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Peer-reviewed author version

HUZAK, Renato (2017) Regular and slow-fast codimension 4 saddle-node bifurcations. In: JOURNAL OF DIFFERENTIAL EQUATIONS, 262 (2), p. 1119-1154.

DOI: 10.1016/j.jde.2016.10.008

Handle: <http://hdl.handle.net/1942/22607>

Regular and slow-fast codimension 4 saddle-node bifurcations

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Abstract

Using geometric singular perturbation theory, including the family blow-up as one of the main techniques, we prove that the cyclicity, i.e. maximum number of limit cycles, in both regular and slow-fast unfoldings of nilpotent saddle-node singularity of codimension 4 is 2. The blow-up technique enables us to use the well known results for slow-fast codimension 1 and 2 Hopf bifurcations, slow-fast Bogdanov-Takens bifurcations and slow-fast codimension 3 saddle and elliptic bifurcations.

1 Introduction

In planar slow-fast systems $X_{\epsilon,\mu}$ a curve of singularities, called the critical curve, appears for $\epsilon = 0$ where ϵ is a singular perturbation parameter and $\mu \in \mathbb{R}^p$, $\mu \sim 0$. The critical curve typically consists of normally hyperbolic singularities (the linearized vector field at a normally hyperbolic singularity has one zero eigenvalue with corresponding eigenvector tangent to the critical curve) and one contact point (often called turning point). We assume the contact point is of nilpotent type, for $\mu = 0$. It is shown in [DMDR11] that any smooth family of planar slow-fast vector fields $X_{\epsilon,\mu}$, locally near the nilpotent contact point for $(\epsilon, \mu) \sim (0, 0)$, is smoothly equivalent (preserving (ϵ, μ)) to the following normal form:

$$\begin{cases} \dot{x} &= y - f(x, \mu) \\ \dot{y} &= \epsilon(g(x, \epsilon, \mu) + (y - f(x, \mu))h(x, y, \epsilon, \mu)) \end{cases} \quad (1)$$

for smooth functions f, g and h and $f(0, 0) = \partial_x f(0, 0) = 0$.

Remark 1. *In this paper we focus on smooth families of vector fields (smooth stands for C^∞ -smoothness).*

In this paper, we assume the nilpotent contact point is of order two ($\frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0$). After a smooth coordinate change and a smooth rescaling of time (see [DMD11b]), the family (1) can be brought into the form

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -xy + \epsilon\tilde{g}(x, \epsilon, \mu) + \epsilon y^2 H(x, y, \epsilon, \mu) \end{cases} \quad (2)$$

where \tilde{g} and H are smooth functions.

We call the order of vanishing of $\tilde{g}(x, 0, 0)$ at $x = 0$, which is ≥ 0 , the *singularity order at the contact point* $(x, y) = (0, 0)$ (see [DMDR11]). The determination of small-amplitude limit cycles (i.e. limit cycles in a fixed neighborhood

of the origin $(x, y) = (0, 0)$ in planar slow-fast systems (2) has recently been the subject of many investigations, and *the main goal of this paper is to give a complete analysis of the small-amplitude limit cycles in (2) when the singularity order at the contact point is 4*. When the contact point is a slow-fast jump point (i.e. the singularity order is 0), then it is easy to see that there are no limit cycles (see [DR96], [KS01], [MKKR94]). If the singularity order is 1, small-amplitude limit cycles may be generated by a (slow-fast) Hopf bifurcation as $\tilde{g}(0, \epsilon, \mu)$ varies through the origin. Small-amplitude limit cycles in a codimension 1 slow-fast Hopf case have been studied in [KS01] generalizing the Van der Pol system (see [DR96]). In [DR09], a slow-fast Hopf point of higher codimensions in Liénard setting ($H \equiv 0$ in (2)) has been dealt with. The main result in [DR09] gives finite upper bounds for the number of small-amplitude limit cycles in analytic families or in smooth families with finite codimension. In a general (“non-Liénard”) setting, a codimension 2 slow-fast Hopf point, in the presence of center, has been treated in [Huz16]. The maximum number of small-amplitude limit cycles in this case is shown to be 2 (we refer to this paper for more details). When the singularity order at the contact point in (2) is 2, we deal with a slow-fast unfolding of a Bogdanov-Takens point, and it is shown that from this point, at most one limit cycle may perturb (see [DMD11a]). This case was easier to treat due to the presence of the symmetry-breaking quadratic term αx^2 ($\alpha \neq 0$) in \tilde{g} . When the singularity order at the contact point is 3, the family (2) is called the slow-fast unfolding of a saddle singularity of codimension 3 (+) or the slow-fast unfolding of an elliptic singularity of codimension 3 (-), depending on the sign in front of the cubic term in \tilde{g} (see [HDMD13]). In analogy with the results for the slow-fast Hopf point, the number of small-amplitude limit cycles in this codimension 3 case depends on the higher order terms in \tilde{g} , and, in the presence of the quartic term αx^4 ($\alpha \neq 0$) in \tilde{g} , it is shown that the maximum number of limit cycles of both the slow-fast saddle point of codimension 3 and the slow-fast elliptic point of codimension 3 is 2. This cyclicity result follows from [HDMD13], [HDMD14] and [Huz16]. The cases with the singularity order at the contact point ≥ 4 have not yet been studied and, as mentioned above, in this paper we investigate the small-amplitude limit cycle phenomenon in the slow-fast codimension 4 case. The reason we study this case is twofold. On one hand, the presence of the quartic term eliminates possibility of symmetric behavior of (2) and therefore simplifies our study, to some extent. On the other hand, we treat the codimension 4 case using a recursive approach which enables us to utilize the well known results for the slow-fast system (2) with the singularity order ≤ 3 .

The recursive approach can be used to treat slow-fast codimension n bifurcations for all $n \geq 5$. This is a topic of further study. Clearly, the study of the codimension n case becomes more difficult as the codimension increases. In order to be able to apply the recursive approach to the higher codimension cases, one has to study the codimension 3 case when the coefficient in front of the quartic term is close to 0 (hence the case not treated in [HDMD13]), one has to deal with a conjecture, formulated by Dumortier and Roussarie in [DR09], which has been solved only in some low codimension cases (we refer to [DR09] and [FTV13] for more details),

We point out that detectable limit cycles (hence not small-amplitude limit cycles) that pass near such a codimension n nilpotent contact point, for all (odd)

n , have been studied in [DMD11b].

The slow-fast unfoldings (2) can be seen as slow-fast variants of “regular” unfoldings of planar nilpotent singularities. In fact, consider a smooth unfolding of the nilpotent singularity, i.e., a smooth family X_μ with X_0 having a nilpotent singularity at the origin of \mathbb{R}^2 . Then X_μ is smoothly conjugate, respecting μ , to $\{\dot{x} = y, \dot{y} = F(x, \mu) + yG(x, \mu) + y^2Q(x, y, \mu)\}$ where F , G and Q are smooth functions, $F(x, 0) = O(x^2)$, $G(x, 0) = O(x)$ and $Q = O(\left(\|(x, y)\| + \|\mu\|\right)^N)$, for N as large as required (see [Tak74] and [Dum]). Like in the slow-fast case, when $\frac{\partial G}{\partial x}(0, 0) \neq 0$, we may assume that $G(x, \mu) = -x$ after a smooth coordinate change and a smooth rescaling of time. We write $F(x, 0) = \rho_2 x^2 + \dots$. If $\rho_2 \neq 0$, we deal with a well known (regular) unfolding of a Bogdanov-Takens point (see [Tak74], [Bog76], [RW95] for more details). When $\rho_2 = 0$ and $\rho_3 \neq 0$, we deal with a nilpotent saddle, focus or elliptic singularity of codimension 3 studied in [DRSZ91]. Note that the focus case cannot be observed in the slow-fast codimension 3 case due to the presence of the small parameter ϵ (see [HDMD13]). As far as we know, the regular cases of codimension ≥ 4 ($\rho_2 = \rho_3 = 0$) have not been treated. We refer to [DDM04] for an overview of the known results and remaining problems. *In the present paper, we also give a complete analysis of the small-amplitude limit cycles in the regular codimension 4 case ($\rho_2 = \rho_3 = 0$ and $\rho_4 \neq 0$).*

Techniques used in the study of regular cases are essentially different from those used in the study of slow-fast cases because in the slow-fast setting, the cyclicity results have to have a uniform limit not only as regular perturbation parameters tend to 0, but also as the singular perturbation parameter $\epsilon \rightarrow 0$. In the codimension 4 case, we will show that techniques from geometric singular perturbation theory can be used to deal not only with the slow-fast case but also with the regular case. This is due to the fact that the regular codimension n nilpotent singularity, for all $n \geq 4$, is of slow-fast type after a suitable blow-up (see [Pan02], [DMD11b]). The blow-up will be explained in detail in Section 2 in the regular codimension 4 case. Like in the slow-fast case, the regular codimension n case, with $n \geq 5$, is a topic of further study.

Now suppose that the singularity order at the contact point in (2) is equal to 4. Then we can write

$$\tilde{g}(x, \epsilon, \mu) = \sum_{k=0}^{k=3} \rho_k(\epsilon, \mu) x^k + \rho_4(\epsilon, \mu) x^4 (1 + O(x))$$

where $\rho_k(0, 0) = 0$, $k \leq 3$, and $\rho_4(0, 0) \neq 0$. We may assume that $\rho_4(\epsilon, \mu) = \pm 1$ after a rescaling of the coordinates (x, y) and a time rescaling. As in [DMD11a] and [HDMD13] we consider the $\rho_k(\epsilon, \mu)$ as new independent parameters, and we denote by λ the parameter (ϵ, μ) appearing inside the functions \tilde{g} and H in (2). For the sake of generality we take λ to be in an arbitrary compact subset Λ of some Euclidean space. Note that a minus sign in front of x^4 can be changed into a plus sign by applying $(x, t) \rightarrow (-x, -t)$. Thus it suffices to study slow-fast system $X_{\epsilon, b, \lambda}$, where $X_{\epsilon, b, \lambda}$ stands for

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -xy + \epsilon(b_0 + b_1x + b_2x^2 + b_3x^3 + x^4 + x^5G(x, \lambda)) + \epsilon y^2 H(x, y, \lambda) \end{cases}$$

where G and H are smooth, $\epsilon \geq 0$ is the singular perturbation parameter close to 0, $b = (b_0, b_1, b_2, b_3)$ is regular perturbation parameter close to 0 and $\lambda \in \Lambda$.

Similarly, in the regular codimension 4 case, it suffices to study $X_{1,b,\lambda}$, under the given conditions on parameters (b, λ) . Thus from now on we focus on the family $X_{\epsilon,b,\lambda}$, with $\epsilon \in [0, M]$, where $M > 1$ is an arbitrarily large fixed real number. When $\epsilon \geq 0$ and $\epsilon \sim 0$, we call the family $X_{\epsilon,b,\lambda}$ *slow-fast codimension 4 saddle-node bifurcations* (in short, the singular case). When $\epsilon > 0$ uniformly, we deal with *regular codimension 4 saddle-node bifurcations* (in short, the regular case). It is not so hard to see that the origin of $X_{\epsilon,0,\lambda}$ is a nilpotent saddle-node, for each $(\epsilon, \lambda) \in]0, M] \times \Lambda$ (see Section 2).

We say that the cyclicity of the origin $(x, y) = (0, 0)$ of $X_{\epsilon,b,\lambda}$ is bounded by N if there exist a neighborhood V of $(x, y) = (0, 0)$ and a neighborhood W of $(0, 0, 0, 0)$ in b -space such that for each $(\epsilon, b, \lambda) \in [0, M] \times W \times \Lambda$ the system $X_{\epsilon,b,\lambda}$ has at most N limit cycles inside V . The minimum of such N is the cyclicity of the origin. We can now state the main result of this paper.

Theorem 1.1. *The cyclicity of the origin of $X_{\epsilon,b,\lambda}$ is 2.*

Theorem 1.1 will be proved in Sections 2 and 3. Though this result is very simple to state, its proof is very involved. In fact, to understand the dynamics near the origin, we have to blow up the family $X_{\epsilon,b,\lambda}$ at the origin using the so-called blowing-up for families of vector fields introduced for the first time in the context of slow-fast systems (see [DR96]). Blowing up the phase coordinates (x, y) and the parameter b , we generate a slow-fast system of type (2) in the new phase coordinates (\bar{x}, \bar{y}) , with a new singular perturbation parameter that is independent of ϵ , and with the singularity order ≤ 3 at the contact point $(\bar{x}, \bar{y}) = (0, 0)$. Thus we may use the known results for small-amplitude limit cycles in the (\bar{x}, \bar{y}) -space (hence limit cycles in a fixed neighborhood of $(\bar{x}, \bar{y}) = (0, 0)$). Besides these small limit cycles, we may encounter so-called detectable canard limit cycles in the (\bar{x}, \bar{y}) -plane, i.e., limit cycles passing from a stable branch of the critical curve to an unstable branch, when crossing the turning point $(\bar{x}, \bar{y}) = (0, 0)$. The detectable canard limit cycles have been studied in [DMD08] (resp. in [DMD10]) when the singularity order at $(\bar{x}, \bar{y}) = (0, 0)$ is 1 (resp. 3). When the singularity order at $(\bar{x}, \bar{y}) = (0, 0)$ is 0 or 2, the detectable canard limit cycles are not possible (see Sections 3.1 and 3.4). Note that both the small-amplitude limit cycles and the detectable canard limit cycles in the (\bar{x}, \bar{y}) -plane become small-amplitude limit cycles of the original family $X_{\epsilon,b,\lambda}$ because their size tends to 0 in the (x, y) -plane as $b \rightarrow 0$.

The proof of Theorem 1.1 consists of two steps:

1. Find the cyclicity of each limit periodic set in the (\bar{x}, \bar{y}) -plane which can bifurcate in the small-amplitude limit cycles or the detectable canard limit cycles. As these bifurcations are of different nature they cannot be studied in a uniform way and we have to use different technical methods.
2. Glue together the different local results to obtain the cyclicity of the origin $(x, y) = (0, 0)$ on which each of these limit periodic sets is blown down.

This is the most common method used in the study of the cyclicity of contact points (see [KS01], [DR09], [DMD11a], [Huz16]). Clearly, this gluing method becomes more difficult to apply as the singularity order at the contact point increases. On one hand, the so-called slow dynamics along the critical curve in the (\bar{x}, \bar{y}) -plane becomes more complex (hence limit periodic sets from which detectable canard limit cycles bifurcate are very diverse), and on the other hand we

have to develop new techniques that enable us to glue together small-amplitude limit cycles and canard limit cycles in the (\bar{x}, \bar{y}) -plane, with the singularity order at $(\bar{x}, \bar{y}) = (0, 0)$ equal to 3 (see Section 3.6).

When $H = 0$ in (2), (2) becomes a family of (generalized) slow-fast Liénard vector fields. A motivation to study the slow-fast Liénard vector fields can be found in [Dum06] and [Rou07]. It is closely related to the second part of Hilbert's 16th problem which is in essence to determine maximal number of limit cycles a planar polynomial vector field may have if the polynomial degree of the vector field is given (see [Sma00]). More precisely, given any polynomial generalized Liénard equation $\{\dot{x} = y, \dot{y} = -f(x)y - g(x)\}$, where $\deg f = n$ and $\deg g = m$, find the uniform bound $\mathcal{L}(m, n)$ on the number of limit cycles in terms of the two degrees. It has been shown that $\mathcal{L}(1, 2) = 1$ ([LdMP77]), $\mathcal{L}(1, 3) = 1$ ([LL12]), $\mathcal{L}(2, 1) = 1$ ([Cop89]), $\mathcal{L}(3, 1) = 1$ ([DL96] and [DR90]) and $\mathcal{L}(2, 2) = 1$ ([DL97]). If we want to contribute to finding $\mathcal{L}(m, 1)$ for $m \geq 4$, then we have to study (see [Dum06]) slow-fast Liénard vector fields $\{\dot{x} = y, \dot{y} = -xy + \epsilon(b_0 + b_1x + \dots + b_{l-1}x^{l-1} \pm x^l + O(x^{l+1}))\}$ where $l = 0, 1, \dots, m-1$, $\epsilon > 0$ is the singular parameter kept small and $(b_0, b_1, \dots, b_{l-1})$ are regular perturbation parameters close to 0. Thus Theorem 1.1 can help us find $\mathcal{L}(5, 1)$. Note that Theorem 1.1 implies that the above slow-fast Liénard equation, with $l = 4$, has at most 2 limit cycles in an arbitrary compact set in the phase space, by taking $(\epsilon, b_0, b_1, b_2, b_3)$ sufficiently small (detectable limit cycles that pass near the contact point are not possible because $l = 4$ is even, see [DMD11b]).

Planar limit cycles of slow-fast type also appear quite often in applications where they have been used to model electrical circuits, (bio)chemical reactions ([Moe02], [GS09], [KS11], ...), predator-prey systems ([BNRS06], [LZ13], ...), neural models ([DMW15], ...), epidemic models ([LLMZ14], ...), etc.

2 Blow-up in the parameter space and family blow-up

2.1 Blow-up in the parameter space and statement of the results

Consider the family $X_{\epsilon, b, \lambda}$ from the previous section. Like in [HDMD13], we first blow up the origin in (b_0, b_1, b_2, b_3) -space by means of a quasi-homogeneous blow-up:

$$(b_0, b_1, b_2, b_3) = (r^4 B_0, r^3 B_1, r^2 B_2, r B_3), \quad r \geq 0, \quad B = (B_0, B_1, B_2, B_3) \in \mathbb{S}^3.$$

Clearly, we can study a complete neighborhood of $b = 0$ by studying each value of (r, B) with $r \sim 0$ and with B on a 3-sphere. Instead of working with the spherical coordinates, in the proof of Theorem 1.1 it is more convenient to use one of the following 8 traditional charts (or regions) covering the 3-sphere:

- **Jump region**

$$(b_0, b_1, b_2, b_3) = (\pm r^4, r^3 B_1, r^2 B_2, r B_3), \quad (B_1, B_2, B_3) \in K_0, \quad \text{where } K_0 \text{ is a sufficiently large compact set in } \mathbb{R}^3.$$

- **Saddle region**

$(b_0, b_1, b_2, b_3) = (r^4 B_0, +r^3, r^2 B_2, r B_3)$, $B_0 \in U_1$, $(B_2, B_3) \in K_1$, where U_1 is a sufficiently small neighborhood of the origin in \mathbb{R} and where K_1 is a sufficiently large compact set in \mathbb{R}^2 .

- **Slow-fast Hopf region**

$(b_0, b_1, b_2, b_3) = (r^4 B_0, -r^3, r^2 B_2, r B_3)$, $B_0 \in U_1$, $(B_2, B_3) \in K_1$, where U_1 is a sufficiently small neighborhood of the origin in \mathbb{R} and where K_1 is a sufficiently large compact set in \mathbb{R}^2 .

- **Slow-fast Bogdanov-Takens region**

$(b_0, b_1, b_2, b_3) = (r^4 B_0, r^3 B_1, \pm r^2, r B_3)$, $(B_0, B_1) \in U_2$, $B_3 \in K_2$, where U_2 is a sufficiently small neighborhood of the origin in \mathbb{R}^2 and where K_2 is a sufficiently large compact set in \mathbb{R} .

- **Slow-fast codimension 3 saddle region**

$(b_0, b_1, b_2, b_3) = (r^4 B_0, r^3 B_1, r^2 B_2, +r)$, $(B_0, B_1, B_2) \in U_3$ where U_3 is a sufficiently small neighborhood of the origin in \mathbb{R}^3 .

- **Slow-fast codimension 3 elliptic region**

$(b_0, b_1, b_2, b_3) = (r^4 B_0, r^3 B_1, r^2 B_2, -r)$, $(B_0, B_1, B_2) \in U_3$ where U_3 is a sufficiently small neighborhood of the origin in \mathbb{R}^3 .

It is obvious that for any small U_1 , U_2 and U_3 we can take K_0 , K_1 and K_2 large enough such that we cover a complete neighborhood of the origin in the b -space by the chosen charts. More precisely, in the proof of Theorem 1.1 we first choose a sufficiently small U_3 and fix it. Then we take a K_2 as large as required and fix it. For the fixed U_3 and K_2 , and for a sufficiently small and fixed U_2 , we choose K_1 as large as needed. Finally, for the fixed U_3 , K_2 , U_2 , K_1 , and for a sufficiently small but fixed U_1 , we take a large K_0 . (See statements of Theorem 2.1–Theorem 2.5.) Note that the compact sets K_0 , K_1 and K_2 become larger as the size of U_1 , U_2 and U_3 tends to 0.

Taking into account this blow-up in the b -space, we arrive at an $(\epsilon, B, r, \lambda)$ -family $X_{\epsilon, B, r, \lambda}$ of planar vector fields where $X_{\epsilon, B, r, \lambda}$ stands for

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -xy + \epsilon(r^4 B_0 + r^3 B_1 x + r^2 B_2 x^2 + r B_3 x^3 + x^4 + x^5 G(x, \lambda)) \\ &+ \epsilon y^2 H(x, y, \lambda). \end{cases}$$

The family $X_{\epsilon, B, r, \lambda}$ can exhibit different kinds of limit cycle bifurcations near the origin in the (x, y) -plane, depending on the region in the parameter space b , i.e., depending on how (b_0, b_1, b_2, b_3) approaches $(0, 0, 0, 0)$. In each region defined above, we find maximum number of small-amplitude limit cycles of $X_{\epsilon, B, r, \lambda}$ (see Theorem 2.1–Theorem 2.5). Theorem 1.1 follows directly from Theorem 2.1–Theorem 2.5.

Theorem 2.1 (The jump region). *Let $B_0 = +1$ or $B_0 = -1$. Given any $B_i^1 > 0$, $i = 1, 2, 3$. There exist a neighborhood V of $(x, y) = (0, 0)$ and $r_0 > 0$ such that the family $X_{\epsilon, B, r, \lambda}$ has no periodic orbits in V for each $(\epsilon, B_1, B_2, B_3, r, \lambda) \in [0, M] \times [-B_1^1, B_1^1] \times [-B_2^1, B_2^1] \times [-B_3^1, B_3^1] \times [0, r_0] \times \Lambda$.*

Theorem 2.1 will be proved in Section 3.1. In the jump region a contact point of jump type described in Section 1 will appear after blowing up the origin in the (x, y, r) -space, in both regular and singular cases.

Theorem 2.2 (The saddle region). *Let $B_1 = +1$. Given any $B_i^1 > 0$, $i = 2, 3$. There exist a neighborhood V of $(x, y) = (0, 0)$, $r_0 > 0$ and $B_0^1 > 0$ such that $X_{\epsilon, B, r, \lambda}$ has no periodic orbits in V for each $(\epsilon, B_0, B_2, B_3, r, \lambda) \in [0, M] \times [-B_0^1, B_0^1] \times [-B_2^1, B_2^1] \times [-B_3^1, B_3^1] \times [0, r_0] \times \Lambda$.*

Theorem 2.2 will be proved in Section 3.2. A contact point of saddle type with the singularity order 1 will appear after blow-up of the origin $(x, y, r) = (0, 0, 0)$, in both regular and singular cases.

Theorem 2.3 (The slow-fast Hopf region). *Let $B_1 = -1$. Given any $B_i^1 > 0$, $i = 2, 3$. There exist a neighborhood V of $(x, y) = (0, 0)$, $r_0 > 0$ and $B_0^1 > 0$ such that $X_{\epsilon, B, r, \lambda}$ has at most 2 limit cycles in V for each $(\epsilon, B_0, B_2, B_3, r, \lambda) \in [0, M] \times [-B_0^1, B_0^1] \times [-B_2^1, B_2^1] \times [-B_3^1, B_3^1] \times [0, r_0] \times \Lambda$.*

Theorem 2.3 will be proved in Section 3.3. The most difficult part of the paper is dealing with the slow-fast Hopf region. After blowing up $(x, y, r) = (0, 0, 0)$, in both regular and singular cases, we find a (slow-fast) Hopf bifurcation of codimension 1 or 2 at the origin in the new phase coordinates (\bar{x}, \bar{y}) , and at $B_0 = 0$. The codimension depends on the parameter B_2 (see Section 3.3). As discribed in Section 1, we also have to consider all limit periodic sets that can generate detectable canard limit cycles in (\bar{x}, \bar{y}) -plane by perturbation. The gluing method, introduced in Section 1, becomes very involved because, to find all the limit periodic sets and to glue them together, we have to vary 2-dimensional parameter (B_2, B_3) kept in a large compact set.

Theorem 2.4 (The slow-fast Bogdanov-Takens region). *Let $B_2 = +1$ or $B_2 = -1$. Given any $B_3^1 > 0$. There exist a neighborhood V of $(x, y) = (0, 0)$, $r_0 > 0$, $B_0^1 > 0$, $B_1^1 > 0$ such that $X_{\epsilon, B, r, \lambda}$ has at most 1 (hyperbolic) limit cycle in V for each $(\epsilon, B_0, B_1, B_3, r, \lambda) \in [0, M] \times [-B_0^1, B_0^1] \times [-B_1^1, B_1^1] \times [-B_3^1, B_3^1] \times [0, r_0] \times \Lambda$.*

Theorem 2.4 will be proved in Section 3.4. Similar to previous cases, a blow-up of $(x, y, r) = (0, 0, 0)$ is needed to find a well-known slow-fast unfolding of a Bogdanov-Takens point described in Section 1, in both regular and singular cases.

Theorem 2.5 (The slow-fast codimension 3 saddle/elliptic region). *Let $B_3 = +1$ or $B_3 = -1$. There exist a neighborhood V of $(x, y) = (0, 0)$, $r_0 > 0$ and a (B_0, B_1, B_2) -neighborhood U_3 of the origin such that $X_{\epsilon, B, r, \lambda}$ has at most 2 limit cycles in V for each $(\epsilon, B_0, B_1, B_2, r, \lambda) \in [0, M] \times U_3 \times [0, r_0] \times \Lambda$.*

Though there is no distinction between the saddle case and the elliptic case in formulation of the statements in Theorem 2.5, the proof of Theorem 2.5 in the saddle case is different from the proof of Theorem 2.5 in the elliptic case. In the elliptic case, after blowing up $(x, y, r) = (0, 0, 0)$, we have to glue together a contact point of codimension 3 elliptic type, described in Section 1, and (detectable) limit periodic sets, in both regular and singular cases. In the saddle case, this gluing is not needed due to a special slow dynamics in (\bar{x}, \bar{y}) -plane. Theorem 2.5 in the saddle case (resp. in the elliptic case) will be proved in Section 3.5 (resp. in Section 3.6).

As mentioned above, one of the crucial steps in proving Theorem 2.1–Theorem 2.5 is the blow-up of the family $X_{\epsilon, B, r, \lambda}$ at the origin $(x, y, r) = (0, 0, 0)$.

2.2 Blow-up of the origin $(x, y, r) = (0, 0, 0)$ in charts

Our goal is to include r in the phase space and blow up the origin in (x, y, r) -space. This allows us to use the recursive approach explained in Section 1. We blow up the origin $(x, y, r) = (0, 0, 0)$ using the blow-up transformation

$$(x, y, r) = (u\bar{x}, u^2\bar{y}, u\bar{r}), \quad u \geq 0, \quad \bar{r} \geq 0, \quad (\bar{x}, \bar{y}, \bar{r}) \in \mathbb{S}^2. \quad (3)$$

This blow-up transforms the r -family of two-dimensional problems $X_{\epsilon, B, r, \lambda}$ into a less degenerate u -singular (see Section 2.2.1) three-dimensional problem, allowing us to replace an r -uniform neighborhood of $(x, y) = (0, 0)$ by a neighborhood inside $\{u \geq 0\}$ of a larger object, the so-called blow-up locus $\{(u, \bar{x}, \bar{y}, \bar{r}); u = 0, \bar{x}^2 + \bar{y}^2 + \bar{r}^2 = 1, \bar{r} \geq 0\}$ (see Figure 1). As usual, we study the dynamics in the blown-up coordinates in different charts.

Remark 2. *Note that in this paper our focus is not on a “classical” blow-up (see [DR96]) of nilpotent contact points in singular perturbation problems, which includes a singular perturbation parameter; that type of desingularization has already been done in [DR09], [DMD11a], [HDMD13], [HDMD14], [Huz16].*

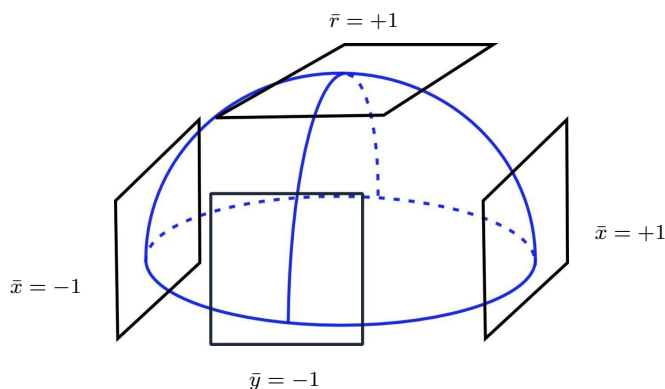


Figure 1: Different charts near the blow-up sphere in (x, y, r) -space. The chart $\{\bar{y} = +1\}$ is not shown: it is on the back side of the sphere and is the symmetric counterpart of $\{\bar{y} = -1\}$.

2.2.1 The family chart

We take $\bar{r} = +1$ in (3) and keep (\bar{x}, \bar{y}) in a large compact set D . In this traditional rescaling chart, after division by $u > 0$, the blown-up field is an $(\epsilon, B, u, \lambda)$ -family of 2-dimensional vector fields $X_{\epsilon, B, u, \lambda}^f$:

$$\begin{cases} \dot{\bar{x}} &= \bar{y} \\ \dot{\bar{y}} &= -\bar{x}\bar{y} + \epsilon u (B_0 + B_1\bar{x} + B_2\bar{x}^2 + B_3\bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x}, \lambda)) \\ &\quad + \epsilon u \bar{y}^2 H(u\bar{x}, u^2\bar{y}, \lambda). \end{cases} \quad (4)$$

The blown-up vector field can be also treated as an (ϵ, B, λ) -family of 3-dimensional vector fields if we add $\dot{u} = 0$ to (4).

Observe that because $u \sim 0$ we deal in (4) with a singular perturbation problem, in both the singular case ($\epsilon \sim 0$) and the regular case ($\epsilon \neq 0$ uniformly). One might think that this singular perturbation problem, in the singular case, is far more degenerate than the original $X_{\epsilon, B, r, \lambda}$ since the family $X_{\epsilon, B, u, \lambda}^f$ in the singular case is singularly perturbed in 2 parameters, u and ϵ . But this is not true. We introduce the following slow-fast system:

$$\begin{cases} \dot{\bar{x}} &= \bar{y} \\ \dot{\bar{y}} &= -\bar{x}\bar{y} + \bar{\epsilon}(B_0 + B_1\bar{x} + B_2\bar{x}^2 + B_3\bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x}, \lambda)) \\ &\quad + \bar{\epsilon}\bar{y}^2 H(u\bar{x}, u^2\bar{y}, \lambda), \end{cases} \quad (5)$$

where $\bar{\epsilon} \sim 0$ is the singular perturbation parameter. We denote the family (5) by $X_{\bar{\epsilon}, B, u, \lambda}^F$. The family $X_{\bar{\epsilon}, B, u, \lambda}^F$ is a special case of (2) with the singularity order at the contact point $(\bar{x}, \bar{y}) = (0, 0)$ at most 3 because $(B_0, B_1, B_2, B_3) \neq (0, 0, 0, 0)$ ($(B_0, B_1, B_2, B_3) \in \mathbb{S}^3$). Thus, we may apply the well-known results described in Section 1. On the other hand, since $X_{\epsilon, B, u, \lambda}^f = X_{\epsilon u, B, u, \lambda}^F$, we obtain complete information about the number of limit cycles in the family $X_{\epsilon, B, u, \lambda}^f$, for $\epsilon \in [0, M]$ and $u \sim 0$, by studying limit cycle bifurcations that may occur in $X_{\bar{\epsilon}, B, u, \lambda}^F$, for $\bar{\epsilon} \sim 0$ and $u \sim 0$. From now on, we thus focus on only the family $X_{\bar{\epsilon}, B, u, \lambda}^F$, for $\bar{\epsilon} \sim 0$ and $u \sim 0$, in the region D .

2.2.2 The phase-directional charts

Since our goal is to study limit cycles of $X_{\epsilon, B, r, \lambda}$ in a neighborhood of the origin in the (x, y) -plane that does not shrink to the origin as $r \rightarrow 0$, we also have to consider the dynamics in the blown-up coordinates in the phase-directional rescaling charts “ $\bar{x} = \pm 1, \bar{y} = \pm 1$ ” (Figure 1). Thus we have to study the dynamics of $X_{\epsilon, B, r, \lambda} + 0 \frac{\partial}{\partial r}$ in the blown-up coordinates near the “equator” $\{(u, \bar{x}, \bar{y}, \bar{r}); u = 0, \bar{r} = 0, \bar{x}^2 + \bar{y}^2 = 1\}$. It will be clear from the following analysis that the dynamics in an (ϵ, B, λ) -uniform neighborhood of the equator is like in Figure 2, where $(\epsilon, B, \lambda) \in]0, M] \times \mathbb{S}^3 \times \Lambda$.

1. The phase-directional chart $\{\bar{x} = +1\}$

In the phase-directional chart $\{\bar{x} = +1\}$ the blow-up map (3) has the form

$$(x, y, r) = (U, U^2\bar{Y}, U\bar{R}), \quad U \geq 0.$$

In these new coordinates $X_{\epsilon, B, r, \lambda} + 0 \frac{\partial}{\partial r}$ becomes (after division by $U > 0$):

$$\begin{cases} \dot{U} &= U\bar{Y} \\ \dot{\bar{R}} &= -\bar{R}\bar{Y} \\ \dot{\bar{Y}} &= -\bar{Y} - 2\bar{Y}^2 + \epsilon U(1 + O(U, \bar{R}, \bar{Y}^2)). \end{cases} \quad (6)$$

It is easy to see the following facts. On $\{U = 0, \bar{R} = 0\}$, (6) has singularities at $\bar{Y} = -\frac{1}{2}$ and $\bar{Y} = 0$. The first one is a hyperbolic (resonant) saddle (the eigenvalues of the linear part at this singularity are $(-\frac{1}{2}, \frac{1}{2}, 1)$), and the second one is a semi-hyperbolic singularity with the \bar{Y} -axis as stable manifold and a 2-dimensional center direction, transverse to the \bar{Y} -axis. An (ϵ, B, λ) -family of center manifolds at $(U, \bar{R}, \bar{Y}) = (0, 0, 0)$ is expressed by

$$\bar{Y} = \epsilon U(1 + O(U, \bar{R})),$$

and the related center behavior is given by

$$\{\dot{U} = \epsilon U^2(1 + O(U, \bar{R})), \dot{\bar{R}} = -\epsilon \bar{R} U(1 + O(U, \bar{R}))\}.$$

2. *The phase-directional chart $\{\bar{x} = -1\}$*

We study the part of the sphere where $\bar{x} \sim -1$ by applying the coordinate change $(U, \bar{R}, t) \rightarrow (-U, -\bar{R}, -t)$ to (6). Note that the directional blow-up formula in the chart $\{\bar{x} = -1\}$ is $(x, y, r) = (-U, U^2 \bar{Y}, U \bar{R})$, $U \geq 0$.

3. *The phase-directional charts $\{\bar{y} = +1\}$ and $\{\bar{y} = -1\}$*

The directional blow-up formula in the charts $\{\bar{y} = +1\}$ and $\{\bar{y} = -1\}$ is given by

$$(x, y, r) = (U \bar{X}, \pm U^2, U \bar{R}), \quad U \geq 0.$$

Besides the singularities we already found in the charts $\{\bar{x} = +1\}$ and $\{\bar{x} = -1\}$, there are no extra singularities in the part of the equator covered by the charts $\{\bar{y} = \pm 1\}$. The dynamics near the equator in the chart $\{\bar{y} = +1\}$ is regular pointing from left to right. Near $\bar{X} = 0$, the dynamics near the equator in the chart $\{\bar{y} = -1\}$ points from right to left.

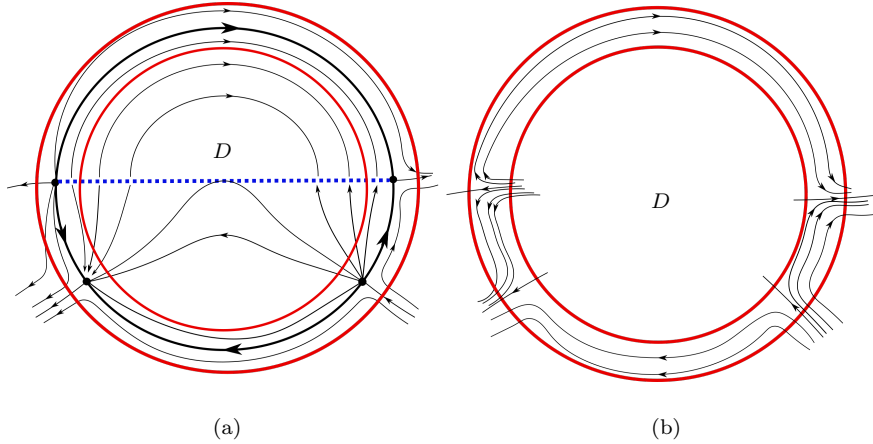


Figure 2: A bird's eye view of the blow-up of $(x, y, r) = (0, 0, 0)$, for $\epsilon \in]0, M]$. (a) The dynamics of $X_{\epsilon, B, r, \lambda} + 0 \frac{\partial}{\partial r}$ in the family chart (the region D) and in the phase-directional charts, for $r = U \bar{R} = 0$. The critical curve of $X_{\epsilon, B, 0, \lambda}^f$ connects two semi-hyperbolic singularities on the equator. (b) The dynamics of $X_{\epsilon, B, r, \lambda} + 0 \frac{\partial}{\partial r}$ in a fixed neighborhood of the equator, for each level $r = U \bar{R} > 0$, outside the region D .

It is clear from Figure 2(a) that the origin $(x, y) = (0, 0)$ of $X_{\epsilon, 0, \lambda}$ is a (nilpotent) saddle-node singularity, for each $(\epsilon, \lambda) \in]0, M] \times \Lambda$.

2.2.3 Combining the family chart and the phase-directional charts. Slow-fast analysis in the family chart

Figure 2 indicates that in order to prove Theorem 2.1–Theorem 2.5, we need to study only singular perturbation problem (5) in an arbitrarily large compact set

D . Indeed, it is obvious that orbits which spend some time in the neighborhood of the equator in Figure 2(b) cannot be closed in a small neighborhood of $(x, y) = (0, 0)$. Therefore all small-amplitude limit cycles have to be confined to D . From this together with (3) we can conclude that the size of the small-amplitude limit cycles in the (x, y) -plane tends to 0 as $r \rightarrow 0$, or equivalently as $b \rightarrow 0$.

The rest of the section is devoted to the study of the slow-fast structure of $X_{\bar{\epsilon}, B, u, \lambda}^F$ given by (5). The curve of singularities of $X_{0, B, u, \lambda}^F$ is $\{\bar{y} = 0\}$. We call the set $\{\bar{y} = 0\}$ the critical curve. As usual, the critical curve consists of semi-hyperbolic singularities, with the exception of the origin $(\bar{x}, \bar{y}) = (0, 0)$, where we have a nilpotent contact point. Note that the curve is normally attracting when $\bar{x} > 0$ and normally repelling when $\bar{x} < 0$. See the region D in Figure 2(a). Clearly, orbits of the so-called fast subsystem $X_{0, B, u, \lambda}^F$ are parabolas $\bar{y} = -\frac{1}{2}\bar{x}^2 + c$, and the dynamics of $X_{\bar{\epsilon}, B, u, \lambda}^F$, away from the critical curve, is given more or less by the dynamics of $X_{0, B, u, \lambda}^F$.

On the other hand, the dynamics of $X_{\bar{\epsilon}, B, u, \lambda}^F$ near the critical curve, for $\bar{x} \neq 0$, can be studied using the so-called slow dynamics. More precisely, center manifolds at the semi-hyperbolic singularity $(\bar{x}, \bar{y}, \bar{\epsilon}) = (\bar{x}, 0, 0)$, $\bar{x} \neq 0$, of $X_{\bar{\epsilon}, B, u, \lambda}^F + 0 \frac{\partial}{\partial \bar{\epsilon}}$ are given by

$$\bar{y} = \bar{\epsilon} \left(\frac{B_0 + B_1 \bar{x} + B_2 \bar{x}^2 + B_3 \bar{x}^3 + \bar{x}^4 + u \bar{x}^5 G(u \bar{x}, \lambda)}{\bar{x}} + O(\bar{\epsilon}) \right).$$

From the first equation in (5) we obtain the dynamics inside these center manifolds:

$$\dot{\bar{x}} = \frac{d\bar{x}}{dt} = \bar{\epsilon} \left(\frac{B_0 + B_1 \bar{x} + B_2 \bar{x}^2 + B_3 \bar{x}^3 + \bar{x}^4 + u \bar{x}^5 G(u \bar{x}, \lambda)}{\bar{x}} + O(\bar{\epsilon}) \right).$$

We find the slow dynamics along the critical curve (but outside the contact point) after dividing the last equation by $\bar{\epsilon}$ and letting $\bar{\epsilon} \rightarrow 0$:

$$\bar{x}' = \frac{d\bar{x}}{d\tau} = \frac{d\bar{x}}{\bar{\epsilon} dt} = \frac{B_0 + B_1 \bar{x} + B_2 \bar{x}^2 + B_3 \bar{x}^3 + \bar{x}^4 + u \bar{x}^5 G(u \bar{x}, \lambda)}{\bar{x}}, \quad \bar{x} \neq 0. \quad (7)$$

The reader is referred to [DR96] for more details about the definition of slow dynamics.

There are essentially two kinds of closed curves (so-called limit periodic sets) that may bifurcate in limit cycles of $X_{\bar{\epsilon}, B, u, \lambda}^F$, for $\bar{\epsilon} > 0$ and $u > 0$: *canard limit periodic sets* $\Gamma_{\bar{y}}$, $\bar{y} > 0$, consisting of the orbit of the fast subsystem through the point $(0, \bar{y})$ and the piece of the critical curve between the α -limit set and the ω -limit set of that fast orbit, and *the nilpotent contact point* $(\bar{x}, \bar{y}) = (0, 0)$ from which so-called small-amplitude limit cycles in the (\bar{x}, \bar{y}) -plane may bifurcate. Clearly, if limit cycles appear near $\Gamma_{\bar{y}}$, then the slow dynamics allows the passage from the attracting part of the critical curve to the repelling part of the critical curve, for some parameters (B, u, λ) . Of course, the passage near the contact point has to be studied separately from the rest of the critical curve since the slow dynamics is not defined at $\bar{x} = 0$. As mentioned in Section 1, the passage near the contact point and the limit cycles bifurcating from the contact point, in the slow-fast system $X_{\bar{\epsilon}, B, u, \lambda}^F$, have already been dealt with before (see Section 3).

3 Proofs of Theorem 2.1–Theorem 2.5

In this section we prove Theorem 2.1–Theorem 2.5. We focus on $X_{\bar{\epsilon}, B, u, \lambda}^F$ in an arbitrarily large (but fixed) compact set D in the (\bar{x}, \bar{y}) -plane (Figure 2) and, depending on the chosen region in the parameter space b (see Section 2.1), we detect all limit periodic sets, for $\bar{\epsilon} = u = 0$, in D that can generate limit cycles by perturbation, for $\bar{\epsilon} > 0$ and $u > 0$. We find maximum number of limit cycles near each such limit periodic set, and we glue together the local results to obtain the cyclicity of D .

3.1 The jump region

In the jump region, we consider the slow-fast system $X_{\bar{\epsilon}, (\pm 1, B_1, B_2, B_3), u, \lambda}^F$ where the parameters (B_1, B_2, B_3) are kept in an arbitrary compact set, $\bar{\epsilon} \sim 0$, $u \sim 0$ and where $\lambda \in \Lambda$. The contact point of $X_{\bar{\epsilon}, (\pm 1, B_1, B_2, B_3), u, \lambda}^F$ is of jump type and, as explained in Section 1, there are no limit cycles. We also refer to [HDMD13], Section 3.4, for a detailed study of the jump case. Indeed, after making a blow-up at the origin $(\bar{x}, \bar{y}, \bar{\epsilon}) = (0, 0, 0)$, we see that there are no singularities of $X_{\bar{\epsilon}, (\pm 1, B_1, B_2, B_3), u, \lambda}^F$ in a fixed neighborhood of the origin in the (\bar{x}, \bar{y}) -plane for $\bar{\epsilon} > 0$. Thus there are no small-amplitude periodic orbits of $X_{\bar{\epsilon}, (\pm 1, B_1, B_2, B_3), u, \lambda}^F$ under the given conditions on the parameters. Since the slow dynamics (7) in the jump region does not point from the right to the left for $\bar{x} \sim 0$ and $\bar{x} \neq 0$ ($\bar{x}' = \frac{\pm 1 + O(\bar{x})}{\bar{x}}$), we have no limit cycles close to canard limit periodic sets $\Gamma_{\bar{y}}$. This concludes the proof of Theorem 2.1.

3.2 The saddle region

The proof of Theorem 2.2 can be found in [HDMD13], Section 3.5. However, for the sake of completeness, we give a sketch of the proof of this result. We consider $X_{\bar{\epsilon}, (B_0, 1, B_2, B_3), u, \lambda}^F$ where the parameters (B_2, B_3) are kept in an arbitrary compact set in \mathbb{R}^2 , $\bar{\epsilon} \sim 0$, $B_0 \sim 0$, $u \sim 0$ and where $\lambda \in \Lambda$. After blowing up the origin $(\bar{x}, \bar{y}, \bar{\epsilon}) = (0, 0, 0)$, it can be easily seen that there is one hyperbolic saddle of $X_{\bar{\epsilon}, (B_0, 1, B_2, B_3), u, \lambda}^F$ in a fixed neighborhood of $(\bar{x}, \bar{y}) = (0, 0)$ for $\bar{\epsilon} \sim 0$, $\bar{\epsilon} > 0$, $B_0 \sim 0$, $u \sim 0$ (see [HDMD13]). Thus we have no small-amplitude periodic orbits under the given conditions on the parameters. Since the slow dynamics (7) in the saddle case points from the left to the right near $\bar{x} = 0$ ($\bar{x}' = 1 + O(\bar{x}) > 0$, for $B_0 = 0$), no limit cycles bifurcate from $\Gamma_{\bar{y}}$.

3.3 The slow-fast Hopf region

In this section we prove Theorem 2.3. Besides the proof of Theorem 2.5 in the elliptic case, the proof of Theorem 2.3 is the technically most difficult part of this paper.

We consider the singular perturbation system $X_{\bar{\epsilon}, (B_0, -1, B_2, B_3), u, \lambda}^F$ and, like in Section 3.2, we suppose that $(\bar{\epsilon}, B_0, u) \sim (0, 0, 0)$, $\lambda \in \Lambda$ and $(B_2, B_3) \in [-B_2^1, B_2^1] \times [-B_3^1, B_3^1]$ where $B_2^1 > 0$ and $B_3^1 > 0$ are arbitrarily large, but fixed, real numbers. We write $\mathcal{B} = [-B_2^1, B_2^1] \times [-B_3^1, B_3^1]$. From Section 3.7 of [HDMD13], it follows that, in order to prove Theorem 2.3, it suffices to study

the singular perturbation system $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$:

$$\begin{cases} \dot{\bar{x}} &= \bar{y} \\ \dot{\bar{y}} &= -\bar{x}\bar{y} + \delta^2(\delta \bar{B}_0 - \bar{x} + B_2\bar{x}^2 + B_3\bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x}, \lambda)) \\ &\quad + \delta^2\bar{y}^2 H(u\bar{x}, u^2\bar{y}, \lambda), \end{cases} \quad (8)$$

where $\delta \sim 0$ is the new singular perturbation parameter, $(\bar{B}_0, u) \sim (0, 0)$, $(B_2, B_3) \in \mathcal{B}$ and $\lambda \in \Lambda$. Indeed, we have to make the following rescaling:

$$(\bar{\epsilon}, B_0) = (\delta^2 E, \delta \bar{B}_0), \quad \delta \geq 0, \quad \delta \sim 0, \quad E \geq 0, \quad (E, \bar{B}_0) \in \mathbb{S}^1.$$

As usual, we use different charts covering the sphere \mathbb{S}^1 . When $(\bar{\epsilon}, B_0) = (\delta^2 E, \pm \delta)$, with $E \sim 0$ and $E \geq 0$, then after a blow-up $(\bar{x}, \bar{y}, \delta) = (v\tilde{x}, v^2\tilde{y}, v\tilde{\delta})$, with $v \geq 0$, $\tilde{\delta} \geq 0$ and $(\tilde{x}, \tilde{y}, \tilde{\delta}) \in \mathbb{S}^2$, at $(\bar{x}, \bar{y}, \delta) = (0, 0, 0)$, the system $X_{\delta^2 E, (\pm \delta, -1, B_2, B_3), u, \lambda}^F$ in the family chart $\{\tilde{\delta} = 1\}$ becomes (after division by v):

$$\begin{cases} \dot{\tilde{x}} &= \tilde{y} \\ \dot{\tilde{y}} &= -\tilde{x}\tilde{y} + E(\pm 1 + O(\tilde{x})) + EO(\tilde{y}^2) \end{cases}$$

The origin $(\tilde{x}, \tilde{y}) = (0, 0)$ is now a jump point for this slow-fast system and limit cycles cannot appear. Therefore it suffices to deal with the chart $\{E = 1\}$ in which we have $(\bar{\epsilon}, B_0) = (\delta^2, \delta \bar{B}_0)$, with \bar{B}_0 in a large compact set. After studying the system $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ in different charts of the blow-up at $(\bar{x}, \bar{y}, \delta) = (0, 0, 0)$, given by

$$(\bar{x}, \bar{y}, \delta) = (v\tilde{x}, v^2\tilde{y}, v\tilde{\delta}), \quad v \geq 0, \quad \tilde{\delta} \geq 0, \quad (\tilde{x}, \tilde{y}, \tilde{\delta}) \in \mathbb{S}^2, \quad (9)$$

we find that limit cycles of $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ occur only if $\bar{B}_0 \sim 0$ (for $\bar{B}_0 = 0$ a center appears on the blow-up locus of the blow-up (9)). We refer to [HDMD13] for more details.

From the following theorem it follows that the limit cycles may bifurcate from the contact point $(\bar{x}, \bar{y}) = (0, 0)$ in the slow-fast system (8), for $\delta > 0$ and $\bar{B}_0 \sim 0$. In other words, at $\bar{B}_0 = 0$, a (slow-fast) Hopf bifurcation takes place.

Theorem 3.1. *(i) Let $B_2^0 > 0$ be any arbitrarily small fixed number and let $K := \mathcal{B} \cap \{|B_2| \geq B_2^0\}$. There exist small $\delta_0 > 0$, $\bar{B}_0^0 > 0$, $u_0 > 0$ and a neighborhood U of $(\bar{x}, \bar{y}) = (0, 0)$ such that the following statements are true.*

- 1 *The family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 1 (hyperbolic) limit cycle in U for each $(\delta, \bar{B}_0, B_2, B_3, u, \lambda) \in [0, \delta_0] \times [-\bar{B}_0^0, \bar{B}_0^0] \times K \times [0, u_0] \times \Lambda$.*
- 2 *When we fix $(\delta, B_2, B_3, u, \lambda) \in [0, \delta_0] \times K \times [0, u_0] \times \Lambda$, the \bar{B}_0 -family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ undergoes, in U and at $\bar{B}_0 = 0$, a Hopf bifurcation of codimension 1. Assume $(B_2, B_3) \in K$ and $B_2 > 0$. When \bar{B}_0 increases there is in U an attracting hyperbolic focus and no limit cycle; when \bar{B}_0 decreases there is in U a repelling hyperbolic focus and an attracting limit cycle of which the size monotonically grows as \bar{B}_0 decreases. Assume $(B_2, B_3) \in K$ and $B_2 < 0$. When \bar{B}_0 decreases there is in U a repelling hyperbolic focus and no limit cycle; when \bar{B}_0 increases there is in U an attracting hyperbolic focus and a repelling limit cycle of which the size monotonically grows as \bar{B}_0 increases.*

- (ii) There exist small $\delta_0 > 0$, $\bar{B}_0^0 > 0$, $B_2^0 > 0$ and $u_0 > 0$ and a neighborhood U of $(\bar{x}, \bar{y}) = (0, 0)$ such that the family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 2 limit cycles in U for each $(\delta, \bar{B}_0, B_2, B_3, u, \lambda) \in [0, \delta_0] \times [-\bar{B}_0^0, \bar{B}_0^0] \times \mathcal{B} \cap \{|B_2| \leq B_2^0\} \times [0, u_0] \times \Lambda$.

Proof. Let's prove (i). The contact point $(\bar{x}, \bar{y}) = (0, 0)$ in slow-fast system (8) is a slow-fast Hopf point of codimension 1 (for the exact definition of the notion of slow-fast Hopf point we refer to [DR09]). To see it we refer to [HDMD13], Section 3.7. Note that the codimension is 1 because the coefficient B_2 in front of the quadratic term $\bar{x}^2 \frac{\partial}{\partial \bar{y}}$ in (8) is nonzero. From [HDMD13] it follows that (8) has at most 1 (hyperbolic) limit cycle in a $(\delta, \bar{B}_0, B_2, B_3, u, \lambda)$ -uniform neighborhood of the contact point, and that, for $\delta > 0$ and at $\bar{B}_0 = 0$, a (slow-fast) Hopf bifurcation of codimension 1 takes place, with the properties described in the statement 2.

The statement (ii) follows from [Huz16], Sections 3–5. For the sake of completeness, we give here a sketch of the proof of (ii). Using the coordinate change $Y = \bar{y} + \frac{1}{2}\bar{x}^2$, we arrive at a representation of (8) in the so-called Liénard plane:

$$\begin{cases} \dot{\bar{x}} &= \bar{y} - \frac{1}{2}\bar{x}^2 \\ \dot{\bar{y}} &= \delta^2(\delta \bar{B}_0 - \bar{x} + B_2 \bar{x}^2 + B_3 \bar{x}^3 + \bar{x}^4 + u \bar{x}^5 G(u \bar{x}, \lambda)) \\ &+ \delta^2(\bar{y} - \frac{1}{2}\bar{x}^2)^2 H(u \bar{x}, u^2(\bar{y} - \frac{1}{2}\bar{x}^2), \lambda), \end{cases} \quad (10)$$

where we denote Y again by \bar{y} . Since we assume $B_2 \sim 0$ and the coefficient in front of the quartic term $\bar{x}^4 \frac{\partial}{\partial \bar{y}}$ in (10) is nonzero, the origin $(\bar{x}, \bar{y}) = (0, 0)$ is a slow-fast Hopf point of codimension 2 (see [DR09]). To prove that the cyclicity of $(\bar{x}, \bar{y}) = (0, 0)$ in (10) is bounded by 2, we need to blow up the origin $(\bar{x}, \bar{y}, \delta) = (0, 0, 0)$ using the blow-up formula (9). In the family directional chart $\{\delta = +1\}$ (10) becomes (after division by $v > 0$):

$$\begin{cases} \dot{\tilde{x}} &= \tilde{y} - \frac{1}{2}\tilde{x}^2 \\ \dot{\tilde{y}} &= \bar{B}_0 - \tilde{x} + B_2 v \tilde{x}^2 + B_3 v^2 \tilde{x}^3 + v^3(\tilde{x}^4 + H(0, 0, \lambda)(\tilde{y} - \frac{1}{2}\tilde{x}^2)^2 + O(v)). \end{cases} \quad (11)$$

We denote this vector field by $X_{v, \bar{B}_0, B_2, B_3, u, \lambda}$. It can be easily seen that the vector field $X_{0, 0, B_2, B_3, u, \lambda}$ is of center type with the center at $(\tilde{x}, \tilde{y}) = (0, 0)$ and that $-e^{-\tilde{y}} X_{0, 0, B_2, B_3, u, \lambda}$ is a Hamiltonian vector field and its Hamiltonian is $\mathcal{H}(\tilde{x}, \tilde{y}) = e^{-\tilde{y}}(\tilde{y} - \frac{1}{2}\tilde{x}^2 + 1)$. We write $J_0(h) = \int_{\gamma_h} e^{-\tilde{y}} d\tilde{x}$, $J_1(h) = \int_{\gamma_h} e^{-\tilde{y}} \tilde{x}^2 d\tilde{x}$ and $J_2(h) = \int_{\gamma_h} e^{-\tilde{y}} \tilde{x}^4 d\tilde{x} + H(0, 0, \lambda) \int_{\gamma_h} e^{-\tilde{y}} (\tilde{y} - \frac{1}{2}\tilde{x}^2)^2 d\tilde{x}$, where $\gamma_h := \{\mathcal{H}(\tilde{x}, \tilde{y}) = h\}$, $h \in]0, 1]$, is a closed curve oriented counter-clockwise. Based on [FTV13], it was shown in Section 4 of [Huz16] that the system $\{\frac{d}{dh} J_0, \frac{d}{dh} J_1, \frac{d}{dh} J_2\}$ of analytic functions is a strict Chebyshev system on $[h_0, 1]$ of degree 2, for all small $h_0 > 0$ (For a definition of a strict Chebyshev system we refer to [DR09]). This implies that any fixed compact set in the (\tilde{x}, \tilde{y}) -plane can produce at most 2 limit cycles (we refer once more to Section 4 of [Huz16] for more details).

The size of these limit cycles tends to 0 as $\delta = v \rightarrow 0$. Therefore we also have to consider limit cycles in the (\bar{x}, \bar{y}) -plane that are unbounded in the (\tilde{x}, \tilde{y}) -plane and close to the origin in the (\bar{x}, \bar{y}) -plane, for which the essential part of the study has to be done in the phase directional charts of (9). They have been studied in Section 3 of [Huz16] by using strict Chebyshev systems of degree 2.

In Section 5 of [Huz16], it has been proven that the cyclicity of $(\bar{x}, \bar{y}) = (0, 0)$ in (10) is bounded by 2 by gluing the local results and constructing a “global” strict Chebyshev system of degree 2. \square

On the other hand, the slow dynamics of (8), which is given by

$$\bar{x}' = -1 + B_2\bar{x} + B_3\bar{x}^2 + \bar{x}^3 + u\bar{x}^4G(u\bar{x}, \lambda), \quad (12)$$

points from the right to the left at least near $\bar{x} = 0$. Thus, canard limit cycles can also arise in the (\bar{x}, \bar{y}) -plane. Now let's detect all the canard limit periodic sets $\Gamma_{\bar{y}}$ in (8) from which the canard limit cycles may bifurcate. Clearly, since $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$ in (8), the limit periodic sets $\Gamma_{\bar{y}}$ have to be studied at $(\delta, \bar{B}_0, u) = (0, 0, 0)$. The following analysis of the slow dynamics (12) will show that the limit periodic sets $\Gamma_{\bar{y}}$ are like in Figure 3.

Let's write

$$d(\bar{x}, B_2, B_3, u, \lambda) = -1 + B_2\bar{x} + B_3\bar{x}^2 + \bar{x}^3 + u\bar{x}^4G(u\bar{x}, \lambda).$$

The discriminant

$$J = B_2^2B_3^2 - 4B_2^3 + 4B_3^3 - 18B_2B_3 - 27$$

of the cubic \bar{x} -polynomial $d(\bar{x}, B_2, B_3, 0, \lambda) \in \mathbb{R}[\bar{x}]$ can be used to find out how many real zeros $d(\bar{x}, B_2, B_3, 0, \lambda)$ has. It is obvious that the cubic polynomial $d(\bar{x}, B_2, B_3, 0, \lambda)$ has at least one positive zero. The set $\{J = 0\}$ in the parameter space (B_2, B_3) is symmetric with respect to the line $B_3 = -B_2$, and consists of two curves: $SN_1 \cup \{(-1, 1)\} \cup SN_2$ and the cusp curve $SN_3 \cup \{(3, -3)\} \cup SN_4$. For each parameter value $(B_2, B_3) \in SN_1 \cup \{(-1, 1)\} \cup SN_2$ (resp. $(B_2, B_3) \in SN_3 \cup SN_4$), $d(\bar{x}, B_2, B_3, 0, \lambda)$ has a negative double zero (resp. a positive double zero). For $(B_2, B_3) = (3, -3)$, $d(\bar{x}, B_2, B_3, 0, \lambda)$ has a (positive) triple zero. When (B_2, B_3) is in the region between the curves $SN_1 \cup \{(-1, 1)\} \cup SN_2$ and $SN_3 \cup \{(3, -3)\} \cup SN_4$ ($J < 0$), the cubic polynomial has one real zero. For each (B_2, B_3) in the region above the curve $SN_1 \cup \{(-1, 1)\} \cup SN_2$ (resp. under the curve $SN_3 \cup \{(3, -3)\} \cup SN_4$), the cubic polynomial has 2 distinct negative zeros and 1 positive zero (resp. 3 distinct positive zeros). We have $J > 0$ for each (B_2, B_3) in these two regions.

Remark 3. *The zeros of $d(\bar{x}, B_2, B_3, 0, \lambda)$ correspond to the singularities of the slow dynamics (12) for $u = 0$. We denote the negative singularities by \bar{x}_L^1, \bar{x}_L^2 and the positive singularities by $\bar{x}_R^1, \bar{x}_R^2, \bar{x}_R^3$.*

The limit periodic sets $\Gamma_{\bar{y}}$ are very diverse and their fast orbit may end up in a regular point of the slow dynamics on both sides of the critical curve, in a singularity of the slow dynamics on one side of the critical curve, or in a singularity on both sides of the critical curve. In order to detect those $\Gamma_{\bar{y}}$ the fast orbit of which ends up in a zero of $d(\bar{x}, B_2, B_3, 0, \lambda)$ on both sides of the critical curve (note that $-\bar{x}_L^1 = \bar{x}_R^1$ in Figure 3(1),(n)), we have to solve the following system:

$$\begin{cases} d(\bar{x}, B_2, B_3, 0, \lambda) = 0 \\ d(-\bar{x}, B_2, B_3, 0, \lambda) = 0, \end{cases} \quad (13)$$

where $\bar{x} > 0$. After adding and subtracting these two equations, we get

$$\bar{x} = \sqrt{-B_2} = \frac{1}{\sqrt{B_3}}, \quad B_2 < 0, B_3 > 0. \quad (14)$$

Thus, the system (13) has a solution $\bar{x} > 0$ if and only if $(B_2, B_3) \in \{(B_2, B_3) | -B_2 B_3 = 1, B_2 < 0, B_3 > 0\} \cap \mathcal{B}$ (see the curve $C_1 \cup \{(-1, 1)\} \cup C_2$ in Figure 3). When the solution exists, it is unique and given by (14). When $(B_2, B_3) \in C_2$, a $\Gamma_{\bar{y}}$ with the above property cannot occur because the zero $\bar{x}_L^1 \in]\bar{x}_L^2, \bar{x}_R^1[$ does not allow the passage from the right to the left on the interval $[\bar{x}_L^2, \bar{x}_R^1]$ in Figure 3(k).

Remark 4. *It can be easily seen that $\bar{x}_1 \neq 0$ is a zero of $d(\bar{x}, B_2, B_3, 0, \lambda)$ if and only if $\frac{1}{\bar{x}_1}$ is a zero of $d(\bar{x}, -B_3, -B_2, 0, \lambda)$. Thus, if we know the limit periodic sets $\Gamma_{\bar{y}}$ for $(B_2, B_3) \in \mathcal{B}$, then we can easily find the limit periodic sets $\Gamma_{\bar{y}}$ for $(-B_3, -B_2) \in \mathcal{B}$.*

The value $\bar{x}_0 = \sqrt{-B_2}$, $B_2 < 0$, tells us when to use Lemma 3.3(iii). More precisely, in Figures 3(g) to 3(o) we find the limit periodic sets $\Gamma_{\bar{y}}$ with the property that $\bar{y} < \frac{-B_2}{2} = \frac{\bar{x}_0^2}{2}$. The cyclicity of those limit periodic sets can be studied by using Lemma 3.3(iii). We refer to Sections 3.3.2–3.3.6 for more details.

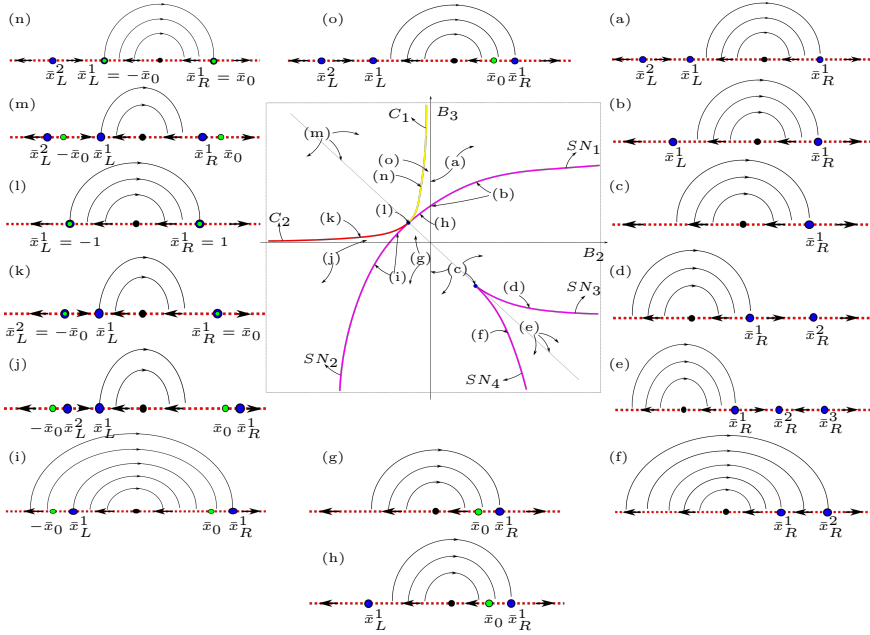


Figure 3: Canard limit periodic sets $\Gamma_{\bar{y}}$, for $(\delta, \bar{B}_0, u) = (0, 0, 0)$ and $(B_2, B_3) \in \mathcal{B}$, with indication of slow dynamics. Singularities of the slow dynamics (12), for $u = 0$, are denoted by $\bar{x}_L^1, \bar{x}_L^2, \bar{x}_R^1, \bar{x}_R^2, \bar{x}_R^3$. The value $\bar{x}_0 = \sqrt{-B_2}$, $B_2 < 0$, tells us where to use Lemma 3.3(iii).

Since the slow dynamics (12) is regular with possible isolated singularities located away from the contact point $(\bar{x}, \bar{y}) = (0, 0)$, to find the cyclicity of

the limit periodic sets $\Gamma_{\bar{y}}$ given in Figure 3, we can use the results given in [DMD08]. Following [DMD08], we study the limit cycles of $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ near $\cup_{\bar{y} \in [\mu, \eta]} \Gamma_{\bar{y}}$, $0 < \mu < \eta$, as zeros of a difference map. We first define a section $S = \{\tilde{x} = 0\}$, parametrized by $\bar{y} \in [\mu, \eta]$, and a second section $T = \{\tilde{x} = 0\}$ that we define along the blow up locus of the blow-up (9) at the origin $(\tilde{x}, \tilde{y}) = (0, 0)$. The section T is located in the family chart $\{\tilde{\delta} = 1\}$ and parametrized by the blow-up coordinate \tilde{y} . By following the orbits of $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ in forward and backward time we can define transition maps from S to T , denoted by respectively F_1 and F_2 . Closed orbits of $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ are given by zeros of the difference map

$$\Delta = F_1 - F_2.$$

Following [DMD08], $\frac{\partial \Delta}{\partial \bar{y}}$ can be given in terms of a divergence integral. More precisely, if we write $\tau = (\delta, \bar{B}_0, B_2, B_3, u, \lambda)$, then we have

$$\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) = -\frac{1}{\delta^4} L_+(\bar{y}, \tau) \exp \mathcal{I}_+(\bar{y}, \tau) - \left(-\frac{1}{\delta^4} L_-(\bar{y}, \tau) \exp \mathcal{I}_-(\bar{y}, \tau) \right) \quad (15)$$

where L_{\pm} are strictly positive functions due to the chosen parameterizations of S and T , and where

$$\mathcal{I}_{\pm}(\bar{y}, \tau) = \int_{\mathcal{O}^{\pm}(\bar{y}, \tau)} \operatorname{div} (\pm X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F) dt$$

where $\mathcal{O}^+(\bar{y}, \tau)$ (resp. $\mathcal{O}^-(\bar{y}, \tau)$) is the orbit through the point $(0, \bar{y}) \in S$, in positive time (resp. in negative time) until it hits the section T . If we introduce the analytic function $A(\alpha, \beta) = \frac{\exp \alpha - \exp \beta}{\alpha - \beta} > 0$ if $\alpha \neq \beta$ and $A(\alpha, \alpha) = \exp \alpha$ (see [DR09]), and if we write $\mathcal{I} = \mathcal{I}_+ - \mathcal{I}_-$, then (15) can be written as

$$\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) = -\frac{1}{\delta^6} A(\alpha, \beta) \left(\delta^2 \mathcal{I}(\bar{y}, \tau) + O(\delta^2) \right) \quad (16)$$

with

$$\alpha = \mathcal{I}_+(\bar{y}, \tau) + \ln(L_+(\bar{y}, \tau))$$

and

$$\beta = \mathcal{I}_-(\bar{y}, \tau) + \ln(L_-(\bar{y}, \tau)).$$

We don't specify the $O(\delta^2)$ -term in (16) since it is not the leading order part in the expression $\delta^2 \mathcal{I} + O(\delta^2)$. We refer to [DMD08] for more details. Using Rolle's theorem, it can be shown that the number of periodic orbits of $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ near the set $\cup_{\bar{y} \in [\mu, \eta]} \Gamma_{\bar{y}}$, at the τ -level, is bounded by 1+ the number of zeros (counting multiplicity) of $\delta^2 \mathcal{I}$ w.r.t. $\bar{y} \in [\mu, \eta]$.

The following is a simple but important observation.

Lemma 3.2. *Suppose that, for a fixed parameter τ , $\delta^2 \mathcal{I}$ has precisely one zero (counting multiplicity) in $[\mu, \eta]$ which we denote by \bar{y}_0 . If $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}_0, \tau) < 0$ and if two limit cycles occur near the set $\cup_{\bar{y} \in [\mu, \eta]} \Gamma_{\bar{y}}$, then the smaller limit cycle (resp. the bigger limit cycle) has to be (hyperbolically) repelling (resp. attracting).*

Proof. The lemma follows directly from (16). \square

Lemma 3.2 helps us glue local cyclicity results together in Sections 3.3.3–3.3.6.

It follows from [DMD08] that the main tool for studying zeros of $\delta^2\mathcal{I}$ is the slow divergence integral along $[-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$:

$$I(\bar{y}, B_2, B_3, u, \lambda) = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{wdw}{d(w, B_2, B_3, u, \lambda)}. \quad (17)$$

Note that the divergence of (8) on the critical curve $\{\bar{y} = 0\}$ is $-\bar{x}$ for $\delta = 0$, while $dt = \frac{dw}{d(w, B_2, B_3, u, \lambda)}$. The α -limit set (resp. the ω -limit set) of the orbit of the fast subsystem $X_{0, B_2, B_3, u, \lambda}^F$ through the point $(0, \bar{y})$ is $\{(-\sqrt{2\bar{y}}, 0)\}$ (resp. $\{(\sqrt{2\bar{y}}, 0)\}$). The slow divergence integral I is well defined for any value $(\bar{y}, B_2, B_3, u, \lambda)$, with the property that $d(w, B_2, B_3, u, \lambda) < 0$ for all $w \in [-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$. Clearly, I is well defined for $\bar{y} \sim 0$.

Lemma 3.3. *The following statements are true:*

- (i) *Take any small $\mu > 0$. There is a small $u_0 > 0$ such that for any value $(\bar{y}_1, B_2, B_3, u, \lambda) \in [\mu, \frac{1}{\mu}] \times \mathcal{B} \times [0, u_0] \times \Lambda$, with the property that $d(w, B_2, B_3, u, \lambda) < 0$ for all $w \in [-\sqrt{2\bar{y}_1}, \sqrt{2\bar{y}_1}]$, the slow divergence integral $I(\bar{y}, B_2, B_3, u, \lambda)$ has at most 1 zero (counting multiplicity) w.r.t $\bar{y} \in [\mu, \bar{y}_1]$.*
- (ii) *Take any small $\mu > 0$. There exist sufficiently small $\rho_0 > 0$, $B_2^0 > 0$ and $u_0 > 0$ such that for any $(\bar{y}, B_2, B_3, u, \lambda) \in [\mu, \frac{1}{\mu}] \times (\mathcal{B} \cap \{B_2 \geq -B_2^0\}) \times [0, u_0] \times \Lambda$, with the property that $d(w, B_2, B_3, u, \lambda) < 0$ for all $w \in [-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$, we have that $I(\bar{y}, B_2, B_3, u, \lambda) < -\rho_0$.*
- (iii) *Let $\mu > 0$ and $B_2^0 > 0$ be arbitrarily small and $\mu < \frac{B_2^0}{4}$. There exist small $\rho_0 > 0$ and $u_0 > 0$ so that for each $(B_2, B_3, u, \lambda) \in (\mathcal{B} \cap \{B_2 \leq -B_2^0\}) \times [0, u_0] \times \Lambda$ and $\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]$, with the property that $d(w, B_2, B_3, u, \lambda) < 0$ for all $w \in [-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$, we have that $I(\bar{y}, B_2, B_3, u, \lambda) > \rho_0$.*

Proof. First of all we notice that

$$\begin{aligned} I(\bar{y}, B_2, B_3, u, \lambda) &= \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{wdw}{d(w, B_2, B_3, u, \lambda)} \\ &= \int_0^{\sqrt{2\bar{y}}} w \left(\frac{1}{d(w, B_2, B_3, u, \lambda)} - \frac{1}{d(-w, B_2, B_3, u, \lambda)} \right) dw \\ &= \int_0^{\sqrt{2\bar{y}}} \frac{-2w^2(B_2 + w^2 + uO(w^3))}{d(w, B_2, B_3, u, \lambda) \cdot d(-w, B_2, B_3, u, \lambda)} dw. \end{aligned} \quad (18)$$

Let's prove the statement (i). Clearly, the derivative of I with respect to \bar{y} is given by

$$\frac{\partial I}{\partial \bar{y}}(\bar{y}, B_2, B_3, u, \lambda) = \frac{-2\sqrt{2\bar{y}}(B_2 + 2\bar{y} + uO((\sqrt{2\bar{y}})^3))}{d(\sqrt{2\bar{y}}, B_2, B_3, u, \lambda) \cdot d(-\sqrt{2\bar{y}}, B_2, B_3, u, \lambda)}. \quad (19)$$

Since $(B_2, B_3) \in \mathcal{B}$, $u \sim 0$ and \bar{y} in a compact set $[\mu, \frac{1}{\mu}]$, the expression (19) has at most 1 zero (counting multiplicity) w.r.t $\bar{y} \in]0, \bar{y}_1]$, where $\bar{y}_1 \in [\mu, \frac{1}{\mu}]$ has the

property from the statement (i). Using Rolle's theorem we find that the slow divergence integral I has at most 2 zeros (counting multiplicity) w.r.t $\bar{y} \in]0, \bar{y}_1]$. Since $I(0, B_2, B_3, u, \lambda) = 0$, the slow divergence integral I has at most 1 zero counting multiplicity w.r.t $\bar{y} \in]0, \bar{y}_1]$, and the first statement of the lemma is proved.

Let's prove the statement (ii). First, we suppose that $B_2 \in K \subset]0, B_2^1]$ where K is compact. Since $B_2 \in K$, $u \sim 0$ and w in a compact set, there is $\rho_1 > 0$ such that $B_2 + w^2 + uO(w^3) \geq \rho_1$ in (18). From (18), we get

$$\begin{aligned} I(\bar{y}, B_2, B_3, u, \lambda) &\leq -\rho_1 \int_0^{\sqrt{2\bar{y}}} \frac{2w^2}{d(w, B_2, B_3, u, \lambda).d(-w, B_2, B_3, u, \lambda)} dw \\ &\leq -\rho_1 \int_0^{\sqrt{2\mu}} \frac{2w^2}{d(w, B_2, B_3, u, \lambda).d(-w, B_2, B_3, u, \lambda)} dw < -\rho_0, \end{aligned}$$

where $\rho_0 > 0$. The last inequality follows from the fact that the last integral is strictly positive, uniformly in $(B_2, B_3) \in \mathcal{B}$, $B_2 \in K$, $u \sim 0$ and $\lambda \in \Lambda$. This concludes the proof of the statement (ii) for $B_2 \in K$.

Suppose now that $B_2 \sim 0$. From (18) we have

$$\begin{aligned} I(\bar{y}, B_2, B_3, u, \lambda) &= \int_0^{\sqrt{2\mu}} \frac{-2w^2(B_2 + w^2 + uO(w^3))}{d(w, B_2, B_3, u, \lambda).d(-w, B_2, B_3, u, \lambda)} dw \\ &\quad + \int_{\sqrt{2\mu}}^{\sqrt{2\bar{y}}} \frac{-2w^2(B_2 + w^2 + uO(w^3))}{d(w, B_2, B_3, u, \lambda).d(-w, B_2, B_3, u, \lambda)} dw. \quad (20) \end{aligned}$$

If we denote by I_1 the first integral on the right of (20), it can be checked that

$$I_1 = -(\sqrt{2\mu})^3 \left(\frac{2B_2}{3} + \mu \left(\frac{4}{5} + \frac{4(B_2^3 + 2B_2B_3)}{5} + O(\mu) \right) \right). \quad (21)$$

Since $\mu > 0$ is small and fixed and B_3 in a compact set, the expression (21) implies existence of $\rho_0 > 0$ such that $I_1 < -\rho_0$, assuming that $B_2 \sim 0$ is sufficiently small. On the other hand, since $B_2 \sim 0$, $u \sim 0$, $\bar{y} \geq \mu$ and w in a compact set, we have that the second integral on the right of (20) is strictly negative. This concludes the proof of the statement (ii) for $B_2 \sim 0$.

Let's prove the statement (iii). Suppose that $(B_2, \bar{y}) \in K$ where K is an arbitrary compact subset of $\{(B_2, \bar{y}) \mid -B_2^1 \leq B_2 < 0, 0 < \bar{y} < -\frac{B_2}{2}\}$. Since $(B_2, \bar{y}) \in K$, $u \sim 0$, and $w \leq \sqrt{2\bar{y}}$, we have that $B_2 + w^2 + uO(w^3) \leq -\rho_1$ in (18), for some $\rho_1 > 0$. In the similar way as in the first part of the proof of the statement (ii), from this inequality we obtain that $I(\bar{y}, B_2, B_3, u, \lambda) > \rho_0$, for some $\rho_0 > 0$. \square

Remark 5. Lemma 3.3 will be used to find out how many limit cycles can occur near a $\Gamma_{\bar{y}}$ the fast orbit of which ends up in a singularity of the slow dynamics on at most one side of the critical curve. To find the number of limit cycles near a $\Gamma_{\bar{y}}$ the fast orbit of which ends up in a singularity of the slow dynamics on both sides of the critical curve, we will investigate the zeros of the "full" divergence integral $\delta^2\mathcal{I}$ using normal-form theory. See [DMD08] for more details.

To finish the proof of Theorem 2.3, we cover the compact set \mathcal{B} in the (B_2, B_3) -space by 7 sets (see Figure 4), and in each of these sets, we combine

Theorem 3.1, Lemma 3.2, Lemma 3.3, [DMD08], [Huz16] to find the number of small-amplitude limit cycles in the original (x, y) -space (see Sections 3.3.1–3.3.7). We point out that the small-amplitude limit cycles in the (x, y) -space cannot be studied uniformly in $(B_2, B_3) \in \mathcal{B}$ and different gluing techniques have to be used in each set.

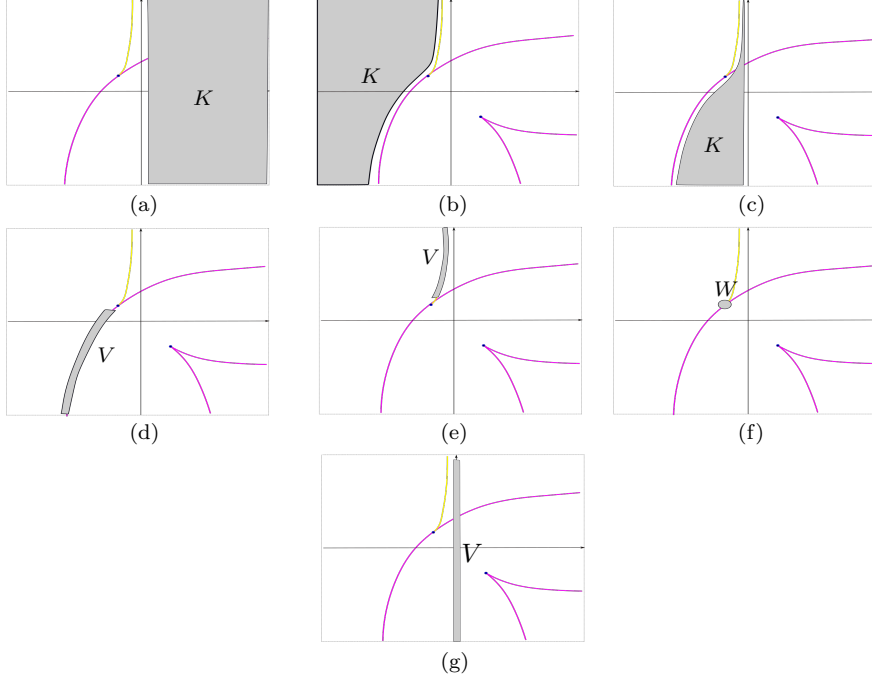


Figure 4: Seven sets covering the compact set \mathcal{B} in (B_2, B_3) -space. For each set, we glue the local cyclicity results near $(\bar{x}, \bar{y}) = (0, 0)$ and $\Gamma_{\bar{y}}$ to obtain the cyclicity of the origin $(x, y) = (0, 0)$ using different gluing techniques. The gluing methods are not uniform in $(B_2, B_3) \in \mathcal{B}$.

First, we take a sufficiently small neighborhood W of $(B_2, B_3) = (-1, 1)$ (see Figure 4(f)). Then we choose “tubular” neighborhoods V of C_1 and SN_2 as in Figures 4(d)–(e) such that we cover a complete (tubular) neighborhood of the curve $C_1 \cup \{(-1, 1)\} \cup SN_2$. We also choose a tubular neighborhood V of the B_3 -axis as in Figure 4(g). Now, if we take compact sets K large enough (see Figures 4(a)–(c)), we cover the compact set \mathcal{B} .

Remark 6. We denote singularities of the slow dynamics (12), for $u > 0$, again by $\bar{x}_L^1, \bar{x}_L^2, \bar{x}_R^1, \bar{x}_R^2, \bar{x}_R^3$ (see Sections 3.3.1–3.3.7). For example, when $\bar{x} = \bar{x}_R^1$ is a hyperbolic singularity of the slow dynamics for $(B_2, B_3, u) = (\tilde{B}_2, \tilde{B}_3, 0)$, the slow dynamics (resp. the vector field $X_{\delta^2, (\delta \tilde{B}_0, -1, B_2, B_3), u, \lambda}^F$) has a persistent hyperbolic singularity (resp. a persistent hyperbolic saddle) near $\bar{x} = \bar{x}_R^1$ (resp. near $(\bar{x}, \bar{y}) = (\bar{x}_R^1, 0)$), for $(B_2, B_3, u) \sim (\tilde{B}_2, \tilde{B}_3, 0)$ (resp. for $(\delta, \tilde{B}_0, B_2, B_3, u) \sim (0, 0, \tilde{B}_2, \tilde{B}_3, 0)$, $\delta > 0$), which we denote again by \bar{x}_R^1 (resp. by $(\bar{x}_R^1, 0)$). For each $(\delta, \tilde{B}_0, B_2, B_3, u) \sim (0, 0, \tilde{B}_2, \tilde{B}_3, 0)$ and $\lambda \in \Lambda$, the stable manifold at the hyperbolic saddle $(\bar{x}_R^1, 0)$ intersects the section $S = \{\bar{x} = 0\}$ at a point de-

noted by $(\bar{x}, \bar{y}) = (0, \frac{1}{2}(\bar{x}_R^1)^2)$. This notation makes sense because the ω -limit set of the fast orbit of the fast subsystem $X_{0, (0, -1, \bar{B}_2, \bar{B}_3), 0, \lambda}^F$ through the point $(\bar{x}, \bar{y}) = (0, \frac{1}{2}(\bar{x}_R^1)^2)$ is $\{(\bar{x}_R^1, 0)\}$. Keeping in mind this notation we study the cyclicity of $\Gamma_{\frac{1}{2}(\bar{x}_R^1)^2}$ by looking at zeros of Δ , $\delta^2\mathcal{I}$ or I for $\bar{y} \sim \frac{1}{2}(\bar{x}_R^1)^2$ and $\bar{y} < \frac{1}{2}(\bar{x}_R^1)^2$.

Remark 7. We suppose that the compact set D , introduced in Section 2.2, is large enough such that all limit periodic sets $\Gamma_{\bar{y}}$ (and singularities of the slow dynamics) in Figures 3(a) to 3(o) are contained in D .

3.3.1 The parameter region $\{B_2 > 0\}$

Let $B_2^0 > 0$ be any arbitrarily small fixed number. We consider slow fast systems $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$, given in (8), with $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in K := \mathcal{B} \cap \{B_2 \geq B_2^0\}$ (see Figure 4(a)) and $\lambda \in \Lambda$. In this section we prove:

- The family $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 1 (hyperbolically attracting) limit cycle in D for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in K$ and $\lambda \in \Lambda$.

Theorem 3.1(i) implies that (8) has at most 1 (hyperbolic and attracting) limit cycle in a $(\delta, \bar{B}_0, B_2, B_3, u, \lambda)$ -uniform neighborhood of the contact point $(\bar{x}, \bar{y}) = (0, 0)$ because the parameter B_2 is strictly positive. Since $(B_2, B_3) \in K$, canard limit periodic sets $\Gamma_{\bar{y}}$ that generate limit cycles by perturbation can be found in Figures 3(a) to 3(f). Since the slow dynamics has no negative singularities in Figures 3(c) to 3(f) and $\bar{x}_R^1 < -\bar{x}_L^1$ in Figures 3(a) to 3(b), the fast orbit of any $\Gamma_{\bar{y}}$ in Figures 3(a) to 3(f) may end up in a singularity of the slow dynamics on at most one side of the critical curve. Thus, to obtain the cyclicity of $\Gamma_{\bar{y}}$, it suffices to deal with the slow divergence integral (17).

By Lemma 3.3(ii), for any $\mu > 0$ small, there exist small $\rho_0 > 0$ and $u_0 > 0$ such that for any $(\bar{y}, B_2, B_3, u, \lambda) \in [\mu, \frac{1}{\mu}] \times K \times [0, u_0] \times \Lambda$, with the property that the slow dynamics $d(w, B_2, B_3, u, \lambda) < 0$ for all $w \in [-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$, we have that $I(\bar{y}, B_2, B_3, u, \lambda) < -\rho_0$. Now, using the results of [DMD08], we find that the cyclicity of set $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$ in Figures 3(a) to 3(e) and the cyclicity of set $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^2)^2]} \Gamma_{\bar{y}}$ in Figure 3(f) is 1 ($I \neq 0$). Since I is strictly negative, the limit cycle generated from such sets has to be hyperbolic and attracting for $\delta \sim 0$, $\delta > 0$ and $(B_2, B_3, u, \lambda) \in K \times [0, u_0] \times \Lambda$. Note that the limit cycle generated from $(\bar{x}, \bar{y}) = (0, 0)$ is also hyperbolically attracting. Thus, at most 1 (hyperbolically attracting) limit cycle can be generated from set $\cup_{\bar{y} \in [0, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$ in Figures 3(a) to 3(e) or from set $\cup_{\bar{y} \in [0, \frac{1}{2}(\bar{x}_R^2)^2]} \Gamma_{\bar{y}}$ in Figure 3(f). This concludes the proof of the statement.

3.3.2 The parameter region above the curve $C_1 \cup \{(-1, 1)\} \cup SN_2$

In this section we suppose that $(B_2, B_3) \in K$, where K is a compact set as in Figure 4(b), and we prove:

- The family $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 1 (hyperbolically repelling) limit cycle in D for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in K$ and $\lambda \in \Lambda$.

The proof of this statement is very similar to the proof of the statement in Section 3.3.1. Since B_2 is strictly negative in K , Theorem 3.1(i) implies that

the family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 1 (hyperbolically repelling) limit cycle in a fixed neighborhood of the contact point. Since the parameter (B_2, B_3) is kept in the compact set K , we deal with limit periodic sets $\Gamma_{\bar{y}}$ in Figures 3(j), 3(k) and 3(m). Since $-\bar{x}_L^1 < \bar{x}_R^1$ uniformly in $(B_2, B_3) \in K$, we consider $\Gamma_{\bar{y}}$, $\bar{y} \in]0, \frac{1}{2}(\bar{x}_L^1)^2]$. Clearly, the α -limit set of the fast orbit of $\Gamma_{\bar{y}}$ may be the (simple) singularity \bar{x}_L^1 of the slow dynamics. As in Section 3.3.1, we use the slow divergence integral I .

Since $-\bar{x}_L^1 < \bar{x}_0 = \sqrt{-B_2}$ uniformly in $(B_2, B_3) \in K$ (i.e. $\frac{1}{2}(\bar{x}_L^1)^2 < \frac{-B_2}{2}$ uniformly in $(B_2, B_3) \in K$), Lemma 3.3(iii) implies that for any $\mu > 0$ small there exist $\rho_0 > 0$ and $u_0 > 0$ sufficiently small such that $I(\bar{y}, B_2, B_3, u, \lambda) > \rho_0$ for each $\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_L^1)^2[$ and $(B_2, B_3, u, \lambda) \in K \times [0, u_0] \times \Lambda$. (Note that near $\bar{x} = \bar{x}_L^1$ the slow dynamics has a persistent simple singularity, for $u \sim 0$ and $u > 0$, that we denote again by \bar{x}_L^1 .) Now, following [DMD08], we find that the cyclicity of $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_L^1)^2]} \Gamma_{\bar{y}}$ is 1 ($I \neq 0$) and that the generated limit cycle is hyperbolically repelling ($I > 0$). Since the small-amplitude limit cycle in the (\bar{x}, \bar{y}) -space is also hyperbolically repelling, we have at most 1 (hyperbolically repelling) limit cycle near $\cup_{\bar{y} \in [0, \frac{1}{2}(\bar{x}_L^1)^2]} \Gamma_{\bar{y}}$.

3.3.3 The parameter region between the curves $C_1 \cup \{(-1, 1)\} \cup SN_2$ and $\{B_2 = 0\}$

We keep the parameter (B_2, B_3) in a compact set K as in Figure 4(c). In this section we will prove:

- *The family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 2 limit cycles in D for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in K$ and $\lambda \in \Lambda$. Fixing $(\delta, B_2, B_3, u, \lambda)$, with $\delta > 0$, the \bar{B}_0 -family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ contains a saddle-node bifurcation of limit cycles, in D and at some $\bar{B}_0^0 > 0$.*

We focus on limit periodic sets $\Gamma_{\bar{y}}$, $\bar{y} \in]0, \frac{1}{2}(\bar{x}_R^1)^2]$, in Figures 3(g), 3(h) and 3(o) ($\bar{x}_R^1 < -\bar{x}_L^1$ uniformly in $(B_2, B_3) \in K$). Clearly, the slow divergence integral I , given in (17), is well defined for $\bar{y} \in]0, \frac{1}{2}(\bar{x}_R^1)^2[$, $(B_2, B_3, \lambda) \in K \times \Lambda$ and $u \sim 0$, and

$$\lim_{\bar{y} \rightarrow \frac{1}{2}(\bar{x}_R^1)^2} I(\bar{y}, B_2, B_3, u, \lambda) = -\infty. \quad (22)$$

On the other hand, since $0 < \frac{-B_2}{2} < \frac{1}{2}(\bar{x}_R^1)^2$ uniformly in $(B_2, B_3) \in K$ (note that $0 < \bar{x}_0 < \bar{x}_R^1$ uniformly in $(B_2, B_3) \in K$), Lemma 3.3(iii) implies that for any $\mu > 0$ small there exist $\rho_0 > 0$ and $u_0 > 0$ sufficiently small such that $I(\bar{y}, B_2, B_3, u, \lambda) > \rho_0$ for each $(B_2, B_3, u, \lambda) \in K \times [0, u_0] \times \Lambda$ and $\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]$. From this together with (22) and Lemma 3.3(i) we conclude that I has precisely 1 (simple) zero in $[\mu, \frac{1}{2}(\bar{x}_R^1)^2[$, which is denoted by \bar{y}_0 , and that $\frac{\partial I}{\partial \bar{y}}(\bar{y}_0, B_2, B_3, u, \lambda) < 0$, for each $(B_2, B_3, u, \lambda) \in K \times [0, u_0] \times \Lambda$. Using the results of [DMD08], we have that the cyclicity of $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$ is bounded by 2 (I has at most 1 zero counting multiplicity). Using the results of [Dum11], we find that the \bar{B}_0 -family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$, for fixed $(B_2, B_3, \lambda) \in K \times \Lambda$, $(\delta, u) \sim (0, 0)$ and $\delta > 0$, undergoes, near $\Gamma_{\bar{y}_0}$ and at $\bar{B}_0 = \bar{B}_0^0 \sim 0$, a saddle-node bifurcation of limit cycles (I has a simple zero). Thus, the cyclicity of the set $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$ is 2.

When 2 limit cycles appear near the set $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$, the smaller one (resp. the bigger one) has to be hyperbolically repelling (resp. hyperbolically attracting) because $\frac{\partial I}{\partial \bar{y}}(\bar{y}_0, B_2, B_3, u, \lambda) < 0$ (see Lemma 3.2). Since a small limit cycle generated by the Hopf bifurcation of codimension 1, near the contact point, is hyperbolically repelling (see Theorem 3.1(i) ($B_2 < 0$)), it is clear that the cyclicity of $\cup_{\bar{y} \in [0, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$ is 2. This implies the statement and $\bar{B}_0^0 > 0$.

3.3.4 The parameter region near the curve SN_2

In this section we show that

- *There exists a sufficiently small tubular neighborhood V of the curve SN_2 as in Figure 4(d) such that $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 2 limit cycles in D for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in V$ and $\lambda \in \Lambda$.*

Taking into account Figure 3(i) we have to study the cyclicity of $\Gamma_{\bar{y}}$ for all $\bar{y} \in]0, \frac{1}{2}(\bar{x}_R^1)^2]$. Note that, since $-\bar{x}_L^1 < \sqrt{-B_2} < \bar{x}_R^1$ for $(B_2, B_3) \in SN_2$, there are no canard limit periodic sets $\Gamma_{\bar{y}}$ the fast orbit of which ends up in a singularity of the slow dynamics on both sides of the critical curve. We use Lemma 3.3(iii) to find that for any small $\mu > 0$ the slow divergence integral I is strictly positive for each (B_2, B_3) near the curve SN_2 , $u \sim 0$, $\lambda \in \Lambda$ and $\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]$, with the property that $d(w, B_2, B_3, u, \lambda) < 0$ for all $w \in [-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$. From this together with Lemma 3.3(i), we have that, if 2 limit cycles occur near $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$, then there exists a simple zero $\bar{y}_0 \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]$ of I (i.e. a simple zero of $\delta^2 \mathcal{I}$) with the property that $\frac{\partial I}{\partial \bar{y}}(\bar{y}_0, B_2, B_3, u, \lambda) < 0$. Like in Section 3.3.3, combining Theorem 3.1(i) ($B_2 < 0$) and Lemma 3.2, at most 2 limit cycles may bifurcate from $\cup_{\bar{y} \in [0, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$.

3.3.5 The parameter region near the curve C_1

In this section our goal is to prove:

- *There is a sufficiently small tubular neighborhood V of the curve C_1 as in Figure 4(e) such that $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 2 limit cycles in D for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in V$ and $\lambda \in \Lambda$.*

For $(\delta, \bar{B}_0, u) = (0, 0, 0)$ and $(B_2, B_3) \in C_1$, we detect canard limit periodic sets $\Gamma_{\bar{y}}$ that can generate limit cycles by perturbation (see Figure 3(n)):

$$\Gamma_{\bar{y}}, \bar{y} \in]0, \frac{-B_2}{2}].$$

Note that the fast orbit of $\Gamma_{\frac{-B_2}{2}}$ connects two hyperbolic singularities $\bar{x} = -\sqrt{-B_2}$ and $\bar{x} = \sqrt{-B_2}$ of the slow dynamics. The family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has a persistent hyperbolic saddle near $(\bar{x}, \bar{y}) = (\sqrt{-B_2}, 0)$ (resp. near $(\bar{x}, \bar{y}) = (-\sqrt{-B_2}, 0)$) with ratio of eigenvalues $-\delta^2 \nu_1$, $\nu_1 = \frac{2(1-B_2\sqrt{-B_2})}{-B_2} + O(\delta, \bar{B}_0, u) > 0$ (resp. with ratio of eigenvalues $-\delta^2 \nu_2$, $\nu_2 = \frac{2(1+B_2\sqrt{-B_2})}{-B_2} + O(\delta, \bar{B}_0, u) > 0$). Note that ν_2 is strictly positive because $-1 < B_2$ uniformly in V . Since $\nu_1 \neq \nu_2$ and the slow dynamics is regular for $(B_2, B_3) \in C_1$, $u = 0$ and $\bar{x} \in]-\sqrt{-B_2}, \sqrt{-B_2}[$, [DMD08] implies that the cyclicity of the slow-fast two-saddle-limit periodic set $\Gamma_{\frac{-B_2}{2}}$ is bounded by 2. More precisely, it has been

proved in [DMD08](Lemma 4.11) that $\delta^2\mathcal{I}$ has at most 1 zero near $\bar{y} = \frac{-B_2}{2}$, $(\delta, \bar{B}_0, u) = (0, 0, 0)$ and $(B_2, B_3) \in C_1$, which is simple. Moreover, since $\nu_1 > \nu_2$, the derivative $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}$, calculated in the simple zero of $\delta^2\mathcal{I}$ (if it exists), is strictly negative.

On the other hand, Lemma 3.3(iii) implies that for any $\mu > 0$ small there exist $\rho_0 > 0$ and $u_0 > 0$ sufficiently small and a small tubular neighborhood V of C_1 such that $I(\bar{y}, B_2, B_3, u, \lambda) > \rho_0$ for each $(B_2, B_3, u, \lambda) \in V \times [0, u_0] \times \Lambda$ and $\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]$. Thus, near $\cup_{\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]} \Gamma_{\bar{y}}$ at most 1 limit cycle may appear.

When $\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]$, we have that $\delta^2\mathcal{I} - I = O(\delta) + O(1) \cdot \delta^2 \log \delta$ where $O(\delta)$ and $O(1)$ are smooth, including at $\delta = 0$. This follows from [DMD05] due to the fact that the slow dynamics is regular along $[-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$ for $\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]$, $u \sim 0$ and for (B_2, B_3) near C_1 . Thus, $\delta^2\mathcal{I}$ is strictly positive for each $\bar{y} \in [\mu, \frac{-B_2}{2} - \mu]$, $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $\delta > 0$, (B_2, B_3) near C_1 and $\lambda \in \Lambda$.

Putting all the informations about $\delta^2\mathcal{I}$ together, we find that $\delta^2\mathcal{I}$ has at most 1 zero (counting multiplicity) w.r.t $\bar{y} \in [\mu, \frac{-B_2}{2}[,$ for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $\delta > 0$, (B_2, B_3) near C_1 and $\lambda \in \Lambda$. Thus, we conclude that the cyclicity of $\cup_{\bar{y} \in [\mu, \frac{-B_2}{2}]} \Gamma_{\bar{y}}$ is bounded by 2. From this together with Theorem 3.1(i)($B_2 < 0$), Lemma 3.2 and with the fact that $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}} < 0$ in a possible simple zero of $\delta^2\mathcal{I}$, we find that the cyclicity of $\cup_{\bar{y} \in [0, \frac{-B_2}{2}]} \Gamma_{\bar{y}}$ is also bounded by 2.

3.3.6 The parameter region near the point $(-1, 1)$

Here we prove:

- *There is a neighborhood W of $(B_2, B_3) = (-1, 1)$ (see Figure 4(f)) such that $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 2 limit cycles in D for all $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in W$ and $\lambda \in \Lambda$.*

In Figure 3(1), we detect canard limit periodic sets $\Gamma_{\bar{y}}$ from which limit cycles may bifurcate for $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $\delta > 0$, $(B_2, B_3) \sim (-1, 1)$ and $\lambda \in \Lambda$:

$$\Gamma_{\bar{y}}, \bar{y} \in]0, \frac{1}{2}].$$

For $u = 0$ and $(B_2, B_3) = (-1, 1)$, the slow dynamics (12) has a singularity of multiplicity 1 at $\bar{x} = 1$ and a singularity of multiplicity 2 at $\bar{x} = -1$. These 2 singularities are connected by the fast orbit of the limit periodic set $\Gamma_{\frac{1}{2}}$. Since we deal with different multiplicities at $\bar{x} = 1$ and $\bar{x} = -1$, Theorem 2.26 of [DMD08] implies that the cyclicity of $\Gamma_{\frac{1}{2}}$ is bounded by 3. In this section our principal goal is to prove that the set $\cup_{\bar{y} \in [0, \frac{1}{2}]} \Gamma_{\bar{y}}$ can produce at most 2 limit cycles by gluing local cyclicity results together. This gluing method will enable us to improve the existing upper bound for the number of limit cycles bifurcating from $\Gamma_{\frac{1}{2}}$, and it will be clear that it can be used in the more general framework of [DMD08].

At $\bar{x} = 1$ and near the parameter value $(\bar{B}_0, B_2, B_3, u) = (0, -1, 1, 0)$ we use a C^k -normal form (see [DMD08]):

$$\begin{cases} \dot{v}_1 &= \delta^2 \nu_1 v_1 \\ \dot{v}_2 &= -v_2, \end{cases} \quad (23)$$

where $-\delta^2\nu_1$, $\nu_1 > 0$, is ratio of eigenvalues of a persistent hyperbolic saddle of $X_{\delta^2, (\delta\bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ near $(\bar{x}, \bar{y}) = (1, 0)$. In these new coordinates the point $(\bar{x}, \bar{y}) = (1, 0)$ is located at $(v_1, v_2) = (0, 0)$. Following [DMD08] or [HDMD13], we have

$$\delta^2\mathcal{I}_+(\bar{y}, \tau) = \int_{\alpha_+(\bar{y}, \tau)}^1 \frac{-1 + \nu_1\delta^2}{\nu_1 v_1} dv_1 + O(1),$$

near $\bar{y} = \frac{1}{2}$. The integral in the expression for $\delta^2\mathcal{I}_+$ is the divergence integral (multiplied by δ^2) calculated in the normal form coordinates from $\{v_2 = 1\}$ to $\{v_1 = 1\}$ where we assume that the orbit $\mathcal{O}^+(\bar{y}, \tau)$ intersects the plane $\{v_2 = 1\}$ in a point with $v_1 = \alpha_+(\bar{y}, \tau)$. Clearly, α_+ is a C^k -diffeomorphism with $\alpha_+(\frac{1}{2}, 0, 0, -1, 1, 0, \lambda) = 0$ and $\frac{\partial\alpha_+}{\partial\bar{y}} < 0$. The $O(1)$ -term represents the contribution in the divergence integral \mathcal{I}_+ (multiplied by δ^2) due to the passage along the fast fiber until we reach $\{v_2 = 1\}$ and the passage along the attracting part of the critical curve between $\{v_1 = 1\}$ and the section T , where the slow dynamics is regular. It is δ -regularly C^k in $(\bar{y}, \bar{B}_0, B_2, B_3, u, \lambda)$, i.e. $O(1)$ and all its derivatives up to order k w.r.t. $(\bar{y}, \bar{B}_0, B_2, B_3, u, \lambda)$ are continuous including at $\delta = 0$.

At $\bar{x} = -1$ and near the parameter value $(\bar{B}_0, B_2, B_3, u) = (0, -1, 1, 0)$ we have a C^k -normal form (see [DMD08]):

$$\begin{cases} \dot{v}_1 &= -\delta^2 h(v_1, \tau) \\ \dot{v}_2 &= v_2, \end{cases} \quad (24)$$

where $h(v_1, 0, 0, -1, 1, 0, \lambda)$ has a zero of multiplicity 2 at $v_1 = 0$. Similarly, we find that

$$\delta^2\mathcal{I}_-(\bar{y}, \tau) = \int_{\alpha_-(\bar{y}, \tau)}^1 \frac{-1 + \delta^2 \frac{\partial h}{\partial v_1}(v_1, \tau)}{h(v_1, \tau)} dv_1 + O(1),$$

near $\bar{y} = \frac{1}{2}$. The integral in the expression for $\delta^2\mathcal{I}_-$ is the divergence integral (multiplied by δ^2) in the normal form coordinates between the section $\{v_2 = 1\}$ and the section $\{v_1 = 1\}$, following orbits in negative time. The orbit $\mathcal{O}^-(\bar{y}, \tau)$ intersects the plane $\{v_2 = 1\}$ in a point with $v_1 = \alpha_-(\bar{y}, \tau)$. The function α_- and the $O(1)$ -term have the same properties like α_+ and the $O(1)$ -term in the expression for $\delta^2\mathcal{I}_+$.

Thus, we get

$$\begin{aligned} \delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}, \tau) &= \frac{(1 - \nu_1 \delta^2) \frac{\partial \alpha_+}{\partial \bar{y}}(\bar{y}, \tau)}{\nu_1 \alpha_+(\bar{y}, \tau)} \\ &\quad - \frac{(1 - \delta^2 \frac{\partial h}{\partial v_1}(\alpha_-(\bar{y}, \tau), \tau)) \frac{\partial \alpha_-}{\partial \bar{y}}(\bar{y}, \tau)}{h(\alpha_-(\bar{y}, \tau), \tau)} + O(1), \end{aligned}$$

near $\bar{y} = \frac{1}{2}$. Using the above expression and the properties of α_{\pm} and h , we finally get

$$\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}, \tau) = \frac{\beta_0 + \beta_1(\bar{y} - \frac{1}{2}) + O((\bar{y} - \frac{1}{2})^2)}{\nu_1 \alpha_+(\bar{y}, \tau) h(\alpha_-(\bar{y}, \tau), \tau)} \quad (25)$$

where $\beta_0 = O(\delta, \bar{B}_0, B_2 + 1, B_3 - 1, u)$ and

$$\begin{aligned} \beta_1 &= -\nu_1 \frac{\partial \alpha_+}{\partial \bar{y}}(\frac{1}{2}, 0, 0, -1, 1, 0, \lambda) \frac{\partial \alpha_-}{\partial \bar{y}}(\frac{1}{2}, 0, 0, -1, 1, 0, \lambda) \\ &\quad + O(\delta, \bar{B}_0, B_2 + 1, B_3 - 1, u). \end{aligned}$$

Clearly, the coefficient β_1 is strictly negative. From this together with (25) and Rolle's theorem we conclude that $\frac{\partial \mathcal{I}}{\partial \bar{y}}$ has at most 1 zero (counting multiplicity) near $\bar{y} = \frac{1}{2}$.

On the other hand, since for any small fixed $\mu > 0$ the slow dynamics is regular along $[-\sqrt{2\bar{y}}, \sqrt{2\bar{y}}]$ for $\bar{y} \in [\mu, \frac{1}{2} - \mu]$ by taking $u \sim 0$ and $(B_2, B_3) \sim (-1, 1)$, we find that $\delta^2 \mathcal{I} = I + O(\delta) + O(1) \cdot \delta^2 \log \delta$ and $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}} = \frac{\partial I}{\partial \bar{y}} + O(\delta) + O(1) \cdot \delta^2 \log \delta$ where $\bar{y} \in [\mu, \frac{1}{2} - \mu]$, $O(\delta)$ and $O(1)$ are smooth functions including at $\delta = 0$ (see [DMD05]). Using these expressions, Lemma 3.3(iii) and (19), we find that

$$\mathcal{I}(\bar{y}, \tau) > 0, \quad \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}, \tau) > 0 \quad (26)$$

for $\bar{y} \in [\mu, \frac{1}{2} - \mu]$, $(\delta, u) \sim (0, 0)$, $\delta > 0$ and $(B_2, B_3) \sim (-1, 1)$.

We will prove that \mathcal{I} has at most 1 zero (multiplicity taken into account) w.r.t. $\bar{y} \in [\mu, \frac{1}{2}]$. Moreover, if \mathcal{I} has a simple zero at $\bar{y} = \bar{y}_0$, then $\frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}_0, \tau) < 0$ (this will follow directly from the first inequality in (26)).

Assume that \mathcal{I} has at least 2 zeros, counting multiplicity, in $[\mu, \frac{1}{2}]$. Then the 2 zeros of \mathcal{I} have to be located near $\bar{y} = \frac{1}{2}$ due to the first inequality in (26). This implies, using (26) and Rolle's theorem, that $\frac{\partial \mathcal{I}}{\partial \bar{y}}$ has at least 2 zeros (counting multiplicity) near $\bar{y} = \frac{1}{2}$. This is in clear contradiction to the fact that $\frac{\partial \mathcal{I}}{\partial \bar{y}}$ has at most 1 zero (counting multiplicity) near $\bar{y} = \frac{1}{2}$.

Like in Sections 3.3.3–3.3.5, we conclude now that the set $\cup_{\bar{y} \in [0, \frac{1}{2}]} \Gamma_{\bar{y}}$ can produce at most 2 limit cycles.

Remark 8. *Let's explain how this technique can be used to prove the statement from Section 3.3.5. At $\bar{x} = \sqrt{-B_2}$ and near the parameter values $(\bar{B}_0, u) = (0, 0)$ and $(B_2, B_3) \in C_1$ we can use the normal form (23). At $\bar{x} = -\sqrt{-B_2}$ we use the normal form $\{\dot{v}_1 = -\delta^2 \nu_2 v_1, \dot{v}_2 = v_2\}$. The expression for $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}$ near $\bar{y} = \frac{-B_2}{2}$ is similar to (25):*

$$\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}} = L \cdot \left(\beta_0 + \beta_1 \left(\bar{y} + \frac{B_2}{2} \right) + O\left(\left(\bar{y} + \frac{B_2}{2} \right)^2 \right) \right)$$

where $L > 0$ and where $\beta_1 < 0$ because $\nu_1 > \nu_2$. Thus, $\frac{\partial \mathcal{I}}{\partial \bar{y}}$ has at most 1 zero (counting multiplicity) near $\bar{y} = \frac{-B_2}{2}$. The rest of the proof of the statement from Section 3.3.5 is now similar to the proof of the statement from this section.

3.3.7 The parameter region near the line $\{B_2 = 0\}$

Our goal is to prove that

- *There is a small tubular neighborhood V of the line $\{B_2 = 0\}$ as in Figure 4(g) such that $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 2 limit cycles in D for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $(B_2, B_3) \in V$ and $\lambda \in \Lambda$.*

In order to prove this statement we use the gluing method developed in [Huz16].

In Figures 3(a) to 3(c), we find $\Gamma_{\bar{y}}$ from which limit cycles can bifurcate for $(\delta, \bar{B}_0, B_2, u) \sim (0, 0, 0, 0)$ and $\delta > 0$:

$$\Gamma_{\bar{y}}, \quad \bar{y} \in]0, \frac{1}{2}(\bar{x}_R^1)^2].$$

By Lemma 3.3(ii), for any $\mu > 0$ small, there exist small $\rho_0 > 0$, $u_0 > 0$ and a tubular neighborhood V of the line $\{B_2 = 0\}$ in the \mathcal{B} such that for any $(\bar{y}, B_2, B_3, u, \lambda) \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2] \times V \times [0, u_0] \times \Lambda$ we have that $I(\bar{y}, B_2, B_3, u, \lambda) < -\rho_0$. Thus, the set $\cup_{\bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$ can produce at most 1 (hyperbolically attracting) limit cycle. Moreover, from (16) we find that

$$\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) > 0 \quad (27)$$

for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $\delta > 0$ and $(\bar{y}, B_2, B_3, \lambda) \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2] \times V \times \Lambda$.

Following Section 5 of [Huz16], for each $(\delta, \bar{B}_0, u) \sim (0, 0, 0)$, $\delta > 0$ and $(B_2, B_3, \lambda) \in V \times \Lambda$, we can extend smoothly the difference map Δ to a smooth section \tilde{S} parametrized by $\bar{y} \in [0, \frac{1}{2}(\bar{x}_R^1)^2]$. More precisely, the section \tilde{S} extends smoothly the section $S = \{\bar{x} = 0, \bar{y} \in [\mu, \frac{1}{2}(\bar{x}_R^1)^2]\}$ up to the focus $(\bar{x}, \bar{y}) = (\bar{x}_\tau, 0) \sim (0, 0)$ of the family $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$. We denote the extension of Δ again by Δ . Since $\bar{y} = 0$ represents the focus, we have

$$\Delta(0, \tau) = 0, \quad (28)$$

under the given conditions on the parameter τ .

Theorem 3.1(ii) implies that $X_{\delta^2, (\delta \bar{B}_0, -1, B_2, B_3), u, \lambda}^F$ has at most 2 limit cycles in a τ -uniform neighborhood of $(\bar{x}, \bar{y}) = (0, 0)$. Thus the difference map Δ has at most 2 zeros (counting multiplicity) on the interval $]0, \mu]$ (under the given conditions on the parameter τ). Moreover, it has been proved in Section 5 of [Huz16], based on the study of Chebyshev systems of degree 2, that $\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau)$ has at most 2 zeros (multiplicity taken into account) on the interval $]0, \mu]$.

If we suppose now that Δ has at least 3 zeros on the interval $]0, \frac{1}{2}(\bar{x}_R^1)^2]$ (hence at least 3 limit cycles occur), then Rolle's theorem and (28) imply that $\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau)$ has at least 3 zeros on the interval $]0, \frac{1}{2}(\bar{x}_R^1)^2]$. These 3 zeros have to be located on the interval $]0, \mu]$ due to (27). This is in contradiction to the fact that $\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau)$ has at most 2 zeros on $]0, \mu]$. Thus the set $\cup_{\bar{y} \in [0, \frac{1}{2}(\bar{x}_R^1)^2]} \Gamma_{\bar{y}}$ can produce at most 2 limit cycles.

3.4 The slow-fast Bogdanov-Takens region

In this section we prove Theorem 2.4. We consider the system $X_{\bar{\epsilon}, (B_0, B_1, \pm 1, B_3), u, \lambda}^F$ where $\bar{\epsilon} \sim 0$, $(B_0, B_1) \sim (0, 0)$, $u \sim 0$, $\lambda \in \Lambda$ and where B_3 is in an arbitrary compact set in \mathbb{R} . This singular perturbation problem represents standard slow-fast Bogdanov-Takens bifurcations which have been studied in [DMD11a]. It is shown in [DMD11a] that $X_{\bar{\epsilon}, (B_0, B_1, \pm 1, B_3), u, \lambda}^F$ has at most one (hyperbolic) limit cycle in a uniform neighborhood of $(\bar{x}, \bar{y}) = (0, 0)$ for $(\bar{\epsilon}, B_0, B_1, u) \sim (0, 0, 0, 0)$. (The size of this small-amplitude limit cycle tends to 0 as $(B_0, B_1) \rightarrow (0, 0)$.) On the other hand, note that for $(B_0, B_1) = (0, 0)$ the slow dynamics near $\bar{x} = 0$ (but $\bar{x} \neq 0$) is given by $\bar{x}' = \pm \bar{x}(1 + O(\bar{x}))$. Thus, the passage from the attracting part of the critical curve to the repelling part of the critical curve is not possible. As a consequence, there are no limit cycles near $\Gamma_{\bar{y}}$. See also [HDMD13], Section 3.6.

3.5 The slow-fast codimension 3 saddle region

In this section our goal is to prove Theorem 2.5 in the saddle case $B_3 = +1$. We consider the family $X_{\bar{\epsilon}, (B_0, B_1, B_2, +1), u, \lambda}^F$:

$$\begin{cases} \dot{\bar{x}} &= \bar{y} \\ \dot{\bar{y}} &= -\bar{x}\bar{y} + \bar{\epsilon}(B_0 + B_1\bar{x} + B_2\bar{x}^2 + \bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x}, \lambda)) \\ &\quad + \bar{\epsilon}\bar{y}^2 H(u\bar{x}, u^2\bar{y}, \lambda), \end{cases} \quad (29)$$

where $(\bar{\epsilon}, B_0, B_1, B_2, u) \sim (0, 0, 0, 0, 0)$ and $\lambda \in \Lambda$. The family (29) represents slow-fast codimension 3 saddle bifurcations which have been studied in [HDMD13] and [Huz16]. More precisely, since the coefficient in front of the term $\bar{x}^4 \frac{\partial}{\partial \bar{y}}$ in (29) is 1 (hence different from zero, uniformly over the parameter $(\bar{\epsilon}, B_0, B_1, B_2, u, \lambda)$), [HDMD13] and [Huz16] imply that (29) has at most two limit cycles in an $(\bar{\epsilon}, B_0, B_1, B_2, u, \lambda)$ -uniform neighborhood of $(\bar{x}, \bar{y}) = (0, 0)$, and that the size of these limit cycles tends to zero as $(B_0, B_1, B_2) \rightarrow (0, 0, 0)$.

For $(B_0, B_1, B_2) = (0, 0, 0)$, the slow dynamics (7) is given by $\bar{x}' = \bar{x}^2(1 + O(\bar{x}))$, clearly pointing from the left to the right near $\bar{x} = 0$. Thus, no limit cycles bifurcate from canard limit periodic sets $\Gamma_{\bar{y}}$. This concludes the proof of Theorem 2.5 in the saddle case.

3.6 The slow-fast codimension 3 elliptic region

In this section our focus is on well-known slow-fast codimension 3 elliptic bifurcations $X_{\bar{\epsilon}, (B_0, B_1, B_2, -1), u, \lambda}^F$ with $(\bar{\epsilon}, B_0, B_1, B_2, u) \sim (0, 0, 0, 0, 0)$, and $\lambda \in \Lambda$. The cyclicity of $(\bar{x}, \bar{y}) = (0, 0)$ (resp. the cyclicity of canard limit periodic sets $\Gamma_{\bar{y}}$) in the slow-fast codimension 3 elliptic bifurcations has been studied in [HDMD13], [HDMD14] and [Huz16] (resp. in [DMD10]). We glue together these local results to obtain the cyclicity of the compact set D in the (\bar{x}, \bar{y}) -plane.

We claim that, in order to prove Theorem 2.5 in the elliptic case $B_3 = -1$, it is sufficient to apply the gluing method to the family $X_{\delta^2, (\delta\tilde{r}^3\tilde{B}_0, -\tilde{r}^2, \tilde{r}\tilde{B}_2, -1), u, \lambda}^F$:

$$\begin{cases} \dot{\bar{x}} &= \bar{y} \\ \dot{\bar{y}} &= -\bar{x}\bar{y} + \delta^2(\delta\tilde{r}^3\tilde{B}_0 - \tilde{r}^2\bar{x} + \tilde{r}\tilde{B}_2\bar{x}^2 - \bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x}, \lambda)) \\ &\quad + \delta^2\bar{y}^2 H(u\bar{x}, u^2\bar{y}, \lambda), \end{cases} \quad (30)$$

where $(\delta, \tilde{B}_0, \tilde{r}, u) \sim (0, 0, 0, 0)$, \tilde{B}_2 is kept in an arbitrary compact subset K of \mathbb{R} and where $\lambda \in \Lambda$. Indeed, in [HDMD13] and [DMD10] the (B_0, B_1, B_2) -parameters were reparametrized by introducing weighted spherical coordinates: $(B_0, B_1, B_2) = (\tilde{r}^3\tilde{B}_0, \tilde{r}^2\tilde{B}_1, \tilde{r}\tilde{B}_2)$, $\tilde{r} \geq 0$, $(\tilde{B}_0, \tilde{B}_1, \tilde{B}_2) \in \mathbb{S}^2$. Like in this paper, different regions in the parameter space (B_0, B_1, B_2) were used: the jump region $\{\tilde{B}_0 = \pm 1\}$, the saddle region $\{\tilde{B}_1 = +1\}$, the slow-fast Hopf region $\{\tilde{B}_1 = -1\}$ and the slow-fast Bogdanov-Takens region $\{\tilde{B}_2 = \pm 1\}$. It was shown in [HDMD13] (resp. in [DMD10]) that in the jump region, the saddle region and in the slow-fast Bogdanov-Takens region the family $X_{\bar{\epsilon}, (B_0, B_1, B_2, -1), u, \lambda}^F$ has at most 1 (hyperbolic) limit cycle near $(\bar{x}, \bar{y}) = (0, 0)$ (resp. no limit cycles Hausdorff-close to $\Gamma_{\bar{y}}$). Thus, for the parameters kept in these regions, the set D can produce at most 1 limit cycle. In the slow-fast Hopf region, in which $(B_0, B_1, B_2) = (\tilde{r}^3\tilde{B}_0, -\tilde{r}^2, \tilde{r}\tilde{B}_2)$ with $\tilde{B}_0 \sim 0$ and $\tilde{B}_2 \in K$, the study of the cyclicity of D is much more delicate because we can find limit cycles not only

near $(\bar{x}, \bar{y}) = (0, 0)$ but also Hausdorff-close to $\Gamma_{\bar{y}}$. Like in Section 3.3, it is possible to show that we have limit cycles in the family $X_{\bar{\epsilon}, (B_0, B_1, B_2, -1), u, \lambda}^F$ with the parameter (B_0, B_1, B_2) kept in the slow-fast Hopf region, only if $(\bar{\epsilon}, \tilde{B}_0) = (\delta^2, \delta \tilde{B}_0)$ where $\tilde{B}_0 \sim 0$. We refer to [HDMD13] or [DMD10] for more details. Thus, in the rest of this section we will focus on the slow-fast system (30).

Remark 9. *When the parameter $\tilde{B}_2 \neq 0$, the family (30) has at most 1 limit cycle near $(\bar{x}, \bar{y}) = (0, 0)$, generated by a Hopf bifurcation of codimension 1 (see Theorem 3.7(i)). When $\tilde{B}_2 \sim 0$, the origin $(\bar{x}, \bar{y}) = (0, 0)$ is shown to produce at most 2 limit cycles (see Theorem 3.7(ii)). If the parameter \tilde{B}_2 is kept in a compact set $K \subset \mathbb{R} \setminus [-2, 2]$ and $(\delta, \tilde{B}_0, \tilde{r}, u) \sim (0, 0, 0, 0)$, then there is no canard explosion in the (\bar{x}, \bar{y}) -plane, i.e., there are no limit cycles near $\Gamma_{\bar{y}}$. Indeed, a saddle-node bifurcation of singularity occurs in the family (30) near $(\bar{x}, \bar{y}) = (\tilde{r}\bar{x}, \tilde{r}^2\bar{y}) = (\tilde{r}(\pm 1), \tilde{r}^2 0)$ and for parameter value $(\delta, \tilde{B}_0, \tilde{B}_2, \tilde{r}, u)$ close to $(0, 0, \pm 2, 0, 0)$, $\delta > 0$ and $\tilde{r} > 0$. When $|\tilde{B}_2| > 2$, two hyperbolic singularities, generated by the saddle-node bifurcation, don't allow passage from the attracting part of the critical curve to the repelling part of the critical curve in the (\bar{x}, \bar{y}) -plane. Thus, we have no limit cycles Hausdorff-close to $\Gamma_{\bar{y}}$. When $|\tilde{B}_2| < 2$, the passage is possible. See [HDMD13] or [DMD10] for more details. Thus, for $\tilde{B}_2 \in K \subset \mathbb{R} \setminus [-2, 2]$, the set D cannot produce more than 2 limit cycles. From now on we keep \tilde{B}_2 in a compact set $K = [-2 - \rho, 2 + \rho]$, where $\rho > 0$.*

We detect canard limit periodic sets $\Gamma_{\bar{y}}$ from which limit cycles of (30) can bifurcate (see Figure 5):

$$\Gamma_{\bar{y}}, \bar{y} \in]0, \frac{1}{2}\bar{x}_R^2].$$

We denote by $\bar{x}_R = \bar{x}_R(u, B_1, B_2, \lambda) \sim 1$ a simple singularity of the slow dynamics of (30):

$$\bar{x}' = B_1 + B_2\bar{x} + \bar{x}^2(-1 + \bar{x} + u\bar{x}^2G(u\bar{x}, \lambda)), \quad (31)$$

where $(B_1, B_2) = (-\tilde{r}^2, \tilde{r}\tilde{B}_2) \sim (0, 0)$. Note that the passage from the attracting part of the critical curve to the repelling part of the critical curve might be possible because for $(B_1, B_2) = (0, 0)$ the slow dynamics (31) is strictly negative for $\bar{x} < \bar{x}_R$, with the exception of the origin $\bar{x} = 0$, where it has a saddle-node singularity.

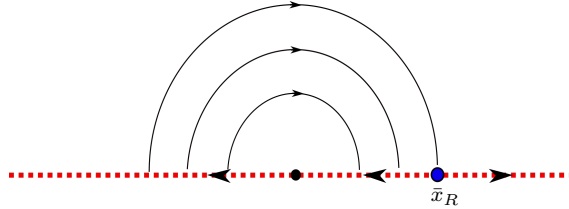


Figure 5: Canard limit periodic sets $\Gamma_{\bar{y}}$, with indication of slow dynamics.

Remark 10. *We cannot use the results of [DMD08] to study the cyclicity of $\Gamma_{\bar{y}}$ because the slow dynamics (31) has a singularity at the contact point for*

$B_1 = B_2 = 0$ and, as a consequence, the slow divergence integral

$$I(\bar{y}, B_1, B_2, u, \lambda) = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{wdw}{d(w, B_1, B_2, u, \lambda)}, \quad \bar{y} \in]0, \frac{1}{2}\bar{x}_R^2[,$$

becomes $\infty - \infty$ as $(B_1, B_2) \rightarrow (0, 0)$. The function $d(\bar{x}, B_1, B_2, u, \lambda)$ is the right hand side of (31). As shown in [DMD10], it is better to deal with $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}$ which is well approximated by the (well defined) derivative of the slow divergence integral

$$\frac{\partial I}{\partial \bar{y}}(\bar{y}, B_1, B_2, u, \lambda) = \frac{-2\sqrt{2\bar{y}}(B_2 + 2\bar{y} + uO((\sqrt{2\bar{y}})^3))}{d(\sqrt{2\bar{y}}, B_1, B_2, u, \lambda) \cdot d(-\sqrt{2\bar{y}}, B_1, B_2, u, \lambda)}.$$

See Remark 12. Clearly, for any fixed small $\mu > 0$ we have that $\frac{\partial I}{\partial \bar{y}}(\bar{y}, B_1, B_2, u, \lambda)$ is strictly negative for all $\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2 - \mu]$ and $\lambda \in \Lambda$, by taking the parameter (B_1, B_2, u) sufficiently small. Thus $\frac{\partial I}{\partial \bar{y}}$ has no zeros on the interval $[\mu, \frac{1}{2}\bar{x}_R^2 - \mu]$ under the given conditions on the parameters, and, by Theorem 5 of [DMD10], we find that the set $\cup_{\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2 - \mu]} \Gamma_{\bar{y}}$ produces at most 2 limit cycles. Since Theorem 5 of [DMD10] cannot be applied to limit periodic sets $\Gamma_{\bar{y}}$ the fast orbit of which ends up in a singularity of the slow dynamics, we have to study separately the cyclicity of $\Gamma_{\frac{1}{2}\bar{x}_R^2}$ (see Section 3.6.1).

Our gluing method consists in studying the graphs of the difference map near the contact point and the graphs of the difference map near the set $\cup_{\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2]} \Gamma_{\bar{y}}$, and finding out how the “local” graphs can be put together to give “global” graphs of the difference map near the set $\cup_{\bar{y} \in [0, \frac{1}{2}\bar{x}_R^2]} \Gamma_{\bar{y}}$. Once we have the global graphs, it will be straightforward to see that the set $\cup_{\bar{y} \in [0, \frac{1}{2}\bar{x}_R^2]} \Gamma_{\bar{y}}$ can produce at most 2 limit cycles.

More precisely, if we denote the parameter $(\delta, \tilde{r}^3 \bar{B}_0, -\tilde{r}^2, \tilde{r} \bar{B}_2, u, \lambda)$ by τ and if we use the notation introduced in Section 3.3, we get the following expression for the derivative of the difference map near $\cup_{\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2]} \Gamma_{\bar{y}}$:

$$\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) = -\frac{1}{\delta^6 \tilde{r}^4} A(\alpha, \beta) \left(\delta^2 \mathcal{I}(\bar{y}, \tau) + O(\delta^2) \right), \quad (32)$$

where $\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2[$, $\delta > 0$ and $\tilde{r} > 0$.

Remark 11. For each τ with $\delta > 0$, the stable manifold at the hyperbolic saddle of (30), near the hyperbolic singularity $\bar{x} = \bar{x}_R$ of the slow dynamics, intersects the section $S = \{\bar{x} = 0\}$ at a point which we denote by $(\bar{x}, \bar{y}) = (0, \frac{1}{2}\bar{x}_R^2)$. Thus, to find the cyclicity of $\Gamma_{\frac{1}{2}\bar{x}_R^2}$, we have to study $\Delta, \delta^2 \mathcal{I}$ for $\bar{y} \sim \frac{1}{2}\bar{x}_R^2$ and $\bar{y} < \frac{1}{2}\bar{x}_R^2$.

If we compare expression (32) to the expression (16), we can see that the extra-term $\frac{1}{\tilde{r}^4}$ appears in (32). This is due to the fact that in the definition of the section T we have to combine now two blow-up constructions: the so-called primary blow-up $(\bar{x}, \bar{y}) = (\tilde{r}\bar{x}, \tilde{r}^2\bar{y})$, and the so-called secondary blow-up $(\bar{x}, \bar{y}) = (\delta\tilde{x}, \delta^2\tilde{y})$ similar to (9). The section $T = \{\tilde{x} = 0\}$ is parametrized by \tilde{y} . We refer to [DMD10] for more details.

Remark 12. From [DMD08] and [DMD10] it follows that Δ and $O(\delta^2)$ in (32) (resp. $\frac{\partial \Delta}{\partial \bar{y}}$ and $\frac{\partial O(\delta^2)}{\partial \bar{y}}$) are C^k -functions in $(\bar{y}, \delta, \tilde{r}, \bar{B}_0, \bar{B}_2, u, \lambda)$ (resp.

C^k -functions in (\bar{y}, τ)) with a C^k -extension to the boundary of their domain. Moreover, it has been proved in [DMD10] that $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}$ is a C^k -function in (\bar{y}, τ) , $\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2 - \mu]$, with a C^k -extension to the boundary of its domain, and equal to $\frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}, B_1, B_2, u, \lambda) + O(\delta)$. Following Remark 10, we obtain now that the derivative of the expression $\delta^2 \mathcal{I}(\bar{y}, \tau) + O(\delta^2)$ in (32) w.r.t. \bar{y} is strictly negative on the segment $\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2 - \mu]$.

In Section 3.6.1 we prove

Proposition 3.4. *Consider the difference map $\Delta(\bar{y}, \tau)$ on the interval $[\mu, \frac{1}{2}\bar{x}_R^2[$, with $\delta > 0$ and $\tilde{r} > 0$. There arise 3 possibilities:*

- (a) $\Delta(\bar{y}, \tau)$ is strictly decreasing ($\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) < 0$ for all $\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2[$).
- (b) There exists $\bar{y}_0 \in [\mu, \frac{1}{2}\bar{x}_R^2[$ such that $\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) < 0$ for all $\bar{y} \in [\mu, \bar{y}_0[$, $\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}_0, \tau) = 0$, $\frac{\partial^2 \Delta}{\partial \bar{y}^2}(\bar{y}_0, \tau) > 0$ and $\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) > 0$ for all $\bar{y} \in]\bar{y}_0, \frac{1}{2}\bar{x}_R^2[$.
- (c) $\Delta(\bar{y}, \tau)$ is strictly increasing ($\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) > 0$ for all $\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2[$).

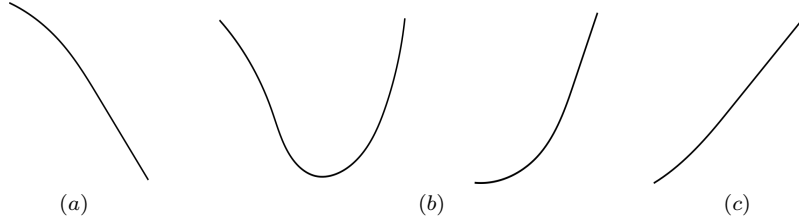


Figure 6: The graphs of the difference map Δ on the interval $[\mu, \frac{1}{2}\bar{x}_R^2[$.

As a simple consequence of Proposition 3.4 and Rolle's theorem, at most 2 limit cycles can bifurcate from the set $\cup_{\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2[} \Gamma_{\bar{y}}$ for $\delta > 0$ and $\tilde{r} > 0$.

Following [Huz16] (Sections 6 and 7), for each fixed value $(\delta, \tilde{r}, \bar{B}_0, \bar{B}_2, u, \lambda)$, $\delta > 0$ and $\tilde{r} > 0$, the difference map Δ can be C^k -extended to a smooth section \tilde{S} parametrized by $\bar{y} \in [0, \frac{1}{2}\bar{x}_R^2[$, where $\bar{y} = 0$ represents a focus $(\bar{x}, \bar{y}) = (\bar{x}_\tau, 0) \sim (0, 0)$ of (30). After the primary blow-up, this focus can be detected near the origin in the (\bar{x}, \bar{y}) -plane. See Figure 7. Like in Section 3.3.7, we have $\Delta(0, \tau) = 0$.

If, for a fixed τ , with $\delta > 0$ and $\tilde{r} > 0$, Δ has no zeros on the interval $]0, \mu]$, then, by Proposition 3.4, the system (30) has at most 2 limit cycles in D . Thus, it suffices to consider only those parameters τ for which Δ has at least 1 zero on the interval $]0, \mu]$. The following proposition will be proved in Section 3.6.2.

Proposition 3.5. *There exist small $\mu > 0$, $\delta_0 > 0$, $\tilde{r}_0 > 0$, $\bar{B}_0^0 > 0$ and $u_0 > 0$ such that for each $(\delta, \tilde{r}, \bar{B}_0, \bar{B}_2, u, \lambda) \in]0, \delta_0] \times]0, \tilde{r}_0] \times [-\bar{B}_0^0, \bar{B}_0^0] \times K \times [0, u_0] \times \Lambda$, with the property that Δ has at least 1 zero (counting multiplicity) on the interval $]0, \mu]$, the graph of Δ on $]0, \mu]$ can be found in Figures 8(a) to 8(j). In each of these figures, Δ and $\frac{\partial \Delta}{\partial \bar{y}}$ have at most 2 zeros (counting multiplicity) w.r.t. $\bar{y} \in]0, \mu]$.*

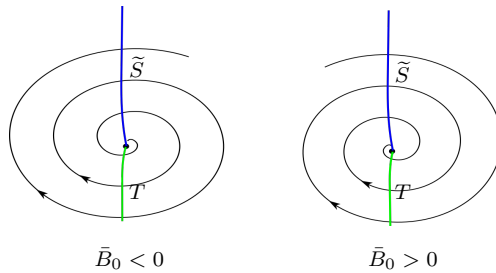


Figure 7: The dynamics of (30) near the focus $(\bar{x}, \bar{y}) = (\bar{x}_\tau, 0) \sim (0, 0)$, for a fixed parameter τ , with $\delta > 0$ and $\tilde{r} > 0$. The section \tilde{S} and the section T in the (\bar{x}, \bar{y}) -plane.

Combining Proposition 3.4 and Proposition 3.5 (i.e., gluing the graphs given in Figure 6 and Figure 8), we see that the difference map Δ (and its derivative $\frac{\partial \Delta}{\partial \bar{y}}$) has at most 2 zeros (counting multiplicity) w.r.t. $\bar{y} \in]0, \frac{1}{2}\bar{x}_R^2[$. This concludes the proof of Theorem 2.5 in the elliptic case.

3.6.1 Proof of Proposition 3.4

In this section we prove that for any fixed small $\mu > 0$ there exist small $\rho_0 > 0$, $\delta_0 > 0$, $u_0 > 0$ and a small neighborhood W of the origin in \mathbb{R}^3 such that $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}$ is a C^k -function in $(\bar{y}, \tau) \in [\mu, \frac{1}{2}\bar{x}_R^2] \times [0, \delta_0] \times W \times [0, u_0] \times \Lambda$, and $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}, \tau) < -\rho_0$ for $(\bar{y}, \tau) \in [\mu, \frac{1}{2}\bar{x}_R^2] \times [0, \delta_0] \times W \times [0, u_0] \times \Lambda$. This statement, the expression (32) and the first part of Remark 12 imply Proposition 3.4.

Taking into account the second part of Remark 12, it suffices to prove the statement for $\bar{y} \sim \frac{1}{2}\bar{x}_R^2$. Since the slow dynamics (31) is regular for $\bar{x} < 0$ and $(B_1, B_2, u) = (0, 0, 0)$ (see Figure 5), [DMD10] implies that $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}, \tau)$ is a bounded C^k -function which we denote by $O(1)$. We don't specify this $O(1)$ -term because it will not be the leading order term in $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y}, \tau)$.

It remains to study $\delta^2 \frac{\partial \mathcal{I}_+}{\partial \bar{y}}(\bar{y}, \tau)$, along the orbit $\mathcal{O}^+(\bar{y}, \tau)$ of (30) from the point $(0, \bar{y})$ to the section T . The following lemma (see [DMD08]) allows us to study the divergence integral in normal form coordinates:

Lemma 3.6. *Let $\Psi : V \subset \mathbb{R}^n \rightarrow V' \subset \mathbb{R}^n : y \mapsto x = \Psi(y)$ be a diffeomorphic transformation between two local charts of an n -dimensional manifold. Let X be a vector field defined on V' and let $Y = \Psi^*(X)$ be the pull back of this vector field on V . Then*

$$\int_{\Psi(\mathcal{O})} \text{div}_{\mathbb{R}^n} X dt = \int_{\mathcal{O}} \text{div}_{\mathbb{R}^n} Y dt + \log \frac{J(y_2)}{J(y_1)}$$

where \mathcal{O} is an orbit of Y from one point y_1 of V to another point y_2 and where $J(y)$ is the Jacobian determinant of the transformation Ψ . Let $h \cdot Y$ be an equivalent vector field on V for some strictly positive function h . Then

$$\int_{\mathcal{O}} \text{div}_{\mathbb{R}^n} (Y) dt = \int_{\mathcal{O}} \text{div}_{\mathbb{R}^n} (hY) dt' - \log \frac{h(y_2)}{h(y_1)},$$

where $dt' = dt/h$.

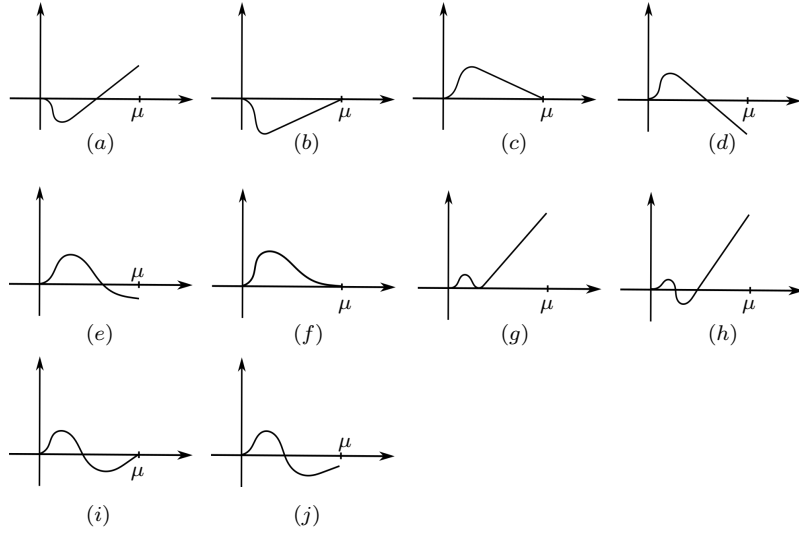


Figure 8: All possible graphs of the difference map Δ on the interval $[0, \mu]$, with the property that Δ has at least 1 zero on the interval $]0, \mu]$. In figures (e) and (f) we have $\frac{\partial \Delta}{\partial \bar{y}}(\mu, \tau) = 0$.

At $\bar{x} = \bar{x}_R$ and near $\tau = (0, 0, 0, 0, 0, \lambda)$ we use the C^k -normal form (23) where $-\delta^2 \nu_1$, $\nu_1 = \nu_1(\tau) > 0$, is ratio of eigenvalues of a hyperbolic saddle of (30) near $(\bar{x}, \bar{y}) = (\bar{x}_R, 0)$. The orbit $\mathcal{O}^+(\bar{y}, \tau)$ intersects the section $\{v_2 = 1\}$ (resp. the section $\{v_1 = 1\}$) in a point with $v_1 = \alpha_+(\bar{y}, \tau)$ (resp. in a point with $v_2 = \alpha_+^0(\bar{y}, \tau)$), where $\alpha_+(\bar{y}, \tau)$ (resp. $\alpha_+^0(\bar{y}, \tau)$) is C^k due to Theorem 2.16 of [DMD08]. Now we split up $\delta^2 \mathcal{I}_+$ in three parts:

$$\delta^2 \mathcal{I}_+(\bar{y}, \tau) = \delta^2 \mathcal{I}_1(\bar{y}, \tau) + \delta^2 \mathcal{I}_2(\bar{y}, \tau) + \delta^2 \mathcal{I}_3(\bar{y}, \tau).$$

The divergence integral $\mathcal{I}_1(\bar{y}, \tau)$ is taken along the fast part of the orbit $\mathcal{O}^+(\bar{y}, \tau)$ between the point $(0, \bar{y})$ and the point $(v_1, v_2) = (\alpha_+(\bar{y}, \tau), 1)$. Since the vector field (30) is (locally) C^∞ -conjugate to a divergence free flow box, along the fast fiber, $\mathcal{I}_1(\bar{y}, \tau)$ is equal to C^k -log-terms in Lemma 3.6. Thus, $\delta^2 \mathcal{I}_1(\bar{y}, \tau)$ and $\delta^2 \frac{\partial \mathcal{I}_1}{\partial \bar{y}}(\bar{y}, \tau)$ are $O(\delta^2)$ and C^k .

The divergence integral $\mathcal{I}_2(\bar{y}, \tau)$ is taken along the orbit $\mathcal{O}^+(\bar{y}, \tau)$ between the point $(v_1, v_2) = (\alpha_+(\bar{y}, \tau), 1)$ and the point $(v_1, v_2) = (1, \alpha_+^0(\bar{y}, \tau))$. Using Lemma 3.6 we find that

$$\delta^2 \frac{\partial \mathcal{I}_2}{\partial \bar{y}}(\bar{y}, \tau) = \frac{(1 - \nu_1 \delta^2) \frac{\partial \alpha_+}{\partial \bar{y}}(\bar{y}, \tau)}{\nu_1 \alpha_+(\bar{y}, \tau)} + O(\delta^2),$$

where $O(\delta^2)$ is C^k . See also Section 3.3.6. Since $\frac{\partial \alpha_+}{\partial \bar{y}} < 0$ uniformly in $\bar{y} \sim \frac{1}{2} \bar{x}_R^2$, $\alpha_+ > 0$ for $\bar{y} < \frac{1}{2} \bar{x}_R^2$ and $\alpha_+ \rightarrow 0$ as $\bar{y} \rightarrow \frac{1}{2} \bar{x}_R^2$, we conclude that $\delta^2 \frac{\partial \mathcal{I}_2}{\partial \bar{y}}(\bar{y}, \tau)$ tends to $-\infty$ as $\bar{y} \rightarrow \frac{1}{2} \bar{x}_R^2$.

The divergence integral $\mathcal{I}_3(\bar{y}, \tau)$ is taken along the orbit $\mathcal{O}^+(\bar{y}, \tau)$ between the point $(v_1, v_2) = (1, \alpha_+^0(\bar{y}, \tau))$ and the section T . We can write \mathcal{I}_3 as

$$\mathcal{I}_3(\bar{y}, \tau) = \tilde{\mathcal{I}}_3(\alpha_+^0(\bar{y}, \tau), \tau),$$

where $\tilde{\mathcal{I}}_3(v_2, \tau)$ is the divergence integral along the orbit $\mathcal{O}^+(\bar{y}, \tau)$ between the point $(v_1, v_2) = (1, v_2)$ and the section T . Since the slow dynamics is regular in this part, [DMD10] implies that $\delta^2 \frac{\partial \tilde{\mathcal{I}}_3}{\partial v_2}(v_2, \tau)$ is $O(1)$ and C^k . Thus, the derivative

$$\delta^2 \frac{\partial \tilde{\mathcal{I}}_3}{\partial \bar{y}}(\bar{y}, \tau) = \delta^2 \frac{\partial \tilde{\mathcal{I}}_3}{\partial v_2}(\alpha_+^0(\bar{y}, \tau), \tau) \frac{\partial \alpha_+^0}{\partial \bar{y}}(\bar{y}, \tau)$$

is $O(1)$ and C^k . Again, there is no need to specify the $O(1)$ -term since it is bounded.

Putting all the informations together we find that $\delta^2 \frac{\partial \tilde{\mathcal{I}}_3}{\partial \bar{y}}(\bar{y}, \tau)$ is C^k and tends to $-\infty$ as $\bar{y} \rightarrow \frac{1}{2} \bar{x}_R^2$. Since $\delta^2 \frac{\partial \tilde{\mathcal{I}}_3}{\partial \bar{y}}(\bar{y}, \tau)$ is bounded we conclude that $\delta^2 \frac{\partial \tilde{\mathcal{I}}_3}{\partial \bar{y}}(\bar{y}, \tau)$ is C^k and strictly negative for $\bar{y} \sim \frac{1}{2} \bar{x}_R^2$ and $\bar{y} < \frac{1}{2} \bar{x}_R^2$.

3.6.2 Proof of Proposition 3.5

The combined results of [HDMD13], [HDMD14] and [Huz16] imply Proposition 3.5. However, for the sake of completeness, we give a sketch of the proof of Proposition 3.5.

Let us recall that the parameter \bar{B}_2 is kept in a compact set.

Theorem 3.7. *The following statements are true.*

(i) *Let $\bar{B}_2^0 > 0$ be any arbitrarily small fixed number. There exist small $\delta_0 > 0$, $\tilde{r}_0 > 0$, $\bar{B}_0^0 > 0$, $u_0 > 0$ and a neighborhood U of $(\bar{x}, \bar{y}) = (0, 0)$ such that the following statements are true.*

1 *The family (30) has at most 1 (hyperbolic) limit cycle in U for each $(\delta, \tilde{r}, \bar{B}_0, \bar{B}_2, u, \lambda) \in [0, \delta_0] \times [0, \tilde{r}_0] \times [-\bar{B}_0^0, \bar{B}_0^0] \times \{|\bar{B}_2| \geq \bar{B}_2^0\} \times [0, u_0] \times \Lambda$.*

2 *When we fix $(\delta, \tilde{r}, \bar{B}_2, u, \lambda) \in [0, \delta_0] \times [0, \tilde{r}_0] \times \{|\bar{B}_2| \geq \bar{B}_2^0\} \times [0, u_0] \times \Lambda$, the \bar{B}_0 -family (30) undergoes, in U and at $\bar{B}_0 = 0$, a Hopf bifurcation of codimension 1. Assume $\bar{B}_2 \geq \bar{B}_2^0$. When \bar{B}_0 increases there is in U an attracting hyperbolic focus and no limit cycle; when \bar{B}_0 decreases there is in U a repelling hyperbolic focus and an attracting limit cycle of which the size monotonically grows as \bar{B}_0 decreases. Assume $\bar{B}_2 \leq -\bar{B}_2^0$. When \bar{B}_0 decreases there is in U a repelling hyperbolic focus and no limit cycle; when \bar{B}_0 increases there is in U an attracting hyperbolic focus and a repelling limit cycle of which the size monotonically grows as \bar{B}_0 increases.*

(ii) *There exist small $\delta_0 > 0$, $\tilde{r}_0 > 0$, $\bar{B}_0^0 > 0$, $\bar{B}_2^0 > 0$, $u_0 > 0$ and a neighborhood U of $(\bar{x}, \bar{y}) = (0, 0)$ such that the family (30) has at most 2 limit cycles in U for each $(\delta, \tilde{r}, \bar{B}_0, \bar{B}_2, u, \lambda) \in [0, \delta_0] \times [0, \tilde{r}_0] \times [-\bar{B}_0^0, \bar{B}_0^0] \times [-\bar{B}_2^0, \bar{B}_2^0] \times [0, u_0] \times \Lambda$.*

Theorem 3.7(i) follows directly from Theorem 2.4 of [HDMD13]. Theorem 3.7(ii) follows from Theorem 2.2 of [Huz16] because the coefficient in front of the term $\bar{x}^4 \frac{\partial}{\partial \bar{y}}$ in (30) is nonzero.

Take a fixed small $\bar{B}_2^0 > 0$. Theorem 3.7(i) implies that there is a small $\mu > 0$ such that for each $(\delta, \tilde{r}, \bar{B}_0, u) \sim (0, 0, 0, 0)$, $\delta > 0$, $\tilde{r} > 0$, $\lambda \in \Lambda$ and $\bar{B}_2 \geq \bar{B}_2^0$ (resp. $\bar{B}_2 \leq -\bar{B}_2^0$), with the property that Δ has at least 1 zero on the interval

$]0, \mu]$, the graph of Δ can be found in Figures 8(a) to 8(b) (resp. Figures 8(c) to 8(d)). To prove that in this case $\frac{\partial \Delta}{\partial \bar{y}}$ has precisely 1 zero on $]0, \mu]$, we need to have a look at the proof of Theorem 2.4 in [HDMD13] (Section 3.7). In fact, using the primary blow-up, it has been proved in Section 3.7 of [HDMD13] that there is a small $M > 0$ such that $\frac{\partial \Delta}{\partial \bar{y}}$ has precisely 1 zero (counting multiplicity) w.r.t. $\bar{y} \in]0, \tilde{r}^2 M]$ (the proof was based on [DR09] and [FTV13]), and that $\frac{\partial \Delta}{\partial \bar{y}}$ is strictly positive (resp. strictly negative) for $\bar{B}_2 \geq \bar{B}_2^0$ (resp. $\bar{B}_2 \leq -\bar{B}_2^0$) and $\bar{y} \in]\tilde{r}^2 M, \mu]$ (the slow divergence integral, obtained after the primary blow-up, is strictly negative (resp. strictly positive) when $\bar{B}_2 \geq \bar{B}_2^0$ (resp. $\bar{B}_2 \leq -\bar{B}_2^0$)).

On the other hand, Theorem 3.7(ii) implies that there is a small $\mu > 0$ such that for each $(\delta, \tilde{r}, \bar{B}_0, \bar{B}_2, u) \sim (0, 0, 0, 0, 0)$, $\delta > 0$, $\tilde{r} > 0$, and $\lambda \in \Lambda$ the difference map Δ has at most 2 zeros (counting multiplicity) w.r.t. $\bar{y} \in]0, \mu]$. Moreover, from the proof of Theorem 2.2 in [Huz16] and from the fact that the coefficient in front of the term $\bar{x}^4 \frac{\partial}{\partial \bar{y}}$ in (30) is strictly positive, it can be seen that, if Δ has at least 1 zero on the interval $]0, \mu]$, then the graph of Δ can be found in Figures 8(a) to 8(j). In fact, the following rescaling in the parameter space (\bar{B}_2, \tilde{r}) was made in [Huz16]:

$$(\bar{B}_2, \tilde{r}) = \tilde{\rho}(\bar{B}_2, \bar{r}), (\bar{B}_2, \bar{r}) \in \mathbb{S}^1, \bar{r} > 0, \tilde{\rho} \sim 0, \bar{\rho} > 0.$$

Let's suppose that Δ has at least 1 zero on the interval $]0, \mu]$. There are 2 possibilities:

1. Let \bar{B}_2 be strictly positive or close to 0. There is a small $M > 0$ such that $\frac{\partial \Delta}{\partial \bar{y}}$ has at most 2 zeros (counting multiplicity) w.r.t. $\bar{y} \in]0, \tilde{r}^2 M[$ (see Section 7.1 in [Huz16]), and $\frac{\partial \Delta}{\partial \bar{y}}$ is strictly positive for $\bar{y} \in]\tilde{r}^2 M, \mu]$ (see Theorem 6.5(a) in [Huz16]). Note that this result implies that, if Δ has precisely 2 zeros (counting multiplicity) on $]0, \mu]$, then Δ is strictly positive for $\bar{y} \sim 0$ and $\bar{y} > 0$.
2. Let \bar{B}_2 be strictly negative. There is a small $M > 0$ such that $\frac{\partial \Delta}{\partial \bar{y}}$ has precisely 1 zero (counting multiplicity) w.r.t. $\bar{y} \in]0, \tilde{r}^2 M[$ and Δ is strictly positive for $\bar{y} \sim 0$ and $\bar{y} > 0$ (see Section 7.1 in [Huz16]), and $\frac{\partial \Delta}{\partial \bar{y}}$ has at most 1 zero (counting multiplicity) w.r.t. $\bar{y} \in]\tilde{r}^2 M, \mu]$ and $\frac{\partial \Delta}{\partial \bar{y}}$ is strictly negative at $\bar{y} = \tilde{r}^2 M$ (see Theorem 6.5 (b) and (c) in [Huz16]).

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