Representations of the small quasi-quantum group Quq(sl2)

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REPRESENTATIONS OF THE SMALL QUASI-QUANTUM GROUP $\mathrm{Qu}_q(\mathfrak{sl}_2)$

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ABSTRACT. The quasi-Frobenius-Lusztig kernel $\mathrm{Qu}_q(\mathfrak{sl}_2)$ associated with $\mathfrak{sl}_2$ has been constructed in [9]. In this paper we study the representations of this small quasi-quantum group. We give a complete list of non-isomorphic indecomposables and the tensor product decomposition rules for simples and projectives. A description of the Grothendieck ring is provided.

1. INTRODUCTION

In [9] the first author introduced a quasi-Hopf version of the small quantum group $\mathbf{u}_q(\mathfrak{sl}_2)$, denoted $\mathrm{Qu}_q(\mathfrak{sl}_2)$, where q is an n^2-th primitive root of unity for some natural number n. It is proved in loc.cit. (see also Lemma 2.1 here) that for odd n, $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is twisted equivalent to the Hopf algebra $\mathbf{u}_q(\mathfrak{sl}_2)$, but for even n it is not! So for even n the $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is a new quasi-Hopf algebra. The purpose of this paper is to study the representations of this new algebra. We restrict, for convenience, to the case $4|n$ while the general case $2|n$ will be remarked.

The representation theory of the small quantum group $\mathbf{u}_q(\mathfrak{sl}_2)$ and the restricted quantum universal enveloping algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ associated to $\mathfrak{sl}_2$ have been studied extensively, cf. [10, 11, 12, 8]. So for us the problem to study the finite dimensional representations of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ arises naturally. We will provide in this paper a complete list of non-isomorphic indecomposable modules. Here a new phenomenon appears: $\mathrm{Qu}_q(\mathfrak{sl}_2)$ has no Steinberg modules (i.e. simple projective modules). Moreover, the dimensions of all simple modules are *odd*. Furthermore, $\mathrm{Qu}_q(\mathfrak{sl}_2)$ has some interesting "symmetry" properties: all the blocks have the same dimension and they are Morita equivalent to one-another. Another new phenomenon of interest appears here: the basic algebra of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ can be equipped with a Hopf algebra structure, whereas we can prove that this does not happen for $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$.

To understand $\mathrm{Qu}_q(\mathfrak{sl}_2)$ further, it is helpful to study the decomposition rules of tensor products of modules, i.e., a version of the Clebsch-Gordan formula for $\mathrm{Qu}_q(\mathfrak{sl}_2)$. After proving that the direct summands of the tensor products of two simple modules are either simple or projective, the decomposition rules for the tensor products of simples and projectives, as well as simples and simples, are given explicitly. From this, the structure of its Grothendieck ring is derived.

The paper is organized as follows. In Section 2, we describe all simple modules and projective modules of $\mathrm{Qu}_q(\mathfrak{sl}_2)$, and consequently the basic algebra of each block may be given by using quiver and relations. In Proposition 2.15, we establish a Hopf algebra structure on the basic algebra of $\mathrm{Qu}_q(\mathfrak{sl}_2)$, and prove in Proposition 2.16 that this is not the case for $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$.

Section 3 is devoted to finding a complete list of non-isomorphic indecomposable modules (Theorem 3.1). Finally in Section 4, we provide the decomposition rules of tensor products (Theorem 4.8) and determine the Grothendieck ring K_0 of $\mathrm{Q}\mathbf{u}_q(\mathfrak{sl}_2)$ (Theorem 4.9). Thus we obtained a fairly complete description of the representation theory of $\mathrm{Q}\mathbf{u}_q(\mathfrak{sl}_2)$, highlighting new interesting phenomena when compared to the existing theory for $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$.

Throughout this paper, we work over a fixed algebraically closed field k of characteristic zero.

2. Simples, projectives and basic algebra

We recall the definition of $\mathrm{Q}\mathbf{u}_q(\mathfrak{sl}_2)$ from [9]. Let n be a natural number and q an n^2-th primitive root of unity. Let $\mathfrak{q} := q^n$. The quasi-Hopf algebra $\mathrm{Q}\mathbf{u}_q(\mathfrak{sl}_2)$ is defined as follows. As an associative algebra, it is generated by four elements $\kappa, \hat{\kappa}, E, F$ satisfying

$$(2.1) \qquad \kappa^n = 1, \quad \hat{\kappa}^n = \kappa^{-2}, \quad \kappa\hat{\kappa} = \hat{\kappa}\kappa,$$

$$(2.2) \qquad \kappa E \kappa^{-1} = \mathfrak{q}E, \quad \kappa F \kappa^{-1} = \mathfrak{q}^{-1}F,$$

$$(2.3) \qquad \hat{\kappa}E\hat{\kappa}^{-1} = \mathfrak{q}q^{-2}E, \quad \hat{\kappa}F\hat{\kappa}^{-1} = \mathfrak{q}^{-1}q^2 F,$$

$$(2.4) \qquad E^{n^2} = F^{n^2} = 0,$$

$$(2.5) \qquad FE - q^{-1}EF = 1 - \kappa^{-1}\hat{\kappa}.$$

Define

$$(2.6) \qquad 1_i := \frac{1}{n}\sum_{j=0}^{n-1}(\mathfrak{q}^{n-i})^j \kappa^j, \quad \flat = \sum_{i=0}^{n-1} q^{-i} 1_i.$$

The reassociator ϕ_s, the comultiplication Δ, the counit ε, the elements α, β and the antipode S are given as follows:

$$(2.7) \qquad \phi_s = \sum_{i,j,k=0}^{n-1} \mathfrak{q}^{-i[\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k,$$

$$(2.8) \qquad \Delta(\kappa) = \kappa \otimes \kappa, \quad \Delta(\hat{\kappa}) = \hat{\kappa} \otimes \hat{\kappa},$$

$$(2.9) \qquad \Delta(E) = E \otimes \flat^{-1} + \kappa^{-1} \otimes 1_0 E + 1 \otimes \sum_{i=1}^{n-1} 1_i E,$$

$$(2.10) \qquad \Delta(F) = F \otimes \flat + \kappa^{-1}\hat{\kappa} \otimes F \sum_{i=1}^{n-1} 1_i + \hat{\kappa} \otimes F 1_0,$$

$$(2.11) \qquad \varepsilon(\kappa) = \varepsilon(\hat{\kappa}) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0,$$

$$(2.12) \qquad \alpha = \kappa, \quad \beta = 1$$

$$(2.13) \qquad S(\kappa) = \kappa^{-1}, \quad S(\hat{\kappa}) = \hat{\kappa}^{-1},$$

$$(2.14) \qquad S(x) = -(\kappa \sum_{i=1}^{n-1} 1_i E + \kappa^2 1_0 E)\flat\kappa^{-1},$$

$$(2.15) \qquad S(F) = -(\kappa^2\hat{\kappa}^{-1}F \sum_{i=1}^{n-1} 1_i + \kappa\hat{\kappa}^{-1}F 1_0)\flat^{-1}\kappa^{-1}.$$

Combining [5, Thm. 4.3] and [9, Thm. 4.1], we have the following.

Lemma 2.1. *(1) If n is odd, then $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is twist equivalent to the Hopf algebra $\mathbf{u}_q(\mathfrak{sl}_2)$.*

(2) If n is even, then $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is not twist equivalent to any Hopf algebra.

By Lemma 2.1, we only need to consider the case where n is even. For convenience, we assume $4|n$ throughout the paper. The results for $2|n$ case will be stated as remarks.

Let $\mathbf{u}^+, \mathbf{u}^-$ and $\mathbf{u}^0$ be the subalgebras of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ generated by E, F and $\{\kappa, \hat{\kappa}\}$ respectively. Then $\mathrm{Qu}_q(\mathfrak{sl}_2)$ has a triangle decomposition

$$\mathrm{Qu}_q(\mathfrak{sl}_2) = \mathbf{u}^-\mathbf{u}^0\mathbf{u}^+.$$

Let $m = \frac{n}{2}$. Define

$$\varphi_{2i} := \frac{1}{mn}\sum_{k=0}^{mn-1} q^{-2ik}(\kappa^{-1}\hat{\kappa})^k, \quad \mathbb{1}_0 := \frac{1}{2}(1 + \kappa\hat{\kappa}^{\frac{n}{2}}), \quad \mathbb{1}_1 := \frac{1}{2}(1 - \kappa\hat{\kappa}^{\frac{n}{2}}).$$

Let $e_{2i,0} = \varphi_{2i}\mathbb{1}_0$ and $e_{2i,1} = \varphi_{2i}\mathbb{1}_1$.

Lemma 2.2. *The set $\{e_{2i,0}, e_{2i,1}|1 \le i \le \frac{n^2}{2}\}$ is a complete set of primitive idempotents of $\mathbf{u}^0$.*

Proof. Let V be the space spanned by $e_{2i,0}, e_{2i,1}$ for $1 \le i \le \frac{n^2}{2}$. It is enough to show that κ and $\hat{\kappa}$ both belong to V. By the definition of $e_{2i,0}, e_{2i,1}$, we know $\kappa^{-1}\hat{\kappa}, \kappa\hat{\kappa}^{\frac{n}{2}} \in V$ and thus $\hat{\kappa}^{\frac{n}{2}+1} \in V$. Clearly, the order of $\hat{\kappa}$ is $\frac{n^2}{2}$. To show $\hat{\kappa} \in V$, it suffices to show that $\frac{n}{2}+1$ is coprime to $\frac{n^2}{2}$. From the identity: $(\frac{n}{2}-1)(\frac{n}{2}+1) = \frac{n^2}{4} - 1$, we know that $l.c.m(\frac{n}{2}+1, \frac{n^2}{2}) = 1$ or 2. If $l.c.m(\frac{n}{2}+1, \frac{n^2}{2}) = 2$, we have $2|(\frac{n}{2}+1)$ which is absurd by our assumption that $4|n$. Therefore, $\hat{\kappa} \in V$ and hence $\kappa \in V$. $\quad\square$

Remark 2.3. If we only assume that $2|n$, we can not assure that Lemma 2.2 is always true. However, if we define $\varphi'_{2i} := \frac{1}{mn}\sum_{k=0}^{mn-1} q^{-2ik}\hat{\kappa}^k$ and $e'_{2i,0} := \varphi'_{2i}\mathbb{1}_0, e'_{2i,1} := \varphi'_{2i}\mathbb{1}_1$. Then one can show that the set $\{e'_{2i,0}, e'_{2i,1}|1 \le i \le \frac{n^2}{2}\}$ is always a complete set of primitive idempotents of $\mathbf{u}^0$.

Lemma 2.4. *The following identities hold in $\mathrm{Qu}_q(\mathfrak{sl}_2)$.*

$$\kappa^{-1}\hat{\kappa}E = q^{-2}E\kappa^{-1}\hat{\kappa}, \quad \kappa^{-1}\hat{\kappa}F = q^2 F\kappa^{-1}\hat{\kappa},$$

$$\kappa\hat{\kappa}^{\frac{n}{2}}E = -E\kappa\hat{\kappa}^{\frac{n}{2}}, \quad \kappa\hat{\kappa}^{\frac{n}{2}}F = -F\kappa\hat{\kappa}^{\frac{n}{2}},$$

$$\kappa^{-1}\hat{\kappa}e_{2i,0} = q^{2i}e_{2i,0}, \quad \kappa^{-1}\hat{\kappa}e_{2i,1} = q^{2i}e_{2i,1},$$

$$\kappa\hat{\kappa}^{\frac{n}{2}}e_{2i,0} = e_{2i,0}, \quad \kappa\hat{\kappa}^{\frac{n}{2}}e_{2i,1} = -e_{2i,1}.$$

Proof. Straightforward. $\quad\square$

For any natural number s, define $s_q := 1 + q + \cdots + q^{s-1}$.

Lemma 2.5. *The following identities hold in $\mathrm{Qu}_q(\mathfrak{sl}_2)$.*

$$(2.16) \qquad\qquad FE^s = q^{-s}E^s F + s_{q^{-1}}E^{s-1} - s_q\kappa^{-1}\hat{\kappa}E^{s-1},$$

$$(2.17) \qquad\qquad E^s F = q^s FE^s + qs_{q^{-1}}E^{s-1}\kappa^{-1}\hat{\kappa} - qs_q E^{s-1},$$

$$(2.18) \qquad\qquad EF^s = q^s F^s E + q^s s_q F^{s-1}\kappa^{-1}\hat{\kappa} - qs_q F^{s-1},$$

$$(2.19) \qquad\qquad F^s E = q^{-s}EF^s + s_{q^{-1}}F^{s-1} - s_q F^{s-1}\kappa^{-1}\hat{\kappa}.$$

Proof. We only prove the first one since the other proofs are similar. It is clear that the formula (2.16) is true for $s = 1$. Now assume that the formula (2.16) holds for s. We show that it holds for $s + 1$. Indeed, we have:

$$
\begin{aligned}
FE^{s+1} &= (q^{-s}E^s F + s_{q^{-1}}E^{s-1} - s_q\kappa^{-1}\hat{\kappa}E^{s-1})E \\
&= q^{-s}E^s(q^{-1}EF + (1 - \kappa^{-1}\hat{\kappa})) + s_{q^{-1}}E^s - s_q\kappa^{-1}\hat{\kappa}E^s \\
&= q^{-(s+1)}E^{s+1}F + q^{-s}E^s - q^{-s}q^{2s}\kappa^{-1}\hat{\kappa}E^s + s_{q^{-1}}E^s - s_q\kappa^{-1}\hat{\kappa}E^s \\
&= q^{-(s+1)}E^{s+1}F + (s+1)_{q^{-1}}E^s - (s+1)_q\kappa^{-1}\hat{\kappa}E^s.
\end{aligned}
$$

$\square$

For $1 \leq i \leq \frac{n^2}{2}$, define

$$(2.20) \qquad\qquad \alpha_{2i,0} := F^{n^2-1}e_{2i,0}, \quad \alpha_{2i,1} := F^{n^2-1}e_{2i,1}.$$

Corollary 2.6. *For* $1 \leq i \leq \frac{n^2}{2}$, $j = 0, 1$ *and* $s \leq n^2 - 1$, *we have:*

$$FE^s\alpha_{2i,j} = s_{q^{-1}}(1 - q^{2i-1-s})E^{s-1}\alpha_{2i,j}, \quad \kappa^{-1}\hat{\kappa}E^s\alpha_{2i,j} = q^{2i-2-2s}E^s\alpha_{2i,j},$$

$$\kappa\hat{\kappa}^{\frac{n}{2}}E^s\alpha_{2i,j} = (-1)^{s+1+j}E^s\alpha_{2i,j}.$$

Proof. These are direct consequences of Formula (2.16) and Lemma 2.4. $\square$

Thus, for any $1 \leq i \leq \frac{n^2}{2}$ and $j = 0, 1$, the n^2-dimensional left ideal $\mathbf{Qu}_q(\mathfrak{sl}_2)\alpha_{2i,j}$ may be represented schematically as:

where $\beta_{2i,j} = E^{2i-2}\alpha_{2i,j}, \widetilde{\alpha}_{2i,j} = E^{2i-1}\alpha_{2i,j}$ and $\widetilde{\beta}_{2i,j} = E^{n^2-1}\alpha_{2i,j}$. Each dot stands for a 1-dimensional subspace and an upward (resp. downward) arrow indicates a nonzero left-multiplication action by F (resp. E). We call $\mathbf{Qu}_q(\mathfrak{sl}_2)\alpha_{2i,j}$ a *Verma* module.

Contrasting to the classical $\mathbf{u}_q(\mathfrak{sl}_2)$ case (e.g. see page 362 in [11]), the single downward arrow *always* appears since $2i - 1$ is odd while n^2 is even. Therefore, we always have $F^{n^2-1}\beta_{2i,j} = 0$ for $1 \leq i \leq \frac{n^2}{2}$ and $j = 0, 1$, and

$$(2.21) \qquad\qquad \beta_{2i,j} = F\gamma_{2i,j},$$

for a unique vector $\gamma_{2i,j}$ of the form $\gamma_{2i,j} = \gamma_{2i,j}^{-}\gamma_{2i,j}^{0}\gamma_{2i,j}^{+}$ for $\gamma_{2i,j}^{\star} \in \mathbf{u}^{\star}$ with $\star = -, 0, +$. Observe that $\gamma_{2i,j}$ and $\widetilde{\alpha}_{2i,j}$ have the same left $\kappa^{-1}\hat{\kappa}$- and $\kappa\hat{\kappa}^{\frac{n}{2}}$-eigenvalues.

Lemma 2.7. *For $1 \leq i \leq \frac{n^2}{2}, j = 0, 1$ and $s \leq n^2 - 1$, we have*

$$FE^s\gamma_{2i,j} - q^{-s}E^s\beta_{2i,j} = s_{q^{-1}}(1 - q^{1-s-2i})E^{s-1}\gamma_{2i,j}.$$

Proof. By Lemma 2.5, we have:

$$
\begin{aligned}
FE^s\gamma_{2i,j} &= (q^{-s}E^sF + s_{q^{-1}}E^{s-1} - s_q\kappa^{-1}\hat{\kappa}E^{s-1})\gamma_{2i,j} \\
&= q^{-s}E^s\beta_{2i,j} + s_{q^{-1}}E^{s-1}\gamma_{2i,j} - q^{-(s-1)}s_{q^{-1}}E^{s-1}\kappa^{-1}\hat{\kappa}\gamma_{2i,j} \\
&= q^{-s}E^s\beta_{2i,j} + s_{q^{-1}}E^{s-1}\gamma_{2i,j} - q^{-(s-1)}s_{q^{-1}}q^{-2i}E^{s-1}\gamma_{2i,j} \\
&= q^{-s}E^s\beta_{2i,j} + s_{q^{-1}}(1 - q^{1-s-2i})E^{s-1}\gamma_{2i,j}.
\end{aligned}
$$

$\square$

Corollary 2.8. *For $1 \leq i \leq \frac{n^2}{2}$ and $j = 0, 1$, $F^{n^2-1}E^{n^2-1}\gamma_{2i,j}$ is a nonzero multiple of $\widetilde{\alpha}_{2i,j}$.*

Proof. Using Lemma 2.7 repeatedly, we have:

$$
\begin{aligned}
F^{n^2-1}E^{n^2-1}\gamma_{2i,j} &= \prod_{t=n^2-1}^{n^2-2i} t_{q^{-1}}(1 - q^{1-t-2i})F^{n^2-2i+1}E^{n^2-2i+1}\gamma_{2i,j} \\
&= q^{-(n^2-2i+1)}\prod_{t=n^2-1}^{n^2-2i} t_{q^{-1}}(1 - q^{1-t-2i})F^{n^2-2i}E^{n^2-2i+1}F\gamma_{2i,j} \\
&= q^{-(n^2-2i+1)}\prod_{t=n^2-1}^{n^2-2i} t_{q^{-1}}(1 - q^{1-t-2i})F^{n^2-2i}\widetilde{\beta}_{2i,j}.
\end{aligned}
$$

Since $F^{n^2-2i}\widetilde{\beta}_{2i,j}$ is clearly a nonzero multiple of $\widetilde{\alpha}_{2i,j}$, the lemma is proved. $\square$

Corollary 2.9. *For $1 \leq i \leq \frac{n^2}{2}$ and $j = 0, 1$, the vectors in $\{E^l\alpha_{2i,j}, E^l\gamma_{2i,j}|0 \leq l \leq n^2 - 1\}$ are linear independent.*

Proof. Corollary 2.8 entails that all vectors in $\{E^l\alpha_{2i,j}, E^l\gamma_{2i,j}|0 \leq l \leq n^2 - 1\}$ are nonzero. By definition, $\gamma_{2i,j}$ and $E^{2i-1}\alpha_{2i,j}$ are linear independent. From this, we deduce that all vectors in $\{E^s\alpha_{2i,j}, E^t\gamma_{2i,j}|0 \leq t \leq n^2 - 2i, \ 2i - 1 \leq s \leq n^2 - 1\}$ are linear independent which implies the desired result since the other vectors have different heights (see the beginning of Section 4 for the definition of height). $\square$

For $1 \leq i \leq \frac{n^2}{2}$ and $j = 0, 1$, the left ideal $Q\mathbf{u}_q(\mathfrak{sl}_2)\gamma_{2i,j}$ may be represented schematically as:
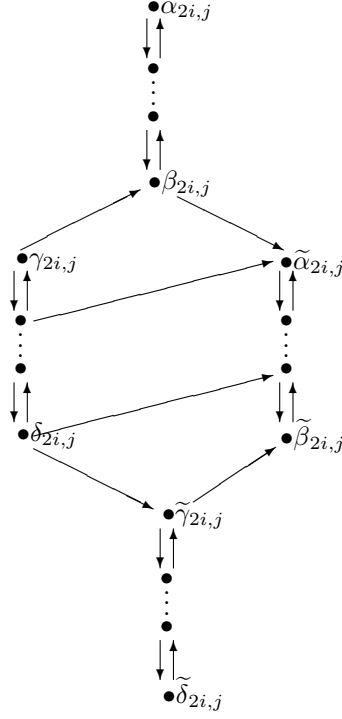
where $\delta_{2i,j} = E^{n^2-2i}\gamma_{2i,j}, \widetilde{\gamma}_{2i,j} = E^{n^2-2i+1}\gamma_{2i,j}$ and $\widetilde{\delta}_{2i,j} = E^{n^2-1}\gamma_{2i,j}$. Put

$$(2.22) \qquad\qquad\qquad \mathrm{P}_{2i,j} := \mathrm{Qu}_q(\mathfrak{sl}_2)\gamma_{2i,j},$$

and

$$(2.23) \qquad\qquad\qquad \mathrm{S}_{2i,j} := \mathrm{Soc}(\mathrm{P}_{2i,j}),$$

the socle of $\mathrm{P}_{2i,j}$. That is, $\mathrm{S}_{2i,j}$ can be represented as:

It is a simple $\mathrm{Qu}_q(\mathfrak{sl}_2)$-module isomorphic to the top of $\mathrm{P}_{2i,j}$.

Lemma 2.10. *For $1 \leq i, i' \leq \frac{n^2}{2}$ and $j, j' = 0, 1$, $\mathrm{S}_{2i,j} \cong \mathrm{S}_{2i',j'}$ if and only if $i = i'$ and $j = j'$.*

Proof. It it not hard to see that $\dim \mathrm{S}_{2i,j} = n^2 - 2i + 1$. If $\mathrm{S}_{2i,j} \cong \mathrm{S}_{2i',j'}$, then $\dim \mathrm{S}_{2i,j} = \dim \mathrm{S}_{2i',j'}$ and so $i = i'$. By comparing the $\kappa\hat{\kappa}^{\frac{n}{2}}$-eigenvectors, we have $j = j'$. $\qquad\square$

Lemma 2.11. *For $1 \leq i \leq \frac{n^2}{2}$ and $j = 0, 1$, we have in $\mathrm{Qu}_q(\mathfrak{sl}_2)$:*

$$E^{n^2-1}\alpha_{2i,j}E^{n^2-2i} \neq 0.$$

Proof. By using the fourth formula in Lemma 2.5 repeatedly, we have:

$$
\begin{aligned}
E^{n^2-1}\alpha_{2i,j}E^{n^2-2i} &= E^{n^2-1}F^{n^2-1}E^{n^2-2i}e_{\overline{n^2-2i},j} \\
&= E^{n^2-1}(q^{-(n^2-1)}EF^{n^2-1}+(n^2-1)_{q^{-1}}F^{n^2-2}-(n-1)_qF^{n^2-2}\kappa^{-1}\hat{\kappa}) \\
&\qquad E^{n^2-2i-1}e_{\overline{n^2-2i},j} \\
&= E^{n^2-1}((n^2-1)_{q^{-1}}-(n^2-1)_qq^{2i+2})F^{n^2-2}E^{n^2-2i-1}e_{\overline{n^2-2i},j} \\
&= (n^2-1)_{q^{-1}}(1-q^{2i})E^{n^2-1}F^{n^2-2}E^{n^2-2i-1}e_{\overline{n^2-2i},j} \\
&= \cdots \\
&= \prod_{t=1}^{n^2-2i}(n^2-t)_{q^{-1}}(1-q^{t-1+2i})E^{n^2-1}F^{2i-1}e_{\overline{n^2-2i},j},
\end{aligned}
$$

where $\bar{i}$ denotes the least positive residue of i modulo n^2. It is not hard to see that $E^{n^2-1}F^{2i-1}e_{\overline{n^2-2i},j} \neq 0$. $\qquad\square$

Corollary 2.12. *The right multiplication by E^h defines an isomorphism $\mathrm{P}_{2i,j} \xrightarrow{\cong} \mathrm{P}_{2i,j}\,E^h$ for $0 \le h \le n^2-2i$.*

Proof. It is enough to show that the right multiplication by E^{n^2-2i} is a monomorphism, but this is a direct consequence of Lemma 2.11. $\qquad\square$

Theorem 2.13. *As a left $\mathrm{Qu}_q(\mathfrak{sl}_2)$-module, we have:*

$$
\mathrm{Qu}_q(\mathfrak{sl}_2) = \bigoplus_{j=0}^{1}\bigoplus_{i=1}^{\frac{n^2}{2}}\bigoplus_{h=0}^{n^2-2i}\mathrm{P}_{2i,j}\,E^h.
$$

Proof. By counting the dimensions of both sides, we only need to show that the sum $\sum_{i,j,h}\mathrm{P}_{2i,j}\,E^h$ is a direct sum.

Claim. $\mathrm{P}_{2i,j}\,E^h \bigcap \mathrm{P}_{2i',j'}\,E^{h'} \neq 0$ if and only if $i=i'$, $j=j'$ and $h=h'$.

Proof of this claim. If $\mathrm{P}_{2i,j}\,E^h \bigcap \mathrm{P}_{2i',j'}\,E^{h'} \neq 0$, then they have the same socle. Since $\mathrm{P}_{2i,j} \cong \mathrm{P}_{2i,j}\,E^h$ and $\mathrm{P}_{2i',j'} \cong \mathrm{P}_{2i',j'}\,E^{h'}$, the socles of $\mathrm{P}_{2i,j}$ and $\mathrm{P}_{2i',j'}$ are isomorphic. Therefore, Lemma 2.10 implies $i=i'$ and $j=j'$. Consequently, $h=h'$.

Inductively, we assume that the sum of any n terms of $\mathrm{P} = \{\mathrm{P}_{2i,j}\,E^{h_i}|1 \le i \le \frac{n^2}{2}, 0 \le h_i \le n^2-2i, 0 \le j \le 1\}$ is a direct sum. We show the conclusion for $n+1$ terms. Take $M_1,\ldots,M_{n+1} \in \mathrm{P}$ and assume that $\sum_{l=1}^{n+1}M_l$ is not a direct sum. Then there is a l, say $n+1$, such that $M_{n+1}\bigcap\sum_{l=1}^{n}M_l = M_{n+1}\bigcap\bigoplus_{l=1}^{n}M_l \neq 0$. Therefore, $\mathrm{Soc}(M_{n+1}) = \mathrm{Soc}(M_l)$ for some $1 \le l \le n$ and thus $M_{n+1}\bigcap M_l \neq 0$, which is absurd by the Claim. $\qquad\square$

Now we have the following conclusions.

Corollary 2.14. (a) $\{\mathrm{P}_{2i,j}|1 \le i \le \frac{n^2}{2}, j=0,1\}$ *forms a complete set of non-isomorphic indecomposable projective $\mathrm{Qu}_q(\mathfrak{sl}_2)$-modules.*

(b) $\{\mathrm{S}_{2i,j}|1 \le i \le \frac{n^2}{2}, j=0,1\}$ *forms a complete set of non-isomorphic simple $\mathrm{Qu}_q(\mathfrak{sl}_2)$-modules.*

(c) $\mathrm{Qu}_q(\mathfrak{sl}_2)$ *has n^2 non-isomorphic indecomposable projective modules and every indecomposable projective module is has dimension $2n^2$.*

(d) $\mathrm{Qu}_q(\mathfrak{sl}_2)$ has n^2 non-isomorphic simple modules and every simple module is of odd dimension.

(e) $\mathrm{P}_{2i,j}$ and $\mathrm{P}_{2i',j'}$ belong to the same block if and only if $2i + 2i' = n^2 + 2$ and $j + j' = 1$.

(f) The number of blocks of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is $\frac{n^2}{2}$ and each block has dimension $2n^4$. Moreover, every block of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is Morita equivalent to the following basic algebra Λ:

$$
\cdot \underset{\overset{x_2}{\xleftarrow{\hspace{1cm}}}}{\overset{\overset{y_1}{\xrightarrow{\hspace{1cm}}}}{\underset{\underset{y_2}{\xleftarrow{\hspace{1cm}}}}{\overset{x_1}{\xrightarrow{\hspace{1cm}}}}}} \cdot \qquad \begin{array}{l} x_s x_t = y_s y_t \\ x_s y_t = y_s x_t = 0 \end{array} \qquad for\ 1 \le s \ne t \le 2.
$$

(g) $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is of tame representation type.

Proof. (a) is a direct consequence of Theorem 2.13. (a) implies (b) since every projective module is also injective (This follows from the fact that every finite-dimensional quasi-Hopf algebra is Frobenius). By Corollary 2.9, $\dim \mathrm{P}_{2i,j} = 2n^2$ and thus we obtain (c). Statement (d) is clear since $\dim \mathrm{S}_{2i,j} = n^2 - 2i + 1$.

Now let J be the Jacobson radical of $\mathrm{Qu}_q(\mathfrak{sl}_2)$. Then we have the following isomorphisms:

$$\mathrm{P}_{2i,0}/J\,\mathrm{P}_{2i,0} \cong \mathrm{S}_{2i,0}, \qquad \mathrm{P}_{2i,1}/J\,\mathrm{P}_{2i,1} \cong \mathrm{S}_{2i,1},$$

$$J\,\mathrm{P}_{2i,0}/J^2\,\mathrm{P}_{2i,0} \cong \mathrm{S}_{n^2-2i+2,1} \oplus \mathrm{S}_{n^2-2i+2,1}, \qquad J\,\mathrm{P}_{2i,1}/J^2\,\mathrm{P}_{2i,1} \cong \mathrm{S}_{n^2-2i+2,0} \oplus \mathrm{S}_{n^2-2i+2,0},$$

$$\mathrm{Soc}(\mathrm{P}_{2i,0}) = J^2\,\mathrm{P}_{2i,0} \cong \mathrm{S}_{2i,0}, \qquad \mathrm{Soc}(\mathrm{P}_{2i,1}) = J^2\,\mathrm{P}_{2i,1} \cong \mathrm{S}_{2i,1}.$$

These imply that $\mathrm{P}_{2i,0}$ and $\mathrm{P}_{n^2-2i+2,1}$ belong to the same block for $1 \le i \le \frac{n^2}{2}$. Therefore, Statement (e) follows.

(c)+(e) implies that the number of blocks of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is $\frac{n^2}{2}$. Denote by $\mathrm{B}_{2i,j}$ the block containing $\mathrm{P}_{2i,j}$. The representation theory of finite-dimensional algebras tells us that the dimension of $\mathrm{B}_{2i,0}$ is equal to

$$\dim \mathrm{S}_{2i,0} \dim \mathrm{P}_{2i,0} + \dim \mathrm{S}_{n^2-2i+2,1} \dim \mathrm{P}_{n^2-2i+2,1} = 2n^2[(n^2 - 2i + 1) + (2i - 1)] = 2n^4.$$

Moreover, the basic algebra of $\mathrm{B}_{2i,0}$ is isomorphic to the opposite algebra of

$$\mathrm{End}_{\mathrm{Qu}_q(\mathfrak{sl}_2)}(\mathrm{P}_{2i,0} \oplus \mathrm{P}_{n^2-2i+1,1}).$$

Parallel to [11, Sec. 5], we can easily show that $\mathrm{End}_{\mathrm{Qu}_q(\mathfrak{sl}_2)}(\mathrm{P}_{2i,0} \oplus \mathrm{P}_{n^2-2i+1,1}) \cong \Lambda$. But the opposite algebra of Λ is isomorphic to itself. Hence, (f) is proved.

Note that the basic algebra Λ was studied extensively, see for example [4, 7, 11, 12]. It is known that Λ is a tame algebra and thus we obtain the last statement (g). $\qquad \square$

Let ζ_l be an l-th primitive root of unity and m a positive integer satisfying $(m, l) = 1$. Denote by $\mathbf{h}(\zeta_l, m)$ the algebra $k\langle y, x, g\rangle/(x^l, y^l, g^l - 1, gx - \zeta_l xg, gy - \zeta_l^m yg, xy - yx)$. This algebra $\mathbf{h}(\zeta_l, m)$ can be equipped with a Hopf algebra structure with comultiplication, antipode and counit given by

$$\Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + g^m \otimes y, \quad \Delta(g) = g \otimes g$$

$$S(x) = -xg^{-1}, \ \ S(y) = -g^{-m}y, \ \ S(g) = g^{-1}, \ \ \varepsilon(x) = \varepsilon(y) = 0.\ \varepsilon(g) = 1.$$

Note that such a Hopf algebra $\mathbf{h}(\zeta_l, m)$ is called a *book algebra* in [1]. It is a basic algebra since $\mathbf{h}(q, m)/J_{\mathbf{h}(q,m)}$ is a commutative semisimple algebra.

Proposition 2.15. *The basic subalgebra* $\mathrm{B}(\mathrm{Qu}_q(\mathfrak{sl}_2))$ *of* $\mathrm{Qu}_q(\mathfrak{sl}_2)$ *has a Hopf algebra structure such that*

$$\mathrm{B}(\mathrm{Qu}_q(\mathfrak{sl}_2)) \cong \boldsymbol{h}(-1, 1) \otimes k\mathbb{Z}_{\frac{n^2}{2}} \quad \text{as Hopf algebras.}$$

Proof. By Corollary 2.14 (f), we have an algebra isomorphism:

$$\mathrm{B}(\mathrm{Qu}_q(\mathfrak{sl}_2)) \cong \Lambda^{(\frac{n^2}{2})}.$$

It is not hard to see that $\Lambda \cong \mathbf{h}(-1, 1)$. This implies that the basic algebra of each block can be equipped with a Hopf algebra structure, i.e., the book algebra $\mathbf{h}(-1, 1)$. Therefore, we have an isomorphism of Hopf algebras:

$$\mathrm{B}(\mathrm{Qu}_q(\mathfrak{sl}_2)) \cong \mathbf{h}(-1, 1) \otimes k\mathbb{Z}_{\frac{n^2}{2}}.$$

$\square$

Contrasting to this, we have the following.

Proposition 2.16. *There is no Hopf algebra structure on the basic algebras* $\mathrm{B}(\mathbf{u}_q(\mathfrak{sl}_2))$ *and* $\mathrm{B}(\mathfrak{U}_q(\mathfrak{sl}_2))$.

Proof. Let H be a finite-dimensional Hopf algebra such that the underlying algebra is basic. By [6, Thm. 2.3], the Ext-quiver of H must be a *covering quiver* (see [6] for the definition), or equivalently a *Hopf quiver* (see [2]). By the definition of the covering quiver, we know that the Ext-quivers of all blocks of H are isomorphic as direct graphs.

Now let $H = \mathbf{u}_q(\mathfrak{sl}_2)$ or $H = \mathfrak{U}_q(\mathfrak{sl}_2)$. Then it is well-known that H contains a Steinberg module. Therefore, the block B_s containing this Steinberg module is Morita equivalent to k. Thus its Ext-quiver is just a point. So if there is a Hopf structure on $\mathrm{B}(H)$, then all blocks are simple algebras by the foregoing argument. It follows that H is semisimple, which is absurd. $\square$

Remark 2.17. (a) In the classical $\mathbf{u}_q(\mathfrak{sl}_2)$ case or the restricted quantum universal enveloping algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ case, the order of the group-like element K (see [10, 11] for the definitions of $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$) is high enough to distinguish between different vectors in an indecomposable projective module and differentiate one projective module from another. However, in the $\mathrm{Qu}_q(\mathfrak{sl}_2)$ case, we lose this convenient tool partly because the orders of the group-like elements κ and $\hat{\kappa}$ are not high enough. Fortunately, we can still use them to differentiate two non-isomorphic projective modules.

(b) If $2|n$ while $4 \nmid n$, one can use $e'_{2i,0}, e'_{2i,1}$ defined in Remark 2.3 and the same procedure we developed to define projective modules and simple modules. Moreover all conclusions in Corollary 2.14 still hold. We leave the proof for the interested reader.

(c) There are two apparent differences between the representations of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ and those of the classical $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$. Namely, (I): $\mathrm{Qu}_q(\mathfrak{sl}_2)$ has no Steinberg modules; (II): all simple modules of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ are of odd dimensions. On the other hand, it is well-known that both $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$ have Steinberg modules and the dimensions of the simple modules may be even.

3. Indecomposable modules

Let Λ be the basic algebra given in Corollary 2.14 (f). The Auslander-Reiten quiver Γ_Λ of Λ is known. Doubling the following picture we obtain Γ_Λ:
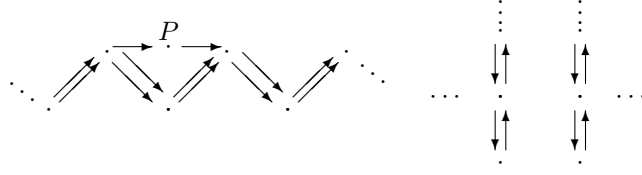
A $\mathbb{P}^1 k$ family of homogeneous tubes

To give all indecomposables, we only need to construct indecomposable $\mathbf{Qu}_q(\mathfrak{sl}_2)$-modules corresponding to dots in Γ_Λ by Corollary 2.14(f). The construction is parallel to [12, Sec. 4] and [11, Sec. 5]. So we omit the proof here and state the results directly.

3.1. **The indecomposable modules** $V_l^{2i,j}$. For any non-negative integer l and $1 \leq i \leq \frac{n^2}{2}, j = 0, 1$, the indecomposable module $V_l^{2i,j}$ has a basis:

$$\{a_u(m-1), e_v(m)|0 \leq m \leq l,\ 0 \leq u \leq n^2 - 2i,\ 1 \leq v \leq 2i - 1\}$$

with the action given by

$$\kappa^{-1}\hat{\kappa}e_v(m) = q^{2i-2v}e_v(m),\ \ \kappa\hat{\kappa}^{\frac{n}{2}}e_v(m) = (-1)^{v+j}e_v(m)$$
$$Ee_v(m) = e_{v+1}(m),$$
$$Fe_v(m) = (v-1)_{q^{-1}}(1-q^{2i-v})e_{v-1}(m) + \delta_{v,1}a_{n^2-2i}(m-1)$$

and

$$\kappa^{-1}\hat{\kappa}a_u(m-1) = q^{-2i-2u}a_u(m-1),\ \ \kappa\hat{\kappa}^{\frac{n}{2}}a_u(m-1) = (-1)^{u+j}a_u(m-1)$$
$$Ea_u(m-1) = a_{u+1}(m-1),$$
$$Fa_u(m-1) = (u+2i-1)_{q^{-1}}(1-q^{-u})a_{u-1}(m-1),$$

where $a_{n^2-2i+1}(m-1) = a_{-1}(m-1) = a_u(-1) = e_0(m) = 0$ and $e_{2i}(m) = a_0(m)$. It may be useful to depict this module by means of diagram:
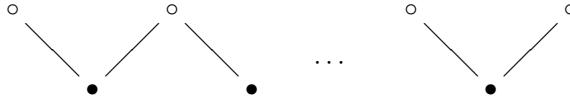
There are l copies of $\bullet$ and $l+1$ copies of $\circ$. In comparing with the diagrams displayed earlier, $\bullet$ and $\circ$ stand for $S_{2i,j}$ and $S_{n^2-2i+2,j'}$ respectively where $j'+j = 1$. The lines / and \ may be understood as the actions of E and F respectively.

They form all syzygies of simple modules. Indeed, we have:

$$\Omega^l(S_{2i,j}) = \begin{cases} V_l^{2i,j}, & l \text{ is odd} \\ V_l^{n^2-2i+2,j'}, & l \text{ is even,} \end{cases}$$

where j' is determined by requiring $j' + j = 1$. Here $\Omega^l(S_{2i,j})$ denotes the l-th syzygy of $S_{2i,j}$.

3.2. **The indecomposable modules $\widetilde{V}_n^{2i,j}$.** For any non-negative integer l and $1 \leq i \leq \frac{n^2}{2}, j = 0, 1$, the indecomposable modules $\widetilde{V}_n^{2i,j}$ has a basis:

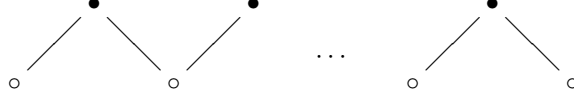$$\{a_u(m+1), e_v(m) | 0 \leq m \leq l, \ 0 \leq u \leq n^2 - 2i, \ 1 \leq v \leq 2i - 1\}$$

with the action given by

$$\kappa^{-1}\hat{\kappa}e_v(m) = q^{2i-2v}e_v(m), \quad \kappa\hat{\kappa}^{\frac{n}{2}}e_v(m) = (-1)^{v+j}e_v(m),$$

$$Ee_v(m) = e_{v+1}(m),$$

$$Fe_v(m) = (v-1)_{q^{-1}}(1 - q^{2i-v})e_{v-1}(m),$$

and

$$\kappa^{-1}\hat{\kappa}a_u(m+1) = q^{-2i-2u}a_u(m+1), \quad \kappa\hat{\kappa}^{\frac{n}{2}}a_u(m+1) = (-1)^{u+j}a_u(m+1),$$

$$Ea_u(m+1) = a_{u+1}(m+1),$$

$$Fa_u(m+1) = (u+2i-1)_{q^{-1}}(1-q^{-u})a_{u-1}(m+1) + \delta_{u,0}e_{2i-1}(m),$$

where $a_{-1}(m) = a_u(0) = e_{-1}(m) = e_{2i}(m) = 0$ and $a_{n^2-2i+2}(m) = e_1(m)$. This module can be described schematically as follows:

There are l copies of $\bullet$ and $l+1$ copies of $\circ$. The relation with cosyzygies is:

$$\Omega^{-l}(\mathrm{S}_{2i,j}) = \begin{cases} \widetilde{V}_l^{2i,j} & l \text{ is odd} \\ \widetilde{V}_l^{n^2-2i+2,j'} & l \text{ is even,} \end{cases}$$

where j' is determined by the requirement $j' + j = 1$. Here $\Omega^{-l}(\mathrm{S}_{2i,j})$ denotes the l-th cosyzygy of $\mathrm{S}_{2i,j}$.
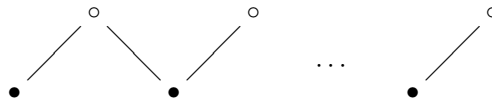
3.3. **The indecomposable modules $W_l^{2i,j}$.** For any positive integer l and $1 \leq i \leq \frac{n^2}{2}, j = 0, 1$, one has a basis of $W_l^{2i,j}$ as follows:

$$\{e_u(m) | 1 \leq m \leq l, \ 1 \leq u \leq n^2\}$$

with the action given by

$$\kappa^{-1}\hat{\kappa}e_u(m) = q^{2i-2u}e_u(m), \quad \kappa\hat{\kappa}^{\frac{n}{2}}e_u(m) = (-1)^{u+j}e_u(m),$$

$$Ee_u(m) = e_{u+1}(m),$$

$$Fe_u(m) = (u-1)_{q^{-1}}(1 - q^{2i-u})e_{u-1}(m) + \delta_{u,1}e_{n^2}(m-1),$$

where $e_{n^2+1}(m) = e_0(m) = e_u(0) = 0$. The diagram of this module is:

There are l copies of $\bullet$ and l copies of $\circ$. The modules constructed in this subsection correspond to those parametrized by $\lambda = [1, 0] \in \mathbb{P}^1 k$.
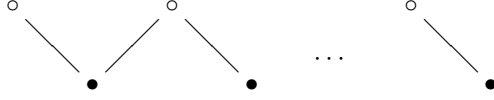
3.4. **The indecomposable modules** $\widetilde{W}_l^{2i,j}$. For any positive integer l and $1 \leq i \leq \frac{n^2}{2}, j = 0, 1$, one has a basis of $\widetilde{W}_l^{2i,j}$:

$$\{f_u(m)|1 \leq m \leq l-1,\ 1 \leq u \leq n^2\} \cup \{f_u(n)|1 \leq u \leq 2i-1\} \cup \{f_u(0)|2i \leq u \leq n^2\}$$

with the action given by

$$\kappa^{-1}\hat{\kappa}f_u(m) = q^{2i-2u}f_u(m),\ \ \kappa\hat{\kappa}^{\frac{n}{2}}f_u(m) = (-1)^{u+j}f_u(m),$$
$$Ef_u(m) = f_{u+1}(m),$$
$$Ff_u(m) = (u-1)_{q^{-1}}(1-q^{2i-u})f_{u-1}(m) + \delta_{u,1}f_{n^2}(m-1),$$

where $f_{n^2+1}(m) = f_0(m) = f_{2i}(n) = 0$. The diagram of this module is give by

There are l copies of $\bullet$ and l copies of $\circ$. The modules constructed in this subsection correspond to those parametrized by $\lambda = [0,1] \in \mathbb{P}^1 k$.

3.5. **The indecomposable modules** $T_l^{2i,j}(\lambda)$. For any positive integer l, $1 \leq i \leq \frac{n^2}{2}, j = 0, 1$ and $\lambda \in k^*$, the indecomposable modules $T_l^{2i,j}(\lambda)$ has a basis:

$$\{a_u(m), e_v(m)|1 \leq m \leq l,\ 0 \leq u \leq n^2 - 2i,\ 1 \leq v \leq 2i-1\}$$

with the action given by

$$\kappa^{-1}\hat{\kappa}e_v(m) = q^{2i-2v}e_v(m),\ \ \kappa\hat{\kappa}^{\frac{n}{2}}e_v(m) = (-1)^{v+j}e_v(m),$$
$$Ee_v(m) = \begin{cases} e_{v+1}(m), & v \neq 2i-1 \\ a_0(m), & v = 2i-1, \end{cases}$$
$$Fe_v(m) = \begin{cases} (v-1)_{q^{-1}}(1-q^{2i-v})e_{v-1}(m), & v \neq 1 \\ \lambda a_{n^2-2i}(m) + a_{n^2-2i}(m-1), & v = 1, \end{cases}$$

and

$$\kappa^{-1}\hat{\kappa}a_u(m) = q^{-2i-2u}a_u(m),\ \ \kappa\hat{\kappa}^{\frac{n}{2}}a_u(m) = (-1)^{u+j}a_u(m),$$
$$Ea_u(m) = \begin{cases} a_{u+1}(m), & u \neq n^2 - 2i \\ 0, & u = n^2 - 2i, \end{cases}$$
$$Fa_u(m) = \begin{cases} (u+2i-1)_{q^{-1}}(1-q^{-u})a_{u-1}(m), & u \neq 0 \\ 0, & u = 0, \end{cases}$$

where $a_u(-1) = 0$ for all $0 \leq u \leq n^2 - 2i$. The indecomposable modules $T_l^{2i,j}(\lambda)$ correspond to those parametrized by $\lambda' = [1, \lambda] \in \mathbb{P}^1 k$.

3.6. **Auslander-Reiten sequences.** We have the following Auslander-Reiten sequences:

$$0 \longrightarrow V_{l+2}^{2i,j} \longrightarrow V_{l+1}^{2i,j} \oplus V_{l+1}^{2i,j} \longrightarrow V_l^{2i,j} \longrightarrow 0,$$

$$0 \longrightarrow \widetilde{V}_l^{2i,j} \longrightarrow \widetilde{V}_{l+1}^{2i,j} \oplus \widetilde{V}_{l+1}^{2i,j} \longrightarrow \widetilde{V}_{l+2}^{2i,j} \longrightarrow 0,$$

$$0 \longrightarrow V_1^{2i,j} \longrightarrow \mathrm{P}_{2i,j} \oplus \mathrm{S}_{n^2-2i+2,j'} \oplus \mathrm{S}_{n^2-2i+2,j'} \longrightarrow \widetilde{V}_1^{2i,j} \longrightarrow 0,$$

$$0 \longrightarrow W_l^{2i,j} \longrightarrow W_{l+1}^{2i,j} \oplus W_{l-1}^{2i,j} \longrightarrow W_l^{2i,j} \longrightarrow 0,$$

$$0 \longrightarrow \widetilde{W}_l^{2i,j} \longrightarrow \widetilde{W}_{l+1}^{2i,j} \oplus \widetilde{W}_{l-1}^{2i,j} \longrightarrow \widetilde{W}_l^{2i,j} \longrightarrow 0,$$

$$0 \longrightarrow W_l^{2i,j}(\lambda) \longrightarrow W_{l+1}^{2i,j}(\lambda) \oplus W_{l-1}^{2i,j}(\lambda) \longrightarrow W_l^{2i,j}(\lambda) \longrightarrow 0.$$

Comparing with the Auslander-Reiten quiver Γ_Λ, we obtain the following:

Theorem 3.1. *The modules*

- $\mathrm{P}_{2i,j}, \quad 1 \le i \le \frac{n^2}{2}, \ j = 0,1;$
- $V_l^{2i,j}, \ \widetilde{V}_l^{2i,j} \quad 1 \le i \le \frac{n^2}{2}, \ j = 0,1, \ l \ge 0;$
- $W_l^{2i,j}, \ \widetilde{W}_l^{2i,j} \quad 1 \le i \le \frac{n^2}{2}, \ j = 0,1, \ l \ge 1;$
- $T_l^{2i,j}(\lambda), \quad 1 \le i \le \frac{n^2}{2}, \ j = 0,1, \ l \ge 0, \ \lambda \in k^*$

form a complete list of finite-dimensional indecomposable $\mathrm{Qu}_q(\mathfrak{sl}_2)$-modules.

4. Tensor products

The aim of this section is to give the tensor product decomposition formula for simple modules and projective modules. Looking at the definition of $\mathrm{Qu}_q(\mathfrak{sl}_2)$, one may find that its comultiplication is more complicated than the one of $\mathbf{u}_q(\mathfrak{sl}_2)$. However, we still have that this comultiplication preserves some kinds of "grading", described in detail as follows:

Let ζ_{2n^2} be a $2n^2$-th primitive root of unity. We define an algebraic automorphism σ of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ as follows:

$$\sigma(\kappa) := \kappa, \ \ \sigma(\hat{\kappa}) := \hat{\kappa}, \ \ \sigma(E) = \zeta_{2n^2}E, \ \ \sigma(F) = \zeta_{2n^2}^{-1}F.$$

The promised grading is the decomposition of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ into σ-eigenspaces:

$$\mathrm{Qu}_q(\mathfrak{sl}_2) = \bigoplus_{s=-(n^2-1)}^{n^2-1} \mathrm{Qu}_q(\mathfrak{sl}_2)_s$$

where

$$\mathrm{Qu}_q(\mathfrak{sl}_2)_s := \bigoplus_{b-a=s, 0 \le a,b \le n^2-1} F^a \mathbf{u}^0 E^b.$$

We say that the elements in $\mathrm{Qu}_q(\mathfrak{sl}_2)_s$ have *height s*. As usual, the grading on $\mathrm{Qu}_q(\mathfrak{sl}_2) \otimes \mathrm{Qu}_q(\mathfrak{sl}_2)$ is given by

$$\mathrm{Qu}_q(\mathfrak{sl}_2) \otimes \mathrm{Qu}_q(\mathfrak{sl}_2) = \bigoplus_s (\mathrm{Qu}_q(\mathfrak{sl}_2) \otimes \mathrm{Qu}_q(\mathfrak{sl}_2))_s := \bigoplus_{u+v=s} \mathrm{Qu}_q(\mathfrak{sl}_2)_u \otimes \mathrm{Qu}_q(\mathfrak{sl}_2)_v.$$

The following conclusion is clear.

Lemma 4.1. *For* $1 - n^2 \le s \le n^2 - 1$, *we have:*

$$\Delta(\mathbf{Qu}_q(\mathfrak{sl}_2)_s) \subseteq (\mathbf{Qu}_q(\mathfrak{sl}_2) \otimes \mathbf{Qu}_q(\mathfrak{sl}_2))_s.$$

Recall the definition of $\mathrm{P}_{2i,j}$, the set $\{E^l\alpha_{2i,j}, E^l\gamma_{2i,j} | 0 \le l \le n^2 - 1\}$ forms a basis of $\mathrm{P}_{2i,j}$. We already know that $E^l\alpha_{2i,j}, E^l\gamma_{2i,j}$ are $\kappa^{-1}\hat{\kappa}, \kappa\hat{\kappa}^{\frac{n}{2}}$-eigenvectors. Moreover, it is not hard to see that $E^l\alpha_{2i,j} \in \mathbf{Qu}_q(\mathfrak{sl}_2)_{l+1-n^2}$ and $E^l\gamma_{2i,j} \in \mathbf{Qu}_q(\mathfrak{sl}_2)_{l+2i-n^2}$. The same argument can be applied to $\mathrm{S}_{2i,j}$. Therefore, by regarding $\mathrm{P}_{2i,j}$ and $\mathrm{S}_{2i,j}$ as subspaces of $\mathbf{Qu}_q(\mathfrak{sl}_2)$, one sees that they consist of homogeneous elements and thus they are graded according to the heights. It follows that, for $1 \le i_1, i_2 \le \frac{n^2}{2}$ and $0 \le j_1, j_2 \le 1$, we have:

$$\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} = \bigoplus_s (\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2})_s.$$

By Lemma 4.1, E (resp. F) maps $(\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2})_s$ to $(\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2})_{s+1}$ (resp. $(\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2})_{s-1}$) by left multiplication.

Let H be a finite-dimensional quasi-Hopf algebra and M_1, M_2 two H-modules. It is well-known that $M_1 \otimes M_2$ is projective if either M_1 or M_2 is projective. This implies that $\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$ is a direct sum of indecomposable projective modules. So we have the following *method* to compute $\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$: A $\kappa^{-1}\hat{\kappa}, \kappa\hat{\kappa}^{\frac{n}{2}}$-eigenvector v of lowest height determines an indecomposable projective module P_v, which is a summand of $\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$. We then delete the corresponding $\kappa^{-1}\hat{\kappa}, \kappa\hat{\kappa}^{\frac{n}{2}}$-eigenvectors in P_v. Continue this process until there is no any other $\kappa^{-1}\hat{\kappa}, \kappa\hat{\kappa}^{\frac{n}{2}}$-eigenvector.

In the rest of this section, $\bar{i}$ denotes the least non-negative residue of i modulo 2 and $2\mathrm{P}$ stands for $\mathrm{P} \oplus \mathrm{P}$. Using the method we just introduced, we obtain the following decomposition rules.

Theorem 4.2. *For* $1 \le i_1, i_2 \le \frac{n^2}{2}$ *and* $0 \le j_1, j_2 \le 1$, *we have:*

(a) *If* $2i_1 - 1 \ge n^2 - 2i_2 + 1$ *and* $i_1 \le i_2$, *then*

$$\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} \cong \bigoplus_{l=0}^{n^2 - 2i_2} \mathrm{P}_{n^2 + 2i_1 - 2i_2 - 2l, \overline{j_1 + j_2 + l}}.$$

(b) *If* $2i_1 - 1 \ge n^2 - 2i_2 + 1$ *and* $i_1 > i_2$, *then*

$$\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$$
$$\cong \bigoplus_{l=0}^{i_1 - i_2 - 1} 2\,\mathrm{P}_{2i_1 - 2i_2 - 2l, \overline{j_1 + j_2 + l}} \oplus \bigoplus_{l=0}^{n^2 - 2i_1} \mathrm{P}_{n^2 - 2i_1 + 2i_2 - 2l, \overline{j_1 + j_2 + l}}.$$

(c) *If* $2i_1 - 1 < n^2 - 2i_2 + 1$ *and* $i_1 \le i_2$, *then*

$$\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$$
$$\cong \bigoplus_{l=0}^{2i_1 - 2} \mathrm{P}_{n^2 + 2i_1 - 2i_2 - 2l, \overline{j_1 + j_2 + l}} \oplus \bigoplus_{l=0}^{\frac{n^2}{2} - i_1 - i_2} 2\,\mathrm{P}_{n^2 - 2i_1 - 2i_2 + 2 - 2l, \overline{j_1 + j_2 + l - 1}}.$$

(d) *If $2i_1 - 1 < n^2 - 2i_2 + 1$ and $i_1 > i_2$, then*

$$P_{2i_1,j_1} \otimes S_{2i_2,j_2}$$

$$\cong \bigoplus_{l=0}^{i_1-i_2-1} 2\,P_{2i_1-2i_2-2l,\overline{j_1+j_2+l}} \oplus \bigoplus_{l=0}^{2i_2-2} P_{n^2-2i_1+2i_2-2l,\overline{j_1+j_2+l}}$$

$$\oplus \bigoplus_{l=0}^{\frac{n^2}{2}-i_1-i_2} 2\,P_{n^2-2i_1-2i_2+2-2l,\overline{j_1+j_2+l-1}}\,.$$

Proof. We want to equip every module considered with a formal function that can reflect heights and $\kappa^{-1}\hat{\kappa}, \kappa\hat{\kappa}^{\frac{n}{2}}$-eigenvalues. Let

$y^?$:   height of a vector;

$(q^{-2})^?$: $\kappa^{-1}\hat{\kappa}$ − eigenvalue of a vector;

$(-1)^?$: $\kappa\hat{\kappa}^{\frac{n}{2}}$ − eigenvalue of a vector.

Then the formal function $\eta(S_{2i,j})$ associated to $S_{2i,j}$ is

$$\eta(S_{2i,j}) \quad = \quad \sum_{l=0}^{n^2-2i} (q^{-2i-2l}y^l, (-1)^l y^l).$$

And

$$\eta(P_{2i,j}) = (1 + y^{n^2})\eta(S_{n^2-2i+2,j'}) + 2y^{2i-1}\eta(S_{2i,j}).$$

In case $2i_1 - 1 \geq n^2 - 2i_2 + 1$ and $i_1 \leq i_2$, it is immediate that we have:

$$\eta(P_{2i_1,j_1})\eta(S_{2i_2,j_2}) = \sum_{l=0}^{n^2-2i_2} y^l \eta(P_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}}).$$

This gives the decomposition rules in Part (a).

For Part (b), we have the equation of the height functions:

$$\eta(P_{2i_1,j_1})\eta(S_{2i_2,j_2}) \quad = \quad (1 + y^{n^2}) \sum_{l=0}^{i_1-i_2-1} y^l \eta(P_{2i_1-2i_2-2l,\overline{j_1+j_2+l}})$$

$$+ \sum_{l=0}^{n^2-2i_1} y^{2i_1-2i_2+l} \eta(P_{n^2-2i_1+2i_2-2l,\overline{j_1+j_2+l}}).$$

For Part (c), the following equation holds:

$$\eta(P_{2i_1,j_1})\eta(S_{2i_2,j_2}) \quad = \quad \sum_{l=0}^{2i_1-2} y^l \eta(P_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}})$$

$$+2 \sum_{l=0}^{\frac{n^2}{2}-i_1-i_2} y^{2i_1-1+l} \eta(P_{n^2-2i_1-2i_2+2-2l,\overline{j_1+j_2+l-1}}).$$

For Part (d), we have:

$$
\begin{aligned}
\eta(\mathrm{P}_{2i_1,j_1})\eta(\mathrm{S}_{2i_2,j_2}) \;=\;& (1+y^{n^2})\sum_{l=0}^{i_1-i_2-1} y^l \eta(\mathrm{P}_{2i_1-2i_2-2l,\overline{j_1+j_2+l}}) \\
&+y^{2i_1-2i_2}\sum_{l=0}^{2i_2-2} y^l \eta(\mathrm{P}_{n^2-2i_1+2i_2-2l,\overline{j_1+j_2+l}}) \\
&+2y^{2i_1-1}\sum_{l=0}^{\frac{n^2}{2}-i_1-i_2} y^l \eta(\mathrm{P}_{n^2-2i_1-2i_2+2-2l,\overline{j_1+j_2+l-1}}).
\end{aligned}
$$

$\square$

Remark 4.3. Let H be a finite-dimensional quasi-Hopf algebra and assume $0 \to M_1 \to M_2 \to M_3 \to 0$ to be a short exact sequence of H-modules. Tensoring the sequence with a projective H-module P, we get a split exact sequence: $0 \to P{\otimes}M_1 \to P{\otimes}M_2 \to P{\otimes}M_3 \to 0$. Therefore, Theorem 4.2 gives the decomposition formulas for the tensor product of a projective $\mathbf{Qu}_q(\mathfrak{sl}_2)$-module with any other finite-dimensional $\mathbf{Qu}_q(\mathfrak{sl}_2)$-module. In fact, let M be an arbitrary $\mathbf{Qu}_q(\mathfrak{sl}_2)$-module and $\bigoplus_l \mathrm{S}_l$ the direct sum of its composition factors. Then we have:

$$
\mathrm{P}\otimes M \cong \bigoplus_l \mathrm{P}\otimes \mathrm{S}_l\,.
$$

Denote by K_0 the Grothendieck ring of $\mathbf{Qu}_q(\mathfrak{sl}_2)$. We want to characterize the ring structure of K_0.

Lemma 4.4. *Let $1 \le i_1, i_2 \le \frac{n^2}{2}$ be two positive integers and $0 \le j_1, j_2 \le 1$. If $2i_1-1 \ge n^2-2i_2+1$ and $i_1 \le i_2$, then*

$$
\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} \cong \bigoplus_{l=0}^{n^2-2i_2} \mathrm{S}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}}\,.
$$

Proof. The notation introduced in the proof of Theorem 4.2 will be used freely. Since $\mathrm{Soc}(\mathrm{P}_{2i_1,j_1}) = \mathrm{S}_{2i_1,j_1}$, we have a natural embedding:

$$
\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} \hookrightarrow \mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} \cong \bigoplus_{l=0}^{n^2-2i_2} \mathrm{P}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}}\,.
$$

Thanks to the embedding, we have the equation:

$$
(4.1) \qquad y^{2i_1-1}\eta(\mathrm{S}_{2i_1,j_1})\eta(\mathrm{S}_{2i_2,j_2}) = \sum_{l=0}^{n^2-2i_2} y^l \eta(\mathrm{S}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}}).
$$

It follows that

$$
[\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}] = [\bigoplus_{l=0}^{n^2-2i_2} \mathrm{S}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}}]
$$

in K_0. Assume that $\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} = \bigoplus_t M_t$ is the decomposition of $\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$ into indecomposables. So it is enough to show that every M_t is simple. Otherwise, there would be a M_t containing at least two simples as composition factors. Since M_t is a submodule of $\bigoplus_{l=0}^{n^2-2i_2} \mathrm{P}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}}$, M_t contains a Verma submodule. But every Verma submodule of $\bigoplus_{l=0}^{n^2-2i_2} \mathrm{P}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}}$ must contain a $\kappa^{-1}\hat\kappa, \kappa\hat\kappa^{\frac{n}{2}}$-eigenvector v with height $\le n^2 - 2i_2$. This contradicts to Equation (4.1) since the height of any $\kappa^{-1}\hat\kappa, \kappa\hat\kappa^{\frac{n}{2}}$-eigenvector in $\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$ is at least $2i_1 - 1$ which is bigger than $n^2 - 2i_2$. $\square$

As a consequence of Lemma 4.4, we obtain the following.

Corollary 4.5. *For $0 \leq j_1, j_2 \leq 1$ and $2 \leq i \leq \frac{n^2}{2} - 1$, we have:*

$$\mathrm{S}_{2i,j_1} \otimes \mathrm{S}_{n^2-2,j_2} \cong \mathrm{S}_{2i+2,\overline{j_1+j_2}} \oplus \mathrm{S}_{2i,\overline{j_1+j_2+1}} \oplus \mathrm{S}_{2i-2,\overline{j_1+j_2}} \, .$$

Lemma 4.6. *For $0 \leq j_1, j_2 \leq 1$, we have:*

$$\begin{aligned}
\mathrm{S}_{n^2,j_1} \otimes \mathrm{S}_{n^2-2,j_2} &\cong \mathrm{S}_{n^2-2,\overline{j_1+j_2}} \\
\mathrm{S}_{2,j_1} \otimes \mathrm{S}_{n^2-2,j_2} &\cong \mathrm{S}_{4,\overline{j_1+j_2}} \oplus \mathrm{P}_{2,\overline{j_1+j_2+1}} \, .
\end{aligned}$$

Proof. The first isomorphism is clear and we prove the second one. Using the same method demonstrated in the proof of Lemma 4.4, we obtain:

$$[\mathrm{S}_{2,j_1} \otimes \mathrm{S}_{n^2-2,j_2}] = [\mathrm{S}_{4,\overline{j_1+j_2}}] + [\mathrm{P}_{2,\overline{j_1+j_2+1}}]$$

in K_0. Since $\mathrm{S}_{4,\overline{j_1+j_2}}$ and $\mathrm{P}_{2,\overline{j_1+j_2+1}}$ belong to different blocks, we know that $\mathrm{S}_{4,\overline{j_1+j_2}}$ is a direct summand of $\mathrm{S}_{2,j_1} \otimes \mathrm{S}_{n^2-2,j_2}$. That is,

$$\mathrm{S}_{2,j_1} \otimes \mathrm{S}_{n^2-2,j_2} \cong \mathrm{S}_{4,\overline{j_1+j_2}} \oplus M$$

with $[M] = [\mathrm{P}_{2,\overline{j_1+j_2+1}}]$. By Theorem 4.2 (c), we have:

$$\mathrm{P}_{4,\overline{j_1+j_2}} \oplus 2\,\mathrm{P}_{2,\overline{j_1+j_2+1}} \cong \mathrm{P}_{2,j_1} \otimes \mathrm{S}_{n^2-2,j_2} \twoheadrightarrow \mathrm{S}_{2,j_1} \otimes \mathrm{S}_{n^2-2,j_2} \cong \mathrm{S}_{4,\overline{j_1+j_2}} \oplus M.$$

Since $\mathrm{P}_{4,\overline{j_1+j_2}}$ and $\mathrm{P}_{2,\overline{j_1+j_2+1}}$ belong to different block, we obtain:

$$2\,\mathrm{P}_{2,\overline{j_1+j_2+1}} \twoheadrightarrow M.$$

If $\mathrm{P}_{2,\overline{j_1+j_2+1}} \not\twoheadrightarrow M$, then $M/JM \cong 2\,\mathrm{P}_{2,\overline{j_1+j_2+1}} / J(2\,\mathrm{P}_{2,\overline{j_1+j_2+1}})$. Here J denotes the Jacobson radical of $\mathrm{Qu}_q(\mathfrak{sl}_2)$. Thus $\dim M/JM = 2(n^2 - 1)$ implying $\mathrm{S}_{n^2,\overline{j_1+j_2}} \subseteq \mathrm{Soc}(M)$. This is impossible since we also have $\mathrm{Soc}(M) \hookrightarrow 2\,\mathrm{P}_{2,\overline{j_1+j_2+1}}$. So $\mathrm{P}_{2,\overline{j_1+j_2+1}} \twoheadrightarrow M$. It follows that $\mathrm{P}_{2,\overline{j_1+j_2+1}} \cong M$ since $\dim \mathrm{P}_{2,\overline{j_1+j_2+1}} = \dim M$. $\square$

We obtain the following basic observation, which is consistent with the classical $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$ cases.

Proposition 4.7. *Let $1 \leq i_1, i_2 \leq \frac{n^2}{2}$ be two positive integers and $0 \leq j_1, j_2 \leq 1$. The indecomposable direct summands of $\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$ are either simple or projective.*

Proof. By Corollary 4.5, both S_{2i_1,j_1} and S_{2i_2,j_2} occur as direct summands of suitable tensor powers of $\mathrm{S}_{n^2-2,j}$ for $j = 0$ or $j = 1$. Thus by combining Corollary 4.5 and Lemma 4.6, we get the desired conclusion. $\square$

Combing Theorem 4.2 with Proposition 4.7, we obtain the following refined decomposition rules:

Theorem 4.8. *For $1 \leq i_1, i_2 \leq \frac{n^2}{2}$ and $0 \leq j_1, j_2 \leq 1$, we have:*

(a) *If $2i_1 - 1 \geq n^2 - 2i_2 + 1$ and $i_1 \leq i_2$, then*

$$\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} \cong \bigoplus_{l=0}^{n^2-2i_2} \mathrm{S}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}} \, .$$

(b) *If $2i_1 - 1 \geq n^2 - 2i_2 + 1$ and $i_1 > i_2$, then*

$$\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2} \cong \bigoplus_{l=0}^{n^2-2i_1} \mathrm{S}_{n^2-2i_1+2i_2-2l,\overline{j_1+j_2+l}}.$$

(c) *If $2i_1 - 1 < n^2 - 2i_2 + 1$ and $i_1 \leq i_2$, then*

$$\mathrm{S}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$$
$$\cong \bigoplus_{l=0}^{2i_1-2} \mathrm{S}_{n^2+2i_1-2i_2-2l,\overline{j_1+j_2+l}} \oplus \bigoplus_{l=0}^{\frac{n^2}{2}-i_1-i_2} \mathrm{P}_{n^2-2i_1-2i_2+2-2l,\overline{j_1+j_2+l-1}}.$$

(d) *If $2i_1 - 1 < n^2 - 2i_2 + 1$ and $i_1 > i_2$, then*

$$\mathrm{P}_{2i_1,j_1} \otimes \mathrm{S}_{2i_2,j_2}$$
$$\cong \bigoplus_{l=0}^{2i_2-2} \mathrm{S}_{n^2-2i_1+2i_2-2l,\overline{j_1+j_2+l}} \oplus \bigoplus_{l=0}^{\frac{n^2}{2}-i_1-i_2} \mathrm{P}_{n^2-2i_1-2i_2+2-2l,\overline{j_1+j_2+l-1}}.$$

Now we are ready to describe the Grothendieck ring K_0. Let $\mathbb{Z}[g,x]$ be the polynomial algebra over $\mathbb{Z}$ in two variables g and x. We define polynomials $f_{2m,j} \in \mathbb{Z}[g,x]$ inductively for $0 \leq m \leq \frac{n^2}{2} - 1$, $j = 0,1$ as follows:

$$f_{0,0} = 1, \quad f_{0,1} = g, \quad f_{2,0} = x, \quad f_{2,1} = xg,$$
$$f_{4,0} = x^2 - xg - 1, \quad f_{4,1} = x^2g - x - g.$$
Assume $\quad f_{2m,j} = x^m g^j - a_1 f_{2(m-1),\overline{j+1}} - a_2 f_{2(m-2),\overline{j+2}} - \cdots - a_m f_{0,\overline{j+m}}.$ Define
$$f_{2(m+1),j} = x^{m+1}g^j - (1+a_1)f_{2m,\overline{j+1}}$$
$$- \sum_{l=1}^{m-1}(a_{l-1}+a_l+a_{l+1})f_{2(m-l),\overline{j+l+1}} - a_{m-1}f_{0,\overline{j+m+1}},$$

where $a_0 = 1$. Let I be the ideal of $\mathbb{Z}[g,x]$ generated by $g^2 - 1$ and $f_{n^2-2,0}x - 2f_{n^2-2,1} - f_{n^2-4,0} - 2$. We have the following main result:

Theorem 4.9. *The Grothendieck ring K_0 of $\mathrm{Qu}_q(\mathfrak{sl}_2)$ is isomorphic to the quotient ring $\mathbb{Z}[g,x]/I$.*

Proof. Define a ring map

$$\Upsilon: \mathbb{Z}[g,x] \longrightarrow K_0, \quad g \mapsto [\mathrm{S}_{n^2,1}], \quad x \mapsto [\mathrm{S}_{n^2-2,0}].$$

The map Υ is well-defined since K_0 is commutative by Theorem 4.8. Using the first isomorphism in Lemma 4.6 repeatedly, we obtain:

$$\Upsilon(f_{2m,j}) = [\mathrm{S}_{n^2-2m,j}],$$

for $0 \leq m \leq \frac{n^2}{2} - 1$, $j = 0,1$. It follows that Υ is surjective. Now $\mathrm{S}_{n^2,1} \otimes \mathrm{S}_{n^2,1} = \mathrm{S}_{n^2,0}$ implies that $\Upsilon(g^2 - 1) = 0$. Applying the second isomorphism in Lemma 4.6, we obtain $\Upsilon(f_{n^2-2,0}x - 2f_{n^2-2,1} - f_{n^2-4,0} - 2) = 0$. Therefore, Υ induces an ephimorphism of rings:

$$\bar{\Upsilon}: \mathbb{Z}[g,x]/I \twoheadrightarrow K_0.$$

It remains to show that Υ is injective. For convenience, we denote the generators of $\mathbb{Z}[g,x]/I$ still by g and x. Observe that $\{[\mathrm{S}_{n^2,1}]^j[\mathrm{S}_{n^2-2,0}]^i|0 \le j \le 1, 0 \le i \le \frac{n^2}{2}-1\}$ is a $\mathbb{Z}$-basis of K_0. From this, we can define a $\mathbb{Z}$-linear map:

$$\Psi: \; K_0 \longrightarrow \mathbb{Z}[g,x]/I, \qquad [\mathrm{S}_{n^2,1}]^j[\mathrm{S}_{n^2-2},0]^i \mapsto g^j x^i.$$

It is not hard to check that $\Psi\bar{\Upsilon} = \mathrm{id}$. Hence, $\bar{\Upsilon}$ is injective. $\qquad\square$

Acknowledgments

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