Partial orders for zero-sum arrays with applications to network theory
Non Peer-reviewed author version

Highlights

- We introduce several partial orders in the set of zero-sum arrays
- This leads to theories of hierarchy and domination
- This theory is applied to directed, acyclic networks

Partial orders for zero-sum arrays
with applications to network theory

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Abstract

In this contribution we study partial orders in the set of zero-sum arrays. Concretely, these partial orders relate to local and global hierarchy and dominance theories. The exact relation between hierarchy and dominance curves is explained. Based on this investigation we design a new approach for measuring dominance or stated otherwise, power structures, in networks. A new type of Lorenz curve to measure dominance or power is proposed, and used to illustrate intrinsic characteristics of networks. The new curves, referred to as D-curves are partly concave and partly convex. As such they do not satisfy Dalton's transfer principle.

It is shown that D-curves have several properties making them suitable to measure dominance. If dominance and being a subordinate are reversed, the dominance structure in a network is also reversed.

Keywords: zero-sum arrays, dominance, hierarchy, power structures, pseudo-Lorenz curves, acyclic digraphs

Introduction

Lorenz curves were introduced at the beginning of the 20th century as a graphical device to show the intrinsic inequality among a set of cells (Lorenz, 1905). Since their introduction, it has become clear that they constitute a powerful device that can be used in many fields and for many applications (Marshall, Olkin & Arnold, 2011). Moreover, many variations on the basic construction have been introduced. For an overview we refer to (Rousseau, 2011). Classically the values associated with cells used in the construction of a Lorenz curve are positive. Yet in applications cells may have positive as well as negative values. For instance, Lorenz curves representing wealth may include people who are in debt, hence have negative wealth. For this reason Lorenz curves are sometimes allowed to include cells with negative values. This case is studied, e.g. in (Cowell & Van Kerm, 2015; Egghe, 2002, section 2.1). Yet, when the sum of all data is zero this approach cannot be used. This is the main motivation for this study of zero-sum arrays. This contribution consists of two main parts: one studying zero-sum arrays in general (part A) and one about applications in directed networks (part B). We make a distinction between hierarchical structures and dominance structures. Besides in information science where one studies networks such as article citation networks, we also have in mind future applications in food webs (Cohen, 1978; Garlaschelli et al., 2003), disease networks (Goh et al., 2007), social networks (Shizuka & McDonald, 2012), innovation networks (Guan et al., 2015), river flows (Poff et al., 2003), epidemiology (Moslonka-Lefebvre et al., 2011), management (Montgomery & Oliver, 2007) and complex networks in general (Strogatz, 2001; Newman, 2003). We like to point out that in this contribution we study structures, not single elements. For instance, when applied to networks we attach a value to the dominance structure as a whole and do not intend to determine a value representing the power of a node, as done e.g. in (van den Brink & Gilles, 2000).

A. Zero-sum arrays

A1. Basic definitions

If X is a (finite) array, i.e. an N-tuple, then the j-th element of X is denoted as $(X)_j = x_j$, where x_j is a real number. In this investigation the components of any array are assumed to be ranked in decreasing order. If X is an array then $-X$ denotes the array where every component x_j is replaced by $-x_j$. Note that also the components of $-X$ are ranked in decreasing order. Hence: $(-X)_j = -(X)_{N-j+1}$. This array is called the opposite array of X. Clearly, the opposite of the opposite is again the original. Hence, taking the opposite array is an involutive operation, e.g. when applied twice it returns the original object. An array is said to be symmetric if, after re-ranking from largest to smallest, X = -X.

Examples. If X = (3,2,-6) then –X = (6,-2,-3). The array S = (3,1,0,-1,-3) is symmetric.

Definition: zero-sum array

If X = (x_1, …, x_N) is a real-valued array (N-tuple), such that $\sum_{i=1}^{N} x_i = 0$, then X is called a zero-sum array. The set of all zero-sum arrays is denoted as **Z;** its subset of arrays of length N, namely $\textbf{\textit{Z}} \cap \mathbb{R}^N$ is denoted as **Z**$_N$. The symmetric array S from the previous example is also an example of a zero-sum array. Actually all symmetric arrays are zero-sum arrays. Y_1 = (1,1,-2) and Y2 = (0.4, 0.3, 0, 0, -0.7) are examples of a non-symmetric zero-sum arrays.

Operations on zero-sum arrays

We mention the following elementary properties.

1. If X is a zero-sum array of length N, then cX, with c a real number, is again a zero-sum array of length N.
2. **Z**$_N$ is a convex cone over the real numbers. This means that if X and Y are zero-sum arrays of length N and α and β are positive real numbers, then αX + βY is again a zero-sum array of length N.

A2. Two types of pseudo-Lorenz curves for zero-sum arrays

Given a zero-sum array X, we set

$$I_+(X) = \{i \in \{1,...,N\} \text{ such that } x_i > 0\}$$
$$I_0(X) = \{i \in \{1,...,N\} \text{ such that } x_i = 0\} \text{ and}$$
$$I_-(X) = \{i \in \{1,...,N\} \text{ such that } x_i < 0\}.$$

We assume from now on that X is not the trivial zero-array, hence $I_0 \neq \{1,…,N\}$. This implies that $I_+(X)$ and $I_-(X)$ are always non-empty, but they may have different numbers of elements. When it is clear about which array we are talking or when it does not matter we simply write I_+, I_0 or I_-.

As $\sum_{i=1}^{N} x_i = 0$ we see that $\sum_{i \in I_+} x_i + \sum_{i \in I_0} x_i + \sum_{i \in I_-} x_i = \sum_{i \in I_+} x_i + \sum_{i \in I_-} x_i = 0$. Hence

$$\sum_{i \in I_+} x_i = - \sum_{i \in I_-} x_i .$$

Next we put $\Sigma_+ = \sum_{i \in I_+} x_i$ and $\forall i = 1,...,N : a_i = \dfrac{x_i}{\Sigma_+}$. With each zero-sum array X,

we associate a corresponding A-array, denoted A_X, and equal to $A_X = (a_1, \ldots, a_N)$. Of course, also A_X is a zero-sum array and $A_{A_X} = A_X$ (this is a projection

property). Next we form the array P_X of partial sums: $(P_X)_j = p_j = \sum_{k=1}^{j} a_k$. Clearly

$p_N = 0$ and $p_{N-|I_-|} = 1$. Similarly we form the array Q_X, with $(Q_X)_j = q_j = \sum_{k=1}^{j} |a_k|$.

For i=1,…,N-|I-|: $p_i = q_i$.
Note that, for

$$k \in I_- : q_k = \sum_{j=1}^{k} |a_j| = \sum_{j=1}^{N-|I_-|} a_j + \sum_{j=N-|I_-|+1}^{k} |a_j| = 1 - \sum_{j=N-|I_-|+1}^{k} a_j = 1 - p_k + p_{N-I_-} = 2 - p_k .$$

Hence $q_N = 2 - 0 = 2$.

The two types of pseudo-Lorenz curves that we will study will be called H-curves and D-curves. They are not really Lorenz curves as they do not satisfy the transfer property (Dalton, 1920; Egghe & Rousseau, 1990, p. 364), as shown further on, which explains the use of the term pseudo-Lorenz curve instead of Lorenz curves.

A.3 Construction of an H-curve (Egghe, 2002; Egghe & Rousseau, 2004)

The H-curve (H for hierarchy, see section A7 and part B for an explanation of this term) of the zero-sum array X, denoted as $H_x(t)$, $t \in [0,1]$ is a polygonal

curve connecting $(0,0)$, $(1/N, a_1)$, $(2/N, a_1 + a_2)$ and so on till $\left(\dfrac{|I_+|}{N}, 1\right)$. From

this point on, the curve continues to the point $(\dfrac{N-|I_-|}{N}, 1)$, …, $(i/N, \sum_{k=1}^{i} a_k)$

ending in (1,0). If $I_0 = \varnothing$, hence $\dfrac{|I_+|}{N} = \dfrac{N-|I_-|}{N}$, there is just one point of

intersection between the H- curve and the line y=1. If $I_0 \neq \varnothing$, then

$$\frac{|I_+|}{N} < \frac{N-|I_-|}{N}, \text{ and from } x = \frac{|I_+|}{N} \text{ to } x = \frac{N-|I_-|}{N}, \text{ the H-curve coincides with}$$

the line y=1. As $\sum_{i=1}^{N} a_i = 0$, an H-curve always ends in the point (1,0). The

H-curve of the N-tuple X can also be considered as the graph of a function. This function is denoted as $H_X(t)$, $0 \leq t \leq 1$, and is defined as follows:

for $t \in \left[\dfrac{i}{N}, \dfrac{i+1}{N}\right]$

$$H_X(t) = \begin{cases} Na_1 t & i=0 \\ \displaystyle\sum_{k=1}^{i} a_k + Na_{i+1}\left(t - \dfrac{i}{N}\right), & i=1,...,N-1 \end{cases}$$

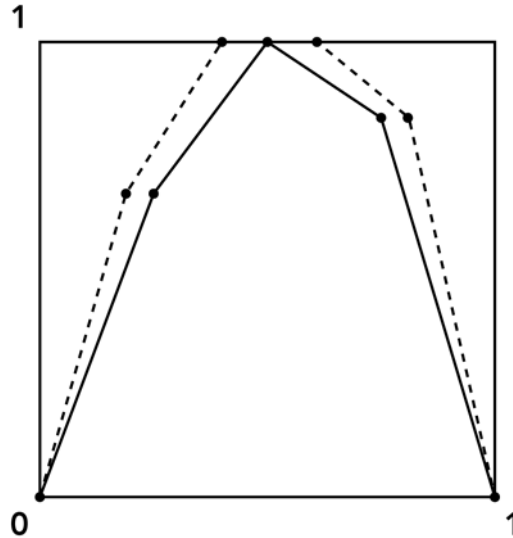H-curves are like bridges always starting in (0, 0) and ending at (1, 0). Fig.1 shows two H-curves.

Fig.1. H- curves of the arrays (4,2,-1,-5) and (4,2,0,-1,-5).

A4. Construction of a D-curve

Contrary to the H-curve, which was introduced by Egghe (2002), the D-curve (D for dominance, see section A7 and part B for an explanation of this term) of a zero-sum array X, is new. It is defined as the polygonal line connecting the points

$$(0,0) \rightarrow \left(\frac{1}{N}, a_1 \right) \rightarrow ... \rightarrow \left(\frac{i}{N}, \sum_{j=1}^{i} a_j \right) \rightarrow ... \rightarrow \left(\frac{|I_+|}{N}, 1 \right) \rightarrow$$

$$\left(\frac{N-|I_-|}{N}, 1 \right) \rightarrow ... \rightarrow \left(\frac{k}{N}, \sum_{j=1}^{k} |a_j| \right) \rightarrow ... \rightarrow (1,2)$$

where $i \in I_+$, $k \in I_-$.

Moreover, this curve can be described as a function, denoted as $D_X(t)$, as follows: for $t \in [0,1]$, we have:

$$D_X(t) = \begin{cases} Na_1 t, & t \in [0, \frac{1}{N}] \\ \sum_{k=1}^{i} a_k + Na_{i+1}\left(t - \frac{i}{N} \right), & t \in [\frac{i}{N}, \frac{i+1}{N}], \quad i=1,...,N-|I_-|-1 \\ \sum_{k=1}^{i} |a_k| + N|a_{i+1}|\left(t - \frac{i}{N} \right), & t \in [\frac{i}{N}, \frac{i+1}{N}], \quad i = N-|I_-|,...,N-1 \end{cases}$$

Contrary to other forms of Lorenz curves and the H-curve (Lorenz, 1905; Egghe, 2002; Egghe & Rousseau, 2004), the D-curve is partly concave (namely when the a's are positive), and partly convex (the part where the a's are negative), see Fig.2.

If $|I_+| = N - |I_-|$ then the D-curve has no horizontal part. If $I_0 \neq \varnothing$ then it has a horizontal middle part, at y=1. Recall that this remark also holds for the H-curve.

An example

The D-curve for (4,2,0,-1,-5), N=5, has A-array (array of a-values) equal to A = $\left(\frac{4}{6}, \frac{2}{6}, 0, \frac{-1}{6}, \frac{-5}{6} \right)$, hence connects points with ordinates $\left(0, \frac{4}{6}, \frac{6}{6}, \frac{6}{6}, \frac{7}{6}, \frac{12}{6}=2 \right)$.

The D-curve of (4,2,-1,-5), N=4, connects points with ordinates

$$\left(0, \frac{4}{6}, \frac{6}{6}, \frac{7}{6}, \frac{12}{6} = 2\right).$$ These two D-curves are shown in Fig. 2.
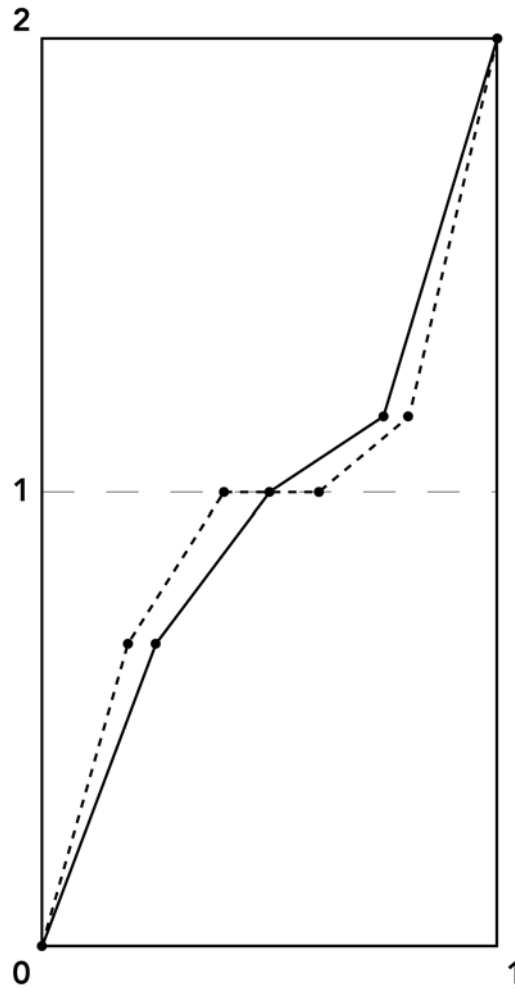

Fig.2. D- curves of the arrays (4,2,-1,-5) and (4,2,0,-1,-5).

If $n_0 = |I_+|$, the number of elements in I_+, then $D_X(n_0/N) = H_X(n_0/N) = 1$. In the previous example (Fig.2) $n_0=2$ and the points with abscissae 2/5 and 2/4 indeed have an ordinate equal to 1.

We mention the following lemmas without proof.

Lemma A. The H-curve of –X is obtained from that of X by a reflection with respect to the line x = 0.5. For every t in [0,1]: $H_X(t) = H_{(-X)}(1-t)$.

Lemma B. The D-curve of –X is obtained from that of X by a reflection with respect to the point (0.5, 1). For every t in [0,1]: $D_X(t) + D_{(-X)}(1-t) = 2$.

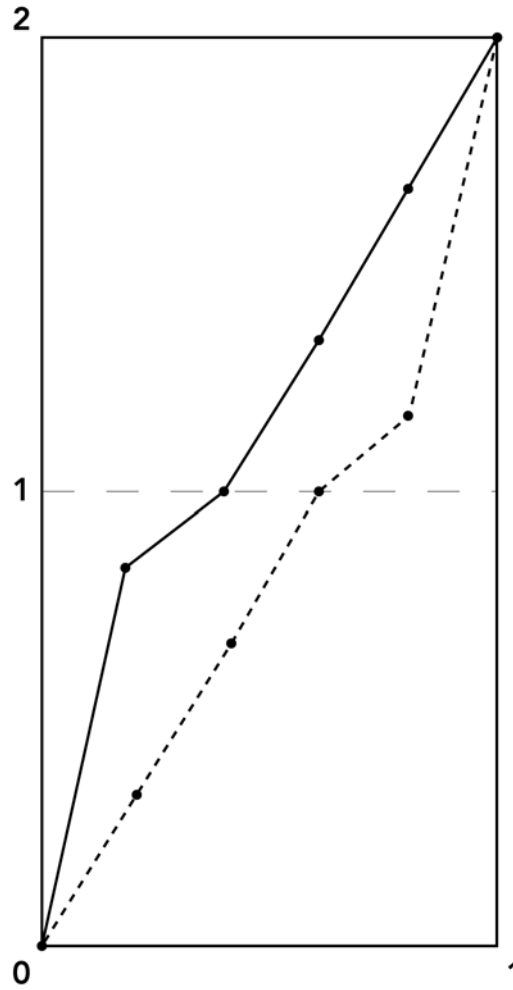Lemma B is illustrated, for X = (5,1,-2,-2,-2) and –X = (2,2,2,-1,-5) in Figure 3.

Fig. 3. The D-curve of the array (5,1,-2,-2,-2) and of its opposite.

Definition. Equivalent zero-sum arrays

Zero-sum arrays that lead to the same H-curve (or equivalently, same D-curve) are said to be equivalent. The arrays (4,2,0,-1,-5), (8,4,0,-2,-10) and $\left(\frac{4}{6},\frac{2}{6},0,\frac{-1}{6},\frac{-5}{6}\right)$ are equivalent. Equivalent zero-sum arrays of length N all have the same A-array. Also (1,1,0,-2) and (1,1,1,1,0,0,-2,-2) are equivalent zero-sum arrays.

A5. Partial orders for zero-sum arrays

Definition: the hierarchical relation $\geq_H$ in Z (Egghe, 2002)

Let X and Y be zero-sum arrays, not necessarily of the same length, then we say that Y is H-larger than X, denoted as $Y \geq_H X$, if, for each $t \in [0,1]$, $H_Y(t) \geq$

$H_X(t)$. Y is strictly H-larger than X, denoted as $Y >_H X$, if, for each $t \in [0,1]$, $H_Y(t)$ ≥ $H_X(t)$ and there is at least one point t_0, hence infinitely many, where $H_Y(t_0) > H_X(t_0)$. If Y is H-larger than X then X is H-smaller than Y. Formally we write:

$$X \leq_H Y \quad \text{if and only if} \quad \forall t \in [0,1]: H_X(t) \leq H_Y(t)$$

Moreover, X = Y (as equivalence classes) if and only if $\forall t \in [0,1]: H_X(t) = H_Y(t)$.

If X and Y have the same length we can easily express this relation as an inequality: let $X = (x_1,\ldots,x_N)$ and $Y = (y_1,\ldots,y_N)$ be zero-sum arrays. Then Y H-majorizes X, denoted as $X \leq_H Y$, if the following inequality is satisfied:

$$\text{for every} \quad i = 0,1,\ldots,N: \quad \sum_{j=1}^{i} a_j \leq \sum_{j=1}^{i} a'_j \qquad a_i = \frac{x_i}{\Sigma_+} \qquad a'_i = \frac{y_i}{\Sigma'_+}$$

(recall that always: $x_1 \geq x_2 \geq \ldots \geq x_N$; and $y_1 \geq y_2 \geq \ldots \geq y_N$). The relation $\leq_H$ determines a partial order in the set of all equivalence classes of zero-sum arrays. Recall that a partial order in a set P is a binary relation over this set which is reflexive, antisymmetric, and transitive. A set equipped with a partial order is called a poset. When $X \leq_H Y$ it is clear that the H-curve of X lies nowhere strictly above the H-curve of Y. Of course, these curves may coincide over some region. This observation is the core of the hierarchical theory (Egghe, 2002).

If $Y \geq_H X$ and not $Y >_H X$ then $H_X = H_Y$, but from this equality we may not conclude that X = Y. We can only conclude that X and Y belong to the same equivalence class. If moreover X and Y have the same dimension (N) we may conclude that the A-arrays of X and Y coincide. Hence there exists c > 0 such that Y = cX. We observe that when talking about a partial order in the set of zero-sum arrays, this is a slight misuse of terminology, as we actually mean a partial order in the set of equivalence classes of zero-sum arrays.

We recall that in a poset two elements may not be comparable. In such case we say that these two elements are intrinsically incomparable for this partial order.

Definition: the dominance relation $\geq_D$ in Z

Let X and Y be zero-sum arrays, not necessarily of the same length, then we say that Y is D-larger than X, denoted as $Y \geq_D X$, if, for each $t \in [0,1]$, $D_Y(t) \geq$

$D_X(t)$. Y is strictly D-larger than X, denoted as Y $>_D$ X, if, for each $t \in [0,1]$, $D_Y(t)$ $\geq D_X(t)$ and there is at least one point t_0, hence infinitely many, where $D_Y(t_0) > D_X(t_0)$. If Y is D-larger than X then X is D-smaller than Y. When X $\leq_D$ Y it is clear that the D-curve of X lies nowhere strictly above the D-curve of Y. The relation $\leq_D$ determines a partial order in the set of all equivalence classes of zero-sum arrays. Formally we write:

$$X \leq_D Y \quad \text{if and only if} \quad \forall t \in [0,1] : D_X(t) \leq D_Y(t)$$

Moreover, X = Y (as equivalence classes) if and only if $\forall t \in [0,1] : D_X(t) = D_Y(t)$.

In case X= $(x_1,\ldots,x_N)$ and Y = $(y_1,\ldots,y_N)$ have the same length also this relation can easily be expressed as an inequality: Y D-dominates X, denoted as X $\leq_D$ Y, if the following relation is satisfied:

$$\text{for every} \quad i = 0,1,\ldots,N: \quad \sum_{j=1}^{i} |a_j| \leq \sum_{j=1}^{i} |a_j'| \quad \text{with} \quad a_i = \frac{x_i}{\Sigma_+} \quad \text{and} \quad a'_i = \frac{y_i}{\Sigma_+'}$$

For reasons that will be explained further on (see A7 and part B) we refer to this partial order as a dominance relation, not to be confused with the standard majorization relation for Lorenz concentration theory (Marshall et al., 2011). As for the H-partial order, here too we misuse the terminology when we say that array X is D-smaller than array Y (when we actually mean equivalence classes).

Definition: the replication operator R_K

For $K \in \mathbb{N}$ the replication operator R_K is defined as:

$$Z_N \to Z_{KN} : X \to R_K(X) = \left(\underbrace{x_1,\ldots,x_1}_{K \text{ times}}, \underbrace{x_2,\ldots,x_2}_{K \text{ times}},\ldots,\underbrace{x_N,\ldots,x_N}_{K \text{ times}} \right)$$

Hence the j-th component of $R_k(X)$ is equal to x_i, where the index i is obtained as follows:
$$\begin{cases} i = \dfrac{j}{K}, & \text{if this quotient is a natural number} \\ i = \left\lfloor \dfrac{j}{K} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

Here $\lfloor x \rfloor$ denotes the floor function, defined as the largest integer smaller than or equal to x.

If X is a zero-sum array then also $R_K(X)$ is a zero-sum array. Clearly X and $R_K(X)$ are H- and D-equivalent as they lead to the same H-curves and D-curves (proof omitted).

Proposition 1

a) If $X \leq_H Y$, then $-X \leq_H -Y$.

b) If $X \leq_D Y$, then $-Y \leq_D -X$.

Proof: a) This follows immediately from lemma A.
b) This follows immediately from lemma B.

Corollary

Any two non-equivalent symmetric zero-sum arrays are intrinsically incomparable for the D-partial order. Hence, their D-curves always intersect.

Proof

If the equivalence classes of arrays X and Y are different and symmetric, and if they are comparable, then $X \leq_D Y$ implies $-Y \leq_D -X$ by the previous proposition. But, by the symmetry property $X \leq_D Y$ is the same as $-X \leq_D -Y$. Hence the D-curves of X and Y are the same, which is a contradiction. We conclude that symmetric zero-sum arrays are incomparable for the D-partial order.

A.6 Maximum and minimum curves

We now consider some properties of H- and D-curves focusing on maximum and minimum curves. Yet, we note first that if $X \leq_H Y$ then X and Y are incomparable in the D-partial order and similarly, if $X \leq_D Y$ then X and Y are incomparable in the H-partial order. This follows from the fact that the negative parts of the H- and of the D-curve are each other's mirror image with respect to the line y=1.

Maximum H-curves
For fixed N the maximum H-curve occurs when (0,0) is immediately connected to the point with coordinates (1/N,1), connected to the point ((N-1)/N,1), via the line y=1, and then finally connected to the point (1,0). This H-curve

corresponds to all zero-sum arrays of the form X = (s,0,…,0,-s), with s > 0. The A-array corresponding to a maximum H-curve of length N is called an H-maximum N array.

Clearly, considering N as a variable, the line y = 1, from (0,1) to (1,1) is an upper bound for all H-maximum N-curves.

Maximum D-curves

For fixed N the maximum D-curve occurs when (0,0) is linearly connected to the point with coordinates (1/N,1), and then linearly connected to the endpoint (1,2). This D-curve corresponds to all zero-sum arrays of the form X = (s,-t,…,-t), with s, t > 0 and s = (N-1)t, by the requirement that X must be a zero-sum array. The N-array corresponding to a D-maximum N-curve of length N is called a D-maximum N-array.

Clearly, considering N as a variable, the line y = x + 1, passing through the points (0,1) and (1,2) is an upper bound for all D-maximum N-curves.

Minimum H-curves

For fixed N, there is no minimum H-curve, recalling that the zero-array is excluded. Yet, Egghe (2002) characterizes all minimal H-curves, i.e. all H-curves L for which there does not exist another H-curve L' such that L'< L. This characterization is provided in the following theorem (Egghe, 2002, Theorem 2.2).

Theorem. Let N be a fixed natural number larger than 1. Let X = (x_1, …,x_N) be a decreasing zero-sum array. Then L_X is minimal if and only if L_X consists of two straight lines, one connecting (0,0) with (i/N,1) and one connecting (i/N,1) with (1,0), i=1, …, N-1. Such a curve is obtained for

$$X = \left(\underbrace{s,…,s}_{j}, \underbrace{-t,…,-t}_{N-j} \right)$$

where s, t > 0 and js = (N-j)t. Hence, there are exactly N-1 minimal curves (N fixed). There is no minimum H-curve, except when N=2.

Minimum D-curves

For fixed N, a minimum D-curve is obtained by connecting the origin (0,0) linearly to the point with coordinates $\left(\dfrac{N-1}{N}, 1 \right)$, and then further linearly to the point (1,2). This minimum D-curve corresponds to all arrays of the form Y =

(u,…, u, -v), with u, v > 0 and v = (N-1)u. Clearly, if X is a maximum N-array then –X is a minimum. The line y = x, passing through the origin and the point (1,1) is a lower bound for all these minimum D-curves.

Figure 4 provides an example of a maximum and a minimum D-curve for the case N =5. These curves correspond to the arrays X = (4,-1.-1,-1,-1) and –X = (1,1,1,1,-4).
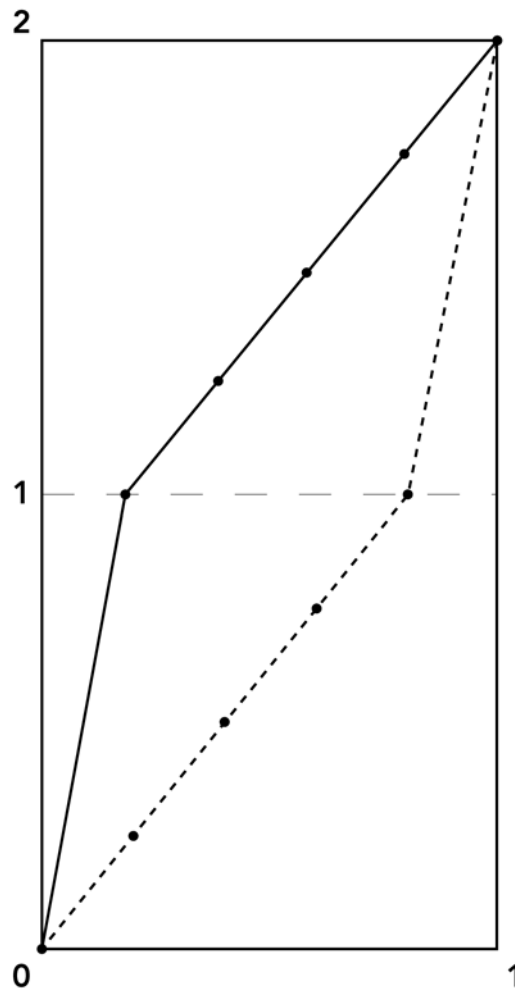

Fig.4. Maximum and minimum D-curves.

Proposition 2

If N increases then the maximum H- or D-curve becomes larger too (in the partial order of H- or D-curves). Similarly, minimum D-curves become smaller.

This proposition follows immediately from the visual representation of the H- and D-curves. It expresses a nice monotonous change property, which is to be expected in a hierarchical or a dominance theory. Recall that in general there are no minimum H-curves.

A7 Understanding the notion of *dominance* and *hierarchy* and their relation

In this section we will show that the terms *dominance* and *hierarchy* as used in this article are different notions, each with their own characteristics.

We consider the pseudo-Lorenz curves H_X and D_X for a zero-sum array X as defined before. The H- and the D-curves are the definitive tools for describing hierarchy (resp. dominance), as will be clear from the sequel. We say that H_X is smaller than H_Y on the interval $I \subset [0, 1]$ if for al $t \in I$: $H_X(t) \leq H_Y(t)$. Similarly:

D_X is smaller than D_Y on the interval $I \subset [0, 1]$ if for al $t \in I$: $D_X(t) \leq D_Y(t)$. We next express the relation between the D-curves and the corresponding H-curves using I_+, I_0 and I_- intervals. Figure 5 illustrates the Proposition 3.

Proposition 3
$$D_X(t) \leq D_Y(t) \text{ on } [0,1]$$
$$\Leftrightarrow$$

$H_X(t) \leq H_Y(t)$ on $I_+(X)$ and $H_X(t) \geq H_Y(t)$ on $I_-(X) \cup I_0(X)$.

$$\Leftrightarrow$$

$H_X(t) \leq H_Y(t)$ on $I_+(Y) \cup I_0(Y)$ and $H_X(t) \geq H_Y(t)$ on $I_-(Y)$.

Proof

If $D_X(t) \leq D_Y(t)$ on [0,1] then $H_X(t) \leq H_Y(t)$ on $I_+(X)$ since on $I_+(X)$ they are the same. Similarly $D_X(t) \leq D_Y(t)$ on [0,1] implies that $H_X(t) \geq H_Y(t)$ on $I_-(X)$ since on $I_-(X)$ they are mirror images of each other (through a reflection over y =1). On $I_0(X)$, $D_X(t) \leq D_Y(t)$ also results in $H_X(t) \geq H_Y(t)$.
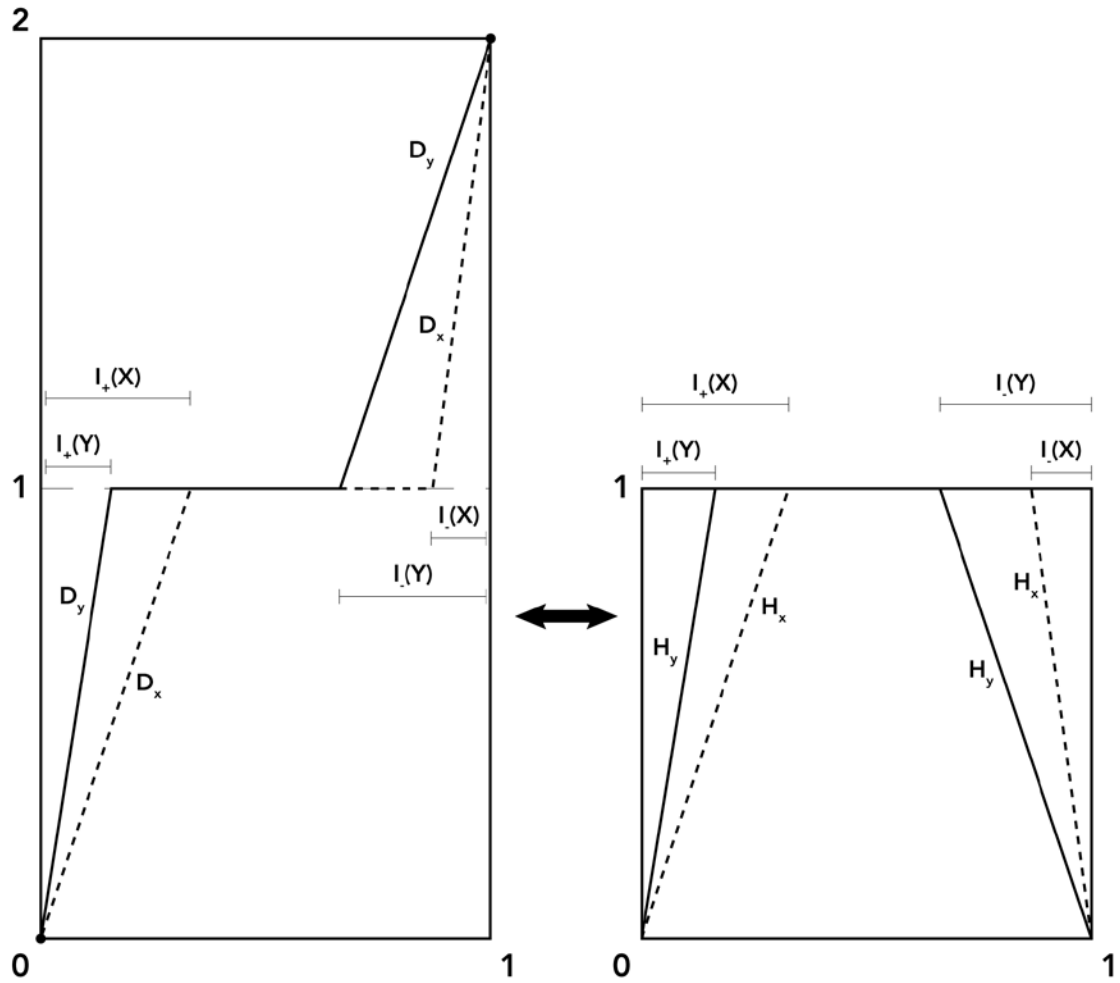
Fig. 5. Illustration of Proposition 3.

Similarly, but slightly differently, we have

Proposition 4 (the proof is left to the reader)

$$H_X(t) \leq H_Y(t) \text{ on } [0,1]$$
$$\Leftrightarrow$$

$D_X(t) \leq D_Y(t)$ on $I_+(X)$ and $D_X(t) \geq D_Y(t)$ on $I_-(X)$; $D_X(t) = D_Y(t) = 1$ on $I_0(X)$

$$\Leftrightarrow$$

$D_X(t) \leq D_Y(t)$ on $I_+(X) \cup I_0(X)$ hence certainly on $I_+(Y)$ and $D_X(t) \geq D_Y(t)$ on

$I_-(X)$ hence certainly on $I_-(Y)$.

Note that an H-curve is a classical Lorenz-curve on $I_+ \cup I_0$ (upon normalization of the abscissae) and also a classical Lorenz-curve on $I_- \cup I_0$ (upon reversing the order of the abscissae followed by a normalization). As a consequence, we have the following result:

Proposition 5

$D_X(t) \leq D_Y(t) \Leftrightarrow H_X(t) \leq H_Y(t)$ on $I_+(X) \cup I_0(X)$ and $H_Y(t) \leq H_X(t)$ on

$I_-(X) \cup I_0(X)$, in the sense of Lorenz-curves (majorization order).

Hence, if the H-curves of X and Y coincide on $I_-(X) \cup I_0(X)$ then their dominance and hierarchy relation on [0,1] are completely determined by the relation of X and Y on $I_+(X) = I_+(Y)$. Concretely, under these circumstances, X≤$_H$Y if and only if X ≤$_D$Y. If, the H-curves of X and Y coincide on $I_+(X) \cup I_0(X) = I_+(Y) \cup I_0(Y)$ then X≤$_H$Y if and only if X ≥$_D$Y. This clearly shows the difference between *dominance* and *hierarchy*.

Consequences for some special zero-sum arrays

(i) Let us consider the special zero-sum array

$$\left(\underbrace{x,....,x}_{i(>0)\ times}, \underbrace{0,...,0}_{j(\geq 0)\ times}, \underbrace{-y,....,-y}_{N-i-j\ times} \right) \tag{1}$$

where $x, y \geq 0$. Since (1) is a zero-sum array, we have, obviously that

$ix = (N - i - j)y$. The H-curve of (1) increases linearly from $(0,0)$ to $\left(\dfrac{i}{N}, 1 \right)$, stays constant, equal to 1,from this point on until $\left(\dfrac{i+j}{N}, 1 \right)$ and then decreases linearly to $(1,0)$. The D-curve is the same as the H-curve on $\left[0, \dfrac{i+j}{N} \right]$ and increases linearly from $\left(\dfrac{i+j}{N}, 1 \right)$ to $(1,2)$.

It is clear that the H-curve increases in j (for i fixed) while for j fixed there is neither an increase nor a decrease in i. For the D-curve we have that for i fixed it decreases in j; while for j fixed it decreases in i. Since, in the above, $i > 0$ and $j \geq 0$, we have that, for (1) the H-curve is maximal for $j = N - 2$ (the maximal value, since $i > 0$) (corresponding to the array $(x, 0, ..., 0, -x)$) and that the H-curve is minimal (for every i) for $j = 0$ (corresponding to the array

$$\left(\underbrace{x, ..., x}_{i\ times}, \underbrace{-y, ..., -y}_{N-i\ times} \right)$$). These results were already known. Further we have that

the D-curve is maximal for $i = 1$ and $j = 0$, hence for $(x, -y, ..., -y)$ and is minimal for $i = N - 1$ (hence $j = 0$) which results in $(x, ..., x, -y)$. Also these results were already obtained.

This result too clarifies the difference between the notions *hierarchy* and *dominance*.

(ii) A variant of (1) is obtained as follows, for the zero-sum array

$$\left(\underbrace{x, ..., x}_{N-i-j\ times}, \underbrace{0, ..., 0}_{j\ times}, \underbrace{-y, ..., -y}_{i\ times} \right) \tag{2}$$

Now the H-curve increases linearly from $(0, 0)$ to $\left(\dfrac{N-i-j}{N}, 1 \right)$, is 1 from this

point to $\left(\dfrac{N-i}{N}, 1 \right)$ and decreases linearly from this point to $(1, 0)$.

For the D-curve, we have that it is the same as the H-curve on $\left[0, \dfrac{N-i}{N} \right]$ and

increases linearly from $\left(\dfrac{N-i}{N}, 1 \right)$ to $(1, 2)$.

We have that, for fixed i, the H-curve decreases in j while for fixed j there is neither an increase or a decrease in i.

For the D-curve we have that for j fixed, it increases in i and for fixed i it increases in *j,* showing again the different nature of *hierarchy* and *dominance*.

Finally, for the extreme cases the N-dependence is as follows:

One obtains maxima for H for arrays of the form $(x,...,-x)$: the *hierarchy* is increasing in N;

One obtains minima for H for arrays of the form $\left(\underbrace{x,...,x}_{i\ times}, \underbrace{-y,...,-y}_{N-i\ times} \right)$: there is no relation of *Hierarchy* with respect to N ;

One obtains maxima for D for arrays of the form $(x,-y,...,-y)$: *Dominance* is increasing in N;

One obtains minima for D for arrays of the form $(x,...,x,-y)$: the *Dominance* is decreasing in N.

A.8 The standard Lorenz majorization order compares arrays with a different concentration. Preserving this order is a necessary property of acceptable concentration measures. Consequently also here we introduce a similar principle.

A measure respecting the hierarchical relation ≥$_H$ in Z (Egghe, 2002)

It is obvious that the area, AR$_H$, under the H-curve and above the horizontal line is a measure which respects the hierarchical relation ≥$_H$. The value of AR$_H$ for a minimal H-curve is 0.5. For fixed N the value for the maximum H-curve is 1 − 1/N. Hence, for every zero-sum array X we have: 0.5 ≤ AR$_H$(X) < 1. If one prefers values between 0 and 1 leading to a normalized area, one should use 2 AR$_H$(X) − 1. It is clear that

$$AR_H(X) = \frac{1}{N}\sum_{i=1}^{N} p_i$$

where the p-values are the components of the array P$_X$. This measure is referred to as the H-measure.

An example: for X = (4,2,-3,-3) the H-measure is equal to:

$$AR_H(X) = \frac{1}{4}\left(\frac{4}{6} + \frac{6}{6} + \frac{3}{6} + 0 \right) = \frac{13}{24}$$

A measure respecting the dominance relation ≥$_D$ in Z

We observe that the area between the D-curve and the line y = x respects the D partial order. This area is denoted as $AR_D(X)$. For any zero-sum array this measure takes values on the interval]0,1[. We will refer to this measure as the D-measure. The D-measure is denoted AR_D and is calculated as:

$$AR_D(X) = \frac{1}{N}\left(\sum_{i=1}^{N} q_i\right) - \frac{N+2}{2N}.$$

where the q-values are the components of the array Q_X.

An example: for X = (4,2,-3,-3) the D-measure is equal to:

$$AR_D(X) = \frac{1}{4}\left(\frac{4}{6} + \frac{6}{6} + \frac{9}{6} + \frac{12}{6}\right) - \frac{6}{8} = \frac{13}{24}$$

The H-measure of a zero-sum array and its opposite are equal, but their D-measures are usually not. This is obvious when considering Fig.3. For this example, namely X = (4,2,-3,-3), we find

$$AR_H(X) = \frac{1}{4}\left(\frac{4}{6} + \frac{6}{6} + \frac{3}{6} + \frac{0}{6}\right) = \frac{13}{24}$$

$$AR_H(-X) = \frac{1}{4}\left(\frac{3}{6} + \frac{6}{6} + \frac{4}{6} + \frac{0}{6}\right) = \frac{13}{24}$$

$$AR_D(X) = \frac{1}{4}\left(\frac{4}{6} + \frac{6}{6} + \frac{9}{6} + \frac{12}{6}\right) - \frac{6}{8} = \frac{13}{24}$$

$$AR_D(-X) = \frac{1}{4}\left(\frac{3}{6} + \frac{6}{6} + \frac{8}{6} + \frac{12}{6}\right) - \frac{6}{8} = \frac{11}{24}$$

Proposition 6. $AR_D(X) + AR_D(-X) = 1$

Considering the D-curves for X and –X we see that the area between the D-curve of X and the line y=x is equal to the area between the D-curve of –X and the line y = x+1. Hence $AR_D(X) + AR_D(-X) = 1$, as illustrated in the previous example. The hatched area under the curve with full lines is equal to $AR_D(-X)$. It is exactly equal to the area above $D_X(t)$, the striped curve, and under y = x+1. This is illustrated by Fig. 6.
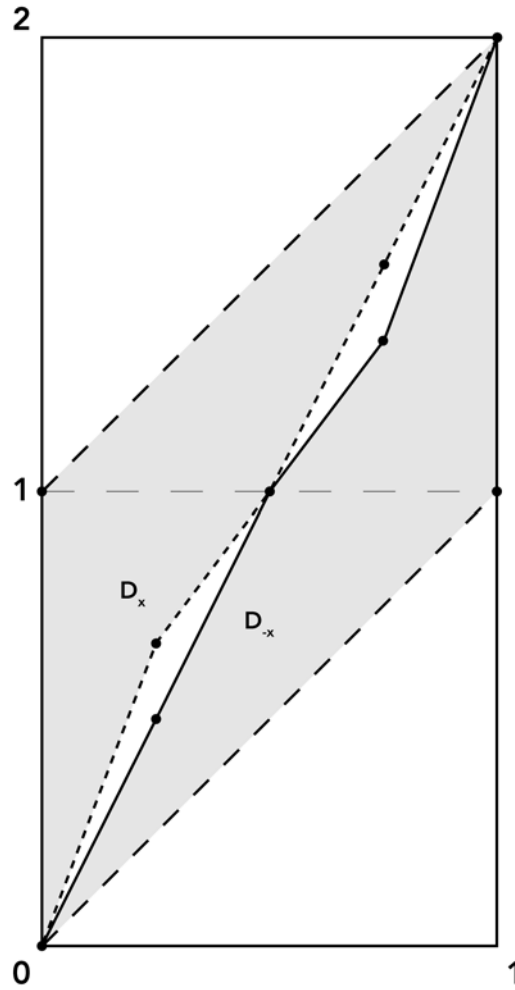
Fig. 6. X = (4,2,-3,-3): the hatched area under $D_{(-X)}(t)$ (full line) and above y=x is equal to the hatched area above $D_X(t)$ (dotted line) and under y=x+1

H-measures and D-measures for maximum and minimum curves.

The maximum array for the hierarchy order is (s,0,…,0, -s), s > 0.

Such arrays have H-measure equal to $\frac{N-1}{N}$ and D-measure equal to $\frac{1}{2}$.

Maximum arrays for the hierarchy order are symmetric, hence they coincide with their opposite.

There is no minimum H-curve, but arrays of the form $X_j = \left(\underbrace{s,...,s}_{j}, \underbrace{-t,...,-t)}_{N-j} \right)$ j

= 1,…, N-1, with s, t > 0 and js = (N-j)t correspond to minimal H-curves.

These curves all have H-measures equal to $\frac{1}{2}$, but their D-measures are different and equal to $\frac{N-j}{N}$. The opposite of a minimum H-curve is again a minimum H-curve.

Maximum D-curves correspond to (s,-t,…,-t), with s, t > 0. These are actually minimal H-curves with j = 1. Consequently, they have H-measures equal to $\frac{1}{2}$ and D-measures equal to $\frac{N-1}{N}$.

Minimum D-curves correspond to Y = (u,…, u, -v), u, v > 0. These too are minimum H-curves, namely for j = N-1. Consequently, they have H-measures equal to $\frac{1}{2}$ and D-measures equal to $\frac{1}{N}$.

Minimum D-curves are opposites of maximum D-curves. Clearly their D-measures sum to 1.

Transfer property for H- and D-curves

We recall that Dalton's transfer property (Dalton, 1920) states that if one takes from a poorer and gives to a richer the concentration increases. In terms of Lorenz curves it states that such a transfer increases the Lorenz curve (assuming arrays are ranked in decreasing order). Obviously, the transfer principle does not hold for H- or for D-curves. If it were this would imply that a transfer from a poorer (smaller value) to a richer (larger value) would yield a curve that is strictly situated above the original one. It suffices to give one counterexample. Consider X = (5,1,0,-2,-4) and take one unit from the 4th component and give it to the first. This leads to X'= (6,1,0,-3,-4). The positive part of the H-curve of X' is now situated above that of X; yet, the negative part of the H-curve of X' is situated under that of X. Hence the two H-curves are not comparable. This transfer is not a counterexample for D-curves. However, transferring one unit from the fifth component and giving it to the first leads to X"= (6,1,0,-2,-5) which has a D-curve that is situated strictly above the original on the positive part and is strictly under the original on the negative part.

However, a transfer property holds for the positive part and the negative part separately. More precisely, if one takes from a poorer and gives to a richer in the positive part, such that the poorer one still has a non-negative value then the curves take up a higher position and the corresponding measures increase.

Similarly, if one removes a positive amount from a negative item and gives to one that is less negative, but still stays non-positive, then the H-curve increases and the corresponding H-measure increases. This operation, however, decreases the D-curve and hence the D-measure decreases too. In this sense we refer to the results of such a transfer as an opposite transfer principle.

B Zero-sum theory applied to directed networks

B1 Basic graph theoretical definitions

We briefly recall the basic graph theoretical terminology needed in the sequel and note that in this article the terms graph and network are considered to be synonyms. A directed graph (in short: digraph), denoted G(V,E) consists of a set of vertices or nodes, denoted as V or V(G) and a set of edges or links, denoted as E or E(G). Nodes will be denoted by minuscules (lower case letters) such as a, b, c, i, j. An edge is an ordered pair of the form (i,j) where i and j are nodes, hence belong to V. Node i is called the initial node and node j is the terminal node of edge (i,j). A directed path, or chain, from node i to node k is a set of edges $(v_n)_{n=1,...,M}$ such that the terminal node of edge v_n coincides with the initial node of edge v_{n+1} and such that node i is the initial node of edge v_1 and node k is the terminal node of edge v_M. If node i coincides with node k the directed path is a directed circuit or loop. A directed graph is called acyclic or loopless if it contains no directed circuits. If a graph is acyclic and there is a link from node A to node B, then there certainly is no link from node B to node A, as otherwise there would exist a cycle from A to B to A. A directed graph is weakly connected if a path exists between any two nodes in the underlying undirected graph. We will always assume that the graphs we study have a finite number of nodes, at least two, and are acyclic and weakly connected.

In- and out degree (Chen, 1971).

The number α_j^+ of edges in the digraph G having node j as their initial node is

called the out-degree of node j. Similarly, the number α_j^- of edges in G having

node j as their terminal node is called the in-degree of node j. As in (Egghe &

Rousseau, 2004) we put $\alpha_j = \alpha_j^+ - \alpha_j^-$. This parameter characterizes the flow

through node j. If it is positive there are more edges leaving node j than reaching it; when this parameter is negative the opposite holds. Since every edge is outgoing from a node and terminating at another, the number of edges in G, denoted as ε, is related to the degrees of its nodes by the following equation:

$$\varepsilon = \sum_j \alpha_j^+ = \sum_j \alpha_j^- \quad \text{or} \quad \sum_j \alpha_j = 0$$

where the summation is over all nodes of graph G. Hence the sequence $(\alpha_j)_j$ forms a zero-sum array derived from the network. Hence we can apply the zero-sum theory developed in the previous part A. The number α_j is called a local flow number (in short: the local flow) and the corresponding zero-sum array is called a local flow array. In a digraph, the number of outgoing links minus the number of incoming links is also known as the degree of the node. In this contribution we will use the terminology of local flow to contrast it to the global flow, defined below.

Similarly, we consider a global theory, where we use arrays of the form $\Sigma = (\sigma_1, \sigma_2, ..., \sigma_N)$, defined as follows:

$$\sigma_i = \sigma_i^+ - \sigma_i^-$$

$\sigma_i^+ = $ the sum of the lengths of all chains that start in node i

$\sigma_i^- = $ the sum of the lengths of all chains that end in node i

From these definitions we see that also Σ leads to a zero-sum array and hence also here we can apply the theory developed in the previous part. Such a zero-sum array will be referred to as a global flow array. This array consists of global flow numbers.

Consequently, in an acyclic directed network we can study four aspects: local and global hierarchy theory and local and global dominance theory. Global hierarchy theory (GHT) uses Σs and H-type curves and has been studied in (Egghe, 2002); local hierarchy theory (LHT) uses local flow numbers and H-type curves and has been studied in (Egghe & Rousseau, 2004; Berliner et al., 2007). Global and local dominance theory (denoted respectively as GDT and LDT) have not been studied yet and are the main focus of this contribution. We note that in the same vein as (Egghe, 2002; Egghe & Rousseau, 2004) we do not include a global theory based on shortest distances, as studied e.g. in (Botafogo, Rivlin & Shneiderman, 1992; De Bra, 2000; Egghe & Rousseau, 2003) admitting though that this would constitute yet another two options.

We point out that these four types of study lead to four partial orders among networks, not nodes! The difference between hierarchical and dominance networks lies only in how nodes with negative flow values are treated, as nodes with positive flow values are treated in exactly the same way.

Not every zero-sum array can be derived from a network. Consider a 3-node

network, and the zero-sum array X = (4,-1,-3). Then 4 cannot possibly be a local flow number as it is impossible that 4 links leave a node; it can also not be a global flow number as in a 3-node network a chain has at most length 2 and there cannot be two chains as this would require at least 4 different nodes. This case is rather trivial as 4 > 3. Let us now consider a 5-node network and the zero-sum array Y=(3,3,0,-3,-3). It is not difficult to see that this array cannot be a local or global array corresponding to a 5-node network (we leave the details to the reader). From this we observe that the study of hierarchies or dominance leads to a subset of all zero-sum arrays, and hence to the open problem: which zero-sum arrays occur as local or global flow arrays from a digraph?

We want to characterize a weakly connected, acyclic network in terms of how much dominance is globally (or locally) present. Recall that the analogous hierarchical theories (local and global) have already been studied in (Egghe, 2002) and (Egghe and Rousseau, 2004). We recall that if the arrays deduced from two networks are both symmetric and non-equivalent, then they are D-incomparable.

B2 Some further notions from graph theory

Definition: a local source of a digraph is a node having in-degree zero, and strictly positive out-degree.
If a local source can reach any other node in a digraph it is called a network source. Since we have assumed that there are no loops in a network, we see that if a network source in a digraph exists it is necessarily unique hence it becomes the network source.

Definition: a local sink of a digraph is a node having out-degree zero, and strictly positive in-degree.
If a local sink can be reached by any other node it is called a network sink. Also a network sink, if it exists, is necessarily unique and hence is referred to as the network sink.

Lemma: an acyclic digraph has at least one local source and one local sink.

Proof. Assume there is no local source. Take now any node, say a, in the graph. If it has in-degree zero then it must have out-degree zero (otherwise it would be a local source and we assume that there is no local source). But this implies that the node is isolated and hence the graph is not weakly connected, which is a contradiction.
Assume now that node a has a strictly positive in-degree. Then we consider a node b linked to a. As it is assumed that there is no local source, b's in-degree is not zero. Then we consider a node c linked to b. However, after a finite

number of steps we must end up in a node we have visited already as the graph has only a finite number of nodes. Yet, if we end up in a node we have visited already we have gone through a cycle. This leads to a contradiction as the graph is assumed to be acyclic. This proves that any (finite) acyclic digraph has a local source. One may similarly show that any acyclic digraph has a local sink.

It is obvious that a local source has a strictly positive flow number and that a local sink has a strictly negative flow number. Moreover, adding a node in a digraph which is linked only to the network source, makes this new node the network source.

A local source in a tree structure is the network source and is also a root; in a tree local sinks are terminal nodes. An N-node network may have N-1 local sources or N-1 local sinks. In this context we like to mention the following open question: can it have **any** number in between?

Before continuing we introduce the following definitions.

Definition: dominance nodes
A node with the highest global flow in a D-graph is called a global dominance node.
A node with the highest local flow in a D-graph is called a local dominance node.
A network source is always a global dominance node.

Proposition 7

Given a graph G and let $X = (x_1, \ldots, x_N)$ be one of the zero-sum arrays derived from this graph, as in the previous part. If we reverse the direction of every link in the network we obtain the "opposite" graph of G, and the corresponding zero-sum array X becomes the opposite array: $-X = (-x_N, \ldots, -x_1)$.

Remark: $X = (x_1, \ldots, x_N)$ and $-X = (-x_N, \ldots, -x_1)$ are not directly corresponding to the opposite digraph since we can deduce the array $X = (x_1, \ldots, x_N)$ from many digraphs.

There exist four different acyclic weakly connected graphs on three points, see Fig.7.
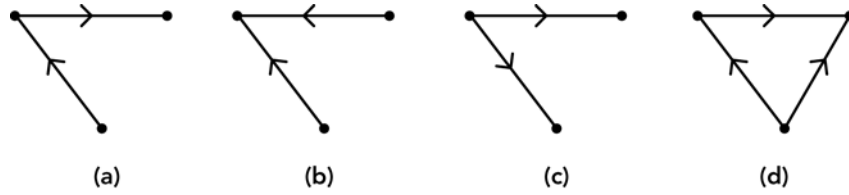
Fig. 7. All possible cases (D-order) for an acyclic weakly connected graph on three nodes.

Local D-arrays for (a),(b),(c) and (d) are respectively: (1,0,-1), (1,1,-2), (2,-1,-1) and (2,0,-2). Global arrays, in the same order, are: (3,0,-3), (1,1,-2), (2,-1,-1), (4,0,-4).

This leads to the following D-rankings:

Locally: (1,1,-2) $\leq_D$ (1,0,-1) ≡ (2,0,-2) $\leq_D$ (2,-1,-1)
Globally: (1,1,-2) $\leq_D$ (3,0,-3) ≡ (4,0,-4) $\leq_D$ (2,-1,-1)

It so happens that in the case of Fig.7 the values for the local and the global D-measures are equal. We find: AR_D(2,-1,-1) = 4/6; AR_D(1,0,-1) = AR_D(3,0,-3) = 3/6 and AR_D(1,1,-2) = 2/6.

Note: non-isomorphic graphs can have the same H- and D-curves. This is illustrated in Figs. 7 and 8. Graph (a) has local array (1,0,-1) and global array (3,0,-3); graph (d) has local array (2,0,-2) and global array (4,0,-4).
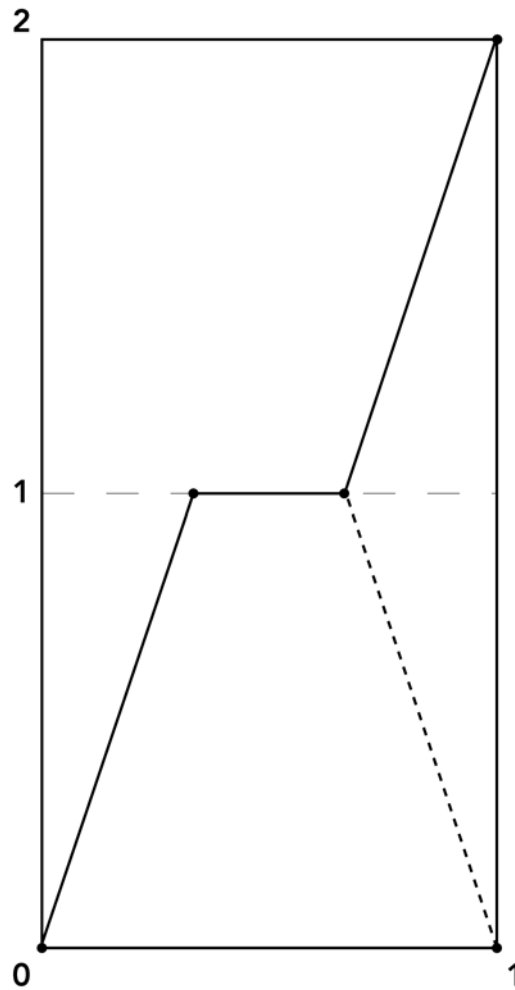
Fig. 8. Graph corresponding to the networks (a) and (d) shown in Figure 7. Clearly local and global D-curves coincide; similarly local and global H-curves coincide (dotted part right part).

B3 Graphs for maximum and minimum D-curves in LDT and GDT.

Proposition 8

For fixed N, the graph shown in Fig.9 yields the only graph corresponding with a maximum global D-array.

Before proceeding with the proof, we recall the following lemma (Egghe, 2002, Lemma III.1)

Lemma. If a and b are vertices in an acyclic, weakly connected digraph, and if the edge (a, b) belongs to the edge set of this graph, then $\sigma_a > \sigma_b$.

This relation does not generally hold for local flow numbers.

Proof of Proposition 8.

We know that a maximum D-curve corresponds to an array X = (s,-t,…,-t), with s, t > 0 and s = (N-1)t. We also see that the graph shown in Fig. 9 has corresponding zero-sum array $(N-1,\underbrace{-1,-1...-1}_{N-1})$ and this locally as well as globally. This array is of form (s, -t,……-t), with s = (N-1) and t=1.

We still have to prove that Fig. 9 - and similar ones for other values of N – provides the only possible maximum D-network, at least according to our definition. We know that an acyclic digraph has at least one local source, i.e. a node with in-degree zero, and strictly positive out-degree. Fig.9 has exactly one node with in-degree zero and strictly positive out-degree, so this node is the source of an acyclic digraph.

Next we show that for a maximum global D-network there cannot be a link between nodes that are not sources. Indeed, if there were a link between nodes a and b (both not local sources) then $\sigma_a \neq \sigma_b$, and hence the coordinates corresponding to a and b in the array would not be equal, which is a contradiction. Hence, Fig. 9 yields the only graph corresponding to a maximum global D-array.
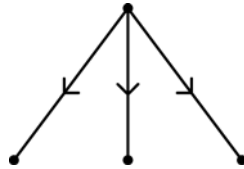

Fig.9. N-node graph corresponding to a maximum D-curve (N=4).

In a citation graph where an arrow indicates 'is cited by' the situation illustrated by Fig.9 is the one where one person (journal) is cited by all others. Of course this can only happen if we are considering a subnetwork of a larger citation network. Such studies are most interesting when the target article is of special importance. An example could be Hirsch's original article on the h-index (Hirsch, 2005). Yet, even then it is probably more interesting to incorporate several citation generations, including indirect influences.

Proposition 9
The maximum D-graph for LDT is the same as that for GDT.

Proof.

Consider an arbitrary node j. Using Fig. 9 as an illustration, we see that $\sigma_j^+ = \alpha_j^+, \sigma_j^- = \alpha_j^-$, so $\sigma_j = \alpha_j$. Hence the maximum graph for GDT is also a maximum graph for LDT. To show that this is the only one we remark that maximum zero-sum arrays of length N are of the form (s, -t, -t, …, -t) with s = (N-1)t. If this array is derived from local flow numbers then we have a digraph with N nodes. If now t=2 (or larger), then s = 2(N-1) (or larger), meaning that there is a node having (at least) degree 2N − 2 . This is not possible in an N-node digraph. Hence the array corresponding to a maximum local graph is (N-1, -1, …, -1), which is the same as for the global theory.

The graph for a minimum D-curve in LDT and GDT.

Proposition 10

For fixed N, the graph shown in Fig.10 yields the only graph corresponding to a minimum D-array.

Proof.
The array corresponding to the graph shown in Fig. 10 is $(\underbrace{1,1...1}_{N-1},-(N-1))$. This is of the form (s, s,…,s,-t), with t=(N-1)s. This shows that Fig. 10 represents a digraph which is a minimum in the local and in the global dominance theory. Now we have to show that this is the only possibility (for fixed N).
From the lemma (Egghe, 2002, Lemma III.1), we know that there is not a link among any two of N-1 nodes that have 1 out-link because if there is an edge between a and b, $\sigma_a \neq \sigma_b$, which does not lead to a minimum array. This proves the case for the global theory. To show that this is the only one, for the local case as well, we remark that minimum zero-sum arrays of length N are of the form (s, s, …, s, -t) with t = (N-1)s, t, s > 0. If this array is derived from local flow numbers then we have a digraph with N nodes. If now s=2 (or larger), then t = (N-1)2 (or larger), meaning that there is a node having (at least) degree 2N − 2 . This is not possible in an N-node digraph. Hence the array corresponding to a minimum local graph is (1,1,…,1,-(N-1)), which is the same as for the global theory. This is illustrated in Figure 10.
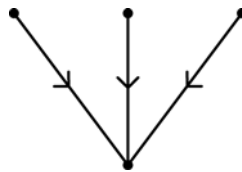

Fig.10. A graph corresponding to the minimum (local and global) D-curve (N=4).

Minimum curves can be derived by reversing the direction of the arrows of maximum curves. Fig.10 can be interpreted as a 'is cited by' graph. Every journal (scientist), except one, is cited, and because cycles are not allowed, and the graph must be connected this can only happen if there is one journal (scientist) which does all the citing (and receives no citations).

B4 Terminology and meaning: hierarchies versus power (dominance)

Let us reconsider the following digraph (Fig.11), which is similar to the one shown in Fig.9.
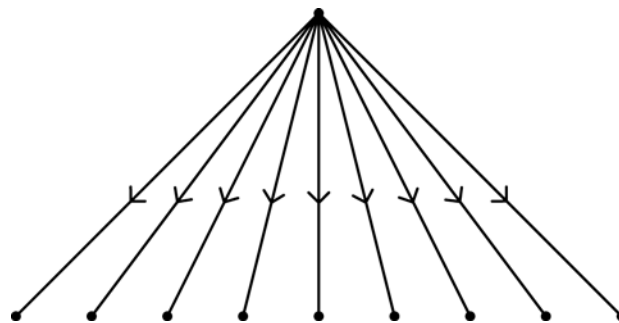

Fig. 11. A strong dominance structure.

There is not much hierarchical structure in this digraph, but it reflects a very strong power structure: one ruler and many equally powerless subordinates.

The H- and D-curves of this graph reflect these features. This illustrates the use of the terms H-curves and D-curves for *Hierarchy* and *Dominance*. The H-curve corresponding to Fig. 11 is a minimal H-curve, while its D-curve is maximum for fixed N. In applications of D-curves to institutes, research groups or scientists as nodes we want to gauge the power structure that is present. The more inequality among the positive nodes the more powerful the order relation is. But also: the more even the negative nodes (in the sense of evenness (Nijssen et al., 1998)), the more powerful the order. In Fig. 11 there is no difference among the local and the global point of view. Generally, however, our interpretation applies only to one of the two perspectives. When applying Fig. 11 to a citation networks one should realize that citers have the most power: they decide to cite or not to cite their collleagues. So, when there is only one citer in a subnetwork this is the most dominating network.

We note that studying the dominance structure and its opposite, the subordinance structure, entails studying X and −X. We already observed that X and −X do not have the same D-measure AR_D, but that $AR_D(X) + AR_D(-X) = 1$. Moreover it follows that when a directed network and its opposite are isomorphic, then their D-measures are equal to 0.5. Hence the difference

between $AR_D(X)$ and $AR_D(-X)$ is an indication of a difference in standpoint: a view from the top or a view from the bottom. The larger the difference between $AR_D(X)$ and $AR_D(-X)$ the more asymmetric the network. In particular for a maximum D-network and its opposite, which is a minimum D-network (locally and globally), the difference is $\frac{N-1}{N} - \frac{1}{N} = \frac{N-2}{N}$. This difference tends to 1 with increasing N. Figure 12 illustrates the fact that for symmetric networks the D-measure is equal to 0.5. The local D-array of the network shown in Fig. 12 is (3,1,1,1,0,0,-1,-1,-1,-3) and its D-measure is: $\frac{1}{10}\left(\frac{66}{6}\right) - \frac{12}{20} = \frac{5}{10} = 0.5$. Its global D-array is (21,11,1,1,0,0,-1,-1,-11,-21) and the corresponding D-measure is: $\frac{1}{10}\left(\frac{374}{34}\right) - \frac{12}{20} = \frac{5}{10} = 0.5$.
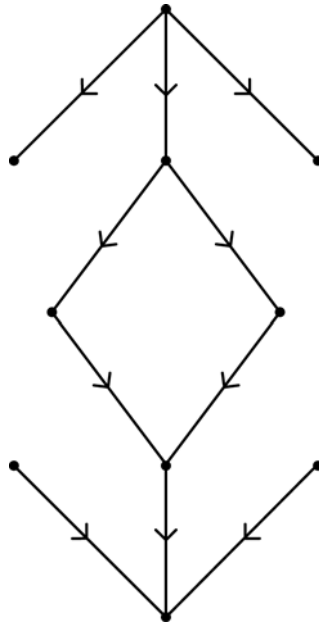

Fig. 12. Symmetric network.

Conclusion

In this contribution we studied partial orders in zero-sum arrays. Concretely, they relate to local and global hierarchy and dominance theories. Based on this investigation we designed a new approach for measuring dominance or stated otherwise, power structures, in networks. A variant of the classical Lorenz curve leads to a partial order among networks, described via zero-sum arrays. These arrays consist of positive and negative values. The new curves, referred to as D-curves are partly concave and partly convex. As such they do not satisfy Dalton's transfer principle. Yet, it follows from the construction of H- and D-curves that other properties such as the replication property,

permutation and scale-invariance are satisfied. Moreover, the transfer principle is satisfied among elements with positive values and a kind of opposite transfer principle is satisfied among elements with negative values.

From the characteristics of this transfer principle, we derive some special properties of the D-curve: given a fixed number of nodes with equal positive flow and a fixed number of nodes with equal negative flow, the more concentrated the nodes with positive flow and the more even the nodes with negative flow in a D-array, the higher the dominance structure. Similarly, the fewer the relative number of nodes with equal positive flows and the more nodes with equal negative flows in a D-array, the higher the dominance structure. Continuing in this way, for fixed N, the maximum D-array corresponds to all zero-sum arrays of the form $X = (s, -t, ..., -t)$ with $s > 0, t > 0$ and $s = (N-1)t$. The corresponding D-curve linearly connects (0,0) to the point with coordinates $(\frac{1}{N}, 1)$ and is then linearly connected to the endpoint (1,2). Similarly, for fixed N, minimum D-arrays corresponds to zero-sum arrays of the form $X = (u, ...u, -v)$ $u > 0, v > 0$ and $v = (N-1)u$. The corresponding D-curve connects the point (0,0) linearly to the point with coordinates $\left(\frac{N-1}{N}, 1\right)$ and is then connected to the endpoint (1,2).

D-curves have more properties making them suitable to measure dominance. For instance: if the length of an array, denoted as N, increases then the maximum D-curve becomes larger in the partial order of D-curves, and the minimum D-curves become smaller, corresponding to the fact that in applications dominance increases when there are more subordinates and decreases when there are less subordinates.

If $X \leq_D Y$, then $-Y \leq_D -X$, which means that if dominance and being a subordinate are reversed, the dominance structure is also reversed.

We applied dominance theory to acyclic directed networks in which the nodes may represent individuals in the system and arrows linking nodes in the network denote antagonistic interaction in the sense that the direction of the arrow is from 'dominator' to subordinate. Yet, reversing the direction of the arrow leads to a study of the opposite relation. A source in an acyclic digraph is a node with in-degree zero and strictly positive flow. We tried to measure to what extent a network source in an acyclic directed network dominates the whole network, and this in the local and in the global sense. We found that only the acyclic directed network corresponding to a maximum D-curve is the

network in which a source has advantage over all other nodes, and the acyclic directed network corresponding to a minimum D-curve is that in which a sink is dominated by all other nodes in the network. These properties show exactly that the D-curve can be used to measure the partial order of dominance.

Acknowledgements

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