Noncommutative motives of separable algebras

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NONCOMMUTATIVE MOTIVES OF SEPARABLE ALGEBRAS

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ABSTRACT. In this article we study in detail the category of noncommutative motives of separable algebras $\mathrm{Sep}(k)$ over a base field k. We start by constructing four different models of the full subcategory of commutative separable algebras $\mathrm{CSep}(k)$. Making use of these models, we then explain how the category $\mathrm{Sep}(k)$ can be described as a "fibered $\mathbb{Z}$-order" over $\mathrm{CSep}(k)$. This viewpoint leads to several computations and structural properties of the category $\mathrm{Sep}(k)$. For example, we obtain a complete dictionary between directs sums of noncommutative motives of central simple algebras (=CSA) and sequences of elements in the Brauer group of k. As a first application, we establish two families of motivic relations between CSA which hold for every additive invariant (*e.g.* algebraic K-theory, cyclic homology, and topological Hochschild homology). As a second application, we compute the additive invariants of twisted flag varieties using solely the Brauer classes of the corresponding CSA. Along the way, we categorify the cyclic sieving phenomenon and compute the (rational) noncommutative motives of purely inseparable field extensions and of dg Azumaya algebras.

1. INTRODUCTION

Noncommutative motives. A *dg category* $\mathcal{A}$, over a base field k, is a category enriched over complexes of k-vector spaces; see §4. Every (dg) k-algebra A naturally gives rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$. In what follows, $\mathrm{dgcat}(k)$ denotes the category of dg categories.

Invariants such as algebraic K-theory, cyclic homology, and topological Hochschild homology, extend naturally from k-algebras to dg categories. In order to study them simultaneously the notion of *additive invariant* was introduced in [34] and the *universal additive invariant* $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ was constructed; consult §5.1-5.2 for details. Given any additive category D, there is an induced equivalence

$$(1.1) \qquad U^* : \mathrm{Fun}(\mathrm{Hmo}_0(k), \mathrm{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}),$$

where the left-hand-side denotes the category of additive functors and the right-hand-side the category of additive invariants. Because of this universal property, which is reminiscent from the yoga of motives, $\mathrm{Hmo}_0(k)$ is called the category of *noncommutative motives*. The tensor product of k-algebras extends also naturally to dg categories. It gives rise to a symmetric monoidal structure on $\mathrm{dgcat}(k)$ which descends to $\mathrm{Hmo}_0(k)$ making the functor U symmetric monoidal.

Following Kontsevich [16, 17, 18], a dg category $\mathcal{A}$ is called *smooth* if it is perfect as a bimodule over itself and *proper* if $\sum_i \dim H^i \mathcal{A}(x, y) < \infty$ for every ordered pair of objects (x, y). Examples include the finite dimensional k-algebras of finite global dimension (when k is perfect) and the dg categories $\mathrm{perf}_{\mathrm{dg}}(X)$ associated to smooth projective k-schemes X. The category of *noncommutative Chow motives* $\mathrm{NChow}(k)$ was introduced in [32] as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)$ consisting of the smooth proper dg categories. By construction, $\mathrm{NChow}(k)$ is additive and rigid symmetric monoidal; consult the survey [33, §4].

Motivating goal. Given an additive rigid symmetric monoidal category $\mathcal{C}$, its $\otimes$-ideal $\mathcal{N}$ is defined by the following formula

$$\mathcal{N}(a, b) := \{ f : a \to b \,|\, \forall\, g : b \to a \ \text{we have} \ \mathrm{tr}(g \circ f) = 0 \} \,,$$

where tr stands for the categorical trace. Motivated by André-Kahn's description of the category of numerical motives (see [3, Example 7.1.2]), the category of *noncommutative numerical motives* $\mathrm{NNum}(k)$ was introduced in [20] as the idempotent completion of the quotient category $\mathrm{NChow}(k)/\mathcal{N}$. Our first result is the following:

Theorem 1.2. *When k is of characteristic zero, the Hom-groups of the additive category $\mathrm{NNum}(k)$ are finitely generated abelian groups.*

Intuitively speaking, Theorem 1.2 shows that the category $\mathrm{NNum}(k)$ encode only a finite amount of data. This motivates the following ambitious goal:

Goal: Construct a simple and explicit model of the category $\mathrm{NNum}(k)$.

Since $\mathrm{NNum}(k)$ contains information about *all* smooth proper dg categories, the above goal seems completely out of reach at the present time. In this article we give the first step towards its solution by addressing the case of the full subcategory of separable algebras. Already in this case some surprisingly interesting phenomena occur! For example, the latter category is strongly related with the classical theory of $\mathbb{Z}$-orders; see Proposition 2.26. For a number of applications, please consult §3.

2. STATEMENT OF RESULTS

Throughout the article, except in the above Theorem 1.2 and in Appendix B, k will be a field of arbitrary characteristic. Unless stated differently, all tensor products will be taken over k. Let $G := \mathrm{Gal}(k_{\mathrm{sep}}/k)$ be the absolute Galois group of k, equipped with its profinite topology. Given a central simple k-algebra A, we will write $\deg(A)$ for its degree, $\mathrm{ind}(A)$ for its index, $\mathrm{per}(A)$ for its period, and $[A]$ for its class in the Brauer group $\mathrm{Br}(k)$ of k; consult [13, §4.4-§4.5].

Commutative separable algebras. Recall from [19, §III Prop. 4.1] that a commutative k-algebra A is separable if and only if it is isomorphic to a finite product of finite separable field extensions of k. Thanks to [19, §III Thm. 1.4(1) and Prop. 3.2], every (commutative) separable k-algebra is smooth and proper as a dg category. Let us then denote by $\mathrm{CSep}(k)$ the full subcategory of $\mathrm{NChow}(k)$ consisting of the objects $U(A)$ with A a commutative separable k-algebra. As mentioned in §5.2, $U(\mathcal{A}) \oplus U(\mathcal{B}) \simeq U(\mathcal{A} \times \mathcal{B})$ for all dg categories $\mathcal{A}$ and $\mathcal{B}$. This implies that the category $\mathrm{CSep}(k)$ is additive. Since (commutative) separable k-algebras are stable under tensor product (see [19, §III Prop. 1.7]) and the universal additive invariant U is symmetric monoidal, $\mathrm{CSep}(k)$ is moreover symmetric monoidal. The following result is a special case of Proposition 2.10 stated below.

Proposition 2.1. *The quotient functor* $\mathrm{CSep}(k) \to \mathrm{NNum}(k)$ *is fully-faithful.*

Intuitively speaking, Proposition 2.1 shows that $\mathrm{CSep}(k)$ is insensitive to the numerical equivalence relation. Consider now the following four categories:

(i) Recall from [2, §4] that the category of Chow motives $\mathrm{Chow}(k)$ comes equipped with a symmetric monoidal functor $M : \mathrm{SmProj}(k)^{\mathrm{op}} \to \mathrm{Chow}(k)$ defined on smooth projective k-schemes. Let us denote by $\mathrm{Etale}(k)$ the full subcategory of $\mathrm{Chow}(k)$ consisting of the objects $M(X)$ with X a finite étale k-scheme. Note that $\mathrm{Etale}(k)$ is an additive symmetric monoidal category.

(ii) Given a finite G-set S, let $\mathrm{Map}^G(S, \mathbb{Z})$ be the set of G-invariant functions $\alpha : S \to \mathbb{Z}$. The *convolution category* $\mathrm{Cov}(G)$ is defined as follows: the objects are the finite G-sets S; the morphisms $\mathrm{Hom}_{\mathrm{Cov}(G)}(S_1, S_2)$ are the G-invariant functions $\mathrm{Map}^G(S_1 \times S_2, \mathbb{Z})$; the composition law is the convolution product

$$\mathrm{Map}^G(S_1 \times S_2, \mathbb{Z}) \times \mathrm{Map}^G(S_2 \times S_3, \mathbb{Z}) \longrightarrow \mathrm{Map}^G(S_1 \times S_3, \mathbb{Z})$$

where

$$(\alpha, \beta) \mapsto (\alpha * \beta)(s_1, s_3) := \sum_{s_2 \in S_2} \alpha(s_1, s_2) \cdot \beta(s_2, s_3) \,;$$

the identities are the G-invariant functions $S \times S \to \mathbb{Z}$ which are equal to 1 on the diagonal and 0 elsewhere. The disjoint union and the cartesian product of finite G-sets makes $\mathrm{Cov}(G)$ into an additive symmetric monoidal category.

(iii) Given closed subgroups $H, K \subseteq G$ of finite index, let $H\backslash G/K$ be the set of (H, K) double cosets in G and $\mathrm{Map}(H\backslash G/K, \mathbb{Z})$ the set of functions $\alpha : H\backslash G/K \to \mathbb{Z}$. The category $\mathrm{Heck}(G)$ is defined as follows: the objects are the closed subgroups $H \subseteq G$ of finite index; the morphisms $\mathrm{Hom}_{\mathrm{Heck}(G)}(H, K)$ are the functions $\mathrm{Map}(H\backslash G/K, \mathbb{Z})$; the composition law is the convolution product

$$\mathrm{Map}(H\backslash G/K, \mathbb{Z}) \times \mathrm{Map}(K\backslash G/L, \mathbb{Z}) \longrightarrow \mathrm{Map}(H\backslash G/L, \mathbb{Z}) \,,$$

where

$$(\alpha, \beta) \mapsto (\alpha \star \beta)(\overline{g}) := \sum_{\overline{h} \in G/K} \alpha(h^{-1}) \cdot \beta(hg) \,;$$

the identities are the characteristic functions $\delta_{H1H} : H\backslash G/H \to \mathbb{Z}$. Note that $\mathrm{End}_{\mathrm{Heck}(G)}(H, H)$ identifies with the classical Hecke algebra of the pair (G, H). In the particular case where H is a normal subgroup of G, the latter Hecke algebra reduces to the group ring $\mathbb{Z}[G/H]$. The *Hecke category* $\mathrm{Hecke}(G)$ is defined as the closure of $\mathrm{Heck}(G)$ under (formal) finite direct sums. By construction, $\mathrm{Hecke}(G)$ is an additive category.

(iv) A right $\mathbb{Z}[G]$-module N is called a *permutation G-module* if it admits a finite $\mathbb{Z}$-basis which is invariant under the G-action. Equivalently, N is of the form $\mathbb{Z}[S]$ for some finite G-set S. Let us denote by $\mathrm{Perm}(G)$ the full subcategory of those right $\mathbb{Z}[G]$-modules which are permutation G-modules. Note that $\mathrm{Perm}(G)$ is an additive category.

By combining Merkurjev-Panin's work [24, Prop. 1.7] with [31, Thm. 6.10], we obtain the following additive equivalence of categories

$$(2.2) \qquad \mathrm{CSep}(k) \overset{\simeq}{\longrightarrow} \mathrm{Perm}(G) \qquad U(A) \mapsto K_0(A \otimes k_{\mathrm{sep}}) \,.$$

Other models of the category $\mathrm{CSep}(k)$ are provided by the following result:

Proposition 2.3. *The following additive categories are equivalent:*

$$(2.4) \qquad \mathrm{CSep}(k) \quad \mathrm{Etale}(k) \quad \mathrm{Cov}(G) \quad \mathrm{Hecke}(G) \quad \mathrm{Perm}(G) \,.$$

Remark 2.5. Recall from [2, §4.1.6] that the idempotent completion of $\mathrm{Etale}(k)$ identifies with the category of (integral) Artin motives $\mathrm{AM}(k)$. Thanks to Proposition 2.3, the same holds for all the other categories (2.4).

By combining Proposition 2.3 and Remark 2.5 with some examples arising from integral representation theory, we obtain the following (surprising) remarks:

Remark 2.6. Let $H := \langle a, b, c \,|\, a^8 = b^2 = c^2 = abc \rangle$ be the generalized quaternion group of order 32. R. Swan constructed in [30] a non-free projective left ideal $I \subset \mathbb{Z}[H]$ such that $I \oplus \mathbb{Z}[H] \simeq \mathbb{Z}[H] \oplus \mathbb{Z}[H]$ as $\mathbb{Z}[H]$-modules. We claim that I is *not* a permutation H-module. Assume that $I \simeq \mathbb{Z}[S]$ for some finite H-set S. Since H is a 2-group, $(\mathbb{Z}/2\mathbb{Z})[H]$ is a local ring. This implies that I is indecomposable and consequently that S has a single H-orbit. Making use of the equality $\mathrm{rank}_{\mathbb{Z}}(I) = \mathrm{rank}_{\mathbb{Z}}(\mathbb{Z}[H])$, we hence conclude that $S \simeq H$. This contradicts the fact that I is non-free. As an application, we obtain the following results:

  (i) The above categories (2.4) are, in general, *not* idempotent complete! Let k be a base field for which we can choose a Galois field extension l/k inside k_{sep} with Galois group H. Via the induced group homomorphism $G \twoheadrightarrow H$, $\mathrm{Perm}(H)$ identifies with a full subcategory of $\mathrm{Perm}(G)$. Consequently, the idempotent in $\mathrm{End}_{\mathrm{Perm}(G)}(\mathbb{Z}[H] \oplus \mathbb{Z}[H])$ corresponding to the projection $\mathbb{Z}[H] \oplus \mathbb{Z}[H] \to I$ does not split in $\mathrm{Perm}(G)$.
 (ii) The category of (integral) Artin motives $\mathrm{AM}(k)$ does *not* satisfy cancellation[1]!

Remark 2.7. Let $H := PSL(2, \mathbb{F}_{29})$. L. Scott constructed in [29] non-conjugate subgroups $L_1, L_2 \subset H$ such that $\mathbb{Z}[H/L_1] \simeq \mathbb{Z}[H/L_2]$ as $\mathbb{Z}[H]$-modules. Let k be a base field for which we can choose a Galois field extension l/k inside k_{sep} with Galois group H; write l_i for the associated field extension l^{L_i}. On the one hand, since L_1 is non-conjugate to L_2, we have $l_1 \not\simeq l_2$. On the other, since the above equivalence of categories (2.2) sends l_i to the permutation G-module $\mathbb{Z}[H/L_i]$, we have $U(l_1) \simeq U(l_2)$.

We finish this subsection with a result concerning inseparable field extensions:

Theorem 2.8. *Given a purely inseparable field extension l/k of degree p^r, we have an isomorphism $U(k)_R \simeq U(l)_R$ for every commutative ring R containing $1/p$.*

Intuitively speaking, Theorem 2.8 shows that the noncommutative motives of purely inseparable field extensions only contain torsion information.

Corollary 2.9. *Let l/k be a purely inseparable field extension of degree p^r and $E : \mathrm{dgcat}(k) \to \mathrm{D}$ an additive invariant with values in a $\mathbb{Z}[1/p]$-linear category. Under these assumptions, we have an isomorphism $E(k) \simeq E(l)$.*

Proof. It follows from the combination of Theorem 2.8 with equivalence (5.4). $\square$

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[1]It seems reasonable to conjecture that the above categories (2.4) also do *not* satisfy cancellation. Surprisingly, it appears that this problem (in the particular case of the category $\mathrm{Perm}(G)$) has not been studied; we have consulted a few experts on this matter.

Separable algebras. Recall from [19, §III Thm. 3.1] that a k-algebra A is separable if and only if it is isomorphic to a finite product of matrix algebras $M_{r \times r}(D)$. Here, D is a finite dimensional division k-algebra with center $Z(D)$ a finite separable field extension of k. Let us denote by $\mathrm{Sep}(k)$ the full subcategory of $\mathrm{NChow}(k)$ consisting of the objects $U(A)$ with A a separable k-algebra. As explained above, $\mathrm{Sep}(k)$ is an additive symmetric monoidal subcategory of $\mathrm{NChow}(k)$. The following result generalizes Proposition 2.1.

Proposition 2.10. *The quotient functor* $\mathrm{Sep}(k) \to \mathrm{NNum}(k)$ *is fully-faithful.*

Proof. Since the category $\mathrm{Sep}(k)$ is rigid symmetric monoidal, it suffices to show that the composition bilinear pairings (see §5.2)

$$\mathrm{Hom}_{\mathrm{Sep}(k)}(U(k), U(A)) \times \mathrm{Hom}_{\mathrm{Sep}(k)}(U(A), U(k)) \longrightarrow \mathrm{Hom}_{\mathrm{Sep}(k)}(U(k), U(k))$$

are non-degenerate; see [3, Lem. 7.1.1]. As mentioned above, $U(\mathcal{A} \times \mathcal{B}) \simeq U(\mathcal{A}) \oplus U(\mathcal{B})$ for all dg categories $\mathcal{A}$ and $\mathcal{B}$. Moreover, thanks to Morita invariance (see §4 and §5.1), we have $U(M_{r \times r}(D)) \simeq U(D)$. Hence, it suffices to treat the particular case where A is a division k-algebra D. In this case, since every finitely generated projective right D-module is free and $K_0(D^{\mathrm{op}}) \simeq K_0(D)$, the above pairing reduces to $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, (m, n) \mapsto m \cdot \dim(D) \cdot n$. This pairing is clearly non-degenerate. $\quad\square$

Notation 2.11. Given a finite G-set S, let us write k_S for the commutative separable k-algebra $\mathrm{Hom}_G(S, k_{\mathrm{sep}})$. Note that $k_{S_1} \otimes k_{S_2} \simeq k_{S_1 \times S_2}$. In the same vein, given $s \in S$, let $k_s := \mathrm{Hom}_G(Gs, k_{\mathrm{sep}})$. Note that $k_s = k_{\mathrm{sep}}^H$, where H is the stabilizer of s. Finally, given an Azumaya algebra A over k_S (see [19, §III Thm. 5.1]), let us write A_s for the central simple k_s-algebra $A \otimes_{k_S} k_s$ and $\mathrm{ind}_s(A)$ for the index of A_s.

Consider now the following two categories:
(i) Given finite G-sets S_1, S_2 and Azumaya algebras A, B over k_{S_1} and k_{S_2}, respectively, let $\mathrm{Map}^{G,A,B}(S_1 \times S_2, \mathbb{Z})$ be the subset of $\mathrm{Map}^G(S_1 \times S_2, \mathbb{Z})$ consisting of those G-invariant functions $\alpha : S_1 \times S_2 \to \mathbb{Z}$ such that $\alpha((s_1, s_2)) \in \mathrm{ind}_{(s_1, s_2)}(A^{\mathrm{op}} \otimes B) \cdot \mathbb{Z}$ for every $(s_1, s_2) \in S_1 \times S_2$. The category $\mathrm{Cov}'(G)$ is defined as follows: the objects are the pairs (S, A) with S is a finite G-set and A an Azumaya algebra over k_S; the morphisms $\mathrm{Hom}_{\mathrm{Cov}'(G)}((S_1, A), (S_2, B))$ are the functions $\mathrm{Map}^{G,A,B}(S_1 \times S_2, \mathbb{Z})$; the composition law and the identities are the same as those of the convolution category. Finally, the definitions

$$(S_1, A) \oplus (S_2, B) := (S_1 \amalg S_2, A \times B) \quad (S_1, A) \otimes (S_2, B) := (S_1 \times S_2, A \otimes B)$$

endow $\mathrm{Cov}'(G)$ with an additive symmetric monoidal structure.
(ii) Let $H, K \subseteq G$ be closed subgroups of finite index and A, B finite dimensional division k-algebras with centers k_{sep}^H and k_{sep}^K, respectively. Galois theory gives rise to the following isomorphism

$$k_{\mathrm{sep}}^H \otimes k_{\mathrm{sep}}^K \simeq \prod_{\overline{g} \in H \backslash G / K} l_{\overline{g}} \quad \text{where} \quad l_{\overline{g}} := k_{\mathrm{sep}}^{g^{-1}Hg \cap K}.$$

Making use of it, we conclude that

$$
\begin{aligned}
A^{\mathrm{op}} \otimes B & \simeq & A^{\mathrm{op}} \otimes_{k_{\mathrm{sep}}^H} (k_{\mathrm{sep}}^H \otimes k_{\mathrm{sep}}^K) \otimes_{k_{\mathrm{sep}}^K} B \\
& \simeq & \prod_{\overline{g} \in H\backslash G/K} A^{\mathrm{op}} \otimes_{k_{\mathrm{sep}}^H} l_{\overline{g}} \otimes_{k_{\mathrm{sep}}^K} B \\
& \simeq & \prod_{\overline{g} \in H\backslash G/K} (l_{\overline{g}} \otimes_{k_{\mathrm{sep}}^H} A)^{\mathrm{op}} \otimes_{l_{\overline{g}}} (l_{\overline{g}} \otimes_{k_{\mathrm{sep}}^K} B) \, .
\end{aligned}
$$

(2.12)

The $\overline{g}$-factor of (2.12) is a central simple $l_{\overline{g}}$-algebra. Thanks to the Wedderburn theorem, it can be written as $M_{r_{\overline{g}} \times r_{\overline{g}}}(D_{\overline{g}})$ for a unique integer $r_{\overline{g}} \geq 1$ and finite dimensional division k-algebra $D_{\overline{g}}$ with center $l_{\overline{g}}$. Let $\mathrm{Map}^{A,B}(H\backslash G/K, \mathbb{Z})$ be the subset of $\mathrm{Map}(H\backslash G/K, \mathbb{Z})$ consisting of those functions $\alpha : H\backslash G/K \to \mathbb{Z}$ such that $\alpha(\overline{g}) \in \mathrm{ind}_{l_{\overline{g}}}(D_{\overline{g}}) \cdot \mathbb{Z}$ for every $\overline{g} \in H\backslash G/K$. The category $\mathrm{Heck}'(G)$ is defined as follows: the objects are the pairs (H, A) with $H \subseteq G$ a closed subgroup of finite index and A a central simple k_{sep}^H-algebra; the morphisms $\mathrm{Hom}_{\mathrm{Heck}'(G)}((H, A), (K, B))$ are the functions $\mathrm{Map}^{A,B}(H\backslash G/K, \mathbb{Z})$; the composition law and the identities are the same as those of the category $\mathrm{Heck}(G)$. The closure of $\mathrm{Heck}'(G)$ under (formal) finite direct sums will be denoted by $\mathrm{Hecke}'(G)$. By construction, $\mathrm{Hecke}'(G)$ is an additive category.

The next result extends the equivalences $\mathrm{CSep}(k) \simeq \mathrm{Cov}(G) \simeq \mathrm{Hecke}(G)$ of Proposition 2.3 to possibly noncommutative separable k-algebras.

Theorem 2.13. (i) *The above data gives rise to well-defined additive categories*
 $\mathrm{Cov}'(k)$ *and* $\mathrm{Hecke}'(G)$, *with* $\mathrm{Cov}'(k)$ *being moreover symmetric monoidal.*
 (ii) *The additive categories* $\mathrm{Sep}(k), \mathrm{Cov}'(G)$ *and* $\mathrm{Hecke}'(G)$ *are equivalent.*

In what follows, we will equip $\mathrm{Hecke}'(G)$ with the symmetric monoidal structure inherited from $\mathrm{Cov}'(G)$ under the equivalence of Theorem 2.13.

Corollary 2.14. *The assignment* $A \mapsto Z(A)$ *(where* $Z(A)$ *denotes the center of* A) *can be extended to an additive symmetric monoidal functor* $Z : \mathrm{Sep}(k) \to \mathrm{CSep}(k)$. *This functor is moreover a retraction of the inclusion* $\mathrm{CSep}(k) \subset \mathrm{Sep}(k)$.

Proof. Note that we have the canonical forgetful functor

(2.15) $\mathrm{Cov}'(G) \longrightarrow \mathrm{Cov}(G) \qquad (S, A) \mapsto S$

as well as the inclusion of categories

(2.16) $\mathrm{Cov}(G) \longrightarrow \mathrm{Cov}'(G) \qquad S \mapsto (S, k_S) \, .$

Under the equivalences $\mathrm{CSep}(k) \simeq \mathrm{Cov}(G)$ and $\mathrm{Sep}(k) \simeq \mathrm{Cov}'(G)$, (2.15) (resp. (2.16)) identifies with the assignment $U(A) \mapsto U(Z(A))$ on objects (resp. with the inclusion of categories $\mathrm{CSep}(k) \subset \mathrm{Sep}(k)$). Therefore, the claim follows from the fact that the functors (2.15)-(2.16) are additive symmetric monoidal and from the equality (2.15) ∘ (2.16) = Id. $\square$

Central simple algebras. Let $\mathrm{CSA}(k)$ be the full subcategory of $\mathrm{Sep}(k)$ consisting of the objects $U(A)$ with A a central simple k-algebra, and $\mathrm{CSA}(k)^{\oplus}$ its closure under finite direct sums. Note that $\mathrm{CSA}(k)^{\oplus}$ is an additive symmetric monoidal subcategory of $\mathrm{Sep}(k)$. Moreover, by unraveling the above definitions, it is easy to

see that we have the following 2-cartesian square of categories:

(2.17)
$$\begin{array}{ccc} \{U(k)^{\oplus n} \,|\, n \geq 0\} & \longrightarrow & \mathrm{CSA}(k)^{\oplus} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{CSep}(k) & \longrightarrow & \mathrm{Sep}(k)\,. \end{array}$$

Intuitively speaking, (2.17) shows that the categories $\mathrm{CSep}(k)$ and $\mathrm{CSA}(k)^{\oplus}$ are "orthogonal", encoding respectively the commutative and the noncommutative information. As a consequence of [35, Thm. 2.1], we have $U(l)_{\mathbb{Q}} \simeq U(B)_{\mathbb{Q}}$ for every finite separable field extension l/k and central simple l-algebra B. Therefore, we obtain the following equivalences of categories:

$$\{U(k)_{\mathbb{Q}}^{\oplus n} \,|\, n \geq 0\} \simeq \mathrm{CSA}(k)_{\mathbb{Q}}^{\oplus} \qquad \mathrm{CSep}(k)_{\mathbb{Q}} \simeq \mathrm{Sep}(k)_{\mathbb{Q}}\,.$$

Roughly speaking, the rational noncommutative information disappears!

Remark 2.18 (Dg Azumaya algebras). The classical notion of Azumaya algebra can be generalized to the differential graded setting; see Appendix B. In *loc. cit.* we establish some properties of these dg Azumaya algebras and compute their noncommutative motives. In particular, we show that in this generality the rational noncommutative information does *not* disappear.

Recall from [13, Prop. 4.1.16] that every central simple k-algebra A admits a p-primary decomposition $A = \otimes_{p \in \mathcal{P}} A^p$. Here, $\mathcal{P}$ stands for the prime numbers and A^p is characterized by the fact that its index is the p-primary component of $\mathrm{ind}(A)$.

As proved in [31, Thm. 9.1], the following equivalence holds

(2.19)
$$U(A) \simeq U(B) \Leftrightarrow [A] = [B]$$

for any two central simple k-algebras A and B. The following result extends the above equivalence (2.19) to a complete dictionary between objects of the category $\mathrm{CSA}(k)^{\oplus}$ and sequences of elements in the Brauer group $\mathrm{Br}(k)$.

Theorem 2.20. *The following holds:*
(i) *The category $\mathrm{CSA}(k)^{\oplus}$ is idempotent complete.*
(ii) *The indecomposable objects of the category $\mathrm{CSA}(k)^{\oplus}$ are of the form $U(B)$, with B a central simple k-algebra.*
(iii) *Given central simple k-algebras $A_1, \ldots, A_n$, the indecomposable direct summands of $U(A_1) \oplus \cdots \oplus U(A_n)$ are of the form $U(B)$ with B a central simple k-algebra satisfying the following condition: for every $p \in \mathcal{P}$ there exists an integer $\varrho_p \in \{1, \ldots, n\}$ such that $[B^p] = [A_{\varrho_p}^p]$. Consequently, the Brauer class $[B]$ belongs to the subgroup of $\mathrm{Br}(k)$ generated by $\{[A_1], \ldots, [A_n]\}$.*
(iv) *Given central simple k-algebras $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, the following two conditions (a)-(b) are equivalent:*
   (a) *We have an isomorphism of noncommutative motives:*

(2.21)
$$U(A_1) \oplus \cdots \oplus U(A_n) \simeq U(B_1) \oplus \cdots \oplus U(B_m)\,.$$

   (b) *The equality $n = m$ holds and for every $p \in \mathcal{P}$ there exists a permutation σ_p (which depends on p) such that $[B_i^p] = [A_{\sigma_p(i)}^p]$ for every $1 \leq i \leq n$.*

Corollary 2.22. *The above isomorphism (2.21) implies that the Brauer classes $[A_1], \ldots, [A_n]$ and $[B_1], \ldots, [B_n]$ generate the same subgroup of $\mathrm{Br}(k)$. Consequently, we obtain a well-defined map $\mathrm{Iso}(\mathrm{CSA}(k)^{\oplus}) \to \{\text{subgroups of } \mathrm{Br}(k)\}$.*

Proof. Recall from [13, Prop. 4.5.13] that the index and the period of a central simple k-algebra have the same prime factors. Hence, the proof follows from the combination of Theorem 2.20(iv)(b) with the p-primary decomposition $\mathrm{Br}(k) = \prod_{p \in \mathcal{P}} \mathrm{Br}(k)\{p\}$ of the Brauer group; see [13, Prop. 4.5.16]. $\square$

Corollary 2.23. *Given central simple k-algebras $A_1, \ldots, A_n, B_1, \ldots, B_n, C$, we have the following cancellation property:*

$$\oplus_{i=1}^n U(A_i) \oplus U(C) \simeq \oplus_{i=1}^n U(B_i) \oplus U(C) \Rightarrow \oplus_{i=1}^n U(A_i) \simeq \oplus_{i=1}^n U(B_i).$$

Proof. The proof follows from the fact that for every $p \in \mathcal{P}$ the permutation σ_p of item (iv)(b) of Theorem 2.20 can be choosen to fix the Brauer class $[C^p]$; consult the proof of Theorem 2.20(iv) for details. $\square$

Remark 2.24. Thanks to Theorem 2.20 and Corollary 2.23, the additive category $\mathrm{CSA}(k)^{\oplus}$ has none of the pathologies described in Remark 2.6.

We will prove Theorem 2.20 via a ring theoretic description of certain full subcategories of $\mathrm{CSA}(k)^{\oplus}$. Given central simple k-algebras $A_1, \ldots, A_n$, let us denote by $\mathrm{CSA}(A_1, \ldots, A_n)$ the full subcategory of $\mathrm{Sep}(k)$ consisting of the objects $U(A_1), \ldots, U(A_n)$. Its closure under finite direct sums (resp. finite direct sums and direct factors) will be denoted by $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus}$ (resp. $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus,\natural}$). Consider the following ring $\Lambda(A_1, \ldots, A_n)$ of $n \times n$ matrices

$$\Lambda(A_1, \ldots, A_n)_{i,j} := \mu_{ij}\mathbb{Z} \qquad 1 \leq i, j \leq n,$$

where $\mu_{ij} := \mathrm{ind}(A_i^{\mathrm{op}} \otimes A_j)$. Since $\mu_{ij} = \mu_{ji}$ and $\mu_{ii} = 1$, we have

$$(2.25) \qquad \Lambda(A_1, \ldots, A_n) = \begin{pmatrix} \mathbb{Z} & \mu_{12}\mathbb{Z} & \cdots & \mu_{1n}\mathbb{Z} \\ \mu_{12}\mathbb{Z} & \mathbb{Z} & \cdots & \mu_{2n}\mathbb{Z} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{1n}\mathbb{Z} & \mu_{2n}\mathbb{Z} & \cdots & \mathbb{Z} \end{pmatrix}_{n \times n}.$$

Proposition 2.26. *The following holds:*
 (i) *We have a ring isomorphism $\mathrm{End}(\oplus_i U(A_i)) \simeq \Lambda(A_1, \ldots, A_n)$.*
 (ii) *We have an additive equivalence[2] of categories*

$$\varphi : \mathrm{CSA}(A_1, \ldots, A_n)^{\oplus,\natural} \simeq \mathrm{Proj}(\Lambda(A_1, \ldots, A_n)) \qquad U(A) \mapsto \mathrm{Hom}(\oplus_i U(A_i), U(A)),$$

   where Proj *stands for the category of finitely generated projective right modules.*

Remark 2.27. The ring $\Lambda(A_1, \ldots, A_n)$ is a so-called $\mathbb{Z}$-*order*, i.e. it is free of finite type as a $\mathbb{Z}$-module and the quotient ring is a central simple $\mathbb{Q}$-algebra. This class of rings plays a central role in integral representation theory; see Curtis-Reiner [9] for instance. We were quite intrigued by the above connection between the classical theory of $\mathbb{Z}$-orders and the recent theory of noncommutative motives.

Assume now that the Brauer classes $[A_1], \ldots, [A_n]$ form a subgroup H of $\mathrm{Br}(k)$. This implies that the category $\mathrm{CSA}(A_1, \ldots, A_n)$ is symmetric monoidal and that $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus}$ and $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus,\natural}$ are additive symmetric monoidal.

Proposition 2.28. *Under the above assumption, the indecomposable objects in $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus,\natural}$ are of the form $U(A_i)$. Consequently, $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus}$ is idempotent complete.*

---

[2]This equivalence holds also when n is infinite, provided we replace the ring $\Lambda(A_1, \ldots, A_n)$ by a category. We leave these straightforward generalizations to the reader.

Under the above assumption, we can give an alternative description of the category $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus}$ which is compatible with the symmetric monoidal structure. Consider the following H-graded subring of $\mathbb{Z}H$

$$(2.29) \qquad \Sigma(A_1, \ldots, A_n) := \bigoplus_{i=1}^{n} \mu_i \mathbb{Z}[A_i] \subset \mathbb{Z}H \,,$$

where $\mu_i := \mathrm{ind}(A_i)$. Note that (2.29) is indeed a subring of $\mathbb{Z}H$ since $\mathrm{ind}(A_i \otimes A_j)$ divides $\mathrm{ind}(A_i) \cdot \mathrm{ind}(A_j)$. Given any H-graded commutative ring Σ and $h \in H$, we will write $\Sigma(h)$ for the graded projective Σ-bimodule $\Sigma(h)_g := \Sigma_{hg}$.

Proposition 2.30. *Assume that* $[A_1], \ldots, [A_n]$ *is a subgroup of the Brauer group. Then, we have an additive* symmetric monoidal *equivalence*[3] *of categories*

$$\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus} \simeq \mathrm{Proj}_{\mathrm{gr}}(\Sigma(A_1, \ldots, A_n)) \qquad U(A_i) \mapsto \Sigma(A_1, \ldots, A_n)([A_i]) \,.$$

Let A be a central simple k-algebra. As explained above, the (idempotent complete) category $\mathrm{CSA}(k, A, A^{\otimes 2}, \ldots, A^{\otimes(\mathrm{per}(A)-1)})^{\oplus}$ is additive symmetric monoidal. Hence, we can consider the associated Grothendieck ring

$$(2.31) \qquad K_0(\mathrm{CSA}(k, A, A^{\otimes 2}, \ldots, A^{\otimes(\mathrm{per}(A)-1)})^{\oplus}) \,.$$

Theorem 2.32. *Let* $\prod_i p_i^{r_i}$ *be the p-primary decomposition of* $\mathrm{per}(A)$. *Under these notations, the assignment* $t \mapsto U(A)$ *gives rise to a ring isomorphism*

$$\mathbb{Z}[t]/\langle (1-t) \prod_i (1 + t + \cdots + t^{p_i^{r_i}-1}) \rangle \simeq (2.31) \,.$$

Thanks to Theorem 2.32, the rank of (2.31) is equal to $\sum_i (p_i^{r_i} - 1) + 1$. Note that this number is strictly inferior to the period $\mathrm{per}(A) = \prod_i p_i^{r_i}$ whenever there are at least two distinct prime numbers.

3. APPLICATIONS

Motivic relations. Let A be a central simple k-algebra, $\prod_i p_i^{r_i}$ the p-primary decomposition of $\mathrm{per}(A)$, and $p_A(t)$ the polynomial $\prod_i (1 + t + \cdots + t^{p_i^{r_i}-1})$.

Notation 3.1. Given a polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in \mathbb{N}[t]$ and an additive invariant $E : \mathrm{dgcat}(k) \to \mathrm{D}$, let $E(A; p(t))$ be the following direct sum

$$(3.2) \qquad E(k)^{\oplus a_0} \oplus E(A)^{\oplus a_1} \oplus E(A^{\otimes 2})^{\oplus a_2} \oplus \cdots \oplus E(A^{\otimes n})^{\oplus a_n} \in \mathrm{D} \,.$$

Our first family of motivic relations is the following:

Proposition 3.3. *Under the above notations, we have* $E(A; p_A(t)) \simeq E(A; t p_A(t))$.

Proof. Consider the noncommutative motives $U(A; p_A(t))$ and $U(A; t p_A(t))$. Theorem 2.32 implies that their classes in the Grothendieck ring (2.31) are the same. Hence, making use of the cancellation property of Corollary 2.23, we conclude that their are isomorphic; see [37, II §2 Cor. 1.2]. The proof follows now from the equivalence of categories (1.1). $\square$

Example 3.4. Let A be the reader's favorite central simple k-algebra with $\mathrm{per}(A) = 6$. In this particular case, $p_A(t) = (1+t)(1+t+t^2)$. By combining Proposition 3.3 with Corollary 2.23, we hence obtain $E(k) \oplus E(A) \simeq E(A^{\otimes 3}) \oplus E(A^{\otimes 4})$.

---

[3]This equivalence holds also when n is infinite.

Our second family of motivic relations, which greatly generalizes the preceding Example 3.4, is the following:

Proposition 3.5. *Let A, B, C be central simple k-algebras and $E : \mathrm{dgcat}(k) \to \mathrm{D}$ an additive invariant. Assume that $\mathrm{ind}(A)$ and $\mathrm{ind}(B)$ are coprime. Under these assumptions, we have an isomorphism*

$$(3.6) \qquad E(C) \oplus E(A \otimes B \otimes C) \simeq E(A \otimes C) \oplus E(B \otimes C) \,.$$

Proof. Let $\prod_i p_i^{r_i}$ (resp. $\prod_j q_j^{s_j}$) be the p-primary decomposition of $\mathrm{ind}(A)$ (resp. $\mathrm{ind}(B)$). Since $\mathrm{ind}(A)$ and $\mathrm{ind}(B)$ are coprime, $p_i \neq q_j$ for every i and j. Consequently, we have $A^{q_j} = k$ and $B^{p_i} = k$. This implies that $(A \otimes B \otimes C)^{q_j} = B^{q_j} \otimes C^{q_j}$ and $(A \otimes B \otimes C)^{p_i} = A^{p_i} \otimes C^{p_i}$. Making use of Theorem 2.20(iv)(b), we hence obtain an isomorphism $U(C) \oplus U(A \otimes B \otimes C) \simeq U(A \otimes C) \oplus U(B \otimes C)$. The proof follows now from the equivalence of categories (1.1). $\square$

Example 3.7. Wedderburn theorem implies that $A \simeq M_{r \times r}(D_A), B \simeq M_{s \times s}(D_B)$, and $A \otimes B \simeq M_{t \times t}(D_{A \otimes B})$, for unique integers $r, s, t \geq 1$ and division k-algebras D_A, D_B and $D_{A \otimes B}$. Therefore, the above isomorphism (3.6), with $C = k$ and E equal to the first algebraic K-theory group (see [13, §2.8]), reduces to

$$(3.8) \qquad k^\times \times D_{A \otimes B}^\times / [D_{A \otimes B}^\times, D_{A \otimes B}^\times] \simeq D_A^\times / [D_A^\times, D_A^\times] \times D_B^\times / [D_B^\times, D_B^\times] \,.$$

To the best of the authors knowledge, all the above relations (3.6) (in particular (3.8)) are new in the literature. More generally, given central simple k-algebras $A_1, \ldots, A_n, B_1, \ldots, B_n$ satisfying Theorem 2.20(iv)(b), we have an isomorphism

$$(3.9) \qquad E(A_1) \oplus \cdots \oplus E(A_n) \simeq E(B_1) \oplus \cdots \oplus E(B_n) \,.$$

These latter relations can be nevertheless recovered from the previous ones:

Proposition 3.10. (i) *The above relations (3.6) imply all the relations (3.9).*
 (ii) *When E is moreover symmetric monoidal, the above relations (3.6) with $C = k$ imply all the relations (3.9).*

Severi-Brauer varieties. Let A be a central simple k-algebra and $\mathrm{SB}(A)$ the associated Severi-Brauer variety. As proved in [6, Prop. 2.8], we have an isomorphism

$$(3.11) \qquad U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{SB}(A))) \simeq U(k) \oplus U(A) \oplus U(A)^{\otimes 2} \oplus \cdots \oplus U(A)^{\otimes (\deg(A) - 1)} \,.$$

As an application of the above Theorem 2.20(iv), we obtain the following result:

Theorem 3.12. *Given central simple k-algebras A and B with the same degree, the following three conditions are equivalent:*
 (i) *We have an isomorphism $U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{SB}(A))) \simeq U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{SB}(B)))$.*
 (ii) *The Brauer classes $[A]$ and $[B]$ generate the same subgroup of $\mathrm{Br}(k)$.*
 (iii) *There exists an integer i, coprime to $\mathrm{per}(A)$, such that $[B] = [A]^i$.*

Remark 3.13. N. Karpenko proved in [14, Criterion 7.1] that we have an isomorphism between Chow motives $M(\mathrm{SB}(A)) \simeq M(\mathrm{SB}(B))$ if and only if A is isomorphic to B or to B^{op}. Roughly speaking, this shows that conditions (ii) $\Leftrightarrow$ (iii) of Theorem 3.12 are truly a noncommutative phenomenon.

Corollary 3.14. *Let A be a central simple k-algebra, i an integer coprime to $\mathrm{per}(A)$, and $E : \mathrm{dgcat}(k) \to \mathrm{D}$ an additive invariant. Under these assumptions, we have an isomorphism $E(\mathrm{SB}(A)) \simeq E(\mathrm{SB}(B))$, where B stands for the unique central simple k-algebra satisfying the conditions $\deg(B) = \deg(A)$ and $[B] = [A]^i$.*

The following result relates noncommutative motives with birationality:

Proposition 3.15. *The universal additive invariant U (and therefore every additive invariant) is a birational invariant of Severi-Brauer varieties.*

Proof. As proved by Amitsur in [1, §9], if two Severi-Brauer varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(B)$ are birational, then A and B have the same degree and $[A]$ and $[B]$ generate the same subgroup of $\mathrm{Br}(k)$. The proof follows then from Theorem 3.12. $\qquad\square$

Remark 3.16 (Amitsur's conjecture). S. Amitsur conjectured in [1] that if two central simple k-algebras A and B have the same degree and $[A]$ and $[B]$ generate the same subgroup of $\mathrm{Br}(k)$, then $\mathrm{SB}(A)$ and $\mathrm{SB}(B)$ are birational. Thanks to Proposition 3.15, Amitsur's conjecture can be reformulated as follows:

Conjecture: U is a complete birational invariant of Severi-Brauer varieties.

Twisted flag varieties. Let A be a central simple k-algebra, $d_1, \ldots, d_m$, $m \geq 1$, positive integers such that $\sum_i d_i = \deg(A)$, and $\mathrm{Flag}(d_1, \ldots, d_m; A)$ the associated twisted flag variety. Recall that when $m = 1$, $\mathrm{Flag}(d_1; A)$ identifies with the twisted Grassmanian variety $\mathrm{Gr}(d_1; A)$ and when $d_1 = 1$, $\mathrm{Gr}(d_1; A)$ reduces to $\mathrm{SB}(A)$.

Notation 3.17. Given an additive symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, an object $b \in \mathcal{C}$, and a polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in \mathbb{N}[t]$, let $p(b)$ be the direct sum $\mathbf{1}^{\oplus a_0} \oplus b^{\oplus a_1} \oplus (b^{\otimes 2})^{\oplus a_2} \oplus \cdots \oplus (b^{\otimes n})^{\oplus a_n} \in \mathcal{C}$. Note that the above direct sum (3.2) reduces to $p(E(A))$ when E is moreover symmetric monoidal.

Theorem 3.18. *We have the following motivic decomposition*

$$(3.19) \qquad U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Flag}(d_1, \ldots, d_m; A))) \simeq \binom{\deg(A)}{d_1 \cdots d_m}_{U(A)},$$

where $\binom{\deg(A)}{d_1 \cdots d_m}_t$ stands for the Gaussian polynomial; see Example A.1.

Note that (3.19) generalizes the above motivic decomposition (3.11). In the same vein, the following results generalize Theorem 3.12 and Corollary 3.14.

Theorem 3.20. *Given central simple k-algebras A and B with the same degree, the following three conditions are equivalent:*

(i) *We have an isomorphism of noncommutative motives*

$$U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Flag}(d_1, \ldots, d_m; A))) \simeq U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Flag}(d_1, \ldots, d_m; B))).$$

(ii) *The Brauer classes $[A]$ and $[B]$ generate the same subgroup of $\mathrm{Br}(k)$.*
(iii) *There exists an integer i, coprime to $\mathrm{per}(A)$, such that $[B] = [A]^i$.*

Proof. Assume condition (i). Thanks to isomorphism (3.19) and Corollary 2.22, the Brauer classes $[k], [A], \ldots, [A^{\otimes(\mathrm{per}(A)-1)}]$ and $[k], [B], \ldots, [B^{\otimes(\mathrm{per}(B)-1)}]$ generate the same subgroup of $\mathrm{Br}(k)$. This implies condition (ii). Conditions (ii) and (iii) are clearly equivalent. Assume now condition (iii). Thanks to the equivalence

(2.19), we have $U(B) \simeq U(A)^{\otimes i}$. Hence, condition (i) follows from

$$
\begin{aligned}
U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Flag}(d_1, \ldots, d_m; B))) &\simeq \begin{pmatrix} \deg(B) \\ d_1 \cdots d_m \end{pmatrix}_{U(B)} \\
&\simeq \begin{pmatrix} \deg(A) \\ d_1 \cdots d_m \end{pmatrix}_{U(A)^{\otimes i}} \\
&\simeq \begin{pmatrix} \deg(A) \\ d_1 \cdots d_m \end{pmatrix}_{U(A)} \\
&\simeq U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Flag}(d_1, \ldots, d_m; A))),
\end{aligned}
$$

(3.21)

where (3.21) follows from the cyclic sieving phenomenon; see Corollary A.4.   □

Corollary 3.22. *Let A be a central simple k-algebra, $d_1, \ldots, d_m, m \geq 1$, positive integers such that $\sum_i d_i = \deg(A)$, i an integer coprime to $\mathrm{per}(A)$, and $E : \mathrm{dgcat}(k) \to \mathrm{D}$ an additive invariant. Under these assumptions, we have*

$$
E(\mathrm{Flag}(d_1, \ldots, d_m; A)) \simeq E(\mathrm{Flag}(d_1, \ldots, d_m; B)),
$$

where B stands for the unique central simple k-algebra satisfying the conditions $\deg(B) = \deg(A)$ and $[B] = [A]^i$.

Proof. It follows from the combination of Theorem 3.20 with equivalence (1.1).   □

4. Background on dg categories

Let $\mathcal{C}(k)$ be the symmetric monoidal category of cochain complexes of k-vector spaces. A *dg category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [15] for further details.

Let $\mathcal{A}$ be a dg category. Its opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A right $\mathcal{A}$-module is a dg functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k-vector spaces. Let $\mathcal{C}(\mathcal{A})$ be the category of right $\mathcal{A}$-modules. Following [15, §3], the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of objectwise quasi-isomorphisms. Its full subcategory of compact objects will be denoted by $\mathcal{D}_c(\mathcal{A})$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence $\mathcal{D}(\mathcal{A}) \overset{\sim}{\to} \mathcal{D}(\mathcal{B})$ on derived categories; see [15, §4.6]. As proved in [34, Thm. 5.3], $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let $\mathrm{Hmo}(k)$ be the homotopy category hence obtained.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A} \otimes \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$. As explained in [15, §2.3], this construction gives rise to symmetric monoidal categories $(\mathrm{dgcat}(k), -\otimes-, k)$ and $(\mathrm{Hmo}(k), -\otimes-, k)$. Finally, given dg categories $\mathcal{A}$ and $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule is a dg functor $\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$, *i.e.* a right $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module. Associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$, we have the $\mathcal{A}$-$\mathcal{B}$-bimodule

(4.1) $_F\mathcal{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}(k)$ $(x, w) \mapsto \mathcal{B}(w, F(x))$

as well as the $\mathcal{B}$-$\mathcal{A}$-bimodule

$$(4.2) \qquad \mathcal{B}_F : \mathcal{B} \otimes \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}(k) \qquad (w, x) \mapsto \mathcal{B}(F(x), w) \,.$$

5. Background on noncommutative motives

5.1. **Additive invariants.** Given a dg category $\mathcal{A}$, let $T(\mathcal{A})$ be the dg category of pairs (i, x), where $i \in \{1, 2\}$ and $x \in \mathcal{A}$. The complex of morphisms in $T(\mathcal{A})$ from (i, x) to (i', x') is given by $\mathcal{A}(x, x')$ if $i' \geq i$ and is zero otherwise. Composition is induced by $\mathcal{A}$. Intuitively speaking, $T(\mathcal{A})$ "dg categorifies" the notion of upper triangular matrix. Note that we have two inclusion dg functors $i_1, i_2 : \mathcal{A} \hookrightarrow T(\mathcal{A})$.

Recall from [34] that a functor $E : \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category D, is called an *additive invariant* if it satisfies the following conditions:

(i) it sends Morita equivalences to isomorphisms;

(ii) given a dg category $\mathcal{A}$, the inclusion dg functors i_1, i_2 induce an isomorphism[4]

$$E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(T(\mathcal{A})) \,.$$

Examples of additive invariants include algebraic K-theory, cyclic homology (and all its variants), topological Hochschild homology, etc; consult the survey [33]. When applied to the dg categories $\mathrm{perf}_{\mathrm{dg}}(X)$, these invariants agree with the corresponding invariants of the quasi-compact quasi-separated k-schemes X.

5.2. **Universal additive invariant.** Given dg categories $\mathcal{A}$ and $\mathcal{B}$, let $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every $x \in \mathcal{A}$ the associated right $\mathcal{B}$-module $\mathrm{B}(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. As proved in [34, Cor. 5.10], there is a bijection $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Iso} \, \mathrm{rep}(\mathcal{A}, \mathcal{B})$ under which the composition law of $\mathrm{Hmo}(k)$ corresponds to the (derived) tensor product of bimodules. Since the above $\mathcal{A}$-$\mathcal{B}$-bimodules (4.1) clearly belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, we hence obtain a well-defined symmetric monoidal functor

$$(5.1) \qquad \mathrm{dgcat}(k) \longrightarrow \mathrm{Hmo}(k) \qquad F \mapsto {}_F\mathcal{B} \,.$$

The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects as $\mathrm{Hmo}(k)$ and with morphisms given by $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) := K_0\mathrm{rep}(\mathcal{A}, \mathcal{B})$, where $K_0\mathrm{rep}(\mathcal{A}, \mathcal{B})$ stands for the Grothendieck group of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. The composition law is induced by the (derived) tensor product of bimodules and the symmetric monoidal structure extends by bilinearity from $\mathrm{Hmo}(k)$ to $\mathrm{Hmo}_0(k)$. Note that we have also a well-defined symmetric monoidal functor

$$(5.2) \qquad \mathrm{Hmo}(k) \longrightarrow \mathrm{Hmo}_0(k) \qquad \mathrm{B} \mapsto [\mathrm{B}] \,.$$

As proved in [34, Thms. 5.3 and 6.3], the universal additive invariant U is obtained by composing the above functors (5.1) and (5.2). Finally, recall from [34, Lem. 4.2] that $U(\mathcal{A}) \oplus U(\mathcal{B}) \simeq U(\mathcal{A} \times \mathcal{B}) \simeq U(\mathcal{A} \amalg \mathcal{B})$ for any two dg categories $\mathcal{A}$ and $\mathcal{B}$.

---

[4]Condition (ii) can be equivalently formulated in terms of semi-orthogonal decompositions in the sense of Bondal-Orlov [7]; see [34, Thm. 6.3(4)].

5.3. **Coefficients.** Let R be a commutative ring. The R-*linearization* of $\mathrm{Hmo}_0(k)$ is the R-linear category $\mathrm{Hmo}_0(k)_R$ obtained by tensoring the morphisms of $\mathrm{Hmo}_0(k)$ with R. Note that $\mathrm{Hmo}_0(k)_R$ inherits a R-linear symmetric monoidal structure and that we have a well-defined symmetric monoidal functor

$$(5.3) \qquad \mathrm{Hmo}_0(k) \longrightarrow \mathrm{Hmo}_0(k)_R \qquad [\mathrm{B}] \mapsto [\mathrm{B}] \otimes_{\mathbb{Z}} R \,.$$

The *universal additive invariant with R-coefficients* $U(-)_R$ is obtained by composing the universal additive invariant U with (5.3). Given any R-linear additive category D, there is an induced equivalence of categories

$$(5.4) \qquad U(-)_R^* : \mathrm{Fun}(\mathrm{Hmo}_0(k)_R, \mathrm{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}) \,,$$

where the left-hand side denotes the category of R-linear additive functors.

6. Proof of Theorem 1.2

Let $\mathcal{A}$ be a smooth proper dg category. Since $\mathrm{NNum}(k)$ is a rigid symmetric monoidal category, it suffices to show that $\mathrm{Hom}_{\mathrm{NNum}(k)}(U(k), U(\mathcal{A}))$ is a finitely generated abelian group. Recall from [22, §4] that the Grothendieck group $K_0(\mathcal{A}) := K_0(\mathcal{D}_c(\mathcal{A}))$ of $\mathcal{A}$ comes equipped with the bilinear form:

$$\chi : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \longrightarrow \mathbb{Z} \qquad (P, Q) \mapsto \sum_i (-1)^i \dim \mathrm{Hom}_{\mathcal{D}_c(\mathcal{A})}(P, Q[i]) \,.$$

This form is in general not symmetric neither anti-symmetric. However, as proved in [22, Thm. 4.3], the left and right kernels agree; let $\mathrm{Ker}(\chi)$ be the resulting kernel. Note now that the proof of [22, Thm. 1.1] (with F replaced by $\mathbb{Z}$) implies that

$$(6.1) \qquad \mathrm{Hom}_{\mathrm{NNum}(k)}(U(k), U(\mathcal{A})) \simeq K_0(\mathcal{A})/_{\mathrm{Ker}(\chi)} \,.$$

Hence, it is enough to show that the right-hand side of (6.1) is finitely generated. As explained in [23, §4], the $\otimes$-ideal $\mathcal{N}$ is compatible with extension of scalars. Consequently, we have isomorphisms of $\mathbb{Q}$-vector spaces:

$$(K_0(\mathcal{A})/_{\mathrm{Ker}(\chi)})_{\mathbb{Q}} \simeq K_0(\mathcal{A})_{\mathbb{Q}}/_{\mathrm{Ker}(\chi_{\mathbb{Q}})} \simeq \mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, U(\mathcal{A})_{\mathbb{Q}}) \,.$$

Since by assumption k is of characteristic zero, the category $\mathrm{NNum}(k)_{\mathbb{Q}}$ is abelian semi-simple; see [20, Thm. 1.10]. This implies in particular that the above $\mathbb{Q}$-vector spaces are finite dimensional. Consider now the injective group homomorphism

$$(6.2) \qquad K_0(\mathcal{A})/_{\mathrm{Ker}(\chi)} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{A})/_{\mathrm{Ker}(\chi)}, \mathbb{Z})$$

induced by the above bilinear form χ. Since the $\mathbb{Q}$-vector space $(K_0(\mathcal{A})/_{\mathrm{Ker}(\chi)})_{\mathbb{Q}}$ is finite dimensional, the right-hand side of (6.2) is finitely generated. Hence, we conclude finally that the abelian group $K_0(\mathcal{A})/_{\mathrm{Ker}(\chi)}$ is also finitely generated.

7. Proof of Proposition 2.3

Consider the following composition

$$(7.1) \qquad \mathrm{Etale}(k) \subseteq \mathrm{Chow}(k) \xrightarrow{\pi} \mathrm{Chow}(k)/_{-\otimes \mathbb{Z}(1)} \,,$$

where $\mathbb{Z}(1)$ stands for the Tate motive and $\mathrm{Chow}(k)/_{-\otimes \mathbb{Z}(1)}$ for the orbit category; see [32, §7]. As explained in the proof of [21, Thm. 1.1], the functor (7.1) is not only additive and symmetric monoidal but moreover fully-faithful. Recall now from [6,

Props. 4.1 and 6.6] the construction of the additive symmetric monoidal functor Ψ making the following diagram commute

$$(7.2) \qquad \begin{array}{ccc} \text{etale}(k)^{\mathrm{op}} & = & \text{etale}(k)^{\mathrm{op}} \\ \scriptstyle{M}\downarrow & & \\ \text{Etale}(k) & & \downarrow \scriptstyle{X \mapsto U(\mathrm{perf}_{\mathrm{dg}}(X))} \\ \scriptstyle{(7.1)}\downarrow & & \\ \text{Chow}(k)/_{-\otimes\mathbb{Z}(1)} & \xleftarrow{\Psi} & \text{CSep}(k) \,, \end{array}$$

where $\text{etale}(k)$ stands for the category of finite étale k-schemes. Since Ψ is symmetric monoidal we conclude automatically from (7.2) that $\text{Etale}(k) \simeq \text{CSep}(k)$.

Let us now prove that $\text{Cov}(G) \simeq \text{Perm}(G)$. Given finite G-sets S_1 and S_2, a simple verification shows that the assignments $S_1 \mapsto \mathbb{Z}[S_1]$ and

$$\text{Map}^G(S_1 \times S_2, \mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[S_1], \mathbb{Z}[S_2]) \qquad \alpha \mapsto (s_1 \mapsto \sum_{s_2 \in S_2} \alpha(s_1, s_2)s_2)$$

give rise to a functor $\text{Cov}(G) \to \text{Perm}(G)$. Thanks to the natural identifications

$$\mathbb{Z}[S] \overset{s \mapsto \delta_s}{\simeq} \text{Fun}(S, \mathbb{Z}) \qquad \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2] \simeq \mathbb{Z}[S_1 \amalg S_2] \,,$$

this latter functor is moreover fully-faithful and additive. By definition of $\text{Perm}(G)$, we conclude that it is furthermore essentially surjective and hence and equivalence.

Let us now prove that $\text{Hecke}(G) \simeq \text{Cov}(G)$. Given closed subgroups $H, K \subseteq G$ of finite index, we claim that the assignments $H \mapsto G/H$ and

$$(7.3) \quad \text{Map}(H\backslash G/K, \mathbb{Z}) \longrightarrow \text{Map}^G(G/H \times G/K, \mathbb{Z}) \quad \alpha \mapsto \alpha'(\overline{g_1}, \overline{g_2}) := \alpha(g_1^{-1}g_2)$$

give rise to a fully-faithful functor $\text{Heck}(G) \to \text{Cov}(G)$. Clearly, the identities are preserved. Given closed subgroups $H, K, L \subseteq G$ and functions $\alpha \in \text{Map}(H\backslash G/K, \mathbb{Z})$ and $\beta \in \text{Map}(K\backslash G/L, \mathbb{Z})$, the value of $\alpha' * \beta'$ (resp. $(\alpha \star \beta)'$) at $(\overline{g_1}, \overline{g_3})$ equals

$$\sum_{\overline{g_2} \in G/K} \alpha(g_1^{-1}g_2) \cdot \beta(g_2^{-1}g_3) \qquad (\text{resp. } \sum_{\overline{h} \in G/K} \alpha(h^{-1}) \cdot \beta(hg_1^{-1}g_3)) \,.$$

Since the right-hand side can be obtained from the left-hand side via the substitution $\overline{g_2} \mapsto g_1\overline{h^{-1}}$, we hence conclude that $\alpha' * \beta' = (\alpha \star \beta)'$, i.e. that the composite law is also preserved. In what concerns fully-faithfulness, note that (7.3) is an isomorphism; it inverse is given by $\gamma \mapsto \gamma(\overline{1}, -)$. This implies our claim. Now, consider the (unique) fully-faithful functor $\text{Hecke}(G) \to \text{Cov}(G)$ which extends the above functor and preserves finite direct sums. Since every finite G-set S decomposes into the disjoint union $\amalg_i G/H_i$ of its orbits, we conclude (using Galois theory) that the latter functor is moreover essentially surjective and hence an equivalence.

Remark 7.4. As the above proof shows, given an arbitrary group G, the categories $\text{Cov}(G), \text{Hecke}(G)$ and $\text{Perm}(G)$ are equivalent.

Remark 7.5. By composing $\text{Cov}(G) \simeq \text{Perm}(G)$ with the inverse of (2.2), we obtain the additive symmetric monoidal equivalence of categories (see Notation 2.11):

$$(7.6) \qquad\qquad \text{Cov}(G) \overset{\simeq}{\longrightarrow} \text{CSep}(k) \qquad S \mapsto U(k_S) \,.$$

Given finite G-sets S_1 and S_2, the induced isomorphism

$$\text{Map}^G(S_1 \times S_2, \mathbb{Z}) \overset{\sim}{\longrightarrow} \text{Hom}_{\text{CSep}(k)}(U(k_{S_1}), U(k_{S_2})) \simeq K_0(k_{S_1 \times S_2})$$

sends the characteristic function $\delta_{(s_1,s_2)}$ of the G-orbit of $(s_1,s_2) \in S_1 \times S_2$ to the class $[k_{(s_1,s_2)}]$ of the finitely generated projective right $k_{S_1 \times S_2}$-module $k_{(s_1,s_2)}$.

8. Proof of Theorem 2.8

Note first that every ring homomorphism $R \to R'$ gives rise to a well-defined functor $\mathrm{Hmo}_0(k)_R \to \mathrm{Hmo}_0(k)_{R'}$. Hence, since $\mathbb{Z}[1/p]$ is initial among all the rings containing $1/p$, it suffices to prove the particular case $R = \mathbb{Z}[1/p]$.

Let us denote by $\iota : k \to l$ the field extension homomorphism. Recall from (4.1)-(4.2) that ι gives rise to a k-l-bimodule ${}_\iota l$ and also to a l-k-bimodule l_ι. Note that since the extension l/k is finite, $l_\iota \in \mathrm{rep}(l,k)$. Consider now the composition

$$(8.1) \qquad U(k) \xrightarrow{[{}_\iota l]} U(l) \xrightarrow{[l_\iota]} U(k).$$

By definition of the category $\mathrm{Hmo}_0(k)$, (8.1) identifies with the Grothendieck class $[l] \in K_0\mathrm{rep}(k,k) \simeq K_0(k) \simeq \mathbb{Z}$. Hence, since $[l:k] = p^r$, we conclude that (8.1) is equal to $p^r \cdot \mathrm{id}_{U(k)}$. Consider now the other composition

$$(8.2) \qquad U(l) \xrightarrow{[l_\iota]} U(k) \xrightarrow{[{}_\iota l]} U(l).$$

In this case, (8.2) identifies with the class $[l \otimes_k l] \in K_0\mathrm{rep}(l,l) \simeq K_0(l \otimes l)$. Since by hypothesis l/k is a purely inseparable field extension, [35, Lem. 9.6] implies that the k-algebra $l \otimes_k l$ is local. Hence, $K_0(l \otimes_k l) \simeq \mathbb{Z}$. Using the equality

$$\frac{\dim_k(l \otimes_k l)}{\dim_k(l)} = \frac{p^{r^2}}{p^r} = p^r$$

and the fact that the identity endomorphism of $U(l)$ corresponds to the class $[l] \in K_0\mathrm{rep}(l,l)$, we conclude that (8.2) is equal to $p^r \cdot \mathrm{id}_{U(l)}$. The proof follows now from the fact that the image of (8.1) (or (8.2)) under the functor (5.3) (with $R = \mathbb{Z}[1/p]$) is an isomorphism in $\mathrm{Hmo}_0(k)_{\mathbb{Z}[1/p]}$.

9. Proof of Theorem 2.13

For technical reasons, we will replace the categories of separable k-algebras and G-sets by their skeletons. Clearly, this procedure preserves the associated motivic categories up to equivalence. We treat first the case of the category $\mathrm{Cov}'(G)$. Consider it as a *graph*, *i.e.* as a category without units and composition. Let us start by constructing a graph isomorphism $Q : \mathrm{Cov}'(G) \xrightarrow{\cong} \mathrm{Sep}(k)$. On objects we set $Q((S,A)) := U(A)$. This assignment establishes a bijection (because we are working with skeletal categories!) between the objects of $\mathrm{Cov}'(G)$ and the objects of $\mathrm{Sep}(k)$. Its inverse is given by $U(A) \mapsto (\mathrm{Hom}_{k\text{-alg}}(Z(A), k_{\mathrm{sep}}), A)$, where $Z(A)$ stands for the center of A. Given objects (S_1, A) and (S_2, B) of $\mathrm{Cov}'(G)$ and $(s_1, s_2) \in S_1 \times S_2$, consider the following G-invariant map

$$\delta'_{(s_1,s_2)}(t_1,t_2) := \begin{cases} \mathrm{ind}_{(s_1,s_2)}(A^{\mathrm{op}} \otimes B) & \text{when } (t_1,t_2) \in G(s_1,s_2) \\ 0 & \text{otherwise}. \end{cases}$$

Note that these maps form a $\mathbb{Z}$-basis of $\mathrm{Map}^{G,A,B}(S_1 \times S_2, \mathbb{Z})$. Making use of them, we set $Q(\delta'_{(s_1,s_2)}) := [I_{(s_1,s_2)}] \in K_0(A^{\mathrm{op}} \otimes B)$, where $I_{(s_1,s_2)}$ stands for the minimal ideal of the central simple $k_{(s_1,s_2)}$-algebra $(A^{\mathrm{op}} \otimes B)_{(s_1,s_2)}$. Thanks to Lemma 9.3(i) below (with A replaced by $A^{\mathrm{op}} \otimes B$), we obtain an isomorphism

$$\mathrm{Map}^{G,A,B}(S_1 \times S_2, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sep}(k)}(U(A), U(B)) \simeq K_0(A^{\mathrm{op}} \otimes B).$$

This concludes the construction of the graph isomorphism Q. To every dg category $\mathcal{A}$ we can associate the dg k_{sep}-linear category $\mathcal{A} \otimes k_{\mathrm{sep}}$ obtained by tensoring the cochain complexes of k-vector spaces $\mathcal{A}(x,y)$ with k_{sep}. This assignment is functorial on $\mathcal{A}$ and, as proved in [21, §7], gives rise to an additive symmetric monoidal functor $- \otimes k_{\mathrm{sep}} : \mathrm{NChow}(k) \to \mathrm{NChow}(k_{\mathrm{sep}})$. Making use of it, consider now the following diagram of graphs

$$(9.1) \qquad \begin{array}{ccccccc} \mathrm{Cov}'(G) & \xrightarrow[\simeq]{Q} & \mathrm{Sep}(k) & \xrightarrow{-\otimes k_{\mathrm{sep}}} & \mathrm{Sep}(k_{\mathrm{sep}}) & \xrightarrow{\simeq} & \mathrm{CSep}(k_{\mathrm{sep}}) \\ {\scriptstyle (S,A)\mapsto S}\downarrow & & \downarrow{\scriptstyle \phi} & & & & \| \\ \mathrm{Cov}(G) & \xrightarrow[(7.6)]{\simeq} & \mathrm{CSep}(k) & \xrightarrow{\qquad\qquad -\otimes k_{\mathrm{sep}} \qquad\qquad} & & & \mathrm{CSep}(k_{\mathrm{sep}})\,, \end{array}$$

where ϕ is the unique graph morphism making the left-hand side square commute. Given an object (S, A) of $\mathrm{Cov}'(G)$, we have $U(A \otimes k_{\mathrm{sep}}) \simeq U(k_S \otimes k_{\mathrm{sep}})$. Moreover, thanks to Lemma 9.3(ii) below (with A replaced by $A^{\mathrm{op}} \otimes B$), the following equality

$$[I_{(s_1,s_2)} \otimes k_{\mathrm{sep}}] = \mathrm{ind}_{(s_1,s_2)}(A^{\mathrm{op}} \otimes B) \cdot [k_{(s_1,s_2)} \otimes k_{\mathrm{sep}}]$$

holds in the Grothendieck group $K_0((A^{\mathrm{op}} \otimes B) \otimes k_{\mathrm{sep}})$. These two facts imply that the above diagram (9.1) is commutative. Note that all the graph morphisms of the right-hand side rectangle are functors, except a priori ϕ, and that $- \otimes k_{\mathrm{sep}}$ is moreover faithful. This implies that ϕ is also a functor. Using the left-hand side commutative square, we hence conclude that $\mathrm{Cov}'(G)$ is a well-defined additive symmetric monoidal category and that Q is an equivalence of categories.

The case of the category $\mathrm{Hecke}'(G)$ is similar. Let us start by constructing a graph isomorphism $Q' : \mathrm{Hecke}'(G) \xrightarrow{\simeq} \mathrm{Cov}'(G)$. On objects we set $Q'((H, A)) := (G/H, A)$. On morphisms we observe that (7.3) restricts to an isomorphism

$$\mathrm{Map}^{A,B}(H \backslash G / K, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Map}^{G,A,B}(G/H \times G/K, \mathbb{Z})\,.$$

Consider now the following commutative diagram of graphs:

$$(9.2) \qquad \begin{array}{ccc} \mathrm{Hecke}'(G) & \xrightarrow[\simeq]{Q'} & \mathrm{Cov}'(G) \\ {\scriptstyle (H,A)\mapsto H}\downarrow & & \downarrow{\scriptstyle (S,A)\mapsto S} \\ \mathrm{Hecke}(G) & \xrightarrow[H\mapsto G/H]{\simeq} & \mathrm{Cov}(G)\,. \end{array}$$

As above, (9.2) allows us to conclude that $\mathrm{Hecke}'(G)$ is a well-defined additive category and that Q' is an equivalence of categories.

Lemma 9.3. *Let S be a finite G-set and A an Azumaya algebra over k_S; see Notation 2.11. Given $s \in S$, let us write I_s for the minimal ideal of the central simple k_s-algebra A_s. Under these notations, the following holds:*

(i) *The Grothendieck group $K_0(A)$ is freely generated by the classes $[I_s]$ of the finitely generated projective right A-modules I_s, where s runs through a set of representatives of the G-orbits in S.*

(ii) *The image of $[I_s]$ under the group homomorphism*

$$K_0(A) \xrightarrow{-\otimes k_{\mathrm{sep}}} K_0(A \otimes k_{\mathrm{sep}}) \xrightarrow[\simeq]{\mathrm{Morita}} K_0(k_S \otimes k_{\mathrm{sep}})$$

identifies with $\mathrm{ind}_s(A) \cdot [k_s \otimes k_{\mathrm{sep}}]$.

Proof. The Azumaya k_S-algebra A is Morita equivalent to the product $\prod_s A_s$, where s runs through a set of representatives of the G-orbits in S. Therefore, the proof of item (i) follows from the fact that the class $[I_s]$ is a generator of the Grothendieck group $K_0(A_s) \simeq \mathbb{Z}$. In what concerns item (ii), consider the following commutative diagrams

$$
\begin{array}{ccccc}
K_0(A) & \xrightarrow{-\otimes k_{\mathrm{sep}}} & K_0(A \otimes k_{\mathrm{sep}}) & \xrightarrow[\sim]{\mathrm{Morita}} & K_0(k_S \otimes k_{\mathrm{sep}}) \\
{\scriptstyle -\otimes_A A_{s'}}\Big\downarrow & & {\scriptstyle (-\otimes_A A_{s'})\otimes k_{\mathrm{sep}}}\Big\downarrow & & \Big\downarrow{\scriptstyle (-\otimes_{k_S} k_{s'})\otimes k_{\mathrm{sep}}} \\
K_0(A_{s'}) & \xrightarrow{-\otimes k_{\mathrm{sep}}} & K_0(A_{s'} \otimes k_{\mathrm{sep}}) & \xrightarrow[\mathrm{Morita}]{\sim} & K_0(k_{s'} \otimes k_{\mathrm{sep}}) \, ,
\end{array}
$$

where s' runs through a set of representatives of the G-orbits in S. Under the canonical isomorphism $K_0(k_S \otimes k_{\mathrm{sep}}) \simeq \prod_{s'} K_0(k_{s'} \otimes k_{\mathrm{sep}})$, the right vertical map corresponds to the projection into the s'^{th}-factor. Moreover, $[I_s] \otimes_A A_{s'}$ is equal to $[I_s] \in K_0(A_{s'})$ when $s' = s$ and equal to 0 when $s' \neq s$. Therefore, the proof of item (ii) follows from the fact that the image of $[I_s] \in K_0(A_s)$ under the bottom horizontal composition agrees with $\mathrm{ind}(A_s) \cdot [k_s \otimes k_{\mathrm{sep}}]$. $\qquad\square$

10. Proof of Proposition 2.26

Under the equivalence $\mathrm{Sep}(k) \simeq \mathrm{Cov}'(G)$ of Theorem 2.13, $\mathrm{CSA}(k)$ identifies with the full subcategory of $\mathrm{Cov}'(G)$ consisting of those objects (S, A) with S a fixed singleton $\{s\}$. Consequently, we obtain the following identifications

$$(10.1) \qquad \mathrm{Hom}_{\mathrm{CSA}(k)}(U(A_i), U(A_j)) \simeq \mathrm{ind}(A_i^{\mathrm{op}} \otimes A_j) \cdot \mathbb{Z} \, ,$$

under which the composition law of $\mathrm{CSA}(k)$ corresponds to multiplication. Since $\mu_{ij} := \mathrm{ind}(A_i^{\mathrm{op}} \otimes A_j)$, the proof of item (i) follows then from the definition of $\Lambda(A_1, \ldots, A_n)$. The proof of item (ii) is standard and we leave it to the reader.

11. Proof of Theorem 2.20 and Proposition 2.28

Let $A_1, \ldots, A_n$ be central simple k-algebras as in Proposition 2.26 and $\Lambda := \Lambda(A_1, \ldots, A_n)$. We start by specializing Arnold's results [5] to our situation.

Theorem 11.1. *(see* [5, Thm. I]*) Let* $P_1, \ldots, P_n \in \mathrm{Proj}(\Lambda)$ *be the right Λ-modules given by the rows of the matrix representation* (2.25) *of Λ, and* $Q \in \mathrm{Proj}(\Lambda)$ *an indecomposable right Λ-module. Then, for every* $p \in \mathcal{P}$ *there exists an integer* $\varrho_p \in \{1, \ldots, n\}$ *such that* $Q_{(p)} \simeq (P_{\varrho_p})_{(p)}$.

Remark 11.2. Note that $P_1, \ldots, P_n$ are the images of $U(A_1), \ldots, U(A_n)$ under the equivalence of categories φ of Proposition 2.26(ii).

Arnold's result [5, Thm. I] also applies to each one of the rings $\Lambda_{(p)}$. In these cases, we obtain the following result:

Proposition 11.3. *The indecomposable finitely generated projective right $\Lambda_{(p)}$-modules are of the form* $(P_i)_{(p)}$ *with* $i \in \{1, \ldots, n\}$.

Arnold's work [5] gives also rise to the following results:

Proposition 11.4. *Assume given for any* $p \in \mathcal{P}$ *a right $\Lambda_{(p)}$-module* $P_p \in \mathrm{Proj}(\Lambda_{(p)})$ *of $\mathbb{Z}_{(p)}$-rank rn. Then, there exists a right Λ-module*[5] $P \in \mathrm{Proj}(\Lambda)$ *such that* $P_{(p)} = P_p$ *for every* $p \in \mathcal{P}$.

---

[5]Thanks to Theorem 11.5, the right Λ-module P is moreover unique.

Proof. Making use of Proposition 11.3, we can assume without loss of generality that $r = 1$, i.e. that all the P_p's are indecomposable. Given $p \in \mathcal{P}$, let $\varrho_p \in \{1, \ldots, n\}$ be such that $(P_{\varrho_p})_{(p)} = P_p$; see Theorem 11.1. We need to construct a right Λ-module $P \in \mathrm{Proj}(\Lambda)$ with the property that $P_{(p)} = (P_{\varrho_p})_{(p)}$ for every $p \in \mathcal{P}$. Using the matrix representation (2.25) of Λ and the above Remark 11.2, we observe that all the P_p's may be viewed as submodules of finite index of the right Λ-module $[\mathbb{Z} \cdots \mathbb{Z}]$. It then suffices to take $P := \bigcap_p (P_{\varrho_p})_{(p)}$, where the intersection takes place inside $[\mathbb{Q} \cdots \mathbb{Q}]$; see [10, Prop. I. 5.2]. $\qquad\square$

Theorem 11.5. *(see* [5, Cor. II]*) Given right Λ-modules $P, Q \in \mathrm{Proj}(\Lambda)$, one has $P \simeq Q$ if and only if $P_{(p)} \simeq Q_{(p)}$ for every $p \in \mathcal{P}$.*

Corollary 11.6. *Given right Λ-modules $P, Q \in \mathrm{Proj}(\Lambda)$, Q is a direct summand of P if and only if $Q_{(p)}$ is a direct summand of $P_{(p)}$ for every $p \in \mathcal{P}$.*

Proof. We focus ourselves on the non-obvious implication. Assume that $Q_{(p)}$ is a direct summand of $P_{(p)}$ for every $p \in \mathcal{P}$. For each such p choose a complement $Q_{(p)} \oplus R_p \simeq P_{(p)}$. Using Proposition 11.4, we hence obtain a well-defined right Λ-module $R \in \mathrm{Proj}(\Lambda)$ such that $R_{(p)} = R_p$ for every $p \in \mathcal{P}$. Theorem 11.5 allows us then to conclude that $Q \oplus R \simeq P$. $\qquad\square$

Given a prime number p, let us denote by $\mathrm{Hmo}_0(k)_{(p)}$ the $\mathbb{Z}_{(p)}$-linear category $\mathrm{Hmo}_0(k)_{\mathbb{Z}_{(p)}}$, by $(-)_p$ the functor (5.3) with $R = \mathbb{Z}_{(p)}$, and by $U(-)_{(p)}$ the functor $U(-)_{\mathbb{Z}_{(p)}}$. Under these notations, we have the following combative diagram

$$(11.7) \qquad \begin{array}{ccc} \mathrm{CSA}(A_1, \ldots, A_n)^{\oplus, \natural} & \xrightarrow[\simeq]{\varphi} & \mathrm{Proj}(\Lambda) \\ {\scriptstyle (-)_{(p)}} \downarrow & & \downarrow {\scriptstyle (-)_{(p)}} \\ \mathrm{CSA}(A_1, \ldots, A_n)^{\oplus, \natural}_{(p)} & \xrightarrow[\varphi_p]{\simeq} & \mathrm{Proj}(\Lambda_{(p)}), \end{array}$$

where φ_p stands for the equivalence $\mathrm{Hom}(\oplus_i U(A_i)_{(p)}, -)$; see Proposition 2.26(ii). In what follows, we write $\mathrm{CSA}(k)^{\oplus, \natural}$ from the closure of $\mathrm{CSA}(k)$ under finite direct sums and direct summands.

Lemma 11.8. *Given noncommutative motives $M, M' \in \mathrm{CSA}(k)^{\oplus, \natural}$, one has $M \simeq M'$ if and only if $M_{(p)} \simeq M'_{(p)}$ for every $p \in \mathcal{P}$.*

Proof. We focus ourselves on the non-obvious implication. Assume that $M_{(p)} \simeq M'_{(p)}$ for every $p \in \mathcal{P}$. Since $M, M' \in \mathrm{CSA}(A_1, \ldots, A_n)^{\oplus, \natural}$ for suitable central simple k-algebras $A_1, \ldots, A_n$, the above commutative diagram (11.7) implies that

$$\varphi(M)_{(p)} \simeq \varphi_p(M_{(p)}) \simeq \varphi_p(M'_{(p)}) \simeq \varphi(M')_{(p)} \in \mathrm{Proj}(\Lambda_{(p)})$$

for every $p \in \mathcal{P}$. Using Theorem 11.5 and the fact that φ is an equivalence of categories, we hence conclude that $M \simeq M'$. $\qquad\square$

Lemma 11.9. *Given central simple k-algebras A and B, the following holds:*
(i) *For every $p \in \mathcal{P}$, one has $U(A)_{(p)} = U(A^p)_{(p)}$.*
(ii) *One has $U(A)_{(p)} \simeq U(B)_{(p)}$ for every $p \in \mathcal{P}$ if and only if $[A] = [B]$.*

Proof. Recall from [35, Thm. 2.1] that given a central simple k-algebra C whose index is a prime power p^r, we have $U(k)_R \simeq U(C)_R$ for every commutative ring R containing $1/p$. In particular, $U(k)_{(q)} \simeq U(C)_{(q)}$ for every prime number $q \neq p$.

Consider the p-primary decomposition $A = \otimes_{p \in \mathcal{P}} A^p$. Since the functor $U(-)_{(p)}$ is symmetric monoidal, we hence conclude that $U(A)_{(p)} \simeq U(A^p)_{(p)}$. This proves item (i). In what concerns item (ii), the non-obvious implication follows from the combination of Lemma 11.8 with Equivalence (2.19). $\qquad\square$

Remark 11.10. It is possible to prove Lemma 11.9(ii) without invoking the results of Arnold [5] (*i.e.* Theorem 11.5). We focus ourselves in the non-obvious implication. Assume that $[A] \neq [B]$ or equivalently that $U(A) \not\simeq U(B)$; see Equivalence (2.19). Thanks to Lemma 11.9(i), we can assume without loss of generality that $\mathrm{ind}(A)$ and $\mathrm{ind}(B)$ are powers of a prime number p. Hence, we obtain the identification

$$(11.11) \qquad \mathrm{Hom}_{\mathrm{CSA}(k)}(U(A), U(B)) \stackrel{(10.1)}{\simeq} \mathrm{ind}(A^{\mathrm{op}} \otimes B) \cdot \mathbb{Z} = p^s \mathbb{Z}$$

for some integer $s \geq 1$. Under (11.11), the composition bilinear pairing

$$\mathrm{Hom}_{\mathrm{CSA}(k)}(U(A), U(B)) \times \mathrm{Hom}_{\mathrm{CSA}(k)}(U(B), U(A)) \longrightarrow \mathrm{Hom}_{\mathrm{CSA}(k)}(U(A), U(B))$$

identifies with the multiplication pairing $p^s \mathbb{Z} \times p^s \mathbb{Z} \to \mathbb{Z}$. The analogous composition pairing, with $U(-)$ replaced by $U(-)_{(p)}$, identifies also with the multiplication pairing $p^s \mathbb{Z}_{(p)} \times p^s \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$. Since $s \geq 1$, the element $1 \in \mathbb{Z}_{(p)}$ is not in the image of the latter pairing. This allows us to conclude that $U(A)_{(p)} \not\simeq U(B)_{(p)}$.

Proof of Theorem 2.20.

Item (i). Given $M \in \mathrm{CSA}(k)^{\oplus, \natural}$, we need to prove that $M \in \mathrm{CSA}(k)^{\oplus}$. Clearly, we can assume that $M \in \mathrm{CSA}(A_1, \ldots, A_n)^{\oplus, \natural}$ for suitable central simple k-algebras $A_1, \ldots, A_n$. Making use of the equivalence of categories φ, we have $\varphi(M) \simeq Q_1 \oplus \cdots \oplus Q_m$ with $Q_1, \ldots, Q_m \in \mathrm{Proj}(\Lambda)$ indecomposable right Λ-modules. Without loss of generality, we can also assume that $m = 1$; let $Q := Q_1$.

Now, recall from Theorem 11.1 that for every $p \in \mathcal{P}$ there exists an integer $\varrho_p \in \{1, \ldots, n\}$ such that $Q_{(p)} \simeq (P_{\varrho_p})_{(p)}$. Consequently, we obtain the identifications

$$\varphi_p(M_{(p)}) \stackrel{(a)}{\simeq} Q_{(p)} \simeq (P_{\varrho_p})_{(p)} \stackrel{(b)}{\simeq} \varphi(U(A_{\varrho_p}))_{(p)} \stackrel{(c)}{\simeq} \varphi_p(U(A^p_{\varrho_p})_{(p)}),$$

where (a) follows from the commutative diagram (11.7), (b) from Remark 11.2, and (c) from Lemma 11.9(i) and (11.7). Using the fact that φ_p is an equivalence of categories, we hence conclude that $M_{(p)} \simeq U(A^p_{\varrho_p})_{(p)}$. Let $B := \otimes_{p \in \mathcal{P}} A^p_{\varrho_p}$. Since $M_{(p)} \simeq U(B)_{(p)}$ for every $p \in \mathcal{P}$, Lemma 11.8 implies that $M \simeq U(B)$. As a consequence, $M \in \mathrm{CSA}(k)^{\oplus}$.

Item (ii). Every object in $\mathrm{CSA}(k)^{\oplus}$ is of the form $U(A_1) \oplus \cdots \oplus U(A_n)$. Therefore, the indecomposable objects must be of the form $U(B)$ with B a central simple k-algebra. Since the endomorphism rings $\mathrm{End}_{\mathrm{CSA}(k)}(U(B)) \simeq \mathbb{Z}$ have no non-trivial idempotents, we conclude that the objects $U(B)$ are indeed indecomposable.

Item (iii). Let B be a central simple k-algebra satisfying the following condition: for every $p \in \mathcal{P}$ there exists an integer $\varrho_p \in \{1, \ldots, n\}$ such that $[B^p] = [A^p_{\varrho_p}]$. We need to prove that $U(B)$ is an indecomposable direct summand of $U(A_1) \oplus \cdots \oplus U(A_n)$. Consider the equivalence of categories

$$\varphi : \mathrm{CSA}(A_1, \ldots, A_n, B)^{\oplus, \natural} \simeq \mathrm{Proj}(\Lambda(A_1, \ldots, A_n, B)).$$

Let us denote by $P_1, \ldots, P_n, Q$ the images of $U(A_1), \ldots, U(A_n), U(B)$ under φ. Under these notations, we have the following identifications

$$Q_{(p)} \overset{(a)}{\simeq} \varphi_p(U(B^p)_{(p)}) \overset{(b)}{\simeq} \varphi_p(U(A_{\varrho_p}^p)_{(p)}) \overset{(c)}{\simeq} \varphi(U(A_{\varrho_p}))_{(p)} = (P_{\varrho_p})_{(p)} \,,$$

where (a) and (c) follow from the commutative diagram (11.7) and Lemma 11.9(i), and (b) from Lemma 11.9(ii). Corollary 11.6 implies then that Q is a direct summand of $P_1 \oplus \cdots \oplus P_n$. Using the fact that φ is an equivalence of categories, we hence conclude that $U(B)$ is a direct summand of $U(A_1) \oplus \cdots \oplus U(A_n)$. Finally, as explained in item(ii), $U(B)$ is moreover indecomposable.

Now, let M be an indecomposable direct summand of $U(A_1) \oplus \cdots \oplus U(A_n)$. Proceeding as in the proof of item (i), we conclude that $M \simeq U(B)$ with $B = \otimes_{p \in \mathcal{P}} A_{\varrho_p}^p$. Clearly, for every $p \in \mathcal{P}$ there exists an integer $\varrho_p \in \{1, \ldots, n\}$ such that $[B^p] = [A_{\varrho_p}^p]$. This concludes the proof.

Item (iv). We start by proving (a) implies (b). Clearly, condition (a) implies that

$$(11.12) \qquad U(A_1)_{(p)} \oplus \cdots \oplus U(A_n)_{(p)} \simeq U(B_1)_{(p)} \oplus \cdots \oplus U(B_m)_{(p)}$$

for every $p \in \mathcal{P}$. Consider the equivalence of categories

$$(11.13) \quad \mathrm{CSA}(A_1, \ldots, A_n, B_1, \ldots, B_m)_{(p)} \overset{\varphi_p}{\simeq} \mathrm{Proj}(\Lambda(A_1, \ldots, A_n, B_1, \ldots, B_m)_{(p)}) \,.$$

Since the right-hand side of (11.13) is a Krull-Schmidt category, we hence conclude that $n = m$ and that there exists a permutation σ_p (which depends on p) such that $U(B_i)_{(p)} \simeq U(A_{\sigma_p(i)})_{(p)}$ for every $1 \leq i \leq n$. Thanks to Lemma 11.9, the latter condition is equivalent to condition (b).

Let us now prove the converse implication. Thanks once again to Lemma 11.9, condition (b) implies that $n = m$ and that the above isomorphism (11.12) holds for every $p \in \mathcal{P}$. By applying Lemma 11.8 to the left and right-hand side of (11.12), we hence obtain condition (a). This concludes the proof.

Proof of Proposition 2.28. As explained in the proof of Theorem 2.20(ii), the objects $U(A_i)$ are indecomposable. We now prove the converse. Let M be an indecomposable object in $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus, \natural}$. Note that in the proof of Theorem 2.20(i) we can assume without loss of generality that the Brauer classes $\{[A_1], \ldots, [A_n]\}$ of the suitable central simple k-algebras $A_1, \ldots, A_n$ form a subgroup of $\mathrm{Br}(k)$. Therefore, as in *loc. cit.*, we conclude that $M \simeq U(B)$ with $B = \otimes_{p \in \mathcal{P}} A_{\varrho_p}^p$. Since the class $[A_{\varrho_p}^p]$ belongs to the cyclic subgroup of $\mathrm{Br}(k)$ generated by $[A_{\varrho_p}]$, $[B] \in \{[A_1], \ldots, [A_n]\}$. Hence, the proof follows now automatically from Equivalence (2.19).

12. Proof of Proposition 2.30

The objects $\{U(A_i)\}_{1 \leq i \leq n}$ and $\{\Sigma(A_1, \ldots, A_n)([A_i])\}_{1 \leq i \leq n}$ form a set of generators of the additive categories $\mathrm{CSA}(A_1, \ldots, A_n)^{\oplus}$ and $\mathrm{Proj}_{\mathrm{gr}}(\Sigma(A_1, \ldots, A_n))$, respectively. Since the functor $U(A_i) \mapsto \Sigma(A_1, \ldots, A_n)([A_i])$ is not only additive but also symmetric monoidal, it suffices to show that the induced homomorphisms

$$\mathrm{Hom}(U(A_i), U(A_j)) \longrightarrow \mathrm{Hom}(\Sigma(A_1, \ldots, A_n)([A_i]), \Sigma(A_1, \ldots, A_n)([A_j]))$$

are invertible. As explained above, the left-hand side identifies with $\mathrm{ind}(A_i^{\mathrm{op}} \otimes A_j) \cdot \mathbb{Z}$. In what concerns the right-hand side, it identifies with

$$\mathrm{Hom}(\Sigma(A_1, \ldots, A_n)([k]), \Sigma(A_1, \ldots, A_n)([A_i^{\mathrm{op}} \otimes A_j]))$$
$$\simeq \quad \Sigma(A_1, \ldots, A_n)([A_i^{\mathrm{op}} \otimes A_j])_{[k]} = \mathrm{ind}(A_i^{\mathrm{op}} \otimes A_j) \cdot \mathbb{Z}$$

because $\Sigma(A_1, \ldots, A_n)([A_i])$ is a strongly dualizable object of $\mathrm{Proj}_{\mathrm{gr}}(\Sigma(A_1, \ldots, A_n))$ with dual $\Sigma(A_1, \ldots, A_n)([A_i^{\mathrm{op}}])$. This completes the proof.

13. Proof of Theorem 2.32

Let us prove first the particular case where $\mathrm{per}(A)$ is a prime power p^r. Since $(1 - t)(1 + t + \cdots + t^{p^r - 1}) = (1 - t^{p^r})$, we need then to show that the assignment $t \mapsto U(A)$ gives rise to a ring isomorphism

$$(13.1) \qquad \mathbb{Z}[t]/\langle(1 - t^{p^r})\rangle \xrightarrow{\sim} K_0(\mathrm{CSA}(k, A, A^{\otimes 2}, \ldots, A^{\otimes(p^r - 1)})^{\oplus}).$$

Thanks to the cancellation property of Corollary 2.23, the elements of the right-hand side of (13.1) are formal differences (not just equivalence classes)

$$(13.2) \qquad [\oplus_j U(A^{\otimes s_j})] - [\oplus_{j'} U(A^{\otimes s_{j'}})] \qquad 0 \leq s_j, s_{j'} \leq p^r - 1.$$

This implies that the above homomorphism (13.1) is surjective. Now, recall that (13.2) is trivial if and only if there exists an isomorphism

$$(13.3) \qquad \oplus_j U(A^{\otimes s_j}) \simeq \oplus_{j'} U(A^{\otimes s_{j'}}).$$

Since $\mathrm{per}(A)$ and $\mathrm{ind}(A)$ have the same prime factors, $A^p = A$ and $A^q = k$ for every prime number $q \neq p$. Hence, by combining Theorem 2.20(iv) with the equivalence (2.19), we conclude that (13.3) holds if and only if $j = j'$ and there exists a permutation σ such that $U(A^{\otimes s_{j'}}) \simeq U(A^{\otimes s_{\sigma(j)}})$ for every j. This implies that the above homomorphism (13.1) is moreover injective, and therefore an isomorphism.

Let us now prove the general case where $\mathrm{per}(A) = \prod_i p_i^{r_i}$. As above, the assignment $t \mapsto U(A)$ gives rise to a surjective ring homomorphism

$$(13.4) \qquad \eta : \mathbb{Z}[t]/\langle(1 - t^{\mathrm{per}(A)})\rangle \twoheadrightarrow K_0(\mathrm{CSA}(k, A, A^{\otimes 2}, \ldots, A^{\otimes(\mathrm{per}(A) - 1)})^{\oplus}).$$

For every prime power $p_i^{r_i} \in \mathrm{per}(A)$ consider the following commutative diagram[6]:

$$\begin{array}{ccc}
\mathbb{Z}[t]/\langle(1 - t^{\mathrm{per}(A)})\rangle & \xrightarrow{\quad \eta \quad} & K_0(\mathrm{CSA}(k, A, A^{\otimes 2}, \ldots, A^{\otimes(\mathrm{per}(A) - 1)})^{\oplus}) \\
\Big\downarrow{\scriptstyle t \mapsto t} & & \Big\downarrow{\scriptstyle A \mapsto A^{p_i}} \\
\mathbb{Z}[t]/\langle(1 - t^{p_i^{r_i}})\rangle & \xrightarrow[t \mapsto U(A^{p_i})]{\sim} & K_0(\mathrm{CSA}(k, A^{p_i}, (A^{p_i})^{\otimes 2}, \ldots, (A^{p_i})^{\otimes(p_i^{r_i} - 1)})^{\oplus}).
\end{array}$$

The lower horizontal homomorphism is an isomorphism (as proved above) and the kernel of the left-hand side vertical homomorphism is given by $\langle(1 - t^{p_i^{r_i}})\rangle$. The commutativity of the above square implies then that $\mathrm{Ker}(\eta) \subseteq \bigcap_i \langle(1 - t^{p_i^{r_i}})\rangle$. Let us now prove the converse inclusion, or equivalently that the intersection of the kernels of the right-hand side vertical homomorphisms is trivial. Recall from above that the elements of the right-hand side of (13.4) are formal differences

$$(13.5) \qquad [\oplus_j U(A^{\otimes s_j})] - [\oplus_{j'} U(A^{\otimes s_{j'}})] \qquad 0 \leq s_j, s_{j'} \leq \mathrm{per}(A) - 1.$$

On one hand, (13.5) is trivial if and only if there exists an isomorphism $\oplus_j U(A^{\otimes s_j}) \simeq \oplus_{j'} U(A^{\otimes s_{j'}})$. On the other hand, as explained above, (13.5) belongs to the kernel

---

[6]Thanks to Theorem 2.20(iv) that the right-hand side vertical homomorphism is well-defined.

of the right-hand side vertical morphisms if and only if $j = j'$ and there exists a permutation σ_{p_i} (which depends on p_i) such that $U((A^{p_i})^{\otimes s_{j'}}) \simeq U((A^{p_i})^{\otimes s_{\sigma_{p_i}(j)}})$ for every j. Thanks to Theorem 2.20(iv), these two conditions are equivalent. Hence, $\mathrm{Ker}(\eta) = \bigcap_i \langle (1 - t^{p_i^{r_i}}) \rangle$. Finally, making use of the equalities $1 - t^{p_i^{r_i}} = (1-t)(1 + t + t^2 + \cdots + t^{p_i^{r_i}-1})$, we conclude that

$$\mathrm{Ker}(\eta) = \bigcap_i \langle (1 - t^{p_i^{r_i}}) \rangle = \langle (1-t) \prod_i (1 + t + t^2 + \cdots + t^{p_i^{r_i}-1}) \rangle .$$

This concludes the proof.

14. PROOF OF PROPOSITION 3.10

Making use of Theorem 2.20(iv)(b) and the fact that every permutation σ_p can be written as a composition of transpositions, it suffices to prove the following claim: given central simple k-algebras $D_1 = \otimes_{q \in \mathcal{P}} D_1^q$ and $D_2 = \otimes_{q \in \mathcal{P}} D_2^q$, the relations (3.6) gives rise to an isomorphism between the following objects ($\mathcal{P}' := \mathcal{P} \backslash \{p\}$):

$$(14.1) \qquad E(D_1^p \otimes (\otimes_{q \in \mathcal{P}'} D_1^q)) \oplus E(D_2^p \otimes (\otimes_{q \in \mathcal{P}'} D_2^q))$$

$$(14.2) \qquad E(D_2^p \otimes (\otimes_{q \in \mathcal{P}'} D_1^q)) \oplus E(D_1^p \otimes (\otimes_{q \in \mathcal{P}'} D_2^q)) .$$

The relation (3.6) applied to the central simple k-algebras $A := (D_1^p)^{\mathrm{op}} \otimes D_2^p$, $B := (\otimes_{q \in \mathcal{P}'} D_1^q) \otimes (\otimes_{q \in \mathcal{P}'} D_2^q)$ and $C := D_1$, implies that (14.1)-(14.2) are isomorphic. This proves item (i). In what concerns item (ii), note that by tensoring (14.1)-(14.2) with the object $E(D_1^{\mathrm{op}})$, we obtain:

$$(14.3) \qquad E(k) \oplus E((D_1^p)^{\mathrm{op}} \otimes D_2^p \otimes (\otimes_{q \in \mathcal{P}'} D_1^q)^{\mathrm{op}} \otimes (\otimes_{q \in \mathcal{P}'} D_2^q))$$

$$(14.4) \qquad E((D_1^p)^{\mathrm{op}} \otimes D_2^p) \oplus E((\otimes_{q \in \mathcal{P}'} D_1^q)^{\mathrm{op}} \otimes (\otimes_{q \in \mathcal{P}'} D_2^q)) .$$

Hence, relation (3.6) applied to the central simple k-algebras $A := (D_1^p)^{\mathrm{op}} \otimes D_2^p$, $B := (\otimes_{q \in \mathcal{P}'} D_1^q)^{\mathrm{op}} \otimes (\otimes_{q \in \mathcal{P}'} D_2^q)$ and $C := k$ implies that (14.3)-(14.4) are isomorphic. Finally, using the fact that the object $E(\mathcal{D}_1^{\mathrm{op}})$ is $\otimes$-invertible, we conclude that (14.1)-(14.2) are also isomorphic. This proves item (ii).

15. PROOF OF THEOREM 3.18

Recall from [24, §1] the construction of Merkurjev-Panin's motivic category $\underline{\mathcal{C}}$ and of the symmetric monoidal functors $\Phi : \mathrm{SmProj}(k)^{\mathrm{op}} \to \underline{\mathcal{C}}$ and $\Psi : \mathrm{sep}(k) \to \underline{\mathcal{C}}$, where $\mathrm{sep}(k)$ stands for the category of separable k-algebras. Roughly speaking, the objects of $\underline{\mathcal{C}}$ are the pairs (X, A), with X a smooth projective k-scheme and A a separable k-algebra, the morphisms of $\underline{\mathcal{C}}$ are given by the Grothendieck groups $\mathrm{Hom}_{\underline{\mathcal{C}}}((X, A), (Y, B)) := K_0 \mathrm{Vect}(X \times Y, A^{\mathrm{op}} \otimes B)$, where $\mathrm{Vect}(X \times Y, A^{\mathrm{op}} \otimes B)$ stands for the exact category of those right $(\mathcal{O}_{X \times Y} \otimes (A^{\mathrm{op}} \otimes B))$-modules which are locally free and of finite rank as $\mathcal{O}_{X \times Y}$-modules, and the composition law of $\underline{\mathcal{C}}$ is induced by the tensor product of modules.

Proposition 15.1. *We have an isomorphism (see Notation 3.17)*

$$(15.2) \qquad \Phi(\mathrm{Flag}(d_1, \ldots, d_m; A)) \simeq \binom{\deg(A)}{d_1 \cdots d_m}_{\Psi(A)} .$$

Proof. Recall that $G := \mathrm{Gal}(k_{\mathrm{sep}}/k)$. We start by recalling some results from Panin's work [25]. Let $\tilde{H}$ be a split semi-simple simply connected algebraic group (defined over k) and $\tilde{T} \subset \tilde{P} \subset \tilde{H}$ a split maximal torus and a parabolic subgroup. The corresponding representation rings are denoted by $R(\tilde{T}), R(\tilde{P})$, and $R(\tilde{H})$,

respectively. Let $\tilde{Z}$ be the center of $\tilde{H}$ and $H := \tilde{H}/\tilde{Z}$ the corresponding adjoint group. We write $\mathrm{Ch} = \mathrm{Hom}(\tilde{Z}, \mathbb{G}_m)$ for the character group. The representation rings introduced above are canonically Ch-graded. A representation V has degree $\mathcal{X} : \tilde{Z} \to \mathbb{G}_m$ if $\tilde{Z}$ acts on V through the character $\mathcal{X}$. Fix an element $\gamma \in H^1(G, H(k_{\mathrm{sep}}))$. It gives rise to a canonical group homomorphism $\beta_\gamma : \mathrm{Ch} \to \mathrm{Br}(k)$ which sends $\mathcal{X} : \tilde{Z} \to \mathbb{G}_m$ to the image of γ under the composed homomorphism

$$H^1(G, H(k_{\mathrm{sep}})) \xrightarrow{\partial} H^2(G, \tilde{\mathcal{Z}}(k_{\mathrm{sep}})) \xrightarrow{\mathcal{X}} H^2(G, \mathbb{G}_m(k_{\mathrm{sep}})) = \mathrm{Br}(k).$$

Let $\mathcal{F} := \tilde{H}/\tilde{P}$. Since the $\tilde{H}$-action on $\mathcal{F}$ factors through H we have a corresponding twisted homogeneous space ${}_\gamma\mathcal{F}$. For any Ch-homogeneous basis $\theta_1, \ldots, \theta_q$ of $R(\tilde{P})$ over $R(\tilde{H})$, Panin constructs an isomorphism in the category $\mathcal{C}$

$$(15.3) \qquad \bigoplus_{i=1}^q \Psi(\beta_\gamma(|\theta_i|)) \simeq \Phi({}_\gamma\mathcal{F}),$$

where $|\theta_i| \in \mathrm{Ch}$ is the degree of θ_i. This construction follows from the combination of [25, Thm. 6.7] with the functor F_γ introduced in the proof of [25, Lem. 6.5].

We now specialize[7] the above constructions to $\tilde{H} = \mathrm{Sl}_n$, where $n := \deg(A)$. In this case, $\tilde{\mathcal{Z}}$ consists of the constant diagonal matrices with entries the n-roots of unity. Hence, Ch has a canonical generator given by the inclusion of $\tilde{\mathcal{Z}}$ inside the diagonal matrices of Gl_n. This implies that canonically $\mathrm{Ch} = \mathbb{Z}/n\mathbb{Z}$. Let $\tilde{P} \subset \tilde{H}$ be the standard parabolic subgroup of elements preserving the flag $(0 \subset k^{d_1} \subset k^{d_1+d_2} \subset \cdots \subset k^{d_1+\cdots+d_n} = k^n)$. In this case, we have ${}_\gamma\mathcal{F} = \mathrm{Flag}(d_1, \ldots, d_m; A)$, where $[A] = \beta_\gamma(\bar{1})$. Let $\tilde{T} \subset \tilde{P}$ be the maximal torus of diagonal matrices. If we denote by t_i the ith diagonal entry, then t_i is an element of $R(\tilde{T}) = \mathrm{Hom}(\tilde{T}, \mathbb{G}_m)$. Under the above choices, the associated Weyl groups $\mathcal{W}_{\tilde{P}}$ and $\mathcal{W}_{\tilde{H}}$ of $\tilde{P}$ and $\tilde{H}$ are given, respectively, by $S_{d_1} \times \cdots \times S_{d_m}$ and S_n. Therefore, we have the equalities

$$
\begin{aligned}
R(\tilde{T}) &= \tilde{R}(\tilde{T})/(t_1 \ldots t_n - 1) \\
R(\tilde{P}) &= R(\tilde{P})^{\mathcal{W}_{\tilde{P}}} = \tilde{R}(\tilde{P})/(\sigma_{d_1}^{(1)}\sigma_{d_2}^{(2)} \ldots \sigma_{d_m}^{(m)} - 1) \\
R(\tilde{H}) &= R(\tilde{H})^{\mathcal{W}_{\tilde{H}}} = \tilde{R}(\tilde{H})/(\sigma_n - 1),
\end{aligned}
$$

where $\tilde{R}(\tilde{T})$, $\tilde{R}(\tilde{P})$, and $\tilde{R}(\tilde{H})$, are given respectively by

$$\mathbb{Z}[t_1, \ldots, t_n] \quad \mathbb{Z}[\sigma_1^{(1)}, \ldots, \sigma_{d_1}^{(1)}, \ldots \sigma_1^{(m)}, \ldots, \sigma_{d_m}^{(m)}] \quad \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$$

for the appropriate symmetric functions $\sigma_i^{(j)}$ and σ_k in $t_1, \ldots, t_n$. The Ch-grading on the above is obtained from the $\mathbb{Z}$-grading on $\tilde{R}(\tilde{T})$ given by $|t_i| = 1$.

Now, fix a $\mathbb{Z}$-graded basis $\tilde{\theta}_1, \ldots, \tilde{\theta}_q$ of $\tilde{R}(\tilde{P})$ over $\tilde{R}(\tilde{H})$. Tensoring $- \otimes_{\tilde{R}(\tilde{T})} R(\tilde{T})$ yields a Ch-graded basis $\theta_1, \ldots, \theta_q$ of $R(\tilde{P})$ over $R(\tilde{H})$. Since $A^{\otimes n}$ is Morita equivalent to k we hence obtain the following equalities:

$$(15.4) \qquad \Psi(\beta_\gamma(|\theta_i|)) = \Psi(\beta_\gamma(\bar{1})^{|\tilde{\theta}_i|}) = \Psi([A]^{|\tilde{\theta}_i|}) = \Psi(A)^{\otimes|\tilde{\theta}_i|},$$

---

[7]It would be more convenient to use $\tilde{H} = \mathrm{Gl}_n$ but this is not a semi-simple group so it does not fall literally under Panin's setting [25].

where $|\tilde{\theta}_i| \in \mathbb{Z}$ stands for the degree of $\tilde{\theta}_i$. Finally, by combining (15.4) into (15.3) and with the Poincaré series computation

$$\sum_i t^{|\tilde{\theta}_i|} = \binom{n}{d_1 \cdots d_m}_t,$$

we obtain the desired isomorphism (15.2). This concludes the proof. $\qquad\square$

As proved in [31, Thm. 6.10], there exists an additive fully-faithful symmetric monoidal functor $\Theta : \underline{\mathcal{C}} \to \mathrm{NChow}(k) \subset \mathrm{Hmo}_0(k)$ making the diagrams commute:

$$
\begin{array}{ccc}
\mathrm{SmProj}(k)^{\mathrm{op}} \xrightarrow{X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)} \mathrm{dgcat}(k) & \qquad & \mathrm{sep}(k) \xrightarrow{A \mapsto A} \mathrm{dgcat}(k) \\
\Phi \downarrow \qquad\qquad\qquad \downarrow U & & \Psi \downarrow \qquad\qquad \downarrow U \\
\underline{\mathcal{C}} \xrightarrow{\quad \Theta \quad} \mathrm{Hmo}_0(k) & & \underline{\mathcal{C}} \xrightarrow{\quad \Theta \quad} \mathrm{Hmo}_0(k) \,.
\end{array}
$$

As a consequence, the searched motivic decomposition (3.19) is obtained by applying the functor Θ to the isomorphism (15.2). This achieves the proof.

Appendix A. Cyclic sieving phenomenon

Let C be a cyclic group of order n acting on a finite set X and $p(t) \in \mathbb{Z}[t]$ a polynomial with integer coefficients. Given an element $c \in C$, we write X^c for the subset of c-fixed points of X. Recall from [27] that the triple $(X, p(t), C)$ *exhibits the cyclic sieving phenomenon* if $|X^c| = p(\omega)$ for every $c \in C$ and root of unit ω having the same order as c. Intuitively speaking, the polynomial $p(t)$ works as a generating function for the set X.

Example A.1. Given non-negative integers $m \leq n$, let X be the set of all m-element subsets of $\{1, 2, \ldots, n\}$ and $p(t)$ the *Gaussian polynomial*

$$\binom{n}{m}_t := \frac{(1 - t^n)(1 - t^{n-1}) \cdots (1 - t^{n-m+1})}{(1 - t)(1 - t^2) \cdots (1 - t^m)} \in \mathbb{N}[t] \,.$$

If $c \in C$ acts on X by cycling the elements of a m-subset modulo n, then the triple $(X, p(t), C)$ exhibits the cyclic sieving phenomenon; see [27, Thm. 1.1(b)].

Lemma A.2. *Let $(X, p(t), C)$ be a triple exhibiting the cyclic sieving phenomenon. Given a divisor $l | n$ and a coprime integer $(i, l) = 1$, we have the congruence relation*

(A.3) $$p(t^i) \cong p(t) \text{ modulo } (t^l - 1) \,.$$

Proof. It suffices to show that $p(\omega^i) = p(\omega)$ for every ω such that $\omega^l = 1$. Since by hypothesis $(i, l) = 1$, the roots of unit ω and ω^i have the same order. Consequently, $p(\omega^i) = |X^c| = p(\omega)$ for some $c \in C$. This achieves the proof. $\qquad\square$

Corollary A.4. *Given non-negative integers $m \leq n$, an additive symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, and an object $b \in \mathcal{C}$ such that $b^{\otimes l} \simeq \mathbf{1}$, we have an induced isomorphism $\binom{n}{m}_{b^{\otimes i}} \simeq \binom{n}{m}_b$; see Notation 3.17.*

Proof. Since $b^{\otimes l} \simeq \mathbf{1}$, we have the following evaluation map

$$\mathbb{N}[t]/(t^l = 1) \longrightarrow \mathrm{Iso}(\mathcal{C}) \qquad p(t) \mapsto p(b) \,.$$

Consequently, the proof follows from the above congruence (A.3). $\qquad\square$

Appendix B. Noncommutative motives of dg Azumaya algebras

In this appendix we assume that k is a commutative ring; let s be the number of components of the associated affine k-scheme $\mathrm{Spec}(k)$. Recall from [15, 33, 34] that all constructions and results of §4-5 hold also in this generality; simply replace the tensor product by its derived version $- \otimes^{\mathbf{L}} -$.

DG Azumaya algebras. A dg k-algebra A is called a *dg Azumaya algebra* if:
   (i) The underlying complex of k-modules is a compact generator of $\mathcal{D}(k)$.
   (ii) The canonical morphism $A^{\mathrm{op}} \otimes^{\mathbf{L}} A \to \mathbf{R}\mathrm{Hom}(A, A)$ in $\mathcal{D}(k)$ is an isomorphism.

Example B.1.   (i) The ordinary Azumaya algebras (see [12]) are the dg Azumaya algebras whose underlying complex is k-flat and concentrated in degree zero.
   (ii) When k is a field, every dg Azumaya algebra is isomorphic in the homotopy category $\mathrm{Hmo}(k)$ to an ordinary Azumaya algebra; see [36, Prop. 2.12].
   (iii) For every non-torsion étale cohomology class $\alpha \in H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m)$ there exists a dg Azumaya algebra A_α (representing this class α) which is *not* isomorphic in $\mathrm{Hmo}(k)$ to an ordinary Azumaya algebra; see [36, page 584].

The *derived Brauer group* $\mathrm{dBr}(k)$ *of* k is the set of isomorphism classes of dg Azumaya algebras in $\mathrm{Hmo}(k)$. The group structure is induced by the derived tensor product. B. Toën constructed in [36, Cor. 3.8] (see also [4]) a bijective map

$$(B.2) \qquad \psi : \mathrm{dBr}(k) \longrightarrow H^1_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{Z}) \times H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m) \,.$$

We will now describe this map in down-to-earth terms, avoiding the language of derived stacks. For notational reasons, we will follow the geometric setting.

Let $X = \mathrm{Spec}(k)$, $\mathcal{L}(X)$ the set of locally constant functions $X \to \mathbb{Z}$, and $\mathrm{DPic}(X)$ the derived Picard group of X (which we consider as a 2-group). As proved by Rouquier-Zimmermann in [28, §3], we have the following equivalence

$$(B.3) \qquad \mathcal{L}(X) \times \mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{DPic}(X) \qquad (\underline{n}, L) \mapsto L[\underline{n}] \,,$$

where $L[\underline{n}]$ is such that $L[\underline{n}]_{|U} := L_{|U}[n_U]$ for every connected component $U \subset X$.

Now, let A be a dg Azumaya algebra over X. Following [36, Prop. 1.14], there exists an étale cover $f : Y \to X$ such that $f^*(A) \simeq \mathbf{R}\mathrm{End}_Y(P)$ for some perfect complex on Y. Let us write $(Y/X)_\bullet$ for the associated hypercovering with $Y_n := (Y/X)_n := Y^{\times_X^n}$. Under these notations, we have the following isomorphisms:

$$(B.4) \quad \mathbf{R}\mathrm{End}_{Y_2}(\mathrm{pr}_1^*(P)) \simeq \mathrm{pr}_1^* \mathbf{R}\mathrm{End}_Y(P) \simeq \mathrm{pr}_2^* \mathbf{R}\mathrm{End}_Y(P) \simeq \mathbf{R}\mathrm{End}_{Y_2}(\mathrm{pr}_2^*(P)) \,.$$

In what follows, we will use the notations $(-)_{i \cdots j} := \mathrm{pr}_{i \cdots j}^*(-)$. Thanks to Morita theory, (B.4) is induced from an isomorphism

$$(B.5) \qquad \gamma : L \otimes^{\mathbf{L}}_{Y_2} P_1 \xrightarrow{\sim} P_2 \qquad L \in \mathrm{DPic}(Y) \,,$$

which is unique up to multiplication with an element of $\mathcal{O}^*_{Y_2}$. By pulling-back γ to Y_3, in three different ways, we obtain once again by Morita theory an isomorphism $\phi : L_{23} \otimes^{\mathbf{L}}_{Y_3} L_{12} \xrightarrow{\sim} L_{13}$ in $\mathrm{DPic}(Y)$ making the following diagram commute:

$$(B.6) \qquad
\begin{array}{ccc}
L_{23} \otimes^{\mathbf{L}}_{Y_3} L_{12} \otimes^{\mathbf{L}}_{Y_3} P_1 & \xrightarrow{\ \mathrm{Id} \otimes \gamma_{12}\ } & L_{23} \otimes^{\mathbf{L}}_{Y_3} P_2 \\
{\scriptstyle \phi \otimes \mathrm{Id}} \downarrow & & \downarrow {\scriptstyle \gamma_{23}} \\
L_{13} \otimes^{\mathbf{L}}_{Y_3} P_1 & \xrightarrow[\ \gamma_{13}\ ]{} & P_3 \,.
\end{array}$$

The morphism ϕ satisfies the standard cocycle condition when pulled-back to Y_4. Moreover, ϕ is well-defined up to an obvious type of coboundary. Let $\bar{\phi}$ be the corresponding equivalence class. We call $(Y, L, \bar{\phi})$ a set of *Picard data* on Y.

Thanks to the above isomorphism (B.3), we have $L = L'[\underline{n}]$ with $L' \in \mathrm{Pic}(Y)$. Moreover, the isomorphism ϕ implies that $\underline{n}_{12} + \underline{n}_{23} = \underline{n}_{13}$ on Y_2. Hence, $\underline{n}$ defines an element of $\check{H}^1((Y/X)_\bullet, \mathbb{Z})$. Now, let $g : Z \to Y_2$ be an étale cover[8] such that $g^*(L') \simeq \mathcal{O}_Z$ in $\mathrm{Pic}(Z)$. In what follows, we fix such a trivialization. Let $S_{1,\bullet}$ be the truncated hypercovering $Z \rightrightarrows Y \to X$, $S_\bullet$ the coskeleton $\mathrm{cosk}\, S_{1,\bullet}$, and $g : S_\bullet \to (Y/X)_\bullet$ the induced map of hypercoverings. Using the chosen trivializations, the following morphism (deduced from ϕ)

$$g^*(\phi) : g^*(L'_{23}) \otimes_Z^{\mathbf{L}} g^*(L'_{12}) \longrightarrow g^*(L'_{13})$$

is given by multiplication with an element $\tilde{\phi} \in \Gamma(Z, \mathcal{O}_Z^*)$. Since $\tilde{\phi}$ still satisfies the cocycle condition, it defines an element in $\check{H}^2(S^\bullet, \mathbb{G}_m)$; one checks that this element does not depend on the chosen trivialization neither on the isomorphism γ.

The map (B.2) can now be explicitly described as the image of $(\underline{n}, \tilde{\phi})$ under

$$\check{H}^1((Y/X)_\bullet, \mathbb{Z}) \times \check{H}^2(S_\bullet, \mathbb{G}_m) \longrightarrow H^1_{\mathrm{et}}(X, \mathbb{Z}) \times H^2_{\mathrm{et}}(X, \mathbb{G}_m).$$

Lemma B.7. *Let A be a dg Azumaya algebra over X and $(Y, L, \bar{\phi})$ a set of Picard data for A. Then, A is trivial if and only if $(Y, L, \bar{\phi})$ satisfies the following condition: there exists an element $K \in \mathrm{DPic}(Y)$ and an isomorphism $\theta : L \xrightarrow{\sim} K_2 \otimes_{Y_2}^{\mathbf{L}} K_1^{-1}$ making the following diagram commute:*

(B.8)
$$
\begin{array}{ccc}
L_{23} \otimes_{Y_3}^{\mathbf{L}} L_{12} & \xrightarrow{\quad\phi\quad} & L_{13} \\
{\scriptstyle \theta_{23}\otimes\theta_{12}}\downarrow & & \downarrow{\scriptstyle \theta_{13}} \\
(K_3 \otimes_{Y_3}^{\mathbf{L}} K_2^{-1}) \otimes_{Y_3}^{\mathbf{L}} (K_2 \otimes_{Y_3}^{\mathbf{L}} K_1^{-1}) & \xrightarrow[\text{canonical}]{} & K_3 \otimes_{Y_3}^{\mathbf{L}} K_1^{-1}\,.
\end{array}
$$

Proof. Let $(Y, L, \bar{\phi})$ be a Picard data for A satisfying the above condition(s). In this case, we can replace the L in the isomorphism $\phi : L \otimes_{Y_2}^{\mathbf{L}} P_1 \xrightarrow{\sim} P_2$ by $K_1 \otimes_{Y_2}^{\mathbf{L}} K_2^{-1}$. Via the above procedure, using the lower row of (B.8), one observes that $\check{\psi}(A)$ is trivial. Therefore, the injectivity of (B.2) implies that A is trivial.

Let us now prove the converse. Assume that A is trivial. In this case, $A \simeq \mathbf{R}\mathrm{End}_X(Q)$ with Q a perfect complex on X. Consider the isomorphisms

(B.9) $$\mathbf{R}\mathrm{End}_Y(P) \simeq f^*(A) = \mathbf{R}\mathrm{End}_X(f^*(Q)).$$

Thanks to Morita theory, (B.9) is induced from an isomorphism $f^*(Q) \simeq K \otimes_Y^{\mathbf{L}} P$ with $K \in \mathrm{DPic}(Y)$. Since $(f^*(Q))_1 \simeq Q_{12} \simeq (f^*(Q))_2$ on Y_2, we hence obtain an isomorphism $K_1 \otimes_{Y_2}^{\mathbf{L}} P_1 \simeq K_2 \otimes_{Y_2}^{\mathbf{L}} P_2$. This implies that $L \simeq K_1 \otimes_{Y_2}^{\mathbf{L}} K_2^{-1}$. The commutativity of (B.8) follows from Morita theory. This concludes the proof. $\square$

Definition B.10. Given a dg Azumaya algebra A over X, let $\deg(A) := \sqrt{\mathrm{rank}(A)}$, where $\mathrm{rank}(A)$ is defined by the usual alternating sum of ranks in each degree.
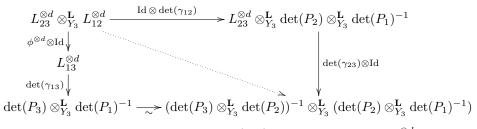
We consider $\deg(A)$ as a locally constant function on X. Note that the local étale triviality of A implies that $\deg(A)$ takes integer values. The following result plays a key role in the case of ordinary Azumaya algebras; see [19, §IV Thm. 6.1].

---

[8]Of course, the "étale" cover can be taken to be a Zariski cover.

Theorem B.11. *The dg Azumaya algebra $A^{\otimes \deg(A)}$ is trivial.*

Proof. Let $d := \deg A$ and $B := A^{\otimes d}$. If $(Y, L, \overline{\phi})$ is a set of Picard data for A, we need to show that the set of Picard data $(Y, L^{\otimes d}, \overline{\phi}^{\otimes d})$ for B satisfies the conditions of Lemma B.7. By taking the determinant of (B.5) we obtain an isomorphism

$$\det(\gamma) : L^{\otimes d} \otimes^{\mathbf{L}}_{Y_2} \det(P_1) \overset{\sim}{\to} \det(P_2) \quad \Leftrightarrow \quad \det(\gamma) : L^{\otimes d} \overset{\sim}{\to} \det(P_2) \otimes^{\mathbf{L}}_{Y_2} \det(P_1)^{-1}\,.$$

Similarly, by first taking the determinant of (B.6), and then tensoring the result with $\det P_1^{-1}$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
L^{\otimes d}_{23} \otimes^{\mathbf{L}}_{Y_3} L^{\otimes d}_{12} & \xrightarrow{\mathrm{Id} \otimes \det(\gamma_{12})} & L^{\otimes d}_{23} \otimes^{\mathbf{L}}_{Y_3} \det(P_2) \otimes^{\mathbf{L}}_{Y_3} \det(P_1)^{-1} \\
{\scriptstyle \phi^{\otimes d} \otimes \mathrm{Id}} \downarrow & & \\
L^{\otimes d}_{13} & & \downarrow {\scriptstyle \det(\gamma_{23}) \otimes \mathrm{Id}} \\
{\scriptstyle \det(\gamma_{13})} \downarrow & & \\
\det(P_3) \otimes^{\mathbf{L}}_{Y_3} \det(P_1)^{-1} & \xrightarrow{\ \sim\ } & (\det(P_3) \otimes^{\mathbf{L}}_{Y_3} \det(P_2))^{-1} \otimes^{\mathbf{L}}_{Y_3} (\det(P_2) \otimes^{\mathbf{L}}_{Y_3} \det(P_1)^{-1})
\end{array}
$$

Note that the lower triangle is precisely (B.8) with L replaced by $L^{\otimes d}$ and K by $\det(P)$. This concludes the proof. $\square$

The following result[9] sheds some new light on dg Azumaya algebras.

Theorem B.12. *Let A be a dg Azumaya algebra of rank $(r_1, \ldots, r_s)$. When $r_1, \ldots, r_s \neq 0$, A is isomorphic in $\mathrm{Hmo}(k)$ to an ordinary Azumaya algebra.*

Proof. We may assume without loss of generality that $\mathrm{Spec}(k)$ is indecomposable. In this case, $s = 1$; let $r := r_1$. Thanks to Theorem B.11, the class $\psi(A) \in H^1_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{Z}) \times H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m)$ is r-torsion. Making use of the short exact sequence of constant sheaves $0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Q}} \to \underline{\mathbb{Q}/\mathbb{Z}} \to 0$, we observe that $H^1_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{Z})$ is contained in $H^1_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{Q})$. In particular, it is torsion-free. This hence implies that $\psi(A) \in H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m)_{\mathrm{tors}}$. Finally, making use of Gabber's result [11, Thm II.1], we conclude that A is isomorphic in $\mathrm{Hmo}(k)$ to an ordinary Azumaya algebra. $\square$

Corollary B.13. *Let A_α be a dg Azumaya algebra as in Example B.1(iii) of rank $(r_1, \ldots, r_s)$. Then, there exists an integer $i \in \{1, \ldots, s\}$ such that $r_i = 0$.*

Corollary B.13 shows that the dg Azumaya algebras A_α associated to non-torsion étale cohomology classes $\alpha \in H^2_{\mathrm{et}}(\mathrm{Spec}(k), \mathbb{G}_m)$ have always trivial rank.

Noncommutative motives. By combining the above Theorem B.12 with [35, Thm. 2.1], we obtain the following computation:

Corollary B.14. *Given a dg Azumaya algebra A as in Theorem B.12, we have a canonical isomorphism $U(k)_R \simeq U(A)_R$ for every commutative ring R containing $1/r$ with $r := r_1 \times \cdots \times r_s$.*

Intuitively speaking, Corollary B.14 shows that the difference between the noncommutative motives of A and k is a torsion phenomenon. As the next result shows, this is *not* the case when we consider the dg Azumaya algebras of Example B.1(iii):

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[9]As mentioned by the referee, Theorem B.12 can alternatively be obtained by combining Theorem B.11 with the material of [4, §7.4-§7.5].

Theorem B.15. *Let A be a dg Azumaya algebra which is not isomorphic in $\mathrm{Hmo}(k)$ to an ordinary Azumaya algebra. Then, we have $U(k)_{\mathbb{Q}} \not\simeq U(A)_{\mathbb{Q}}$ in $\mathrm{Hmo}_0(k)_{\mathbb{Q}}$.*

Proof. Let us assume that $U(k)_{\mathbb{Q}} \simeq U(A)_{\mathbb{Q}}$. Thanks to the construction of the $\mathbb{Q}$-linear category $\mathrm{Hmo}_0(k)_{\mathbb{Q}}$ (see §5.3) and to the fact that A is smooth and proper (see [36, Prop. 2.5]), there exists a right A-module $P \in \mathcal{D}_c(A)$, a right A^{op}-module $Q \in \mathcal{D}_c(A^{\mathrm{op}})$, and positive integers $m, n > 0$ satisfying the following equalities:

$$[P \otimes_A^{\mathbf{L}} Q] = n \cdot [k] \in K_0 \mathcal{D}_c(k) \qquad [Q \otimes^{\mathbf{L}} P] = m \cdot [A] \in K_0 \mathcal{D}_c(A^{\mathrm{op}} \otimes^{\mathbf{L}} A) \,.$$

Now, choose an étale extension $k \to k'$ making $A' := A \otimes_k k'$ and k' isomorphic in $\mathrm{Hmo}(k')$; see [36, Prop. 2.14]. Once again by construction of the category $\mathrm{Hmo}(k)$, there exists a right A'-module $R \in \mathcal{D}_c(A')$, a right A'^{op}-module S, and isomorphisms

$$S \otimes_{k'}^{\mathbf{L}} R \simeq A' \in \mathcal{D}_c(A'^{\mathrm{op}} \otimes^{\mathbf{L}} A) \qquad R \otimes_{A'}^{\mathbf{L}} S \simeq k' \in \mathcal{D}_c(k') \,,$$

giving rise the following equalities:

$$[(P' \otimes_{A'}^{\mathbf{L}} S) \otimes_{k'}^{\mathbf{L}} (R \otimes_{A'}^{\mathbf{L}} Q')] = n \cdot [k'] \qquad [(R \otimes_{A'}^{\mathbf{L}} Q') \otimes_{k'}^{\mathbf{L}} (P' \otimes_{A'}^{\mathbf{L}} S)] = m \cdot [k'] \,.$$

This clearly implies that the rank of $P' \otimes_{A'} S$ is non-trivial. Thanks to Lemma B.16 below, $P' \otimes_{A'} S$ is a compact generator of $\mathcal{D}(k')$. Hence, since S induces an isomorphism in $\mathrm{Hmo}(k')$ between A' and k', P' is a compact generator of $\mathcal{D}(A')$. Making use of Lemma B.17 below, we conclude moreover that P is a generator of A. We can therefore consider the dg Azumaya k-algebra $B := \mathbf{R}\mathrm{End}_A(P)$. Note that B is isomorphic in $\mathrm{Hmo}(k)$ to the dg Azumaya k-algebra A. Using the equalities

$$\mathrm{rank}(B) = \mathrm{rank}(\mathbf{R}\mathrm{End}_A(P)) = \mathrm{rank}(\mathbf{R}\mathrm{End}_{A'}(P')) = \mathrm{rank}(\mathbf{R}\mathrm{End}_{k'}(P' \otimes_{A'}^{\mathbf{L}} S)) \,,$$

we observe that the rank of B is also non-trivial. The above Theorem B.12 hence implies that B (and consequently A) is isomorphic in $\mathrm{Hmo}(k)$ to an ordinary Azumaya algebra. This contradiction achieves the proof. $\square$

Lemma B.16. *Let k be a commutative ring and $P \in \mathcal{D}_c(k)$. If $\mathrm{rank}(P) \neq 0$, then P is a (compact) generator of $\mathcal{D}(k)$.*

Proof. Let us assume first that k is Noetherian. If $\mathrm{rank}(P) \neq 0$, then $\mathrm{rank}(P \otimes_k^{\mathbf{L}} k(\mathfrak{p})) \neq 0$ for every $\mathfrak{p} \in \mathrm{Spec}(k)$. This automatically implies that $P \otimes_k^{\mathbf{L}} k(\mathfrak{p}) \neq 0$ for every $\mathfrak{p} \in \mathrm{Spec}(k)$. Using Neeman's work [26, Thm 2.8], we hence conclude that the triangulated localizing subcategory of $\mathcal{D}(k)$ generated by P agrees with $\mathcal{D}(k)$.

Let us now prove the general case. Represent P by a complex of finitely generated projective k-modules. Thanks to Noetherian approximation, there exists a Noetherian subring k_0 of k and a complex of finitely generated projective k_0-modules P_0 such that $P \simeq P_0 \otimes_{k_0} k$; note that since P_0 is a complex of projective k_0-modules, we have also $P \simeq P_0 \otimes_{k_0}^{\mathbf{L}} k$. The (locally constant) function $\mathrm{rank}(P)$ on $\mathrm{Spec}(k)$ is the pull-back of the (locally constant) function $\mathrm{rank}(P_0)$ on $\mathrm{Spec}(k_0)$. Since k_0 is a subring of k the map $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$ is surjective on connected components and we obtain $\mathrm{rank}(P_0) \neq 0$. Since the ring k_0 is Noetherian, the preceding considerations hence imply that P_0 is a (compact) generator of $D(k_0)$. The proof follows now automatically from the combination of the classical adjunction

$$\begin{array}{c} \mathcal{D}(k) \\ {\scriptstyle -\otimes_{k_0}^{\mathbf{L}} k} \uparrow \downarrow {\scriptstyle \mathrm{Res}} \\ \mathcal{D}(k_0) \end{array}$$

with the fact that the restriction functor Res is conservative. $\square$

Lemma B.17. *Let A be a dg k-algebra, $P \in \mathcal{D}_c(A)$, and k'/k a faithfully flat extension. If P' is a generator of $\mathcal{D}(A')$, then P is a (compact) generator of $\mathcal{D}(A)$.*

Proof. Let $Q \in \mathcal{D}(A)$ such that $\mathrm{RHom}_A(P, Q) = 0$. Since the extension k'/k is flat and P is a perfect complex, we have $\mathrm{RHom}_{A'}(P', Q') = \mathrm{RHom}_A(P, Q)' = 0$. Consequently, $Q' = 0$. Using the fact that the extension k'/k is faithfully flat, we hence conclude that $Q = 0$. $\square$

Remark B.18. Let A be a dg Azumaya algebra. Similarly to the case of ordinary Azumaya algebras, we have the following equivalence of categories:

$$\mathcal{D}_c(k) \xrightarrow{\simeq} \mathcal{D}_c(A^{\mathrm{op}} \otimes^{\mathbf{L}} A) \qquad P \mapsto P \otimes^{\mathbf{L}} A \,.$$

Hence, since the Hochschild homology[10] $HH_*(A)$ and the Hochschild cohomology $HH^*(A)$ of A can be recovered from $\mathcal{D}_c(A^{\mathrm{op}} \otimes^{\mathbf{L}} A)$, we obtain induced isomorphisms

$$k \simeq HH_*(k) \simeq HH_*(A) \qquad k \simeq HH^*(k) \simeq HH^*(A) \,.$$

Note that Theorem B.15 implies that these isomorphisms are *not* motivic, *i.e.* they are not induced from an isomorphism in $\mathrm{Hmo}_0(k)$.

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