{L} \otimes_{A} N)]) \\ &\cong \operatorname{S}^{-2} \operatorname{HH}_{d-i}(A,U \otimes_{A} N). \end{aligned}$$

In the rest of this section, we work with the category of graded modules. Let A be a graded algebra, and let A-GrMod denote the category of graded left A-modules and graded homomorphisms of degree zero. For any  $M, N \in A$ -GrMod,  $\operatorname{Hom}_A(M, N)$  is the graded vector space consisting of graded A-module homomorphisms. That is,

$$\operatorname{Hom}_{A}(M, N) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{A\operatorname{-GrMod}}(M, N(i)).$$

Let H be a Hopf algebra. We say that a graded algebra A is a left graded H-module algebra if it is a left H-module algebra such that each  $A_i$  is an H-module. Let  $\sigma$  is a 2-cocycle on H. The cocycle deformation  $H^{\sigma}$  is a Hopf algebra. If A is a left graded  $H^{\sigma}$ -module algebra, then we have the algebra  $A \# H^{\sigma}$ . Moreover, we can construct the algebra  $A \#_{\sigma} H$  by Remark 1.7. It is easy to see that both  $A \# H^{\sigma}$  and  $A \#_{\sigma} H$  have natural graded algebra structures.

Now, we fix a Hopf algebra H and a 2-cocycle  $\sigma$  on H. Let V be a left  $H^{\sigma}$ -module and  $A = T(V)/\langle R \rangle$  an N-Koszul graded  $H^{\sigma}$ -module algebra. The dual  $V^*$  is a right  $H^{\sigma}$ -module with the module structure given by

(18)  $(\alpha \lhd h)(x) = \alpha(h \cdot x).$ 

for  $\alpha \in V^*$ ,  $h \in H$  and  $x \in V$ .

**Remark 2.7.** Let  $\{e_1, e_2, \dots, e_n\}$  be a basis of V. Suppose that  $h \cdot e_i = \sum_{j=1}^n c_{ji}^h e_j$  with  $c_{ji}^h \in \mathbb{k}$ . Then we have  $e_i^* \triangleleft h = \sum_{j=1}^n c_{jj}^h e_j^*$ .

We extend the action " $\triangleleft$ " on  $V^*$  to  $(V^*)^{\otimes n}$ :

 $(\alpha_n \otimes \alpha_{n-1} \otimes \cdots \otimes \alpha_1) \triangleleft h = (\alpha_n \triangleleft h_n) \otimes (\alpha_{n-1} \triangleleft h_{n-1}) \otimes \cdots \otimes (\alpha_1 \triangleleft h_1).$ 

It is easy to check that  $R^{\perp} \triangleleft h \subseteq R^{\perp}$ . Consequently,  $A^{!}$  is a right  $H^{\sigma}$ -module algebra with the action " $\triangleleft$ ". In fact, one can make  $A^{!}$  into a left  $H^{\sigma}$ -module algebra as follows:

(19) 
$$h \cdot \beta = \beta \lhd (S^{\sigma^{-1}}h),$$

for any  $\beta \in A^!$  and  $h \in H$ .

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Thanks to Lemma 2.5, we obtain the following proposition generalizing [23, Proposition 2.2].

**Proposition 2.8.** Let H be a Hopf algebra,  $\sigma$  a 2-cocycle on H, and A a left graded  $H^{\sigma}$ -module algebra. If A is an N-Koszul graded twisted CY algebra of dimension d with Nakayama automorphism  $\mu$ , then as  $A^e \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ -modules

$$\operatorname{Ext}_{A^{e}}^{i}(A, A^{e}) \cong \begin{cases} 0, & i \neq d; \\ A_{\mu} \otimes A_{\mathbf{n}(d)}^{!}, & i = d, \end{cases}$$

where the  $A^e \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ -module structure on  $A_{\mu} \otimes A^!_{\mathbf{n}(d)}$  is given by

(20) 
$$(a \otimes b \otimes h)(x \otimes \alpha) = a((S^{\sigma^{-2}}h_1) \cdot x)\mu(b) \otimes h_2 \cdot \alpha,$$

for any  $a \otimes b \otimes h \in A^e \rtimes_{S^{-2}} H$  and  $x \otimes \alpha \in A \otimes A^!_{\mathbf{n}(d)}$ .

Proof. The algebra  $H^{\sigma}$  is a Hopf algebra and the algebra A is a left  $H^{\sigma}$ -module algebra. Proposition 2.1 in [23] shows that the  $A^{e}$ -projective resolution  $\mathbf{K}_{\mathbf{b}}(\mathbf{A}) \to A \to 0$  of A is an  $A^{e} \rtimes H^{\sigma}$ -module complex. The  $A^{e} \rtimes H^{\sigma}$ -module structure is defined as follows. Each term in  $\mathbf{K}_{\mathbf{b}}(\mathbf{A})$  is of the form  $A \otimes (A_{p}^{!})^{*} \otimes A$ . Since  $A_{p}^{!}$  is a right  $H^{\sigma}$ -module with the action " $\triangleleft$ " defined in (18),  $(A_{p}^{!})^{*}$  is a natural left  $H^{\sigma}$ -module. That is,

$$(h \cdot \omega)(x) = \omega(x \lhd h),$$

for any  $h \in H^{\sigma}$ ,  $\omega \in (A_p^!)^*$  and  $x \in A_p^!$ . Each  $A \otimes (A_p^!)^* \otimes A$  is an  $A^e \rtimes H^{\sigma}$ module with the module structure defined by

(21) 
$$(a \otimes b \otimes h) \cdot (x \otimes \omega \otimes y) = a(h_1 \cdot x) \otimes h_2 \cdot \omega \otimes (h_3 \cdot y)b,$$

where  $a \otimes b \otimes h \in A^e \rtimes H$  and  $x \otimes \omega \otimes y \in A \otimes (A_p^!)^* \otimes A$ .

Now we recall another bimodule complex constructed in [7]. First, we define two N-differentials:

$$\delta_l, \delta_r: A \otimes A_p^! \otimes A \to A \otimes A_{p+1}^! \otimes A$$

as follows:

$$\delta_l(x \otimes \alpha \otimes y) = \sum_{i=1}^n x e_i \otimes e_i^* \alpha \otimes y$$
, and  $\delta_r(x \otimes \alpha \otimes y) = \sum_{i=1}^n x \otimes \alpha e_i^* \otimes e_i y$ ,

for  $x \otimes \alpha \otimes y \in A \otimes A_p^! \otimes A$ . It is easy to check that  $\delta_l$  and  $\delta_r$  commute. Fix a primitive *N*-th root of unity *q*. The complex

$$\mathbf{L}_{\mathbf{l}-\mathbf{r}}(\mathbf{A}): A \otimes A \xrightarrow{\delta_r - \delta_l} A \otimes V^* \otimes A \xrightarrow{\delta_r - q\delta_l} \cdots \xrightarrow{\delta_r - q^{N-1}\delta_l} A \otimes A_N^! \otimes A \xrightarrow{\delta_r - \delta_l} \cdots$$

is an *N*-complex. The complex  $\mathbf{L}_{\mathbf{b}}(\mathbf{A})$  is the contraction of  $\mathbf{L}_{\mathbf{l}-\mathbf{r}}(\mathbf{A})$ . It is obtained by keeping the arrow  $A \otimes A \xrightarrow{\delta_r - \delta_l} A \otimes V^* \otimes A$  at the far left, then putting together the N - 1 following ones, and continuing alternately:

$$\mathbf{L}_{\mathbf{b}}(\mathbf{A}): A \otimes A \xrightarrow{\delta} A \otimes V^* \otimes A \xrightarrow{\delta^{N-1}} A \otimes A_N^! \otimes A \xrightarrow{\delta} A \otimes A_{N+1}^! \otimes A \xrightarrow{\delta^{N-1}} \cdots,$$

where  $\delta = \delta_r - \delta_l$  and  $\delta^{N-1} = \delta_r^{N-1} + \delta_r^{N-2}\delta_l + \cdots + \delta_r\delta_l^{N-2} + \delta_l^{N-1}$ . When the Hopf algebra  $H^{\sigma}$  is involutory, Proposition 2.2 in [23] shows that the complex Hom<sub>A<sup>e</sup></sub>(**K**<sub>b</sub>(**A**),  $A^e$ ) and the complex **L**<sub>b</sub>(**A**) are isomorphic as  $A^e \rtimes H^{\sigma}$ complexes.

When  $H^{\sigma}$  is not involutory,  $\operatorname{Hom}_{A^{e}}(\mathbf{K}_{\mathbf{b}}(\mathbf{A}), A^{e})$  is a complex of  $A^{e} \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ modules by Lemma 2.5. In this case,  $\operatorname{Hom}_{A^{e}}(\mathbf{K}_{\mathbf{b}}(\mathbf{A}), A^{e})$  and  $\mathbf{L}_{\mathbf{b}}(\mathbf{A})$  are isomorphic as  $A^{e} \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ -module complexes. The  $A^{e} \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ -module structure of each term  $A \otimes A_{p}^{!} \otimes A$  in  $\mathbf{L}_{\mathbf{b}}(\mathbf{A})$  is given by

$$(a \otimes b \otimes h) \cdot (x \otimes \alpha \otimes y) = a((S^{\sigma^{-2}}h_1) \cdot x) \otimes h_2 \cdot \alpha \otimes (h_3 \cdot y)b,$$

for any  $a \otimes b \otimes h \in A^e \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$  and  $x \otimes \alpha \otimes y \in A \otimes A_p^! \otimes A$ .

Now we can use the complex  $\mathbf{L}_{\mathbf{b}}(\mathbf{A})$  to compute  $\operatorname{Ext}_{A^e}^*(A, A^e)$ . The method is the same as the one in the proof of Proposition 2.2 in [23].

Since the algebra A is an N-Koszul graded twisted CY algebra, A is AS-regular (see [32, Lemma 1.2]). The Ext algebra E(A) of A is graded Frobenius by Corollary 5.12 in [7]. Thus, there exists an automorphism  $\phi$  of E(A), such that

$$E(A)_{\phi} \cong E(A)^*(-d)$$

as E(A)-bimodules.

Let  $\{e_1, e_2, \dots, e_n\}$  be a basis of  $A_1 = V$ , and  $\{e_1^*, e_2^*, \dots, e_n^*\}$  the corresponding dual basis. Suppose that  $\phi$  is given by

$$\phi(e_1^*, e_2^*, \cdots, e_n^*) = (e_1^*, e_2^*, \cdots, e_n^*)Q,$$

for some invertible matrix Q. Define an automorphism of A via

$$\varphi(e_1, e_2, \cdots, e_n) = (e_1, e_2, \cdots, e_n)Q^T,$$

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where  $Q^T$  is the transpose of Q. It is obvious that the restriction of  $\phi$  to  $V^*$ and the restriction of  $\varphi$  to V are dual to each other.

Let  $\epsilon$  be the automorphism of A defined by  $\epsilon(a) = (-1)^i a$  for any homogeneous element  $a \in A_i$ . By assumption, we have  $\operatorname{Ext}_{A^e}^i(A, A^e) = 0$  for  $i \neq d$ . Now we compute  $\operatorname{Ext}_{A^e}^d(A, A^e)$ . Suppose  $N \geq 3$ . Then the dimension d must be odd. We consider the following sequence

(22) 
$$A \otimes A^!_{\mathbf{n}(d)-1} \otimes A \xrightarrow{\delta} A \otimes A^!_{\mathbf{n}(d)} \otimes A \xrightarrow{u} A_{\mu} \otimes A^!_{\mathbf{n}(d)} \to 0,$$

where  $\mu = \epsilon^{d+1}\varphi$  and the morphism u is given by  $u(x \otimes \alpha \otimes y) = x\mu(y) \otimes \alpha$ , for any  $x \otimes \alpha \otimes y \in A \otimes A^!_{\mathbf{n}(d)} \otimes A$ . Since E(A) is Frobenius with Nakayama automorphism  $\phi$ , by [7, Proposition 3.1], we have  $e^*_i \alpha = \alpha \phi(e^*_i)$ , for any  $\alpha \in A^!_{\mathbf{n}(d)-1}$ . Now for any  $x \otimes \alpha \otimes y \in A \otimes A^!_{\mathbf{n}(d)-1} \otimes A$ , we have:

$$\begin{split} u\delta(x\otimes\alpha\otimes y) &= u(\sum_{i=1}^{n}x\otimes\alpha e_{i}^{*}\otimes e_{i}y-\sum_{i=1}^{n}xe_{i}\otimes e_{i}^{*}\alpha\otimes y)\\ &= \sum_{i=1}^{n}x\mu(e_{i}y)\otimes\alpha e_{i}^{*}-\sum_{i=1}^{n}xe_{i}\mu(y)\otimes e_{i}^{*}\alpha\\ &= \sum_{i=1}^{n}x\mu(e_{i})\mu(y)\otimes\alpha e_{i}^{*}-\sum_{i=1}^{n}xe_{i}\mu(y)\otimes\alpha\phi(e_{i}^{*})\\ &= \sum_{i=1}^{n}x\mu(e_{i})\mu(y)\otimes\alpha e_{i}^{*}-\sum_{i=1}^{n}xe_{i}\mu(y)\otimes\alpha(\sum_{j=1}^{n}q_{ji}e_{j}^{*})+\\ &= \sum_{i=1}^{n}x\mu(e_{i})\mu(y)\otimes\alpha e_{i}^{*}-\sum_{i=1}^{n}\sum_{j=1}^{n}q_{ji}xe_{i}\mu(y)\otimes\alpha e_{j}^{*}\\ &= \sum_{i=1}^{n}x\mu(e_{i})\mu(y)\otimes\alpha e_{i}^{*}-\sum_{i=1}^{n}x\varphi(e_{i})\mu(y)\otimes\alpha e_{i}^{*}\\ &= \sum_{i=1}^{n}(-1)^{d+1}x\varphi(e_{i})\mu(y)\otimes\alpha e_{i}^{*}-\sum_{i=1}^{n}x\varphi(e_{i})\mu(y)\otimes\alpha e_{i}^{*}\\ &= 0. \end{split}$$

Therefore, the sequence (22) is a complex. Hence, it is exact by [7, Proposition 4.1].

Similar to the proof of [23, Prop 2.2], we can show that (20) defines an  $A^e \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ -module structure on  $A \otimes A^!_d$  and u is an  $A^e \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ -homomorphism. Therefore,  $\operatorname{Ext}^d_{A^e}(A, A^e) \cong A_\mu \otimes A^!_{\mathbf{n}(d)}$  as  $A^e \rtimes_{S^{\sigma^{-2}}} H^{\sigma}$ -modules.

For the case N = 2, the proof is similar.

Let H be a Hopf algebra,  $\sigma$  a 2-cocycle on H, and A a graded  $H^{\sigma}$ -module algebra. Let P be an  $A^e \rtimes H^{\sigma}$ -module. Hom<sub> $A^e$ </sub> $(P, A^e)$  is a right  $H^{\sigma}$ -module as defined in (14). Then we can define a right H-module structure on Hom<sub> $A^e$ </sub> $(P, A^e) \otimes_{\sigma} H \otimes_{\sigma} H$ :

(23) 
$$(f \otimes k \otimes l) \leftarrow h = f \leftarrow h_2 \otimes (S_{1,\sigma}h_1) \bullet_{\sigma} k \otimes l \bullet_{\sigma} h_3$$

for all  $f \otimes k \otimes l \in \operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes_{\sigma} H$  and  $h \in H$ . Recall that H can be viewed as the algebra H(1, 1). Here  $h_1 \otimes h_2 \otimes h_3 = (\Delta_{1,\sigma}^{\sigma} \otimes \operatorname{id}) \Delta_{1,1}^{\sigma}(h)$ . Both  $\Delta_{1,\sigma}^{\sigma}$  and  $\Delta_{1,1}^{\sigma}$  are algebra homomorphisms. So this H-module is well-defined. We denote this H-module by  $\operatorname{Hom}_{A^e}(P, A^e)_* \otimes_{*\sigma} H \otimes_{\sigma} H_*$ . The right *H*-module structure of *H* induces a natural *H*-module structure on  $\operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H$ . That is,

$$(24) (f \otimes k \otimes l) \leftarrow h = f \otimes k \otimes lh$$

for all  $f \otimes k \otimes l \in \operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H$  and  $h \in H$ . We denote this *H*-module by  $\operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H_*$ .

We can define an  $(A \#_{\sigma} H)^{e}$ -module structure on  $\operatorname{Hom}_{A^{e}}(P, A^{e}) \otimes_{\sigma} H \otimes H_{*}$  as follows:

(25) 
$$(a\#h) \cdot (f \otimes k \otimes l) = a((S^{\sigma^2}h_1) \rightharpoonup f) \otimes S_{1,\sigma}(S_{\sigma,1}(h_2)) \bullet_{\sigma} k \otimes h_3l,$$
$$(f \otimes k \otimes l) \cdot (b\#g) = f(k_1 \cdot b) \otimes k_2 \bullet_{\sigma} g \otimes l,$$

for any a # h,  $b \# g \in A \#_{\sigma} H$  and  $f \otimes k \otimes l \in \operatorname{Hom}_{A^{e}}(P, A^{e}) \otimes_{\sigma} H \otimes H_{*}$ . Recall that the left  $H^{\sigma}$ -module structure of  $\operatorname{Hom}_{A^{e}}(P, A^{e})$  is defined in (12). Here  $h_{1} \otimes h_{2} \otimes h_{3} = (\Delta_{\sigma,1}^{\sigma} \otimes \operatorname{id}) \Delta_{\sigma,1}^{\sigma}(h)$  and  $k_{1} \otimes k_{2} = \Delta_{\sigma,1}^{\sigma}(k)$ . We first check that the left  $A \#_{\sigma} H$ -module structure is well-defined. We have the following equations:

$$\begin{array}{ll} (b\#g) \cdot [(a\#h) \cdot (f \otimes k \otimes l)] \\ = & (b\#g) \cdot [a((S^{\sigma^2}h_1) \rightarrow f) \otimes S_{1,\sigma}(S_{\sigma,1}(h_2)) \cdot \sigma k \otimes h_3 l] \\ = & b[(S^{\sigma^2}g_1) \rightarrow (a((S^{\sigma^2}h_1) \rightarrow f))] \otimes S_{1,\sigma}(S_{\sigma,1}(g_2)) \cdot \sigma S_{1,\sigma}(S_{\sigma,1}(h_2)) \cdot \sigma k \otimes g_3 h_3 l \\ \stackrel{(11)}{=} & b[(g_1 \cdot a)((S^{\sigma^2}g_2) * (S^{\sigma^2}h_1)) \rightarrow f] \otimes S_{1,\sigma}(S_{\sigma,1}(g_3)) \cdot \sigma S_{1,\sigma}(S_{\sigma,1}(h_2)) \cdot \sigma k \otimes g_4 h_3 l \\ = & b(g_1 \cdot a)((S^{\sigma^2}g_2) * (S^{\sigma^2}h_1)) \rightarrow f \otimes S_{1,\sigma}(S_{\sigma,1}(g_3)) \cdot \sigma S_{1,\sigma}(S_{\sigma,1}(h_2)) \cdot \sigma k \otimes g_4 h_3 l \\ = & b(g_1 \cdot a)((S^{\sigma^2}(g_2 * h_1)) \rightarrow f \otimes S_{1,\sigma}(S_{\sigma,1}(g_3) \cdot \sigma S_{1,\sigma}(S_{\sigma,1}(h_2)) \cdot \sigma k \otimes g_4 h_3 l \\ = & b(g_1 \cdot a)(S^{\sigma^2}(g_2 * h_1)) \rightarrow f \otimes S_{1,\sigma}(S_{\sigma,1}(g_3 \cdot \sigma h_2)) \cdot \sigma k \otimes g_4 h_3 l \\ = & [b(g_1 \cdot a)\#g_2 \cdot \sigma h] \cdot (f \otimes k \otimes l) \\ = & [(b\#g)(a\#h)] \cdot (f \otimes k \otimes l). \end{array}$$

By Lemma 1.4 we know that  $S_{1,\sigma} \circ S_{\sigma,1}$  is an algebra homomorphism of  ${}_{\sigma}H$ . Therefore, the fifth equation holds. The sixth equation follows from the fact that  $\Delta_{\sigma,1}^{\sigma}$  is an algebra homomorphism. It follows that  $\operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H_*$  is a left  $A \#_{\sigma} H$ -module. Similarly, we can see that  $\operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H_*$  is a right  $A \#_{\sigma} H$ -module and for any  $a \# h, b \# g \in A \#_{\sigma} H$ , and  $f \otimes k \otimes l \in \operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H_*$ ,

$$[(a\#h)(f \otimes k \otimes l)](b\#g) = (a\#h)[(f \otimes k \otimes l)(b\#g)].$$

In conclusion,  $\operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H_*$  is indeed an  $(A \#_{\sigma} H)^e$ -module as defined in (25).

The module  $\operatorname{Hom}_{A^e}(P, A^e)_* \otimes_{*\sigma} H \otimes_{\sigma} H_*$  is also an  $(A \#_{\sigma} H)^e$ -module with the module structure defined by

(26) 
$$(a\#h) \cdot (f \otimes k \otimes l) = (S^{\sigma^{-1}}(h_1l_1) \cdot a)f \otimes k \otimes h_2 \cdot_{\sigma} l_2, (f \otimes k \otimes l) \cdot (b\#g) = f(k_1 \cdot b) \otimes k_2 \cdot_{\sigma} g \otimes l,$$

where  $h_1 \otimes h_2 = \Delta_{\sigma,1}^{\sigma}(h), \ l_1 \otimes l_2 = \Delta_{\sigma,1}^{\sigma}(l)$  and  $k_1 \otimes k_2 = \Delta_{\sigma,1}^{\sigma}(k)$ .

Now both  $\operatorname{Hom}_{A^e}(P, A^e)_* \otimes_{*\sigma} H \otimes H_*$  and  $\operatorname{Hom}_{A^e}(P, A^e) \otimes_{\sigma} H \otimes H_*$  are right  $H \otimes (A \#_{\sigma} H)^e$ -modules.

**Lemma 2.9.** Let H be a Hopf algebra,  $\sigma$  a 2-cocycle on H, and A a graded left  $H^{\sigma}$ -module algebra. If P is an  $A^e \rtimes H^{\sigma}$ -module, then the following  $\Psi$  and  $\Phi$  are  $H \otimes (A \#_{\sigma} H)^e$ -module isomorphisms

$$\operatorname{Hom}_{A^{e}}(P, A^{e})_{*} \otimes_{*\sigma} H \otimes_{\sigma} H_{*} \underset{\Phi}{\overset{\Psi}{\rightleftharpoons}} \operatorname{Hom}_{A^{e}}(P, A^{e}) \otimes H \otimes H_{*},$$

where the module structures are given by (23), (24), (25) and (26),  $\Psi$  and  $\Phi$  are defined as follows:

$$\begin{split} \Psi(f \otimes k \otimes l) &= f \leftarrow S^{\sigma}(l_1) \otimes S_{1,\sigma}(S_{\sigma,1}(l_2)) \bullet_{\sigma} k \otimes l_3, \\ \Phi(f \otimes k \otimes l) &= f \leftarrow l_2 \otimes S_{1,\sigma}(l_1) \bullet_{\sigma} k \otimes l_3. \end{split}$$

Moreover,  $\Psi$  and  $\Phi$  are inverse to each other.

**Lemma 2.10.** Let H be a Hopf algebra,  $\sigma$  a 2-cocycle on H, and A a graded left  $H^{\sigma}$ -module algebra. Let P be an  $A^{e} \rtimes H^{\sigma}$ -module, and M an  $(A \#_{\sigma} H)^{e}$ bimodule. Then  $\operatorname{Hom}_{A^{e}}(P, M)$  is a right H-module defined by

$$(f \leftarrow h)(x) = S_{1,\sigma}(h_1)f(h_2x)h_3$$

for any  $h \in H$ ,  $f \in \text{Hom}_{A^e}(P, M)$  and  $x \in P$ . Here  $h_1 \otimes h_2 \otimes h_3 = (\Delta_{1,\sigma}^{\sigma} \otimes \text{id})\Delta_{1,1}^{\sigma}(h)$ .

*Proof.* For any  $h, k \in H$  and  $f \in \text{Hom}_{A^e}(P, M)$ , the following equations hold:

$$\begin{split} [(f \leftarrow h) \leftarrow k](x) &= S_{1,\sigma}(k_1)(f \leftarrow h)(k_2x)k_3 \\ &= S_{1,\sigma}(k_1)[S_{1,\sigma}(h_1)f(h_2(k_2(x)))h_3]k_3 \\ &= [S_{1,\sigma}(k_1) \cdot_{\sigma} S_{1,\sigma}(h_1)]f((h_2 * k_2)(x))(h_3 \cdot_{\sigma} k_3) \\ &= [S_{1,\sigma}(h_1 \cdot_{\sigma^{-1}} k_1)]f((h_2 * k_2)(x))(h_3 \cdot_{\sigma} k_3) \\ &= [f \leftarrow (hk)](x). \end{split}$$

The third equation holds since M is an  $A\#_{\sigma}M$ -bimodule. The fourth equation follows from Lemma 1.4(i). The last equation follows from the fact that both  $\Delta_{1,\sigma}^{\sigma}$  and  $\Delta_{1,1}^{\sigma}$  are algebra homomorphisms.

**Remark 2.11.** Since A is a graded left  $H^{\sigma}$ -module algebra, A is naturally an  $A^e \rtimes H^{\sigma}$ -module. Hence,  $\operatorname{Hom}_{A^e}(A, M)$  is a right H-module for any  $(A \#_{\sigma} H)^e$ -bimodule M. H is just the algebra H(1, 1). From the fact that  $S_{1,\sigma}(h_1)h_2 = \varepsilon(h)$  for any  $h \in H$ , it is easy to check that

 $\operatorname{Hom}_{H}(\Bbbk, \operatorname{Hom}_{A^{e}}(A, M)) \cong \operatorname{Hom}_{(A \#_{\sigma} H)^{e}}(A \#_{\sigma} H, M),$ 

for any  $(A \#_{\sigma} H)^{e}$ -bimodule M.

From Lemma 2.10 we see that  $\operatorname{Hom}_{A^e}(P, (A\#_{\sigma}H)^e)$  is a right *H*-module. Moreover, the inner structure of  $(A\#_{\sigma}H)^e$  induces a right  $(A\#_{\sigma}H)^e$ -module structure on  $\operatorname{Hom}_{A^e}(P, (A\#_{\sigma}H)^e)$ . That is,

$$[f \cdot (a\#h) \otimes (b\#g)](x) = f(x)_1(a\#h) \otimes (b\#g)f(x)_2$$

for any  $f \in \operatorname{Hom}_{A^e}(P, (A \#_{\sigma} H)^e)$  and  $a \# h, b \# g \in A \#_{\sigma} H$ .

**Lemma 2.12.** Let P be an  $A^e \rtimes H^{\sigma}$ -module.

(i) There is a right  $H \otimes (A \#_{\sigma} H)^{e}$ -module homomorphism

$$\Theta: \operatorname{Hom}_{A^e}(P, A^e)_* \otimes_{*\sigma} H \otimes_{\sigma} H_* \to \operatorname{Hom}_{A^e}(P, (A \#_{\sigma} H)^e)$$
$$f \otimes k \otimes l \mapsto \Theta(f \otimes k \otimes l)$$

where  $\Theta(f \otimes k \otimes l)(x) = f(x)_1 \# k \otimes l_1 f(x)_2 \# l_2$  for any  $x \in P$ . Here  $l_1 \otimes l_2 = \Delta_{\sigma,1}^{\sigma}(l)$ .

 (ii) If P is finitely generated projective when viewed as an A<sup>e</sup>-module, then Θ is an isomorphism.

In [34], Stefan showed the relation between the Hochschild cohomologies of A and B, where B/A is a Hopf-Galois extension. When  $B = A \#_{\sigma} H$  is a cleft extension, we have the following lemma:

**Lemma 2.13.** [34, Theorem 3.3] Let H be a Hopf algebra,  $\sigma$  a 2-cocycle on H. Let A be a graded  $H^{\sigma}$ -module algebra and N an  $(A \#_{\sigma} H)^{e}$ -bimodule. Then there is a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_{H^e}^p(H, \operatorname{Ext}_{A^e}^q(A, N)) \Longrightarrow \operatorname{Ext}_{(A\#_{\sigma}H)^e}^{p+q}(A\#_{\sigma}H, N)$$

which is natural in N. The right H-module  $\operatorname{Ext}_{A^e}^q(A, N)$  is viewed as  $H^e$ -module via the trivial action on the left side.

**Lemma 2.14.** Let H be a Hopf algebra,  $\sigma$  a 2-cocycle on H and A a left  $H^{\sigma}$ -module algebra. If both A and H are homologically smooth, then so is  $A \#_{\sigma} H$ .

*Proof.* Let I be an injective  $A \#_{\sigma} H$ -module. Hom<sub> $A^e$ </sub>(A, I) is a right H-module by Remark 2.11. From the proof of [34, Proposition 3.2], we see that Hom<sub> $A^e$ </sub>(A, I) is an injective H-module. Moreover, we see in Remark 2.11 that

 $\operatorname{Hom}_{H}(\Bbbk, \operatorname{Hom}_{A^{e}}(A, M)) \cong \operatorname{Hom}_{(A \#_{\sigma} H)^{e}}(A \#_{\sigma} H, M)$ 

for any  $A\#_{\sigma}H$ -bimodule M. Now the proof of Proposition 2.11 in [23] is valid for the cleft extension  $A\#_{\sigma}H$ . We obtain that  $A\#_{\sigma}H$  is homologically smooth.

The following lemma is probably well-known, for the convenience of the reader, we provide a proof here.

**Lemma 2.15.** Let H be an augmented algebra such that H is a twisted CY algebra of dimension d with Nakayama automorphism  $\nu$ . Then H is of global dimension d. Moreover, there is an isomorphism of right H-modules

$$\operatorname{Ext}_{H}^{i}({}_{H}\mathbb{k},{}_{H}H) \cong \begin{cases} 0, & i \neq d; \\ k_{\xi}, & i = d, \end{cases}$$

where  $\xi : H \to \mathbb{k}$  is the homomorphism defined by  $\xi(h) = \varepsilon(\nu(h))$  for any  $h \in H$ .

*Proof.* If H is an augmented algebra, then  ${}_{H}\mathbb{k}$  is a finite dimensional module. By [9, Remark 2.8], H has global dimension d.

It follows from [9, Proposition 2.2] that H admits a projective bimodule resolution

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to H \to 0,$$

where each  $P_i$  is finitely generated as an *H*-*H*-bimodule. Tensoring with functor  $\otimes_H \mathbb{k}$ , we obtain a projective resolution of  $_H \mathbb{k}$ :

$$0 \to P_d \otimes_H \Bbbk \to \dots \to P_1 \otimes_H \Bbbk \to P_0 \otimes_H \Bbbk \to {}_H \Bbbk \to 0$$

Since each  $P_i$  is finitely generated, the following isomorphisms of right *H*-modules holds:

$$\mathbb{k} \otimes_H \operatorname{Hom}_{H^e}(P_i, H^e) \cong \operatorname{Hom}_H(P_i \otimes_H \mathbb{k}, H).$$

Therefore, the complex  $\operatorname{Hom}_H(P_{\bullet} \otimes_H \Bbbk, H)$  is isomorphic to the complex  $\Bbbk \otimes_H$  $\operatorname{Hom}_{H^e}(P_{\bullet}, H^e)$ . The algebra H is twisted CY with Nakayama automorphism  $\nu$ . So the following H-H-bimodule complex is exact,

$$0 \to \operatorname{Hom}_{H^e}(P_0, H^e) \to \cdots \to \operatorname{Hom}_{H^e}(P_{d-1}, H^e) \to \operatorname{Hom}_{H^e}(P_d, H^e) \to H^{\nu} \to 0.$$

Thus the complex  $\mathbb{k} \otimes_H \operatorname{Hom}_{H^e}(P_{\bullet}, H^e)$  is exact except at  $\mathbb{k} \otimes_H \operatorname{Hom}_{H^e}(P_d, H^e)$ , whose homology is  $\mathbb{k} \otimes_H H^{\nu}$ . It is easy to see that  $\mathbb{k} \otimes_H H^{\nu} \cong \mathbb{k}_{\xi}$ , where  $\xi : H \to \mathbb{k}$  is the algebra homomorphism defined by  $\xi(h) = \varepsilon(\nu(h))$  for any  $h \in H$ . In conclusion, we obtain the following isomorphisms right H-modules

$$\operatorname{Ext}_{H}^{i}(_{H}\mathbb{k}, _{H}H) \cong \begin{cases} 0, & i \neq d; \\ \mathbb{k}_{\xi}, & i = d. \end{cases}$$

**Remark 2.16.** In a similar way, we can also obtain the following isomorphisms of left *H*-modules:

$$\operatorname{Ext}_{H}^{i}(\mathbb{k}_{H}, H_{H}) \cong \begin{cases} 0, & i \neq d; \\ \eta k, & i = d, \end{cases}$$

where  $\eta: H \to \mathbb{k}$  is the homomorphism defined by  $\eta = \varepsilon \circ \nu^{-1}$ . Therefore, if H is a twisted CY augmented algebra, then H has finite global dimension and satisfy the AS-Gorenstein condition. However, H is not necessarily Noetherian. It is not AS-regular in the sense of Definition 1.11. We still call  $\operatorname{Ext}_{H}^{i}(_{H}\mathbb{k}, _{H}H)$  and  $\operatorname{Ext}_{H}^{d}(\mathbb{k}_{H}, H_{H})$  left and right homological integral of H and denoted them by  $\int_{H}^{l}$  and  $\int_{H}^{r}$  respectively.

**Lemma 2.17.** Let H be a twisted CY Hopf algebra with homological integral  $\int_{H}^{l} = \mathbb{k}_{\xi}$ , where  $\xi : H \to \mathbb{k}$  is an algebra homomorphism. Then the Nakayama automorphism  $\nu$  of H is given by  $\nu(h) = \xi(h_1)S^2(h_2)$  for any  $h \in H$ . If the right homological integral of H is  $\int_{H}^{l} = {}_{\eta}\mathbb{k}$ , then  $\eta = \xi \circ S$ .

Proof. Proposition 4.5(a) in [10] holds true when the Hopf algebra is not necessarily Noetherian. So we obtain that the Nakayama automorphism  $\nu$  satisfies  $\nu(h) = \xi(h_1)S^2(h_2)$  for any  $h \in H$ . From Remark 2.16, we see that  $\eta = \varepsilon \circ \nu^{-1}$ . Note that for every  $h \in H$ ,  $\nu^{-1}(h) = \xi(Sh_1)S^{-2}(h_2)$  and  $\xi \circ S^2(h) = \xi(h)$ . Therefore, we obtain that  $\eta = \xi \circ S$ .

**Theorem 2.18.** Let H be a twisted CY Hopf algebra with homological integral  $\int_{H}^{l} = \mathbb{k}_{\xi}$ , where  $\xi : H \to \mathbb{k}$  is an algebra homomorphism and let  $\sigma$  be a 2-cocycle on H. Let A be an N-Koszul graded twisted CY algebra with Nakayama automorphism  $\mu$  such that A is a left graded  $H^{\sigma}$ -module algebra. Then  $A \#_{\sigma} H$  is a graded twisted CY algebra with Nakayama automorphism  $\rho$  defined by

$$\rho(a\#h) = \mu(a)\# \operatorname{hdet}_{H^{\sigma}}(h_1)(S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(h_2)))\xi(h_3)$$

for all  $a \# h \in A \#_{\sigma} H$ .

Proof. Assume that the CY dimensions of H and A are  $d_1$  and  $d_2$  respectively. Take the Koszul complex  $\mathbf{K}_{\mathbf{b}}(\mathbf{A}) \to A \to 0$ . In the proof of Proposition 2.8, we see that  $\mathbf{K}_{\mathbf{b}}(\mathbf{A}) \to A \to 0$  is a complex of  $A^e \rtimes H^{\sigma}$ -modules. It follows from Lemma 2.9 and Lemma 2.12 that the following isomorphisms of  $H \otimes (A \# H)^e$ -module complexes hold:

$$\operatorname{Hom}_{A^{e}}(\mathbf{K}_{\mathbf{b}}(A), (A \#_{\sigma} H)^{e}) \cong \operatorname{Hom}_{A^{e}}(\mathbf{K}_{\mathbf{b}}(A), (A^{e}))_{*} \otimes_{*\sigma} H \otimes_{\sigma} H_{*}$$
$$\cong \operatorname{Hom}_{A^{e}}(\mathbf{K}_{\mathbf{b}}(A), (A^{e})) \otimes_{\sigma} H \otimes H_{*}.$$

After taking cohomologies, we obtain that

$$\operatorname{Ext}_{A^e}^q(A, (A\#_{\sigma}H)^e) \cong \operatorname{Ext}_{A^e}^q(A, A^e) \otimes_{\sigma} H \otimes H_*$$

as  $H \otimes (A \#_{\sigma} H)^e$ -modules, for any  $q \ge 0$ .

If we view the right *H*-module  $\operatorname{Ext}_{A^e}^q(A, (A\#_{\sigma}H)^e)$  as  $H^e$ -module via the trivial action on the left side, then

$$\begin{aligned} \operatorname{Ext}_{H^{e}}^{p}(H, \operatorname{Ext}_{A^{e}}^{q}(A, (A \# H)^{e})) &\cong \operatorname{Ext}_{H}^{p}(\Bbbk, \operatorname{Ext}_{A^{e}}^{q}(A, (A \#_{\sigma} H)^{e})) \\ &\cong \operatorname{Ext}_{H}^{p}(\Bbbk, \operatorname{Ext}_{A^{e}}^{q}(A, A^{e}) \otimes_{\sigma} H \otimes H_{*}) \\ &\cong \operatorname{Ext}_{A^{e}}^{q}(A, A^{e}) \otimes_{\sigma} H \otimes \operatorname{Ext}_{H}^{p}(\Bbbk_{H}, H_{H}). \end{aligned}$$

By Lemma 2.13,  $\text{Ext}^{i}_{(A\#H)^{e}}(A\#H, (A\#H)^{e})) = 0$ , for  $i \neq d_{1} + d_{2}$  and

$$\operatorname{Ext}_{(A\#H)^e}^{d_1+d_2}(A\#H, (A\#H)^e)) \cong \operatorname{Ext}_{A^e}^{d_2}(A, A^e) \otimes_{\sigma} H \otimes \operatorname{Ext}_{H}^{d_1}(\Bbbk_H, H_H).$$

It is an isomorphism of  $(A \#_{\sigma} H)^e$ -bimodules if the  $(A \#_{\sigma} H)^e$ -bimodule on  $\operatorname{Ext}_{A^e}^{d_2}(A, A^e) \otimes_{\sigma} H \otimes \operatorname{Ext}_{H}^{d_1}(\Bbbk, H)$  is given by

$$(a\#h) \cdot (x \otimes k \otimes l) = a((S^{\sigma^2}h_1) \rightharpoonup x) \otimes S_{1,\sigma}(S_{\sigma,1}(h_2)) \bullet_{\sigma} k \otimes \xi(Sh_3)l,$$
  
$$(x \otimes k \otimes l) \cdot (b\#g) = x(k_1 \cdot b) \otimes k_2 \bullet_{\sigma} g \otimes l,$$

for any  $a \# h, b \# g \in A \#_{\sigma} H$  and  $x \otimes k \otimes l \in \operatorname{Ext}_{A^{e}}^{d_{2}}(A, A^{e}) \otimes_{\sigma} H \otimes \operatorname{Ext}_{H}^{d_{1}}(\Bbbk_{H}, H_{H})$ . Note that  $\operatorname{Ext}_{H}^{d_{1}}(\Bbbk_{H}, H_{H}) \cong_{\eta} \Bbbk$ , where  $\eta = \xi \circ S$  (Lemma 2.17).

By Proposition 2.8, we obtain the following isomorphism:

$$\operatorname{Ext}_{(A\#H)^e}^{d_1+d_2}(A\#H, (A\#H)^e)) \cong A_{\mu} \otimes A_{d_2}^! \otimes H \otimes_{\xi \circ S} \Bbbk.$$

Since the algebra A is N-Koszul graded twisted CY of dimension  $d_2$ , it is ASregular of global dimension  $d_2$ . By [22, Lemma 5.10], we obtain that  $A_{d_2}^! \cong \operatorname{Ext}_A^{d_2}(\Bbbk, \Bbbk)$  is one dimensional. Let t be a nonzero element in  $A_{d_2}^!$ . The left  $H^{\sigma}$ -action on  $A_{d_2}^!$  is given by

$$h \cdot t = \operatorname{hdet}(S^{\sigma^{-1}}h)t,$$

for any  $h \in H$ . Therefore, the  $(A \# H)^e$ -module structure on  $A_\mu \otimes A_{d_2}^! \otimes H \otimes_{\xi \circ S} \Bbbk$  is given by

(27) 
$$(a\#h) \cdot (x \otimes t \otimes k \otimes y)$$
$$= a(h_1 \cdot x) \otimes hdet_{H^{\sigma}}(S^{\sigma}h_2)t \otimes (S_{1,\sigma}(S_{\sigma,1}h_3)) \bullet_{\sigma} k \otimes \xi(Sh_4)y$$
$$(x \otimes t \otimes k \otimes y) \cdot (b\#g)$$
$$= x\mu(k_1 \cdot b) \otimes t \otimes k_2 \bullet_{\sigma} g \otimes y,$$

for  $(x \otimes t \otimes k \otimes y) \in A_{\mu} \otimes A^!_{d_2} \otimes H \otimes_{\xi \circ S} \Bbbk$  and  $a \# h, b \# g \in A \# H$ .

Now we prove that  $A_{\mu} \otimes A_{d_2}^! \otimes_{\sigma} H \otimes_{\xi \circ S} \mathbb{k} \cong (A \#_{\sigma} H)^{\rho}$  as  $(A \#_{\sigma} H)^e$ -modules for some automorphism  $\rho$  of  $A \#_{\sigma} H$ .

It is straightforward to check that for any  $x \in A, k \in H$ , we have:

$$x \otimes t \otimes k \otimes 1 = [x \# \operatorname{hdet}_{H^{\sigma}}(k_1) S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(k_2)) \xi(k_3)] \cdot (1 \otimes t \otimes 1 \otimes 1)$$
  
=  $(1 \otimes t \otimes 1 \otimes 1) \cdot (\mu^{-1}(x) \# k).$ 

This implies that  $(1 \otimes t \otimes 1 \otimes 1)$  is a left and right  $A \#_{\sigma} H$ -module generator of  $A_{\mu} \otimes A_{d_2}^! \otimes_{\sigma} H \otimes_{\xi \circ S} \Bbbk$ . The same formula implies that no nonzero element of A # H annihilates  $(1 \otimes t \otimes 1 \otimes 1)$ . Therefore,  $A_{\mu} \otimes A_{d_2}^! \otimes_{\sigma} H \otimes_{\xi \circ S} \Bbbk$  is a free  $A \#_{\sigma} H$ -module of rank 1 on each side. So  $A_{\mu} \otimes A_{d_2}^! \otimes_{\sigma} H \otimes_{\xi \circ S} \Bbbk \cong (A \# H)^{\rho}$ as  $(A \#_{\sigma} H)^e$ -modules for some automorphism  $\rho$  of  $A \#_{\sigma} H$ . Next we compute  $\rho$ . For any  $h \in H$ ,

$$(1 \otimes t \otimes 1 \otimes 1) \cdot (1 \# h) = 1 \otimes t \otimes h \otimes 1 = (1 \# \operatorname{hdet}_{H^{\sigma}}(h_1) S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(h_2)) \xi(h_3)) \cdot (1 \otimes t \otimes 1 \otimes 1).$$

This shows that  $\rho(h) = \operatorname{hdet}(h_1)(S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(h_2)))\xi(h_3).$ 

On the other hand, for any  $a \in A$ , we have:

$$(1 \otimes t \otimes 1 \otimes 1) \cdot (a\#1) = \mu(a) \otimes t \otimes 1 \otimes 1 = (\mu(a)\#1) \cdot (1 \otimes t \otimes 1 \otimes 1).$$

So  $\rho(a) = \mu(a)$ . It follows that the automorphism  $\rho$  of  $A \#_{\sigma} H$  is give by

$$\rho(a\#h) = \mu(a)\# \operatorname{hdet}_{H^{\sigma}}(h_1)(S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(h_2)))\xi(h_3)$$

for any  $a \# h \in A \# H$  and  $A_{\mu} \otimes A_{d_2}^! \otimes_{\sigma} H \otimes \Bbbk_{\xi} \cong (A \#_{\sigma} H)^{\rho}$ . To summarize, we obtain the following isomorphisms of  $(A \# H)^e$ -modules:

$$\operatorname{Ext}^{i}_{(A\#_{\sigma}H)^{e}}(A\#_{\sigma}H, (A\#_{\sigma}H)^{e}) \cong \begin{cases} 0, & i \neq d_{1} + d_{2}; \\ (A\#_{\sigma}H)^{\rho}, & i = d_{1} + d_{2}. \end{cases}$$

By Lemma 2.14,  $A \#_{\sigma} H$  is homologically smooth. The proof is completed.  $\Box$ Let H be a Hopf algebra. For an algebra homomorphism  $\xi : H \to \mathbb{k}$ , We write  $[\xi]^l$  for the *left winding homomorphism* of  $\xi$  defined by

$$[\xi]^{l}(h) = \xi(h_1)h_2,$$

for any  $h \in H$ . The right winding automorphism  $[\xi]^r$  of  $\xi$  can be defined similarly. It is well-known that both  $[\xi]^l$  and  $[\xi]^r$  are algebra automorphisms of H. In Theorem 2.18, if we take the 2-cocycle to be trivial, we obtain the following result about smash products.

**Theorem 2.19.** Let H be a twisted CY Hopf algebra with homological integral  $\int_{H}^{l} = \mathbb{k}_{\xi}$ , where  $\xi : H \to \mathbb{k}$  is an algebra homomorphism and A an N-Koszul graded twisted CY algebra with Nakayama automorphism  $\mu$  such that A is a left graded H-module algebra. Then A # H is a twisted CY algebra with Nakayama automorphism  $\rho = \mu \# (S^{-2} \circ [\text{hdet}_{H}]^{l} \circ [\xi]^{r}).$ 

*Proof.* From Theorem 2.18, we see that A#H is a graded twisted CY algebra with Nakayama automorphism  $\rho$  defined by

$$\rho(a\#h) = \mu(a)\# \operatorname{hdet}_H(h_1)(S^{-2}(h_2))\xi(h_3)$$

for all  $a \# h \in A \#_{\sigma} H$ . That is,  $\rho = \mu \# (S^{-2} \circ [\operatorname{hdet}_H]^l \circ [\xi]^r)$ .

**Corollary 2.20.** With the same assumption as in Theorem 2.19, the algebra A#H is a CY algebra if and only if  $hdet_H = \xi \circ S$  and  $\mu # S^{-2}$  is an inner automorphism of A#H.

*Proof.* Since  $\mu \# (S^{-2} \circ [\operatorname{hdet}_H]^l \circ [\xi]^r) = (\mu \# S^{-2}) \circ (\operatorname{id} \# ([\operatorname{hdet}_H]^l \circ [\xi]^r))$ , the sufficiency part is clear.

In the proof of Theorem 2.18, if we let the cocycle  $\sigma$  be trivial, then the proof is just a modification of the proof of the sufficiency part of [23, Theorem 2.12]. If we modify the proof of the necessary part, we obtain that  $\xi \star \text{hdet}_H = \varepsilon$ , where  $\star$  stands for the convolution product. It is easy to see that  $\xi \circ S$  and  $\xi$ are inverse to each other with respect to the convolution product. Therefore, we obtain that  $\text{hdet}_H = \xi \circ S$ . Now  $\mu \# (S^{-2} \circ [\text{hdet}_H]^l \circ [\xi]^r) = \mu \# S^{-2}$ . It follows from Theorem 2.19 that  $\mu \# S^{-2}$  is an inner automorphism.  $\Box$ 

In case A is an N-Koszul graded CY algebra and H is a CY Hopf algebra, we have the following consequence.

**Corollary 2.21.** Let H be a CY Hopf algebra, and let A be an N-Koszul graded CY algebra and a left graded H-module algebra. Then A#H is a graded CY algebra if and only if the homological determinant of the H-action on A is trivial and id  $#S^2$  is an inner automorphism of A#H.

*Proof.* Since H is a CY Hopf algebra, by Lemma 1.15 (ii), the algebra H satisfies  $\int_{H}^{l} = \mathbb{k}$ . Now the corollary follows immediately from Corollary 2.20.

**Remark 2.22.** From Lemma 2.15 and Lemma 2.17, it is not hard to see that if H is CY Hopf algebra, then  $S^2$  is an inner automorphism of H. However, id  $\#S^2$  is not necessarily an inner automorphism of A#H even if A#H is CY. Example 4.2 in Section 4 is a counterexample. It also shows that the smash product A#H could be a CY algebra when A itself is not.

In Theorem 2.18, if we let the algebra A be k, then we obtain the following result about the twisted CY property of cleft objects.

**Theorem 2.23.** Let H be a twisted CY Hopf algebra with  $\int_{H}^{l} = \xi \mathbb{k}$ . Suppose  $_{\sigma}H$  is a right cleft object of H. Then  $_{\sigma}H$  is a twisted CY algebra with Nakayama automorphism  $\mu$  defined by

$$\mu(x) = S_{\sigma,1}^{-1}(S_{1,\sigma}^{-1}(x_1))\xi S(x_2)$$

for any  $x \in {}_{\sigma}H$ .

3. CLEFT OBJECTS OF  $U(\mathcal{D}, \lambda)$ 

The pointed Hopf algebras  $U(\mathcal{D}, \lambda)$  introduced in [5] are generalizations of the quantized enveloping algebras  $U_q(\mathfrak{g})$ , where  $\mathfrak{g}$  is a finite dimensional semisimple Lie algebra. Chelma showed that the algebras  $U_q(\mathfrak{g})$  are CY algebras [11, Theorem 3.3.2]. The CY property of the algebras  $U(\mathcal{D}, \lambda)$  were discussed in [39]. In this section we will show that the cleft objects of the algebras  $U(\mathcal{D}, \lambda)$  are all twisted CY algebras.

3.1. The Hopf algebra  $U(\mathcal{D}, \lambda)$ . We refer to [3] for a detailed discussion about braided Hopf algebras and Yetter-Drinfeld modules. For a group  $\Gamma$ , we denote by  $_{\Gamma}^{\Gamma} \mathcal{YD}$  the category of Yetter-Drinfeld modules over the group algebra  $\mathbb{k}\Gamma$ . If  $\Gamma$  is an abelian group, then it is well-known that a Yetter-Drinfeld module over the algebra  $\mathbb{k}\Gamma$  is just a  $\Gamma$ -graded  $\Gamma$ -module.

We fix the following terminology.

- a free abelian group  $\Gamma$  of finite rank s;
- a Cartan matrix  $\mathbb{A} = (a_{ij}) \in \mathbb{Z}^{\theta \times \theta}$  of finite type, where  $\theta \in \mathbb{N}$ . Let  $(d_1, \dots, d_{\theta})$  be a diagonal matrix of positive integers such that  $d_i a_{ij} = d_j a_{ji}$ , which is minimal with this property;
- a set  $\mathcal{X}$  of connected components of the Dynkin diagram corresponding to the Cartan matrix  $\mathbb{A}$ . If  $1 \leq i, j \leq \theta$ , then  $i \sim j$  means that they belong to the same connected component;
- a family  $(q_I)_{I \in \mathcal{X}}$  of elements in k which are *not* roots of unity;
- elements  $g_1, \dots, g_{\theta} \in \Gamma$  and characters  $\chi_1, \dots, \chi_{\theta} \in \hat{\Gamma}$  such that

(28) 
$$\chi_j(g_i)\chi_i(g_j) = q_I^{d_i a_{ij}}, \ \chi_i(g_i) = q_I^{d_i}, \text{ for all } 1 \leq i, j \leq \theta, \ I \in \mathcal{X}.$$

For simplicity, we write  $q_{ji} = \chi_i(g_j)$ . Then Equation (28) reads as follows:

(29) 
$$q_{ii} = q_I^{d_i} \text{ and } q_{ij}q_{ji} = q_I^{d_i a_{ij}} \text{ for all } 1 \leq i, j \leq \theta, I \in \mathcal{X}.$$

Let  $\mathcal{D}$  be the collection  $\mathcal{D}(\Gamma, (a_{ij})_{1 \leq i,j \leq \theta}, (q_I)_{I \in \mathcal{X}}, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta})$ . A linking datum  $\lambda = (\lambda_{ij})$  for  $\mathcal{D}$  is a collection of elements  $(\lambda_{ij})_{1 \leq i < j \leq \theta, i \neq j} \in \mathbb{K}$  such that  $\lambda_{ij} = 0$  if  $g_i g_j = 1$  or  $\chi_i \chi_j \neq \varepsilon$ . We write the datum  $\lambda = 0$ , if  $\lambda_{ij} = 0$  for all  $1 \leq i < j \leq \theta$ . The datum  $(\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), q_I, (g_i), (\chi_i), (\lambda_{ij}))$  is called a generic datum of finite Cartan type for group  $\Gamma$ .

A generic datum of finite Cartan type for a group  $\Gamma$  defines a Yetter-Drinfeld module over the group algebra  $\Bbbk\Gamma$ . Let V be a vector space with basis  $\{x_1, x_2, \dots, x_{\theta}\}$ . We set

$$|x_i| = g_i, \ g(x_i) = \chi_i(g)x_i, \ 1 \leq i \leq \theta, g \in \Gamma,$$

where  $|x_i|$  denote the degree of  $x_i$ . This makes V a Yetter-Drinfeld module over the group algebra  $\Bbbk\Gamma$ . We write  $V = \{x_i, g_i, \chi_i\}_{1 \leq i \leq \theta} \in {}^{\Gamma}_{\Gamma} \mathcal{YD}$ . The braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \ 1 \leqslant i, j \leqslant \theta$$

The tensor algebra T(V) on V is a natural graded braided Hopf algebra in  $_{\Gamma}^{\Gamma} \mathcal{YD}$ . The smash product  $T(V) \# \Bbbk \Gamma$  is a usual Hopf algebra. It is also called a bosonization of T(V) by  $\Bbbk \Gamma$ .

**Definition 3.1.** Given a generic datum of finite Cartan type  $(\mathcal{D}, \lambda)$  for a group  $\Gamma$ . Define  $U(\mathcal{D}, \lambda)$  as the quotient Hopf algebra of the smash product  $T(V) \# \mathbb{k} \Gamma$  modulo the ideal generated by

$$(\mathrm{ad}_{c}x_{i})^{1-a_{ij}}(x_{j}) = 0, \ 1 \leqslant i \neq j \leqslant \theta, \ i \sim j,$$
$$x_{i}x_{j} - \chi_{j}(g_{i})x_{j}x_{i} = \lambda_{ij}(g_{i}g_{j} - 1), \ 1 \leqslant i < j \leqslant \theta, \ i \nsim j$$

where  $ad_c$  is the braided adjoint representation defined in [5, Sec. 1].

The algebra  $U(\mathcal{D}, \lambda)$  is a pointed Hopf algebra with

$$\Delta(g) = g \otimes g, \ \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \ g \in \Gamma, 1 \leqslant i \leqslant \theta.$$

To present the CY property of the algebras  $U(\mathcal{D}, \lambda)$ , we recall the concept of root vectors. Let  $\Phi$  be the root system corresponding to the Cartan matrix  $\mathbb{A}$ with  $\{\alpha_1, \dots, \alpha_{\theta}\}$  a set of fix simple roots, and  $\mathcal{W}$  the Weyl group. We fix a reduced decomposition of the longest element  $w_0 = s_{i_1} \cdots s_{i_p}$  of  $\mathcal{W}$  in terms of the simple reflections. Then the positive roots are precisely the followings,

$$\beta_1 = \alpha_{i_1}, \ \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}).$$

For  $\beta_i = \sum_{i=1}^{\theta} m_i \alpha_i$ , we write

$$g_{\beta_i} = g_1^{m_1} \cdots g_{\theta}^{m_{\theta}}$$
 and  $\chi_{\beta_i} = \chi_1^{m_1} \cdots \chi_{\theta}^{m_{\theta}}$ .

Lusztig defined the root vectors for a quantum group  $U_q(\mathfrak{g})$  in [26]. Up to a non-zero scalar, each root vector can be expressed as an iterated braided commutator. In [4, Sec. 4.1], the root vectors were generalized on a pointed Hopf algebras  $U(\mathcal{D}, \lambda)$ . For each positive root  $\beta_i$ ,  $1 \leq i \leq p$ , the root vector  $x_{\beta_i}$  is defined by the same iterated braided commutator of the elements  $x_1, \dots, x_{\theta}$ , but with respect to the general braiding.

**Remark 3.2.** If  $\beta_j = \alpha_l$ , then we have  $x_{\beta_j} = x_l$ . That is,  $x_1, \dots, x_{\theta}$  are the simple root vectors.

**Lemma 3.3.** Let  $(\mathcal{D}, \lambda)$  be a generic datum of finite Cartan type for a group  $\Gamma$ , and H the Hopf algebra  $U(\mathcal{D}, \lambda)$ . Let s be the rank of  $\Gamma$  and p the number of the positive roots of the Cartan matrix.

- (i) The algebra H is Noetherian AS-regular of global dimension p+s. The left homological integral module ∫<sub>H</sub><sup>l</sup> of H is isomorphic to k<sub>ζ</sub>, where ζ : H → k is an algebra homomorphism defined by ζ(g) = (∏<sub>i=1</sub><sup>p</sup> χ<sub>βi</sub>)(g) for all g ∈ Γ and ζ(x<sub>k</sub>) = 0 for all 1 ≤ k ≤ θ.
- (ii) The algebra H is twisted CY with Nakayama automorphism  $\mu$  defined by  $\mu(x_k) = q_{kk}x_k$ , for all  $1 \leq k \leq \theta$ , and  $\mu(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$  for all  $g \in \Gamma$ .
- (iii) The algebra H is CY if and only if  $\prod_{i=1}^{p} \chi_{\beta_i} = \varepsilon$  and  $S^2$  is an inner automorphism.

*Proof.* (i) This is Theorem 2.2 in [39].

(ii) By Lemma 1.15(i), we conclude that the algebra H is twisted CY with Nakayama automorphism  $\mu$  defined by  $\mu(x_k) = S^{-2}(x_k) = q_{kk}x_k$  for  $1 \leq k \leq \theta$ and  $\mu(g) = \xi(g)g = (\prod_{i=1}^p \chi_{\beta_i})(g)g$  for  $g \in \Gamma$ .

(iii) This follows directly from (i) and Lemma 1.15 (ii).

**Remark 3.4.** Theorem 2.3 in [39] showed that the Nakayama automorphism of the algebra  $U(\mathcal{D}, \lambda)$  is the algebra automorphism  $\nu$  defined by  $\nu(x_k) = \prod_{i=1, i \neq j_k}^p \chi_{\beta_i}(g_k) x_k$ , for all  $1 \leq k \leq \theta$ , and  $\nu(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$  for all  $g \in \Gamma$ , where each  $j_k$ ,  $1 \leq k \leq \theta$ , is the integer such that  $\beta_{j_k} = \alpha_k$ . Now we show that the algebra automorphisms  $\mu$  and  $\nu$  only differ by an inner automorphism.

By a similar discussion to the one in the proof of Lemma 4.1 in [39], we see that

$$\prod_{i=1,i\neq j_k}^p \chi_{\beta_i}(g_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^p \chi_{\beta_i}(g_k)) = \prod_{i=1,i\neq j_k}^p \chi_k^{-1}(g_{\beta_i})$$

for each  $1 \leq k \leq \theta$ . Therefore,

$$[\prod_{i=1}^{p} g_{\beta_i}]^{-1}(\mu(x_k))[\prod_{i=1}^{p} g_{\beta_i}] = \prod_{i=1}^{p} \chi_k^{-1}(g_{\beta_i})q_{kk}x_k = \prod_{i=1, i \neq j_k}^{p} \chi_k^{-1}(g_{\beta_i})x_k = \prod_{i=1, i \neq j_k}^{p} \chi_{\beta_i}(g_k) = \nu(x_k)$$

for  $1 \leq k \leq \theta$ . Moreover,  $\Gamma$  is abelian, so  $[\prod_{i=1}^{p} g_{\beta_i}]^{-1}(\mu(g))[\prod_{i=1}^{p} g_{\beta_i}] = \mu(g) = \nu(g)$  for all  $g \in \Gamma$ . This shows that  $\mu$  and  $\nu$  indeed differ by an inner automorphism.

In [28], the author classified the cleft objects of a class of pointed Hopf algebras. This class of algebras contains the algebras  $U(\mathcal{D}, \lambda)$ .

Now we fix a generic datum of finite Cartan type

$$(\mathcal{D},\lambda) = (\Gamma, (a_{ij})_{1 \leqslant i, j \leqslant \theta}, (q_I)_{I \in \mathcal{X}}, (g_i)_{1 \leqslant i \leqslant \theta}, (\chi_i)_{1 \leqslant i \leqslant \theta}, (\lambda_{ij})_{1 \leqslant i < j \leqslant \theta, i \nsim j}),$$

where  $\Gamma$  is a free abelian group of rank s.

Let  $\sigma \in Z^2(\Bbbk\Gamma)$  be a 2-cocycle for the group algebra  $\Bbbk\Gamma$ . Define  $\chi_i^{\sigma}(g) = \frac{\sigma(g,g_i)}{\sigma(g_i,g)}\chi_i(g)$ . From [28, Proposition 1.11], we obtain that

$${}_{\sigma}V = \{x_i, g_i, \chi_i^{\sigma}\}_{1 \leq i \leq \theta} \in {}_{\Gamma}^{\Gamma} \mathcal{YD}.$$

The associated braiding is given by

$$c^{\sigma}(x_i \otimes x_j) = q_{ij}^{\sigma} x_j \otimes x_i,$$

where  $q_{ij}^{\sigma} = \frac{\sigma(g_i, g_j)}{\sigma(g_j, g_i)} q_{ij}$ .

Define

$$\Xi(\sigma) = \{(i,j) \mid i < j, i \nsim j, \chi_i^{\sigma} \chi_j^{\sigma} = 1\}.$$

Given the braided vector space  ${}_{\sigma}V$ , we have the tensor algebra  $T({}_{\sigma}V)$  and the smash product  $T({}_{\sigma}V)\#\Bbbk\Gamma$ . The 2-cocycle  $\sigma$  for the group algebra  $\Bbbk\Gamma$  can be regarded as a 2-cocycle for  $T({}_{\sigma}V)\#\Bbbk\Gamma$  through the projection  $T({}_{\sigma}V)\#\Bbbk\Gamma \rightarrow$  $\Bbbk\Gamma$ . Then we have the crossed product  $T({}_{\sigma}V)\#_{\sigma}\Bbbk\Gamma$ . The difference between the crossed product and the smash product  $T({}_{\sigma}V)\#\Bbbk\Gamma$  is given by

$$\overline{g}\overline{g'} = \sigma(g,g')\overline{gg'}, \ g,g' \in \Gamma, \forall g \in G.$$

Here  $g \in T({}_{\sigma}V) \# \Bbbk \Gamma$  is denoted by  $\overline{g} \in T({}_{\sigma}V) \#_{\sigma} \Bbbk \Gamma$  to avoid confusion.

**Definition 3.5.** Given  $\pi = (\pi_{ij}) \in \mathbb{k}^{\Xi(\sigma)}$ . Define  $B^{\lambda}(\sigma, \pi)$  to be the quotient algebra of  $T(\sigma V) \#_{\sigma} \mathbb{k} \Gamma$  modulo the ideal generated by

$$(\mathrm{ad}_{c^{\sigma}} x_i)^{1-a_{ij}}(x_j) = 0, \ 1 \leq i \neq j \leq \theta, \ i \sim j,$$
$$(\mathrm{ad}_{c^{\sigma}} x_i)(x_j) - \lambda_{ij}\overline{g}_i\overline{g}_j + \pi_{ij} = 0, \ 1 \leq i < j \leq \theta, i \nsim j,$$

where we set  $\pi_{ij} = 0$  if  $(i, j) \notin \Xi(\sigma)$ .

Let  $\mathcal{Z} = \mathcal{Z}(\Gamma, \Xi, \mathbb{k})$  denote the set of all pairs  $(\sigma, \pi)$ , where  $\sigma \in Z^2(\mathbb{k}\Gamma)$  and  $\pi = (\pi_{ij}) \in \mathbb{k}^{\Xi(\sigma)}$ . For two pairs  $(\sigma, \pi)$  and  $(\sigma', \pi')$ , define  $(\sigma, \pi) \sim (\sigma', \pi')$ , if there is an invertible map  $f : \mathbb{k}\Gamma \to \mathbb{k}$  such that

$$\sigma'(g,h) = f^{-1}(g)f^{-1}(h)\sigma(g,h)f(gh), \quad g,h \in \Gamma;$$
$$\pi'_{ij} = f^{-1}(g_i)f^{-1}(g_j)\pi_{ij}, \quad (i,j) \in \Xi(\sigma).$$

This defines an equivalence relation on  $\mathcal{Z}$ . We write  $\mathcal{H}(\Gamma, \Xi, \Bbbk) = \mathcal{Z}/\sim$ .

The following Lemma is the right version of Theorem 6.3 in [28]. It describes the isomorphism classes of right cleft objects of the algebras  $U(\mathcal{D}, \lambda)$ .

Lemma 3.6. The map defined by

$$\begin{array}{ccc} \mathcal{H}(\Gamma,\Xi,\Bbbk) & \longrightarrow & \operatorname{Cleft}(U(\mathcal{D},\lambda)) \\ (\sigma,\pi) & \longmapsto & B^{\lambda}(\sigma,\pi) \end{array}$$

is a bijection, where  $\text{Cleft}(U(\mathcal{D}, \lambda))$  denotes the set of the isomorphism classes the right cleft objects of  $U(\mathcal{D}, \lambda)$ .

**Proposition 3.7.** Given a pair  $(\sigma, \pi) \in \mathcal{Z}(\Gamma, \Xi, \mathbb{k})$ . The algebra  $B^{\lambda}(\sigma, \pi)$  is twisted CY with Nakayama automorphism defined by  $\mu(x_k) = q_{kk}x_k$  for all  $1 \leq k \leq \theta$  and  $\mu(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$  for all  $g \in \Gamma$ .

In particular, the algebra  $B^{\lambda}(\sigma,\pi)$  is CY if and only if there is en element  $h \in \mathbb{k}\Gamma$  such that  $\frac{\sigma(h,g)}{\sigma(g,h)} = (\prod_{i=1}^{p} \chi_{\beta_i})(g)$ , for all  $g \in \Gamma$  and  $(\prod_{i=1, i \neq j_k}^{p} \chi_{\beta_i})(g)\chi_k(h) = 1$  for each  $1 \leq k \leq \theta$ , where each  $j_k$ ,  $1 \leq k \leq \theta$ , is the integer such that  $\beta_{j_k} = \alpha_k$ .

*Proof.* Let  $H = U(\mathcal{D}, \lambda)$ . Without loss of generality, we may assume that  $\sigma$  satisfies that

$$\sigma(g, g^{-1}) = \sigma(g^{-1}, g) = 1$$

for all  $g \in \Gamma$ . This follows from Lemma 3.6 and the fact that for each pair  $(\sigma, \pi)$ , there is a pair  $(\sigma', \pi')$  such that  $(\sigma, \pi) \sim (\sigma', \pi')$  and  $\sigma'$  satisfies  $\sigma'(g, g^{-1}) = \sigma'(g^{-1}, g) = 1$  for all  $g \in \Gamma$ . The algebra  $B_q^{\lambda}(\sigma, \pi)$  is a cleft object of H. Then  $B_q^{\lambda}(\sigma, \pi) \cong_{\tau} H$ , for some 2-cocycle  $\tau$ . The 2-cocycle  $\tau$  can be calculated using Lemma 1.9. We conclude that  $\tau$  satisfies the following:

$$\begin{aligned} \tau(g,g') &= \sigma(g,g'), \\ \tau(g,x_i) &= \tau(x_i,g) = 0, \ 1 \leq i \leq \theta, g, g' \in \Gamma. \\ \tau(x_i,x_j) &= \begin{cases} \lambda_{ij}\sigma(g_i,g_j) - \pi_{ij}, & i < j, i \nsim j \\ 0, & otherwise. \end{cases} \end{aligned}$$

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Lemma 3.3 shows that the algebra  $H = U(\mathcal{D}, \lambda)$  is Noetherian AS-regular. The left homological integral module  $\int_{H}^{l}$  of H is isomorphic to  $\mathbb{k}_{\zeta}$ , where  $\zeta : H \to \mathbb{k}$  is an algebra homomorphism defined by  $\zeta(g) = (\prod_{i=1}^{p} \chi_{\beta_{i}})(g)$  for all  $g \in \Gamma$  and  $\zeta(x_{k}) = 0$  for all  $1 \leq k \leq \theta$ .

Since *H* is AS-regular, by Theorem 2.23,  $B_q(\sigma, \pi) \cong {}_{\tau}H$  is a twisted CY algebra. Its Nakayama automorphism can be calculated as follows. For  $g \in \Gamma$ ,

$$\begin{split} \mu(g) &= S_{\tau,1}^{-1}(S_{1,\tau}^{-1}(g))\zeta(g) = S_{\tau,1}^{-1}(g^{-1}\sigma(g^{-1},g))\zeta(g) \\ &= S_{\tau,1}^{-1}(g^{-1})\zeta(g) = \sigma(g,g^{-1})g\zeta(g) = \zeta(g)g \\ &= (\prod_{i=1}^{p}\chi_{\beta_i})(g)g. \end{split}$$

For each  $1 \leq k \leq \theta$ ,

$$\begin{split} \mu(x_k) &= S_{\tau,1}^{-1}(S_{1,\tau}^{-1}(x_k)) = S_{\tau,1}^{-1}(-g_k^{-1}x_k\sigma(g_k^{-1},g_k)) \\ &= S_{\tau,1}^{-1}(-g_k^{-1}x_k) = \sigma(g_k^{-1},g_k)q_{kk}x_k \\ &= q_{kk}x_k. \end{split}$$

The algebra  $B^{\lambda}(\sigma, \pi)$  is CY if and only if the algebra automorphism  $\mu$  is inner. Since the algebra  $U(\mathcal{D}, \lambda)$  is a domain [5, Theorem 4.3], the invertible elements of  $B^{\lambda}(\sigma, \pi)$  fall in  $\mathbb{k}\Gamma$ . In  $B^{\lambda}(\sigma, \pi)$ , for  $l, g \in \Gamma$  and  $1 \leq k \leq \theta$ , we have

$$\bar{l}\bar{g} = \frac{\sigma(l,g)}{\sigma(g,l)}\bar{g}\bar{l}, \ \bar{l}x_k = \chi_k^{\sigma}(l)x_k\bar{l} = \frac{\sigma(l,g_k)}{\sigma(g_k,l)}\chi_k(l)x_k\bar{l}.$$

With these facts, we see that the automorphism  $\mu$  is an inner automorphism if and only if there exists an element  $h \in \Bbbk\Gamma$  such that

(30) 
$$\frac{\sigma(h,g)}{\sigma(g,h)} = (\prod_{i=1}^{p} \chi_{\beta_i})(g), \quad \frac{\sigma(h,g_k)}{\sigma(g_k,h)} \chi_k(h) = q_{kk},$$

for all  $g \in \Gamma$  and  $1 \leq k \leq \theta$ . Note that if  $\frac{\sigma(h,g)}{\sigma(g,h)} = (\prod_{i=1}^p \chi_{\beta_i})(g)$  holds for any  $g \in \Gamma$ , then  $\frac{\sigma(h,g_k)}{\sigma(g_k,h)} = (\prod_{i=1}^p \chi_{\beta_i})(g)$ . So the condition (30) is equivalent to

$$\frac{\sigma(h,g)}{\sigma(g,h)} = (\prod_{i=1}^p \chi_{\beta_i})(g), (\prod_{i=1,i\neq j_k}^p \chi_{\beta_i})(g)\chi_k(h) = 1$$

for all  $g \in \Gamma$  and  $1 \leq k \leq \theta$ , where each  $j_k$ ,  $1 \leq k \leq \theta$ , is the integer such that  $\beta_{j_k} = \alpha_k$ .

We end this section by giving some examples. We first need the following lemma.

**Lemma 3.8.** Let  $\Gamma$  be an abelian group,  $\sigma$  a 2-cocycle for the group algebra  $\Bbbk\Gamma$ . For any  $g, k, h \in \Gamma$ , we have

$$\frac{\sigma(gk,h)}{\sigma(h,gk)} = \frac{\sigma(g,h)}{\sigma(h,g)} \frac{\sigma(k,h)}{\sigma(h,k)}$$

*Proof.* Since  $\sigma$  is a 2-cocycle, the following equations hold for any  $g, h, k \in \Gamma$ .

(31) 
$$\sigma(g,k)\sigma(gk,h) = \sigma(k,h)\sigma(g,kh)$$

(32) 
$$\sigma(g,k)\sigma(h,gk) = \sigma(h,g)\sigma(hg,k)$$

(33)  $\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk)$ 

(34) 
$$\sigma(h,k)\sigma(g,hk) = \sigma(g,h)\sigma(gh,k)$$

By (31) and (32), we obtain

$$\frac{\sigma(gk,h)}{\sigma(h,gk)} = \frac{\sigma(k,h)\sigma(g,kh)}{\sigma(h,g)\sigma(hg,k)} \\
\stackrel{(33,34)}{=} \frac{\sigma(k,h)\frac{\sigma(g,h)\sigma(gh,k)}{\sigma(h,k)}}{\sigma(h,g)\frac{\sigma(h,k)\sigma(g,hk)}{\sigma(g,h)}} \\
= \frac{\sigma(g,h)}{\sigma(h,g)}\frac{\sigma(k,h)}{\sigma(h,k)}\frac{\sigma(g,h)\sigma(gh,k)}{\sigma(h,k)\sigma(g,hk)} \\
\stackrel{(33)}{=} \frac{\sigma(g,h)}{\sigma(h,g)}\frac{\sigma(k,h)}{\sigma(h,k)}.$$

Now we give an example in which the algebra  $U(\mathcal{D}, \lambda)$  is CY, but the algebra  $B^{\lambda}(\sigma, \pi)$  is not necessarily CY.

**Example 3.9.** Let  $(\mathcal{D}, \lambda)$  be the datum given by

- $\Gamma = \langle y_1, y_2 \rangle$ , a free abelian group of rank 2;
- The Cartan matrix is of type  $A_2 \times A_2$ ;
- $g_1 = g_3 = y_1, g_2 = g_4 = y_2;$
- $\chi_1(y_1) = q^2, \chi_1(y_2) = q^{-1}, \chi_2(y_1) = q^{-1}, \chi_2(y_2) = q^{-2}$ , and  $\chi_3 = \chi_1^{-1}, \chi_4 = \chi_2^{-1}$ , where q is not a root of unity;
- $\lambda = (\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}) = (0, 1, 1, 0).$

Then the algebra  $U(\mathcal{D}, \lambda)$  is just the quantized enveloping algebra  $U_q(\mathfrak{g})$ , where  $\mathfrak{g}$  is the simple Lie algebra corresponding to the Cartan matrix of type  $A_2$ . Therefore,  $U(\mathcal{D}, \lambda)$  is CY ([11, Theorem 3.3.2]). In fact, we have that

$$\beta_1 = \alpha_1, \ \beta_2 = \alpha_1 + \alpha_2, \ \beta_3 = \alpha_2, \ \beta_4 = \alpha_3, \ \beta_5 = \alpha_3 + \alpha_4, \ \beta_6 = \alpha_4$$

are the positive roots, where  $\alpha_i$   $(1 \leq i \leq 4)$  are the simple roots. Hence  $\prod_{i=1}^{6} \chi_{\beta_i} = \chi_1^2 \chi_2^2 \chi_3^2 \chi_4^2 = \varepsilon$ . Moreover,  $(y_1^{-2} y_2^{-2}) x_i (y_1^2 y_2^2) = q_{ii}^{-1} x_i = S^2(x_i)$  for  $1 \leq i \leq 4$ .

Let  $\sigma$  be a 2-cocycle such that  $u_{12} = \frac{\sigma(y_2,y_1)}{\sigma(y_1,y_2)}$  is not a root of unity. Let  $u_{21} = u_{12}^{-1}$ . We claim that the algebra  $B^{\lambda}(\sigma,\pi)$  can not be a CY algebra. Otherwise, by Proposition 3.7, there is an element  $y_1^i y_2^j \in \Gamma$  such that for any  $y_1^k y_2^l \in \Gamma$ ,  $\frac{\sigma(y_1^i y_2^j, y_1^k y_2^l)}{\sigma(y_1^k y_2^l, y_1^i y_2^j)} = u_{21}^{il} u_{12}^{jk} = 1$ , where the first equation follows from

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Lemma 3.8 and the second equation holds because  $\prod_{i=1}^{6} \chi_{\beta_i} = \varepsilon$ . Now let k = l = 1. We obtain that  $u_{21}^i u_{12}^j = u_{12}^{i-j} = 1$ , Since  $u_{12}$  is not a root of unity, we have that i = j. Then  $u_{21}^{il} u_{12}^{jk} = u_{12}^{k-l}$  can not equal to 1 when  $k \neq l$ . This is a contradiction.

The next example shows that the algebra  $U(\mathcal{D}, \lambda)$  is not CY, but some cleft objects are CY.

**Example 3.10.** Let  $(\mathcal{D}, \lambda)$  be the datum given by

- $\Gamma = \langle y_1, y_2 \rangle$ , a free abelian group of rank 2;
- The Cartan matrix  $\mathbb{A}$  is of type  $A_1 \times A_1$ ;
- $g_1 = y_1, g_2 = y_2;$
- $\chi_1(g_1) = q^2, \chi_1(g_2) = q^{-4}, \chi_2(g_1) = q^4, \chi_2(g_2) = q^{-2}$ , where q is not a root of unity;
- $\lambda = 0$ .

The positive roots of  $\mathbb{A}$  are just the simple roots. Since  $\chi_1 \chi_2 \neq \varepsilon$ , the algebra  $H = U(\mathcal{D}, \lambda)$  is not CY (Lemma 3.3 (c)).

Let  $B^0(\sigma, \pi)$  be a cleft object of H such that the 2-cocycle  $\sigma$  satisfies  $u_{12} = \frac{\sigma(g_2,g_1)}{\sigma(g_1,g_2)} = q^3$ . We also put  $u_{21} = u_{12}^{-1}$ . Choose an element  $h = g_1^2 g_2^2 \in \Gamma$ . Then

$$\frac{\sigma(h,g_1)}{\sigma(g_1,h)} = \frac{\sigma(g_1^2g_2^2,g_1)}{\sigma(g_1,g_1^2g_2^2)} = u_{12}^2 = q^6 = \chi_1\chi_2(g_1),$$

where the second equation also follows from Lemma 3.8. Similarly,

$$\frac{\sigma(h,g_2)}{\sigma(g_2,h)} = \frac{\sigma(g_1^2g_2^2,g_2)}{\sigma(g_2,g_1^2g_2^2)} = u_{21}^2 = q^{-6} = \chi_1\chi_2(g_2).$$

Moreover,

$$\begin{split} \chi_2(g_1)\chi_1(h) &= \chi_2(g_1)\chi_1(g_1^2g_2^2) = 1, \\ \chi_1(g_2)\chi_2(h) &= \chi_1(g_2)\chi_2(g_1^2g_2^2) = 1. \end{split}$$

By Proposition 3.7, the algebra  $B^0(\sigma, \pi)$  is a CY algebra.

## 4. More Examples

In this section, we give some examples of Theorem 2.18.

The following example shows that it is possible that the crossed product of CY algebras might be a CY algebra, while their smash product is not CY.

**Example 4.1.** Let  $A = k \langle x_1, x_2 \rangle / (x_1 x_2 - x_2 x_1)$  be the polynomial algebra with two variables. Then A is a CY algebra. Let  $\Gamma$  be the free abelian group of rank 2 with generators  $g_1$  and  $g_2$ . There is a  $\Gamma$ -action on A as follows:

$$g_1 \cdot x_1 = qx_1, \quad g_2 \cdot x_1 = q^{-1}x_1, \ g_1 \cdot x_2 = qx_2, \quad g_2 \cdot x_2 = q^{-1}x_2,$$

where q is not a root of unity. The homological determinant of this  $\Gamma$ -action is not trivial, namely,  $hdet(g_1) = q^2$ ,  $hdet(g_2) = q^{-2}$ . The algebra  $A \# \Bbbk \Gamma$  is not a CY algebra by Theorem 2.12 in [23].

Let  $\sigma$  be a 2-cocycle on  $\Gamma$  such that  $\frac{\sigma(g_2,g_1)}{\sigma(g_1,g_2)} = q$ . Without loss of generality, we may assume that  $\sigma(g, g^{-1}) = \sigma(g^{-1}, g) = 1$  for  $g \in \Gamma$ . Then the algebra  $A\#_{\sigma}\Bbbk\Gamma$  is a twisted CY algebra with Nakayama automorphism  $\rho$  defined by  $\rho(a\#g) = \mathrm{hdet}(h)a\#g$  for any  $a\#g \in A\#_{\sigma}\Bbbk\Gamma$ . Choose an element  $h = g_1^2g_2^2 \in \Gamma$ . By Lemma 3.8,

$$\frac{\sigma(h,g_1)}{\sigma(g_1,h)} = \frac{\sigma(g_1^2g_2^2,g_1)}{\sigma(g_1,g_1^2g_2^2)} = \left(\frac{\sigma(g_2,g_1)}{\sigma(g_1,g_2)}\right)^2 = q^2 = \operatorname{hdet}(g_1),$$
  
$$\frac{\sigma(h,g_2)}{\sigma(g_2,h)} = \frac{\sigma(g_1^2g_2^2,g_2)}{\sigma(g_2,g_1^2g_2^2)} = \left(\frac{\sigma(g_1,g_2)}{\sigma(g_2,g_1)}\right)^2 = q^{-2} = \operatorname{hdet}(g_2).$$

Moreover,  $h \cdot x_i = x_i$ ,  $1 \leq i \leq 2$ . Therefore,  $\rho(a \# g) = h(a \# g)h^{-1}$ , for any  $a \# g \in A \#_{\sigma} \Bbbk \Gamma$ . The automorphism  $\rho$  is an inner automorphism. So the algebra  $A \#_{\sigma} \Bbbk \Gamma$  is a CY algebra.

In the followings, we provide some examples involving the algebras  $U(\mathcal{D}, \lambda)$ . The definitions of algebras  $U(\mathcal{D}, \lambda)$  are recalled in Section 3.1.

The following example shows that the smash product A#H is a CY algebra while A itself is not.

**Example 4.2.** Let *H* be  $U(\mathcal{D}, \lambda)$  with the datum  $(\mathcal{D}, \lambda)$  given by

- $\Gamma = \langle g \rangle$ , a free abelian group of rank 1;
- The Cartan matrix is of type  $A_1 \times A_1$ ;
- $g_1 = g_2 = g;$
- $\chi_1(g) = q^2$ ,  $\chi_2(g) = q^{-2}$ , where q is not a root of unity;
- $\lambda_{12} = \frac{1}{q-q^{-1}}$ .

The algebra H is isomorphic to the quantum enveloping algebra  $U_q(\mathfrak{sl}_2)$ .

Let  $A = k \langle u, v \rangle / (uv - qvu)$  be the quantum plane. There is an *H*-action on *A* as follows:

$$x_1 \cdot u = 0, \quad x_2 \cdot u = qv, \quad g \cdot u = qu, x_1 \cdot v = u, \quad x_2 \cdot v = 0, \quad g \cdot v = q^{-1}v.$$

The algebra A # H is isomorphic to the quantized symplectic oscillator algebra of rank 1 [17].

It is well known that the algebra A is a twisted CY algebra with Nakayama automorphism  $\mu$  given by

$$\mu(u) = qu, \ \mu(v) = q^{-1}v$$

and the algebra H is a CY Hopf algebra ([11, Theorem 3.3.2]). One can easily check that the homological determinant of the H-action is trivial and for any  $x \in A \# H$ ,  $[\mu \# S^{-2}](x) = gxg^{-1}$ . That is, the automorphism  $\mu \# S^{-2}$  is an inner automorphism. Therefore, A # H is a CY algebra.

The invertible elements of A # H are  $\{g^m\}_{m \in \mathbb{Z}}$ . Therefore, one can see that the automorphism id  $\#S^2$  of A # H can not be an inner automorphism, although,  $S^2$  is an inner automorphism of H.

More generally, we have the following example.

**Example 4.3.** Let *H* be  $U(\mathcal{D}, \lambda)$  with the datum  $(\mathcal{D}, \lambda)$  given by

- $\Gamma = \langle y_1, y_2, \cdots, y_n \rangle$ , a free abelian group of rank n;
- The Cartan matrix  $\mathbb{A}$  is of type  $A_n \times A_n$ ;
- $g_i = g_{n+i} = y_i, \ 1 \leq i \leq n;$
- $\chi_i(g_j) = q^{a_{ij}}, \ \chi_{n+i}(g_j) = q^{-a_{ij}}, \ 1 \leq i \leq n$ , where q is not a root of unity;

• 
$$\lambda_{ij} = \delta_{n+i,j} \frac{1}{q-q^{-1}}, \ 1 \leq i < j \leq 2n.$$

Then H is isomorphic to the algebra  $U_q(\mathfrak{sl}_n)$ . It is also a CY Hopf algebra.

Let A be the quantum polynomial algebra

$$\mathbb{k}\langle u_1, u_2, \cdots, u_{n+1} \mid u_j u_i - q u_i u_j, 1 \leq i < j \leq n+1 \rangle.$$

There is an H-action on A as follows:

$$\begin{split} x_{i} \cdot u_{j} &= \delta_{ij} u_{i+1}, 1 \leqslant i \leqslant n; \\ y_{i} \cdot u_{j} &= \begin{cases} q^{-1} u_{j}, & j = i; \\ q x_{j}, & j = i+1; \\ x_{j}, & otherwise. \end{cases} x_{i} \cdot u_{j} &= \delta_{i+1,j} q u_{i}, n+1 \leqslant i \leqslant 2n \\ \\ \end{cases}$$

It is well known that the algebra A is a twisted CY algebra with Nakayama automorphism  $\mu$  given by  $\mu(u_i) = q^{n+2-2i}u_i, 1 \leq i \leq n+1.$ 

One can also check that the homological determinant of the *H*-action is trivial. The automorphism  $\mu \# S^{-2}$  is an inner automorphism. For any  $x \in A \# H$ ,  $[\mu \# S^{-2}](x) = gxg^{-1}$ , where  $g = y_1^n y_2^{2n-2} \cdots y_i^{in-i(i-1)} \cdots y_n^{n^2-n(n-1)}$ . Therefore, A # H is a CY algebra.

Let  $H^0$  be the algebra  $U(\mathcal{D}, 0)$ . The algebra H is a cocycle deformation of  $U(\mathcal{D}, 0)$ . Actually,  $H \cong (H^0)^{\sigma}$ , where  $\sigma$  is a 2-cocycle on  $H^0$  such that  $\sigma(h_1, h_2) = 1$ ,  $\sigma(x_i, h_1) = \sigma(h_2, x_i) = 0$ , for all  $h_1, h_2 \in \Gamma$  and  $1 \leq i \leq n+1$ , and

$$\sigma(x_i, x_j) = \begin{cases} \lambda_{ij}, & j = n + i; \\ 0, & otherwise. \end{cases}$$

Then we have the crossed product  $A\#_{\sigma}H^{0}$ . By Theorem 2.18,  $A\#_{\sigma}H^{0}$  is a twisted CY algebra with Nakayama automorphism  $\eta$  defined by  $\eta(a\#h) = \mu(a)\#h$ , for all  $a\#h \in A\#H$ . In fact,  $\eta$  is an inner automorphism. For any  $x \in A\#_{\sigma}H^{0}$ ,  $\eta(x) = gxg^{-1}$ . So  $A\#_{\sigma}H^{0}$  is also a CY algebra.

**Example 4.4.** Let  $H = U(\mathcal{D}, \lambda)$ , where  $(\mathcal{D}, \lambda)$  is the datum given by

- $\Gamma = \langle y_1, y_2 \rangle$ , a free abelian group of rank 2;
- The Cartan matrix  $\mathbb{A}$  is of type  $A_1 \times A_1$ ;
- $g_1 = y_1, g_2 = y_2;$
- $\chi_1(g_1) = q^2, \chi_1(g_2) = q^{-4}, \chi_2(g_1) = q^4, \chi_2(g_2) = q^{-2}$ , where q is not a root of unity;
- $\lambda = 0.$

The algebra H is a twisted CY algebra with homological integral  $_{\xi_1}$ k, where  $\xi_1$  is the algebra homomorphism given by

$$\xi_1(g_1) = q^6 g_1, \xi_1(g_2) = q^{-6} g_2, \text{ and } \xi_1(x_i) = 0, i = 1, 2.$$

Let  $\sigma$  be a 2-cocycle on H such that  $\frac{\sigma(g_1,g_2)}{\sigma(g_2,g_1)} = q^3$ ,  $\sigma(x_i,g_j) = \sigma(g_j,x_i) = 0$ ,  $1 \leq i, j \leq 2$ , and  $\sigma(x_1,x_2) = \frac{1}{q-1}, \sigma(x_2,x_1) = 0$ . Then the cocycle deformation  $H^{\sigma}$  is just the algebra  $U(\mathcal{D}',\lambda')$ , where  $(\mathcal{D}',\lambda')$  is the datum given by

- $\Gamma = \langle y_1, y_2 \rangle$ , a free abelian group of rank 2;
- The Cartan matrix is of type  $A_1 \times A_1$ ;
- $g_1 = y_1, g_2 = y_2;$
- $\chi_1(g_1) = q^{-2}, \chi_1(g_2) = q, \chi_2(g_1) = q^{-1}, \chi_2(g_2) = q^2$ , where q is not a root of unity;
- $\lambda_{12} = \frac{1}{q-1}$ .

The algebra  $H^{\sigma}$  is a twisted CY algebra with homological integral  $_{\xi_2}$ k, where  $\xi_2$  is the algebra homomorphism given by

$$\xi_2(g_1) = q^{-3}g_1, \xi_2(g_2) = q^3g_2$$
, and  $\xi_2(x_i) = 0, i = 1, 2.$ 

Let  $A = k \langle u, v \rangle / (uv - q^2 vu)$  be the quantum plane. There is an  $H^{\sigma}$ -action on A as follows:

$$\begin{aligned} x_1 \cdot u &= 0, \quad x_2 \cdot u = v, \quad g_1 \cdot u = q^{-1}u, \quad g_2 \cdot u = q^2u \\ x_1 \cdot v &= u, \quad x_2 \cdot v = 0, \quad g_1 \cdot v = qv, \quad g_2 \cdot v = q^{-2}v. \end{aligned}$$

We have mentioned in Example 4.2 that A is a twisted CY algebra with Nakayama automorphism  $\mu$  given by

$$\mu(u) = q^2 u, \ \mu(v) = q^{-2} v.$$

One can check that the homological determinant of the H action is trivial. Now we can form the algebras  $A\#H^{\sigma}$  and  $A\#_{\sigma}H$ . By Theorem 2.19, the algebra  $A\#H^{\sigma}$  is a twisted CY algebra with Nakayama automorphism  $\mu\#(S^{-2}\circ[\xi]^r)$ . This automorphism cannot be an inner automorphism. That is,  $A\#H^{\sigma}$  is not a CY algebra. Theorem 2.18 shows that the algebra the algebra  $A\#_{\sigma}H$  is a twisted CY algebra with Nakayama automorphism  $\rho$  defined by  $\rho(a) = \mu(a)$ ,  $a \in A, \ \rho(x_1) = q^{-2}x_1, \ \rho(x_2) = q^2x_2$ , and  $\rho(g_i) = \xi(g_i)g_i, \ i = 1, 2$ . The automorphism  $\rho$  is an inner automorphism. For any  $x \in A\#_{\sigma}H, \ \rho(x) =$  $(g_1^2g_2^2)^{-1}x(g_1^2g_2^2)$ . Therefore, the algebra  $A\#_{\sigma}H$  is a CY algebra.

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