Quasi-Frobenius-Lusztig kernels for simple Lie algebras

Peer-reviewed author version

Liu, Gongxiang; Van Oystaeyen, Fred & ZHANG, Yinhuo (2017)
Quasi-Frobenius-Lusztig kernels for simple Lie algebras. In: TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, 369(3), p. 2049-2086.

DOI: 10.1090/tran/6731
Handle: http://hdl.handle.net/1942/22931

QUASI-FROBENIUS-LUSZTIG KERNELS FOR SIMPLE LIE ALGEBRAS

GONGXIANG LIU, FRED VAN OYSTAEYEN, AND YINHUO ZHANG

ABSTRACT. In [19], the quasi-Frobenius-Lusztig kernel associated with $\mathfrak{sl}_2$ was constructed. In this paper we construct the quasi-Frobenius-Lusztig kernels associated with any simple Lie algebra $\mathfrak{g}$.

1. INTRODUCTION

Considering a product on representations of an algebra, an idea useful in physics, leads to the consideration of a coproduct on the algebra and hence to the study of a bialgebra, or more in particular, a Hopf algebra structure. The theory of algebraic groups is dual to the theory of commutative and cocommutative Hopf algebras. More general Hopf algebras then fit in a theory of quantum groups as defined by Drinfeld [6, 7], Jimbo [17], Lusztig [20, 21] and others. By allowing non-canonical isomorphisms for triple products of representations, leading to so-called associators, one obtains a generalization of a Hopf algebra to a quasi-Hopf algebra, termed quasi-algebra for short. This raises the natural question whether it is possible to find essentially new quasi-quantum groups corresponding to such quasi-algebras? Now, for a simple finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, a result of Drinfeld [8, Prop 3.16] states that a quasitriangular quantized quasi-Hopf enveloping algebra $U\mathfrak{g}[[h]]$ is twist equivalent to the usual quantum group $U_h\mathfrak{g}$. This means that the quasi-quantum group associated to a simple finite dimensional Lie algebra is essentially not new. But what happens in the restricted case? In other words, does there exist a quasi-algebra analogue for Lusztig's definition of a small quantum group, that is, do we have quasi-Frobenius-Lusztig kernels?

A remarkable recent development in Hopf algebra theory is the Andruskiewitsch-Schneider's classification of finite dimensional pointed Hopf algebras, cf. [1]; here the Frobenius-Lusztig kernels play a dominant role. So it is reasonable to expect that the theory of quasi-FL kernels (short for Frobenius-Lusztig kernels) will provide insight in the structure of finite dimensional quasi-Hopf algebras. Another direction relates to Conformal Field Theory (CFT). It has been established by Majid, cf. [23], that there is a quasitriangular quasi-algebra associated to a Topological Field Theory (TFT, for short). The relevance of quasi-Hopf algebras in TFT has been studied by Dijkgraaf, Pasquier, and Roche [5]; in *loc.cit.* a new class of semisimple quasitriangular quasi-Hopf algebras, denoted by $D^{\omega}(G)$, has been constructed. Further development in CFT, in particular Logorithmic Conformal Field Theory, are pressing for the systematic construction and deeper study of finite dimensional quasitriangular quasi-algebras, in particular to look at nonsemisimple ones. We refer to [9] and references therein for more detail. It is fair to say that in the present situation there is a lack of such examples.

However, the answer to the question about the existence of quasi-FL kernels is positive! The simplest quasi-FL kernel has been constructed in [19]. It was denoted by $\mathrm{Q}\mathbf{u}_q(\mathfrak{sl}_2)$ and it was associated to $\mathfrak{sl}_2$. The aim of the present paper is to define $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$, the quasi-FL kernel associated to an arbitrary simple finite dimensional Lie algebra $\mathfrak{g}$, extending the ideas found in [19]. Inspired by the classical FL kernel theory, one may believe that the quasi-FL kernel associated to a finite dimensional Lie algebra $\mathfrak{g}$ should be the Drinfeld double of the half small quasi-quantum group as defined in [12]. Our primary mission is to compute them and to make a comparison with the Hopf algebra case. It turns out that the computation of quasi-FL kernels is really much more difficult than in the Hopf case.

Half small quasi-quantum groups appeared in the work of Etingof and Gelaki [12] and the notation used for them was $A_q(\mathfrak{g})$ where q is an n^2-th primitive root of unity. In case n is odd and prime to the determinant of the Cartan matrix, they established that $D(A_q(\mathfrak{g}))$ is twist equivalent to $\mathbf{u}_q(\mathfrak{g})$ [11] (in case $\mathfrak{g}$ is not of type G_2). We go on to show that the conditions cannot be removed. More precisely, we establish that $D(A_q(\mathfrak{g}))$ is not twist equivalent to any Hopf algebra in many cases. This leads to new examples of nonsemisimple quasitriangular quasi-Hopf algebras and their corresponding braided tensor categories, which have independent interest by themselves.

In Section 2, we include some preliminaries including the definition of $A_q(\mathfrak{g})$, some facts about quiver Majid algebras and a useful criterion for a 3-cocycle to be a 3-coboundary. The Majid algebra $M_q(\mathfrak{g}) := (A_q(\mathfrak{g}))^*$ is studied in detail in Section 3, and we pay particular attention to the Serre relation in Proposition 3.4. Section 4 is devoted to the computation of the Drinfeld double $D(A_q(\mathfrak{g}))$. The computations are explicit and some of them are rather tedious. This makes for the technical heart of the paper.

In Section 5 we then go on to provide a presentation for $D(A_q(\mathfrak{g}))$ in terms of generators and relations. We discover some similarities between $D(A_q(\mathfrak{g}))$ and $\mathbf{u}_q(\mathfrak{g})$. In particular in Theorem 5.3, we obtain that $D(A_q(\mathfrak{g})) \cong \mathrm{Q}\mathbf{u}_q(\mathfrak{g})$. The final Section 6 is devoted to detecting when $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is twist equivalent to a Hopf algebra and the cases where this does not happen are identified.

Throughout, we work over an algebraically closed field $\mathbb{k}$ of characteristic 0 and $[\frac{\bullet}{\bullet}]$ stands for the floor function, that is, for any natural numbers a, b, $[\frac{a}{b}]$ denotes the biggest integer which is not bigger than $\frac{a}{b}$. For general background knowledge, the reader is referred to [8] for quasi-Hopf algebras, to [3, 18] for general theory about tensor categories, and to [16] for pointed Majid algebras.

2. Preliminaries

In this section we will recall the Etingof-Gelaki's constructions of half quasi-quantum groups [12], quiver Majid algebras [14, 16], and the Drinfeld double of a quasi-Hopf algebra [13, 22, 24]. Then we formulate a criterion for a 3-cocycle of a finite abelian group to be a 3-coboundary.

2.1. **Half small quasi-quantum group $A_q(\mathfrak{g})$.** A quasi-bialgebra $(H, \mathrm{M}, \mu, \Delta, \varepsilon, \phi)$ is a $\mathbb{k}$-algebra (H, M, μ) with algebra morphisms $\Delta: H \to H \otimes H$ (the comultiplication) and $\varepsilon: H \to \mathbb{k}$ (the counit), and an invertible element $\phi \in H \otimes H \otimes H$ (the reassociator), such that:

$$(id \otimes \Delta)\Delta(a)\phi = \phi(\Delta \otimes id)\Delta(a), \quad a \in H,$$
$$(id \otimes id \otimes \Delta)(\phi)(\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi)(id \otimes \Delta \otimes id)(\phi)(\phi \otimes 1),$$
$$(\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta,$$
$$(id \otimes \varepsilon \otimes id)(\phi) = 1 \otimes 1.$$

Denote $\sum X^i \otimes Y^i \otimes Z^i$ by ϕ and $\sum \overline{X}^i \otimes \overline{Y}^i \otimes \overline{Z}^i$ by ϕ^{-1}. A quasi-bialgebra H is called a quasi-Hopf algebra if there are a linear algebra antimorphism $S: H \to H$ (called the antipode) and two elements $\alpha, \beta \in H$ satisfying for all $a \in H$:

$$\sum S(a_{(1)})\alpha a_{(2)} = \alpha\varepsilon(a), \quad \sum a_{(1)}\beta S(a_{(2)}) = \beta\varepsilon(a),$$
$$\sum X^i \beta S(Y^i)\alpha Z^i = 1 = \sum S(\overline{X^i})\alpha \overline{Y^i}\beta S(\overline{Z^i}).$$

Here and below we use the Sweedler sigma notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ (or $a' \otimes a''$) for the comultiplication and $a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+1)}$ for the n-iterated coproduct $\Delta^n(a)$ of a. We call an invertible element $J \in H \otimes H$ a (Drinfeld) *twist* of H if it satisfies $(\varepsilon \otimes id)(J) = (id \otimes \varepsilon)(J) = 1$. For a twist $J = \sum f_i \otimes g_i$ with inverse $J^{-1} = \sum \overline{f_i} \otimes \overline{g_i}$, let:

$$\alpha_J := \sum S(\overline{f_i})\alpha \overline{g_i}, \quad \beta_J := \sum f_i \beta S(g_i). \tag{2.1}$$

Given a twist J of H, if β_J is invertible, then one can construct a new quasi-Hopf algebra structure $H_J = (H, \Delta_J, \varepsilon, \phi_J, S_J, \beta_J\alpha_J, 1)$ on the algebra H, where:

$$\Delta_J(a) = J\Delta(a)J^{-1}, \quad a \in H,$$

$$\phi_J = (1 \otimes J)(id \otimes \Delta)(J)\phi(\Delta \otimes id)(J^{-1})(J \otimes 1)^{-1}$$

and,

$$S_J(a) = \beta_J S(a)\beta_J^{-1}, \quad a \in H.$$

Next we will define the quasi-Hopf algebra $A_q(\mathfrak{g})$. Given an $m \times m$ Cartan matrix (a_{ij}) of finite type, it is known that there is a vector $(d_1, \ldots, d_m)$ with integer entries $d_i \in \{1, 2, 3\}$ such that the matrix $(d_i a_{ij})$ is symmetric. Let $n \geq 2$ be a natural number and q be an n^2-th primitive root of unity.

Let $N, M,\, d \geq 0$ be integers. Following Gauss, we define:

$$[N]^!_d = \prod_{h=1}^{N} \frac{q^{dh} - q^{-dh}}{q^d - q^{-d}}, \qquad \begin{bmatrix} M+N \\ N \end{bmatrix}_d = \frac{[M+N]^!_d}{[M]^!_d [N]^!_d}.$$

Let H be a finite dimensional Hopf algebra generated by grouplike elements g_i and skew-primitive elements e_i, $i = 1, \ldots, m$, such that:

$$g_i^{n^2} = 1, \;\; g_i g_j = g_j g_i, \;\; g_i e_j g_i^{-1} = q^{\delta_{i,j}} e_j, \;\; e_i^{l_i} = 0,$$
$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} e_i^r e_j e_i^s = 0, \;\; \text{if} \;\; i \neq j,$$

and

$$\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i$$

where $l_i = \mathrm{ord}(q^{d_i a_{ii}})$, the order of $q^{d_i a_{ii}}$, and $K_i := \prod_j g_j^{d_i a_{ij}}$. From now on, we use

$$c_{ij} := d_i a_{ij} \tag{2.2}$$

to denote the entries of the symmetrized Cartan matrix.

Consider the subalgebra $A \subset H$ generated by g_i^n, e_i for $i = 1, \ldots, m$. It is clear that A is not a Hopf subalgebra. However, we will see that it is a quasi-Hopf subalgebra of H_J for some twist J of H.

Let $\{1_a | a = (a_1, \ldots, a_m) \in (\mathbb{Z}_{n^2})^m\}$ be the set of primitive idempotents of $\Bbbk(\mathbb{Z}_{n^2})^m$. Define $1_k^i := \frac{1}{n^2} \sum_{j=0}^{n^2-1} (q^{n^2-k})^j g_i^j$, and denote by $\epsilon_i \in (\mathbb{Z}_{n^2})^m$ the vector with 1 in the i-th place and 0 otherwise. Note that

$$1_a = 1^1_{a_1} 1^2_{a_2} \cdots 1^m_{a_m}, \quad 1_a g_i = q^{a_i} 1_a, \quad 1_a e_i = e_i 1_{a-\epsilon_i}. \tag{2.3}$$

Let

$$\mathfrak{q} := q^n, \;\; h_i := g_i^n.$$

So the subgroup generated by h_i is isomorphic to $(\mathbb{Z}_n)^m$. Similarly, let $\{\mathbf{1}_a | a = (a_1, \ldots, a_m) \in (\mathbb{Z}_n)^m\}$ be the set of primitive idempotents of $\Bbbk(\mathbb{Z}_n)^m$, $\mathbf{1}_k^i := \frac{1}{n} \sum_{j=0}^{n-1} (\mathfrak{q}^{n-k})^j h_i^j$ and $\epsilon_i \in (\mathbb{Z}_n)^m$ the vector with 1 in the i-th place and 0 otherwise. For later use, we let $\mathbf{1}_0$ stand for the element $\prod_{i=1}^m \mathbf{1}_0^i$. Then we have the following identities:

$$\mathbf{1}_k^i = \sum_{s=0}^{n-1} 1^i_{k+sn}, \;\; \mathbf{1}_a h_i = \mathfrak{q}^{a_i} \mathbf{1}_a, \quad \mathbf{1}_a e_i = e_i \mathbf{1}_{a-\epsilon_i}. \tag{2.4}$$

For any natural number x, y, define $c(x, y) := q^{-x(y-y')}$, where y' denotes the remainder after dividing y by n. Let

$$J := \sum_{a,b \in (\mathbb{Z}_{n^2})^m} \prod_{i,j=1}^{m} c(a_i, b_j)^{c_{ij}} 1_a \otimes 1_b. \tag{2.5}$$

Define $d(J) := (1 \otimes J)(id \otimes \Delta)(J)(\Delta \otimes id)(J^{-1})(J \otimes 1)^{-1}$, *the differential* of J. The following result is a combination of Lemma 4.2 and Theorem 4.3 in [12].

Lemma 2.1. (1) $d(J)=\sum_{a,b,c\in(\mathbb{Z}_n)^m}(\prod_{i,j=1}^m \mathfrak{q}^{-c_{ij}a_i[\frac{b_j+c_j}{n}]})\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c$.

(2) *The subalgebra A generated by $h_i=g_i^n$ and $e_i, i=1,\cdots,m$, is a quasi-Hopf subalgebra of H_J with coproduct Δ_J and reassociator $\phi=d(J)$.*

Definition 2.2. *The quasi-Hopf algebra in Lemma 2.1 is called the half small quasi-quantum group of $\mathfrak{g}$, denoted by $A_q(\mathfrak{g})$.*

For simplicity, we introduce two more notations:

$$\flat_i:=\sum_{a\in(\mathbb{Z}_n)^m}\prod_{j=1}^m q^{-c_{ij}a_j}\mathbf{1}_a,\quad H_i=\prod_{j=1}^m h_j^{c_{ji}}=\prod_{j=1}^m h_j^{c_{ij}}. \tag{2.6}$$

In [11, 12], there are no explicit formulas for the coproduct, the elements α,β and the antipode for $A_q(\mathfrak{g})$. In fact, they have the following expressions.

Lemma 2.3. *For the quasi-Hopf algebra $A_q(\mathfrak{g})$, we have, for $i=1,\ldots,m$,*

$$\begin{array}{ll}\Delta_J(e_i)=e_i\otimes\flat_i^{-1}+1\otimes\sum_{j=1}^{n-1}\mathbf{1}_j^ie_i+H_i^{-1}\otimes\mathbf{1}_0^ie_i, & \Delta_J(h_i)=h_i\otimes h_i,\\ \alpha=\sum_{a\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^m\mathfrak{q}^{c_{st}a_s[\frac{n-1+a_t}{n}]}\mathbf{1}_a, & \beta=1,\\ S(e_i)=-(\alpha\sum_{j=1}^{n-1}\mathbf{1}_j^ie_i+H_i\alpha\mathbf{1}_0^ie_i)\flat_i\alpha^{-1}, & S(h_i)=h_i^{-1}.\end{array}$$

Proof. We have:

$$\begin{aligned}\Delta_J(e_i) &= J\Delta(e_i)J^{-1}\\ &= \sum_{a,b\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^m c(a_s,b_t)^{c_{st}}1_a\otimes 1_b(e_i\otimes K_i+1\otimes e_i)\\ &\quad\times\sum_{c,d\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^m c(c_s,d_t)^{-c_{st}}1_c\otimes 1_d\\ &= \sum_{c,b\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^m c((c+\epsilon_i)_s,b_t)^{c_{st}}\prod_{s,t=1}^m c(c_s,b_t)^{-c_{st}}q^{\sum_j c_{ij}b_j}1_{c+\epsilon_i}e_i\otimes 1_b\\ &\quad+\sum_{a,d\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^m c(a_s,(d+\epsilon_i)_t)^{c_{st}}\prod_{s,t=1}^m c(a_s,d_t)^{-c_{st}}1_a\otimes 1_{d+\epsilon_i}e_i\\ &= \sum_{c,b\in(\mathbb{Z}_{n^2})^m}\prod_{s\neq i,t}c(c_s,b_t)^{c_{st}-c_{st}}\prod_{t=1}^m c(c_i+1,b_t)^{c_{it}}c(c_i,b_t)^{-c_{it}}q^{\sum_j c_{ij}b_j}1_{c+\epsilon_i}e_i\otimes 1_b\\ &\quad+\sum_{a,d\in(\mathbb{Z}_{n^2})^m}\prod_{s,t\neq i}c(a_s,d_t)^{c_{st}-c_{st}}\prod_{s=1}^m c(a_s,d_i+1)^{c_{si}}c(a_s,d_i)^{-c_{si}}1_a\otimes 1_{d+\epsilon_i}e_i\end{aligned}$$

$$
\begin{aligned}
&= \sum_{c,b\in(\mathbb{Z}_{n^2})^m} q^{\sum_j(-c_{ij}b_j+c_{ij}b'_j)}q^{\sum_j c_{ij}b_j}1_{c+\epsilon_i}e_i\otimes 1_b\\
&\quad + \sum_{a,d\in(\mathbb{Z}_{n^2})^m}\prod_{s=1}^{m} c(a_s,d_i+1)^{c_{si}}c(a_s,d_i)^{-c_{si}}1_a\otimes 1_{d+\epsilon_i}e_i\\
&= \sum_{a,b\in(\mathbb{Z}_{n^2})^m}\prod_{t=1}^{m} q^{c_{it}b'_t}1_{a+\epsilon_i}e_i\otimes 1_b + \sum_{a,b\in(\mathbb{Z}_{n^2})^m}\prod_{s=1}^{m} q^{c_{si}a_s((b_i+1)'-b'_i-1)}1_a\otimes 1_{b+\epsilon_i}e_i\\
&= e_i\otimes \sum_{b\in(\mathbb{Z}_n)^m}\prod_{t=1}^{m} q^{c_{it}b_t}\mathbf{1}_b + \sum_{a,b\in(\mathbb{Z}_n)^m}\prod_{s=1}^{m} q^{c_{si}a_s((b_i+1)'-b'_i-1)}\mathbf{1}_a\otimes \mathbf{1}^i_{b_i+1}e_i\\
&= e_i\otimes \flat_i^{-1} + \sum_{a\in(\mathbb{Z}_n)^m}\mathbf{1}_a\otimes\sum_{b_i\neq n-1}\mathbf{1}^i_{b_i+1}e_i + \sum_{a\in(\mathbb{Z}_n)^m}\prod_{s=1}^{m}\mathfrak{q}^{-c_{si}a_s}\mathbf{1}_a\otimes\mathbf{1}^i_0 e_i\\
&= e_i\otimes\flat_i^{-1} + 1\otimes\sum_{j=1}^{n-1}\mathbf{1}^i_j e_i + H_i^{-1}\otimes\mathbf{1}^i_0 e_i.
\end{aligned}
$$

By definition:

$$\alpha_J=\sum S(\overline{f}_i)\overline{g}_i=\sum_{a\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^{m}c(-a_s,a_t)^{-c_{st}}1_a=\sum_{a\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^{m}q^{-c_{st}a_s(a_t-a'_t)}1_a,$$

$$\beta_J=\sum f_iS(g_i)=\sum_{a\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^{m}c(a_s,-a_t)^{c_{st}}1_a=\sum_{a\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^{m}q^{-c_{st}a_s(-a_t-(n^2-a_t)')}1_a,$$

and so:

$$
\begin{aligned}
\alpha &= \alpha_J\beta_J\\
&= \sum_{a\in(\mathbb{Z}_{n^2})^m}\prod_{s,t=1}^{m}q^{c_{st}a_s(a'_t+(n^2-a_t)')}1_a\\
&= \sum_{a\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^{m}q^{c_{st}a_s(a_t+(n-a_t)')}\mathbf{1}_a\\
&= \sum_{a\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^{m}\mathfrak{q}^{c_{st}a_s[\frac{n-1+a_t}{n}]}\mathbf{1}_a.
\end{aligned}
$$

By the comultiplication formula for e_i and the definition of the antipode, we obtain:

$$S(e_i)\alpha\flat_i^{-1}+\alpha\sum_{j=1}^{n-1}\mathbf{1}^i_je_i+H_i\alpha\mathbf{1}^i_0e_i=\alpha\varepsilon(e_i)=0.$$

It follows that $S(e_i)=-(\alpha\sum_{j=1}^{n-1}\mathbf{1}^i_je_i+H_i\alpha\mathbf{1}^i_0e_i)\flat_i\alpha^{-1}$. The formulas for elements h_i are obvious. □

Remark 2.4. In [12] Etingof and Gelaki used the Cartan matrix (a_{ij}) to define the half small quasi-quantum group $A_q(\mathfrak{g})$. In order to keep the consistency with Lusztig's definition [21], we use the symmetrized Cartan matrix (c_{ij}) instead of (a_{ij}) throughout this paper. To show the difference, we use $A'_q(\mathfrak{g})$ to denote Etingof-Gelaki's half small quasi-quantum group. The two are equal in the simply laced case. But, in general, $A_q(\mathfrak{g}) \ncong A'_q(\mathfrak{g})$ and they are not even twist equivalent (see Section 6 for the definition of twist equivalence). The reason is that they have different reassociators. For example, take the Cartan matrix of type G_2 and assume that they are twist equivalent. Denote by ϕ_A (resp. $\phi_{A'}$) the reassociator of $A_q(\mathfrak{g})$ (resp. $A'_q(\mathfrak{g})$). If the representation categories of $A_q(\mathfrak{g})$ and $A'_q(\mathfrak{g})$ are monoidal equivalent, then their tensor subcategories generated by simple objects are also monoidal equivalent. This implies that $((\Bbbk G)^*, \phi_A)$ is twist equivalent to $((\Bbbk G)^*, \phi_{A'})$, where $G \cong (\mathbb{Z}_n)^m$ is the set of group-like elements of both $A_q(\mathfrak{g})$ and $A'_q(\mathfrak{g})$. However, $((\Bbbk G)^*, \phi_A)$ and $((\Bbbk G)^*, \phi_{A'})$ are twist equivalent if and only if ϕ_A and ϕ'_A are cohomologous cocycles up to automorphisms of G. But clearly this is not always the case (e.g., Taking $n = 3$ and so $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, one can find that ϕ_A is just corresponding to a 3-cocycle over $\mathbb{Z}_3$). So $A_q(\mathfrak{g})$ and $A'_q(\mathfrak{g})$ are not twist equivalent.

2.2. **Quiver Majid algebras.** A dual quasi-bialgebra, or Majid bialgebra for short, is a coalgebra (H, Δ, ε) equipped with a compatible quasi-algebra structure. Namely, there exist two coalgebra homomorphisms:

$$\mathrm{M} : H \otimes H \to H,\ a \otimes b \mapsto ab, \quad \mu : \Bbbk \to H,\ \lambda \mapsto \lambda 1_H$$

and a convolution-invertible map $\Phi : H^{\otimes 3} \to \Bbbk$ called a reassociator, such that for all $a, b, c, d \in H$ the following equalities hold:

$$\begin{gathered} a_{(1)}(b_{(1)}c_{(1)})\Phi(a_{(2)}, b_{(2)}, c_{(2)}) = \Phi(a_{(1)}, b_{(1)}, c_{(1)})(a_{(2)}b_{(2)})c_{(2)}, \\ 1_H a = a = a 1_H, \\ \Phi(a_{(1)}, b_{(1)}, c_{(1)}d_{(1)})\Phi(a_{(2)}b_{(2)}, c_{(2)}, d_{(2)}) \\ = \Phi(b_{(1)}, c_{(1)}, d_{(1)})\Phi(a_{(1)}, b_{(2)}c_{(2)}, d_{(2)})\Phi(a_{(3)}, b_{(1)}, c_{(3)}), \\ \Phi(a, 1_H, b) = \varepsilon(a)\varepsilon(b). \end{gathered}$$

H is called a Majid algebra if, in addition, there exist a coalgebra antimorphism $S : H \to H$ and two functionals $\alpha, \beta : H \to \Bbbk$ such that for all $a \in H$,

$$\begin{gathered} S(a_{(1)})\alpha(a_{(2)})a_{(3)} = \alpha(a)1_H, \quad a_{(1)}\beta(a_{(2)})S(a_{(3)}) = \beta(a)1_H, \\ \Phi(a_{(1)}, S(a_{(3)}), a_{(5)})\beta(a_{(2)})\alpha(a_{(4)}) = \\ \Phi^{-1}(S(a_{(1)}), a_{(3)}, S(a_{(5)}))\alpha(a_{(2)})\beta(a_{(4)}) = \varepsilon(a). \end{gathered}$$

A Majid algebra H is said to be pointed, if the underlying coalgebra is pointed. Given a pointed Majid algebra $(H, \Delta, \varepsilon, \mathrm{M}, \mu, \Phi, S, \alpha, \beta)$, we let $\{H_n\}_{n \geq 0}$ be its coradical filtration, and $\operatorname{gr} H = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$ the associated graded coalgebra. Then $\operatorname{gr} H$ possesses an (induced) graded Majid algebra structure. The corresponding graded reassociator $\operatorname{gr}\Phi$ satisfies $\operatorname{gr}\Phi(\bar{a}, \bar{b}, \bar{c}) = 0$ for all $\bar{a}, \bar{b}, \bar{c} \in \operatorname{gr} H$ unless they all lie in H_0. Similar condition holds for $\operatorname{gr}\alpha$ and $\operatorname{gr}\beta$. In particular, H_0 is a Majid subalgebra and it coincides with the group algebra $\Bbbk G$ of the group $G = G(H)$, the set of group-like elements of H.

Now assume that H is a Majid algebra with reassociator Φ. A linear space M is called an H-Majid bimodule, if M is an H-bicomodule with structure maps $(\delta_{_L}, \delta_{_R})$, and there are two H-bicomodule morphisms:

$$\mu_L : H \otimes M \longrightarrow M,\ h \otimes m \mapsto h \cdot m, \quad \mu_R : M \otimes H \longrightarrow M,\ m \otimes h \mapsto m \cdot h$$

such that for all $g, h \in H, m \in M$, the following equalities hold:

$$1_H \cdot m = m = m \cdot 1_H, \tag{2.7}$$

$$g_{(1)} \cdot (h_{(1)} \cdot m_0)\Phi(g_{(2)}, h_{(2)}, m_1) = \Phi(g_{(1)}, h_{(1)}, m^{-1})(g_{(2)}h_{(2)}) \cdot m^0, \tag{2.8}$$

$$m_0 \cdot (g_{(1)}h_{(1)})\Phi(m_1, g_{(2)}, h_{(2)}) = \Phi(m^{-1}, g_{(1)}, h_{(1)})(m^0 \cdot g_{(2)}) \cdot h_{(2)}, \tag{2.9}$$

$$g_{(1)} \cdot (m_0 \cdot h_{(1)})\Phi(g_{(2)}, m_1, h_{(2)}) = \Phi(g_{(1)}, m^{-1}, h_{(1)})(g_{(2)} \cdot m^0) \cdot h_{(2)}, \tag{2.10}$$

where we use the Sweedler notation:

$$\delta_{_L}(m) = m^{-1} \otimes m^0, \quad \delta_{_R}(m) = m_0 \otimes m_1$$

for the comodule structure maps. Since we only consider Majid bimodules over $(\Bbbk G, \Phi)$, it is convenient to rewrite formulas (2.8)-(2.10). Assume M is a Majid bimodule over $(\Bbbk G, \Phi)$ and so $M = \bigoplus_{g,h\in G} {}^gM^h$, where:

$${}^gM^h = \{m \in M \mid \delta_{_L}(m) = g \otimes m,\ \delta_{_R}(m) = m \otimes h\} .$$

Formulas (2.8)-(2.10) can be simplified as:

$$e \cdot (f \cdot m) = \frac{\Phi(e, f, g)}{\Phi(e, f, h)}(ef) \cdot m, \tag{2.11}$$

$$(m \cdot e) \cdot f = \frac{\Phi(h, e, f)}{\Phi(g, e, f)} m \cdot (ef), \tag{2.12}$$

$$(e \cdot m) \cdot f = \frac{\Phi(e, h, f)}{\Phi(e, g, f)} e \cdot (m \cdot f), \tag{2.13}$$

for all $e, f, g, h \in G$ and $m \in {}^gM^h$.

Now let us recall some basic definitions about quivers. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t :\ Q_1 \longrightarrow Q_0$ are two maps assigning respectively the source and the target for each arrow. A path of length $l \geq 1$ in the quiver Q is a finitely ordered sequence of l arrows $a_l \cdots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $1 \leq i \leq l-1$. By convention, a vertex is said to be a *trivial path* of length 0. For a quiver Q, the associated path coalgebra $\Bbbk Q$ is the $\Bbbk$-space spanned by the set of paths, with counit and comultiplication maps defined by $\varepsilon(g) = 1,\ \Delta(g) = g \otimes g$ for each $g \in Q_0$, and for each nontrivial path $p = a_n \cdots a_1,\ \varepsilon(p) = 0$,

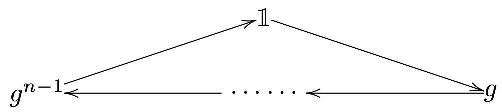
$$\Delta(a_n \cdots a_1) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p\ .$$

The lengths of paths give a natural gradation to the path coalgebra. Let Q_n denote the set of paths of length n in Q. Then $\Bbbk Q = \oplus_{n\geq 0}\Bbbk Q_n$ and $\Delta(\Bbbk Q_n) \subseteq \oplus_{n=i+j}\Bbbk Q_i \otimes \Bbbk Q_j$. It is clear that $\Bbbk Q$ is pointed with the set of group-likes $G(\Bbbk Q) = Q_0$, and has the following coradical filtration

$$\Bbbk Q_0 \subseteq \Bbbk Q_0 \oplus \Bbbk Q_1 \subseteq \Bbbk Q_0 \oplus \Bbbk Q_1 \oplus \Bbbk Q_2 \subseteq \cdots .$$

Hence $\Bbbk Q$ is coradically graded.

In this paper, we consider a special kind of quiver, that is, a *Hopf quiver* [4] defined via a group and its ramification datum. Let G be a group and denote by $\mathcal{C}$ its set of conjugacy classes. A ramification datum R of G is a formal sum $\sum_{C\in\mathcal{C}} R_C C$ of conjugacy classes with coefficients R_C in $\mathbb{N} = \{0, 1, 2, \cdots\}$. The corresponding Hopf quiver $Q = Q(G, R)$ is defined as follows: the set of vertices Q_0 is G, and for each $x \in G$ and $c \in C$, there are R_C arrows going from x to cx. For example, let $G = \mathbb{Z}_n = \langle g \rangle$ and $R = g$, the corresponding Hopf quiver is:



It is called a *basic cycle of length* n.

It is shown in [15] that the path coalgebra $\Bbbk Q$ admits a graded Majid algebra structure if and only if the quiver Q is a Hopf quiver. Moreover, for a given Hopf quiver $Q = Q(G, R)$, if we fix a Majid algebra structure on $\Bbbk Q_0 = (\Bbbk G, \Phi)$ with quasi-antipode (S, α, β), then the set of graded Majid algebra structures on $\Bbbk Q$ with $\Bbbk Q_0 = (\Bbbk G, \Phi, S, \alpha, \beta)$ is in one-to-one correspondence with the set of $(\Bbbk G, \Phi)$-Majid bimodule structures on $\Bbbk Q_1$. We need to recall this correspondence here. One direction is clear. That is, given a graded Majid algebra structure on the path coalgebra $\Bbbk Q$, then $\Bbbk Q_1$ is a $\Bbbk Q_0$-Majid bimodule with module and comodule structures respectively defined by the multiplication and the comultiplication of $\Bbbk Q$.

Conversely, assume that $\Bbbk Q_1$ is a $\Bbbk Q_0$-Majid bimodule. We need to define a multiplication for any two paths in $\Bbbk Q$, which can be obtained as follows. Let $p \in \Bbbk Q$ be a path. An *n-thin split* of p is a sequence $(p_1, \cdots, p_n)$ of vertices and arrows such that the concatenation $p_n \cdots p_1$ is exactly p. These n-thin splits are in one-to-one correspondence with the n-sequences of $(n-l)$ 0's and l 1's. Denote by D_l^n the set of such sequences. Clearly $|D_l^n| = \binom{n}{l}$. For $d = (d_1, \cdots, d_n) \in D_l^n$, the corresponding n-thin split is written as $dp = ((dp)_1, \cdots, (dp)_n)$, in which $(dp)_i$ is a vertex if $d_i = 0$ and an arrow if $d_i = 1$. Let $\alpha = a_m \cdots a_1$ and $\beta = b_n \cdots b_1$ be paths of length m and n respectively. Given $d \in D_m^{m+n}$, we let $\bar{d} \in D_n^{m+n}$ be the complement sequence of d obtained by replacing each 0 by 1 and each 1 by 0. Define an element

$$(\alpha \cdot \beta)_d = [(d\alpha)_{m+n} \cdot (\bar{d}\beta)_{m+n}] \cdots [(d\alpha)_1 \cdot (\bar{d}\beta)_1]$$

in $\Bbbk Q_{m+n}$, where $[(d\alpha)_i \cdot (\bar{d}\beta)_i]$ is understood as the action of the $\Bbbk Q_0$-Majid bimodule on $\Bbbk Q_1$ and the terms in different brackets are put together by cotensor product, or equivalently by concatenation. In terms of this notation, the formula of the product of α and β is given as follows:

$$\alpha \cdot \beta = \sum_{d \in D_m^{m+n}} (\alpha \cdot \beta)_d \ . \tag{2.14}$$

Now let $H = H_0 \oplus H_1 \oplus \cdots$ be a coradically graded pointed Majid algebra. The *Gabriel quiver* $Q(H)$ is defined as follows. Its vertices are group-like elements of H, and the number of arrows between two group-like elements, say g and h, is equal to the number of linear independent non-trivial (h, g)-skew primitive elements. Recall that x is an (h, g)-skew primitive element if $\Delta(x) = g \otimes x + x \otimes h$ and is *trivial* if $x = c(g - h)$ for some $c \in \Bbbk$. The Gabriel quiver $Q(H)$ possesses the following properties:

- $Q(H)$ is a Hopf quiver;
- The H_0-Majid bimodule structure on H_1 induces a $\Bbbk Q(H)_0$-Majid bimodule structure on $\Bbbk Q(H)_1$, and $\Bbbk Q(H)$ is hence a Majid algebra;
- (Theorem of Gabriel's Type) H is a large Majid subalgebra of $\Bbbk Q(H)$. By "a large subalgebra" we mean that it contains the set of vertices and arrows of the Hopf quiver.

One may refer to [15] for more detail. The formula (2.14) can help us to determine the multiplication of any two elements of H. We shall use this observation to study the structure of $A_q(\mathfrak{g})^*$ in the next section.

2.3. **Drinfeld double of a quasi-Hopf algebra.** The construction of the Drinfeld double of a quasi-Hopf algebra is not a trivial generalization from Hopf to the quasi-Hopf case. The double of a Hopf algebra H is defined on $H \otimes H^*$, with H and H^* being subalgebras. If H is a quasi-Hopf algebra, then H^* is not an associative algebra. Thus, one is at a loss to look for an associative algebra structure on $H \otimes H^*$, and might expect that the double should be some kind of hybrid object. Majid solved this problem in [22]. He showed that there exists a quantum double $D(H)$ as a quasi-Hopf algebra defined on $H \otimes H^*$. Hausser and Nill [13] gave a computable realization of $D(H)$ on $H \otimes H^*$. An explicit construction of $D(H)$ was obtained by Schauenburg [24]. Here we recall Schauenburg's construction.

Let $(H, \mathrm{M}, \mu, \Delta, \varepsilon, \phi, S, \alpha, \beta)$ be a finite dimensional quasi-Hopf algebra. Let $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)} = \sum X^i \otimes Y^i \otimes Z^i$ and $\phi^{-1} = \phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)} = \sum \overline{X}^i \otimes \overline{Y}^i \otimes \overline{Z}^i$. Define

$$\gamma := \sum (S(U^i) \otimes S(T^i))(\alpha \otimes \alpha)(V^i \otimes W^i), \tag{2.15}$$

$$\mathbf{f} := \sum (S \otimes S)(\Delta^{op}(\overline{X}^i)) \cdot \gamma \cdot \Delta(\overline{Y}^i \beta S(\overline{Z}^i)), \tag{2.16}$$

$$\chi := (\phi \otimes 1)(\Delta \otimes id \otimes id)(\phi^{-1}), \tag{2.17}$$

$$\omega := (1 \otimes 1 \otimes 1 \otimes \tau(\mathbf{f}^{-1}))(id \otimes \Delta \otimes S \otimes S)(\chi)(\phi \otimes 1 \otimes 1), \tag{2.18}$$

where $(1 \otimes \phi^{-1})(id \otimes id \otimes \Delta)(\phi) = \sum T^i \otimes U^i \otimes V^i \otimes W^i$ and τ is the usual twist, i.e., $\tau(a \otimes b) = b \otimes a$.

As a linear space, $D(H) = H \otimes H^*$ and we write $h \bowtie \psi := h \otimes \psi \in D(H)$. There are two canonical actions, denoted by $\rightharpoonup$, $\leftharpoonup$, of H on H^*. By definition, for any $a, b \in H$ and $\psi \in H^*$

$$\rightharpoonup: \ H \otimes H^* \longrightarrow H^*, \quad (a \rightharpoonup \psi)(b) = \psi(ba),$$

$$\leftharpoonup: \ H^* \otimes H \longrightarrow H^*, \quad (\psi \leftharpoonup a)(b) = \psi(ab).$$

Define a map $\mathbf{T}: \ H^* \to D(H)$ by

$$\mathbf{T}(\psi) = \phi^{(1)}_{(2)} \bowtie S(\phi^{(2)})\alpha(\phi^{(3)} \rightharpoonup \psi \leftharpoonup \phi^{(1)}_{(1)}). \tag{2.19}$$

With the above preparations, we are now able to describe $D(H)$.

Theorem 2.5. [24, Thm. 6.3, 9.3] *Let H be a finite dimensional quasi-Hopf algebra. The quasi-Hopf structure on $D(H) = H \otimes H^*$, such that H is a quasi-Hopf subalgebra via the embedding $h \hookrightarrow h \bowtie \varepsilon$, is determined by the following:*

(i) *As an associative algebra, $D(H)$ is generated by H and $\mathbf{T}(H^*)$, and its multiplication is given by:*

$$\begin{aligned}&(g \bowtie \varphi)(h \bowtie \psi)\\ &= gh_{(1)(2)}\omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \psi \leftharpoonup \omega^{(1)})(\omega^{(4)}S(h_{(2)}) \rightharpoonup \varphi \leftharpoonup h_{(1)(1)}\omega^{(2)}); \quad (\star)\end{aligned}$$

as a quasi-coalgebra, the comultiplication of $D(H)$ is given by:

$$\begin{aligned}\Delta_D(\mathbf{T}(\psi)) &= \tilde{\phi}^{(2)}\mathbf{T}(\psi_{(1)} \leftharpoonup \tilde{\phi}^{(1)})\phi^{(-1)}\phi^{(1)}\\ &\otimes \ \tilde{\phi}^{(3)}\phi^{(-3)}\mathbf{T}(\phi^{(3)} \rightharpoonup \psi_{(2)} \leftharpoonup \phi^{(-2)})\phi^{(2)}, \quad (\star\star)\end{aligned}$$

for $g, h \in H$ and $\varphi, \psi \in H^*$, where $\tilde{\phi}$ is another copy of ϕ.

(ii) *The reassociator ϕ_D, the counit ε_D, the elements α_D, β_D and the antipode S_D are given by:*

$$\phi_D = \phi \bowtie \varepsilon, \varepsilon_D(\mathbf{T}(\psi)) = \psi(\phi^{(1)}S(\phi^{(2)})\alpha\phi^{(3)}), \tag{2.20}$$

$$\alpha_D = \alpha \bowtie \varepsilon, \quad \beta_D = \beta \bowtie \varepsilon, \tag{2.21}$$

$$S_D(\mathbf{T}(\psi)) = \mathbf{f}^{(2)}\mathbf{T}(\mathbf{f}^{(-2)} \rightharpoonup S^{-1}(\psi) \leftharpoonup \mathbf{f}^{(1)})\mathbf{f}^{(-1)}, \tag{2.22}$$

for $\psi \in H^*$.

Remark 2.6. (1) *It is easy to see that $1 \bowtie \varepsilon$ is the unit element of $D(H)$ by the formula $(\star)$. Moreover, as a special case of the product, we have:*

$$(1 \bowtie \varphi)(h \bowtie \varepsilon) = h_{(1)(2)} \bowtie S(h_{(2)}) \rightharpoonup \varphi \leftharpoonup h_{(1)(1)}, \tag{2.23}$$

for $h \in H$ and $\varphi \in H^*$.

(2) *In the process of our computations, we find that there are some errors or misprints in [24] and [13]. Especially, there are misprints in the expression of the element $\mathbf{f}$ given both in [24] and [13], in the expression of the element χ given in [24] and in the expression of the comultiplication formula given in [24]. The correct versions are (2.16), (2.17) and $(\star\star)$.*

2.4. **3-cocycles.** Let G be a group and $(B_\bullet, \partial_\bullet)$ its bar resolution. By applying $\mathrm{Hom}_{\mathbb{Z}G}(-, \Bbbk^*)$ we get a complex $(B_\bullet^*, \partial_\bullet^*)$, where $\Bbbk^* = \Bbbk \setminus \{0\}$ is a trivial G-module. Later on, we will encounter the following problem: Given a 3-cocycle of the complex $(B_\bullet^*, \partial_\bullet^*)$, we have to determine whether it is a 3-coboundary or not. In this subsection, we solve this problem in case G is a finite abelian group.

Now let G be a finitely generated abelian group. Thus $G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$. For every $\mathbb{Z}_{m_i}$, we fix a generator g_i throughout of this paper for $1 \leq i \leq k$. It is known that the following periodic sequence is a projective resolution for the trivial $\mathbb{Z}_{m_i}$-module $\mathbb{Z}$ [25, Sec. 6.2]:

$$\cdots \longrightarrow \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{T_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{N_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{T_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{N_i} \mathbb{Z} \longrightarrow 0, \tag{2.24}$$

where $T_i = g_i - 1$ and $N_i = \sum_{j=0}^{m_i-1} g_i^j$.

We construct the tensor product of the above periodic resolutions for G. Let $K_\bullet$ be the following complex of projective (in fact, free) $\mathbb{Z}G$-modules. For each sequence $a_1, \ldots, a_k$ of nonnegative integers, let $\Psi(a_1, \ldots, a_k)$ be a free generator in degree $a_1 + \cdots + a_k$. Define:

$$K_m := \bigoplus_{a_1+\cdots+a_k=m} (\mathbb{Z}G)\Psi(a_1, \ldots, a_k),$$

and

$$d_i(\Psi(a_1, \ldots, a_k)) = \begin{cases} 0, & a_i = 0, \\ (-1)^{\sum_{l<i} a_l} N_i \Psi(a_1, \ldots, a_i - 1, \ldots, a_k), & 0 \neq a_i \text{ even}, \\ (-1)^{\sum_{l<i} a_l} T_i \Psi(a_1, \ldots, a_i - 1, \ldots, a_k), & a_i \text{ odd}, \end{cases}$$

for $1 \leq i \leq k$. The differential d is defined to be $d_1 + \cdots + d_k$.

Lemma 2.7. *$(K_\bullet, d)$ is a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.*

Proof. Observer that $(K_\bullet, d)$ is exactly the tensor product of the complexes (2.24). Thus the lemma follows from the Künneth formula for complexes (see (3.6.3) in [25]). □

For convenience, we fix the following notations. For any $1 \leq r \leq k$, define $\Psi_r := \Psi(0, \ldots, 1, \ldots, 0)$ where 1 lies in the r-th position. For any $1 \leq r \leq s \leq k$, define $\Psi_{r,s} := \Psi(0, \ldots, 1, \ldots, 1, \ldots, 0)$ where 1 lies in both the r-th and the s-th position if $r < s$ and $\Psi_{r,r} := \Psi(0, \ldots, 2, \ldots, 0)$ where 2 lies in the r-th position. Similarly, one can define $\Psi_{r,s,t}, \Psi_{r,s,s}, \Psi_{r,r,s}$ and $\Psi_{r,r,r}$ for $1 \leq r \leq k$, $1 \leq r < s \leq k$ and $1 \leq r < s < r \leq k$. One could even define $\Psi_{i,j,s,t}$, $\Psi_{i,i,j,s}$, $\Psi_{i,j,s,s}$, $\Psi_{i,j,j,s}$, $\Psi_{i,i,j,j}$, $\Psi_{i,i,i,j}$, $\Psi_{i,j,j,j}$, and $\Psi_{i,i,i,i}$ for $1 \leq i \leq k$, $1 \leq i < j \leq k$, $1 \leq i < j < s \leq k$ and $1 \leq i < j < s < t \leq k$ respectively. Now it it clear that any cochain $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_3, \Bbbk^*)$ is uniquely determined by its values on $\Psi_{r,s,t}, \Psi_{r,s,s}, \Psi_{r,r,s}$ and $\Psi_{r,r,r}$ for $1 \leq r \leq k$, $1 \leq r < s \leq k$ and $1 \leq r < s < t \leq k$. For such numbers, we let $f_{r,s,t} = f(\Psi_{r,s,t}), f_{r,s,s} = f(\Psi_{r,s,s}), f_{r,r,s} = f(\Psi_{r,r,s})$ and $f_{r,r,r} = f(\Psi_{r,r,r})$.

Lemma 2.8. *The 3-cochain $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_3, \Bbbk^*)$ is a cocycle if and only if for all $1 \leq r \leq k$, $1 \leq r < s \leq k$ and $1 \leq r < s < t \leq k$,*

$$f_{r,r,r}^{m_r} = 1, \quad f_{r,s,s}^{m_r} f_{r,r,s}^{m_s} = 1, \quad f_{r,s,t}^{m_r} = f_{r,s,t}^{m_s} = f_{r,s,t}^{m_t} = 1. \tag{2.25}$$

Proof. The proof follows direct computations. By definition, f is a 3-cocycle if and only if $1 = d^*(f)(\Psi_{i,j,s,t}) = f(d(\Psi_{i,j,s,t}))$ for all $1 \leq i \leq j \leq s \leq t \leq k$. For any $a \in \Bbbk^*$, it is clear that $T_i \cdot a = 1$ since $\Bbbk^*$ is considered as a trivial G-module. This means that we only need to consider the condition $1 = d^*(f)(\Psi_{i,j,s,t})$ in the cases: $i = j = s = t$, $i = j < s < t$, $i < j = s < t$, $i < j < s = t$ and $i = j < s = t$ respectively. In case $i = j = s = t$, we have $1 = d^*(f)(\Psi_{i,i,i,i}) = f(N_i\Psi_{i,i,i}) = N_i \cdot f_{i,i,i} = f_{i,i,i}^{m_i}$. Similarly, if $i = j < s < t$, we have $f_{i,s,t}^{m_i} = 1$. If $i < j = s < t$, then we have $f_{i,j,t}^{-m_j} = 1$. In case $i < j < s = t$, we then have $f_{i,j,s}^{m_s} = 1$. Finally, if $i = j < s = t$, we have $f_{i,s,s}^{m_i} f_{i,i,s}^{m_s} = 1$. Now it is easy to see that these relations are the same as in Equation (2.25). □

Lemma 2.9. *The 3-cochain $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_3, \Bbbk^*)$ is a coboundary if and only if for all $1 \leq i < j \leq k$, there are $g_{i,j} \in \Bbbk^*$ such that*

$$f_{i,i,j} = g_{i,j}^{m_i}, \quad f_{i,j,j} = g_{i,j}^{-m_j}, \quad \text{and} \quad f_{l,l,l} = 1 \quad f_{r,s,t} = 1, \tag{2.26}$$

for $1 \leq l \leq k$, and $1 \leq r < s < t \leq k$.

Proof. By definition, f is a coboundary if and only if $f = d^*(g)$ for some 2-cochain $g \in \mathrm{Hom}_{\mathbb{Z}G}(K_2, \Bbbk^*)$. For any $1 \leq i \leq j \leq k$, let $g_{i,j} := g(\Psi_{i,j})$. Since $T_l \cdot a = 1$ for any $a \in \Bbbk^*$, we have $d^*(g)(\Psi_{r,s,t}) = d^*(g)(\Psi_{l,l,l}) = 1$ for $1 \leq r < s < t \leq k$ and $1 \leq l \leq k$. Now for all $1 \leq i < j \leq k$, $f_{i,i,j} = d^*(g)(\Psi_{i,i,j}) = g(N_i\Psi_{i,j} + T_j\Psi_{i,i}) = g_{i,j}^{m_i}$ and $f_{i,j,j} = d^*(g)(\Psi_{i,j,j}) = g(T_i\Psi_{j,j} - N_j\Psi_{i,j}) = g_{i,j}^{-m_j}$. □

Lemma 2.9 provides us an easy way to determine when a 3-cocycle of the complex $(K_\bullet^*, d^*)$ is a 3-coboundary. For the bar resolution, it is sufficient to give a chain map from $(K_\bullet, d_\bullet)$ to $(B_\bullet, \partial_\bullet)$. We define the following three morphisms of $\mathbb{Z}G$-modules:

$$\begin{aligned}
F_1: \quad & K_1 \longrightarrow B_1, \quad \Psi_r \mapsto [g_r];\\
F_2: \quad & K_2 \longrightarrow B_2,\\
& \Psi_{r,s} \mapsto [g_r, g_s] - [g_s, g_r],\\
& \Psi_{r,r} \mapsto \sum_{l=0}^{m_r-1} [g_r^l, g_r];\\
F_3: \quad & K_3 \longrightarrow B_3,\\
& \Psi_{r,s,t} \mapsto [g_r, g_s, g_t] - [g_s, g_r, g_t] - [g_r, g_t, g_s],\\
& \qquad\quad [g_t, g_r, g_s] + [g_s, g_t, g_r] - [g_t, g_s, g_r],\\
& \Psi_{r,r,s} \mapsto \sum_{l=0}^{m_r-1} ([g_r^l, g_r, g_s] - [g_r^l, g_s, g_r] + [g_s, g_r^l, g_r]),\\
& \Psi_{r,s,s} \mapsto \sum_{l=0}^{m_s-1} ([g_r, g_s^l, g_s] - [g_s^l, g_r, g_s] + [g_s^l, g_s, g_r]),\\
& \Psi_{r,r,r} \mapsto \sum_{l=0}^{m_r-1} [g_r, g_r^l, g_r],
\end{aligned}$$

for $0 \le r \le k$, $0 \le r < s \le k$ and $0 \le r < s < t \le k$.

Lemma 2.10. *The following diagram is commutative:*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & K_3 & \xrightarrow{d} & K_2 & \xrightarrow{d} & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow F_3 & & \downarrow F_2 & & \downarrow F_1 & & \| & & \| & & \\
\cdots & \longrightarrow & B_3 & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\partial_1} & B_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Proof. The proof is routine and so we omit it. □

Corollary 2.11. *Let $\phi \in B_3^*$ be a 3-cocycle. Then ϕ is a 3-coboundary if and only if $F_3^*(\phi)$ is a 3-coboundary.*

Proof. Follows from the fact that F_3^* induces an isomorphism between 3-cohomology groups. □

3. The Majid algebra $M_q(\mathfrak{g})$

In this section, we characterize the structure of the Majid algebra $M_q(\mathfrak{g}) := A_q(\mathfrak{g})^*$, the dual of $A_q(\mathfrak{g})$. It is clear that $M_q(\mathfrak{g})$ is a coradically graded pointed Majid algebra such that the reassociator Φ is concentrated on $M_q(\mathfrak{g})_0$, that is, $\Phi(x, y, z) = 0$ unless the homogeneous elements x, y, z all lie in $M_q(\mathfrak{g})_0$.

Recall that we used e_i, h_i $(1 \le i \le m)$ to denote the generators of $A_q(\mathfrak{g})$. It is not hard to see that the elements in $\{\mathbf{1}_a e_i^{n_i} | a \in (\mathbb{Z}_n)^m, 0 \le n_i < l_i, 1 \le i \le m\}$ are linear independent (in fact, $A_q(\mathfrak{g})$ is a subalgebra of $\mathbf{u}_q(\mathfrak{b})$, where $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}$) and can be extended to a basis $\{\mathbf{x}_j\}_j$ consisting of homogeneous elements. The dual basis is denoted by $\{(\mathbf{x}_j)^*\}_j$.

We first fix some notations. Let χ_i be the character of the group generated by elements $h_1, \dots, h_m$, defined as follows:

$$\chi_i(h_j) := \mathfrak{q}^{\delta_{ij}}.$$

Therefore, $\chi_i = (\mathbf{1}_{\epsilon_i})^*$. For $a = (a_1, \dots, a_m) \in (\mathbb{Z}_n)^m$, define $\chi_a := \prod_{i=1}^m \chi_i^{a_i}$. For $1 \le i \le m$, and let

$$\Gamma^i = (\mathbf{1}_{\epsilon_i} e_i)^*.$$

Lemma 3.1. *In $M_q(\mathfrak{g})$, we have:*

$$\Delta(\Gamma^j) = \chi_j \otimes \Gamma^j + \Gamma^j \otimes 1, \quad (\chi_i \Gamma^j)\chi_i^{-1} = \mathfrak{q}^{c_{ji}} q^{-c_{ji}} \Gamma^j. \tag{3.1}$$

Proof. Note that $(\mathbf{1}_{\epsilon_j})^* = \chi_j$, $\mathbf{1}_{\epsilon_j} e_j = e_j \mathbf{1}_0 = \mathbf{1}_{\epsilon_j} e_j \mathbf{1}_0$ and $(\mathbf{1}_{\epsilon_j})^2 e_j = \mathbf{1}_{\epsilon_j} e_j$, so we have the first identity. By the expression of the comultiplication of e_j given in Lemma 2.3, we obtain $\chi_i \Gamma^j = (\mathbf{1}_{\epsilon_i + \epsilon_j} e_j)^*$ and $(\mathbf{1}_{\epsilon_i + \epsilon_j} e_j)^* \chi_i^{-1} = q^{c_{ji}(n-1)} (\mathbf{1}_{\epsilon_j} e_j)^* = \mathfrak{q}^{c_{ji}} q^{-c_{ji}} \Gamma^j$. □

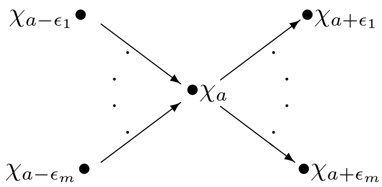
Next, we want to obtain the Gabriel quiver of $M_q(\mathfrak{g})$. We denote this quiver by $Q(M)$. It is not hard to determine the set of vertices of $Q(M)$. Observe that the coradical of $M_q(\mathfrak{g})$ equals $(\Bbbk G, \Phi)$ where $G = \langle \chi_i | 1 \leq i \leq m \rangle \cong (\mathbb{Z}_n)^m$ and we have:

$$\Phi(\chi_a, \chi_b, \chi_c) = \prod_{s,t=1}^{m} \mathfrak{q}^{-c_{st} a_s [\frac{b_t + c_t}{n}]}$$

for $a, b, c \in (\mathbb{Z}_n)^m$. Therefore, $Q(M)_0 = G$. For $1 \leq i \leq m$ and $a = (a_1, \ldots, a_m) \in (\mathbb{Z}_n)^m$, define:

$$\Gamma^i_{\chi_a} := \chi_a \cdot \Gamma^i$$

where '$\cdot$' is the multiplication of $M_q(\mathfrak{g})$. By Lemma 3.1, $\Gamma^i_{\chi_a}$ is a non-trivial $(\chi_a, \chi_{a+\epsilon_i})$- skew primitive element. Clearly $\#\{\Gamma^i_{\chi_a} | 1 \leq i \leq m, a \in (\mathbb{Z}_n)^m\} = mn^m$, which equals $\dim J/J^2$ where J is the Jacobson radical of $A_q(\mathfrak{g})$. The dual relation between the coradical of $M_q(\mathfrak{g})$ and the radical of $A_q(\mathfrak{g})$ guarantees that the set $\{\Gamma^i_{\chi_a} | 1 \leq i \leq m, a \in (\mathbb{Z}_n)^m\}$ forms a basis of $M_q(\mathfrak{g})_1$, leading to the following description of $Q(M)_1$: there is an arrow from χ_a to χ_b if and only if $b = a + \epsilon_i$ for some $1 \leq i \leq m$. In this case, the only arrow is $\Gamma^i_{\chi_a}$. Therefore, locally the quiver $Q(M)$ looks like:



As stated in Subsection 2.2, there is a $\Bbbk Q(M)_0$-Majid bimodule structure on $\Bbbk Q(M)_1$ (from the $M_q(\mathfrak{g})_0$-Majid bimodule structure on $M_q(\mathfrak{g})_1$), which can be described in the following way. We will use the equations (2.11)-(2.13) freely.

Lemma 3.2. *The $\Bbbk Q(M)_0$-Majid bimodule structure on $\Bbbk Q(M)_1$ is given by:*

$$\delta_L : \Bbbk Q(M)_1 \to \Bbbk Q(M)_0 \otimes \Bbbk Q(M)_1, \;\; \Gamma^j_{\chi_a} \mapsto \chi_{a+\epsilon_j} \otimes \Gamma^j_{\chi_a}, \tag{3.2}$$

$$\delta_R : \Bbbk Q(M)_1 \to \Bbbk Q(M)_1 \otimes \Bbbk Q(M)_0, \;\; \Gamma^j_{\chi_a} \mapsto \Gamma^j_{\chi_a} \otimes \chi_a, \tag{3.3}$$

$$\mu_L : \Bbbk Q(M)_0 \otimes \Bbbk Q(M)_1 \to \Bbbk Q(M)_1, \;\; \chi_a \otimes \Gamma^j_{\chi_b} \mapsto \prod_{i=1}^{m} \mathfrak{q}^{-c_{ij} a_i [\frac{b_j+1}{n}]} \Gamma^j_{\chi_{a+b}}, \tag{3.4}$$

$$\mu_R : \Bbbk Q(M)_1 \otimes \Bbbk Q(M)_0 \to \Bbbk Q(M)_1, \;\; \Gamma^j_{\chi_b} \otimes \chi_a \mapsto \prod_{i=1}^{m} q^{c_{ji} a_i} \Gamma^j_{\chi_{a+b}}, \tag{3.5}$$

for $1 \leq j \leq m$, $a, b \in (\mathbb{Z}_n)^m$, where $a + b$ is understood as the addition in $(\mathbb{Z}_n)^m$.

Proof. The bicomodule structure is clear since it is obtained from the comultiplication of $M_q(\mathfrak{g})$ and Lemma 3.1. Due to our choice, the proof of the left module structure is not complicated:

$$\begin{aligned}
\chi_a \cdot \Gamma^j_{\chi_b} &= \chi_a \cdot (\chi_b \cdot \Gamma^j) \\
&= \frac{\Phi(\chi_a, \chi_b, \chi_j)}{\Phi(\chi_a, \chi_b, 1)} (\chi_a \chi_b) \cdot \Gamma^j \\
&= \prod_{i=1}^{m} \mathfrak{q}^{-c_{ij} a_i [\frac{b_j+1}{n}]} \Gamma^j_{\chi_{a+b}}
\end{aligned}$$

where the second equality comes from the definition of Majid bimodule (see Equation (2.11)). In the last equality, we made use of our choice, that is, $\chi_{a+b} \cdot \Gamma^j = \Gamma^j_{\chi_{a+b}}$. We divide the proof of the right module structure into four claims.

Claim 1: $(\chi_i \cdot \Gamma^j_{\chi_b}) \cdot \chi_i^{-1} = \mathfrak{q}^{-c_{ij}[\frac{b_j+1}{n}]} \mathfrak{q}^{c_{ji}} q^{-c_{ji}} \Gamma^j_{\chi_b}$.

Proof of Claim 1. We have

$$\begin{aligned}
(\chi_i \cdot \Gamma^j_{\chi_b}) \cdot \chi_i^{-1} &= \frac{\Phi(\chi_i, \chi_b, \chi_j)}{\Phi(\chi_i, \chi_b, 1)} ((\chi_b \chi_i) \cdot \Gamma^j) \cdot \chi_i^{-1} \\
&= \frac{\Phi(\chi_i, \chi_b, \chi_j)}{\Phi(\chi_i, \chi_b, 1)} \frac{\Phi(\chi_b, \chi_i, 1)}{\Phi(\chi_b, \chi_i, \chi_j)} (\chi_b \cdot (\chi_i \cdot \Gamma^j)) \cdot \chi_i^{-1} \\
&= \frac{\Phi(\chi_i, \chi_b, \chi_j)}{\Phi(\chi_b, \chi_i, \chi_j)} \frac{\Phi(\chi_b, \chi_i, \chi_i^{-1})}{\Phi(\chi_b, \chi_i \chi_j, \chi_i^{-1})} \chi_b \cdot ((\chi_i \cdot \Gamma^j) \cdot \chi_i^{-1}) \\
&= \mathfrak{q}^{-c_{ij}[\frac{b_j+1}{n}]} \mathfrak{q}^{c_{ji}} q^{-c_{ji}} \chi_b \cdot \Gamma^j \\
&= \mathfrak{q}^{-c_{ij}[\frac{b_j+1}{n}]} \mathfrak{q}^{c_{ji}} q^{-c_{ji}} \Gamma^j_{\chi_b},
\end{aligned}$$

where the fourth equality follows from $\frac{\Phi(\chi_i, \chi_b, \chi_j)}{\Phi(\chi_b, \chi_i, \chi_j)} \frac{\Phi(\chi_b, \chi_i, \chi_i^{-1})}{\Phi(\chi_b, \chi_i \chi_j, \chi_i^{-1})} = \mathfrak{q}^{-c_{ij}[\frac{b_j+1}{n}]}$ and Lemma 3.1.

Claim 2: $\Gamma^j_{\chi_b} \cdot \chi_i = q^{c_{ji}} \Gamma^j_{\chi_{b+\epsilon_i}}$.

Proof of Claim 2. Since Δ is an algebra morphism, $\Gamma^j_{\chi_b} \cdot \chi_i$ is a $(\chi_{b+\epsilon_i}, \chi_{b+\epsilon_i+\epsilon_j})$-skew primitive element. So there is a scalar $c \in \Bbbk$ such that $\Gamma^j_{\chi_b} \cdot \chi_i = c\Gamma^j_{\chi_{b+\epsilon_i}}$ since the space of non-trivial $(\chi_{b+\epsilon_i}, \chi_{b+\epsilon_i+\epsilon_j})$-skew primitive elements in $M_q(\mathfrak{g})$ is 1-dimensional. We show that $c = q^{c_{ji}}$. In fact, we have

$$\begin{aligned}
\Gamma^j_{\chi_b} &= \Gamma^j_{\chi_b} \cdot \chi_i^n = \frac{\Phi(\chi_{b+\epsilon_j}, \chi_i, \chi_i^{n-1})}{\Phi(\chi_b, \chi_i, \chi_i^{n-1})} (\Gamma^j_{\chi_b} \cdot \chi_i) \cdot \chi_i^{n-1} \\
&= \mathfrak{q}^{-c_{ji}} c \Gamma^j_{\chi_{b+\epsilon_i}} \cdot \chi_i^{n-1} \\
&= \mathfrak{q}^{-c_{ji}} c \mathfrak{q}^{c_{ij}[\frac{b_j+1}{n}]} (\chi_i \cdot \Gamma^j_{\chi_b}) \cdot \chi_i^{-1} \\
&= c q^{-c_{ji}} \Gamma^j_{\chi_b},
\end{aligned}$$

where the third equality follows from (3.4) and the last equality follows from Claim 1. Therefore, $c = q^{c_{ji}}$.

Claim 3: $\Gamma^j_{\chi_b} \cdot \chi_i^l = q^{c_{ji} l} \Gamma^j_{\chi_{b+l\epsilon_i}}$ for $0 \le l < n$.

Proof of Claim 3. Clearly, one can assume that $l \geq 1$. Inductively, we assume that $\Gamma^j_{\chi_b} \cdot \chi_i^{l-1} = q^{c_{ji}(l-1)}\Gamma^j_{\chi_{b+(l-1)\epsilon_i}}$ for any $b \in (\mathbb{Z}_n)^m$. Then

$$\begin{aligned} \Gamma^j_{\chi_b} \cdot \chi_i^l &= \Gamma^j_{\chi_b} \cdot (\chi_i \chi_i^{l-1}) \\ &= \frac{\Phi(\chi_{b+\epsilon_j}, \chi_i, \chi_i^{l-1})}{\Phi(\chi_b, \chi_i, \chi_i^{l-1})} (\Gamma^j_{\chi_b} \cdot \chi_i) \cdot \chi_i^{l-1} \\ &= q^{c_{ji}} \Gamma^j_{\chi_{b+\epsilon_i}} \cdot \chi_i^{l-1} \\ &= q^{c_{ji} l} \Gamma^j_{\chi_{b+l\epsilon_i}}, \end{aligned}$$

where in the last equality we used the induction hypothesis.

Claim 4: $\Gamma^j_{\chi_b} \cdot \chi_a = \prod_{i=1}^m q^{c_{ji} a_i} \Gamma^j_{\chi_{a+b}}$ for $a = (a_1, \ldots, a_m)$.

Proof of Claim 4. For $1 \leq s \neq t \leq m$ and $0 \leq c_s, c_t \leq n-1$, we have

$$\begin{aligned} \Gamma^j_{\chi_b} \cdot (\chi_s^{c_s} \chi_t^{c_t}) &= \frac{\Phi(\chi_{b+\epsilon_j}, \chi_s^{c_s}, \chi_t^{c_t})}{\Phi(\chi_b, \chi_s^{c_s}, \chi_t^{c_t})} (\Gamma^j_{\chi_b} \cdot \chi_s^{c_s}) \cdot \chi_t^{c_t} \\ &= (\Gamma^j_{\chi_b} \cdot \chi_s^{c_s}) \cdot \chi_t^{c_t}. \end{aligned}$$

That is, the right module structure is associative in case $s \neq t$. In general, one can repeat the above proof to show that:

$$\Gamma^j_{\chi_b} \cdot (\chi_1^{a_1} \chi_2^{a_2} \cdots \chi_m^{a_m}) = (\ldots((\Gamma^j_{\chi_b} \cdot \chi_1^{a_1}) \cdot \chi_2^{a_2}) \cdot \cdots) \cdot \chi_m^{a_m}).$$

Together with Claim 3, this gives the proof of Claim 4. ∎

The above lemma and the formula (2.14) provide us a helpful tool to determine the relations of the generators of $M_q(\mathfrak{g})$. Since the multiplication of $M_q(\mathfrak{g})$ is not associative in general, we need to put parentheses in products. We define:

$$X^{\overrightarrow{l}} := \overbrace{(\cdots(X \cdot X) \cdot X) \cdots X)}^{l}, \quad X^{\overleftarrow{l}} := \overbrace{(X \cdots (X \cdot (X \cdot X)) \cdots)}^{l},$$

for any $X \in M_q(\mathfrak{g})$.

Proposition 3.3. *For $1 \leq i \leq m$, let $l_i = \mathrm{ord}(q^{c_{ii}})$. Then we have*

$$(\Gamma^i)^{\overrightarrow{l_i}} = (\Gamma^i)^{\overleftarrow{l_i}} = 0 \tag{3.6}$$

and $(\Gamma^i)^{\overrightarrow{l}} \neq 0 \neq (\Gamma^i)^{\overleftarrow{l}}$ for $l < l_i$.

Proof. The proof of this result is parallel with the proof of [16, Lem. 3.6] and so we omit the computation. We just explain why the proof of [16, Lem. 3.6] can apply to our case and what results the computation will deliver. Let Q^i be the subquiver of $Q(M)$ defined as follows: the set of vertices is $Q_0^i = \langle \chi_i \rangle \cong \mathbb{Z}_n$ and the set of arrows is $Q_1^i = \{\chi_i^k \cdot \Gamma^i | 0 \leq k \leq n-1\}$. Clearly, Q^i is a basic cycle of length n, which is the case considered in [16, Lem. 3.6]. By the formula (2.14), we find that, to compute $(\Gamma^i)^{\overrightarrow{l}}$ and $(\Gamma^i)^{\overleftarrow{l}}$, we only need to consider the

Majid subalgebra $\Bbbk Q^i$. Therefore, [16, Lem. 3.6] applies. Moreover, if we let $p^l_{t_i}$ be the path starting from χ_i^t with length l in Q^i, then [16, Lem. 3.6] implies that

$$(\Gamma^i)^{\overrightarrow{l}} = l!_{q^{c_{ii}}} p^l_{0_i}, \quad (\Gamma^i)^{\overleftarrow{l}} = \mathbb{q}^{-c_{ii}l'[\frac{l}{n}]} l!_{q^{c_{ii}}} p^l_{0_i}, \tag{3.7}$$

where $l!_{q^{c_{ii}}} = \sum_{j=0}^{l-1} q^{jc_{ii}}$ by definition, and l' is the remainder after dividing l by n. As a consequence, we obtain the desired equations. □

In the following conclusion, there is a delicate point at notation: We will use $\Gamma^j_{\chi_i} \cdot \Gamma^i$ to denote the multiplication in $\Bbbk Q(M)$ while $\Gamma^j_{\chi_i}\Gamma^i$ stands for the connection of arrows (that is, $\Gamma^j_{\chi_i}\Gamma^i$ is the path $1 \to \chi_i \to \chi_i\chi_j$ in $Q(M)$).

Proposition 3.4. *Assume $n \geq 4$. Then for $1 \leq i \neq j \leq m$, we have the following Serre relation:*

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} ((\Gamma^i)^{\overrightarrow{r}} \cdot \Gamma^j) \cdot (\Gamma^i)^{\overleftarrow{s}} = 0.$$

Proof. Since $n \geq 4$, Formula (3.7) implies:

$$(\Gamma^i)^{\overrightarrow{r}} = (\Gamma^i)^{\overleftarrow{r}}, \quad (0 \leq r \leq 1-a_{ij}).$$

So there is no harm in writing $(\Gamma^i)^r$ for both $(\Gamma^i)^{\overrightarrow{r}}$ and $(\Gamma^i)^{\overleftarrow{r}}$. Moreover, by the definition of a Majid algebra, we have

$$((\Gamma^i)^r \cdot \Gamma^j) \cdot (\Gamma^i)^s = \frac{\Phi(1,1,1)}{\Phi(\chi_i^r, \chi_j, \chi_i^s)} (\Gamma^i)^r \cdot (\Gamma^j \cdot (\Gamma^i)^s) = (\Gamma^i)^r \cdot (\Gamma^j \cdot (\Gamma^i)^s).$$

Therefore, the above Serre relation can be written in a more familiar form:

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} (\Gamma^i)^r \cdot \Gamma^j \cdot (\Gamma^i)^s = 0.$$

We will provide a detailed proof of the Serre relation only for Lie algebra $\mathfrak{g}$ of type ADE, where we find out how the product formula (2.14) can be used. For the other types of Lie algebras, we just state the computating results. By (2.14), we have:

$$\begin{aligned} \Gamma^i \cdot \Gamma^j &= [\Gamma^i \cdot \chi_j][1 \cdot \Gamma^j] + [\chi_i \cdot \Gamma^j][\Gamma^i \cdot 1] \\ &= q^{c_{ij}} \Gamma^i_{\chi_j} \Gamma^j + \Gamma^j_{\chi_i} \Gamma^i \end{aligned}$$

and

$$\begin{aligned} \Gamma^j \cdot \Gamma^i &= [\Gamma^j \cdot \chi_i][1 \cdot \Gamma^i] + [\chi_j \cdot \Gamma^i][\Gamma^j \cdot 1] \\ &= q^{c_{ji}} \Gamma^j_{\chi_i} \Gamma^i + \Gamma^i_{\chi_j} \Gamma^j. \end{aligned}$$

Here $\Gamma^j_{\chi_i}\Gamma^i$ is the path $1 \to \chi_i \to \chi_i\chi_j$ and $\Gamma^i_{\chi_j}\Gamma^j$ is the path $1 \to \chi_j \to \chi_i\chi_j$. Therefore, if $a_{ij} = 0$ (which implies $c_{ij} = 0$) then we have $\Gamma^i \cdot \Gamma^j = \Gamma^j \cdot \Gamma^i$. Now consider the case $a_{ij} = -1$.

We compute:

$$\begin{aligned}
\Gamma^i\cdot\Gamma^i_{\chi_j}\Gamma^j &= [\Gamma^i\cdot\chi_i\chi_j][1\cdot\Gamma^i_{\chi_j}][1\cdot\Gamma^j]+[\chi_i\cdot\Gamma^i_{\chi_j}][\Gamma^i\cdot\chi_j][1\cdot\Gamma^j]\\
&\quad+[\chi_i\cdot\Gamma^i_{\chi_j}][\chi_i\cdot\Gamma^j][\Gamma^i\cdot 1]\\
&= q^{c_{ii}+c_{ij}}\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+q^{c_{ij}}\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i\\
&= q^{c_{ij}}(1+q^{c_{ii}})\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i;
\end{aligned}$$

$$\begin{aligned}
\Gamma^i_{\chi_j}\Gamma^j\cdot\Gamma^i &= [\chi_i\chi_j\cdot\Gamma^i][\Gamma^i_{\chi_j}\cdot 1][\Gamma^j\cdot 1]+[\Gamma^i_{\chi_j}\cdot\chi_i][\chi_j\cdot\Gamma^i][\Gamma^j\cdot 1]\\
&\quad+[\Gamma^i_{\chi_j}\cdot\chi_i][\Gamma^j\cdot\chi_i][1\cdot\Gamma^i]\\
&= \Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+q^{c_{ii}}\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+q^{c_{ii}+c_{ji}}\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i\\
&= (1+q^{c_{ii}})\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+q^{c_{ii}+c_{ji}}\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i;
\end{aligned}$$

$$\begin{aligned}
\Gamma^i\cdot\Gamma^j_{\chi_i}\Gamma^i &= [\Gamma^i\cdot\chi_i\chi_j][1\cdot\Gamma^i_{\chi_j}][1\cdot\Gamma^i]+[\chi_i\cdot\Gamma^j_{\chi_i}][\Gamma^i\cdot\chi_i][1\cdot\Gamma^i]\\
&\quad+[\chi_i\cdot\Gamma^j_{\chi_i}][\chi_i\cdot\Gamma^i][\Gamma^i\cdot 1]\\
&= q^{c_{ii}+c_{ij}}\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i+q^{c_{ii}}\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i+\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i\\
&= (1+q^{c_{ii}})\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i+q^{c_{ii}+c_{ij}}\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i,
\end{aligned}$$

and:

$$\begin{aligned}
\Gamma^j_{\chi_i}\Gamma^i\cdot\Gamma^i &= [\chi_i\chi_j\cdot\Gamma^i][\Gamma^j_{\chi_i}\cdot 1][\Gamma^i\cdot 1]+[\Gamma^j_{\chi_i}\cdot\chi_i][\chi_i\cdot\Gamma^i][\Gamma^i\cdot 1]\\
&\quad+[\Gamma^j_{\chi_i}\cdot\chi_i][\Gamma^i\cdot\chi_i][1\cdot\Gamma^i]\\
&= \Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i+q^{c_{ji}}\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i+q^{c_{ji}+c_{ii}}\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i\\
&= q^{c_{ji}}(1+q^{c_{ii}})\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i+\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i.
\end{aligned}$$

Here $\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j$ is the path $1\to\chi_j\to\chi_i\chi_j\to\chi_i^2\chi_j$. The other paths are similar. Thus,

$$\begin{aligned}
\Gamma^i\cdot(\Gamma^i\cdot\Gamma^j) &= \Gamma^i\cdot(q^{c_{ij}}\Gamma^i_{\chi_j}\Gamma^j+\Gamma^j_{\chi_i}\Gamma^i)\\
&= q^{c_{ij}}(q^{c_{ij}}(1+q^{c_{ii}})\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i)\\
&\quad+((1+q^{c_{ii}})\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i+q^{c_{ii}+c_{ij}}\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i)\\
&= q^{2c_{ij}}(1+q^{c_{ii}})\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+q^{c_{ij}}(1+q^{c_{ii}})\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i\\
&\quad+(1+q^{c_{ii}})\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i,
\end{aligned}$$

$$\begin{aligned}
\Gamma^i\cdot(\Gamma^j\cdot\Gamma^i) &= \Gamma^i\cdot(q^{c_{ji}}\Gamma^j_{\chi_i}\Gamma^i+\Gamma^i_{\chi_j}\Gamma^j)\\
&= q^{c_{ji}}((1+q^{c_{ii}})\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i+q^{c_{ii}+c_{ij}}\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i)\\
&\quad+(q^{c_{ij}}(1+q^{c_{ii}})\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i)\\
&= q^{c_{ij}}(1+q^{c_{ii}})\Gamma^i_{\chi_i\chi_j}\Gamma^i_{\chi_j}\Gamma^j+(q^{c_{ii}+c_{ij}+c_{ji}}+1)\Gamma^i_{\chi_i\chi_j}\Gamma^j_{\chi_i}\Gamma^i\\
&\quad+q^{c_{ji}}(1+q^{c_{ii}})\Gamma^j_{\chi_i^2}\Gamma^i_{\chi_i}\Gamma^i,
\end{aligned}$$

and:

$$\begin{aligned}
(\Gamma^j \cdot \Gamma^i) \cdot \Gamma^i &= (q^{c_{ji}} \Gamma^j_{\chi_i} \Gamma^i + \Gamma^i_{\chi_j} \Gamma^j) \cdot \Gamma^i \\
&= q^{c_{ji}} (q^{c_{ji}} (1 + q^{c_{ii}}) \Gamma^j_{\chi_i^2} \Gamma^i_{\chi_i} \Gamma^i + \Gamma^i_{\chi_i \chi_j} \Gamma^j_{\chi_i} \Gamma^i) \\
&\quad + ((1 + q^{c_{ii}}) \Gamma^i_{\chi_i \chi_j} \Gamma^i_{\chi_j} \Gamma^j + q^{c_{ii} + c_{ji}} \Gamma^i_{\chi_i \chi_j} \Gamma^j_{\chi_i} \Gamma^i) \\
&= (1 + q^{c_{ii}}) \Gamma^i_{\chi_i \chi_j} \Gamma^i_{\chi_j} \Gamma^j + q^{c_{ji}} (1 + q^{c_{ii}}) \Gamma^i_{\chi_i \chi_j} \Gamma^j_{\chi_i} \Gamma^i \\
&\quad + q^{2c_{ji}} (1 + q^{c_{ii}}) \Gamma^j_{\chi_i^2} \Gamma^i_{\chi_i} \Gamma^i.
\end{aligned}$$

If $a_{ij} = -1$, then $d_i = 1, c_{ij} = -1$ and so:

$$\Gamma^i \cdot (\Gamma^i \cdot \Gamma^j) = (1 + q^{-2}) \Gamma^i_{\chi_i \chi_j} \Gamma^i_{\chi_j} \Gamma^j + (q + q^{-1}) \Gamma^i_{\chi_i \chi_j} \Gamma^j_{\chi_i} \Gamma^i + (q^2 + 1) \Gamma^j_{\chi_i^2} \Gamma^i_{\chi_i} \Gamma^i,$$

$$\Gamma^i \cdot (\Gamma^j \cdot \Gamma^i) = (q + q^{-1}) \Gamma^i_{\chi_i \chi_j} \Gamma^i_{\chi_j} \Gamma^j + 2 \Gamma^i_{\chi_i \chi_j} \Gamma^j_{\chi_i} \Gamma^i + (q + q^{-1}) \Gamma^j_{\chi_i^2} \Gamma^i_{\chi_i} \Gamma^i,$$

and:

$$(\Gamma^j \cdot \Gamma^i) \cdot \Gamma^i = (q^2 + 1) \Gamma^i_{\chi_i \chi_j} \Gamma^i_{\chi_j} \Gamma^j + (q + q^{-1}) \Gamma^i_{\chi_i \chi_j} \Gamma^j_{\chi_i} \Gamma^i + (1 + q^{-2}) \Gamma^j_{\chi_i^2} \Gamma^i_{\chi_i} \Gamma^i.$$

Therefore, we have the following Serre relation in case $a_{ij} = -1$:

$$\Gamma^i \cdot (\Gamma^i \cdot \Gamma^j) - (q + q^{-1}) \Gamma^i \cdot (\Gamma^j \cdot \Gamma^i) + (\Gamma^j \cdot \Gamma^i) \cdot \Gamma^i = 0.$$

So, the proof for ADE types is done.

For the other types, we need to consider the case $a_{ij} = -2$ or $a_{ij} = -3$. Here we will omit the detail from the computation since they are similar, and we only state the results. Let $\Gamma^{i^r j i^s}$ be the following path in $Q(M)$:

$$1 \to \chi_i \to \cdots \to \chi_i^s \to \chi_i^s \chi_j \to \chi_i^{s+1} \chi_j \to \cdots \to \chi_i^{s+r} \chi_j.$$

Then we have:

$$\begin{aligned}
(\Gamma^i)^3\cdot\Gamma^j &= q^{3c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{i^3j}\\
&+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{i^2ji}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{iji^2}\\
&+(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{ji^3};\\
(\Gamma^i)^2\cdot\Gamma^j\cdot\Gamma^i &= q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{i^3j}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}+2c_{ij}})\Gamma^{i^2ji}\\
&+(1+q^{c_{ii}})(1+q^{c_{ii}+2c_{ij}}+q^{2c_{ii}+2c_{ij}})\Gamma^{iji^2}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{ji^3};\\
\Gamma^i\cdot\Gamma^j\cdot(\Gamma^i)^2 &= q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{i^3j}\\
&+(1+q^{c_{ii}})(1+q^{c_{ii}+2c_{ij}}+q^{2c_{ii}+2c_{ij}})\Gamma^{i^2ji}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}+2c_{ij}})\Gamma^{iji^2}\\
&+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{ji^3};\\
\Gamma^j(\Gamma^i)^3 &= (1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{i^3j}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{i^2ji}\\
&+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{iji^2}\\
&+q^{3c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})\Gamma^{ji^3};
\end{aligned}$$

One can use the above results to build the Serre relation for the Lie algebras of BCF types. We obtain:

$$\begin{aligned}
(\Gamma^i)^4\cdot\Gamma^j &= q^{4c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^4j}\\
&+q^{3c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^3ji}\\
&+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^2ji^2}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{iji^3}\\
&+(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{ji^4};
\end{aligned}$$

$$\begin{aligned}
(\Gamma^i)^3\cdot\Gamma^j\cdot\Gamma^i &= q^{3c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^4j}\\
&+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}+2c_{ij}})\Gamma^{i^3ji}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}+2c_{ij}}+q^{3c_{ii}+2c_{ij}})\Gamma^{i^2ji^2}\\
&+(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}+2c_{ij}}+q^{2c_{ii}+2c_{ij}}+q^{3c_{ii}+2c_{ij}})\Gamma^{iji^3}\\
&+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{ji^4};
\end{aligned}$$

$$\begin{aligned}(\Gamma^i)^2\cdot\Gamma^j\cdot(\Gamma^i)^2 &= q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^4j}\\ &+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}+2c_{ij}}+q^{3c_{ii}+2c_{ij}})\Gamma^{i^3ji}\\ &+(1+q^{c_{ii}})^2(1+q^{c_{ii}+2c_{ij}}+2q^{2c_{ii}+2c_{ij}}+q^{3c_{ii}+2c_{ij}}+q^{4c_{ii}+4c_{ij}})\Gamma^{i^2ji^2}\\ &+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}+2c_{ij}}+q^{3c_{ii}+2c_{ij}})\Gamma^{iji^3}\\ &+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{ji^4};\end{aligned}$$

$$\begin{aligned}\Gamma^i\cdot\Gamma^j\cdot(\Gamma^i)^3 &= q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^4j}\\ &+(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}+2c_{ij}}+q^{2c_{ii}+2c_{ij}}+q^{3c_{ii}+2c_{ij}})\Gamma^{i^3ji}\\ &+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}+2c_{ij}}+q^{3c_{ii}+2c_{ij}})\Gamma^{i^2ji^2}\\ &+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}+2c_{ij}})\Gamma^{iji^3}\\ &+q^{3c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{ji^4};\end{aligned}$$

$$\begin{aligned}\Gamma^j\cdot(\Gamma^i)^4 &= (1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^4j}\\ &+q^{c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^3ji}\\ &+q^{2c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{i^2ji^2}\\ &+q^{3c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{iji^3}\\ &+q^{4c_{ij}}(1+q^{c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}})(1+q^{c_{ii}}+q^{2c_{ii}}+q^{3c_{ii}})\Gamma^{ji^4}.\end{aligned}$$

Similarly, the Serre relation for type G_2 can be obtained from the above equalities. □

4. The Drinfeld double of $A_q(\mathfrak{g})$

In this section, we will determine the structure of $D(A_q(\mathfrak{g}))$. We shall first describe the algebraic structure. To this end, we need to compute the elements $\gamma, \mathbf{f}, \chi$ and ω according to the formulas (2.15)-(2.18). The following easy observation [19, Lem. 3.2] will be used frequently throughout the paper.

Lemma 4.1. *For any two natural numbers i, j, we have following identity:*

$$[\frac{i+j'}{n}] = [\frac{i+j}{n}] - [\frac{j}{n}]. \tag{4.1}$$

Lemma 4.2. *For the quasi-Hopf algebra $A_q(\mathfrak{g})$, we have:*

$$\gamma = \sum_{b,c\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m}\mathfrak{q}^{c_{ij}(-(b_i+c_i)[\frac{b_j+c_j}{n}]+c_i-c_i[\frac{n-b_j}{n}]+b_i[\frac{n-1+b_j}{n}]+c_i[\frac{n-1+c_j}{n}])}\mathbf{1}_b\otimes\mathbf{1}_c,$$

$$\mathbf{f} = \sum_{e,f\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m}\mathfrak{q}^{c_{ij}(-e_i+(e_i+f_i)([\frac{n-(e_j+f_j)'}{n}]-[\frac{e_j+f_j}{n}])-f_i[\frac{n-e_j}{n}]+e_i[\frac{n-1+e_j}{n}]+f_i[\frac{n-1+f_j}{n}])}$$
$$\times\mathbf{1}_e\otimes\mathbf{1}_f,$$

$$\chi = \sum_{a,b,c,d\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m}\mathfrak{q}^{c_{ij}(-a_i[\frac{b_j+c_j}{n}]+(a_i+b_i)[\frac{c_i+d_i}{n}])}\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c\otimes\mathbf{1}_d,$$

and:

$$\omega = \sum_{a,b,c,d,e\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}(-a_i[\frac{b_j+c_j+d_j}{n}]+(a_i+b_i+c_i+d_i+e_i)[\frac{d_j+e_j}{n}]-e_i+e_i[\frac{n-d_j}{n}])} \times \mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c\otimes S(\mathbf{1}_d)\otimes S(\mathbf{1}_e).$$

Proof. By definition, we have

$$\begin{aligned}
& T^i\otimes U^i\otimes V^i\otimes W^i \\
= \; & (1\otimes\phi^{-1})(id\otimes id\otimes\Delta)(\phi) \\
= \; & \sum_{a,b,c\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}a_i[\frac{b_j+c_j}{n}]} 1\otimes\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c \times \\
& \sum_{d,e,f^1,f^2\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{-c_{ij}d_i[\frac{e_j+(f_j^1+f_j^2)'}{n}]} \mathbf{1}_d\otimes\mathbf{1}_e\otimes\mathbf{1}_{f^1}\otimes\mathbf{1}_{f^2} \\
= \; & \sum_{a,b,c,d\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}(a_i[\frac{b_j+c_j}{n}]-d_i[\frac{a_j+(b_j+c_j)'}{n}])} \mathbf{1}_d\otimes\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c
\end{aligned}$$

and so:

$$\begin{aligned}
\gamma = \; & (S(U^i)\otimes S(T^i))(\alpha\otimes\alpha)(V^i\otimes W^i) \\
= \; & \sum_{a,b,c,d\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}(a_i[\frac{b_j+c_j}{n}]-d_i[\frac{a_j+(b_j+c_j)'}{n}])} S(\mathbf{1}_a)\alpha\mathbf{1}_b\otimes S(\mathbf{1}_d)\alpha\mathbf{1}_c \\
= \; & \sum_{a,b,c,d\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}(a_i[\frac{b_j+c_j}{n}]-d_i[\frac{a_j+(b_j+c_j)'}{n}]+b_i[\frac{n-1+b_j}{n}]+c_i[\frac{n-1+c_j}{n}])} \\
& \times S(\mathbf{1}_a)\mathbf{1}_b\otimes S(\mathbf{1}_d)\mathbf{1}_c \\
= \; & \sum_{b,c\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}((n-b_i)'[\frac{b_j+c_j}{n}]-(n-c_i)'[\frac{(n-b_j)'+(b_j+c_j)'}{n}]+b_i[\frac{n-1+b_j}{n}]+c_i[\frac{n-1+c_j}{n}])} \\
& \times\mathbf{1}_b\otimes\mathbf{1}_c \\
= \; & \sum_{b,c\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}(-b_i[\frac{b_j+c_j}{n}]+c_i([\frac{n-b_j+b_j+c_j}{n}]-[\frac{n-b_j}{n}]-[\frac{b_j+c_j}{n}])+b_i[\frac{n-1+b_j}{n}]+c_i[\frac{n-1+c_j}{n}])} \\
& \times\mathbf{1}_b\otimes\mathbf{1}_c \\
= \; & \sum_{b,c\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}(-b_i[\frac{b_j+c_j}{n}]+c_i(1-[\frac{n-b_j}{n}]-[\frac{b_j+c_j}{n}])+b_i[\frac{n-1+b_j}{n}]+c_i[\frac{n-1+c_j}{n}])} \\
& \times\mathbf{1}_b\otimes\mathbf{1}_c \\
= \; & \sum_{b,c\in(\mathbb{Z}_n)^m} \prod_{i,j=1}^m \mathfrak{q}^{c_{ij}(-(b_i+c_i)[\frac{b_j+c_j}{n}]+c_i-c_i[\frac{n-b_j}{n}]+b_i[\frac{n-1+b_j}{n}]+c_i[\frac{n-1+c_j}{n}])} \\
& \times\mathbf{1}_b\otimes\mathbf{1}_c,
\end{aligned}$$

where the fifth equality follows from Lemma 4.1.

Therefore:

$$
\begin{aligned}
\mathbf{f} &= \sum (S\otimes S)(\Delta^{op}(\overline{X}^i))\cdot\gamma\cdot\Delta(\overline{Y}^i\beta S(\overline{Z}^i)) \\
&= \sum_{a^1,a^2,b,c\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{c_{ij}(a_i^1+a_i^2)[\frac{b_j+c_j}{n}]}(S(\mathbf{1}_{a^1})\otimes S(\mathbf{1}_{a^2}))\gamma\Delta(\mathbf{1}_b S(\mathbf{1}_c)) \\
&= \sum_{e,f\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{c_{ij}((n-e_i)'+(n-f_i)')[\frac{(e_j+f_j)'+(n-(e_j+f_j)')'}{n}]}\gamma\mathbf{1}_e\otimes\mathbf{1}_f \\
&= \sum_{e,f\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{-c_{ij}(e_i+f_i)(1-[\frac{n-(e_j+f_j)'}{n}])}\gamma\mathbf{1}_e\otimes\mathbf{1}_f \\
&= \sum_{e,f\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{-c_{ij}(e_i+f_i)(1-[\frac{n-(e_j+f_j)'}{n}])} \\
&\quad \mathrm{q}^{c_{ij}(-(e_i+f_i)[\frac{e_j+f_j}{n}]+f_i-f_i[\frac{n-e_j}{n}]+e_i[\frac{n-1+e_j}{n}]+f_i[\frac{n-1+f_j}{n}])}\mathbf{1}_e\otimes\mathbf{1}_f \\
&= \sum_{e,f\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{c_{ij}(-e_i+(e_i+f_i)([\frac{n-(e_j+f_j)'}{n}]-[\frac{e_j+f_j}{n}])-f_i[\frac{n-e_j}{n}]+e_i[\frac{n-1+e_j}{n}]+f_i[\frac{n-1+f_j}{n}])} \\
&\quad \times\mathbf{1}_e\otimes\mathbf{1}_f,
\end{aligned}
$$

where the fourth equality follows also from Lemma 4.1.

The computation for χ is easy. Indeed:

$$
\begin{aligned}
\chi &= (\phi\otimes 1)(\Delta\otimes id\otimes id)(\phi^{-1}) \\
&= \sum_{a,b,c\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{-c_{ij}a_i[\frac{b_j+c_j}{n}]}\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c\otimes 1 \\
&\quad \sum_{d^1,d^2,e,f\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{c_{ij}(d_i^1+d_i^2)[\frac{e_j+f_j}{n}]}\mathbf{1}_{d^1}\otimes\mathbf{1}_{d^2}\otimes\mathbf{1}_e\otimes\mathbf{1}_f \\
&= \sum_{a,b,c,d\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m} \mathrm{q}^{c_{ij}(-a_i[\frac{b_j+c_j}{n}]+(a_i+b_i)[\frac{c_j+d_j}{n}])}\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c\otimes\mathbf{1}_d.
\end{aligned}
$$

Now we are able to compute the element ω.

$$
\begin{aligned}
\omega &= (1\otimes 1\otimes 1\otimes \tau(\mathbf{f}^{-1}))(id\otimes\Delta\otimes S\otimes S)(\chi)(\phi\otimes 1\otimes 1)\\
&= \sum_{e,f\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^m \mathbb{q}^{c_{ij}(e_i-(e_i+f_i)([\frac{n-(e_j+f_j)'}{n}]-[\frac{e_j+f_j}{n}])+f_i[\frac{n-e_j}{n}]-e_i[\frac{n-1+e_j}{n}]-f_i[\frac{n-1+f_j}{n}])}\\
&\qquad 1\otimes 1\otimes 1\otimes \mathbf{1}_f\otimes \mathbf{1}_e\\
&\quad \sum_{a,b^1,b^2,c,d\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^m \mathbb{q}^{c_{ij}(-a_i[\frac{(b_j^1+b_j^2)'+c_j}{n}]+(a_i+b_i^1+b_i^2)[\frac{c_j+d_j}{n}])}\mathbf{1}_a\otimes\mathbf{1}_{b^1}\otimes\mathbf{1}_{b^2}\otimes S(\mathbf{1}_c)\otimes S(\mathbf{1}_d)\\
&\quad \sum_{a,b^1,b^2\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^m \mathbb{q}^{-c_{ij}a_i[\frac{b_j^1+b_j^2}{n}]}\mathbf{1}_a\otimes\mathbf{1}_{b^1}\otimes\mathbf{1}_{b^2}\otimes 1\otimes 1\\
&= \sum_{a,b^1,b^2,c,d\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^m \mathbb{q}^{c_{ij}(-a_i[\frac{b_j^1+b_j^2+c_j}{n}]+(a_i+b_i^1+b_i^2)[\frac{c_j+d_j}{n}]-d_i-c_i[\frac{n-(n-d_j)'}{n}])}\\
&\quad \mathbb{q}^{c_{ij}((d_i+c_i)([\frac{n-((n-d_j)'+(n-c_j)')'}{n}]-[\frac{(n-d_j)'+(n-c_j)'}{n}])+d_i[\frac{n-1+(n-d_j)'}{n}]+c_i[\frac{n-1+(n-c_j)'}{n}])}\\
&\quad \mathbf{1}_a\otimes\mathbf{1}_{b^1}\otimes\mathbf{1}_{b^2}\otimes S(\mathbf{1}_c)\otimes S(\mathbf{1}_d).
\end{aligned}
$$

Note that we used Lemma 4.1 in the computation in order to obtain:

$$
\mathbb{q}^{-c_{ij}a_i[\frac{b_j^1+b_j^2}{n}]}\mathbb{q}^{c_{ij}(-a_i[\frac{(b_j^1+b_j^2)'+c_j}{n}])}=\mathbb{q}^{-c_{ij}a_i[\frac{b_j^1+b_j^2+c_j}{n}]}.
$$

The following equalities can be verified directly:

$$
\begin{aligned}
[\tfrac{n-((n-d_j)'+(n-c_j)')'}{n}] &= [\tfrac{n-(d_j+c_j)'}{n}],\\
[\tfrac{(n-d_j)'+(n-c_j)'}{n}] &= [\tfrac{2n-d_j-c_j}{n}]-[\tfrac{n-d_j}{n}]-[\tfrac{n-c_j}{n}]\\
&= 1+[\tfrac{n-(d_j+c_j)'}{n}]-[\tfrac{d_j+c_j}{n}]-[\tfrac{n-d_j}{n}]-[\tfrac{n-c_j}{n}],\\
[\tfrac{n-(n-d_j)'}{n}] &= [\tfrac{n-d_j}{n}],\\
[\tfrac{n-1+(n-d_j)'}{n}] &= [\tfrac{n-1+d_j}{n}],\\
[\tfrac{n-1+(n-c_j)'}{n}] &= [\tfrac{n-1+c_j}{n}].
\end{aligned}
$$

Applying these equations to the expression of ω, we obtain:

$$
\begin{aligned}
\omega &= \sum_{a,b^1,b^2,c,d\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^m \mathbb{q}^{c_{ij}(-a_i[\frac{b_j^1+b_j^2+c_j}{n}]+(a_i+b_i^1+b_i^2)[\frac{c_j+d_j}{n}]-d_i-c_i[\frac{n-d_j}{n}])}\\
&\quad \mathbb{q}^{c_{ij}((d_i+c_i)(-1+[\frac{d_j+c_j}{n}]+[\frac{n-d_j}{n}]+[\frac{n-c_j}{n}])+d_i[\frac{n-1+d_j}{n}]+c_i[\frac{n-1+c_j}{n}])}\\
&\quad \mathbf{1}_a\otimes\mathbf{1}_{b^1}\otimes\mathbf{1}_{b^2}\otimes S(\mathbf{1}_c)\otimes S(\mathbf{1}_d)\\
&= \sum_{a,b^1,b^2,c,d\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^m \mathbb{q}^{c_{ij}(-a_i[\frac{b_j^1+b_j^2+c_j}{n}]+(a_i+b_i^1+b_i^2+d_i+c_i)[\frac{c_j+d_j}{n}])}\\
&\quad \mathbb{q}^{c_{ij}(d_i(-1-1+[\frac{n-d_j}{n}]+[\frac{n-c_j}{n}]+[\frac{n-1+d_j}{n}])+c_i(-1+[\frac{n-d_j}{n}]+[\frac{n-c_j}{n}]-[\frac{n-d_j}{n}]+[\frac{n-1+c_j}{n}]))}\\
&\quad \mathbf{1}_a\otimes\mathbf{1}_{b^1}\otimes\mathbf{1}_{b^2}\otimes S(\mathbf{1}_c)\otimes S(\mathbf{1}_d).
\end{aligned}
$$

Now we can apply the following identity to simplify the formula of ω:

$$[\frac{n-1+z}{n}]+[\frac{n-z}{n}]=1, \quad \text{for } 0\leq z<n.$$

It follows that

$$\begin{aligned}\omega &= \sum_{a,b^1,b^2,c,d\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m}\mathfrak{q}^{c_{ij}(-a_i[\frac{b^1_j+b^2_j+c_j}{n}]+(a_i+b^1_i+b^2_i+d_i+c_i)[\frac{c_j+d_j}{n}]-d_i+d_i[\frac{n-c_j}{n}])}\\ &\quad \mathbf{1}_a\otimes\mathbf{1}_{b^1}\otimes\mathbf{1}_{b^2}\otimes S(\mathbf{1}_c)\otimes S(\mathbf{1}_d)\\ &= \sum_{a,b,c,d,e\in(\mathbb{Z}_n)^m}\prod_{i,j=1}^{m}\mathfrak{q}^{c_{ij}(-a_i[\frac{b_j+c_j+d_j}{n}]+(a_i+b_i+c_i+d_i+e_i)[\frac{d_j+e_j}{n}]-e_i+e_i[\frac{n-d_j}{n}])}\\ &\qquad \mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c\otimes S(\mathbf{1}_d)\otimes S(\mathbf{1}_e).\end{aligned}$$

□

The algebraic structure of $D(A_q(\mathfrak{g}))$ can be described by the following three propositions, which can be understood roughly as "the generating relations for $A_q(\mathfrak{g})$", "the generating relations for $M_q(\mathfrak{g})$" and "the generating relations between $A_q(\mathfrak{g})$ and $M_q(\mathfrak{g})$" respectively.

Proposition 4.3. *In $D(A_q(\mathfrak{g}))$, we have the following relations*

$$(h_i\bowtie\varepsilon)^n=1\bowtie\varepsilon,\ \ (h_i\bowtie\varepsilon)(h_j\bowtie\varepsilon)=(h_j\bowtie\varepsilon)(h_i\bowtie\varepsilon), \tag{4.2}$$

$$(h_i\bowtie\varepsilon)(e_j\bowtie\varepsilon)(h_i\bowtie\varepsilon)^{-1}=\mathfrak{q}^{\delta_{i,j}}(e_j\bowtie\varepsilon),\ \ (e_i\bowtie\varepsilon)^{l_i}=0, \tag{4.3}$$

$$\sum_{r+s=1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{d_i}(e_i\bowtie\varepsilon)^r(e_j\bowtie\varepsilon)(e_i\bowtie\varepsilon)^s=0,\ \ \textit{if}\ \ i\neq j. \tag{4.4}$$

for $1\leq i,j\leq m$.

Proof. Follows the fact that $A_q(\mathfrak{g})$ is a quasi-Hopf subalgebra of $D(A_q(\mathfrak{g}))$. □

As a subspace, we always have a natural embedding $M_q(\mathfrak{g})\hookrightarrow D(A_q(\mathfrak{g}))$ through $\varphi\mapsto 1\bowtie\varphi$ for $\varphi\in M_q(\mathfrak{g})$.

Lemma 4.4. *For $h\in A_q(\mathfrak{g})$ and $\varphi\in M_q(\mathfrak{g})$, we have*

$$(h\bowtie\varepsilon)(1\bowtie\varphi)=h\bowtie\varphi.$$

Proof. This is a special case of Remark 6.2 in [24]. The reader also can prove it by using Formula ($\star$) in Theorem 2.5. □

Following Lemma 4.4, we have no worry to write $h\varphi$ for $h\bowtie\varphi$ and just denote by h the element $h\bowtie\varepsilon$ for short. Recall that we have already defined the elements $\flat_i$ and H_i in (2.6). These elements will be used in the following propositions.

Proposition 4.5. *Assume $n \geq 4$. Then we have the following relations in $D(A_q(\mathfrak{g}))$:*

$$(1 \bowtie \chi_i)(1 \bowtie \chi_j) = (1 \bowtie \chi_j)(1 \bowtie \chi_i), \quad (\flat_i \Gamma^i)^{l_i} = 0, \tag{4.5}$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} (\flat_i \Gamma^i)^r (\flat_j \Gamma^j)(\flat_i \Gamma^i)^s = 0, \tag{4.6}$$

for $1 \leq i \neq j \leq m$ and $l_i = \mathrm{ord}(q^{c_{ii}})$.

Proof. For the first part of (4.5), we have:

$$\begin{aligned}
(1 \bowtie \chi_i)(1 \bowtie \chi_j) &= \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \chi_j \leftharpoonup \omega^{(1)})(\omega^{(4)} \rightharpoonup \chi_i \leftharpoonup \omega^{(2)}) \\
&= \sum_{a=\epsilon_j, b=\epsilon_i, c\in(\mathbb{Z}_n)^m, d=(n-1)\epsilon_i, e=(n-1)\epsilon_j} \prod_{s\neq j,t} \mathbb{q}^{c_{st}0} \\
&\quad \prod_{s=j, t\neq i} \mathbb{q}^{c_{jt}0} \mathbb{q}^{c_{ji}(-[\frac{n+c_i}{n}]-(n-1))} \mathbf{1}_c \bowtie (\chi_i \cdot \chi_j) \\
&= 1 \bowtie (\chi_i \cdot \chi_j).
\end{aligned}$$

Similarly, one can show that $(1 \bowtie \chi_j)(1 \bowtie \chi_i) = 1 \bowtie (\chi_j \cdot \chi_i)$ and so $(1 \bowtie \chi_i)(1 \bowtie \chi_j) = (1 \bowtie \chi_j)(1 \bowtie \chi_i)$. Applying the proof of Formula (3.8) in [19, Prop. 3.4], we get $(1 \bowtie \Gamma^i)^{l_i} = 0$. By (4.8) of the next Proposition 4.6, we have:

$$\flat_i \Gamma^i \mathbf{1}_a = \mathbf{1}_{a-\epsilon_i} \flat_i \Gamma^i \quad (\text{and so,} \ \ \Gamma^i \mathbf{1}_a = \mathbf{1}_{a-\epsilon_i} \Gamma^i).$$

Therefore, $(\flat_i \Gamma^i)^{l_i} = 0$.

Finally, we show the Serre relation.

$$\begin{aligned}
(\flat_i \Gamma^i)(\flat_j \Gamma^j) &= \flat_i \sum_{a\in(\mathbb{Z}_n)^m} \prod_{l=1}^{m} q^{-c_{jl}a_l} \mathbf{1}_{a-\epsilon_i} (1 \bowtie \Gamma^i)(1 \bowtie \Gamma^j) \\
&= \flat_i \sum_{a\in(\mathbb{Z}_n)^m} \prod_{l\neq i} q^{-c_{jl}a_l} q^{-c_{ji}(1+a_i)'} \mathbf{1}_a \\
&\quad \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \Gamma^j \leftharpoonup \omega^{(1)})(\omega^{(4)} \rightharpoonup \Gamma^i \leftharpoonup \omega^{(2)}) \\
&= \flat_i \sum_{a\in(\mathbb{Z}_n)^m} \prod_{l\neq i} q^{-c_{jl}a_l} q^{-c_{ji}(1+a_i)'} \mathbf{1}_a \\
&\quad \sum_{a=\epsilon_j, b=\epsilon_i, c\in(\mathbb{Z}_n)^m, d=e=0\epsilon_i} \mathbb{q}^{-c_{ji}[\frac{1+c_i}{n}]} \mathbf{1}_c \bowtie (\Gamma^j \cdot \Gamma^i) \\
&= \flat_i \sum_{a\in(\mathbb{Z}_n)^m} \prod_{l\neq i} q^{-c_{jl}a_l} q^{-c_{ji}(1+a_i)} \mathbf{1}_a \bowtie (\Gamma^j \cdot \Gamma^i) \\
&= q^{-c_{ji}} \flat_i \flat_j (\Gamma^j \cdot \Gamma^i).
\end{aligned}$$

Similarly, we have $(\flat_j\Gamma^j)(\flat_i\Gamma^i) = q^{-c_{ij}}\flat_i\flat_j(\Gamma^i\cdot\Gamma^j)$, and

$$\begin{aligned}(\flat_i\Gamma^i)^2(\flat_j\Gamma^j) &= \flat_i \sum_{a\in(\mathbb{Z}_n)^m}\prod_{l\neq i} q^{-c_{il}a_l}q^{-c_{ii}(1+a_i)'}\mathbf{1}_a \sum_{b\in(\mathbb{Z}_n)^m}\prod_{l\neq i} q^{-c_{jl}b_l}q^{-c_{ji}(2+b_i)'}\mathbf{1}_b \\ &\quad (1\bowtie\Gamma^i)^2(1\bowtie\Gamma^j)\\ &= \flat_i \sum_{a\in(\mathbb{Z}_n)^m}\prod_{l\neq i} q^{-c_{il}a_l}q^{-c_{ii}(1+a_i)'}\mathbf{1}_a \sum_{b\in(\mathbb{Z}_n)^m}\prod_{l\neq i} q^{-c_{jl}b_l}q^{-c_{ji}(2+b_i)'}\mathbf{1}_b\\ &\quad \sum_{c\in(\mathbb{Z}_n)^m} \mathfrak{q}^{-c_{ii}[\frac{1+c_i}{n}]-c_{ji}[\frac{2+c_i}{n}]}\mathbf{1}_c\bowtie\Gamma^j\cdot(\Gamma^i\cdot\Gamma^i)\\ &= q^{-2c_{ji}-c_{ii}}\flat_i^2\flat_j\Gamma^j\cdot(\Gamma^i\cdot\Gamma^i).\end{aligned}$$

In a similar way, we obtain the following identities:

$$(\flat_i\Gamma^i)(\flat_j\Gamma^j)(\flat_i\Gamma^i) = q^{-c_{ii}-c_{ij}-c_{ji}}\flat_i^2\flat_j\Gamma^i\cdot\Gamma^j\cdot\Gamma^i,$$

$$(\flat_i\Gamma^j)(\flat_i\Gamma^i)^2 = q^{-c_{ii}-2c_{ij}}\flat_i^2\flat_j\Gamma^i\cdot\Gamma^i\cdot\Gamma^j.$$

In general, we have the following identity:

$$(\flat_i\Gamma^i)^r(\flat_j\Gamma^j)(\flat_i\Gamma^i)^s = q^{-(r+s)c_{ji}-\frac{(r+s)(r+s-1)}{2}c_{ii}}\flat_i^{r+s}\flat_j(\Gamma^i)^s\cdot\Gamma^j\cdot(\Gamma^i)^r$$

for any two natural numbers r, s satisfying $r+s = 1-a_{ij}$. Therefore, by Proposition 3.4, we have:

$$\begin{aligned}&\sum_{r+s=1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{d_i}((\flat_i\Gamma^i)^r(\flat_j\Gamma^j)(\flat_i\Gamma^i)^s\\ &= q^{-(r+s)c_{ji}-\frac{(r+s)(r+s-1)}{2}c_{ii}}\flat_i^{r+s}\flat_j\sum_{r+s=1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{d_i}(\Gamma^i)^s\cdot\Gamma^j\cdot(\Gamma^i)^r\\ &= 0.\end{aligned}$$

□

Proposition 4.6. *In $D(A_q(\mathfrak{g}))$, we have the following relations:*

$$(1\bowtie\chi_i)h_j = h_j(1\bowtie\chi_i),\quad (\flat_i\chi_i)^n = H_i^{-2}, \tag{4.7}$$

$$h_i(\flat_j\Gamma^j)h_i^{-1} = \mathfrak{q}^{-\delta_{ij}}\flat_j\Gamma^j, \tag{4.8}$$

$$(\flat_i\chi_i)e_j(\flat_i\chi_i)^{-1} = \mathfrak{q}^{c_{ji}}q^{-2c_{ji}}e_j, \tag{4.9}$$

$$(\flat_i\chi_i)(\flat_j\Gamma^j)(\flat_i\chi_i)^{-1} = \mathfrak{q}^{-c_{ji}}q^{2c_{ji}}(\flat_j\Gamma^j), \tag{4.10}$$

$$(\flat_j\Gamma^j)e_i - q^{-c_{ji}}e_i(\flat_j\Gamma^j) = \delta_{ij}(1\bowtie\varepsilon - H_i^{-1}\flat_i\chi_i). \tag{4.11}$$

Proof. By Formula (2.23), we have $(1 \bowtie \chi_i)h_j = h_j \bowtie (h_j^{-1} \rightharpoonup \chi_i \leftharpoonup h_j) = h_j \bowtie \chi_i = h_j(1 \bowtie \chi_i)$. To show the second equation in (4.7), we use the formula ($\star$) in Theorem 2.5. We obtain:

$$\begin{aligned}
&(1 \bowtie \chi_i)(1 \bowtie \chi_i)\\
&= \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \chi_i \leftharpoonup \omega^{(1)})(\omega^{(4)} \rightharpoonup \chi_i \leftharpoonup \omega^{(2)})\\
&= \sum_{a=b=\epsilon_i, c\in(\mathbb{Z}_n)^m, d=e=(n-1)\epsilon_i} \prod_{k,j\neq i} \mathfrak{q}^{c_{kj}0} \prod_{k\neq i, j=i} \mathfrak{q}^{c_{ki}c_k[\frac{(n-1)'+(n-1)'}{n}]}\\
&\qquad \mathfrak{q}^{c_{ii}(-1+c_i[\frac{(n-1)'+(n-1)'}{n}]-(n-1))}\mathbf{1}_c \bowtie \chi_i^2\\
&= \sum_{c\in(\mathbb{Z}_n)^m} \prod_{k=1}^{m} \mathfrak{q}^{c_{ki}c_k}\mathbf{1}_c \bowtie \chi_i^2\\
&= H_i\chi_i^2.
\end{aligned}$$

Here χ_i^2 is the product $\chi_i \cdot \chi_i$ in $M_q(\mathfrak{g})$. Inductively, we have $(1 \bowtie \chi_i)^k = H_i^{k-1}\chi_i^k$ for $1 \le k \le n$. In particular, $(1 \bowtie \chi_i)^n = H_i^{-1}\chi_i^n = H_i^{-1}$. By the first part of (4.7), we obtain:

$$(\flat_i\chi_i)^n = \flat_i^n\chi_i^n = H_i^{-1}H_i^{-1} = H_i^{-2}.$$

For (4.8), it is enough to show that $h_i\Gamma^j h_i^{-1} = \mathfrak{q}^{-\delta_{ij}}(1 \bowtie \Gamma^j)$ because the elements h_i and $\flat_j$ are commutative. Indeed, using Formula (2.23) we obtain:

$$h_i\Gamma^j h_i^{-1} = h_i(h_i^{-1}\chi_j(h_i^{-1})\Gamma^j) = \mathfrak{q}^{-\delta_{ij}}(1 \bowtie \Gamma^j).$$

For (4.9), we have the following:

$$\begin{aligned}
&(\flat_i\chi_i)e_j(\flat_i\chi_i)^{-1}\\
&= \flat_i(e_j)_{(1)(2)}\omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \varepsilon \leftharpoonup \omega^{(1)})(\omega^{(4)}S((e_j)_{(2)}) \rightharpoonup \chi_i \leftharpoonup (e_j)_{(1)(1)}\omega^{(2)})(\flat_i\chi_i)^{-1}\\
&= \flat_i(\sum_{k=1}^{n-1} \mathbf{1}_k^j e_j \bowtie q^{c_{ji}(n-1)}\chi_i + \mathbf{1}_0^j e_j \bowtie \mathfrak{q}^{-c_{ij}}q^{c_{ji}(n-1)}\chi_i)(\flat_i\chi_i)^{-1}\\
&= [q^{c_{ji}(n-1)} \sum_{a\in(\mathbb{Z}_n)^m, a_j\neq 0} \prod_{l=1}^{m} q^{-c_{il}a_l}\mathbf{1}_a e_j \bowtie \chi_i +\\
&\qquad q^{-c_{ji}} \sum_{a\in(\mathbb{Z}_n)^m, a_j=0} \prod_{l=1}^{m} q^{-c_{il}a_l}\mathbf{1}_a e_j \bowtie \chi_i](\flat_i\chi_i)^{-1}.
\end{aligned}$$

Now we need the following identities:

$$\begin{aligned}
&q^{c_{ji}(n-1)} \sum_{a\in(\mathbb{Z}_n)^m, a_j\neq 0} \prod_{l=1}^{m} q^{-c_{il}a_l}\mathbf{1}_a e_j\\
&= q^{c_{ji}(n-1)} \sum_{a\in(\mathbb{Z}_n)^m, a_j\neq 0} \prod_{l=1}^{m} q^{-c_{il}a_l} e_j\mathbf{1}_{a-\epsilon_j}\\
&= q^{c_{ji}(n-1)}q^{-c_{ij}}e_j \sum_{a\in(\mathbb{Z}_n)^m, a_j\neq n-1} \prod_{l=1}^{m} q^{-c_{il}a_l}\mathbf{1}_a,
\end{aligned}$$

and:

$$\begin{aligned}
&q^{-c_{ji}}\sum_{a\in(\mathbb{Z}_n)^m,a_j=0}\prod_{l=1}^{m}q^{-c_{il}a_l}\mathbf{1}_a e_j\\
&=q^{-c_{ji}}\sum_{a\in(\mathbb{Z}_n)^m,a_j=0}\prod_{l=1}^{m}q^{-c_{il}a_l}e_j\mathbf{1}_{a-\epsilon_j}\\
&=q^{-c_{ji}}q^{c_{ij}(n-1)}e_j\sum_{a\in(\mathbb{Z}_n)^m,a_j=n-1}\prod_{l=1}^{m}q^{-c_{il}a_l}\mathbf{1}_a.
\end{aligned}$$

By applying the above identities to the expression of $(\flat_i\chi_i)e_j(\flat_i\chi_i)^{-1}$, we obtain:

$$\begin{aligned}
&(\flat_i\chi_i)e_j(\flat_i\chi_i)^{-1}\\
&=q^{-c_{ji}}q^{c_{ij}(n-1)}e_j(\flat_i\bowtie\chi_i)(\flat_i\chi_i)^{-1}\\
&=\mathfrak{q}^{c_{ji}}q^{-2c_{ji}}e_j.
\end{aligned}$$

To show (4.10), we only need to verify that $(\flat_i\chi_i)(1\bowtie\Gamma^j)(\flat_i\chi_i)^{-1}=\mathfrak{q}^{-c_{ji}}q^{2c_{ji}}(1\bowtie\Gamma^j)$ since the elements χ_i and $\flat_j$ are commutative. Note that $(\flat_i\chi_i)^{-1}=H_i^{-1}\flat_i^{-1}\chi_i^{n-1}$. Thus, we have:

$$\begin{aligned}
&(\flat_i\chi_i)(1\bowtie\Gamma^j)(\flat_i\chi_i)^{-1}\\
&=\flat_i[\omega^{(3)}\bowtie(\omega^{(5)}\rightharpoonup\Gamma^j\leftharpoonup\omega^{(1)})(\omega^{(4)}\rightharpoonup\chi_i\leftharpoonup\omega^{(2)})](\flat_i\chi_i)^{-1}\\
&=(\flat_i\mathfrak{q}^{-c_{ji}}\bowtie(\Gamma^j\cdot\chi_i))H_i^{-1}\flat_i^{-1}\chi_i^{n-1}\\
&=\mathfrak{q}^{-c_{ji}}\flat_i[(1\bowtie\Gamma^j\cdot\chi_i)H_i^{-1}\flat_i^{-1}\chi_i^{n-1}],
\end{aligned}$$

and:

$$\begin{aligned}
&(1\bowtie\Gamma^j\cdot\chi_i)(H_i^{-1}\flat_i^{-1}\bowtie\chi_i^{n-1})\\
&=\sum_{a,b,c,d,e\in(\mathbb{Z}_n)^m}\omega_{a,b,c,d,e}\prod_{l=1}^{m}q^{c_{il}(b_l+c_l+d_l)'}H_i^{-1}\mathbf{1}_c\bowtie(S(\mathbf{1}_e)\rightharpoonup\chi_i^{n-1}\leftharpoonup\mathbf{1}_a)\\
&\qquad(S(H_i^{-1}\mathbf{1}_d)\rightharpoonup\Gamma^j\cdot\chi_i\leftharpoonup H_i^{-1}\mathbf{1}_b)\\
&=\sum_{c\in(\mathbb{Z}_n)^m,a=d=(n-1)\epsilon_i,b=\epsilon_i+\epsilon_j,e=\epsilon_i}\prod_{s\neq i\neq k}\mathfrak{q}^{c_{sk}0}\prod_{s=i\neq k}\mathfrak{q}^{c_{ik}(-(n-1)[\frac{b_k+c_k}{n}])}\\
&\prod_{s\neq i=k}\mathfrak{q}^{c_{si}(b_s+c_s)}\mathfrak{q}^{c_{ii}(1+c_i-1)}\prod_{l\neq j}q^{c_{il}c_l}q^{c_{ij}(1+c_j)'}\mathfrak{q}^{-c_{ji}}H_i^{-1}\mathbf{1}_c\bowtie\chi_i^{-1}\cdot(\Gamma^j\cdot\chi_i)\\
&=\sum_{c\in(\mathbb{Z}_n)^m}\mathfrak{q}^{c_{ij}[\frac{1+c_j}{n}]}\prod_{l\neq j}q^{c_{il}c_l}q^{c_{ij}(1+c_j)'}\prod_{s=1}^{m}\mathfrak{q}^{a_{si}c_s}H_i^{-1}\mathbf{1}_c\bowtie\chi_i^{-1}\cdot(\Gamma^j\cdot\chi_i)\\
&=q^{c_{ij}}\flat_i^{-1}\bowtie\chi_i^{-1}\cdot(\Gamma^j\cdot\chi_i)\\
&=q^{c_{ij}+c_{ji}}\flat_i^{-1}\bowtie\Gamma^j,
\end{aligned}$$

where $\omega_{a,b,c,d,e}$ denotes the coefficient of $\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c\otimes S(\mathbf{1}_d)\otimes S(\mathbf{1}_e)$ in ω. For the third equality we used $\prod_{s=1}^m\mathfrak{q}^{a_{si}c_s}H_i^{-1}\mathbf{1}_c=\mathbf{1}_c$ and Lemma 4.1. Therefore, $(\flat_i\chi_i)(1\bowtie\Gamma^j)(\flat_i\chi_i)^{-1}=\mathfrak{q}^{-c_{ji}}q^{2c_{ji}}(1\bowtie\Gamma^j)$.

We arrive now at the proof of the last equality, (4.11). Using the comultiplication formula for e_i given in Lemma 2.3, we have:

$$\begin{aligned}(\Delta\otimes id)\Delta(e_i) &= e_i\otimes \flat_i^{-1}\otimes \flat_i^{-1} + 1\otimes\sum_{k=1}^{n-1}\mathbf{1}_k^i e_i\otimes \flat_i^{-1} + H_i^{-1}\otimes \mathbf{1}_0^i e_i\otimes \flat_i^{-1}\\ &\quad +1\otimes 1\otimes \sum_{k=1}^{n-1}\mathbf{1}_k^i e_i + H_i^{-1}\otimes H_i^{-1}\otimes \mathbf{1}_0^i e_i.\end{aligned}$$

Substituting the above comultiplication of e_i in the following equation:

$$(1\bowtie\Gamma^j)(e_i\bowtie\varepsilon) = (e_i)_{(1)(2)}\bowtie S((e_i)_{(2)})\rightharpoonup\Gamma^j\leftharpoonup (e_i)_{(1)(1)},$$

we obtain:

$$(1\bowtie\Gamma^j)(e_i\bowtie\varepsilon) = \sum_{k=1}^{n-1}\mathbf{1}_k^i e_i\bowtie\Gamma^j + \mathfrak{q}^{-c_{ji}}\mathbf{1}_0^i e_i\bowtie\Gamma^j,\ \text{for } i\neq j.$$

Now multiplying both sides of the above identity with the element $\flat_j$, we obtain:

$$\begin{aligned}(\flat_j\Gamma^j)e_i &= e_i\sum_{a\in(Z)_n^m, a_i\neq 0}\prod_{l=1}^m q^{-c_{jl}a_l}\mathbf{1}_{a-\epsilon_i}\Gamma^j + \mathfrak{q}^{-c_{ji}}e_i\sum_{a\in(Z)_n^m, a_i=0}\prod_{l=1}^m q^{-c_{jl}a_l}\mathbf{1}_{a-\epsilon_i}\Gamma^j\\ &= q^{-c_{ji}}e_i\sum_{a\in(Z)_n^m, a_i\neq n-1}\prod_{l=1}^m q^{-c_{jl}a_l}\mathbf{1}_a\Gamma^j + q^{-c_{ji}}e_i\sum_{a\in(Z)_n^m, a_i=n-1}\prod_{l=1}^m q^{-c_{jl}a_l}\mathbf{1}_a\Gamma^j\\ &= q^{-c_{ji}}e_i(\flat_j\Gamma^j).\end{aligned}$$

If $i=j$, then we have:

$$(1\bowtie\Gamma^i)(e_i\bowtie\varepsilon) = \flat_i^{-1} + \sum_{k=1}^{n-1}\mathbf{1}_k^i e_i\bowtie\Gamma^i + \mathfrak{q}^{-c_{ii}}\mathbf{1}_0^i e_i\bowtie\Gamma^i - H_i^{-1}\bowtie\chi_i.$$

Similarly, by multiplying both sides with the element $\flat_i$, we obtain:

$$(\flat_i\Gamma^i)e_i = 1\bowtie\varepsilon + q^{-c_{ii}}e_i(\flat_i\Gamma^i) - H_i^{-1}(\flat_i\chi_i).$$

□

The next step is to determine the coalgebraic structure of $D(A_q(\mathfrak{g}))$. We divide it in the following two propositions.

Proposition 4.7. *In $D(A_q(\mathfrak{g}))$, we have:*

$$\Delta(h_i) = h_i\otimes h_i,\ \ \Delta(e_i) = e_i\otimes\flat_i^{-1} + 1\otimes\sum_{j=1}^{n-1}\mathbf{1}_j^i e_i + H_i^{-1}\otimes\mathbf{1}_0^i e_i, \tag{4.12}$$

$$\varepsilon(h_i) = 1,\qquad \varepsilon(e_i) = 0, \tag{4.13}$$

for $1\leq i\leq m$.

Proof. Due to the fact that $A_q(\mathfrak{g})$ is a quasi-Hopf subalgebra of $D(A_q(\mathfrak{g}))$. □

Proposition 4.8. *In $D(A_q(\mathfrak{g}))$, we have:*

$$\Delta(\flat_i\chi_i)=\flat_i\chi_i\otimes\flat_i\chi_i, \tag{4.14}$$

$$\Delta(\flat_i\Gamma^i)=\flat_i\Gamma^i\otimes\flat_i+H_i^{-1}(\flat_i\chi_i)\otimes(\flat_i\Gamma^i)\sum_{j=1}^{n-1}\mathbf{1}_j^i+(\flat_i\chi_i)\otimes(\flat_i\Gamma^i)\mathbf{1}_0^i, \tag{4.15}$$

$$\varepsilon(\flat_i\chi_i)=1,\qquad \varepsilon(\flat_i\Gamma^i)=0, \tag{4.16}$$

for $1\leq i\leq m$.

Proof. Recall that, for any $\psi\in M_q(\mathfrak{g})$, we defined $\mathbf{T}(\psi)=\phi_{(2)}^{(1)}\bowtie S(\phi^{(2)})\alpha\phi^{(3)}\rightharpoonup\psi\leftharpoonup\phi_{(1)}^{(1)}$ (see Equation (2.19) before Theorem 2.5). Thus:

$$\begin{aligned}
\mathbf{T}(\chi_i) &= \phi_{(2)}^{(1)}\bowtie S(\phi^{(2)})\alpha\phi^{(3)}\rightharpoonup\chi_i\leftharpoonup\phi_{(1)}^{(1)}\\
&=\sum_{a^1,a^2,b,c\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^{m}\mathbb{q}^{-c_{st}(a_s^1+a_s^2)[\frac{b_t+c_t}{n}]}\mathbf{1}_{a^2}\bowtie S(\mathbf{1}_b)\alpha\mathbf{1}_c\rightharpoonup\chi_i\leftharpoonup\mathbf{1}_{a^1}\\
&=\sum_{a^1,a^2,b,c\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^{m}\mathbb{q}^{-c_{st}(a_s^1+a_s^2)[\frac{b_t+c_t}{n}]}\mathbb{q}^{c_{st}c_s[\frac{n-1+c_t}{n}]}\mathbf{1}_{a^2}\bowtie S(\mathbf{1}_b)\mathbf{1}_c\rightharpoonup\chi_i\leftharpoonup\mathbf{1}_{a^1}\\
&=\sum_{a^1=c=\epsilon_i,b=(n-1)\epsilon_i,a^2\in(\mathbb{Z}_n)^m}\prod_{s=1}^{m}\mathbb{q}^{-c_{si}a_s^2}\mathbb{q}^{-c_{ii}}\mathbb{q}^{c_{ii}}\mathbf{1}_{a^2}\bowtie\chi_i\\
&=\sum_{a\in(\mathbb{Z}_n)^m}\prod_{s=1}^{m}\mathbb{q}^{-c_{si}a_s}\mathbf{1}_a\bowtie\chi_i\\
&= H_i^{-1}\chi_i.
\end{aligned}$$

Applying formula ($\star\star$) in Theorem 2.5, we have:

$$\begin{aligned}
&\Delta(\mathbf{T}(\chi_i))\\
&=\tilde{\phi}^{(2)}\mathbf{T}(\chi_i\leftharpoonup\tilde{\phi}^{(1)})\phi^{(-1)}\phi^{(1)}\otimes\tilde{\phi}^{(3)}\phi^{(-3)}\mathbf{T}(\phi^{(3)}\rightharpoonup\chi_i\leftharpoonup\phi^{(-2)})\phi^{(2)}\\
&=\sum_{a^1,a^2,a^3,b^1,b^2,b^3,c^1,c^2,c^3\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^{m}\mathbb{q}^{-c_{st}a_s^1[\frac{a_t^2+a_t^3}{n}]+c_{st}b_s^1[\frac{b_t^2+b_t^3}{n}]-c_{st}c_s^1[\frac{c_t^2+c_t^3}{n}]}\\
&\qquad\mathbf{1}_{a^2}\mathbf{T}(\chi_i\leftharpoonup\mathbf{1}_{a^1})\mathbf{1}_{b^1}\mathbf{1}_{c^1}\otimes\mathbf{1}_{a^3}\mathbf{1}_{b^3}\mathbf{T}(\mathbf{1}_{c^3}\rightharpoonup\chi_i\leftharpoonup\mathbf{1}_{b^2})\mathbf{1}_{c^2}\\
&=\sum_{a^1,a^2,a^3,b^2,c^3\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^{m}\mathbb{q}^{-c_{st}a_s^1[\frac{a_t^2+a_t^3}{n}]+c_{st}a_s^2[\frac{b_t^2+a_t^3}{n}]-c_{st}a_s^2[\frac{a_t^3+c_t^3}{n}]}\\
&\qquad\mathbf{T}(\chi_i\leftharpoonup\mathbf{1}_{a^1})\mathbf{1}_{a^2}\otimes\mathbf{T}(\mathbf{1}_{c^3}\rightharpoonup\chi_i\leftharpoonup\mathbf{1}_{b^2})\mathbf{1}_{a^3}\\
&=\sum_{a^2,a^3\in(\mathbb{Z}_n)^m}\prod_{t=1}^{m}\mathbb{q}^{-c_{it}[\frac{a_t^2+a_t^3}{n}]}\prod_{s=1}^{m}\mathbb{q}^{c_{si}a_s^2[\frac{1+a_i^3}{n}]}\prod_{s=1}^{m}\mathbb{q}^{-c_{si}a_s^2[\frac{a_i^3+1}{n}]}\\
&\qquad\mathbf{T}(\chi_i)\mathbf{1}_{a^2}\otimes\mathbf{T}(\chi_i)\mathbf{1}_{a^3}\\
&=\sum_{a^2,a^3\in(\mathbb{Z}_n)^m}\prod_{t=1}^{m}\mathbb{q}^{-c_{it}[\frac{a_t^2+a_t^3}{n}]}\mathbf{T}(\chi_i)\mathbf{1}_{a^2}\otimes\mathbf{T}(\chi_i)\mathbf{1}_{a^3}.
\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}\Delta(\flat_i \mathbf{T}(\chi_i)) &= \Delta(\flat_i)\Delta(\mathbf{T}(\chi_i))\\ &= (\sum_{b,c\in(\mathbb{Z}_n)^m}\prod_{t=1}^m q^{-c_{it}(b_t+c_t)'}\mathbf{1}_b\otimes\mathbf{1}_c)\\ &\quad (\sum_{b,c\in(\mathbb{Z}_n)^m}\prod_{t=1}^m \mathfrak{q}^{-c_{it}[\frac{b_t+c_t}{n}]}\mathbf{T}(\chi_i)\mathbf{1}_b\otimes\mathbf{T}(\chi_i)\mathbf{1}_c)\\ &= \sum_{b,c\in(\mathbb{Z}_n)^m}\prod_{t=1}^m q^{-c_{it}(b_t+c_t)}\mathbf{1}_b\mathbf{T}(\chi_i)\otimes\mathbf{1}_c\mathbf{T}(\chi_i)\\ &= \flat_i\mathbf{T}(\chi_i)\otimes\flat_i\mathbf{T}(\chi_i).\end{aligned}$$

This means that $\flat_i\mathbf{T}(\chi_i)$ is a group-like element. Since $\mathbf{T}(\chi_i)=H_i^{-1}\chi_i$ and H_i^{-1} is a group-like element, $\flat_i\chi_i$ is group-like too. Thus the proof of (4.14) is done.

Using the same method, one can show that:

$$\mathbf{T}(\Gamma^i) = 1 \bowtie \Gamma^i$$

and:

$$\begin{aligned}\Delta(\mathbf{T}(\Gamma^i)) &= \sum_{a^2,a^3\in(\mathbb{Z}_n)^m}\prod_{t=1}^m \mathfrak{q}^{-c_{it}[\frac{a_t^2+a_t^3}{n}]}\mathbf{1}_{a^2}\mathbf{T}(\Gamma^i)\otimes\mathbf{1}_{a^3}\\ &\quad+\sum_{a^2,a^3\in(\mathbb{Z}_n)^m}\prod_{t=1}^m \mathfrak{q}^{-c_{it}[\frac{a_t^2+a_t^3}{n}]}\prod_{s=1}^m \mathfrak{q}^{c_{si}a_s^2[\frac{1+a_i^3}{n}]}\mathbf{1}_{a^2}\mathbf{T}(\chi_i)\otimes\mathbf{1}_{a^3}\mathbf{T}(\Gamma^i).\end{aligned}$$

Thus:

$$\begin{aligned}\Delta(\flat_i\mathbf{T}(\Gamma^i)) &= \Delta(\flat_i)\Delta(\mathbf{T}(\Gamma^i))\\ &= \sum_{a^2,a^3\in(\mathbb{Z}_n)^m}\prod_{t=1}^m q^{-c_{it}(a_t^2+a_t^3)}\mathbf{1}_{a^2}\mathbf{T}(\Gamma^i)\otimes\mathbf{1}_{a^3}\\ &\quad+H_i^{-1}\sum_{a^2\in(\mathbb{Z}_n)^m}\prod_{t=1}^m \mathfrak{q}^{-c_{it}a_t^2}\mathbf{1}_{a^2}\chi_i\otimes\sum_{a^3;a_i^3\neq n-1}\prod_{t=1}^m \mathfrak{q}^{-c_{it}a_t^3}\mathbf{1}_{a^3}\mathbf{T}(\Gamma^i)\\ &\quad+\sum_{a^2\in(\mathbb{Z}_n)^m}\prod_{t=1}^m \mathfrak{q}^{-c_{it}a_t^2}\mathbf{1}_{a^2}\chi_i\otimes\sum_{a^3;a_i^3=n-1}\prod_{t=1}^m \mathfrak{q}^{-c_{it}a_t^3}\mathbf{1}_{a^3}\mathbf{T}(\Gamma^i)\\ &= \flat_i\Gamma^i\otimes\flat_i+H_i^{-1}(\flat_i\chi_i)\otimes(\flat_i\Gamma^i)\sum_{j=1}^{n-1}\mathbf{1}_j^i+(\flat_i\chi_i)\otimes(\flat_i\Gamma^i)\mathbf{1}_0^i.\end{aligned}$$

Now (4.16) is clear. □

Finally, we determine the reassociator ϕ, the elements α,β and the antipode S for $D(A_q(\mathfrak{g}))$.

Proposition 4.9. *In $D(A_q(\mathfrak{g}))$, the reassociator is given by:*

$$\phi = \sum_{a,b,c\in(\mathbb{Z}_n)^m} (\prod_{i,j=1}^{m} \mathfrak{q}^{-c_{ij}a_i[\frac{b_j+c_j}{n}]})\mathbf{1}_a \otimes \mathbf{1}_b \otimes \mathbf{1}_c. \tag{4.17}$$

The elements α, β can be chosen as:

$$\alpha = \sum_{a\in(\mathbb{Z}_n)^m} \prod_{s,t=1}^{m} \mathfrak{q}^{c_{st}a_s[\frac{n-1+a_t}{n}]}\mathbf{1}_a, \quad \beta = 1. \tag{4.18}$$

The antipode S is determined by:

$$S(h_i) = h_i^{-1}, \quad S(\flat_i\chi_i) = (\flat_i\chi_i)^{-1}, \tag{4.19}$$

$$S(e_i) = -(\alpha \sum_{j=1}^{n-1} \mathbf{1}_j^i e_i + H_i\alpha\mathbf{1}_0^i e_i)\flat_i\alpha^{-1}, \tag{4.20}$$

$$S(\flat_i\Gamma^i) = -(H_i(\flat_i\chi_i)^{-1}\alpha(\flat_i\Gamma^i)\sum_{j=1}^{n-1}\mathbf{1}_j^i + (\flat_i\chi_i)^{-1}\alpha(\flat_i\Gamma^i)\mathbf{1}_0^i)\flat_i^{-1}\alpha^{-1}, \tag{4.21}$$

for $1 \le i \le m$.

Proof. By Theorem 2.5 (2) and Lemma 2.3, the reassociator ϕ is given by (4.17), and α, β can be chosen as in (4.18). Since the elements h_i and $\flat_i\chi_i$ are group-like, (4.19) is obvious. Both (4.20) and (4.21) follow directly from the definition of the antipode and the comultiplication formulas for e_i and $\flat_i\Gamma^i$. □

5. Presentation of quasi-Frobenius-Lusztig kernels

In this section, we present $D(A_q(\mathfrak{g}))$ in terms of generators and relations. Let $\mathfrak{g}$ be a simple Lie algebra of finite type, $A = (a_{ij})_{m\times m}$ its Cartan matrix and $C = (d_ia_{ij}) = (c_{ij})$ the symmetrized Cartan matrix. Let n be a natural number ≥ 4, and q an n^2-th primitive root of unity, $\mathfrak{q} = q^n$ and $l_i = \mathrm{ord}(q^{c_{ii}})$.

Definition 5.1. *The quasi-Frobenius-Lusztig kernel $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is a quasi-Hopf algebra defined as follows. As an associative algebra, it is generated by $E_i, F_i, K_i, \hat{K}_i$ $(1 \le i \le m)$ satisfying:*

$$K_iK_j = K_jK_i, \ \ \hat{K}_i\hat{K}_j = \hat{K}_j\hat{K}_i, \ \ K_i\hat{K}_j = \hat{K}_jK_i, \tag{5.1}$$

$$K_i^n = 1, \ \ \hat{K}_i^n = \prod_{l=1}^{m} K_l^{-2c_{il}}, \tag{5.2}$$

$$K_iE_j = \mathfrak{q}^{\delta_{ij}}E_jK_i, \ \ K_iF_j = \mathfrak{q}^{-\delta_{ij}}F_jK_i, \tag{5.3}$$

$$\hat{K}_iE_j = \mathfrak{q}^{c_{ij}}q^{-2c_{ij}}E_j\hat{K}_i, \ \ \hat{K}_iF_j = \mathfrak{q}^{-c_{ij}}q^{2c_{ij}}F_j\hat{K}_i, \tag{5.4}$$

$$F_jE_i - q^{-c_{ij}}E_iF_j = \delta_{ij}(1 - \prod_{l=1}^{m} K_l^{-c_{il}}\hat{K}_i), \tag{5.5}$$

$$E_i^{l_i} = F_i^{l_i} = 0, \tag{5.6}$$

$$(5.7)\qquad \begin{cases} \sum_{r+s=1-a_{ij}}(-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r E_j E_i^s = 0 & i \neq j \\ \sum_{r+s=1-a_{ij}}(-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r F_j F_i^s = 0 & i \neq j. \end{cases}$$

for $1 \leq i, j \leq m$.

Let $\{\mathbf{1}_a | a = (a_1, \ldots, a_m) \in (\mathbb{Z}_n)^m\}$ be the set of primitive idempotents of the group algebra of $\langle K_i | 1 \leq i \leq m \rangle \cong (\mathbb{Z}_n)^m$, $\mathbf{1}_k^i := \frac{1}{n}\sum_{j=0}^{n-1}(\mathfrak{q}^{n-k})^j K_i^j$, $\flat_i := \sum_{a\in(\mathbb{Z}_n)^m}\prod_{j=1}^m q^{-c_{ij}a_j}\mathbf{1}_a$, $H_i := \prod_{j=1}^m K_j^{c_{ji}}$.

The reassociator ϕ*, the comultiplication* Δ*, the counit* ε*, the elements* α, β *and the antipode* S *are given by*

$$(5.8)\qquad \phi = \sum_{a,b,c\in(\mathbb{Z}_n)^m} \left(\prod_{i,j=1}^m \mathfrak{q}^{-c_{ij}a_i[\frac{b_j+c_j}{n}]}\right)\mathbf{1}_a \otimes \mathbf{1}_b \otimes \mathbf{1}_c,$$

$$(5.9)\qquad \Delta(K_i) = K_i \otimes K_i, \quad \Delta(\hat{K}_i) = \hat{K}_i \otimes \hat{K}_i,$$

$$(5.10)\qquad \Delta(E_i) = E_i \otimes \flat_i^{-1} + 1 \otimes \sum_{j=1}^{n-1}\mathbf{1}_j^i E_i + H_i^{-1} \otimes \mathbf{1}_0^i E_i,$$

$$(5.11)\qquad \Delta(F_i) = F_i \otimes \flat_i + H_i^{-1}\hat{K}_i \otimes F_i \sum_{j=1}^{n-1}\mathbf{1}_j^i + \hat{K}_i \otimes F_i \mathbf{1}_0^i,$$

$$(5.12)\qquad \varepsilon(K_i) = \varepsilon(\hat{K}_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$(5.13)\qquad \alpha = \sum_{a\in(\mathbb{Z}_n)^m}\prod_{s,t=1}^m \mathfrak{q}^{c_{st}a_s[\frac{n-1+a_t}{n}]}\mathbf{1}_a, \quad \beta = 1$$

$$(5.14)\qquad S(K_i) = K_i^{-1}, \quad S(\hat{K}_i) = \hat{K}_i^{-1},$$

$$(5.15)\qquad S(E_i) = -(\alpha\sum_{j=1}^{n-1}\mathbf{1}_j^i E_i + H_i\alpha\mathbf{1}_0^i E_i)\flat_i\alpha^{-1},$$

$$(5.16)\qquad S(F_i) = -(H_i\hat{K}_i^{-1}\alpha F_i\sum_{j=1}^{n-1}\mathbf{1}_j^i + \hat{K}_i^{-1}\alpha F_i\mathbf{1}_0^i)\flat_i^{-1}\alpha^{-1}.$$

for $1 \leq i \leq m$.

Lemma 5.2. $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ *is finite dimensional and* $\dim(\mathrm{Q}\mathbf{u}_q(\mathfrak{g})) = (\dim(A_q(\mathfrak{g})))^2$.

Proof. We give a rough proof of this statement. At first, by the relations (5.3)-(5.5), $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ has an triangle decomposition:

$$\mathrm{Q}\mathbf{u}_q(\mathfrak{g}) = \mathbf{u}^+\mathbf{u}^0\mathbf{u}^-$$

where $\mathbf{u}^+$ (resp. $\mathbf{u}^-$) is the subalgebra generated by E_i (resp. F_i) for $1 \leq i \leq m$, and $\mathbf{u}^0$ is the subalgebra generated by $K_i, \hat{K}_i$ for $1 \leq i \leq m$. It is not hard to see that $\dim(\mathbf{u}^0) = n^{2m}$

and $\dim(\mathbf{u}^+)n^m = \dim(\mathbf{u}^-)n^m = \dim(A_q(\mathfrak{g}))$. Therefore,

$$\dim(\mathrm{Q}\mathbf{u}_q(\mathfrak{g})) = (\dim(A_q(\mathfrak{g})))^2.$$

□

The following theorem shows that $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is a quasi-Hopf algebra though one can also verify that (5.1)-(5.16) define a quasi-Hopf algebra.

Theorem 5.3. *As quasi-Hopf algebras, $D(A_q(\mathfrak{g})) \cong \mathrm{Q}\mathbf{u}_q(\mathfrak{g})$.*

Proof. Define a map

$$\Upsilon: \ \mathrm{Q}\mathbf{u}_q(\mathfrak{g}) \longrightarrow D(A_q(\mathfrak{g})), \qquad K_i \mapsto h_i, \ \ \hat{K}_i \mapsto \flat_i \chi_i, \\ E_i \mapsto e_i, \ \ F_i \mapsto \flat_i \Gamma^i.$$

By Propositions 4.3, 4.5 and 4.6, Υ is an algebra morphism. By Propositions 4.7 and 4.8, Υ preserves the comultiplication. Thanks to Theorem 2.5 (1), Υ is surjective, and hence bijective as the dimensions of the two algebras are equal (see Lemma 5.2). □

6. Twist equivalence

In this section, we determine when the quasi-Hopf algebra $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is not twisted equivalent to a Hopf algebra.

Definition 6.1. (1) *We call a quasi-Hopf algebra H* twist equivalent *to another quasi-Hopf algebra K if there is a twist J of H such that $K \cong H_J$ as quasi-bialgebras.*

(2) *A quasi-Hopf algebra H is said to be* genuine *if H is not twist equivalent to any ordinary Hopf algebra.*

We give various sufficient conditions for $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ to be genuine.

Theorem 6.2. *Assume $\mathfrak{g}$ is of type A_m for $m \geq 2$.*

(1) *If $(m+1)|n$, then $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is a genuine quasi-Hopf algebra.*

(2) *If m is odd and $4|n$, then $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is a genuine quasi-Hopf algebra.*

Proof. (1) Let $d = \frac{n}{m+1}$ and $\zeta_{m+1} = \mathrm{q}^d$. Let $G := \langle K_i | 1 \leq i \leq m \rangle$ be the subgroup generated by K_i's in $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$. Consider the following 1-dimensional representation of G:

$$\rho: \ G \longrightarrow \Bbbk, \ \ K_i \mapsto \zeta_{m+1}^i.$$

We show that ρ can be extended to a 1-dimensional representation of $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$, still denoted by ρ. Indeed, we may define:

$$\rho: \ \mathrm{Q}\mathbf{u}_q(\mathfrak{g}) \longrightarrow \Bbbk, \ \ K_i \mapsto \zeta_{m+1}^i, \ \ \hat{K}_i \mapsto 1, \ \ E_i \mapsto 0, \ \ F_i \mapsto 0,$$

for $1 \leq i \leq m$. We need to show that ρ is a well-defined algebra morphism. By our choice, we have $\rho(H_i) = \rho(\prod_{j=1}^{m}(K_j)^{c_{ji}}) = 1$. Therefore, the relations (5.2) and (5.5) are preserved by ρ. The other relations can be checked easily. Thus ρ is well-defined.

Now let X be this 1-dimensional $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module and $\langle X\rangle$ be the tensor subcategory generated by X. Define:

$$X^{\vec{\otimes} l} =: \overbrace{(\cdots(X\otimes X)\otimes X)\cdots)}^{l}.$$

Then the objects of $\langle X\rangle$ are direct sums of elements in $\{X^{\vec{\otimes} l}|0 \leq l < m+1\}$. Now assume that $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is twist equivalent to a Hopf algebra. By the general principle of Tannaka-Krein duality (see, e.g., [3]), there is a fiber functor from the category Rep-$\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ to the category of $\Bbbk$-spaces. Thus its restriction to $\langle X\rangle$ is still a fiber functor. This implies that the restriction of ϕ to $\langle X\rangle$ should come from a 3-coboundary of $(\mathbb{Z}_{m+1})^m$. In fact, by the definition of ρ, $\mathbf{1}_a X \neq 0$ if and only if $kd|a_k$ for $1 \leq k \leq m$, and hence

$$\phi|_{\langle X\rangle} = \sum_{a,b,c\in(\mathbb{Z}_{m+1})^m}\prod_{s,t=1}^{m}\zeta_{m+1}^{-c_{st}sa_s[\frac{tb_t+tc_t}{m+1}]}\mathbf{1}_a\otimes\mathbf{1}_b\otimes\mathbf{1}_c.$$

Here $\mathbf{1}_x$, $x = a, b, c$, denotes a primitive element in $\Bbbk((\mathbb{Z}_{m+1})^m)$. This corresponds to a 3-cocycle Φ over $((\mathbb{Z}_{m+1})^m)^\wedge$, the character group of $(\mathbb{Z}_{m+1})^m$. By definition, $\Phi(\chi_a,\chi_b,\chi_c) = \prod_{s,t=1}^{m}\zeta_{m+1}^{-c_{st}sa_s[\frac{tb_t+tc_t}{m+1}]}$ where $\chi_x, x = a, b, c$, is the dual element of $\mathbf{1}_x$.

Now we show that Φ is not a coboundary and thus we get a contradiction. By Corollary 2.11, it is enough to compute $F_3^*(\Phi)$. We use the same notations as in Subsection 1.4. We have the following:

$$\begin{aligned} f_{1,1,1} &= F_3^*(\Phi)(\Psi_{1,1,1})\\ &= \prod_{l=0}^{m}\Phi(\chi_{\epsilon_1},\chi_{l\epsilon_1},\chi_{\epsilon_1})\\ &= \zeta_{m+1}^{-c_{11}} \neq 1. \end{aligned}$$

By Corollary 2.11 and Lemma 2.9, Φ is not a coboundary.

(2) Consider the following 1-dimensional representation of G:

$$\rho:\ G \longrightarrow \Bbbk,\ \begin{cases} K_i \mapsto 1, & \text{if } i \text{ is even}\\ K_i \mapsto \mathrm{q}^{\frac{n}{4}}, & \text{if } i \text{ is odd.}\end{cases} \tag{6.1}$$

It is not hard to see that $\rho(H_i) = \pm 1$, and that ρ can be also extended to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module by setting:

$$\rho(E_i) = \rho(F_i) = 0,\ \ \rho(\hat{K}_i) = \rho(H_i) \tag{6.2}$$

for $1 \leq i \leq m$. Thus using the same argument developed in the proof of (1), we get the desired result. □

Theorem 6.3. *The following hold.*

(1) *Assume* $\mathfrak{g}$ *is of type* B_m. *If either* $2|m$ *and* $4|n$ *or* $2\nmid m$ *and* $8|n$, *then* $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ *is genuine;*

(2) *Assume* $\mathfrak{g}$ *is of type* C_m, *or* D_m, *or* E_7. *If* $4|n$, *then* $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ *is genuine;*

(3) *Assume* $\mathfrak{g}$ *is of type* E_6. *If* $3|n$, *then* $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ *is genuine.*

Proof. The proof is almost the same as the proof of Theorem 6.2. Therefore, we only provide the construction of the 1-dimensional modules.

The principle for the construction of such a 1-dimensional module is: For a 1-dimensional $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module, the actions of E_i and F_i must be trivial since they are nilpotent. Thus Relation (5.5) implies that the action of $H_i^{-1}\hat{K}_i$ is trivial as well. Applying Relation (5.2), the action of H_i^{-2} is also trivial and hence the action of H_i must be ± 1. So a necessary condition for a 1-dimensional $\Bbbk G$-module to be extended to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module is: the action of H_i is ± 1. Conversely, given a 1-dimensional $\Bbbk G$-module (V,ρ) satisfying $\rho(H_i)=\pm 1$, it can be extended to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module if we set $\rho(E_i)=\rho(F_i)=0$ and $\rho(\hat{K}_i)=\rho(H_i)$.

Type B_m: If m is even and $4|n$, the 1-dim representation ρ in (6.1) extends to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module as in the proof of Theorem 6.2.

If m is odd and $8|n$, define:

$$\rho:\ G\longrightarrow \Bbbk,\ \begin{cases} K_i\mapsto 1, & \text{if } i \text{ is even}\\ K_i\mapsto \mathbb{q}^{\frac{n}{4}}, & \text{if } i \text{ is odd and } i\neq m,\\ K_m\mapsto \mathbb{q}^{\frac{n}{8}}. \end{cases}$$

Then ρ extends to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module by adding (6.2).

Type C_m: If $4|n$, define:

$$\rho:\ G\longrightarrow \Bbbk,\ \begin{cases} K_i\mapsto 1, & \text{for } i<m,\\ K_m\mapsto \mathbb{q}^{\frac{n}{4}}. \end{cases}$$

Then ρ extends to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module by adding (6.2).

Type D_m: If $4|n$, define:

$$\rho:\ G\longrightarrow \Bbbk,\ \begin{cases} K_i\mapsto 1, & \text{if } i<m-1,\\ K_i\mapsto \mathbb{q}^{\frac{n}{4}}, & \text{if } i=m-1,m. \end{cases}$$

Then ρ extends to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module by adding (6.2).

Type E_6: If $3|n$, define:

$$\rho:\ G\longrightarrow \Bbbk,\ \begin{cases} K_i\mapsto 1, & \text{if } i=2,4,\\ K_i\mapsto \mathbb{q}^{\frac{n}{3}}, & \text{if } i=1,5,\\ K_i\mapsto \mathbb{q}^{\frac{2n}{3}}, & \text{if } i=3,6. \end{cases}$$

Then ρ extends to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module by adding (6.2).

Type E_7: If $4|n$, define:

$$\rho: \ G \longrightarrow \Bbbk, \ \begin{cases} K_i \mapsto 1, & \text{if } i = 1,3,4,6, \\ K_i \mapsto \mathfrak{q}^{\frac{n}{4}}, & \text{if } i = 2,5,7. \end{cases}$$

Then ρ extends to a $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$-module by adding (6.2).

□

Remark 6.4. (1) Except type G_2, Etingof and Gelaki proved that $D(A_q(\mathfrak{g}))$ is always twist equivalent to a Hopf algebra provided n is odd and $(n, |(a_{ij})|) = 1$ (for type G_2, they need one more condition, that is, $3 \nmid n$), where $|(a_{ij})|$ is the determinant of the Cartan matrix. It is well-known that the determinant of the Cartan matrix of type A_m (resp. E_6) is $m+1$ (resp. 3). Therefore, our results imply that the condition " n is odd and $(n, |(a_{ij})|) = 1$" can not be removed in general. One could even ask whether such a condition is a necessary condition. But it is not. For example, let $n = 2$. One can use Corollary 2.11 and Lemma 2.9 to show that ϕ is already a coboundary in $A_q(\mathfrak{g})$ and thus $A_q(\mathfrak{g})$ is twist equivalent to a Hopf algebra. Therefore, $D(A_q(\mathfrak{g}))$ is twist equivalent to a Hopf algebra too.

(2) Our methods cannot be applied to Lie algebras of type E_8, F_4 and G_2. We do not know whether there is an n such that $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ is genuine when $\mathfrak{g}$ is one of these types.

Problem 6.5. (1) For type E_8, is $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ twist equivalent to a Hopf algebra? For type F_4, is $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ genuine when $4|n$? For type G_2, is $\mathrm{Q}\mathbf{u}_q(\mathfrak{g})$ genuine when $6|n$?

(2) Give a complete list of genuine quasi-Frobenius-Lusztig kernels.

(3) How can one determine whether a given finite dimensional quasi-Hopf algebra H over $\Bbbk$ is twist equivalent to some Hopf algebra or not?

Acknowledgments

The first author would like thank Department of Mathematics, University of Antwerp for its hospitality. This work is supported both by the NSF of China (No. 11371186) and by a FWO grant.

References

[1] N. Andruskiewitsch, H. -J. Schneider, On the classification of finite dimensional pointed Hopf algebras, *Ann. of Math.* (2) 171 (2010), no. 1, 375-417.

[2] I. E. Angiono, Basic quasi-Hopf algebras over cyclic groups. *Adv. Math.* 225 (2010), no. 6, 3545-3575.

[3] D. Calaque, P. Etingof, Lectures on tensor categories. Quantum groups, 1-38, IRMA *Lect. Math. Theor. Phys.,* 12, Eur. Math. Soc., Zurich, 2008.

[4] C. Cibils, M. Rosso, Hopf quivers. *J. Algebra* 254 (2002), no. 2, 241-251.

[5] R. Dijkgraaf, V. Pasquier and P. Roche, Quasi Hopf algebras, group cohomology and orbifold models. *Nuclear Phys. B Proc. Suppl.* 18B (1990), 60-72.

[6] V.G. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, Vol.1,2 (Berkeley, Calif., 1986), 798-820, Amer. Math. Soc., Providence, RI, 1987.

[7] V.G. Drinfeld, Hopf algebras and the quantum Yang-Baxter equation. (Russian) *Dokl. Akad. Nauk* SSSR 283 (1985), no. 5, 1060-1064.

[8] V.G. Drinfeld, Quasi-Hopf algebras, *Leningrad Math. J.* 1 (1990) 1419-1457.

[9] A.M. Gainutdinov, A.M. Semikhatov, T.Y. Tipunin, B.L. Feigin, The Kazhdan-Lusztig correspondence for the representation category of the triplet W -algebra in logorithmic conformal field theories. (Russian) *Teoret. Mat. Fiz.* 148 (2006), no. 3, 398-427; translation in *Theoret. and Math. Phys.* 148 (2006), no. 3, 1210-1235.

[10] S. Gelaki, Basic quasi-Hopf algebras of dimension n^3. *J. Pure Appl. Algebra* 198 (2005), no. 1-3, 165-174.

[11] P. Etingof, S. Gelaki, The small quantum group as a quantum double. *J. Algebra* 322 (2009), no. 7, 2580-2585.

[12] P. Etingof, S. Gelaki, On radically graded finite dimensional quasi-Hopf algebras. *Mosc. Math. J.* 5 (2005), no. 2, 371-378, 494.

[13] F. Hausser, F. Nill, Doubles of quasi-quantum groups. *Comm. Math. Phys.* 199 (1999), no. 3, 547-589.

[14] H-L. Huang, From projective representations to quasi-quantum groups. *Sci. China Math.* 55 (2012), no. 10, 2067-2080.

[15] H-L. Huang, Quiver approaches to quasi-Hopf algebras, *J. Math. Phys.* 50(4) (2009) 043501-1-043501-9.

[16] H-L. Huang, G. Liu, Y. Ye, Quivers, quasi-quantum groups and finite tensor categories. *Comm. Math. Phys.* 303 (2011), no. 3, 595-612.

[17] M. Jimbo, A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. *Lett. Math. Phys.* 10 (1985), no. 1, 63-69.

[18] C. Kassel, Quantum Groups, Graduate Texts in Math. 155, Springer-Verlag, New York, 1995.

[19] G. Liu, Quasi-Hopf analogue of $\mathbf{u}_q(\mathfrak{sl}_2)$, *Math. Res. Lett.* 21 (2014), no. 3, 585-603..

[20] G. Lusztig, Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra, *J. Amer. Math. Soc.* 3 (1990), no. 1, 257-296.

[21] G. Lusztig, Quantum groups at roots of 1. *Geom. Dedicata* 35 (1990), no. 1-3, 89-113.

[22] S. Majid, Quantum double for quasi-Hopf algebras. *Lett. Math. Phys.* 45 (1998), no. 1, 1-9.

[23] S. Majid, Quasi-quantum groups as internal symmetries of topological quantum field theories. *Lett. Math. Phys.* 22 (1991), no. 2, 83-90.

[24] P. Schauenburg, Hopf modules and the double of a quasi-Hopf algebra. *Trans. Amer. Math. Soc.* 354 (2002), no. 8, 3349-3378.

[25] Charles A. Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

Department of Mathematics, Nanjing University, Nanjing 210093, China

E-mail address: gxliu@nju.edu.cn

Department of Mathematics and Computer Science, University of Antwerp, Antwerp, Belgium

E-mail address: fred.vanoystaeyen@ua.ac.be

Department of Mathematics and Statistics, University of Hasselt, 3590 Diepenbeek, Belgium

E-mail address: yinhuo.zhang@uhasselt.be