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Summability of canard-heteroclinic saddle connections

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Abstract

For a given (real analytic) slow-fast system

$$\left\{ \begin{array}{rcl} \dot{x} & = & \varepsilon f(x,y,\varepsilon) \\ \dot{y} & = & g(x,y,\varepsilon), \end{array} \right.$$

that admits a slow-fast saddle and that satisfies some mild assumptions, the Borel-summability properties of the saddle separatrix tangent in the direction of the critical curve are investigated: 1-summability is shown. It is also shown that slow-fast saddle connections of canard type have summability properties, in contrast to the typical lack of Borel-summability for canard solutions of general equations.

1 Introduction

In this paper we deal with Gevrey-asymptotic methods to state properties of invariant manifolds of certain (real analytic) slow-fast systems. Slow-fast systems, presented in a standard form, are systems that can be represented as

$$\left\{ \begin{array}{rcl} \dot{x} & = & \varepsilon f(x,y,\varepsilon) \\ \dot{y} & = & g(x,y,\varepsilon), \end{array} \right.$$

where ε is a small parameter (typically strictly positive) that identifies the time scale separation between the speed of the x-variable and the speed of the y-variable. Under mild conditions of normal hyperbolicity, Fenichel [1] showed the existence of C^k-smooth invariant manifolds $y = \psi(x,\varepsilon)$, tending towards the zero set of $g(x,y,0)$. In general, these invariant manifolds are not C^∞ smooth, but only C^k for arbitrary k. It does imply that, still under normal hyperbolicity condition, there exists a unique power series expansion $y = \sum_{n=0}^{\infty} \psi_n(x)\varepsilon^n$, which can be the object of study from a formal point of view. It is well-known that the series is typically divergent, and the divergence can be controlled: it is of the order at most Gevrey-1.

Gevrey-1 series can sometimes correspond to unique functions, just like convergent series; in that case the series is called 1-summable (see for example [2]). Unfortunately, Fenichel manifolds are rarely 1-summable, and clearly whenever the invariant manifolds are not infinitely smooth, one cannot expect the asymptotic series to be summed to a smooth object. But even in the more restrictive case where Fenichel manifolds are C^∞, the corresponding power series are typically not 1-summable. This paper deals with isolated situations where the summability can be obtained, of course under some restrictions.

The first restriction that we impose is on the dimensions of the slow and fast variables: we proceed in the context of planar systems, i.e. $x, y \in \mathbb{R}$. In that context, we assume the presence of an analytic curve $y = \psi_0(x)$ which solves the equation $g(x, y, 0) = 0$. This curve is called the *critical curve*. Normal hyperbolicity of the vector field at some point (x_0, y_0) means (besides $y_0 = \psi_0(x_0)$) that

$$\frac{\partial g}{\partial y}(x_0, y_0, 0) \neq 0.$$

This implies, of course, that for each x in a neighbourhood of x_0 the system $\dot{y} = g(x, y, 0)$ has a hyperbolic equilibrium point at $\psi_0(x)$.

The formal invariant manifold, which arises from Fenichel theory, will be denoted $y = \hat{\psi}(x, \varepsilon) = \sum_{n=0}^{\infty} \psi_n(x)\varepsilon^n$. We may now distinguish several cases, depending on the order of zero of the function

$$h(x) = f(x, \psi_0(x), 0) \tag{1}$$

at the point x_0 (which more or less defines the center manifold behaviour):

(1) $h(x_0) \neq 0$. This is the regular case. The slow dynamics shows a drift along the x-axis with nonzero speed.

(2) $h(x_0) = 0$, $h'(x_0) \neq 0$. This is the slow-fast saddle case or slow-fast node case.

(3) $h(x_0) = \cdots = h^{(p)}(x_0) = 0$, $h^{(p+1)}(x_0) \neq 0$. The slow dynamics has a singularity of order $p + 1$, $p \geq 1$.

In case (1), the system does not have a nearby singular point for $\varepsilon \neq 0$. Local Fenichel manifolds are shown to be C^∞ but are in general not summable; it is well-known that summability properties cannot be derived from a local study of the system, but is related to global properties of the system of differential equations (see for example [3]). In case (2), we focus on the slow-fast saddle case, where $h'(x_0) \cdot \frac{\partial g}{\partial y}(x_0, y_0, 0) < 0$. The slow-fast saddle perturbs to a hyperbolic saddle when $\varepsilon > 0$, and the object of study is the ε-family of stable and unstable separatrices. The properties will be explained in detail in Theorem 3.1, here we restrict ourselves to mentioning that the separatrix in the direction of the critical curve is 1-summable in the positive ε-direction. The slow-fast node case will not be considered (from a dynamical systems point of view, it is less relevant).

Case (3) is the more difficult one, since in general up to $p + 1$ singular points may bifurcate from (x_0, y_0) when $\varepsilon > 0$, following elementary catastrophy theory. We do not trace the summability properties of such singularities in generality here, but mention that the restricted case (3) where the singularity of order $p + 1$ is preserved for $\varepsilon > 0$ has been treated before in [4]: the center-separatrix is shown to exhibit "monomial summability" properties, mixing the phase variable x and ε in describing the summability properties.

In both the second and third case, a local study suffices to conclude whether or not 1-summability of $\hat{\psi}(x,\varepsilon)$ is present or not. In the first case however, this is no longer true: in that case it is possible to prove 1-summability of $\hat{\psi}(x,\varepsilon)$ at $x = x_0$ whenever the slow dynamics can take the point in negative time (resp. positive time) towards a point where 1-summability was found, for example near a slow-fast saddle, all the while remaining on an attracting (resp. repelling) part of the critical curve.

Intuitively, the description above says that slow curves that originate from a slow-fast saddle have summability properties. In the case when two such slow curves come together in a contact/turning point, the meaning of which will be explained later on, we will match them together using an extra breaking parameter and thus form canard curves. The main theorem can intuitively be described as follows:

Theorem 1.1. *A heteroclinic saddle connection between two persistent slow-fast saddles on a slow manifold of a real analytic planar slow-fast system is summable w.r.t. the singular parameter (in the positive real direction), uniformly for x in compact subsets of the domain of the critical curve not including the turning point.*

For more precise statements we refer to Theorem 5.11. This theorem can be applied to singular analytic differential equations like

$$\varepsilon(x-c)(d-x)\frac{dy}{dx} = a + x^{m-1}y + \varepsilon F(x,y,\varepsilon,a),\ c < 0 < d,\ m \text{ even}.$$

Theorem 1.1 then shows the existence of a m-summable control curve $a = \mathcal{A}(\varepsilon^{1/m})$ along which the equation has a m-summable solution $y = y(x,\varepsilon^{1/m})$ w.r.t. $\varepsilon^{1/m}$, uniformly on arbitrary compact subsets of $[c,d]$ which do not include the turning point $x = 0$.

The organisation of the paper is as follows: Section 2 starts with some notations and definitions that we will use. In Section 3 we present a local study of the summability near slow-fast saddles; the result is formulated in Theorem 3.1. In Section 4, we show how to use the slow dynamics to extend the local 1-summability properties along the critical curve; the results of that section are gathered in Theorem 4.1. In Section 5, we show how summability properties are carried over to canard points, see Theorem 5.1. The combinations of these three theorems lead to a proof of Theorem 1.1, which is formulated precisely in Theorem 5.11 at the end of the paper.

2 Some definitions and properties

2.1 Gevrey series

Definition 2.1. *Let $V \subset \mathbb{C}^\ell$ be open, $\ell \geq 1$, and let $B > 0$ and $s \geq 0$. A formal series*

$$\hat{f}(x,\varepsilon) = \sum_{n=0}^{\infty} f_n(x)\varepsilon^n$$

is Gevrey-s of type B in ε, uniformly for x in V, if $f_n \in \mathcal{O}(V)$ for all $n \in \mathbb{N}$ and there exists $A > 0$ such that

$$\sup_{x\in V} |f_n(x)| \leq AB^n\Gamma(1+sn).$$

It is easy to see that the set of Gevrey-s series, for a fixed s, forms an algebra. Furthermore, Gevrey-0 series correspond to convergent series and hence define analytic functions on $V \times B\left(0, \frac{1}{B}\right)$. Besides the algebra property, we state without proof the following well-known fact regarding compositions:

Lemma 2.2. *([5]) Suppose that $G(x,y,\varepsilon)$ is an analytic function on $V \times B(0,r_1) \times B(0,r_2)$ and $\hat{f}(x,\varepsilon)$ is a formal Gevrey-s series in ε, uniformly for x in V, and without constant term i.e. $f_0(x) = 0$. Then the formal series $G(x,\hat{f}(x,\varepsilon),\varepsilon)$ is also Gevrey-s.*

2.2 Gevrey functions

Notation 2.3. *For $\theta \in [0,2\pi[$, $\delta \in]0,2\pi[$ and $r > 0$ we denote the (open) sector in the direction θ with opening δ and radius r by*

$$S(\theta,\delta,r) = \left\{ z \in \mathbb{C} \;\middle|\; 0 < |z| < r,\ \mathrm{Arg}(ze^{-i\theta}) \in \left]-\tfrac{\delta}{2}, \tfrac{\delta}{2}\right[\right\}.$$

The infinite sector $\cup_{r>0} S(\theta,\delta,r)$ in the direction θ is denoted by $S(\theta,\delta)$.

Definition 2.4. *Consider some sector S and a subset $V \subset \mathbb{C}^\ell$, $\ell \geq 1$. Let $s \geq 0$ and $\hat{f}(x,\varepsilon) = \sum_{n=0}^{\infty} f_n(x)\varepsilon^n$ a formal series in ε with coefficients in $\mathcal{O}(V)$. We say that a function $f(x,\varepsilon)$, analytic on $V \times S$, is Gevrey-s asymptotic to the formal series $\hat{f}(x,\varepsilon)$, with respect to ε, uniformly for $x \in V$, if for every $\varepsilon \in S$ and every $N \in \mathbb{N}_0$ we have*

$$\sup_{x\in V} \left| f(x,\varepsilon) - \sum_{n=0}^{N-1} f_n(x)\varepsilon^n \right| \leq CD^N\Gamma(1+sN)\,|\varepsilon|^N$$

for certain $C, D > 0$. We denote this by

$$f(x,\varepsilon) \sim_s \sum_{n=0}^{\infty} f_n(x)\varepsilon^n$$

Remark 2.5. The Gevrey property of the series $\sum_n f_n(x)\varepsilon^n$ is an immediate consequence of the fact that some function is Gevrey-asymptotic to it. Additionally, the characterization of Gevrey functions implies C^∞ smoothness at the vertex $\varepsilon = 0$ (this requires a small proof).

We have the following useful result:

Theorem 2.6. *Consider an analytic function $F : D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is an open set, and a function $g : S(\theta, \delta, r) \to D$, Gevrey-s asymptotic to a formal series. The composition function $F \circ g$ is also Gevrey-s asymptotic to a formal series on $S(\theta, \delta, \tilde{r})$, where $\tilde{r} \leq r$.*

2.3 Summability

Definition 2.7. *Given $k > 0$ and a Gevrey-$\frac{1}{k}$ series*

$$\hat{f}(x, \varepsilon) = \sum_{n=0}^{\infty} f_n(x)\varepsilon^n.$$

We say that $\hat{f}$ is Borel k-summable in a direction $\theta \in [0, 2\pi[$ if there exist $r, \tau > 0$ and a function $f(x, \varepsilon)$ analytic on $V \times S(\theta, \frac{\pi}{k} + \tau, r)$ such that $f \sim_{\frac{1}{k}} \hat{f}$.

Remark 2.8. If $k \leq \frac{1}{2}$, the k-sum will be a function defined for ε in a sector on the Riemann surface of the logarithm.

Definition 2.9. *Let, for $k > 0$, $\hat{f}(x, \varepsilon) := \sum_{n=1}^{\infty} f_n(x)\varepsilon^n$ be a Gevrey-$\frac{1}{k}$ series in ε (without constant coefficient), uniformly for $x \in V \subset \mathbb{C}^\ell$. We define the formal Borel transform of order k (with respect to ε) of this series to be*

$$\mathcal{B}_k(\hat{f})(x, \eta) = \sum_{n=1}^{\infty} \frac{f_n(x)}{\Gamma(1 + \frac{(n-1)}{k})} \eta^{n-1}.$$

We see that the formal Borel transform of order k of a type B Gevrey-$\frac{1}{k}$ series is a convergent series for $(x, \eta) \in V \times B(0, 1/B)$ since, for example, the following bound can be found (see [6])

$$\frac{\Gamma(1 + \frac{n}{k})}{\Gamma(1 + \frac{(n-1)}{k})} < \sqrt{\frac{\pi}{e}} \left(\frac{n}{k} + \frac{1}{2}\right)^{\frac{1}{k}}.$$

The following theorem gives an equivalent definition for k-summability.

Theorem 2.10. *([7]) Let $\hat{f}(x, \varepsilon) = \sum_{n=1}^{\infty} f_n(x)\varepsilon^n$ be a Gevrey-$\frac{1}{k}$ series, $k > 0$, uniformly for $x \in V \subset \mathbb{C}^\ell$. For every $\theta \in [0, 2\pi[$, the following two statements are equivalent*

- *The series $\hat{f}$ is Borel k-summable in the direction θ with Borel sum $f(x, \varepsilon)$.*

- *There exists a sector $S(\theta,\tau)$ for $\tau > 0$ such that $\mathcal{B}_k(\hat{f})(x,\eta)$ admits analytic continuation to $S(\theta,\tau)$ of exponential growth at most of order k, i.e. there exist $M, \nu > 0$ such that for all $\eta \in S(\theta,\tau)$*

$$\sup_{x\in V}\left|\mathcal{B}_k(\hat{f})(x,\eta)\right| \leq Me^{\nu|\eta|^k}.$$

Moreover the function f is unique in the case the statements are true.

3 Summability of slow curves near slow-fast saddles

We consider a real analytic slow-fast vector field in standard form, as stated in the introduction:

$$\left\{\begin{array}{lcl}\dot{x} & = & \varepsilon f(x,y,a,\varepsilon)\\ \dot{y} & = & g(x,y,a,\varepsilon).\end{array}\right. \tag{2}$$

We added an extra parameter, in view of its necessity in Section 5, but stress that it will not affect the results here. We assume furthermore

$$f(x_0,y_0,a_0,0) = g(x_0,y_0,a_0,0) = 0, \tag{3}$$

$$\frac{\partial g}{\partial y}(x_0,y_0,a_0,0) < 0, \tag{4}$$

$$\left(\frac{\partial g}{\partial x}.\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}.\frac{\partial f}{\partial x}\right)(x_0,y_0,a_0,0) > 0. \tag{5}$$

These conditions characterize the point (x_0,y_0) as a slow-fast saddle: using the IFT and (3–4) we know that the zero set of $g(x,y,a_0,0) = 0$ is given by a curve $y = \psi_0(x)$ with $\psi_0'(x)$ computable using implicit differentiation. With the expression of $\psi_0'(x)$ in mind, the left hand side of (5) shares the sign with that of $h'(x_0)$, where h is given in (1) in Section 1. Equations (3–5) hence correctly reflect the case (2) of slow-fast saddles reported in Section 1.

One can see, by once again utilizing the IFT, that there exists an analytic parametric curve $(x_s(\varepsilon,a), y_s(\varepsilon,a))$, defined for (ε,a) near $(0,a_0)$ along which system (2) has a singularity (hyperbolic of saddle type when $\varepsilon > 0$). Our main result in this section is:

Theorem 3.1. *Under the conditions (3–5), system (2) has an (ε,a)-family of* analytic *invariant graphs $x = \psi^s(y,\varepsilon,a)$ that correspond to the stable separatrix of the saddle (x_s,y_s) for $\varepsilon > 0$. There is an (ε,a)-family of invariant graphs $y = \psi^u(x,\varepsilon,a)$, defined for $\varepsilon \geq 0$ that correspond to the unstable separatrix of the saddle (x_s,y_s) for $\varepsilon > 0$. It is furthermore* 1*-summable w.r.t. ε in the positive real direction (uniformly for (x,a) in a neighbourhood of (x_0,a_0)).*

The statement regarding the analyticity of the stable separatrix is a classic statement, just added for the sake of completeness and to contrast the properties of the unstable separatrix.

With elementary changes of coordinates and time we can reduce the question concerning the unstable separatrix to a similar one where $f(x, y, a, \varepsilon) = x$, $(x_0, y_0) = (0, 0)$ and $g(x, y, a, \varepsilon) = O(y, \varepsilon)$. (In fact, it amounts to analytically straightening the critical curve and the analytic stable separatrix.) The second statement of Theorem 3.1 is then a direct consequence of:

Theorem 3.2. *Consider the analytic equation*

$$\varepsilon x \frac{\mathrm{d}y}{\mathrm{d}x} = \varphi(x)y + \varepsilon H(x, y, a, \varepsilon), \tag{6}$$

defined in a neighbourhood of $(x, y, a, \varepsilon) = (0, 0, a_0, 0)$, *and satisfying*

(i) φ *is (real) analytic and defined near* 0*;*

(ii) $\varphi(0) < 0$.

This equation has a solution $y(x, a, \varepsilon)$ *defined for* (x, a) *in a polydisc around* $(0, a_0)$ *and for* ε *in a complex sector containing the positive real direction. Furthermore, it is* 1*-summable w.r.t.* ε *in the positive real direction.*

The proof of Theorem 3.2 is spread over the coming subsections.

Remark 3.3. While the positive real direction for ε is dynamically the most relevant one, it should be clear from the proof that the summability also extends to other directions in the (open) right half-plane. This directions in the left half-plane are excluded because in that direction, the Borel transform could exhibit singularities (imposed by the negative linear part of the governing equation) as the example below shows. We note that the negative ε-direction no longer corresponds to a slow-fast saddle, but rather to a node.

Example 3.4. By lemma 3.5 below, the equation

$$\varepsilon x \frac{\mathrm{d}y}{\mathrm{d}x} = -y + \varepsilon \mathrm{Log}\left(\frac{1}{1-x}\right)$$

has a formal series solution which is Gevrey-1 in ε, uniformly for $x \in B(0, \frac{1}{2})$. One can show that the Borel transform (w.r.t. ε), $F(x, \eta)$, of this series satisfies the equation

$$x \int_0^\eta \frac{\partial F}{\partial x}(x, s)\mathrm{d}s + F(x, \eta) = \mathrm{Log}\left(\frac{1}{1-x}\right).$$

The unique solution to this equation is given by $F(x, \eta) = \mathrm{Log}\left(\frac{1}{1-xe^{-\eta}}\right)$ which clearly has singularities when η is taken in a direction in the left half-plane. By theorem 2.10 this implies that there can be no summability in these directions.

3.1 Existence of formal Gevrey-1 solution

Lemma 3.5. *Equation (6) (under the conditions of Theorem 3.2) is formally solved by a unique series*

$$\hat{y}(x, a, \varepsilon) = \sum_{n=1}^{\infty} y_n(x, a)\varepsilon^n.$$

This series is Gevrey-1 in ε, uniformly for $(x,a) \in B(0,s) \times B(a_0,r)$ for sufficiently small $r,s>0$.

Proof. We refer to [8] for a proof based on the method of majorating series and on the use of modified Nagumo norms (we note that in our case the use of standard Nagumo norms is sufficient). For properly referencing [8], the left-hand side of (6) should not have contained the factor x causing an extra degeneracy, but in a remark at the end of Section 4 of [8], the authors do mention that this generalization does not affect their results (which is also quite clear from reading the proof). Please note that, in that reference, the extra parameter a is included for a different reason, but it is clear that it does not have effect on the results. □

Using this result we deduce that the formal Borel transform of order 1,

$$\mathcal{B}_1(\hat{y})(x,a,\eta) = \sum_{n=1}^{\infty} \frac{y_n(x,a)}{(n-1)!}\eta^{n-1}, \tag{7}$$

defines an analytic function on some polydisc around 0. The next section is devoted to discussing its analytic continuation in the η-direction.

3.2 The equation in the Borel plane

The formal expansion obtained in lemma 3.5 will be examined in the Borel plane, by comparing it with a solution of the Borel-transformed version of equation (6). Inspired by [9], The analytic continuation of (7) is achieved by directly applying a fixed point argument in an appropriate Banach space in the Borel plane.

3.2.1 Setting up appropriate Banach spaces

Definition 3.6. *Fix an infinite sector $S = S(\theta,\delta)$ with $\theta \in [0,2\pi[$, $\delta \in]0,2\pi[$, and choose $r>0$. For $\mu>0$ and $f : B(a_0,r)\times S \to \mathbb{C}$ we define*

$$||f||_{\mu,S} = \sup_{(a,\eta)\in B(a_0,r)\times S} |f(a,\eta)|\,(1+\mu^2\,|\eta|^2)e^{-\mu|\eta|}.$$

and we denote the space of analytic functions $B(a_0,r)\times S \to \mathbb{C}$ with a finite bound w.r.t. $\|\cdot\|_{\mu,S}$ by $\mathcal{G}^\mu$.

At first sight, the chosen norm seems somewhat artificial and to some extent it is: the extra factor $1+\mu^2\,|\eta|^2$ is only added so that the norm behaves well w.r.t. convolution product, see below. The exponential factor is added to ensure that functions of at most exponential growth would fit in the space.

Theorem 3.7. *For every $\mu>0$, $\mathcal{G}^\mu$ is a Banach space.*

We will omit a proof but just give as a hint that the introduced norm is equivalent to a sup-norm on bounded subsectors, and therefore the convergence of Cauchy sequences can be done for each bounded subsector. The analyticity of the Cauchy limit would follow from the equivalent property on subsectors.

Properties 3.8. *([8])*

(i) $\|f\|_{\tilde{\mu},S} \leq \|f\|_{\mu,S}$ *if* $\tilde{\mu} \geq \mu$ *and thus* $\mathcal{G}^{\mu} \subset \mathcal{G}^{\tilde{\mu}}$.

(ii) $\|f * g\|_{\mu,S} \leq \frac{4\pi}{\mu} \|f\|_{\mu,S} \|g\|_{\mu,S}$ *with* $f * g$ *the convolution*

$$(f * g)(a, \eta) = \int_0^{\eta} f(a, s) g(a, \eta - s) \mathrm{d}s.$$

Definition 3.9. *Consider the formal series* $F(x, a, \eta) = \sum_{k=0}^{\infty} f_k(a, \eta) x^k$ *with* $f_k \in \mathcal{G}^{\mu}$. *We introduce the norm*

$$\|F\|_{\mu,S} = \sum_{k=0}^{\infty} \|f_k\|_{\mu,S} \mu^{-k},$$

and define

$$\mathcal{G}^{\mu}\{x\} = \{F = \sum_{k=0}^{\infty} f_k(a, \eta) x^k : \, f_k \in \mathcal{G}^{\mu}, \|F\|_{\mu,S} < \infty\}.$$

Each element of $\mathcal{G}^{\mu}\{x\}$ defines an analytic function on $B\left(0, \mu^{-1}\right) \times B(a_0, r) \times S$ of at most of exponential growth as $\eta \to \infty$. Properties 3.8 extend trivially to similar properties on $\mathcal{G}^{\mu}\{x\}$, and we furthermore have

$$\|xF\|_{\mu,S} = \frac{1}{\mu} \|F\|_{\mu,S}, \qquad \forall F \in \mathcal{G}^{\mu}\{x\},$$

which can be simply checked. Like $\mathcal{G}^{\mu}$ is a Banach space, so will $\mathcal{G}^{\mu}\{x\}$ be a Banach space. We don't provide details.

3.2.2 Equation (6) in the Borel plane

Let us now take the Borel transform of equation (6), which we will do term by term. Recall that $\hat{y}$ is the formal solution of this equation and that we have shown the Gevrey-1 property (Theorem 3.5). It follows that $\hat{Y} := \mathcal{B}_1(\hat{y})$ is well-defined near the origin and is a convergent series in the three variables (x, a, η) (recall that η is the coordinate that parameterizes the Borel plane). Cauchy's inequalities imply that also $\frac{\mathrm{d}\hat{y}}{\mathrm{d}x}$ is Gevrey-1, so we already know that (by using the well known fact that the Borel transform of a product of two Gevrey series is given by the convolution product of their respective Borel transforms)

$$\mathcal{B}_1\left(\,\varphi(x)\hat{y}\,\right) = \varphi(x)\hat{Y}, \qquad \mathcal{B}_1\left(\varepsilon x \frac{d\hat{y}}{dx}\right) = x\left(1 * \frac{d\hat{Y}}{dx}\right).$$

In order to transform the third term in the equation, $\varepsilon H(x, \hat{y}, \varepsilon, a)$, (which is Gevrey-1 due to lemma 2.2), we write

$$H(x, y, a, \varepsilon) = \sum_{k=0}^{\infty} H_k(x, a, \varepsilon) y^k \tag{8}$$

and infer that

$$\mathcal{B}_1(\varepsilon H(x,\hat{y},a,\varepsilon)) = \sum_{k=0}^{\infty} \left(\mathcal{B}_1(\varepsilon H_k) * \hat{Y}^{*k}\right), \tag{9}$$

where use the notation $F^{*k} := F*\ldots*F$ (k times) the k-fold convolution product (For $k=0$ the term in the term in the series in (9) evaluates to $\mathcal{B}_1(\varepsilon H_k)$, and not $\mathcal{B}_1(\varepsilon H_k) * 1$, since 1 is not the unity w.r.t. convolution). Statement (9) follows from the following arguments in the rest of this subsection.

It is clear that for every $K \in \mathbb{N}_0$ the formal series

$$\varepsilon H(x, \hat{y}(x,a,\varepsilon), a, \varepsilon)$$

and

$$\sum_{k=0}^{K} \varepsilon H_k(x,a,\varepsilon)\left(\hat{y}(x,a,\varepsilon)\right)^k$$

have equal coefficients for at least $\varepsilon^1, \ldots, \varepsilon^K$. Since they are both Gevrey-1 (lemma 2.2), it is an easy calculation that there exist $A, B > 0$ (independent of K) such that

$$\begin{aligned}
&\left|\mathcal{B}_1\left(\varepsilon H(x,\hat{y},a,\varepsilon)\right)(x,a,\eta) - \sum_{k=0}^{K}\left(\mathcal{B}_1\left(\varepsilon H_k\right) *_\eta \mathcal{B}_1\left(\hat{y}\right)^{*_\eta k}\right)(x,a,\eta)\right| \\
&= \left|\mathcal{B}_1\left(\varepsilon H(x,\hat{y},a,\varepsilon)\right)(x,a,\eta) - \mathcal{B}_1\left(\sum_{k=0}^{K} \varepsilon H_k(x,a,\varepsilon)\hat{y}^k\right)(x,a,\eta)\right| \\
&\leq AB\frac{\left(B\left|\eta\right|\right)^K\left(1+K(1-B\left|\eta\right|)\right)}{\left(B\left|\eta\right|-1\right)^2}.
\end{aligned}$$

This clearly implies (by letting $K \to \infty$) that (9) holds for η in a sufficiently small neighbourhood of 0.

We are now ready to consider the transformed equation of (6):

$$x\left(1 * \frac{\partial \hat{Y}}{\partial x}\right) = \varphi(x)\hat{Y} + \sum_{k=0}^{\infty}\left(\mathcal{B}_1(\varepsilon H_k) * \hat{Y}^{*k}\right). \tag{10}$$

All involved functions in equation are analytic, for $(x,y,\eta) \in B(0,s) \times B(a_0,r) \times B(0,C)$ for some $C > 0$.

3.2.3 Solving the equation in the Borel plane

In this subsection we show the existence of a unique solution of (10) in $\mathcal{G}^\mu\{x\}$. To prepare a fixed point argument, the following formulation of equation (10) will prove to be useful,

$$\hat{Y} = T\left((\varphi(x)-\varphi(0))\hat{Y} + \sum_{k=0}^{\infty}\left(\mathcal{B}_1(\varepsilon H_k) * \hat{Y}^{*k}\right)\right), \tag{11}$$

where T is the linear operator $T : \mathcal{G}^\mu\{x\} \to \mathcal{G}^\mu\{x\} \colon F \mapsto T(F)$, so that $T(F)$ satisfies the linear equation

$$x\left(1 * \frac{\partial}{\partial x} T(F)\right) - \varphi(0)T(F) = F. \tag{12}$$

In order to deal with the fixed-point equation (11), we still have to fix a specific sector in the definition of $\mathcal{G}^\mu$ for our results to hold. Let $0 < \delta < \pi$, we will work in the sector $S(0,\delta)$, which we denote simply by S. We give a few preparatory results.

Proposition 3.10. *For each $k \geq 1$, the function $e^{\frac{k\eta}{\varphi(0)}} \in \mathcal{G}^\mu$ and has norm 1. Furthermore, $1 * ke^{k\eta/\varphi(0)} = \varphi(0)e^{k\eta/\varphi(0)} - \varphi(0)$.*

Proof. Both the kernel function $(1 + \mu^2 |\eta|^2)e^{-\mu|\eta|}$ that is used in the definition of the norm and the function $|e^{\frac{k\eta}{\varphi(0)}}|$ are decreasing as $|\eta|$ is increased. (We have used that $\varphi(0) < 0$ and $0 < \delta < \pi$). So the norm is simply the modulus of the function evaluated at the origin $\eta = 0$. The equality regarding the convolution is just a simple calculation. ☐

Lemma 3.11. *If $F(x,a,\eta) = \sum_{k=0}^\infty f_k(a,\eta)x^k \in \mathcal{G}^\mu\{x\}$ then also*

$$\tilde{F} := \sum_{k=1}^\infty k\left(f_k * e^{\frac{k\eta}{\varphi(0)}}\right)x^k \in \mathcal{G}^\mu\{x\}, \qquad \textit{with } \|\tilde{F}\|_{\mu,S} \leq \frac{|\varphi(0)|}{\left|\cos(\frac{\delta}{2})\right|}\|F\|_{\mu,S}\,.$$

Proof. By the previous lemma and the properties of convolution, it is easy to see that $k(f_k * e^{k\eta/\varphi(0)}) \in \mathcal{G}^\mu$ with norm bounded by $\frac{4\pi}{\mu}\cdot\|f_k\|_{\mu,S}\cdot\|ke^{k\eta/\varphi(0)}\|_{\mu,S}$. However, the last factor grows with k so this bound would not suffice to prove the lemma. We can however improve the bound on the convolution, essentially using the exponentially decaying shape of the second factor in the convolution: for each $(a,\eta) \in B(a_0,r) \times S$

$$\begin{aligned}
\left|k\left(f_k * e^{\frac{k\eta}{\varphi(0)}}\right)\right| &= |\eta|\,k\left|\int_0^1 f_k(a,t\eta)e^{\frac{(1-t)k}{\varphi(0)}\eta}\mathrm{d}t\right| \\
&\leq |\eta|\,k\int_0^1 |f_k(a,t\eta)|\,e^{\frac{(1-t)k}{\varphi(0)}\,\mathrm{Re}(\eta)}\mathrm{d}t \\
&\leq |\eta|\,k\sup_{a,z}|f_k(a,z)|\int_0^1 e^{\frac{(1-t)k}{\varphi(0)}|\eta|\cos(\frac{\delta}{2})}\mathrm{d}t \\
&= \frac{|\varphi(0)|}{\cos(\frac{\delta}{2})}\sup_{a,z}|f_k(a,z)|\left(1 - e^{\frac{k|\eta|\cos(\frac{\delta}{2})}{\varphi(0)}}\right) \leq \frac{|\varphi(0)|}{\cos(\frac{\delta}{2})}\sup_{a,z}|f_k(a,z)|
\end{aligned}$$

where the sup is taken for $z \in S, |z| \leq |\eta|$ and $a \in B(a_0,r)$.

Because $(1+\mu^2|\eta|^2)e^{-\mu|\eta|}$ is decreasing with respect to $|\eta|$, we get

$$\begin{aligned}\left|k\left(f_k * e^{\frac{k\eta}{\varphi(0)}}\right)\right|(1+\mu^2|\eta|^2)e^{-\mu|\eta|} &\leq \left(\frac{|\varphi(0)|}{\cos(\frac{\delta}{2})}\sup_{a,z}|f_k(a,z)|\right)(1+\mu^2|\eta|^2)e^{-\mu|\eta|}\\ &\leq \frac{|\varphi(0)|}{\left|\cos(\frac{\delta}{2})\right|}\sup_{a,z}|f_k(a,z)|(1+\mu^2|z|^2)e^{-\mu|z|}\\ &\leq \frac{|\varphi(0)|}{\left|\cos(\frac{\delta}{2})\right|}\|f_k\|_{\mu,S},\end{aligned}$$

which gives the requested improved bound on $\|k(f_k * e^{k\eta/\varphi(0)})\|_{\mu,S}$ after taking the sup. We conclude that

$$\left\|\sum_{k=1}^{\infty} k\left(f_k * e^{\frac{k\eta}{\varphi(0)}}\right)x^k\right\|_{\mu,S} \leq \frac{|\varphi(0)|}{\left|\cos(\frac{\delta}{2})\right|}\sum_{k=1}^{\infty}\|f_k\|_{\mu,S}\,\mu^{-k} = \frac{|\varphi(0)|}{\left|\cos(\frac{\delta}{2})\right|}\|F\|_{\mu,S}$$

such that $\tilde{F}$ indeed lies in $\mathcal{G}^\mu\{x\}$ with the requested bound on its norm. □

Lemma 3.12. *The linear operator T introduced in (12) is well-defined as an operator from $\mathcal{G}^\mu\{x\}$ into itself, and is given by*

$$T\colon F = \sum_{k=0}^{\infty} f_k(a,\eta)x^k \mapsto \frac{-1}{\varphi(0)}F - \frac{1}{\varphi(0)^2}\sum_{k=1}^{\infty} k(f_k * e^{\frac{k\eta}{\varphi(0)}})x^k,$$

Moreover $\|T\| \leq \frac{1}{|\varphi(0)|}\left(1+\frac{1}{\cos(\frac{\delta}{2})}\right)$.

Proof. The right-hand side is well-defined because of lemma 3.11, and the bound on the operator norm essentially is an easy consequence of the same lemma. Remains to show that $T(F)$ satisfies the convolution equation (12). To do so, it suffices to check this for each f_kx^k, due to the linearity. Let

$$\alpha_k = T(f_kx^k) = \left(\frac{-1}{\varphi(0)}f_k - \frac{1}{\varphi(0)^2}(f_k * ke^{\frac{k\eta}{\varphi(0)}})\right)x^k,$$

Now it is elementary to check that $x(1 * \frac{d\alpha_k}{dx}) = -\frac{1}{\varphi(0)}(f_k * ke^{\frac{k\eta}{\varphi(0)}})x^k$, merely using the associativity and commutativity of $*$ and Proposition 3.10. We directly conclude that $x(1 * \frac{d\alpha_k}{dx}) - \varphi(0)\alpha_k = f_kx^k$. □

Let us now focus our attention again on (11) and define the map

$$\mathcal{V} : \mathcal{G}^\mu\{x\} \to \mathcal{G}^\mu\{x\}\colon F \mapsto T\left((\varphi(x)-\varphi(0))\,F + \sum_{k=0}^{\infty}\mathcal{B}_1(\varepsilon H_k) * F^{*k}\right). \quad (13)$$

It is not yet clear whether or not the expression in the right-hand is a well-defined element of $\mathcal{G}^\mu\{x\}$. This will be a consequence of the next two lemmas.

Lemma 3.13. *The space $\mathcal{G}^{\mu}\{x\}$ is closed under multiplication with analytic functions (in x) with radius of convergence strictly larger than $1/\mu$ i.e. let $F \in \mathcal{G}^{\mu}\{x\}$ and $g(x) \in \mathcal{O}(B(0,r))$ with $r > 1/\mu$ then*

$$g.F \in \mathcal{G}^{\mu}\{x\}, \qquad \text{and } \|g.F\|_{\mu,S} \leq \left(\sum_{k=0}^{\infty} |g_k| \mu^{-k}\right) \|F\|_{\mu,S}.$$

Note that $\varphi(x) - \varphi(0) = O(x)$ so the norm of the linear operator $F \mapsto T((\varphi(x) - \varphi(0))F)$ is of the order $O(\mu^{-1})$, which we may choose as small as required upon increasing μ.

Proposition 3.14. *Let $\gamma(x,a,\varepsilon) = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \gamma_{ni}(a) x^i \varepsilon^n$ be bounded analytic on $B(0,s) \times B(a_0,r) \times B(0,s)$. If $\mu \geq 2/s$, then γ has a Borel transform inside $\mathcal{G}^{\mu}\{x\}$, with*

$$\|\mathcal{B}_1(\varepsilon\gamma)\|_{\mu,S} \leq 32 \sup_{(x,a,\varepsilon)} |\gamma(x,a,\varepsilon)|,$$

where the sup is taken over all $(x,a,\varepsilon) \in B(0,s) \times B(a_0,r) \times B(0,s)$.

Proof. Denote $\|\gamma\| := \sup_{(x,a,\varepsilon)} |\gamma(x,a,\varepsilon)|$. The Borel transform is formally given by

$$\mathcal{B}_1(\varepsilon\gamma) = \sum_{i=0}^{\infty}\left(\sum_{n=0}^{\infty} \frac{\gamma_{ni}(a)}{n!}\eta^n\right) x^i.$$

Using Cauchy's inequalities, we obtain $|\gamma_{ni}(a)| \leq \|\gamma\| s^{-i-n}$. We have

$$\left|\sum_{n=0}^{\infty} \frac{\gamma_{ni}(a)}{n!}\eta^n\right| \leq \frac{\|\gamma\|}{s^i}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{|\eta|}{s}\right)^n = \frac{\|\gamma\|}{s^i} e^{|\eta|/s}$$

which means that $\sum_{n=0}^{\infty} \frac{\gamma_{ni}(a)}{n!}\eta^n$ is an entire analytic function w.r.t. η. Furthermore

$$\left\|\sum_{n=0}^{\infty} \frac{\gamma_{ni}(a)}{n!}\eta^n\right\|_{\mu,S} \leq \sup_{\eta \in S} \frac{\|\gamma\|}{s^i}\left(1 + \mu^2 |\eta|^2\right) e^{|\eta|\left(\frac{1}{s} - \mu\right)} = \frac{\|\gamma\| K(\mu,s)}{s^i}$$

for some positive constant $K(\mu,s)$. In fact, under condition that $\mu \geq \frac{2}{s}$, an easy calculation shows that $K(\mu,s) < 16$. It already implies that all coefficient functions appearing in $\mathcal{B}_1(\varepsilon\gamma)$ are in $\mathcal{G}^{\mu}$, for $\mu > \frac{2}{s}$. Finally,

$$\begin{aligned}\|\mathcal{B}_1(\varepsilon\gamma)\|_{\mu,S} &= \sum_{i=0}^{\infty}\left\|\sum_{n=0}^{\infty}\frac{\gamma_{ni}(a)}{n!}\eta^n\right\|_{\mu,S}\mu^{-i} \\ &\leq 16\|\gamma\|\sum_{i=0}^{\infty}\left(\frac{1}{s\mu}\right)^i = \frac{16}{1 - \frac{1}{s\mu}}\|\gamma\| \leq 32\|\gamma\|,\end{aligned}$$

provided $\mu \geq \frac{2}{s}$. □

Corollary 3.15. *Let H_k be as in (8). There exists a C_0, so that given any $C \geq C_0$, we can choose μ large enough in order that the map*

$$F \mapsto \sum_{k=0}^{\infty} \mathcal{B}_1(\varepsilon H_k) * F^{*k}.$$

is well-defined $B(0,C) \subset \mathcal{G}^{\mu}\{x\} \to B(0,C_0)$. It is then Lipschitz with Lipschitz constant that is $O(\frac{1}{\mu})$ for $\mu \to \infty$.

Proof. Recall that $H(x,y,a,\varepsilon) = \sum_{k=0}^{\infty} H_k(x,a,\varepsilon)y^k$ is bounded analytic near $(0,0,a_0,0)$, say on $B(0,s) \times B(0,R) \times B(a_0,r) \times B(0,s)$ for some $r,s,R > 0$. Denote with $\|H\|$ the sup of $|H(x,y,a,\varepsilon)|$ on this domain. Then Cauchy's estimate shows that

$$\sup_{(x,a,\varepsilon)} |H_k(x,a,\varepsilon)| \leq \|H\| R^{-k}.$$

It follows from lemma 3.14 that $\mathcal{B}_1(\varepsilon H_k) * F^{*k} \in \mathcal{G}^{\mu}\{x\}$ provided F is, and that the norm of $\mathcal{B}_1(\varepsilon H_k)$ is bounded from above by $32\|H\|R^{-k}$, all this as long as $\mu \geq \frac{2}{s}$. To continue, we introduce the shortcut $M_0 := \|\mathcal{B}_1(\varepsilon H_0)\|_{\mu,S}$ and find

$$\begin{aligned}
\left\| \sum_{k=0}^{\infty} \left(\mathcal{B}_1(\varepsilon H_k) * (F)^{*_n k} \right) \right\|_{\mu,S} &\leq M_0 + \sum_{k=1}^{\infty} \left(\frac{4\pi}{\mu} \right)^k \|\mathcal{B}_1(\varepsilon H_k)\|_{\mu,S} \|F\|_{\mu,S}^k \\
&\leq M_0 + 32\,\|H\| \sum_{k=1}^{\infty} \left(\frac{4\pi}{R\mu} \right)^k \|F\|_{\mu,S}^k \\
&= M_0 + 32\,\|H\| \frac{4\pi \|F\|_{\mu,\eta}}{R\mu - 4\pi \|F\|_{\mu,S}}.
\end{aligned}$$

By restricting F to $B(0,C) \subset \mathcal{G}^{\mu}\{x\}$, we find for μ large enough that the image of the map is well-defined in $\mathcal{G}^{\mu}\{x\}$, with norm bounded by $C_0 := 2M_0$. (Note that C_0 does not grow as μ grows.) Finally, we prove the Lipschitz property by estimating the norm difference between the image of two elements F_1 and F_2 in $B(0,C)$, which amounts to bounding

$$32\,\|H\| \sum_{k=1}^{\infty} \frac{4\pi}{\mu} \frac{1}{R^k} \left\| F_1^{*k} - F_2^{*k} \right\|_{\mu,S}$$

By the binomial theorem and using that

$$\|F_1 - F_2\|_{\mu,S} \leq 2C, \quad \|F_2\|_{\mu,S} \leq C \leq 2C$$

we get

$$\begin{aligned}
\left\|F_1^{*k}-F_2^{*k}\right\|_{\mu,S} &= \left\|(F_2+(F_1-F_2))^{*k}-F_2^{*k}\right\|_{\mu,S} \\
&= \left\|\sum_{i=1}^{k}\binom{k}{i}F_2^{*k-i}*(F_1-F_2)^{*i}\right\|_{\mu,S} \\
&\le \left(\frac{4\pi}{\mu}\right)^{k-1}\sum_{i=1}^{k}\binom{k}{i}(2C)^{k-1}\left\|F_1-F_2\right\|_{\mu,S} \\
&\le 2\left(\frac{16\pi C}{\mu}\right)^{k-1}\left\|F_1-F_2\right\|_{\mu,S}
\end{aligned}$$

since $k \ge 1$ and μ is large. We conclude that the Lipschitz constant is bounded by

$$32\left\|H\right\|\sum_{k=1}^{\infty}\left(\frac{4\pi}{\mu}\frac{1}{R^k}\times 2\left(\frac{16\pi C}{\mu}\right)^{k-1}\right)=\frac{256\pi}{\mu R}\left\|H\right\|\sum_{k=1}^{\infty}\left(\frac{16\pi C}{R\mu}\right)^{k-1}.$$

The involved summation is bounded from above by 2 if μ is sufficiently large, implying that the Lipschitz constant is bounded by $\frac{512\pi}{\mu R}$. □

Theorem 3.16. *For sufficiently large μ, the map $\mathcal{V}$ from (13) is well-defined as a map from a closed ball around 0 in $\mathcal{G}^{\mu}\{x\}$ into itself, that is a contraction with Lipschitz constant less than 1. There exists a unique $F \in \mathcal{G}^{\mu}\{x\}$ satisfying*

$$x\left(1*\frac{\partial F}{\partial x}\right)=\varphi(x)F+\sum_{k=0}^{\infty}\left(\mathcal{B}_1(\varepsilon H_k)*F^{*k}\right).$$

Proof. Recall that

$$\mathcal{V}(F)=T\left(\left(\varphi(x)-\varphi(0)\right)F+\sum_{k=0}^{\infty}\left(\mathcal{B}_1(\varepsilon H_k)*F^{*k}\right)\right).$$

Choose C_0 as in the above lemma, and let $C = 2C_0\max\{1,\|T\|\}$. Then for $\|F\| \le C$:

$$\|\mathcal{V}(F)\| \le \|T\|\cdot(O(1/\mu).C+C_0)\le\left(\frac{1}{2}+O(1/\mu)\right)C\le C,$$

whenever μ is large enough. Here we have used the bounds obtained in the above lemmas. At the same time, the contraction property follows from the boundedness of the Lipschitz constant of T together with the fact that the Lipschitz constants of $F\mapsto(\varphi(x)-\varphi(0))F$ and of the map in the above lemma are both $O(\mu^{-1})$.

The second statement follows directly from the Banach fixed point theorem. □

The solution $F \in \mathcal{G}^{\mu}\{x\}$ defines an analytic function $F(x, a, \eta)$ (of at most exponential growth as $|\eta| \to \infty$) on $B(0, \mu^{-1}) \times B(a_0, r) \times S$. We can now easily extend its domain to $B(0, \mu^{-1}) \times B(a_0, r) \times (S \cup B(0, R))$ by using lemma 3.5. Note that a priori it is not directly clear that the solution given in Theorem 3.16 should coincide with the Borel transform $\hat{Y}$ of the formal expansion given in lemma 3.5. We can argue as follows to see that they do coincide.

By reformulating theorem 3.16 to apply on a sector $\tilde{S}$ of finite radius that completely fits inside $B(0, R)$, the unicity given by the fixed-point theorem shows that the restriction of the already known solution F coincides with the solution on the restricted sector. On the other hand, since the fixed-point equation is the Borel-transformed version of the formal equation (6), the Borel transform $\hat{Y}$ of the expansion from lemma 3.5 should also coincide with the given fixed-point solution. Hence our claim is verified.

We have thus shown that the Borel transform of the formal solution found in lemma 3.5 can be continued analytically to an infinite sector in the real direction, with the continuation being of exponential growth at most of order 1. Combining this with theorem 2.10 proves theorem 3.2.

4 Tracing summability along the critical curve

4.1 Statement of the result

In the previous section, summability properties of slow curves near a hyperbolic saddle have been derived. At the same time, we can refer to [4] to conclude that equations of the form

$$\varepsilon x^{p+1} \frac{dy}{dx} = \lambda(x) y + O(y^2, \varepsilon),$$

with $p \geq 1$, enjoy similar results: when $\lambda(0) < 0$, then it is easily seen from the monomial summability (w.r.t. the monomial εx^p) proved in [4] that for a sufficiently small neighbourhood of a compact interval, lying close to 0, on the strictly positive real axis, the equation has a solution that is 1-summable w.r.t. ε in directions close to the real axis.

The question that is answered in this section is whether or not the 1-summability of a slow curve at a given location $x = x_0$ implies the 1-summability of a slow curve at another location. In other words, is the summability information carried along the slow curve? The answer is given by the next theorem. Like in the previous section, an additional parameter a is added for the sake of generality but this parameter does not have any influence on the proof or the result.

Theorem 4.1. *Consider the real analytic slow-fast family of vector fields*

$$\begin{cases} \dot{x} &= \varepsilon f(x, y, a, \varepsilon) \\ \dot{y} &= g(x, y, a, \varepsilon), \end{cases} \tag{14}$$

with a critical curve given by the graph $y = \psi_0(x)$ (for $a = a_0$), $x \in [x_0, x_1] \subset \mathbb{R}$. Suppose that the unperturbed vector field is normally hyperbolically attracting at points of the critical curve, which means

$$\frac{\partial g}{\partial y}(x, \psi_0(x), a_0, 0) < 0, \qquad \forall x \in [x_0, x_1].$$

Suppose furthermore that $f(x, \psi_0(x), a_0, 0) > 0$ for all $x \in [x_0, x_1]$ (in other words the slow dynamics is regular along the critical curve and directed from left to right).

Then 1-summability w.r.t. ε of the formal slow curve at $x \in \mathbb{C}$ near x_0 and for ε in a complex sector containing the positive real axis implies the 1-summability w.r.t. ε of the formal slow curve at $x \in \mathbb{C}$ near $[x_0, x_1]$ and for ε in a (possibly smaller) complex sector containing the positive real axis, all the time keeping a sufficiently close to a_0.

Remark 4.2. Readers who are familiar with the terminology of complex relief functions (see [10] for example) can see that the normally attracting nature of the critical curve and the fact that the theorem is stated on a compact *real* interval means that the straight path from x_0 to x_1 is a descending path according to the complex relief function associated with the slow-fast vector field. It is hence well-known that points close to x_0 and for $\varepsilon > 0$ can be easily integrated towards x_1 without straying from the critical curve. Up to the knowledge of the author, the literature does not contain a statement that carries summability information along a descending path.

It is not hard (in fact this is the topic of the next subsection) to translate the question in Theorem 4.1 to a question regarding analytic differential equations of the form

$$\varepsilon \frac{dy}{dx} = y + \varepsilon H(x, y, a, \varepsilon), \tag{15}$$

defined for (x, y, a, ε) in a complex neighbourhood of $[x_0, x_1] \times \{0\} \times \{a_0\} \times \{0\}$. Using this reduction, theorem 4.1 is a direct consequence of the next theorem. We will elaborate a bit on this in a minute.

Theorem 4.3. *Given the analytic equation (15) defined for (x, y, a, ε) in a complex neighbourhood of $[X_1, 0] \times \{0\} \times \{a_0\} \times \{0\}$, and with $X_1 < 0$.*

Then 1-summability w.r.t. ε of the formal solution at $x \in \mathbb{C}$ near 0 and for ε in a complex sector containing the positive real axis implies the 1-summability w.r.t. ε of the formal slow curve at $x \in \mathbb{C}$ in a neighbourhood, independent of the parameter a, of $[X_1, 0]$ and for ε in a (possibly smaller) complex sector containing the positive real axis, all the time keeping a sufficiently close to a_0.

Remark 4.4. In general, equations of the form (15) will not have a 1-summable solution (not even in isolated directions).
Consider, for example, an entire function h whose set of zeroes is given by

$$\bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k-1} \left\{ k e^{i \frac{2j}{k} \pi} \right\}.$$

Such a function exists by the Weierstrass theorem, see [11].
We claim that the equation

$$\varepsilon\frac{\mathrm{d}y}{\mathrm{d}x} = y + \frac{\varepsilon}{h(x)}$$

has no 1-summable solution in any direction. Indeed, assuming that such a solution does exist, would imply that the Borel transformed equation

$$1 * \frac{\partial Y}{\partial x} = Y + \frac{1}{h(x)}$$

has a solution, $Y(x, \eta)$, which is defined for η in some infinite sector. One can see easily that the unique solution of the above equation is given by $-\frac{1}{h(x+\eta)}$. This function is clearly, by construction of h, not defined on any infinite sector.

4.2 Theorem 4.3 implies Theorem 4.1

Under the conditions of Theorem 4.1, we can make a time rescaling to reduce (14) to

$$\begin{cases} \dot{x} &= \varepsilon \\ \dot{y} &= G(x, y, a, \varepsilon), \end{cases} \qquad \text{with } G(x, y, a, \varepsilon) := \tfrac{g(x,y,a,\varepsilon)}{f(x,y,a,\varepsilon)}.$$

From the conditions imposed on $\frac{\partial g}{\partial y}$ easily follows $\lambda_0(x) := \frac{\partial G}{\partial y}(x, \psi_0(x), a_0, 0) < 0$ for all $x \in [x_0, x_1]$. Let us now extend the critical curve defined for $a = a_0$ to critical curves for nearby values of a, using the implicit function theorem: there exists a unique analytic $\psi(x, a)$ such that $G(x, \psi(x, a), a, 0) = 0$ and $\psi(x, a_0) = \psi_0(x)$. After writing $y = \tilde{y} + \psi(x, a)$, we find

$$\begin{cases} \dot{x} &= \varepsilon \\ \dot{\tilde{y}} &= \lambda(x, a)\tilde{y} + O(\tilde{y}^2) + O(\varepsilon), \end{cases}$$

where $\lambda(x, a) = \frac{\partial G}{\partial y}(x, \psi(x, a), a, 0)$. Note that $\lambda(x, a) = \lambda_0(x) + O(|a - a_0|)$, meaning that we may assume that $\lambda(x, a)$ has a strictly negative real part. Now define

$$u(x, a) = \int_{x_0}^{x} \lambda(s, a)\mathrm{d}s,$$

where we limit this function to a sufficiently small (and simply connected) neighbourhood of $[x_0, x_1]$ and a near a_0. Writing $\tilde{x} = u(x, a)$, we obtain after yet another time rescaling and reversal

$$\begin{cases} \dot{\tilde{x}} &= \varepsilon \\ \dot{\tilde{y}} &= \tilde{y} + O(\tilde{y}^2) + O(\varepsilon), \end{cases}$$

Denote

$$X_1 := u(x_1, a_0) = \int_{x_0}^{x_1} \lambda(s, a_0)\mathrm{d}s < 0.$$

One can see that the mapping $(x, a) \mapsto (u(x, a), a)$ is analytic with an analytic inverse on an environment of $[x_0, x_1] \times \{a_0\}$ mapping this last set onto $[X_1, 0] \times \{a_0\}$. Since the result in theorem 4.3 is obtained on an environment of $[X_1, 0] \times \{a_0\}$, going back to the original variables will yield a result on an environment of $[x_0, x_1] \times \{a_0\}$, which is indeed the goal in theorem 4.1.

Dropping the tildes, invariant manifolds of the above system of differential equations are solution curves of

$$\varepsilon \frac{dy}{dx} = y + y^2 C(x, y, a, \varepsilon) + \varepsilon D(x, y, a, \varepsilon),$$

for some analytic functions C and D. We can now further reduce to a more elementary form with $C = 0$ by applying a singular transformation $y = \varepsilon Y$:

$$\varepsilon \frac{dY}{dx} = Y + \varepsilon Y^2 C(x, y, a, \varepsilon) + D(x, \varepsilon Y, a, \varepsilon) = Y + D(x, 0, a, 0) + O(\varepsilon).$$

The equation in Theorem 4.3 is obtained after a final translation in the Y direction: $Y \mapsto Y + D(x, 0, a, 0)$.

4.3 Proof of Theorem 4.3

We may make the following assumptions about equation (15), which we repeat here for the sake of convenience:

$$\varepsilon \frac{\mathrm{d}y}{\mathrm{d}x} = y + \varepsilon H(x, y, a, \varepsilon). \tag{16}$$

(H_1) H is bounded and analytic on $U \times B(0, r) \times B(a_0, r) \times B(0, r)$ for some $r > 0$ and some open complex neighbourhood U of $[X_1, 0]$.

(H_2) Equation (16) has an (a, ε)-family of bounded analytic solutions $G(x, a, \varepsilon)$ defined for (x, a, ε) in $B(0, s) \times B(a_0, s) \times S(0, \pi + \sigma, s)$ for some $s > 0$ and some $\sigma > 0$. Recalling Definition 2.7, the assumption made here is a consequence of the assumption formulated in Theorem 4.3 regarding the 1-summability w.r.t. ε of a solution of the ode near 0.

The proof of Theorem 4.3 essentially contains two steps. In a first step, we analytically continue the initial solution $G(x, a, \varepsilon)$ defined near 0 towards X_1 (actually a bit further) by using the ode. This will provide a solution near $[X_1, 0]$ and for ε in some sector of opening angle a bit larger than π. In the second and final step, we construct an other solution of the ode near $[X_1, 0]$ but on a complementary complex sector for ε and describe the relation with the analytically continued solution from step 1. We finally apply the well-known Ramis-Sibuya theorem to conclude the 1-summability of the analytically continued solution. This method has been used before in the literature, for example in [3].

Note that lemma 3.5 also applies to this equation thus $G(x, a, \varepsilon) = \mathcal{O}(\varepsilon)$ and we may assume, by choosing s sufficiently small

(H_3) $|G(x, a, \varepsilon)| < \frac{r}{2}$, for all $(x, a, \varepsilon) \in B(0, s) \times B(a_0, s) \times S(0, \pi + \sigma, s)$.

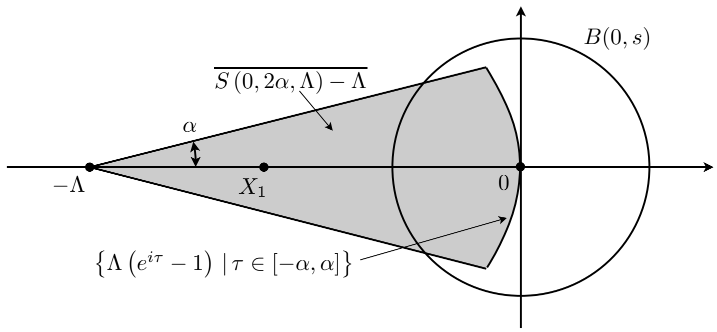

Figure 1: $\overline{S(0,2\alpha,\Lambda)-\Lambda}$

4.3.1 Analytic continuation of the initial solution

We continue with the notations introduced in hypotheses (H_1) and (H_2) above and specify the set on which we want to find a solution to (16).
Choose some $-\Lambda < X_1$ (thus $-\Lambda \in \mathbb{R}$) such that $[-\Lambda, 0] \subset U$. There then exists a small enough half-opening angle $\alpha < \frac{\pi}{2}$ such that

$$\overline{S(0,2\alpha,\Lambda)-\Lambda} \subset U.$$

(see figure 1.) We furthermore assume that

$$\left\{\Lambda\left(e^{i\tau}-1\right) \mid \tau \in [-\alpha,\alpha]\right\} \subset B(0,s).$$

(In other words, the terminating arc of the sector $S(0,2\alpha,\Lambda)-\Lambda$ with vertex $-\Lambda$ lies inside $B(0,s)$, again see figure 1.) Our aim is to analytically continue the initial solution provided in (H_2) on $B(0,s)$ to the domain $S(0,2\alpha,\Lambda)-\Lambda$.

Proposition 4.5. *Let $a, z \in \mathbb{C}$. If $|a| < |z|$ then*

$$|\mathrm{Arg}(z+a) - \mathrm{Arg}(z)| \leq \sin^{-1}\frac{|a|}{|z|}.$$

Lemma 4.6. *Let (H_1), (H_2) and (H_3) be satisfied. Let $0 < \sigma' < \max\{\sigma, \alpha\}$ be fixed. The initial solution $y = G(x, a, \varepsilon)$ of (16) can be analytically continued to a solution defined on*

$$\overline{S(0,2\alpha,\Lambda)-\Lambda} \times B(a_0, s) \times S(0, \pi+\sigma', s'),$$

for sufficiently small $s' > 0$. Moreover this continued solution is bounded by $r/2$.

Proof. Define

$$M = \sup_{x,y,a,\varepsilon} |H(x,y,a,\varepsilon)|,$$

where the supremum is taken for $x \in \overline{S(0, 2\alpha, \Lambda) - \Lambda}$, $|y| < r$, $|a - a_0| < r$, $\varepsilon \in S(0, \pi + \sigma, s)$. We will define conditions on s' so that for any given $\varepsilon \in S(0, \pi + \sigma', s')$, any given $a \in B(a_0, s)$ and any given $z \in \overline{S(0, 2\alpha, \Lambda) - \Lambda}$ it is possible to integrate (16) along a well-chosen path towards z. Independence of path and analytic dependence on parameters and initial conditions ensures that this method yields an analytic solution on the required domain.

In the remainder of the proof we hence fix z, a and ε. The integration path is the linear path from z_0 to z, where $z_0 := \Lambda(e^{i\beta} - 1)$ (β still to be specified, $|\beta| \leq \alpha$, which is located on the terminating arc of the sector $\overline{S(0, 2\alpha, \Lambda) - \Lambda}$) and which lies inside the definition domain of the initial solution defined in (H_2). The ode, restricted to the path from z_0 to z, parameterized by $p(t) = (1-t)z_0 + tz$ is given by:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{z - z_0}{\varepsilon} \left(\gamma + \varepsilon H(p(t), \gamma, a, \varepsilon)\right) \tag{17a}$$

$$\gamma(0) = G(z_0, a, \varepsilon). \tag{17b}$$

It suffices to show that this equation has a maximal solution defined on an interval $]t_2, t_1[$ with $t_1 > 1$. Suppose by contradiction that $t_1 \leq 1$. Clearly the right hand side of (17a) is defined (for all parameters (a, ε)), for (x, γ) on the compact set $[0, 1] \times \overline{B}(0, r/2)$. If we prove that $|\gamma(t)| \leq r/2$ we thus get a contradiction. We prove it by showing that if there exists an $t_* \in]0, 1[$ with $|\gamma(t_*)| = r/2$ we must have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(t \mapsto |\gamma(t)|^2\right)(t_*) < 0$$

which implies what we are aiming for. After some calculations one finds that this derivative is given by

$$2\,\mathrm{Re}\left(\frac{z - z_0}{\varepsilon}\overline{\gamma(t_*)}\left(\gamma(t_*) + \varepsilon H(p(t_*), \gamma(t_*), a, \varepsilon)\right)\right).$$

Consequently it is sufficient to show that

$$\left|\arg\left(\frac{z - z_0}{\varepsilon}\overline{\gamma(t_*)}\left(\gamma(t_*) + \varepsilon H(\sigma(t_*), \gamma(t_*), a, \varepsilon)\right)\right) - \pi\right| < \frac{\pi}{2}.$$

Now define $\rho = \frac{1}{2}(\alpha - \sigma') > 0$ and choose $s' < s$ such that $s' < (r/2M)\sin\rho$ which implies that the next inequality is satisfied (remember that $\varepsilon \in S(0, \pi + \sigma', s')$):

$$|\varepsilon H(p(t_*), \gamma(t_*), a, \varepsilon)| \leq s'M \leq \frac{r}{2}\sin\rho.$$

By proposition 4.5 we then have, since

$$\arg\left(\overline{\gamma(t_*)}\right) = -\arg\left(\gamma(t_*)\right) \text{ and } |\gamma(t_*)| = \frac{r}{2},$$

that

$$\left|\arg\left(\frac{z-z_0}{\varepsilon}\overline{\gamma(t_*)}\left(\gamma(t_*)+\varepsilon H(p(t_*),\gamma(t_*),a,\varepsilon)\right)\right)-\pi\right|$$
$$<\left|\arg\frac{z-z_0}{\varepsilon}-\pi\right|+\rho=\left|\arg\frac{z_0-z}{\varepsilon}\right|+\rho.$$

Given that z lies in a sector with opening angle α and that z_0 can be chosen freely on the ending arc, it is easy to see that the argument of z_0-z can be freely chosen between $-\alpha$ and α.
When the argument of ε is nonnegative we choose

$$\frac{\alpha}{2}<\arg(z_0-z)<\frac{1}{2}(\pi+\sigma'-\alpha),$$

while for $\arg(\varepsilon)<0$ we take

$$-\frac{1}{2}(\pi+\sigma'-\alpha)<\arg(z_0-z)<-\frac{\alpha}{2}.$$

One can see that such a choice can be made by the assumptions $\alpha<\frac{\pi}{2}$, $0<\sigma'$ and that they guarantee that $\left|\arg\frac{z_0-z}{\varepsilon}\right|<\frac{1}{2}(\pi+\sigma'-\alpha)$. It follows that we get

$$\left|\arg\left(\frac{z-z_0}{\varepsilon}\overline{\gamma(t_*)}\left(\gamma(t_*)+\varepsilon H(p(t_*),\gamma(t_*),a,\varepsilon)\right)\right)-\pi\right|$$
$$<\frac{1}{2}(\pi+\sigma'-\alpha)+\rho=\frac{\pi}{2}$$

given the definition of ρ in this proof. □

4.3.2 Gevrey asymptotics of the extension

We are now quite close to showing Theorem 4.3. It remains to show that the analytic continuation provided in Lemma 4.6 is 1-summable w.r.t. ε uniformly for x near $[X_1,0]$.

Let $y=G(x,a,\varepsilon)$ be the continuation provided by Lemma 4.6. We will define a second solution $y=G'(x,a,\varepsilon)$ defined for x near $[-\Lambda,0]$, but for ε on a different sector. We will then consider the difference $G-G'$ for ε in overlapping sectors and show that it is exponentially small w.r.t. $|\varepsilon|$ as $\varepsilon\to 0$. Following the Ramis-Sibuya characterization of Gevrey-1 functions ([12]) we then conclude that both G and G' are Gevrey-1 asymptotic to the same formal power series $\hat{G}(x,a,\varepsilon)$, uniformly for (x,a) given near $(-\Lambda,a_0)$. Furthermore, since the ε-sector of G has opening angle larger than π, G will be 1-summable w.r.t. ε in the bisecting direction.

Lemma 4.7. *Assume (H_1), (H_2) and (H_3) are satisfied. Let $0<\tau<\frac{\pi}{2}$ be fixed. The solution of*

$$\varepsilon\frac{\mathrm{d}y}{\mathrm{d}x}=y+\varepsilon H(x,y,a,\varepsilon)$$
$$y(-\Lambda,a,\varepsilon)=0$$

is defined and analytic on $V \times B(a_0, s) \times S(\pi, \pi - \tau, s'')$ *for some* $s'' > 0$ *and* V *a neighbourhood of* $[-\Lambda, 0]$. *We may assume that the solution is bounded by* $r/2$.

Proof. The proof is completely analogous to the proof of Lemma 4.6. Note that when comparing the situation described in Lemma 4.6 with the one here, it is relevant to see that the real part of ε is negative here, and hence exponential attraction is experienced while continuing the solution at $x = -\Lambda$ to values of x in $[-\Lambda, 0]$ which in essence lie to the right of $-\Lambda$ in the complex plane. □

The following lemma finishes the proof of theorem 4.3.

Lemma 4.8. *Using the notations and assumptions from lemma 4.6, lemma 4.7, together with the extra assumption* $\frac{\sigma'}{2} < \tau < \sigma'$, *we have the following.*

Denote $\nu = \min\{s', s''\}$. *The solution from lemma 4.6, limited to*

$$\widetilde{V} \times B(a_0, s) \times S(0, \pi + \sigma', \nu),$$

with $\widetilde{V}$ *a neighbourhood of* $[X_1, 0]$, *is Gevrey-1 asymptotic, in* ε, *to a formal series, uniformly for* (x, a) *and thus it is 1-summable.*

Remark 4.9. It is possible to prove the above result on (almost) the entire domain of the x variable which was found in lemma 4.6, this is however not necessary for our goal and would make the proof more convoluted.

Proof of lemma 4.8. Denote $G(x, a, \varepsilon)$ the solution found in lemma 4.6 and $G'(x, a, \varepsilon)$ the solution from the above lemma. If we put

$$\Delta(x, a, \varepsilon) = G(x, a, \varepsilon) - G'(x, a, \varepsilon),$$

it satisfies the following equation

$$\varepsilon \frac{\mathrm{d}\Delta}{\mathrm{d}x} = \Delta + \varepsilon \left(H(x, G(x, a, \varepsilon), a, \varepsilon) - H(x, G'(x, a, \varepsilon), a, \varepsilon)\right)$$
$$\Delta(-\Lambda, a, \varepsilon) = G(-\Lambda, a, \varepsilon).$$

Since

$$\begin{aligned} &H(x, G(x, a, \varepsilon), a, \varepsilon) - H(x, G'(x, a, \varepsilon), a, \varepsilon) \\ &= \underbrace{\int_0^1 \frac{\partial H}{\partial y}\left(x, (1-s)G'(x, a, \varepsilon) + sG(x, a, \varepsilon), a, \varepsilon\right) \mathrm{d}s}_{\mathcal{R}(x,a,\varepsilon)} \Delta(x, a, \varepsilon) \end{aligned}$$

it must hold that

$$\Delta(x, a, \varepsilon) = G(-\Lambda, a, \varepsilon) e^{\int_{-\Lambda}^{x} \mathcal{R}(w,a,\varepsilon)\mathrm{d}w} e^{\frac{x+\Lambda}{\varepsilon}}.$$

Denote

$$\tilde{M} = \sup_{x,y,a,\varepsilon} |H(x, y, a, \varepsilon)|,$$

where the supremum is taken for $x \in \overline{S(0,2\alpha,\Lambda)-\Lambda}$, $|y|<r$, $|a-a_0|<r$, $\varepsilon \in S(0,\pi+\sigma',\nu)$

By Cauchy's inequalities it holds that (remember that both G and G' are bounded by $r/2$)

$$|\mathcal{R}(x,a,\varepsilon)| \leq \frac{3}{r}\tilde{M}$$

for all (x,a,ε) in its domain. It follows that

$$\begin{aligned}|\Delta(x,a,\varepsilon)| &\leq |G(-\Lambda,a,\varepsilon)|\, e^{\left|\int_{-\Lambda}^{x}\mathcal{R}(w,a,\varepsilon)\mathrm{d}w\right|} e^{\mathrm{Re}\left(\frac{x+\Lambda}{\varepsilon}\right)} \\ &\leq \frac{r}{2} e^{\frac{3\tilde{M}|x+\Lambda|}{r}} e^{\left|\frac{x+\Lambda}{\varepsilon}\right|\cos\left(\arg\left(\frac{x+\Lambda}{\varepsilon}\right)\right)} \\ &\leq \frac{r}{2} e^{\frac{3\tilde{M}\Lambda}{r}} e^{\left|\frac{x+\Lambda}{\varepsilon}\right|\cos\left(\arg\left(\frac{x+\Lambda}{\varepsilon}\right)\right)}.\end{aligned}$$

To make further estimates we will restrict ourselves to the following domain for the x variable. Define

$$\widetilde{V} := \left(V \cap \overline{S(0,\sigma'-\tau,\Lambda)-\Lambda}\right) \backslash B\left(-\Lambda, \frac{X_1+\Lambda}{2}\right).$$

Notice that for $x \in \widetilde{V}$ we have $|x+\Lambda| \geq \frac{X_1+\Lambda}{2}$. It is furthermore cumbersome but easy to check that for $x \in \tilde{V}$ and $\varepsilon \in S(0,\pi+\sigma',\nu) \cap S(\pi,\pi-\tau,\nu)$,

$$\arg\left(\frac{x+\Lambda}{\varepsilon}\right) \in \left]\frac{\pi}{2} - \frac{\sigma'}{2} + \tau, \frac{3\pi}{2} + \frac{\sigma'}{2} - \tau\right[.$$

Consequently we have

$$|\Delta(x,a,\varepsilon)| \leq \frac{r}{2} e^{\frac{3\tilde{M}\Lambda}{r}} e^{-\frac{X_1+\Lambda}{2|\varepsilon|}\cos\left(\frac{\pi}{2}+\frac{\sigma'}{2}-\tau\right)}$$

for all $(x,a,\varepsilon) \in \widetilde{V} \times B(a_0,s) \times S(0,\pi+\sigma',\nu) \cap S(\pi,\pi-\tau,\nu)$.

The Ramis-Sibuya theorem guarantees the existence of a formal Gevrey-1 series

$$\widehat{G}(x,a,\varepsilon) = \sum_{n=0}^{\infty} g_n(x,a)\varepsilon^n,$$

where the g_n are analytic on $\widetilde{V} \times B(a_0,s)$, such that $G \sim_1 \widehat{G}$ w.r.t $\varepsilon \in S(0,\pi+\sigma',\nu)$ uniformly for $(x,a) \in \widetilde{V} \times B(a_0,s)$. Thus G is 1-summable. □

5 Canard-heteroclinic saddle connections

In this section we will limit ourselves to slow-fast systems with a turning point which can be transformed, locally around the turning point, into a system of the form

$$\left\{\begin{array}{rcl}\dot{x} &=& \varepsilon \\ \dot{y} &=& px^{p-1}y + \varepsilon H(x,y,a,\varepsilon) \\ \dot{\varepsilon} &=& 0.\end{array}\right. \tag{18}$$

Where H is analytic and satisfies $H(0,0,a_0,0)=0$, $\frac{\partial H}{\partial a}(0,0,a_0,0)\neq 0$, notice that p has to be an even number for $x=0$ to be a turning point.

It is shown in [13] that every system of the form

$$\left\{\begin{array}{rcl}\dot{x} &=& \varepsilon\\ \dot{y} &=& \varphi(x)y+\varepsilon H(x,y,a,\varepsilon)\\ \dot{\varepsilon} &=& 0,\end{array}\right.$$

with $\varphi(x)$ a real analytic function with a zero of order $p-1$ at $x=0$ and $H(0,0,a_0,0)=0$, $\frac{\partial H}{\partial a}(0,0,a_0,0)\neq 0$, can be transformed into this form. The authors also give some conditions on more general systems, such that the necessary transformation exists.

Setting $u=\varepsilon^{1/p}$, using the branch of the p-th root for which $1^{1/p}=1$, we prove the following theorem

Theorem 5.1. *Suppose $H(x,y,a,\varepsilon)$ is a bounded analytic function on*

$$B(0,r)\times B(0,r)\times B(a_0,r)\times B\left(0,r\right)$$

with $H(0,0,a_0,0)=0$, $\frac{\partial H}{\partial a}(0,0,a_0,0)\neq 0$. Moreover let there exist invariant manifolds of system (18), $G_1(x,a,\varepsilon)$ and $G_2(x,a,\varepsilon)$, 1-summable in the real direction and defined on

$$B(\mp\lambda,s)\times B(a_0,r)\times S\left(0,\pi+\sigma,r\right)$$

for certain $\lambda,s>0$.

Then there exists a function $a(u)$, p-summable in the real direction, such that the system

$$\left\{\begin{array}{rcl}\dot{x} &=& u^p\\ \dot{y} &=& px^{p-1}y+u^pH(x,y,a(u),u^p)\\ \dot{u} &=& 0,\end{array}\right. \tag{19}$$

has an invariant manifold of the form $y=G(x,u)$, defined for $x\in[-\lambda,\lambda]$ and which is p-summable in the direction 0 in u, uniformly for x in compact sets of $[-\lambda,\lambda]$ which do not include the turning point $x=0$.

Remark 5.2. Notice that by the results from the previous sections the existence of such invariant manifolds is guaranteed if there are slow-fast saddles present on both the attracting an repelling part of the critical curve.

5.1 Extension of invariant manifolds to 0

The general idea of the proof is to again extend the invariant manifolds until they reach $x=0$ and then search for conditions on the parameter a guaranteeing that the two extensions are matched. The continuation of these manifolds will be done under two transformations which resemble, using the terminology of blow-up maps, the phase-directional rescaling and family rescaling chart.

5.1.1 Phase-directional rescaling chart

The first chart we concentrate ourselves upon is a phase-directional rescaling chart, given by

$$x = v$$
$$y = v\overline{y}$$
$$u = v\overline{u}$$

which is clearly an analytic map with an analytic inverse between a domain and its image, provided that the domain does not contain any points where $v = 0$. Applying this transformation to the system (19) gives

$$\begin{cases} \dot{v} &= v^p \overline{u}^p \\ \dot{\overline{y}} &= v^{p-1}\left(p - \overline{u}^p\right)\overline{y} + v^{p-1}\overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right) \\ \dot{\overline{u}} &= -v^{p-1}\overline{u}^{p+1}. \end{cases}$$

Dividing by the common factor v^{p-1} we arrive at

$$\begin{cases} \dot{v} &= v\overline{u}^p \\ \dot{\overline{y}} &= \left(p - \overline{u}^p\right)\overline{y} + \overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right) \\ \dot{\overline{u}} &= -\overline{u}^{p+1}. \end{cases} \tag{20}$$

Since invariant manifolds of the second system will also be invariant manifolds of the first system, we may focus on the second one.

In the following lemma we use the notations, by which we described the domain where equation (19) holds.

Proposition 5.3. *Let p be even, $k \in \{0, \ldots, p-1\}$, $\rho, \theta_1, \theta_2, \Delta > 0$ satisfying $\rho + \theta_1 + \theta_2 + \Delta < \frac{\pi}{2}$, $v_0 \in B(0,r)\backslash\{0\}$, $0 < R < r$ and $K \in \mathbb{C}$ with $|K| < R$.*

There exists a $U > 0$ such that for

$$\overline{u}_1 \in S\left(\frac{2\pi k}{p}, \frac{\pi}{p} - \frac{2}{p}\left(\rho + \theta_1 + \theta_2 + \Delta\right), \sqrt[p]{U}\right)$$
$$v_1 \in \left(\left(v_0^p + S\left(\pi + \arg\left(v_0^p\right), 2\theta_1\right)\right) \cap S\left(\arg\left(v_0^p\right), 2\theta_2\right)\right)^{\frac{1}{p}} = \Omega\left(v_0, \theta_1, \theta_2\right),$$

where the branch of the p-th root is chosen such that $\left(v_0^p\right)^{\frac{1}{p}} = v_0$ and the branch line lies opposite to the point v_0^p, we have the following. (See figure 2 for an example of an $\Omega\left(v_0, \theta_1, \theta_2\right)$.)

The solution of the initial value problem given by equation (20) supplemented with

$$v(0) = v_0\,;\ \overline{y}(0) = \frac{K}{v_0}\,;\ \overline{u}(0) = \frac{v_1\overline{u}_1}{v_0}$$

is defined on $\left[0, \frac{v_1^p - v_0^p}{p(\overline{u}_1 v_1)^p}\right]$ with the endpoint given by

$$\left(v_1, \overline{y}\left(\frac{v_1^p - v_0^p}{p(\overline{u}_1 v_1)^p}\right), \overline{u}_1\right).$$

Moreover $\left|\overline{y}\left(a, \frac{v_1^p - v_0^p}{p(\overline{u}_1 v_1)^p}\right)\right| \leq \frac{R}{|v_0|}$.

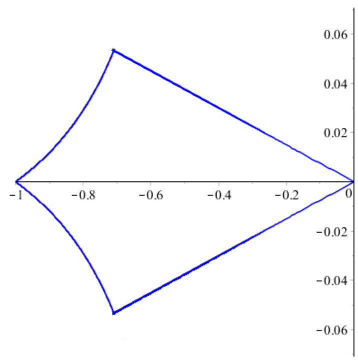

Figure 2: $\Omega(-1, 0.1, 0.3)$

Proof. This can be found in [14], by making some minor modifications to proposition 6.20. $\square$

Consider the invariant manifolds $y = G_1(x, a, u^p)$ and $y = G_2(x, a, u^p)$ of system (19). By restricting them to

$$B(\mp\lambda) \times B(a_0, r) \times S\left(0, \frac{\pi + \sigma}{p}, s\right)$$

for $s > 0$ sufficiently small, we may assume that $|G_{1,2}(x, a, u^p)| < R$ (we use the notations from proposition 5.3). Choose furthermore an $\alpha > 0$ such that $2p\alpha < \pi$ and $\left\{\mp\lambda e^{i\beta} \,|\, \beta \in [-\alpha, \alpha]\right\} \subset B(\mp\lambda)$

Corollary 5.4. *We reuse the notations from proposition 5.3. The extra demand $0 < p\alpha - \rho - \theta_1 - \Delta < \frac{\sigma}{2}$ is also needed.*

Under these conditions, there exists a $U > 0$ such that the system

$$\begin{aligned}
\dot{v} &= v\overline{u}^p \\
\dot{\overline{y}} &= (p - \overline{u}^p)\,\overline{y} + \overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right) \\
\dot{\overline{u}} &= -\overline{u}^{p+1}.
\end{aligned}$$

has two analytic invariant manifolds. The first, $(v, \Upsilon_1(v, a, \overline{u}), \overline{u})$, is defined for $(v, a, \overline{u})$ in

$$\bigcup_{\beta \in [-\alpha, \alpha]} \Omega\left(-\lambda e^{i\beta}, \theta_1, \theta_2\right) \times B(a_0, r_3) \times S\left(\pi, \frac{\pi}{p} - \frac{2}{p}\left(\rho + \theta_1 + \theta_2 + \Delta\right), \sqrt[p]{U}\right)$$

The second, $(v, \Upsilon_2(v, A, \overline{u}), \overline{u})$, is defined for $(v, a, \overline{u})$ in

$$\bigcup_{\beta \in [-\alpha, \alpha]} \Omega\left(\lambda e^{i\beta}, \theta_1, \theta_2\right) \times B(a_0, r_3) \times S\left(0, \frac{\pi}{p} - \frac{2}{p}\left(\rho + \theta_1 + \theta_2 + \Delta\right), \sqrt[p]{U}\right)$$

Moreover, both $|\Upsilon_1(v, a, \overline{u})|$ and $|\Upsilon_2(v, a, \overline{u})|$ are bounded by $\frac{R}{\lambda}$

Proof. Since the proof is analogous for both invariant manifolds, we prove the existence of the manifold Υ_1.

Let $(v_1, \overline{u}_1)$ be elements from the domain specified in the lemma. Consider the following initial value problem

$$\begin{cases} \dot{v} = v\overline{u}^p \\ \dot{\overline{y}} = (p - \overline{u}^p)\,\overline{y} + \overline{u}^p H\left(v, v\overline{y}, a, (v\overline{u})^p\right) \\ \dot{\overline{u}} = -\overline{u}^{p+1}. \end{cases} \tag{21a}$$

$$v(0) = -\lambda\,;\, \overline{y}(0) = -\frac{G_1\left(-\lambda, a, (v_1\overline{u}_1)^p\right)}{\lambda}\,;\, \overline{u}(0) = -\frac{v_1\overline{u}_1}{\lambda}. \tag{21b}$$

We have that

$$\begin{aligned} v(t) &= v_1\overline{u}_1\left(pt + \left(\frac{\lambda}{v_1\overline{u}_1}\right)^p\right)^{\frac{1}{p}} \\ \overline{y}(t) &= \frac{G_1\left(v(t), a, (v_1\overline{u}_1)^p\right)}{v(t)} \\ \overline{u}(t) &= \left(pt + \left(\frac{\lambda}{v_1\overline{u}_1}\right)^p\right)^{-\frac{1}{p}} \end{aligned}$$

is a solution to the above problem and it is defined for t in a neighbourhood of

$$\left\{\frac{\lambda^p}{p(v_1\overline{u}_1)^p}\left(e^{pi\beta} - 1\right) \mid \beta \in [-\alpha, \alpha]\right\}.$$

Using proposition 5.3 we also know that the system consisting of (21a) with initial values

$$v(0) = -\lambda e^{i\tilde{\beta}}\,;\, \overline{y}(0) = -\frac{G_1\left(-\lambda e^{i\tilde{\beta}}, a, (v_1\overline{u}_1)^p\right)}{\lambda e^{i\tilde{\beta}}}\,;\, \overline{u}(0) = -\frac{v_1\overline{u}_1}{\lambda e^{i\tilde{\beta}}},$$

with $\tilde{\beta}$ chosen such that $v_1 \in \Omega\left(-\lambda e^{i\tilde{\beta}}, \theta_1, \theta_2\right)$, has a solution defined for t in a neighbourhood of $\left[0, \frac{v_1^p - \left(\lambda e^{i\tilde{\beta}}\right)^p}{p(v_1\overline{u}_1)^p}\right]$.

Consequently the solution to the problem (21a) with initial values (21b) is defined for t in a neighbourhood of some path between 0 and $\frac{v_1^p - \lambda^p}{p(v_1\overline{u}_1)^p}$. By analytic dependence upon initial values we have that the general solution to (21a); (21b),

$$v(v_1\overline{u}_1; t)\,,\, \overline{y}(v_1\overline{u}_1; t)\,,\, \overline{u}(v_1\overline{u}_1; t),$$

consists of analytic functions in the variables $v_1\overline{u}_1$ and t. It follows that the map

$$(v_1, \overline{u}_1) \mapsto \left(v_1, \overline{y}\left(v_1\overline{u}_1; \frac{v_1^p - \lambda^p}{p(v_1\overline{u}_1)^p}\right), \overline{u}_1\right)$$

is also analytic. The inequality follows readily from proposition 5.3. □

Lemma 5.5. *Take $\tilde{\lambda} > 0$ sufficiently small such that $S\left(\pi, 2\left(\alpha + \frac{\theta_2}{p}\right), \tilde{\lambda}\right)$ is contained in $\bigcup_{\beta\in[-\alpha,\alpha]} \Omega\left(-\lambda e^{i\beta}, \theta_1, \theta_2\right)$. The function $\Upsilon_1(v, a, \overline{u})$ from corollary 5.4 is Gevrey-$\frac{1}{p}$ asymptotic to a formal series for $v \in S\left(\pi, 2\left(\alpha + \frac{\theta_2}{p}\right), \tilde{\lambda}\right)$, uniformly for $(a, \overline{u})$. An analogous statement holds for $\Upsilon_2(v, a, \overline{u})$.*

Proof. The proof can be given in a nearly identical manner as the proofs of proposition 6.24 and theorem 6.25 in [14]. □

Proposition 5.6. *Let*

$$\hat{f}_{1,2}(v, a, \overline{u}) = \sum_{n=0}^{\infty} f_n^{1,2}(a, \overline{u}) v^n$$

be the formal series associated to Υ_1 resp. Υ_2 as in lemma 5.5. The coefficient of v^0 is given by

$$-\frac{H(0, 0, a, 0)}{p} \int_1^{\infty} z^{\frac{1}{p}-1} e^{\frac{1-z}{\overline{u}^p}} \mathrm{d}z$$

for both formal series

Proof. Since the proof is exactly the same for Υ_1 and Υ_2 we only treat Υ_1. Since $(v, \Upsilon_1, \overline{u})$ is an invariant manifold of system (20) it must hold that

$$\begin{aligned} v\overline{u}^p \frac{\partial \Upsilon_1}{\partial v}(v, a, \overline{u}) - \overline{u}^{p+1} \frac{\partial \Upsilon_1}{\partial \overline{u}}(v, a, \overline{u}) \\ = (p - \overline{u}^p)\Upsilon_1(v, a, \overline{u}) + \overline{u}^p H\left(v, v\Upsilon_1(v, a, \overline{u}), a, (v\overline{u})^p\right). \end{aligned}$$

Since $\Upsilon_1(v, a, \overline{u}) \sim_{\frac{1}{p}} \hat{f}_1(v, a, \overline{u})$ w.r.t. v it follows that

$$\begin{aligned} \Upsilon_1(v, a, \overline{u}) &\xrightarrow{v\to 0} f_0^1(a, \overline{u}) \\ \frac{\partial \Upsilon_1}{\partial v}(v, a, \overline{u}) &\xrightarrow{v\to 0} f_1^1(a, \overline{u}) \\ \frac{\partial \Upsilon_1}{\partial \overline{u}}(v, a, \overline{u}) &\xrightarrow{v\to 0} \frac{\partial f_0^1}{\partial \overline{u}}(a, \overline{u}). \end{aligned}$$

Consequently we must have

$$-\overline{u}^{p+1} \frac{\partial f_0^1}{\partial \overline{u}}(a, \overline{u}) = (p - \overline{u}^p) f_0^1(a, \overline{u}) + \overline{u}^p H\left(0, 0, a, 0\right),$$

moreover by proposition 5.3 it must hold that $\lim_{\overline{u}\to 0} f_0^1(a, \overline{u}) = 0$. This implies that the following identity holds

$$\begin{aligned} f_0^1(a, \overline{u}) &= -H(0, 0, a, 0) \int_0^{\overline{u}} t^{-1} e^{\int_t^{\overline{u}} \frac{s^p - p}{s^{p+1}} \mathrm{d}s} \mathrm{d}t \\ &= \overline{u} H(0, 0, a, 0) \int_{\overline{u}}^0 t^{-2} e^{\overline{u}^{-p} - t^{-p}} \mathrm{d}t. \end{aligned}$$

Using the path $\gamma(z) = \overline{u} z^{-\frac{1}{p}}$ with $z \in [1, \infty[$ we get

$$f_0^1(a, \overline{u}) = -\frac{H(0,0,a,0)}{p} \int_1^\infty z^{\frac{1}{p}-1} e^{\frac{1-z}{\overline{u}^p}} \mathrm{d}z.$$

□

5.1.2 Family rescaling chart

To let our two manifolds actually meet each other we will have to switch to another chart, which resembles the family rescaling chart, this is given by

$$\begin{aligned} x &= wX \\ y &= wY \\ u &= w \end{aligned}$$

which is an analytic map with analytic inverse on domains which do not contain $w = 0$. Applying this transformation to our system (19), which we repeat for the sake of convenience,

$$\left\{ \begin{array}{rcl} \dot{x} &=& u^p \\ \dot{y} &=& px^{p-1}y + u^p H(x, y, a, u^p) \\ \dot{u} &=& 0 \end{array} \right.$$

brings us, after dividing by a common factor w^p, to the system

$$\left\{ \begin{array}{rcl} \dot{X} &=& 1 \\ \dot{Y} &=& pX^{p-1}Y + H(wX, wY, a, w^p) \\ \dot{w} &=& 0. \end{array} \right. \tag{22}$$

By corollary 5.4 and lemma 5.5, there exist two invariant manifolds of this system

$$\left(X, X\Upsilon_1\left(wX, a, X^{-1}\right), w\right)$$

defined and analytic on

$$\left\{(X, w) \,\middle|\, X \in S\left(\pi, \frac{\pi}{p} - \frac{2}{p}(\rho + \theta_1 + \theta_2 + \Delta)\right) \Big\backslash \overline{B}\left(0, \frac{1}{\sqrt[p]{U}}\right), \right.$$
$$\left. wX \in S\left(\pi, 2\left(\alpha + \frac{\theta_2}{p}\right), \tilde{\lambda}\right)\right\} \times B(a_0, r_3)$$

and

$$\left(X, X\Upsilon_2\left(wX, a, X^{-1}\right), w\right)$$

defined and analytic on

$$\left\{(X, w) \,\middle|\, X \in S\left(0, \frac{\pi}{p} - \frac{2}{p}(\rho + \theta_1 + \theta_2 + \Delta)\right) \Big\backslash \overline{B}\left(0, \frac{1}{\sqrt[p]{U}}\right), \right.$$
$$\left. wX \in S\left(0, 2\left(\alpha + \frac{\theta_2}{p}\right), \tilde{\lambda}\right)\right\} \times B(a_0, r_3).$$

Moreover if we take some $X_0 \in S\left(0, \frac{\pi}{p} - \frac{2}{p}(\rho+\theta_1+\theta_2+\Delta)\right) \setminus \overline{B}\left(0, \frac{1}{\sqrt[p]{U}}\right)$, both $-X_0\Upsilon_1\left(-wX_0, a, -X_0^{-1}\right)$ and $X_0\Upsilon_2\left(wX_0, a, X_0^{-1}\right)$ are Gevrey-$\frac{1}{p}$ asymptotic to a formal series for $w \in S\left(-\arg(X_0), 2\left(\alpha+\frac{\theta_2}{p}\right), \frac{\tilde{\lambda}}{|X_0|}\right)$.

Proposition 5.7. *For every*

$$X_0 \in S\left(0, \frac{\pi}{p} - \frac{2}{p}(\rho+\theta_1+\theta_2+\Delta)\right) \Big\backslash \overline{B}\left(0, \frac{1}{\sqrt[p]{U}}\right)$$

there exist $\delta_1, \delta_2 > 0$ *such that the solution to*

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}X} &= pX^{p-1}Y + H(wX, wY, a, w^p) \\ Y(X_0, a, w) &= X_0\Upsilon_2\left(wX_0, a, X_0^{-1}\right) \end{aligned}$$

is defined and analytic on

$$[X_0, 0] \times B(a_0, \delta_1) \times S\left(-\arg(X_0), 2\left(\alpha+\frac{\theta_2}{p}\right), \delta_2\right).$$

Furthermore, for the same X_0*, the solution to*

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}X} &= pX^{p-1}Y + H(wX, wY, a, w^p) \\ Y(-X_0, a, w) &= -X_0\Upsilon_1\left(-wX_0, a, -X_0^{-1}\right) \end{aligned}$$

is also defined and analytic on

$$[-X_0, 0] \times B(a_0, \delta_1) \times S\left(-\arg(X_0), 2\left(\alpha+\frac{\theta_2}{p}\right), \delta_2\right).$$

Proof. This can be proved by a rather standard fixed point argument. □

Remark 5.8. It can be shown that if we associate certain δ_1, δ_2 to a X_0 as in the proposition, those same δ_1, δ_2 will also allow us to prove the result for any other

$$\tilde{X}_0 \in S\left(0, \frac{\pi}{p} - \frac{2}{p}(\rho+\theta_1+\theta_2+\Delta)\right) \Big\backslash \overline{B}\left(0, \frac{1}{\sqrt[p]{U}}\right)$$

provided that $\left|\tilde{X}_0\right| = |X_0|$.

Below, we show that the saturations of the invariant manifolds from proposition 5.7 above, can be connected to each other at 0, for a good choice of the parameter a. For this we require a Gevrey version of the implicit function theorem.

Theorem 5.9 (Gevrey implicit function theorem)**.** *([15, 14]) Let* $s > 0$ *and*

$$\hat{f}(a, \varepsilon) = \sum_{n=0}^{\infty} f_n(a)\varepsilon^n$$

be a Gevrey-s series in ε, uniformly for $a \in A$, with $A \subset \mathbb{C}$ open.

Suppose there are $\theta \in [0, 2\pi[$, $\lambda, r > 0$ and $f \in \mathcal{O}(A \times S(\theta, \lambda, r))$ such that $f \sim_s \hat{f}$. If moreover there exists an $a_0 \in A$ with $f_0(a_0) = 0$ and $f_0'(a_0) \neq 0$, we can find an $r' > 0$ and an analytic function

$$\tilde{a} : S(\theta, \lambda, r') \to A$$

such that $\tilde{a}(0) = a_0$ and

$$f(\tilde{a}(\varepsilon), \varepsilon) = 0$$

for all $\varepsilon \in S(\theta, \lambda, r')$.

The function $\tilde{a}$ is also Gevrey-s asymptotic to a formal series

$$\hat{a}(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n.$$

Consider now some $\tilde{X} > \frac{1}{\sqrt[p]{U}}$.

Lemma 5.10. *Let $\beta \in]-\frac{\pi}{2p} + \frac{1}{p}(\rho + \theta_1 + \theta_2 + \Delta), \frac{\pi}{2p} - \frac{1}{p}(\rho + \theta_1 + \theta_2 + \Delta)[$, there exists a analytic function $a_\beta(w)$ defined for $w \in S\left(-\beta, 2\left(\alpha + \frac{\theta_2}{p}\right), \omega_\beta\right)$, for some $\omega_\beta > 0$, with $a_\beta(w) = a_0$ such that $Y_1^\beta(0, a_\beta(w), w) = Y_2^\beta(0, a_\beta(w), w)$. Here Y_1^β and Y_2^β are the solutions associated to $\tilde{X}e^{i\beta}$ as in proposition 5.7.*

Moreover $a_\beta(w)$ is a Gevrey-$\frac{1}{p}$ function.

Proof. We have

$$\begin{aligned} Y_1^\beta(0, a, w) = &- \tilde{X}e^{i\beta} e^{-\left(\tilde{X}e^{i\beta}\right)^p} \Upsilon_1\left(-w\tilde{X}e^{i\beta}, a, -\left(\tilde{X}e^{i\beta}\right)^{-1}\right) \\ &+ \int_{-\tilde{X}e^{i\beta}}^0 H\left(wz, wY_1^\beta(z, a, w), a, w^p\right) e^{-z^p} \mathrm{d}z \end{aligned}$$

and

$$\begin{aligned} Y_2^\beta(0, a, w) = &\tilde{X}e^{i\beta} e^{-\left(\tilde{X}e^{i\beta}\right)^p} \Upsilon_2\left(w\tilde{X}e^{i\beta}, a, \left(\tilde{X}e^{i\beta}\right)^{-1}\right) \\ &+ \int_{\tilde{X}e^{i\beta}}^0 H\left(wz, wY_2^\beta(z, a, w), a, w^p\right) e^{-z^p} \mathrm{d}z. \end{aligned}$$

Consider the time-$(-\tilde{X}e^{i\beta})$ and time-$(\tilde{X}e^{i\beta})$ mappings associated to the analytic differential equation

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = pX^{p-1}Y + H(wX, wY, a, w^p).$$

The above expressions are the images of $-\tilde{X}e^{i\beta}\Upsilon_1(-w\tilde{X}e^{i\beta}, a, -(\tilde{X}e^{i\beta})^{-1})$ resp. $\tilde{X}e^{i\beta}\Upsilon_2(w\tilde{X}e^{i\beta}, a, (\tilde{X}e^{i\beta})^{-1})$ under these mappings. Theorem 2.6 thus shows that these expressions are Gevrey-$\frac{1}{p}$, uniformly in a, for $w \in S(-\beta, 2(\alpha + \frac{\theta_2}{p}), \delta_2)$.

By proposition 5.6 we have

$$\begin{aligned}
&\lim_{w\to 0} Y_2^\beta(0,a,w) - Y_1^\beta(0,a,w)\\
&= H(0,0,a,0)\left(-\frac{2\tilde{X}e^{i\beta}e^{-\left(\tilde{X}e^{i\beta}\right)^p}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{(1-z)\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z\right.\\
&\qquad\left.+\int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right)\\
&= H(0,0,a,0)\left(-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right)
\end{aligned}$$

from which it follows that the coefficient of w^0 of the formal series associated to the Gevrey-$\frac{1}{p}$ function $Y_2^\beta(0,a,w) - Y_1^\beta(0,a,w)$ is given by the expression above. Using the Gevrey implicit function theorem 5.9, we prove the result if we can show that

$$H(0,0,a_0,0)\left(-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right) = 0$$

and

$$\frac{\partial H}{\partial a}(0,0,a_0,0)\left(-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\right) \neq 0.$$

Using our assumption in theorem 5.1 that $H(0,0,a_0,0) = 0$ and $\frac{\partial H}{\partial a}(0,0,a_0,0) \neq 0$, it clearly suffices to check that

$$-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z \neq 0.$$

One can calculate that

$$\begin{aligned}
&-\frac{2\tilde{X}e^{i\beta}}{p}\int_1^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z + \int_{\tilde{X}e^{i\beta}}^{-\tilde{X}e^{i\beta}} e^{-z^p}\mathrm{d}z\\
&\qquad= -\frac{2\tilde{X}e^{i\beta}}{p}\int_0^\infty z^{\frac{1}{p}-1}e^{-z\left(\tilde{X}e^{i\beta}\right)^p}\mathrm{d}z\\
&\qquad= -\frac{2}{p}\int_0^{\infty(p\beta)} z^{\frac{1}{p}-1}e^{-z}\mathrm{d}z\\
&\qquad= -\frac{2}{p}\int_0^{\infty} z^{\frac{1}{p}-1}e^{-z}\mathrm{d}z\\
&\qquad= -\frac{2}{p}\Gamma\left(\frac{1}{p}\right) \neq 0.
\end{aligned}$$

□

We claim that the functions a_β are all analytic continuations of each other. Indeed, suppose that β_1 and β_2 are such that

$$S\left(-\beta_1, 2\left(\alpha + \frac{\theta_2}{p}\right), \omega_{\beta_1}\right) \cap S\left(-\beta_2, 2\left(\alpha + \frac{\theta_2}{p}\right), \omega_{\beta_2}\right) \neq \emptyset,$$

this intersection is then again a sector. By reducing the opening of this sector slightly one can see that $\Upsilon_1(-wX, a, -X^{-1})$ and $\Upsilon_2(wX, a, X^{-1})$ are defined for w in this sector and X in some neighbourhood of $\left\{\tilde{X}e^{i\alpha} \mid \alpha \in [\beta_1, \beta_2]\right\}$. One then sees, using the uniqueness of solutions for analytic initial value problems, that both

$$Y_1^{\beta_1}(0, a_{\beta_1}(w), w) = Y_2^{\beta_1}(0, a_{\beta_1}(w), w)$$

and

$$\begin{aligned} Y_1^{\beta_1}(0, a_{\beta_2}(w), w) &= Y_1^{\beta_2}(0, a_{\beta_2}(w), w) \\ &= Y_2^{\beta_2}(0, a_{\beta_2}(w), w) = Y_2^{\beta_1}(0, a_{\beta_2}(w), w) \end{aligned}$$

hold.

Using the uniqueness part in the Gevrey implicit function theorem we get that a_{β_1} and a_{β_2} are analytic continuations of each other.

Combining our results gives the proof of theorem 5.1.

By collecting the results of theorems 3.1,4.1 and 5.1 we arrive at the following conclusion.

Theorem 5.11. *Consider a real analytic slow-fast family of vector fields*

$$\begin{cases} \dot{x} &= \varepsilon f(x, y, a, \varepsilon) \\ \dot{y} &= g(x, y, a, \varepsilon), \end{cases}$$

with points $x_a, x_t, x_r \in \mathbb{R}$ such that x_t, a turning point, lies in between the two other points, we may assume without loss of generality that $x_a < x_t < x_r$. We furthermore make the following assumptions.

- *There exists a critical curve given by the graph $y = \varphi_0(x)$ (for $a = a_0$), $x \in [x_a, x_r]$ which is hyperbolically attracting to the left of x_t and repelling to the right of this point i.e.*

$$\begin{aligned} \frac{\partial g}{\partial y}(x, \varphi_0(x), a_0, 0) < 0, x \in [x_a, x_t[, \\ \frac{\partial g}{\partial y}(x, \varphi_0(x), a_0, 0) > 0, x \in]x_t, x_r], \\ \frac{\partial g}{\partial y}(x_t, \varphi_0(x_t), a_0, 0) = 0. \end{aligned}$$

- *The points x_a and x_r are slow-fast saddle points with the slow dynamics directed from the attracting to the repelling part of the critical curve, which* *is characterized by*

$$f(x_*, \varphi_0(x_*), a_0, 0) = 0; x_* = x_a, x_r,$$
$$\left(\frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} \cdot \frac{\partial f}{\partial x}\right)(x_*, \varphi_0(x_*), a_0, 0) > 0; x_* = x_a, x_r,$$
$$f(x, \varphi_0(x), a_0, 0) > 0; x \in]x_a, x_r[.$$

- *Locally around the turning point there exists an analytic transformation which transforms the system into the form* (18).

Under these assumptions there exists a function $a(u)$*, p-summable in the real direction such that the system*

$$\begin{cases} \dot{x} &= u^p f(x, y, a(u), u^p) \\ \dot{y} &= g(x, y, a(u), u^p), \end{cases}$$

has an invariant manifold $y = G(x, u)$ *defined for* $[x_a, x_r]$ *which is p-summable in the real direction in* u*, uniformly for* x *in compact sets of* $[x_a, x_r]$ *which do not include the turning point* x_t*.*

Let us conclude by remarking that an alternative method of proving this theorem could have used the technique of combined asymptotic developments, developed in [13].

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